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In Memoriam of Yuri M. Grigor'ev: An Overview of his Research

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Abstract. On August 22, 2023 we lost our dear friend and esteemed colleague Prof. Yuri M. Grigor'ev has passed away. Throughout his scientific career, he made remarkable contributions to the field of hypercomplex analysis, particularly in advancing applications of quaternionic analysis to various problems of mathematical physics. In this paper, we aim to honour his memory by presenting an overview of his contributions to quaternionic analysis.

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1. Life of Yuri Grigor'ev

Yuri Michailovich Grigor'ev was born on July 18, 1959 in the Nyurbinsky District of the Sakha Republic, Russia. He attended school in this district, but because of his exceptional talent in physics and mathematics, he was invited to continue his studies at the Republican Physics-Mathematics School, which was affiliated with the State University of Yakutsk. Yuri Grigor'ev successfully graduated from the school in 1976 and subsequently enrolled in the Faculty of Physics at the State University of Yakutsk.

After completing his university study in physics in 1976, he obtained a PhD position at the Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences, which is a part of the Novosibirsk State University. In 1981, Yuri Grigor'ev successfully defended his doctoral dissertation (Candidate of sciences in Russian academic system), which focused on the application of quaternionic functions to solving boundary value

This article is part of the topical collection in honor of Yuri Grigoriev, edited by Dmitrii Legatiuk and Heikki Orelma.

In memory of our dear friend and colleague, Yuri Grigor'ev.

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Figure 1. Photo of Prof. Yuri Grigor'ev (Source: http://yakutia.science/news/64e563c991e831d952d7c94d/Grigorev-Yurii-Mihailovich--18-07-1959-22-08-2023-)

problems of continuum mechanics. It is worth highlighting that this was an extraordinary achievement, especially given that quaternionic and Clifford analysis were still in their early stages of development in Europe and the United States at that time (Fig. 1).

From 1985, Yuri Grigor'ev became a staff member of the State University of Yakutsk (renamed North-Eastern Federal University in 2010), where he spent the remained of his career, progressing from an early-career research to becoming the head of the Chair of Theoretical Physics. In 1999, he defended his habilitation dissertation (Doctor of Sciences in Russian academic system) at the Institute of Computational Mathematics and Mathematical Geophysics of the Siberian Branch of the Russian Academy of Sciences in Novosibirsk. In his habilitation dissertation, Yuri continued to develop the theory of quaternionic functions and their applications to problems of mathematical physics. In particular, in this work a spatial analogue of the famous Kolosov–Muskhelishvili formulae has been introduced. It must be noted, that Yakutsk is geographically remote from Europe, limiting Yuri access to the latest publications in his field. Hence, in some cases, Yuri and his co-authors were developing completely independent parallel theories for the results, which were already known in the wester hypercomplex community. This sometimes led also to amusing situations, as illustrated by the following story from our private communications with him:

> Once Yuri has submitted a paper to a Russian journal published in Moscow. The topic of the paper war related to constructing series expansions in quaternionic analysis. After a while, Yuri received a reviewer report back saying that the paper is very interesting, but unfortunately the results are already known from the book of F. Brackx, R. Delanghe, and F. Sommen [2]. As Yuri said, this was first time he heard about this book and realised that there is a huge community working on hypercomplex analysis.

During his scientific career, Yuri Grigor'ev engaged in various activities at both national and international levels. In 2006, he became the youngest member of the Academy of Sciences of Sakha Republic, which is a branch of the Russian Academy of Sciences. He also served as a member of several scientific councils in Russia. Throughout his career, Yuri supported also the school education in mathematics and physics: he was actively involved in the organisation of national and international Olympiads, as well as supported involvement of people from school into scientific research of the "Small" academy of science (academy of science targeting the school education). From 2013, Yuri was actively involved in the international hypercomplex community: he participated in numerous conferences and workshops, performed several research stays, became an active member of the International Society for Analysis, its Applications and Computation (ISAAC), and published papers in international journals.

In terms of scientific interests, Yuri's work extended beyond hypercomplex analysis. In addition to his previously mentioned contributions, he actively worked also on topics of applied continuum mechanics to address practical problems in permafrost regions; he modelled and developed technologies for cutting diamonds; addressed problems of geophysics. Throughout his scientific career he authored 153 paper as catalogued by the Russian Science Library, 38 paper indexed by Scopus, and 8 listed in zbMath. The discrepancy among these numbers arise from the fact that many of his works were published in Russian and appeared in local journals, which unfortunately were not translated for the international audience.

