Function Theoretic Methods for the Analytical and Numerical Solution of Some Non-linear Boundary Value Problems with Singularities

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Abstract

The p-Laplace equation is a nonlinear generalization of the well-known Laplace equation. It is often used as a model problem for special types of nonlinearities, and therefore it can be seen as a bridge between very general nonlinear equations and the linear Laplace equation, too. It appears in many problems for instance in the theory of non-Newtonian fluids and fluid dynamics or in rockfill dam problems, as well as in special problems of image restoration and image processing.

The aim of this thesis is to solve the p-Laplace equation for $1 < p < 2$, as well as for $2 < p < 3$ and to find strong solutions in the framework of Clifford analysis. The idea is to apply a hypercomplex integral operator and special function theoretic methods to transform the p-Laplace equation into a p-Dirac equation. We consider boundary value problems for the p-Laplace equation and transfer them to boundary value problems for a p-Dirac equation. These equations will be solved iteratively by applying Banach's fixed-point principle. Applying operator-theoretical methods for the p-Dirac equation, the existence and uniqueness of solutions in certain Sobolev spaces will be proved.

In addition, using a finite difference approach on a uniform lattice in the plane, the fundamental solution of the Cauchy-Riemann operator and its adjoint based on the fundamental solution of the Laplacian will be calculated. Besides, we define generalized discrete Teodorescu transform operators, which are right-inverse to the discrete Cauchy-Riemann operator and its adjoint in the plane. Furthermore, a new formula for generalized discrete boundary operators (analogues of the Cauchy integral operator) will be considered. Based on these operators a new version of discrete Borel-Pompeiu formula is formulated and proved.

This is the basis for an operator calculus that will be applied to the numerical solution of the p-Dirac equation. Finally, numerical results will be presented showing advantages and problems of this approach.

Kurzfassung

Die p-Laplace-Gleichung ist eine nichtlineare Verallgemeinerung der wohlbekannten Laplace-Gleichung Die p-Laplace-Gleichung wird häufig als Referenzbeispiel für spezielle Typen von Nichtlinearitäten benutzt und kann daher auch als Brücke zwischen sehr allgemeinen nichtlinearen partiellen Differentialgleichungen und der linearen Laplace-Gleichung gesehen werden. Sie ist darüber hinaus auch das mathematische Modell für eine Reihe praxisrelevanter Probleme, wie z. B. in der Theorie nicht-newtonscher Flüssigkeiten, der Strömungsmechanik, der Durchfeuchtung von Schüttdämmen und auch ein wichtiges Werkzeug zur Behandlung spezieller Probleme der Bildrekonstruktion und Bildverarbeitung.

Das Ziel dieser Arbeit ist es, die p-Laplace-Gleichung sowohl für $1 < p < 2$ als auch ür $2 < p < 3$ zu lösen. Strenge Lösungen werden unter Benutzung der Clifford-Analysis konstruiert. Die Idee ist dabei, einen hyperkomplexen Integraloperator und funktionentheoretische Methoden auf die p-Laplace-Gleichung anzuwenden und diese Gleichung dadurch in eine p-Dirac-Gleichung zu transformieren, die dann besser gelöst werden kann. Es werden spezielle Randwertprobleme für die p-Laplace-Gleichung in Dirichlet-Probleme für die p-Dirac-Gleichung transformiert und dabei die Ordnung der Differentialgleichung reduziert. Die Randwertprobleme für die p-Dirac-Gleichung werden mit Hilfe des Banachschen Fixpunktprinzips iterativ analytisch gelöst. Durch Anwendung operator-theoretischer Methoden kann die Existenz und Eindeutigkeit der Lösung in bestimmten Sobolev-Räumen nachgewiesen werden.

Darüber hinaus wird eine Finite Differenzenmethode auf einem gleichmäßigen Gitter in der Ebene angewandt, um die Fundamentallösung des diskreten Laplace-Operators numerisch zu berechnen. In der Folge werden daraus Fundamentallösungen des diskreten Cauchy-Riemann-Operators und seines adjungierten Operators erzeugt. Auf dieser Grundlage werden über Faltungen mit den Fundamentallösungen diskrete Teodorescu-Operatoren definiert, die rechtsinvers zum diskreten Cauchy-Riemann-Operator bzw. zum adjungierten diskreten Cauchy-Riemann-Operator sind. Weiterhin werden diskrete Randoperatoren, die analog zum Cauchyschen Integraloperator sind, eingeführt. Alle vorgenannten Operatoren werden in einer neuen Version einer diskreten Borel-Pompeiu-Formel zusammengeführt und bilden die Grundlage für eine diskrete Operatorenrechnung. Diese Untersuchungen erweitern bekannte Resultate auf wesentlich größere Funktionenklassen als bisher möglich waren. Die diskrete Operatorenrechnung wird benutzt, um die diskretisierten Randwertprobleme für die p-Dirac-Gleichung numerisch zu lösen. Numerische Resultate werden vorgestellt und diskutiert. Dabei wird auf Vor- und Nachteile der entwickelten Methode eingegangen.

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1 Introduction and Motivation

1.1 Background

Over the past few decades, there has been a growing interest for the study of the p-Laplacian operator or the p-Laplace operator Δ_p, which is a quasilinear elliptic partial differential operator of second order. That interest rests on the multi-applicability of the p-Laplacian operator or the p-Laplace operator. Indeed, many nonlinear problems in the field of physics, mechanics, technological sciences and engineering are formulated in equations that contain the p-Laplacian, where the p-Laplacian operator is defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \tag{1.1}$$

where u is a scalar-valued function and $1 \leq p < \infty$. In fluid mechanics, for example, a relation of the form $\vec{\tau}(x) = a(x)\nabla_p u(x)$, where $\vec{\tau}$ is the shear stress and $\nabla_p u = |\nabla u|^{p-2}\nabla u$ is the velocity gradient, for a certain fluid obey this form. The constant p here is an arbitrary real number greater than 1. The case $p = 2$ corresponds to a Newtonian fluid, the case $p < 2$ corresponds to a pseudoplastic fluid and $p > 2$ corresponds to a dilatant fluid. As a result, the equations of motion, when a is a constant, involve div $(a\,\nabla_p u)$, that reduces to $a\,\Delta_p u = a$ div $(\nabla_p u)$.

When looking to nonlinear problems, one of the most important problems is the p-Laplace equation. This strong generalization of the Laplace equation is commonly used as a model problem for types of nonlinearities like the Navier-Stokes equation. It may also be used for more complicated equations like the magneto-hydrodynamics equation (MHD). There has been a surge of interest in the p-Laplacian in many different contexts, from game theory to mechanics, image processing and non-Newtonian fluids. For $p \in (1, 4/3]$, the p-Laplacian appears in the study of a mathematical model of glacier flow (glaciology) see [Pelissier and Reynaud, 1974]. For the case $p = 1.5$, the p-Laplacian appears in the study of flow through porous media, see

[Showalter and Walkington, 1991]. The p-Laplace problem appears also in torsional creep problems [Kawohl, 1990]. In addition, it appears in the study of climatology balance models [Diaz and Hernandez, 1997]. In [Carstensen and Klose, 2003] three a-posteriori errors estimates were discussed for the p-Laplace problem, for $p > 2$.

However, there are many unsettled existence, uniqueness and regularity issues. Currently, many works propose weak solutions of the p-Laplace equation, one can see [Adamowicz, 2009], [Edquist and Lindgren, 2009], [Adamowicz and Kalamajska, 2010], [Ciarlet, 2013], [Diening et al., 2013] and [Adamowicz et al., 2014].

This is closely connected with the fact that the p-Laplace equation can be introduced by minimizing the p-Dirichlet integral $\int_\Omega |\text{grad } u|^p \, dx$. This explains, on the one hand, a lot of similar properties of the solutions, compared with harmonic functions, and on the other hand, it seems to be natural to apply variational methods for the solution. Due to the fact that the p-Laplace equation degenerates at the zeros of u or will have singularities, depending on p, there are not so many works on strong solutions of the p-Laplace equation. One way of treating the p-Laplace equation is its transformation into a p-Dirac equation. This leads to the question of generalizing the problem to the p-Laplace or the p-Dirac equation for vector-valued functions. The idea behind is that for equations for vector-valued functions, often, function theoretic methods can be applied.

Monogenic functions are defined as null-solutions of a generalized Cauchy-Riemann system or of a Dirac equation. In this sense, the theory of monogenic functions has to be seen as a generalization of the complex function theory (of holomorphic functions). Monogenic functions are harmonic in all components and in this way the theory can be understood also as a refinement of the classical harmonic analysis (with strong relations to potential theory). Indeed, monogenic functions share a lot of properties with holomorphic functions [Gürlebeck et al., 2008]. They can be expressed by a Cauchy integral formula. There is a Borel-Pompeiu formula and they can be developed explicitly in Taylor expansions. Borel-Pompeiu formula is named after the French mathematician Émile Borel (1871-1956) and the Romanian mathematician Dimitrie Pompeiu (1873-1954). In [Bernstein, 2000] a Borel-Pompeiu formula for functions in several complex variables using Clifford analysis was developed. [Brackx et al., 1982] and [Gürlebeck and Sprößig, 1990] published a discrete Borel-Pompeiu formula.

After some preliminary works by Moisil/Teodorescu, the Swiss mathematician Fueter established in the 1930's the basics of a higher dimensional function theory for functions defined in $\mathbb{R}^4$ with values in the space of quaternions. Much later beginning in the 1980's such theories were extended [Brackx et al., 1982] to the case of functions defined in $\mathbb{R}^N$ with values in a Clifford algebra. The first applications in the framework of a self-contained theory were considered in [Gürlebeck and Sprößig, 1990]. Based on an operator theoretical approach, elliptic boundary value problems could be studied (existence, uniqueness, regularity and integral representations). Recently, this approach has been extended to parabolic and (partially) to the treatment of hyperbolic equations. For a long time, the practical problem was the numerical evaluation of the integral representations. This problem could be overcome by the idea (notion) of systems of monogenic Appell polynomials. This was introduced in [Falcáo and Malonek, 2007], completely solved for the dimensions 2, 3, 4 in [Bock, 2009] and was extended in [Bock et al., 2012].

These systems allow the exact evaluation of representation formulas for the solutions to linear boundary value problems, if they have a representation in terms of monogenic functions. In [Gürlebeck and Bock, 2009] and [Bock and Gürlebeck, 2010] this was applied to the Lame-Navier equations of linear elasticity in the case of smooth solutions in a ball. In [Gürlebeck and Sprößig, 1997] and [Gürlebeck et al., 2016] one can find a lot of integral representation formulas for the solutions to elliptic boundary value problems and applications of hypercomplex analysis to boundary value and initial-boundary value problems from various areas of mathematical physics. Some nonlinear cases like the Navier-Stokes equations and similar equations, containing nonlinearities of the type $u \cdot \text{grad } u$ or u^2 could also be treated by this concept.

The study of p-Laplace and p-Dirac equations began with C. Nolder and J. Ryan [Nolder and Ryan, 2009]. They introduced non-linear Dirac operators in $\mathbb{R}^N$, associated to the p-Laplace equation [Nolder and Ryan, 2009]. C. Nolder explained in [Nolder, 2010] how p-harmonic equations arise from Dirac systems, and the main purpose of his work is to elucidate the connection between the theories of p-harmonic functions and Dirac analysis. The regularity of the p-Laplace equation in the plane has been studied for $p \geq 2$, by E. Lindgren and P. Lindqvist [Lindgren and Lindqvist, 2013]. The infinite Dirac operator has been defined and some of its key properties have been explored by T. Bieske and J. Ryan [Bieske and Ryan, 2010]. P. Lindqvist has also made many notes on the p-Laplace equation [Lindqvist, 2006].