In this paper, we would like to highlight contributions of Yuri Grigor'ev to the field of quaternionic analysis by recalling his quaternionic analogue of Kolosov-Muskhlishvili formulae. This topic is a bright example of very innovative ideas and independent development in hypercomplex analysis.

2. Preliminaries

As we have already discussed, one the main research field of Yuri Grigor'ev was quaternionic analysis and, particularly, its applications in the theory of linear elasticity. The main focus in these works was to study the classical Lamé equation in $\mathbb{R}^3$:

$$L\mathbf{u}:=(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})-\mu\nabla\times(\nabla\times\mathbf{u})=0,$$

where L is the Lamé operator, μ and λ are the Lamé parameters, which are related to the classical Young modulus and Poisson's ratio. The Lamé equation is also often referred to as the *elastostatic equation* as it models the equilibrium state of an elastic isotropic body $\mathcal{B} \subset \mathbb{R}^3$ without external loading.

The unknown in the Lamé equation is the displacement field $\mathbf{u}\colon \mathcal{B} \to \mathbb{R}^3$, which is expressed in component form as follows

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}.$$

The strong solution of the Lamé equation requires that $\mathbf{u} \in C^2(\mathcal{B})$, which implies that the real-valued component functions u_x, u_y, and u_z are twice continuously differentiable.

To address the Lamé equation in quaternionic setting, Yuri Grigor'ev worked with the following algebraic for the basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ of the space $\mathbb{R}^3$ are given by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Consequently, the set $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ generates the well-known Hamiltonian algebra of quaternions $\mathbb{H}$. The algebra of quaternions is a 4-dimensional associative unitary algebra that differs algebraically from complex numbers in that it is not commutative. A general element of quaternion algebra is of the form

$$q = q_0 + q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k},$$

where q_0, q_x, q_y, q_z are real numbers. The vector part of the quaternion q is $\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$ and it is identified with the usual vector in $\mathbb{R}^3$. So the quaternion can be briefly represented as

$$q = q_0 + \mathbf{q}.$$

The scalar and vector parts of a quaternion q are denoted by $\mathrm{Sc}(q) = q_0$ and $\mathrm{Vec}(q) = \mathbf{q}$, respectively. The conjugation of a quaternion q is defined by

$$\bar{q} = q_0 - \mathbf{q}.$$

The quaternion conjugate behaves similarly to the complex conjugate, with the key difference being that $\overline{pq} = \bar{q}\bar{p}$.

Remark 2.1. In his papers, Grigor'ev used the notation $\widetilde{q}$ for the conjugate. In this paper, we use the notation $\bar{q}$, which is nowadays well-established notation for the quaternion conjugate.

Typically, in the papers of Yuri Grigor'ev, the quaternions were introduced up to this point, but implicitly further quaternionic constructions were also used. Therefore, let us provide a few extra definitions. First, the norm can be defined in analogy with complex analysis as

$$|q|^2 = q\bar{q} = \bar{q}q = q_0^2 + q_x^2 + q_y^2 + q_z^2.$$

It follows that the inverse of a non-zero quaternion can be calculated by

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

The product of vectors $\mathbf{p}$ and $\mathbf{q}$ can be decomposed into symmetric and antisymmetric forms by

$$\mathbf{pq} = -\mathbf{p}\cdot\mathbf{q} + \mathbf{p}\times\mathbf{q},$$

where the symmetric part is the (minus) inner product

$$\mathbf{p}\cdot\mathbf{q} = -\frac{1}{2}(\mathbf{pq}+\mathbf{qp}) = p_x q_x + p_y q_y + p_z q_z$$

and the antisymmetric part is the cross product

$$\mathbf{p}\times\mathbf{q} = \frac{1}{2}(\mathbf{pq}-\mathbf{qp}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix}.$$

This allows us to express the product of the quaternions $p = p_0 + \mathbf{p}$ and $q = q_0 + \mathbf{q}$ in the following form

$$pq = p_0 q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}. \tag{2.1}$$

Thus, algebraically speaking, quaternions are a minimal associative division algebra generated by the vector space $\mathbb{R}^3$. Computationally, quaternion calculus can be thought of as optimal extension of Gibbs' vector calculus.

3. Quaternionic Analysis and Radial Integration Operators

In this section, we recall basic ideas of Yuri Grigor'ev's approach to the quaternionic analysis and discuss his innovative tool called *radial integration operators*, which he has used to study partial differential equations in the quaternionic context.

3.1. Quaternionic Analysis

The common style of Yuri Grigor'ev, which originates from the engineering background of problems, is to denote points in space $\mathbb{R}^3$ by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Hence, if we consider a domain $\Omega \subset \mathbb{R}^3$, then a quaternion-valued function $f\colon \Omega \to \mathbb{H}$ can be written in the form

$$f(\mathbf{r}) = f_0(\mathbf{r}) + \mathbf{f}(\mathbf{r}) = f_0(\mathbf{r}) + f_x(\mathbf{r})\mathbf{i} + f_y(\mathbf{r})\mathbf{j} + f_z(\mathbf{r})\mathbf{k}. \tag{3.1}$$

Remark 3.1. Quaternionic analysis in $\mathbb{R}^3$ can be generally constructed in two ways: one way is the approach us Yuri Grigor'ev described above, and another commonly used way is to represent the points of the space $\mathbb{R}^3$ using so-called *reduced quaternions* as follows

$$r = x + y\mathbf{i} + z\mathbf{j},$$

which has its own advantages. However, Yuri Grigor'ev's way of using vector variables is more suited to the nature of quaternion algebra, with all components being algebraically in the same position.

Suppose now that $f \in C^1(\Omega, \mathbb{H})$, implying that $f_0, f_x, f_y, f_z \in C^1(\Omega)$, that is, the components are differentiable real valued functions. The vector derivative

$$\nabla := \mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z$$

is called the *Dirac operator* or the *generalised Cauchy-Riemann operator*. The Dirac operator acts on a real-valued function analogously to the classical gradient. Application of this operator on a quaternion-valued function (3.1) leads to

$$\nabla f = \nabla f_0 + \nabla f_x \mathbf{i} + \nabla f_y \mathbf{j} + \nabla f_z \mathbf{k},$$

after which the products are calculated in quaternion algebra. The action of the Dirac operator can be rewritten using (2.1) in the following way

$$\nabla f = -\nabla \cdot \mathbf{f} + \nabla f_0 + \nabla \times \mathbf{f},$$

in other words, as the sum of the ordinary divergence, gradient, and rotor.

The central function class in quaternioninc analysis is defined as follows:

Definition 3.2. A function $f \in C^1(\Omega, \mathbb{H})$ is called *(left-) regular* if $\nabla f = 0$ in Ω.

Regular functions generalise the concept of holomorphic functions in complex analysis, with the key difference that the powers of the variable $\mathbf{r}$ are not regular, i.e., $\nabla \mathbf{r} = -3 \neq 0$. Regularity can be characterized in the following ways:

Theorem 3.3. *Let $f \in C^1(\Omega, \mathbb{H})$. The following conditions are equivalent:*

(a) $\nabla f = 0$
(b) $f = f_0 + \mathbf{f}$ *satisfies the system*

$$\begin{cases} \nabla \cdot \mathbf{f} = 0, \\ \nabla f_0 + \nabla \times \mathbf{f} = 0. \end{cases}$$

(c) The components of the function satisfies the Moisil-Theodoresco system (MTS)

$$\begin{cases} \dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_y}{\partial y} + \dfrac{\partial f_z}{\partial z} = 0, \\[2ex] \dfrac{\partial f_0}{\partial x} + \dfrac{\partial f_z}{\partial y} - \dfrac{\partial f_y}{\partial z} = 0, \\[2ex] \dfrac{\partial f_0}{\partial y} + \dfrac{\partial f_x}{\partial z} - \dfrac{\partial f_3}{\partial x} = 0, \\[2ex] \dfrac{\partial f_0}{\partial z} + \dfrac{\partial f_y}{\partial x} - \dfrac{\partial f_x}{\partial y} = 0. \end{cases}$$

An important property of the Dirac operator is that it squares to a negative Laplace operator:

$$\Delta = -\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

It follows that the component functions of a regular function are harmonic, and every harmonic function h defines a regular function ∇h.