For the linear case, $p = 2$, the Laplace operator can be factorized by means of two Cauchy-Riemann operators (first order operators). These first order equations can, then, be solved efficiently by applying function theoretical methods. The non-linear analogue to the Cauchy-Riemann or Dirac equation is the p-Dirac equation. This equation has a meaning by itself, but for this thesis, the main interest lies in the efficient solution of the p-Dirac equation as a tool for solving the p-Laplace equation. It is also meant to find out how this equation simplifies the solution of the p-Laplace equation. The mathematical equations which are used in this work are shown in the diagram below:

$$\begin{tikzcd}[column sep=large, row sep=large]
\boxed{\begin{array}{c} p\text{-Poisson equation } (f \neq 0),\\ p\text{-Laplace equation } (f = 0)\\ \operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right) = f \end{array}} \arrow[r, "\text{Substitution}", "\nabla u = w"'] \arrow[d, "\text{Generalization}"'] & \boxed{\begin{array}{c} p\text{-Dirac equation}\\ \operatorname{div}\left(|w|^{p-2}\,w\right) = f \end{array}} \arrow[d, "\text{Generalization}"'] \\
\boxed{\begin{array}{c} \text{Generalized } p\text{-Poisson } (f \neq 0),\\ \text{Generalized } p\text{-Laplace } (f = 0)\\ D\left(|Du|^{p-2}\,Du\right) = f \end{array}} \arrow[r, "\text{Substitution}", "Du = w"'] \arrow[d, "p=2"'] & \boxed{\begin{array}{c} \text{Generalized } p\text{-Dirac}\\ D\left(|w|^{p-2}\,w\right) = f \end{array}} \arrow[d, "p=2"'] \\
\boxed{\begin{array}{c} \text{Poisson equation } (f \neq 0),\\ \Delta u = f \text{ or } D^2 u = f\\ \text{Laplace equation } (f = 0) \end{array}} \arrow[r, "\text{Factorization}"] & \boxed{\begin{array}{c} \text{Dirac equation}\\ Dw = \sum\limits_{i=1}^{N} e_i \frac{\partial w}{\partial x_i} = f \end{array}}
\end{tikzcd}$$

where u is a vector-valued function and f is the non-homogeneous part.

These substitutions look obvious for the first look at it, but they are deeply related to the factorization of the Laplace operator one time by *div grad* and second time by DD.

To that end, the scale of Sobolev spaces shall be used. As it is well known, Sobolev spaces were introduced mainly for the theory of partial differential equations. Indeed, Sobolev spaces are examples of Banach spaces or, in some cases, Hilbert spaces are interesting object for themselves. However, they are of paramount importance given that the theory of partial differential equations can be easily developed in such spaces. This is because partial differential operators are quite well located in Sobolev spaces. By contrast, the spaces of continuous (or of class C^k) functions are not quite appropriate for studying partial differential equations. Instead, Hölder continuous spaces or Hölder continuous along with derivatives need to be considered.

1.2 Models Related to p-Poisson Equation and Physical Motivation

The main goal of this thesis is to study this question or this class of problem under a mathematical point of view. But here it should be also mentioned the physical motivation of the nonlinear problems.

The p-Laplace operator appears in many second order nonlinear equations, one of the important nonlinear elliptic equations is

$$-\text{div}\ (|\nabla u|^{p-2}\nabla u) + \lambda u = 0,\ \ p > 1,\ \ \lambda > 0, \tag{1.2}$$

where ∇u denotes the gradient of u and $|\nabla u|$ is the Euclidean norm in $\mathbb{R}^N$ of the vector ∇u.

Indeed, in the study of fluid media laws of motion, Newtonian fluids are usually considered as the fluids for which the relation between the shear stress τ and the velocity gradient $\frac{du}{dx}$ takes the form

$$\tau = \mu \frac{du}{dx}. \tag{1.3}$$

For the sake of being simple, we shall restrict ourselves here to the plane case. However, this is satisfactory only for a limited number of actual fluid media. For instance, dispersive media treated according to a continuum model does not obey the law given in Equation (1.3). The motions of such a type of non-Newtonian fluids are studied in rheology (see e.g. [Astarita and Marrucci, 1974]).

In general, Equation (1.3) is replaced by the power rheological law

$$\tau = \mu \left| \frac{du}{dx} \right|^{p-2} \frac{du}{dx}, \quad p > 1. \tag{1.4}$$

The quantities μ and p are the rheological characteristics of the medium. Non-Newtonian flow where $p > 2$ is referred to as "dilatant fluids", and the ones where $p < 2$ are called "pseudoplastics". When $p = 2$, they are called Newtonian fluids. The study of non-Newtonian flow properties of media having conductivity in electromagnetic fields leads to an equation similar to (1.2).

Let us take an example where the conducting fluid moves in a flat channel, $-L < x < L$ whose non-conducting walls move adjacent to the x-axis with a velocity $\pm u_0$, (magnetohydrodynamic Couette flow). In the movement of the fluid, there is no electric field, nor pressure gradient, and the outer magnetic field of induction appears to be perpendicular vis-a-vis the walls. By normalizing the problem, one can get another problem, formulated as follows

$$-\frac{d}{dx}\left(\left| \frac{du}{dx} \right|^{p-2} \frac{du}{dx} \right) + \lambda u = 0 \quad \text{in } G = (-1, 1), \tag{1.5}$$

$$u(\pm 1) = 1, \tag{1.6}$$

in which $\lambda > 0$ (the generalized Hartmann number). Regarding the dilatant fluids $p > 2$, and only for them, and λ big enough, the physical meaning of the problem (see [Martinson and Pavlov, 1966]) reveals that flow zones appear where there are fluid velocity movements that disappear over the cross-section of the channel.

The operator Δ_p denotes the *pseudo-Laplacian operator* defined, for $p > 1$, by

$$\Delta_p u = \text{div } (|\nabla u|^{p-2} \nabla u) = \sum_{i=1}^{N} \frac{\partial}{\partial u_i} (|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}),$$

nowadays it is called the *p-Laplacian operator*.

It is possible to find the operator Δ_p with $p \neq 2$ in many other important problems (in addition to the non-Newtonian fluids). The p-Laplacian is also involved in some reaction-diffusion problems (see,[Aris, 1975] p.207), and in flow through porous media (one example is the flow through rockfill dams, [Ahmed and Sunada, 1969] or [Volker, 1969]). Besides the applications in stellar dynamic structures, the p-Laplacian has interesting applications in flows through porous media. In two dimensions space boundary-value problems for the p-Laplace equations are considered which relate to notch and crack problems in the theory of deformation plasticity under conditions of anti-plane strain and they are studied in [Atkinson and Champion, 1984]. Also, the p-Laplace equation appears in nonlinear theories of diffusion and filtration and in the study of fluid flows through natural rocks (see [Atkinson and Champion, 1984] and [Barenblatt et al., 1990]).

Rockfill dam is an important example for civil engineering. A dam problem can be represented as follows:

Example 1. *[Wei, 1989]*

Considering a flow of water through a rockfill dam having a lower hermetic boundary γ_1 *as shown in Figure 1.1, where* G_1 *refers to the wet region,* γ^* *is the seepage "free boundary", and the overtop fluid contacted comes as* $\gamma_{3,1}$*, and the lower contacted boundary as* $\gamma_{3,2}$*,* γ_2 *the dry boundary, and* G *the inside of the rockfill dam (where* $G = G_1 \cup G_2 \cup \gamma^*$*).*

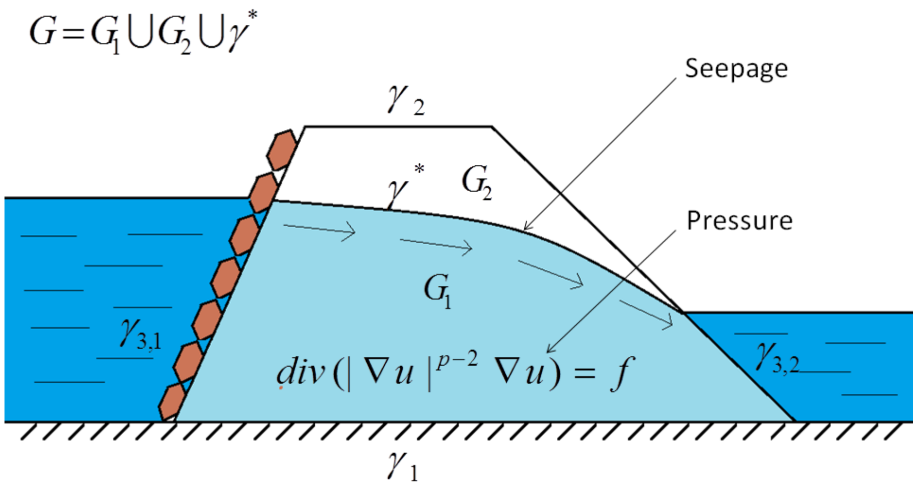


Figure 1.1: Rockfill dam

The problem is to determine a pair (u, G_1) satisfying the following equations:

$$div\left(|\nabla u|^{p-2}\,\nabla u\right) = f \qquad in\ G_1 \tag{1.7}$$

$$u = u_i \qquad on\ \gamma_{3,i},\ i = 1, 2, \tag{1.8}$$

$$u = y \qquad on\ \gamma^*, \tag{1.9}$$

$$\frac{\partial u}{\partial n} = 0 \qquad on\ \gamma^* \cup \gamma_1, \tag{1.10}$$

Given that on γ^* the fluid comes without any pressure (pressure free), there is a boundary condition as evidenced in (1.9). Indeed, $u = \ pressure + y$, and thus $u \geq y$, $G_1 = \{(x, y) : u(x, y) > y\}$, $G_1^c = \{(x, y) : u(x, y) = y$ (pressure free region)$\}$.

Some important examples of such problems were considered as well in [Kythe, 1996]. It also appears in nonlinear elasticity (e.g. [Oden, 1978]), in nonlinear problems in glaciology [Pelissier, 1975], and petroleum extraction [Schoenauer, 1983]. Regularity for a class of non-linear elliptic systems, which contain the operator Δ_p, was studied for $p \geq 2$ in [Uhlenbeck, 1977]. In [Díaz, 1985], the second order nonlinear elliptic equations where Δ_p appears (like non-newtonian fluids) have been been studied in detail in the first and second dimension.

In the two dimensional space, G. Aronsson and U. Janfalk analysed Hele-Shaw flow of a power-law fluid, which is a specific case of a generalized Newtonian fluid, and they showed that in Hele-Shaw flow of a power-law fluid, the equations for the pressure and the stream function are p-Laplace functions in [Aronsson and Janfalk, 1992]. Later, [Drost and Westerweel, 2013] described a novel approach to determine the flow behavior index of a power-law fluid by means of a microfluidic device, by fitting the p-Laplace equation to the velocity field obtained from a micro particle image velocimetry measurement of the flow. They determined the flow behavior index of the fluid.