Regular functions form a function theory, which is similar to classical theory of complex holomorphic functions, however, with minor differences due to non-commutativity. The Cauchy integral formula play a central role in the classical complex analysis, and, consequently, the spatial generalisations of complex analysis are also built around Cauchy integral formulae. In the theory of regular functions, the Cauchy integral formula is provided in the following theorem:

Theorem 3.4. *([12]) Let f be a regular in a domain $\Omega \subset \mathbb{R}^3$ with a piecewise smooth boundary $\partial\Omega$, then for any $\mathbf{r} \in \Omega$ the next formula holds*

$$f(\mathbf{r}) = \frac{1}{4\pi} \oint_{\partial\Omega} M(\mathbf{r} - \mathbf{r}')\mathbf{dS}_{\mathbf{r}'} f(\mathbf{r}')$$

where $M(\mathbf{r} - \mathbf{r}')$ is the Cauchy kernel and $\mathbf{dS}_\mathbf{r}$ is the vector element of the surface $\partial\Omega$ at the point $\mathbf{r} \in \partial\Omega$ directed along the outer normal of $\partial\Omega$.

Let us discuss the implications of this result. The function f should be interpreted so that $f \in C^1(U)$ is regular in some open set $\Omega \subset U$, so that integration over the boundary $\partial\Omega$ can be performed without problems. It is worth noting, that the Cauchy kernel implies that the function

$$\mathbf{r} \mapsto M(\mathbf{r} - \mathbf{r}')$$

is regular in the set $\mathbb{R}^3 \backslash \{\mathbf{r}'\}$, so the integral indeed defines a regular function in the domain Ω. The assumption of the surface element $\mathbf{dS}_\mathbf{r}$ in turn means that it can be written in the form

$$\mathbf{dS}_\mathbf{r} = \mathbf{n}(\mathbf{r})dS(\mathbf{r})$$

where $\mathbf{n}(\mathbf{r})$ is the unit external normal at the point $\mathbf{r}$ of a smooth peace of the surface $\partial\Omega$, and $dS(\mathbf{r})$ is the classical scalar surface element.

3.2. Radial Integration Operators

The theory of radial integral operators is a useful technical tool that Yuri Grigor'ev used in various contexts covering star-symmetric domains and their complements. Let $\Omega^* \subset \mathbb{R}^3$ be a star-shaped domain with respect to the origin. In practical problems, we assume that the domain is bounded. If $f \in C(\Omega^*, \mathbb{H})$, the *radial integration operator in* Ω^* is defined (see [9]) by

$$I^\alpha f(\mathbf{r}) := \int_0^1 t^\alpha f(t\mathbf{r})dt$$

and if $f \in C(\Omega', \mathbb{H})$, the *radial integral operator in the (open) complement* $\Omega' := \mathbb{R}^3 \backslash (\Omega^* \cup \partial\Omega^*)$ is defined (see [27]) by

$$J^\alpha f(\mathbf{r}) := -\int_1^\infty t^\alpha f(t\mathbf{r})dt.$$

Naturally, the convergence of the latter operator is not immediately clear. To solve this problem, Yuri Grigor'ev, together with his PhD student Andrei Yakovlev, defined the following function classes

$$C^n_\alpha(\Omega',\mathbb{H}){:=}\{f\in C^n(\Omega',\mathbb{H})\colon \lim_{r\to\infty} r^{\alpha+1}\partial_x^a\partial_y^b\partial_z^c f(\mathbf{r})=0,\ \forall a+b+c\in\overline{0,n}\},$$

where C^n is the usual class of n times differentiable functions and $r=|\mathbf{r}|$. Hence, if $f\in C^0_\alpha(\Omega)$ then $J^\alpha f$ exists (cf. [27]).

Radial integral operators provide a convenient way to represent general solutions to partial differential equations:

Theorem 3.5. *([9,27]) Let us consider the biharmonic equation*

$$\Delta\Delta u=0.$$

(a) The general solution of the biharmonic equation in Ω^* has the form

$$u(\mathbf{r})=v(\mathbf{r})+\frac{1}{4}r^2 I^{1/2}w(\mathbf{r}),$$

where $u,w\in C^2(\Omega',\mathbb{H})$ are arbitrary harmonic functions with $\Delta u=w$.

(b) The general solution of the biharmonic equation in Ω' has the form

$$u(\mathbf{r})=v(\mathbf{r})+\frac{1}{4}r^2 J^{1/2}w(\mathbf{r}),$$

where $u,w\in C^2_{1/2}(\Omega',\mathbb{H})$ are arbitrary harmonic functions with $\Delta u=w$.

The differentiability argument required in the previous theorem follows from the smoothness of harmonic functions.

Theorem 3.6. *([9,27]) Let $v=v_0+\mathbf{v}$ be a given harmonic function with $\nabla\cdot\mathbf{v}=0$, and let us consider the system*

$$\begin{cases}\nabla\cdot\mathbf{u}=v_0,\\ \nabla\times\mathbf{u}=\mathbf{v}.\end{cases}$$

(a) In Ω^*, if $v\in C^2(\Omega^*,\mathbb{H})$, the general solution has the form

$$\mathbf{u}(\mathbf{r})=-\mathbf{r}\times I^1\mathbf{v}(\mathbf{r})+\frac{1}{4}\nabla\Big(r^2I^{1/2}\big(v_0(\mathbf{r})-\mathbf{r}\cdot I^2(\nabla\times\mathbf{v}(\mathbf{r}))\big)\Big)+\nabla g_0\mathbf{r},$$

where g_0 is and arbitrary harmonic function in Ω^*.

(b) In Ω', if $v\in C^2_1(\Omega',\mathbb{H})$, the general solution has the form

$$\mathbf{u}(\mathbf{r})=-\mathbf{r}\times J^1\mathbf{v}(\mathbf{r})+\frac{1}{4}\nabla\Big(r^2J^{1/2}\big(v_0(\mathbf{r})-\mathbf{r}\cdot J^2(\nabla\times\mathbf{v}(\mathbf{r}))\big)\Big)+\nabla g_0\mathbf{r},$$

where g_0 is an arbitrary harmonic function in Ω'.

Theorem 3.7. *([9,27]) Let us consider the Dirac equation*

$$\nabla f=0.$$

(a) In Ω^* a general solution has the form

$$f(\mathbf{r})=\varphi_0(\mathbf{r})+\mathbf{r}\times I^1\nabla\varphi_0(\mathbf{r})+\nabla\psi_0(\mathbf{r}),$$

where $\varphi_0,\psi_0\in C^2(\Omega^*,\mathbb{H})$ are arbitrary harmonic functions.

(b) In Ω' a general solution has the form

$$f(\mathbf{r}) = \varphi_0(\mathbf{r}) + \mathbf{r} \times J^1 \nabla \varphi_0(\mathbf{r}) + \nabla \psi_0(\mathbf{r}),$$

where $\varphi_0 \in C_1^2(\Omega', \mathbb{H})$ and $\psi_0 \in C^2(\Omega^*, \mathbb{H})$ are arbitrary harmonic functions.

If f is a regular function, then the function F that satisfies the equation $\nabla F = f$ is called a *primitive function.*

Theorem 3.8. *([9]) If $f \in C^1(\Omega^*, \mathbb{H})$ is a regular function, then a general form for the primitive in Ω^* has the form*

$$F(\mathbf{r}) = \chi_0(\mathbf{r}) + \mathbf{r} \times I^1\big(\Delta\chi(\mathbf{r}) - \mathbf{f}(\mathbf{r})\big) - \frac{1}{2}\nabla\big(r^2 I^{1/2} I^1 f_0(\mathbf{r})\big) + \nabla g_0(\mathbf{r}),$$

where χ_0 and g_0 are arbitrary real valued harmonic functions in Ω^*.

Remark 3.9. Yuri Grigor'ev did not publish the formula of the previous theorem for the complement, but it is clear that the formula is also valid in that case. More precisely, this is because the operators I^α and J^α have identical calculus rules, see Theorem 1 in [27] and Section 3 in [9]. Thus, all proofs are automatically valid in both cases.

Radial integrator operators have been examined in at least the works [5,6,9,26,27].

4. Applications to Problems of Linear Elasticity

In this section, we provide a brief overview of Yuri Grigor'ev's ideas related to the applications of quaternionic function theory to problems of linear elasticity.