More general applications to quantum mechanics and quantum physics involving the p-Laplacian, for $p = 6$, have been studied in [Benci et al., 1998]. Existing results for eigenvalue problems involving the p-Laplacian were studied by several authors (see, for instance [Bonanno and Giovannelli, 2005] and [Lê, 2006]), on unbounded domains have been proven in [Montefusco and Rǒdulescu, 2001].

Let us consider the issue of restoring a picture that is damaged. The restoration of the picture can be done by interpolating data from the leftover of the picture. Such a process can be performed by making use of the p-Laplacian. For further information on this interpolation method and for understanding the clear motivation behind the use of the p-Laplacian in this problem, reference can be made to [Caselles et al., 1998]. Another contribution with that respect is the thesis by A. Almansa (see [Almansa, 2002]), where a p-Laplacian model is used for the interpolation of terrain elevation maps.

Applications for the p-Laplacian and equations of p-Laplace type were considered in [Lundström, 2011]. They discussed the power-laws and their connection to the p-Laplacian, image restoring, stochastic characterization and minimization problems. The case $p = 1$ is also very important and used for example in image processing and TV denoising. For example, in [Zhang et al., 2012] and [Gomathi and Kumar, 2014] the p-Laplace equation appears in TV inpainting model and TV wavelet inpainting model. Image inpainting is the process of reconstructing lost or deteriorated parts of images or modifying the damaged images. The physical characteristic of TV model and p-Laplacian operator for damaged wavelet coefficients were considered in [Gomathi, 2014]. Wavelet inpainting models based on p-Laplace operator have been presented in [Hong-Ying et al., 2007].

There is a lot of investigation by Bombieri, De Giorgi and Giusti about the 1-Laplace problem in $\mathbb{R}^N$ for $N > 7$ (see [Bombieri et al., 1969]). Also, in the paper of S. Wei, the $1-$harmonic function has been studied [Wei, 2007]. In other contexts, however, the study of the limiting cases has opened the door to a better understanding of the situation $p \in (1, \infty)$. For example, the study of viscosity solutions is quite suitable for large values of p, see [Diening et al., 2013]. [Moll and Petitta, 2015] give a general condition on the absorption term of the 1-Laplace elliptic equation for the existence of suitable large solutions. They also provide conditions that guarantee uniqueness of solutions to the 1-Laplace problem. They obtained the existence of solutions as limit for $p \to 1$. There are many results of the existence, uniqueness and the regularity results issues for the limiting cases $p = 1$ or $p = \infty$. One example for studying large values of p is the concept of viscosity solutions.

Finally, the equations that involve the p-Laplace operator attracted much attention from researchers as seen in the literature. The p-Laplacian has a wide range of applications in physics and related sciences like chemical reaction design, biophysics and plasma physics. This operator appears in models which describe a variety of phenomena in nature including:
(1) Rheology; (2) Fluid dynamics; (3) Flow through porous media (for instance in flow

through rockfill dams); (4) Nonlinear elasticity; (5) Glaciology; (6) Stellar dynamic structures; (7) Image restoration; (8) Electrorheological fluids; (9) Nonlinear Darcy's law in porous medium; (10) Climatology; (11) Radiation of heat; and (12) Plastic moulding; (13) Finally, it appears in astrophysical application.

The thesis is addressed to mathematicians, physicists, and engineers who are interested in the analysis of nonlinear partial differential equations together with their applications in signal processing, mathematical physics and fluid mechanics.
There are two parts in this work. On the one hand, we have the theoretical part where our representation is quite complete about the p-Laplace and p-Dirac problems with proofs of existence and uniqueness for the solution. We will solve the p-Dirac equation for $1 < p < 2$, as well as for $2 < p < 3$ and search for strong solutions. The new strategy is to apply a generalized Teodorescu transform to the p-Laplace equation and to transform it into a p-Dirac equation by applying ideas from Clifford analysis. The obtained p-Dirac equation will be solved iteratively by using Banach's fixed-point theorem.

On the other hand, we have the numerical analysis and discrete function theory part, which contains important discrete operators including corresponding discrete mathematical methods and functional analysis in the frame of Clifford structures. Another important aspect is that the generalized discrete Borel-Pompeiu formula for a discrete complex-valued function, defined on a bounded domain in the two dimensional space will be proved. That this result is extending corresponding results in [Gürlebeck and Hommel, 2003], which were obtained under much stronger assumptions. Often, we can only explain the main ideas via examples. However, sometimes, we can refer to the literature. The numerical studies were performed for simplicity only for the two dimensional case. Finally, the numerical results of discrete nonlinear problems will be discussed.

1.3 The Structure of the Thesis

The thesis is organized as follows:

Chapter 2: This chapter presents an introduction to the Sobolev spaces and the norm in these spaces. An important part of this chapter deals with Sobolev and Hölder's inequalities, presenting the main notations first. After introducing the space $W^{m,p}(G)$, some famous and important theorems about the product of functions in Sobolev spaces will be considered. In the second part of this chapter, some basics from Clifford analysis and some important operators of Clifford analysis will be introduced to keep the thesis self-contained. It should be mentioned that these operators are important in the theoretical and numerical investigation.

Chapter 3: This chapter deals with boundary value problems for generalized p-Poisson equation and its transformation to p-Dirac equation by applying a Teodorescu transform T_G and ideas from Clifford analysis. We will prove the existence and uniqueness of the solution for the p-Dirac equation for $1 < p < 2$ and $2 < p < 3$. This will be done in two different ways. One way is to prove norm-based results in the scale of Sobolev spaces. The second way is related to point-wise convergence results. The obtained p-Dirac equation will be solved iteratively by using Banach's fixed-point theorem. Finally, the last part of this chapter presents a generalized normalized p-Laplace equation and a generalized normalized p-Dirac equation. It also contains 1-Laplace, ∞-Laplace, 1-Dirac and ∞-Dirac equations, and the relation between them.

Chapter 4: Deals with the applied method for discretizing the analytical problems based on a discrete function theory. The analytical theory of the discrete fundamental solution is well known and the existence of the fundamental solution is well investigated. The problem is the numerically stable calculation of the fundamental solutions which is new. Firstly, we calculate the discrete fundamental solution of Laplacian by using three methods. Then, we calculate the discrete fundamental solution of Cauchy-Riemann operator and its adjoint in the plane. This was an important task because while implementing the algorithms, it was recognized that not all results from the literature about the discrete Teodorescu operator are applicable. It became necessary to generalize them and to get a new formula to generalized discrete Teodorescu transform T_h^1, T_h^2 operators, which are right-inverse to the discrete Cauchy-Riemann operator and its adjoint in the plane. Furthermore, we get a new formula to generalized discrete boundary operators F_h^1, F_h^2. Using discrete T_h and F_h operators, we formulate in a theorem a discrete Borel-Pompeiu formula and we prove it.

Additionally, we define the discrete orthoprojections operators $\mathbf{P}_h$ and $\mathbf{Q}_h$ in the two dimensional case. Initially, it seems to be only a theoretical result with proofs of discretization methods. But, we will get representation formulas which can be directly used in numerical procedures and we will consider some examples to construct discrete holomorphic complex-valued functions and apply a generalized boundary operator F_h to these functions. Finally, we construct a discrete complex-valued function from the image of the orthoprojection $\mathbf{Q}_h$.

Chapter 5: This chapter deals with the discrete nonlinear p-Dirac equations. We use an iteration procedure to describe the solution of the p-Dirac equations, using Banach's fixed-point theorem in the plane. We will calculate the approximate solution for $1 < p < 2$ and $2 < p < 3$. Finally, in order to illustrate our numerical approach, we present numerical examples in form of a discrete approximation of the p-Dirac problem, and show its convergence to the solution of the corresponding continuous problem.

Chapter 6: This chapter contains the summary of the study and suggestions for future work and outlook.

2 Sobolev Spaces and Clifford Analysis

In solving several problems in mathematical physics and variational calculus, it is not enough to deal with classical solutions of differential equations. Therefore, the need for introducing the notion of weak derivatives and working on the so called Sobolev spaces arises. One of these problems is the nonlinear p-Laplace equation. Also, the p-Laplace operator appears in many nonlinear equations. In the early 1900's, the first research on the development of functional analysis began. In the 1920's, G. Hardy and J. Littlewood developed the Hardy-Littlewood inequality. During the 1930's S. Sobolev introduced the first concept for weak solutions, which was later developed in more details by L. Schwartz.

S. Sobolev developed the classical notion of differentiation further, which in turn, expanded the range of applications of Newton's and Leibniz's technique. The theory of Sobolev spaces is the basis for studying the existence, the regularity and the uniqueness of the solutions of nonlinear partial differential equations. Later, Sobolev spaces became one of the central important notions of mathematics, physics and engineering. With the pioneer work of Laurent Schwartz, the development of the theory of distributions gave a clear and definite meaning to such new objects; for example, the Dirac delta distribution.
For the Sobolev spaces, there are many important properties and these properties can be found in [Adams, 1975] and [Mazja, 1985] for further details.

The theory of Sobolev spaces originated from the Russian mathematician S. Sobolev around 1938 [Sobolev, 1938]. That theory was introduced with respect to the theory of partial differential equations. In fact, taking into account that elements of such spaces are special classes of distributions, the theory of Sobolev spaces has a close connection with the theory of distributions.

It is important to underline that the weak derivative is solely determined up to a set of measure zero. Nevertheless, a function is weakly differentiable if all its first order partial derivatives exist and α times weakly differentiable if all its weak derivatives exist for orders up to and including α. Thus, the concept of weak derivatives broadens the concept of classical derivatives, as weak derivatives are special kinds of distributional derivatives. For lack of time, distributional derivatives will not gain attention in this work. For more information we refer to [Ziemer, 1989].

The first section of this chapter is an introduction to the Sobolev spaces and the norm in these spaces. An important part of this section deals with Sobolev and Hölder's inequalities, presenting the main notations first. After introducing the space $W^{m,p}(G)$, we will consider some famous and important theorems about the product functions in the Sobolev space and later apply them. These theorems are very useful to study the mapping properties of the problem that we will discuss in later chapters. In the second section of this chapter, we will deal with Clifford algebra and some operators of Clifford analysis.

The main focus is to solve the p-Poisson and the p-Dirac equations. They will be studied in the scale of Sobolev spaces. The p-Poisson equation is studied in the form

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f, \tag{2.1}$$

where u and f are scalar-valued functions, defined in a domain $G \subset \mathbb{R}^N$, $N \geq 2$ and $1 \leq p < \infty$.

In this chapter, we are concerned with the regularity of solutions regarding the p-Poisson equation (2.1) and p-Dirac equation, in the scale of Sobolev spaces. The natural space for such a solution is the Sobolev space $W^{m,p}(G)$ with $m \geq 0$ and $p \geq 1$.

2.1 Sobolev Spaces and Embedding Theorems

The Sobolev norms are widely used in the study of p-Poisson and p-Dirac equations. The structure of the differential equations shows that some knowledge about the regularity of powers and products of functions belonging to certain Sobolev spaces is necessary. By definition, the weak solutions of the p-harmonic equation are part of the Sobolev space $W^{m,p}_{loc}(G)$. So, specifically, a weak solution can be redefined in a set of Lebesgue measure zero.

Obtaining the weak solutions Hölder's continuity goes through distinguishing between two cases, depending on the value of p. It is worth mentioning that N is the dimension. So, in the case where $p > N$, every function in $W^{1,p}(G)$ is continuous. But the case $p < N$ is much harder. For the values $1 < p < 2$ in the two-dimensional case we must study the product of the functions in detail.