4.1. Three-Dimensional Quaternionic Analogue of the Kolosov–Muskhelisvili Formula

In the following, we establish a connection between regular functions and solutions of the Lamé equation. We define

$$\begin{cases} f_0 := (\lambda + 2\mu)\nabla \cdot \mathbf{u}, \\ \mathbf{f} := -\mu \nabla \times \mathbf{u}. \end{cases} \tag{4.1}$$

It follows that the functions satisfy the system

$$\begin{cases} \nabla \cdot \mathbf{f} = 0, \\ \nabla f_0 + \nabla \times \mathbf{f} = 0, \end{cases}$$

and by Theorem 3.3 the function $f = f_0 + \mathbf{f}$ is regular. There exists a connection between the solutions of the Lamé equation and regular functions in both directions, as shown in the following diagram:

$$
\begin{tikzcd}
L\mathbf{u}=0 \arrow[dr, "(3)"] & (3) \arrow[l, "\text{Theorem 3.5}"'] \\
 & \nabla\cdot\mathbf{f}=0 \arrow[u, "\text{Theorem 3.6}"]
\end{tikzcd}
$$

It is also important to note that when applying Theorem 3.6, the condition

$$\nabla \cdot \mathbf{f} = -\mu \nabla \cdot (\nabla \times \mathbf{u}) = 0.$$

must hold. In other words, when applying the formula in Theorem 3.7, the harmonic function φ_0 must satisfy an additional condition

$$\nabla \cdot \big(\mathbf{r} \times I^1 \nabla \varphi_0(\mathbf{r})\big) = 0.$$

Based on this consideration, it can be concluded that a general solution to the Lamé equation can be expressed in terms of regular functions. This connection is called the *Kolosov-Muskhelisvili formula*:

Theorem 4.1. (Kolosov-Muskhelisvili formula, [9]) *The general solution to the Lamé equation in Ω^* is expressed in terms of two regular functions φ and ψ in the form*

$$2\mu\mathbf{u}(\mathbf{r}) = \kappa\Phi(\mathbf{r}) - \mathbf{r}\overline{\varphi(\mathbf{r})} - \overline{\psi(\mathbf{r})},$$

where

$$\kappa = -\frac{3\lambda + 7\mu}{\lambda + \mu}.$$

The function Φ is a primitive of function φ, having subordinated ψ to the condition $\kappa\Phi_0 = \mathbf{r} \cdot \boldsymbol{\varphi} + \psi_0$.

Remark 4.2. The representation contains conjugates of a regular function that are not, strictly speaking, regular. Let φ be a regular function, i.e. $\nabla\varphi = 0$. Taking the conjugate of this expression, we get

$$0 = \overline{\nabla\varphi} = -\overline{\varphi}\nabla.$$

Thus, a conjugated left-regular function is referred to as a right-regular function.

4.2. Regular Quaternionic Polynomials

In order to apply the representation formulae presented in the previous section to practical problems, it is necessary to represent regular functions using the simplest possible functions. This problem was addressed by Yuri Grigor'ev in [12]. This work is significant because, in practical applications, it is not customary to represent polynomials using vector variables; instead, reduced quaternions are used, refer to Remark 3.1.

As a starting point, Yuri Grigor'ev defines first-order regular polynomials as

$$P^{1,0}(\mathbf{r}):=x+z\mathbf{j} \quad \text{and} \quad P^{0,1}(\mathbf{r}):=y-z\mathbf{i},$$

which are used to define $n=l+m$-homogeneous polynomials by

$$P^{l,m}(\mathbf{r}):=P^{1,0}(\mathbf{r})P^{l-1,m}(\mathbf{r})+P^{0,1}(\mathbf{r})P^{l,m-1}(\mathbf{r}),$$

where $P^{l,m}(\mathbf{r})=0$ if $l<0$ or $m<0$. In Theorem 3.1 of [12], it was proved that the polynomials satisfy the differentiation formulae

$$\frac{\partial}{\partial x}P^{l,m}(\mathbf{r})=(l+m)P^{l-1,m}(\mathbf{r}),$$

$$\frac{\partial}{\partial y}P^{l,m}(\mathbf{r})=(l+m)P^{l,m-1}(\mathbf{r}),$$

$$\frac{\partial}{\partial z}P^{l,m}(\mathbf{r})=(l+m)\big(-\mathbf{i}P^{l,m-1}(\mathbf{r})+\mathbf{j}P^{l-1,m}(\mathbf{r})\big).$$