For a better understanding, we repeat here the needed theorems without proofs. For the proofs and more details we refer to [Adams, 1975], [Reed and Simon, 1978], [Valent, 1985], [Ziemer, 1989], [Hunter and Nachtergaele, 2001], [Wong, 2010], [John, 2013] and [Demengel and Demengel, 2012].
Let us recall some definitions and notations.

Definition 1. *An open connected set $G \subset \mathbb{R}^N$, $N \geq 1$ is called a domain. By $\overline{G}$ we denote the closure of G; $\Gamma = \partial G$ is the boundary.*

Definition 2. *We say that a domain $G' \subset G \subset \mathbb{R}^N$ is a strictly interior subdomain of G, if $\overline{G'} \subset G$.*

We use the following notations:

$$x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N, \quad \partial_j u = \frac{\partial u}{\partial x_j},$$

$$\alpha = (\alpha_1, \alpha_2, \ldots \alpha_N) \in \mathbb{Z}^N \text{ is a multi-index.}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N, \quad \partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}$$

$$\nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_N u), \quad |\nabla u| = \left(\sum_{j=1}^{N} |\partial_j u|^2 \right)^{1/2}.$$

Definition 3 (Continuously differentiable functions)**.** *For any non-negative integer m, $C^m(G)$ denotes the space of all functions whose partial derivatives, up to and including order m, are all continuous on G.*

Definition 4. *The spaces $C^0(G) = C(G)$ and $C^1(G)$ denote the spaces of continuous functions and continuously differentiable functions, respectively, on G. The space*

$$C^\infty(G) := \bigcap_{m=0}^{\infty} C^m(G),$$

denotes the space of smooth functions on G.

The subspace of $C(\overline{G})$ containing the continuous functions with vanishing boundary values is denoted by $C_0(G)$. Finally, we introduce $\mathcal{D}(G)$ as the set of $C^\infty(G)$ functions with compact support in G. The dual space of $\mathcal{D}(G)$ will be denoted by $\mathcal{D}'(G)$.

Definition 5 (Lebesgue integrable functions)**.** *For any real number q, $1 \leq q < \infty$, the Lebesgue space $L_q(G)$ is the space of equivalent classes of q-integrable functions on G, i.e. it is the set of all Lebesgue measurable functions $u(x)$ defined on G such that the norm*

$$||u||_{0,q} = \left(\int_G |u(x)|^q dx \right)^{1/q}$$

is finite, where $||.||_{0,q}$ is the usual norm of $L_q(G)$.

The space $L_q(G)$ for $q \geq 1$ is a Banach space. Moreover for $q = 2$, we have a Hilbert space with the scalar product

$$(u, v)_2 := \int_G \overline{u(x)} v(x) dx$$

Definition 6 (Bounded function)**.** *A measurable function u on the domain G is essentially bounded if there exists a constant k such that $|u(x)| \leq k$ almost everywhere on G. The greatest lower bound of such constants k is called the essential supremum (or essential maximum) of $|u|$ on G, and is denoted by ess $sup_{x\in G}|u(x)|$. We denoted by $L_\infty(G)$ the vector space of all functions u that are essentially bounded on G. The functional $||.||_{0,\infty}$ defined by*

$$||u||_{0,\infty} := \ ess\ sup_{x\in G}|u(x)|$$

is a norm on $L_\infty(G)$.

Definition 7 (Local integrability)**.** *A function u defined almost everywhere on G is locally L_q-integrable on G provided $u \in L_q(\Omega)$ for every measurable Ω such that $\overline{\Omega} \subseteq G$ and $\overline{\Omega}$ is compact in $\mathbb{R}^N$. We write $u \in L_q^{loc}(G)$.*

Definition 8. *$L_q^{loc}(G)$, $1 \le q < \infty$, is the set of all measurable functions $u(x)$ in G such that $\int_{G'} |u(x)|^q dx < \infty$ for any bounded strictly interior subdomain $G' \subset\subset G$.*

Definition 9 (Weak derivatives)**.** *Let $G \subset \mathbb{R}^N$, $N \ge 1$, $u, v \in L_p(G)$, and α be a multi-index. Then v is the weak α-th partial derivative of u if*

$$\int_G u(x)\partial^\alpha \varphi(x)\,dx = (-1)^{|\alpha|}\int_G v(x)\varphi(x)\,dx$$

for all $\varphi \in C_0^\infty(G)$.

Definition 10 (Sobolev spaces)**.** *Let $G \subset \mathbb{R}^N$, $N \ge 1$, be a domain and m be a positive integer. The space, denoted by $W^{m,p}(G)$, containing all equivalence classes of functions $u \in L_p(G)$ whose derivatives (in the distributional sense) $\partial^\alpha u$ for $|\alpha| \le m$ belong to $L_p(G)$ is called Sobolev space.*

The Sobolev space $W^{m,p}(G)$ is a Banach space with the norm

$$\|u\|_{m,p} := \left(\sum_{0\le|\alpha|\le m} \|\partial^\alpha u\|_{0,p}^p\right)^{1/p}, \text{ for } 1 \le p < +\infty, \tag{2.2}$$

and

$$\|u\|_{m,\infty} := \max_{0\le|\alpha|\le m} \|\partial^\alpha u\|_{0,\infty}, \text{ for } p = +\infty. \tag{2.3}$$

For $m = 0$ we have

$$W^{0,p}(G) = L_p(G).$$

The Sobolev space $W^{m,p}(G)$ for $p = 2$ is the Hilbert space $H^m(G)$.

The spaces $W_0^{m,p}(G)$ are the closure of $C_0^m(G)$ in the Sobolev space $W^{m,p}(G)$.

Theorem 1. *Let G be an arbitrary open subset of $\mathbb{R}^N$, $N \ge 1$. The subspace $C^\infty(G) \cap W^{m,p}(G)$ is dense in $W^{m,p}(G)$, for $p < \infty$.*

Example 2. *Let $G = (-1, 1)$. Considering the function $f(x) = |x|^\alpha$, for $0 < \alpha < 1$. This function is obviously not differentiable at zero (at the origin), but for any $1 \le p < 1/(1-\alpha)$, it belongs to the Sobolev space $W^{1,p}(G)$.*
Generally, let us assume that G is an open unit ball in $\mathbb{R}^N$, $N \ge 1$ centered at the origin. Then, the function $|x|^\alpha \in W^{1,p}(G)$ if and only if

$$\alpha > 1 - \frac{N}{p}.$$

To prove this, we can first calculate

$$f_{x_i}(x) = \frac{\alpha x_i}{|x|^{2-\alpha}} \quad , \textit{for } x \neq 0, i = 1, 2, \cdots, N$$

and

$$|\nabla f(x)| = \frac{\alpha}{|x|^{1-\alpha}}.$$

From the above, one can see that $|\nabla f(x)|$ *belongs to* $L_p(G)$ *if and only if* $\alpha > 1 - \frac{N}{p}$.

Dual space: [Adams, 1975] The dual space of the space $W^{m,p}(G)$ is defined as $W^{-m,p'}(G)$, the conjugate exponent for given p, p', where

$$p' = \begin{cases} \infty & \text{if } p = 1 \\ p/(p-1) & \text{if } 1 < p < \infty \\ 1 & \text{if } p = \infty. \end{cases}$$

We will consider the embedding theorem, which gives the relationship between different functional spaces and consider the embedding of Sobolev spaces.

Definition 11. *Let* X *and* Y *be two Banach spaces. The space* X *is embedded into space* Y *and written* $X \hookrightarrow Y$*, if for any* $u \in X$*, we have* $u \in Y$ *and*

$$\|u\|_Y \leq C\|u\|_X,$$

where C *is a constant not dependent on* u*.*

The usefulness in analysis of the Sobolev spaces results from the embedding characteristics of the spaces, especially for studying differential and integral operators. Indeed, embedding theorems appear to be instrumental in modern analysis and in boundary value problems. For more information and several types of embedding theorems reference can be made to [Adams, 1975].

Theorem 2 (Sobolev inequality)**.** *Let* $G \subset \mathbb{R}^N$ *be a bounded domain, and* $1 \leq p < \infty$*, then we have*

$$W_0^{m,p}(G) \subset L_q(G), \qquad \forall\, 1 \leq q \leq \frac{Np}{N - mp} < \infty, \quad N > mp,$$

moreover, for $u \in W_0^{m,p}(G)$*, we have*

$$\|u\|_{0,q} \leq C \|u\|_{m,p} \,,$$

where $q \leq \frac{Np}{N-mp}$ *is compact and* $C = C(N, G, q, m, p)$.

Theorem 3. *Let* $G \subset \mathbb{R}^N$ *be a Lipschitz domain, not necessarily bounded, and* $1 \leq p < \infty$*, the following embeddings are continuous*

$$W^{m,p}(G) \subset L_q(G), \qquad q = \frac{Np}{N - mp}, \quad N > mp.$$

[Reed and Simon, 1978], M. Reed and B. Simon considered the generalized Hölder's inequality.

Theorem 4 (Hölder's inequality)**.** *Let* $1 \leq p, q, r \leq \infty$*. We have*

$$\|u_1 u_2\|_{0,r} \leq \|u_1\|_{0,p} \|u_2\|_{0,q}$$

if $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$*, for all* $u_1 \in L_p(G)$ *and* $u_2 \in L_q(G)$*, where* $G \subseteq \mathbb{R}^N$ *and* $N \geq 1$.

Now, we will focus our attention on the multiplication of functions in Sobolev spaces: Putting together Sobolev and Hölder's inequalities, we refer to the following theorem.

Theorem 5 (Sobolev product estimates)**.** *[Wong, 2010] Let* $u_1 \in W^{m_1,p_1}(\mathbb{R}^N)$ *and* $u_2 \in W^{m_2,p_2}(\mathbb{R}^N)$*, where* $N \geq 1$*. Then, the product of* $u_1 u_2$ *belongs to* $W^{m,p}(\mathbb{R}^N)$*, where* $m \leq \min(m_1, m_2)$ *and*

$$\frac{1}{p} - \frac{m}{N} > \frac{1}{p_1} - \frac{m_1}{N} + \frac{1}{p_2} - \frac{m_2}{N}.$$

Moreover, there is a constant $C = C(m_1, m_2, m, p_1, p_2, p, N)$ *such that*

$$\|u_1 u_2\|_{m,p} \leq C \|u_1\|_{m_1,p_1} \|u_2\|_{m_2,p_2} \,.$$

Remark 1. *[Wong, 2010] Theorem 5 can be considered for* $u_1 \in W_0^{m_1,p_1}(G)$ *and* $u_2 \in W_0^{m_2,p_2}(G)$*, where* $G \subset \mathbb{R}^N$ *and with the same conditions for all parameters. Then, the product of* $u_1 u_2$ *belongs to* $W_0^{m,p}(G)$.

The next theorem describes the products of functions belonging to Sobolev spaces in bounded domains without the restriction of vanishing boundary values, which is the most important theorem.

Theorem 6 (Products of functions in Sobolev Spaces)**.** *[Valent, 1985] Let G be a bounded domain with the cone property in $\mathbb{R}^N$, $N \geq 1$. Let $m \geq 0$ and let p, q, $r \geq 1$ and $p \geq r$, $q \geq r$ and*

$$\frac{m}{N} > \frac{1}{p} + \frac{1}{q} - \frac{1}{r}. \tag{2.4}$$

If $u \in W^{m,p}(G)$ and $v \in W^{m,q}(G)$, then, we have $uv \in W^{m,r}(G)$ and there is a positive number c independent of u and v such that

$$\|uv\|_{m,r} \leq c \|u\|_{m,p} \|v\|_{m,q} .$$

If we have $q > r$ ($p > r$), let us consider the real number α (β) such that:

$$\frac{1}{\alpha} + \frac{1}{q} = \frac{1}{r}, \qquad (\text{resp. } \frac{1}{p} + \frac{1}{\beta} = \frac{1}{r})$$

That is $\alpha = qr/(q-r)$ and $\beta = pr/(p-r)$). By using Hölder's inequality, we will get the following implications:

$$\begin{aligned} u \in L_p(G), \quad p > r, \quad v \in L_\beta(G) \quad &\Rightarrow uv \in L_r(G), \\ \|uv\|_{0,r} \quad &\leq c_1 \|u\|_{0,p} \|v\|_{0,\beta} , \\ u \in L_q(G), \quad q > r, \quad v \in L_\alpha(G) \quad &\Rightarrow uv \in L_r(G), \\ \|uv\|_{0,r} \quad &\leq c_1 \|u\|_{0,q} \|v\|_{0,\alpha} , \end{aligned} \tag{2.5}$$

where the constant $c_1 > 0$ is independent of u and v.

Hence, by the embedding theorem of Sobolev space (see [Adams, 1975]) under the condition $p \leq N$ ($q \leq N$)

$$\begin{aligned} u \in L_p(G), \quad v \in W^{1,q}(G) \ &\Rightarrow uv \in L_r(G), \\ \|uv\|_{0,r} \ &\leq c_2 \|u\|_{0,p} \|v\|_{1,q} , \\ u \in L_q(G), \quad v \in W^{1,p}(G) \ &\Rightarrow uv \in L_r(G), \\ \|uv\|_{0,r} \ &\leq c_2 \|u\|_{0,q} \|v\|_{1,p} , \end{aligned} \tag{2.6}$$

where the constant $c_2 > 0$ is independent of u and v.

T. Valent, in [Valent, 1985] considers unbounded domains in conjunction with bounded domains.

In [Tao, 2010], T. Tao considered an instructive special case.

Theorem 7. *Let $m \geq 1$, $1 < p$, $q < N/m$, $N \geq 1$ and $1 < r < \infty$, such that*

$$\frac{1}{p} + \frac{1}{q} - \frac{m}{N} = \frac{1}{r}.$$

If $u_1 \in W^{m,p}(G)$, $u_2 \in W^{m,q}(G)$ and $G \subset \mathbb{R}^N$, then the product $u_1 u_2$ belongs to $W^{m,r}(G)$, and

$$\|u_1 u_2\|_{m,r} \leq C \, \|u_1\|_{m,p} \, \|u_2\|_{m,q} \, ,$$

this constant C depends only on p, q, m, N and r.

Remark 2. *The formulations of the previous theorems are very similar. The concrete result depends on the domain (bounded or unbounded) and on the regularity of the boundary.*

In [John, 2013], continuous embeddings are considered of Sobolev spaces in different cases:
(1) with the same order of the derivative m and different integration powers. (2) with different order of the derivative and same order of the integration power p.

Theorem 8. *[The Sobolev embedding theorem] [Adams, 1975] [John, 2013]. Let G be a bounded domain in $\mathbb{R}^N$, $N \geq 1$.*
(1)Let $m \geq 0$, and $p, q \in [1, \infty]$ with $q > p$. Then, $W^{m,q}(G) \subset W^{m,p}(G)$.
(2)Let $p \in [1, \infty]$ and $m \leq k$, then $W^{k,p}(G) \subset W^{m,p}(G)$.

We prove a new Lemma and use it later in the mapping properties.

Lemma 1 (The power of modulus in L_p norm)**.** *Let $v \in L_p(G)$, where $1 < p < 2$, then, $|v|^{2-p} \in L_{p^*}(G)$, where $p^* = \frac{p}{2-p}$ and*

$$\|v\|_{0,p}^{2-p} = \left\| |v|^{2-p} \right\|_{0,p^*} .$$

Proof. Since $v \in L_p(G)$, we have by the definition of L_p norm:

$$\left(\|v\|_{0,p}\right)^{2-p} = \left(\left(\int\limits_G |v|^p \, dG \right)^{\frac{1}{p}} \right)^{2-p} .$$

Multiplying the power of $|v|$ by $\frac{2-p}{2-p}$, we get:

$$\left(\int\limits_G \left(|v|^{2-p}\right)^{\frac{p}{2-p}} dG \right)^{\frac{2-p}{p}} = \left(\int\limits_G \left(|v|^{2-p}\right)^{p^*} dG \right)^{\frac{1}{p^*}} = \left\| |v|^{2-p} \right\|_{0,p^*} .$$

This completes the proof of the lemma. □

Remark 3. *In a similar way, we can find* $|v|^{p-1} \in L_{\tilde{p}}(G)$*, where* $\tilde{p} = \frac{p}{p-1}$.

In [Demengel and Demengel, 2012] the next theorem and lemma were considered.

Theorem 9 (Density of the Regular Functions)**.** *Let* G *be an open subset of* $\mathbb{R}^N$ *for* $N \geq 1$*, then, for any* p *with* $1 < p < \infty$*, the space* $\mathcal{D}(G)$ *is dense in the norm of the space* $L_p(G)$*.*

Lemma 2. *Let* $u \in W^{1,q}(G)$*, where* $q \in (1, \infty)$*. Then* $|u|^{q-1} u$ *and* $|u|^q$ *both belong to* $W^{1,1}(G)$*, while*

$$\nabla(|u|^{q-1} u) = q\,|u|^{q-1} \nabla u$$

and

$$\nabla(|u|^q) = q\,|u|^{q-2} u \nabla u.$$

We have to adapt this lemma to the conditions which will be used later in this chapter.

Lemma 3. *Let* v *be an element of* $W^{1,q}(G)$ *for* $q \in (1, \infty)$*. Then,* $|v|^{q-1}$ *belongs to* $W^{1,q/q-1}(G)$ *and*

$$\nabla(|v|^{q-1}) = (q-1)\,|v|^{q-3} v \nabla v$$

Proof. We will follow the idea of the proof of Lemma 2, see [Demengel and Demengel, 2012]. By Theorem 1, there is a sequence $\{v_n\} \subset C^\infty(G) \cap W^{1,q}(G)$ which converges to v in $W^{1,q}(G)$. We can easily show that the gradient of $|v_n|^{q-1}$ is given by

$$\nabla(|v_n|^{q-1}) = (q-1)\,|v_n|^{q-3} v_n \nabla v_n.$$

The sequence $|v_n|^{q-3} v_n$ converges to $|v|^{q-3} v$ in $L_{q/(q-2)}$ and ∇v_n converges to ∇v in L_q. Then it follows that $(q-1)\,|v_n|^{q-3} v_n \nabla v_n$ converges to $(q-1)\,|v|^{q-3} v \nabla v$ in $L_{q/(q-1)}$. Furthermore, $|v_n|^{q-1}$ converges to $|v|^{q-1}$ in $L_{q/(q-1)}$, and consequently also in the sense of distributions. By a property of distributions stated in Section (1.4.8) of [Demengel and Demengel, 2012], $\nabla(|v_n|^{q-1}) \to \nabla(|v|^{q-1})$ in the sense of distributions. Likewise, as $|v_n|^{q-1} \to |v|^{q-1}$ in the sense of distributions also, the convergence holds in $\mathcal{D}'(G)$. Consequently, $\nabla(|v_n|^{q-1})$ converges to $\nabla(|v|^{q-1})$ in $\mathcal{D}'(G)$.
Therefore, taking the limit provides us with the identity and applying Theorem 9 it follows

$$\nabla(|v|^{q-1}) = (q-1)\,|v|^{q-3} v \nabla v.$$

Finally, we have proved that $|v|^{q-1} \in W^{1,q/q-1}(G)$. $\square$

Lemma 4. *Let $u \in W^{1,q}(G)$ for $q \in (1, \infty)$ and G be a bounded domain in $\mathbb{R}^N$; $N \geq 1$, then*

$$|u|^{p_1} \in W^{1,r}(G) \text{ for } r < \frac{qN}{Np_1 - q} \text{ and } Np_1 > q.$$

Proof. We know that $|u|^{p_1-1}$ and $|u|$ both belong to $W^{1,q/p_1-1}(G)$ and $W^{1,q}(G)$ respectively. By using Theorem 6 (Equation (2.4)), we can conclude that

$$\frac{p_1 - 1}{q} + \frac{1}{q} - \frac{1}{r} < \frac{1}{N} \Rightarrow \frac{p_1}{q} - \frac{1}{N} < \frac{1}{r} \Rightarrow r < \frac{qN}{Np_1 - q},$$

and

$$|u|^{p_1-1} |u| = |u|^{p_1}.$$

That means, we have proved $|u|^{p_1} \in W^{1,r}(G)$. $\square$

These results will be used in Chapter 3, to prove existence and regularity results. In the next part of this chapter, we will introduce the basic concepts of Clifford algebras and their associated function theory.

2.2 Clifford Algebra and Basic Operators of Clifford Analysis

The main idea when we study the p-Poisson equation is to substitute the gradient by the most important operator of hypercomplex function theory which is called a generalized Cauchy-Riemann operator to get the generalized p-Poisson equation which is studied in the form

$$\Delta_p u = D(|Du|^{p-2} Du) = f,$$

where u and f are scalar-valued functions, defined in a domain $G \subset \mathbb{R}^N$, $N \geq 2$ and $1 \leq p \leq \infty$. If $f = 0$ then, the equation is called p-Laplace equation.

Substituting Du by w, the p-Dirac equation

$$D(|w|^{p-2} w) = f \tag{2.7}$$

will be obtained. Here, w is a vector-valued function. This leads to the idea of working from the beginning until the end with vector-valued functions u. By replacing the application of the divergence operator and the gradient by the application of a generalized Cauchy-Riemann operator, one obtains a p-Laplace (Poisson) equation for vector-valued functions and a corresponding p-Dirac equation.

The aim of this section is to present the principle facts of hypercomplex analysis as well as some operators from spatial function theory.

2.2.1 Introduction to Clifford Algebras

William K. Clifford, a famous English mathematician and scientific philosopher of the 19^{th} Century, worked extensively in many important branches of pure mathematics and classical mechanics. The unification of geometric algebra into a scheme now referred to as Clifford algebra (named in his honour) is one of the important achievements of W. Clifford.

The exterior algebra of a vector space is called the Grassmann algebra. Grassmann's ideas were finally picked up by W. K. Clifford who turned Grassmann algebras into Clifford algebras. Geometric algebra is used in engineering mechanics (particularly in robotics), as well as in mathematics and physics.

"*The linear extension theory*" known in German as "*Die lineale Ausdehnungslehre*" by the German mathematician H. Grassmann developed in 1844 has paved the way to the birth of the Clifford algebras in 1878. Indeed W. Clifford has embedded the quaternions into the Grassmann system in his paper "*Applications of Grassmann's extensive algebra*", see [Clifford, 1878]. Clifford himself, refers to his algebra as "Geometric algebra".

In 1982, the book "*Clifford Analysis*" was written by F. Brackx, R. Delanghe and F. Sommen [Brackx et al., 1982]. Subsequently, the title of the book became an integral part of higher dimensional analysis, which is characterized by Clifford algebra and its associated applications.