From this, it follows that they are regular. Moreover, they form a basis for the space of n-homogeneous $\mathbb{H}$-valued regular polynomials. Regular quaternion-valued polynomials are then represented component-wise as

$$P^{l,m}(\mathbf{r})=T^{l,m}(\mathbf{r})+X^{l,m}(\mathbf{r})\mathbf{i}+Y^{l,m}(\mathbf{r})\mathbf{j}+Z^{l,m}(\mathbf{r})\mathbf{k},$$

where $T^{l,m},X^{l,m},Y^{l,m},Z^{l,m}$ are n-homogeneous real valued harmonic polynomials. However, this form is overly general, and Theorem 3.4 states that every regular polynomial can be represented as

$$P^{l,m}(\mathbf{r})=T^{l,m}(\mathbf{r})+X^{l,m}(\mathbf{r})\mathbf{i}-X^{l-1,m+1}(\mathbf{r})\mathbf{j},$$

and that the T and X polynomials can be computed recursively

$$\begin{aligned}T^{l,m}(\mathbf{r})&=xX^{l-1,m}(\mathbf{r})+yX^{l,m-1}(\mathbf{r})-zT^{l,m-1}(\mathbf{r}),\\ X^{l,m}(\mathbf{r})&=xT^{l-1,m}(\mathbf{r})+yT^{l,m-1}(\mathbf{r})+z\big(X^{l,m-1}(\mathbf{r})-X^{l-2,m+1}(\mathbf{r})\big).\end{aligned}$$

These polynomials form a basis for the space of n-homogeneous real-valued polynomials.

In general, these formulae are sufficient for calculating explicit polynomial representations. However, Yuri Grigor'ev employs a more sophisticated method based on the Taylor formula, as outlined in Theorem 6.2 of [12].

In section 7 of the article [12], polynomial bases are applied to approximate solutions to the Lamé equation in a star-shaped domain.

4.3. Quaternion Boundary Element Method

It is important to underline, that Yuri Grigor'ev not just developed general representation formulae for the solutions of boundary value problems, but also constructed numerical schemes. In his paper with Vladislav Alehin [16], the boundary element method based on the Cauchy integral formula for regular functions was introduced.

The starting point is a differentiable quaternion-valued function f with the real part $\mathrm{Sc} f = f_0$ and the vector part $\mathrm{Vec} f = \mathbf{f}$. Let S be a sufficiently smooth surface and define the integral operator

$$K_S f(\mathbf{r}) := \frac{1}{2\pi} \oint_S M(\mathbf{r}-\mathbf{r}')\mathbf{dS}_{\mathbf{r}'} f(\mathbf{r}').$$

This integral represents a form of the Cauchy integral formula with a coefficient of $\frac{1}{2}$. If the point $\mathbf{r}'$ belongs to the surface S the operator is a singular integral operator. Let Ω be an open domain such that $\partial\Omega = S$. The operator has the important property that when a Hölder-continuous function $f \in C^\alpha(S)$ satisfies the equation

$$f(\mathbf{r}) - K_S f(\mathbf{r}) = 0, \tag{4.2}$$

then it is necessary and sufficient for f to represent the boundary values of a regular function

$$g(\mathbf{r}) = \frac{1}{4\pi} \oint_S M(\mathbf{r}-\mathbf{r}')\mathbf{dS}_{\mathbf{r}'} f(\mathbf{r}')$$

for defined in Ω (see e.g. [3], Corollary 7.19).

Yuri Grigor'ev and Vladislav Alehin formulated the following problem:

> When the real part f_0 and the normal component f^n of a function are known on the surface S, determine the corresponding regular function f on the set Ω that is determined by these components.