Analysis of functions with values in a Clifford algebra is a unifying language for mathematics, and a revealing language for physics. It is the best approach for the treatment of non-linear boundary value problems (for example p-Laplace and p-Dirac problems) in higher dimensions. There is a numerical analysis which solved many non-linear problems, that is supported by discrete Clifford calculus.

The Clifford analysis encompasses many other theories, namely: operator theory,

potential theory, analytic number theory, boundary value problems of partial differential equations, differential geometry and harmonic analysis.
Clifford analysis plays an increasing role in different areas such as mathematics, physics, computer science and engineering.

The works by [Gürlebeck and Sprößig, 1990], [Gürlebeck and Sprößig, 1997] and [Gürlebeck et al., 2016] are utilized, in the following Subsections 2.2.2 and 2.2.3.

2.2.2 Principles of the Hypercomplex Analysis

Let us consider $N \geq 1$ and $\{e_0, e_1, \ldots, e_N\}$ being an orthonormal basis of $\mathbb{R}^{N+1}$. Then, the multiplication rules of the basis vectors e_i are defined by:

$$\begin{aligned}
e_0 e_i &= e_i e_0 = e_i,\ i = 0, \ldots, N, \\
e_0^2 &= 1, \quad e_i^2 = -1,\ i = 1, \ldots, N, \\
e_i e_j &= -e_j e_i,\ i \neq j,\ i, j = 1, \ldots, N, \\
e_i e_j + e_j e_i &= -2\,\delta_{ij,}\ i, j = 1, \ldots, N,
\end{aligned}$$

where δ_{ij} is the Kronecker symbol. The Clifford algebra $C\ell_{0,N}$ constructed over the Euclidean space $\mathbb{R}^{N+1}$ with the multiplication of basis vectors, where e_j $(j = 0, \cdots, N)$ are unit vectors of $\mathbb{R}^{N+1}$. In this way, the algebra $C\ell_{0,N}$ is completely described. It is called *(universal) Clifford algebra*. Addition and multiplication with real numbers are coordinate-wisely given. A basis for the algebra $C\ell_{0,N}$ is determined by

$$e_0; e_1, \ldots, e_N; e_1e_2, \ldots, e_{N-1}e_N; e_1e_2e_3, \ldots; \ldots; e_1e_2 \ldots e_N,$$

with e_0 as unit element. An arbitrary *Clifford number* is given by

$$x = x_0 + \sum_{k=1}^{N} \sum_{0<i_1<\ldots<i_k\leq N} x_{i_1\ldots i_k} e_{i_1\ldots i_k} = \sum_{\mathcal{A}\in\mathcal{P}_N} x_{\mathcal{A}} e_{\mathcal{A}},$$

with the abbreviations

$$e_{i_1 i_2 \ldots i_k} := e_{i_1} e_{i_2} \ldots e_{i_k}.$$

$\mathcal{P}_N$ contains subsets $\mathcal{A}$ of $\{1, \ldots N\}$, where in the subsets, the numbers are naturally ordered according to the size. $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. The dimension of $C\ell_{0,N}$ is equal to 2^N. A Clifford number with scalar values and vector values only is called *a paravector*. For any $x \in C\ell_{0,N}$, *the absolute value* or *modulus* is defined by

$$|x| := \left(\sum_{\mathcal{A} \in \mathcal{P}_N} x_{\mathcal{A}}^2 \right)^{\frac{1}{2}}.$$

Therefore, one can consider the Clifford algebra as a Euclidean space of dimension 2^N furnished with the Euclidean metric. The functions with values in Clifford algebra can be seen as vector-valued functions. Let $\lambda \in \mathbb{R}$ and $x, y \in C\ell_{0,N}$, then, it holds

$$||x| - |y|| \leq |x - y| \leq |x| + |y| \quad \text{and} \quad |\lambda x| = |\lambda||x|.$$

It should be mentioned that $C\ell_{0,1}$ is isomorphic to the complex numbers $\mathbb{C}$, and $C\ell_{0,2} = \mathbb{H}$, which leads to the real quaternions. We have an increasing tower $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset C\ell_{0,3} \subset \cdots$.

Analogously to the complex case, a conjugation can be defined by

$$\bar{e}_0 = e_0, \, \bar{e}_i = -e_i \quad \text{for } i = 1, \dots N.$$

$N > 1$ leads to the relation $\overline{xy} = \overline{y}\,\overline{x}$. It is easy to show that

$$\overline{\overline{x}} = x \text{ and } \overline{x + y} = \overline{x} + \overline{y}.$$

Again, analogously to the complex case, one can find that

$$\bar{x}x = x\bar{x} = |x|^2$$

and for $x \neq 0$ being a paravector, there is an inverse given by

$$x^{-1} = \frac{\overline{x}}{x\overline{x}}.$$

For a paravector $x = x_0 + \sum_{i=1}^{N} x_i e_i$, we will call

$$\text{Sc } x =: x_0 e_0 = x_0$$

the scalar part of x and

$$\text{Vec } x := \underline{x} = x - x_0 e_0$$

is the vector part of x.

2.2.3 Some Basic Facts and Formulae from Spatial Function Theory

- The Dirac-type operator D in $\mathbb{R}^N$ is the first order differential operator which is defined as follows
$$D = \sum_{i=1}^{N} e_i \frac{\partial}{\partial x_i},$$
which acts on the space $C^1(G, C\ell_{0,N})$, where $G \subset \mathbb{R}^N$ and $x = \sum\limits_{i=1}^{N} x_i e_i$.

  A function $u \in C^1(G, C\ell_{0,N})$ is called left Clifford regular (right Clifford regular) if and only if
$$Du = 0 \ (uD = 0).$$
A function which is both left and right Clifford regular is called both-sided Clifford regular. The solutions of $Du = 0$ play the same role as the holomorphic functions in complex analysis.

- The Cauchy kernel $\mathbf{e}$ in $\mathbb{R}^N \backslash \{0\}$ is defined by
$$\mathbf{e}(x) = \frac{1}{\sigma_N} \frac{\overline{x}}{|x|^N} \ , (x \neq 0)$$
with $x = \sum\limits_{i=1}^{N} x_i e_i$. Here, σ_N denotes the surface area of the unit ball in $\mathbb{R}^N$.

  Using the Cauchy kernel $\mathbf{e}$, we can introduce the next two integral operators.

- Let $u \in C(G, C\ell_{0,N})$ and let G be a domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial G = \Gamma$. The Teodorescu transform over G is defined by
$$(T_G u)(x) = (Tu)(x) = -\int\limits_G \mathbf{e}(x-y) u(y) dy.$$
The Teodorescu transform is a weakly singular operator and the most important algebraic property is that it is a right inverse of the Dirac operator D.
If G is a bounded domain, then
$$T_G : L_p(G) \longrightarrow W^{1,p}(G)$$

is continuous.

- Let $u \in C^1(G, C\ell_{0,N}) \cap C(\overline{G}, C\ell_{0,N})$. The Cauchy-Bitsadze operator or Cauchy-Fueter operator is given by

$$(F_\Gamma u)(x) = (Fu)(x) = \int\limits_\Gamma \mathbf{e}(x-y)\alpha(y)u(y)d\Gamma_y,$$

where α stands for the outward pointing unit normal vector at $y \in \Gamma$. A sufficient smoothness of the boundary $\Gamma = \partial G$ has to be assumed.
Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then, the Cauchy-Fueter operator F_Γ is a continuous map

$$F_\Gamma : W^{m-\frac{1}{p},p}(\Gamma, C\ell_{0,N}) \longrightarrow W^{m,p}(G, C\ell_{0,N}) \cap (\ker\, D)(G, C\ell_{0,N}).$$

- The Cauchy operator F_Γ, the Dirac operator D and its right inverse T_G are connected through the Borel-Pompeiu formula in the domain $G \subset \mathbb{R}^N$, $N \geq 1$, with smooth boundary Γ

$$(F_\Gamma u)(x) + (T_G D u)(x) = \begin{cases} u(x), & x \in G \\ 0, & x \in \mathbb{R}^N \backslash \overline{G}. \end{cases}$$

If u is a $C\ell$-holomorphic function, it means that $F_\Gamma u = u$ and the Cauchy integral represents the $C\ell$-holomorphic function from its boundary values.

- In [Gürlebeck and Sprößig, 1990], it has been proven that operators $\mathbf{P}$ and $\mathbf{Q}$ are orthoprojections onto the subspaces $\ker D \cap L_2(G)$ and $DW_0^{1,2}(G)$ of $L_2(G)$ respectively. More precisely, we have the orthoprojections

$$\mathbf{P} : L_2(G) \longrightarrow \ker D \cap L_2(G) \quad \text{and} \quad \mathbf{Q} : L_2(G) \longrightarrow DW_0^{1,2}(G).$$

These operators have the representations

$$\mathbf{P} = F_\Gamma (tr_\Gamma T_G F_\Gamma)^{-1} tr_\Gamma T_G,$$

$$\mathbf{Q} = I - \mathbf{P}.$$

Here, tr_Γ denotes the restriction operator (trace operator) onto the boundary Γ, and I is the identity operator. This orthogonal decomposition is the so-called Bergman-Hodge decomposition, important for the solution of elliptic equations.

There is an important property of the orthoprojection $\mathbf{Q}$, that should be mentioned. The characterization of a function from the image of $\mathbf{Q}$ can be given in the next lemma and we will use this property in the next chapter.

Lemma 5. *[Gürlebeck and Sprößig, 1990] [Gürlebeck and Sprößig, 1997]. A function u belongs to $Im\,\mathbf{Q}$ if and only if $tr_\Gamma T_G u = 0$, where $Im\,\mathbf{Q}$ is the image of the orthoprojection operator $\mathbf{Q}$ and $tr_\Gamma u$ is the trace or restriction of u onto Γ.*

3 Boundary Value Problems for Nonlinear p-Poisson and p-Dirac Equations

In this chapter, firstly, we will solve the p-Poisson equation for $1 < p < 2$, as well as for $2 < p < 3$ and search for strong solutions. The new strategy is to apply a hypercomplex integral operator (a generalized Teodorescu transform) and spatial function theoretic methods to the generalized p-Poisson equation and transform it into the p-Dirac equation by applying ideas from Clifford analysis.

The obtained p-Dirac equation will be solved iteratively by using Banach's fixed-point theorem (The contraction mapping theorem), which is one of most useful methods for solving linear and non-linear problems (see [Petersen, 1999]). Finally, we will prove the existence and uniqueness of solutions to the p-Dirac equations.

The last part of this chapter deals with ∞-Laplace, ∞-Dirac, 1-Laplace and 1-Dirac equations and the connections between them. The normalized generalized p-Laplacian Δ_p^c will be considered.

The structure of this chapter is as follows:

- Firstly, Section 3.1 deals with Banach's fixed-point theorem.
- In Section 3.2 the generalized p-Poisson equation is considered in terms of the Dirac operator in $\mathbb{R}^N$, $N \geq 1$.
- In Section 3.3 the main result on existence and uniqueness of the solutions of nonlinear p-Dirac problem is shown in different ways in $\mathbb{R}^N$ for $N \geq 2$ and $1 < p < 2$.

- Section 3.4 deals with p-Dirac problem in $\mathbb{R}^N$ for $N \geq 2$ and $2 < p < 3$.
- Finally, in Section 3.5 the normalized generalized p-Laplace and the normalized generalized p-Dirac equations will be considered.

3.1 Banach's Fixed-Point Theorem

In 1920 Banach proved the renowned result which is well-known in the literature as the *Banach's fixed-point theorem* or the *Contraction mapping principle.*

Definition 12. *[Petersen, 1999] Let (X, d) be a metric space. Let $F : X \to X$ be a mapping from the set X to itself, which is called a contraction mapping of (X, d), if there exists a constant C with $0 \leq C < 1$, such that*

$$d(F(x), F(y)) \leq C\, d(x, y), \qquad \forall x, y \in X.$$

This constant is called the contractivity coefficient.

Theorem 10 (Banach's fixed-point theorem)**.** *[Petersen, 1999] Let (X, d) be a Banach space and let $F : X \to X$ be a contraction mapping with contractivity coefficient C. For any initial guess $x_0 \in X$, we define*

$$x_n = F(x_{n-1}) = F^n(x_0), \quad n \geq 1.$$

Then the mapping F has a unique fixed point x and the sequence x_n converges to this point with

$$d(x_n, x) \leq C^n d(x_0, x).$$

The a-priori and a-posteriori error estimates are given by

$$d(x_n, x) \leq \frac{C^n}{1 - C}\, d(x_0, x_1), \quad n = 0, 1, 2, \ldots$$

and

$$d(x_n, x) \leq \frac{C}{1 - C}\, d(x_{n-1}, x_n), \quad n = 0, 1, 2, \ldots.$$

3.2 The Generalized p-Poisson Equation in Terms of the Dirac Operator D

Mostly, the p-Poisson equation is studied in the form

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f,$$

where u and f are scalar-valued functions, defined in a domain $G \subset \mathbb{R}^N$, $N \geq 1$ with sufficiently smooth boundary $\partial G = \Gamma$ and $1 \leq p \leq \infty$. If $f = 0$ then, the equation is called p-Laplace equation.

Considering a (Neumann-type) boundary value problem for the p-Poisson equation and rewriting it in terms of the Dirac operator D , we will get

$$\begin{aligned} \Delta_p u = \operatorname{div}\left(|Du|^{p-2}\, Du\right) &= f \quad \text{in } G, \\ Du &= 0 \quad \text{on } \Gamma. \end{aligned}$$

The classical factorization $\Delta = \operatorname{div}\operatorname{grad}$ will be replaced by the symmetric factorization $-\Delta = D^2$.
Replacing the application of the divergence operator and the gradient by the application of a Dirac operator one obtains a generalized p-Poisson equation for vector-valued functions

$$\Delta_p u = D\left(|Du|^{p-2}\, Du\right) = f \quad \text{in } G, \tag{3.1}$$
$$Du = 0 \quad \text{on } \Gamma. \tag{3.2}$$

By substituting Du by w, we obtain the nonlinear p-Dirac equation

$$D\left(|w|^{p-2}\, w\right) = f \qquad \text{in } G \tag{3.3}$$
$$w = 0 \qquad \text{on } \Gamma. \tag{3.4}$$

Let $u \in W^{m,p}(G)$; $m = 1, 2$ for $1 < p < \infty$. Note that $D^2 = -\Delta$ is the Laplacian in $\mathbb{R}^N$.
In the next sections, we will solve the nonlinear boundary problem (3.3), (3.4), firstly for $1 < p < 2$ and then for $2 < p < 3$.

In [Al-Yasiri and Gürlebeck, 2016], the main result of the next Subsection 3.3 has been published.

3.3 Boundary Value Problem for $1 < p < 2$

Let G be a bounded domain with smooth boundary $\partial G = \Gamma$ in $\mathbb{R}^N,\ N \geq 2$. Find $w \in W^{m,p}(G)$ $(m = 1; 1 < p < 2)$ satisfying

$$D\left(|w|^{p-2}\, w\right) = f \qquad \text{in } G \tag{3.5}$$

$$w = 0 \qquad \text{on } \Gamma. \tag{3.6}$$

In addition to the non-homogenous part f, we assume some necessary conditions:

$$f \in L_{\tilde{p}}(G) \cap L_q^{loc}(G), \text{ with } \tilde{p} = \frac{p}{p-1},\ q > N \text{ and } \mathrm{tr}_\Gamma T_G f = 0.$$

We note that the last condition is equivalent to $f \in \mathrm{Im}\,\mathbf{Q}$. The condition $q > N$ is needed for technical reasons because it guarantees that $T_G f \in C(G)$.
We can now apply the Teodorescu operator T_G to Equation (3.5), resulting in

$$T_G(D(|w|^{p-2}\, w)) = T_G f.$$

By applying the Borel-Pompeiu formula, one obtains

$$T_G(D(|w|^{p-2}\, w)) = |w|^{p-2}\, w - F_\Gamma(|w|^{p-2}\, w) = T_G f.$$

From the boundary condition $w|_\Gamma = 0$ we get

$$|w|^{p-2}\, w|_\Gamma = 0 \Rightarrow F_\Gamma(|w|^{p-2}\, w) = 0.$$

Then, we have

$$|w|^{p-2}\, w = T_G f.$$

Multiplying the previous equation by $|w|^{2-p}$ we obtain the equation

$$w = |w|^{2-p}\, T_G f. \tag{3.7}$$

This equation will be iterated.

We study, at first, the mapping properties of the mapping defined by Equation (3.7). To be precise, we consider that

$$u \in W^{2,p}(G), \text{ for } 1 < p < 2.$$

Then, $Du = w$ belongs to $W^{1,p}(G)$. We can use the result of Lemma 3 to show that

$$|w|^{2-p} \in W^{1,\frac{p}{2-p}}(G).$$

Because of our basic assumption for the right hand side $f \in L_{\tilde{p}}(G)$, where $\tilde{p} = \frac{p}{p-1}$, we can conclude that

$$T_G f \in W^{1,\tilde{p}}(G).$$

By using Theorem 6 (Equation (2.4)), we conclude that

$$|w|^{2-p} T_G f \in W^{1,r}(G), \text{ for } r > p.$$

Using the embedding theorem 8 we get that $W^{1,r}(G) \subset W^{1,p}(G)$. The above calculation shows that both, w and $|w|^{2-p} T_G f$ belong to the same Sobolev space $W^{1,p}(G)$, where $1 < p < 2$.

Now, let us look at the iteration procedure for solving Equation (3.7). Starting from an initial guess w_0 we define for $n = 1, 2, ...$

$$w_n = |w_{n-1}|^{2-p} T_G f. \tag{3.8}$$

Taking the L_p norm in Equation (3.8), one obtains

$$\|w_n\|_{0,p} = \left\| |w_{n-1}|^{2-p} T_G f \right\|_{0,p} \leq \left\| |w_{n-1}|^{2-p} \right\|_{0,\frac{p}{2-p}} \|T_G f\|_{1,\tilde{p}}. \tag{3.9}$$

By using Lemma 1, we calculate

$$\left\| |w_{n-1}|^{2-p} \right\|_{0,\frac{p}{2-p}} = \left(\|w_{n-1}\|_{0,p} \right)^{2-p},$$

finally, we have

$$\|w_n\|_{0,p} \leq \left(\|w_{n-1}\|_{0,p} \right)^{2-p} \|T_G f\|_{1,\tilde{p}}. \tag{3.10}$$

Our goal is to apply Banach's fixed-point theorem.

In the next part, we will discuss two approaches in different regions and different sets of assumptions:

In 3.3.1 we will consider two sets of conditions, one related to the norm condition and the other to the point-wise behavior of the functions for $1 < p < 2$.
An important part of the Subsection 3.3.2 deals with the special case for $p = \frac{3}{2}$, we will consider also two sets of conditions, one related to the point-wise behavior of the functions and the second to the norm condition.
Finally, Subsection 3.3.3 deals with a better regularity of the solution w, where $1 < p < 2$ and $N = 3$.

All conditions will depend on the nonhomogeneous part f and on p. The most important results of this section is the existence and uniqueness of the solution.

Let us make some remarks before dealing with the contractivity of the mapping.

Remark 4. ***(a)*** *The absolute value function $f(x) = |x|$ is Lipschitz continuous, with Lipschitz constant equal to one*

$$|f(x) - f(y)| = ||x| - |y|| \leq |x - y|\,.$$

(b) *The product of square roots can be estimated as follows (i.e. the geometric mean is never greater than the arithmetic mean):*

$$\sqrt{ab} \leq (a+b)/2 \Leftrightarrow (a+b) \geq 2\sqrt{ab},$$

where a and b are positive numbers.

The next remark is related to a special value of $p = 3/2$.

Remark 5. *By following the proof of either Lemma 2 given in [Demengel and Demengel, 2012] or the Lemma 3 we can find*

$$|T_G f|^2 \in W^{1,p}(G).$$

If $f \in L_{\tilde{p}}(G)$, then $T_G f \in W^{1,\tilde{p}}(G)$ where $\tilde{p} = \frac{p}{p-1}$ and $p = 3/2$, while

$$\nabla(|T_G f|^2) = 2\,|T_G f|\,\nabla T_G f.$$

3.3.1 General Case for $1 < p < 2$

This case contains point-wise conditions and norm condition:

3.3.1.1 Point-wise Conditions for $1 < p < 2$

In the first part, only a point-wise condition will be considered. The first condition on the initial guess is

$$|w_0(x)| \leq |T_G f(x)| \leq 1. \tag{3.11}$$

We will prove that this condition remains true for the whole iterated sequence. Let us assume that $|w_k(x)| \leq |T_G f(x)|$ is true for $k = 0, \ldots n-1$. We check that this inequality holds also for $k = n$. By taking the modulus in the Equation (3.8), we get

$$|w_k(x)| \leq |w_{k-1}(x)|^{2-p} |T_G f(x)|$$

and by substituting $|w_{k-1}(x)| \leq |T_G f(x)|$ in the last inequality, we get

$$|w_k(x)| \leq |T_G f(x)|^{2-p} |T_G f(x)| = |T_G f(x)|^{3-p}$$

Indeed, if $1 < p < 2$ and $|T_G f(x)| \leq 1$, one immediately obtains

$$|w_n(x)| \leq |T_G f(x)|^{3-p} \leq |T_G f(x)| . \tag{3.12}$$

We know that $|T_G f(x)| \in W^{1,\tilde{p}}(G)$ and $|w_n(x)| \in W^{1,p}(G)$, where $\tilde{p} = \frac{p}{p-1}$. Applying the embedding theorem 8 yields $W^{1,\tilde{p}}(G) \subset W^{1,p}(G)$, where $\tilde{p} > p$. Taking the L_p norm in Inequality (3.12) proves

$$\|w_n(x)\|_{0,p} \leq \|T_G f(x)\|_{0,p} .$$

Now, we will add a second condition for the initial guess w_0 that allows us later on to fix more specifically, where the solution is by finding lower and upper norm estimates for the solution. Let us assume that

$$|T_G f(x)|^{\frac{1}{p-1}} \leq |w_0(x)| . \tag{3.13}$$

Let us assume that $|T_G f(x)|^{\frac{1}{p-1}} \leq |w_{k-1}(x)|$ is true for $k = 1, \cdots n-1$ and we prove the same inequality for $k = n$.
By taking the $(2-p)$-th power on both sides, we get

$$|T_G f(x)|^{\frac{2-p}{p-1}} \leq |w_{k-1}(x)|^{2-p} .$$

Multiplying the last inequality by $|T_G f(x)|$ results

$$|T_G f(x)|^{\frac{2-p}{p-1}} |T_G f(x)| = |T_G f(x)|^{\frac{1}{p-1}} \leq |w_{k-1}(x)|^{2-p} |T_G f(x)| = |w_k(x)| .$$

This means that $|T_G f(x)| \leq |w_n(x)|^{p-1}$ for $1 < p < 2$.