Consequently, the solution to this problem is based on solving Eq. (4.2. It must first be assumed that $f_0, f^n \in C^\alpha(S)$, and an approximate solution is being sought. Next, the boundary S is approximated by the union of flat triangles S_m, where $m = 1, 2, \ldots, N$, that is,

$$S \approx S' = \bigcup_{m=1}^{N} S_m.$$

Let Ω' denote the domain whose boundary is the surface S'. The boundary values on each element are approximated by local quaternion functions $f_m(\mathbf{r})$, where $\mathbf{r} \in S_m$. Hence

$$f(\mathbf{r}) \approx \frac{1}{4\pi} \sum_{m=1}^{N} \oint_{S_m} M(\mathbf{r}-\mathbf{r}')\mathbf{dS}_{\mathbf{r}'} f_m(\mathbf{r}'). \tag{4.3}$$

This formula provides an approximate value of the unknown function in the domain Ω'. Assume that each triangular element S_m contains only one node, which is located at its centre of mass $\mathbf{r}_m$. Consider the simplest case of zero-order elements, where $f_m = \text{const}$ within each element S_m. In this case, the known quantities are the nodal values of the scalar part

$$f_m^0 := f_0(\mathbf{r}_m)$$

and the nodal values of the normal component of the vector part

$$f_m^n := \mathbf{n}(\mathbf{r}_m) \cdot \mathbf{f}(\mathbf{r}_m)$$

evaluated at the centre of mass $\mathbf{r}_m$.

The unknowns are the nodal values f_{xm}, f_{ym}, f_{zm} of the components of the vector part. Formula (4.3) can be used to form algebraic equations to determine unknown node values. Yuri Grigoriev and Vladislav Alehin presented the so-called direct method, where the unknown node values are obtained from the equations

$$\begin{aligned}
f_m^n &= \frac{1}{4\pi} \sum_{m=1}^{N} \lim_{\mathbf{r}\to\mathbf{r}_m} \operatorname{Sc} \oint_{S_m} M(\mathbf{r}-\mathbf{r}')\mathbf{dS}_{\mathbf{r}'} f_m, \\
f_m^n &= \frac{1}{4\pi} \sum_{m=1}^{N} \lim_{\mathbf{r}\to\mathbf{r}_m} \operatorname{Vec} \oint_{S_m} M(\mathbf{r}-\mathbf{r}')\mathbf{dS}_{\mathbf{r}'} f_m, \\
f_m^n &= \mathbf{n}_m \cdot \mathbf{f}_m = n_{xm} f_{xm} + n_{xm} f_{ym} + n_{xm} f_{ym},
\end{aligned}$$

where $\mathbf{n}_m = \mathbf{n}(\mathbf{r}_m)$ is the unit normal at $\mathbf{r}_m$. Thus, we get $3N$ unknowns and the same number of linear equations to determine them. Since the coefficients f_m are constants, the integrals

$$I_m(\mathbf{r}) := \oint_{S_m} M(\mathbf{r}-\mathbf{r}')\mathbf{dS}_{\mathbf{r}'}$$

need to be computed for practical calculations, that they can be implemented meaningfully. The integrals for the above-mentioned triangular elements were calculated in the article [16], see p. 50.

4.4. Micropolar Elasticity

In his last years, Yuri Grigor'ev, in collaboration with Anna Gavrilieva, published a series of research articles on the theory of micropolar elasticity [17–19]. Micropolar elasticity theory considers problems in elasticity where, in addition to the displacements $\mathbf{u}$ at points on a body, there is also a possible rotational component $\boldsymbol{\omega}$. The quantities are determined from the following system of coupled partial differential equations

$$\begin{aligned}
&(2\mu+\lambda)\operatorname{grad}\operatorname{div}\mathbf{u} - (\mu+\alpha)\operatorname{rot}\operatorname{rot}\mathbf{u} + 2\alpha\operatorname{rot}\boldsymbol{\omega} + \mathbf{X} = \rho\ddot{\mathbf{u}}, \\
&(2\gamma+\beta)\operatorname{grad}\operatorname{div}\boldsymbol{\omega} - (\gamma+\varepsilon)\operatorname{rot}\operatorname{rot}\boldsymbol{\omega} + 2\alpha\operatorname{rot}\mathbf{u} + \mathbf{Y} = j\ddot{\boldsymbol{\omega}}.
\end{aligned}$$

The articles are limited to the plane case, and, therefore, do not use the methods of quaternionic analysis. Micropolar elasticity theory is specifically formulated for three-dimensional space, and Yuri Grigor'ev likely intended to extend this work to that case. Unfortunately, the work remained unfinished and was subsequently carried forward by other researchers. However, the mathematical framework developed by Yuri Grigoriev provides an excellent starting point for further investigation.

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