Lemma 3 shows that $|T_G f(x)|^{\frac{1}{p-1}} \in W^{1,p}(G)$ and $|w_n(x)| \in W^{1,p}(G)$. This allows to take the L_p norm in the last inequality, resulting in

$$\left\| |T_G f|^{\frac{1}{p-1}} \right\|_{0,p} \leq \|w_n\|_{0,p} .$$

Now, we have proved that

$$\left\| |T_G f|^{\frac{1}{p-1}} \right\|_{0,p} \leq \|w_n\|_{0,p} \leq \|T_G f\|_{0,p} . \tag{3.14}$$

Here, we are going to show that the norms of w_n form a decreasing sequence in $L_p(G)$. This is not necessary for the convergence proof but it is interesting for getting sharper norm estimates for the solution as well as for numerical procedures. Recalling Equation (3.8)

$$w_n(x) = |w_{n-1}(x)|^{2-p} T_G f(x)$$

and substituting $|T_G f(x)| \leq |w_{n-1}(x)|^{p-1}$ from the inequality above, we obtain

$$|w_n(x)| \leq |w_{n-1}(x)|^{2-p} |w_{n-1}(x)|^{p-1} = |w_{n-1}(x)| . \tag{3.15}$$

Taking the L_p norm in Inequality (3.15) yields

$$\|w_n\|_{0,p} \leq \|w_{n-1}\|_{0,p} , \tag{3.16}$$

which is necessary to find the contractivity coefficient.

3.3.1.2 Norm Conditions for $1 < p < 2$

We will consider two initial conditions in this part. The first condition is the norm condition

$$\|w_0\|_{0,p} \leq \|T_G f\|_{1,\tilde{p}} \leq 1, \tag{3.17}$$

where $\tilde{p} = \frac{p}{p-1}$.

We will prove that this condition remains true for the whole iterated sequence. Let us assume that $\|w_k\|_{0,p} \leq \|T_G f\|_{1,\tilde{p}}$ is true for $k = 0, \dots n-1$. We check that this condition holds also for $k = n$. Recalling Inequality (3.10)

$$\|w_k\|_{0,p} \leq \left(\|w_{k-1}\|_{0,p}\right)^{2-p} \|T_G f\|_{1,\tilde{p}},$$

and substituting $\|w_{k-1}\|_{0,p} \leq \|T_G f\|_{1,\tilde{p}}$ in the inequality above, we obtain

$$\|w_k\|_{0,p} \leq \left(\|T_G f\|_{1,\tilde{p}}\right)^{2-p} \|T_G f\|_{1,\tilde{p}} = \left(\|T_G f\|_{1,\tilde{p}}\right)^{3-p}.$$

We know that $\|T_G f\|_{1,\tilde{p}} \leq 1$ and $1 < p < 2$, then one can obtain

$$\|w_k\|_{0,p} \leq \left(\|T_G f\|_{1,\tilde{p}}\right)^{3-p} \leq \|T_G f\|_{1,\tilde{p}}.$$

Finally, it proves that:

$$\|w_n\|_{0,p} \leq \|T_G f\|_{1,\tilde{p}}.$$

In the next step, we are going to show that the norms of w_n form a decreasing sequence in $L_p(G)$ by using the norm condition of the initial initial guess.
Recalling Inequality (3.10)

$$\|w_n\|_{0,p} \leq \left(\|w_{n-1}\|_{0,p}\right)^{2-p} \|T_G f\|_{1,\tilde{p}}$$

and substituting $\|T_G f\|_{1,\tilde{p}} \leq \left(\|w_{n-1}\|_{0,p}\right)^{p-1}$ in the above inequality, we get

$$\|w_n\|_{0,p} \leq \left(\|w_{n-1}\|_{0,p}\right)^{2-p} \left(\|w_{n-1}\|_{0,p}\right)^{p-1} = \|w_{n-1}\|_{0,p}.$$

We have proved that

$$\|w_n\|_{0,p} \leq \|w_{n-1}\|_{0,p}, \tag{3.18}$$

which is necessary to find the contractivity coefficient.

We still need to prove that the sequence w_n is bounded from below by using the norm condition, but we are not able to prove it directly. For proving that, we need to consider point-wise condition (3.13) and this condition is read as:

$$|T_G f(x)|^{\frac{1}{p-1}} \leq |w_0(x)|.$$

We proved in the previous subsection that $|T_G f(x)|^{\frac{1}{p-1}} \leq |w_n(x)|$ and $\left\| |T_G f|^{\frac{1}{p-1}} \right\|_{0,p} \leq \|w_n\|_{0,p}$ for $1 < p < 2$.

Remark 6. *If we start with point-wise conditions (3.11) and (3.13), we can obtain the norm conditions (3.14) and (3.16) which are necessary to find the contractivity constant. But if we consider only norm condition (3.17), we are not able to prove all necessary assumptions and we need point-wise condition (3.13).*

For $1 < p < 2$ and $|T_G f(x)| \leq 1$ we have already proved the following properties:

$$|T_G f(x)|^{\frac{1}{p-1}} \leq |w_n(x)| \leq |T_G f(x)| \,, \quad \left\| |T_G f|^{\frac{1}{p-1}} \right\|_{0,p} \leq \|w_n\|_{0,p} \leq \|T_G f\|_{0,p} \,,$$

$$\|w_n\|_{0,p} \leq \|w_{n-1}\|_{0,p} \,.$$

Now, we will find an estimate for the contractivity constant. From Equation (3.8) we conclude the absolute difference of $w_n(x)$ and $w_{n-1}(x)$

$$|w_n(x) - w_{n-1}(x)| = |T_G f(x)| \left| |w_{n-1}(x)|^{2-p} - |w_{n-2}(x)|^{2-p} \right| . \tag{3.19}$$

Considering the function $g(s) = s^{2-p}$ for $s > 0$. Fixing $x \in G$ and applying the mean value theorem to

$$g(|w_{n-1}(x)|) = |w_{n-1}(x)|^{2-p} \text{ and } g(|w_{n-2}(x)|) = |w_{n-2}(x)|^{2-p} .$$

With $|\zeta| \in [|w_{n-1}(x)|, |w_{n-2}(x)|]$ and $g'(|\zeta|) = (2-p)\, |\zeta|^{-p} \zeta = \frac{(2-p)\zeta}{|\zeta|^p}$, we have

$$\left| |w_{n-1}(x)|^{2-p} - |w_{n-2}(x)|^{2-p} \right| = \left| \frac{(2-p)}{|\zeta|^{p-1}} \right| \left| |w_{n-1}(x)| - |w_{n-2}(x)| \right| .$$

By using the inequality $|T_G f(x)|^{\frac{1}{p-1}} \leq |w_n(x)|$ and $|\zeta| \in [|w_{n-1}(x)|, |w_{n-2}(x)|]$, we get

$$|\zeta|^{p-1} \geq |w_{n-1}(x)|^{p-1} \geq |T_G f(x)| \quad \text{or} \quad |\zeta|^{p-1} \geq |w_{n-2}(x)|^{p-1} \geq |T_G f(x)| .$$

Then, we have

$$\frac{1}{|\zeta|^{p-1}} \leq \frac{1}{|T_G f|} .$$

Substituting this inequality in Equation (3.19) we get

$$|w_n(x) - w_{n-1}(x)| \leq \frac{|T_G f|\,(2-p)\, \left| |w_{n-1}(x)| - |w_{n-2}(x)| \right|}{|T_G f|}$$

$$\leq (2-p)\left||w_{n-1}(x)| - |w_{n-2}(x)|\right|. \tag{3.20}$$

Finally, we obtain

$$|w_n(x) - w_{n-1}(x)| \leq (2-p)\,|w_{n-1}(x) - w_{n-2}(x)|\,.$$

Taking the L_p norm in the last inequality, we find

$$\|w_n - w_{n-1}\|_{0,p} \leq c_2 \,\|w_{n-1} - w_{n-2}\|_{0,p}\,.$$

With the contractivity constant

$$c_2 = (2-p),$$

Banach's fixed-point theorem can now be applied.

Theorem 11. *Let G be a bounded domain with smooth boundary $\partial G = \Gamma$ in $\mathbb{R}^N$, $N \geq 2$. Let $1 < p < 2$, $f \in L_{\tilde{p}}(G) \cap L_q^{loc}(G)$ with $\tilde{p} = \frac{p}{p-1}$, $q > N$, $tr_\Gamma T_G f = 0$ and $|T_G f(x)|^{\frac{1}{p-1}} \leq |w_0(x)| \leq |T_G f(x)| \leq 1$. Then the nonlinear boundary value problem*

$$D\left(|w|^{p-2}\, w\right) = f \quad \text{in } G, \tag{3.21}$$

$$w = 0 \quad \text{on } \Gamma, \tag{3.22}$$

has a unique solution $w \in L_p(G)$ and the sequence defined by $w_n = |w_{n-1}|^{2-p} T_G f$ for $n \in \mathbb{N}$ converges in $L_p(G)$ to this solution.

Remark 7. *We get from (3.11),(3.13) and (3.20) also statements on point-wise convergence or uniform convergence, respectively.*

Remark 8. *(i) In Theorem 11 it is necessary to have a bound for the norm of the right hand side $|T_G f(x)| \leq 1$. But in the next subsection for $p = \frac{3}{2}$ such a restriction is not needed.*
(ii) We will give examples of G and f, where this theorem is applicable, in subsequent parts.

3.3.2 Special Case $p = 3/2$

We will consider the same general assumptions as mentioned in the previous subsection but work with different conditions:

3.3.2.1 Point-wise Conditions for $p = 3/2$

Firstly, we will consider a point-wise condition

$$|w_0(x)|^{2-p} \leq |T_G f(x)| \Leftrightarrow |w_0(x)| \leq |T_G f(x)|^{1/(2-p)}, \tag{3.23}$$

where $p = 3/2$. To avoid zeros of the iterates w_n in G, let us assume that the initial guess satisfies

$$|w_0(x)| \geq k_1 |T_G f(x)|^{1/(2-p)}$$

for all $x \in G$ with $k_1 < 1$. It will be proved that Inequality (3.23) is satisfied for all n. Let us assume that the condition $|w_k(x)| \leq |T_G f(x)|^{\frac{1}{2-p}}$ is true for $k = 0, \ldots, n-1$ and we prove that it holds also for $k = n$.
Recalling Equation (3.8) with the index k

$$|w_k(x)| = |w_{k-1}(x)|^{2-p} |T_G f(x)|,$$

and substituting $|w_{k-1}(x)| \leq |T_G f(x)|^{\frac{1}{2-p}}$ in Equation (3.8) one obtains that

$$|w_k(x)| \leq \left(|T_G f(x)|^{\frac{1}{2-p}}\right)^{2-p} |T_G f(x)| = |T_G f(x)|^2.$$

If $p = 3/2$, this proves that

$$|w_n(x)| \leq |T_G f(x)|^{\frac{1}{2-p}} = |T_G f(x)|^2. \tag{3.24}$$

Also, this can be written as:

$$|w_n(x)|^{2-p} \leq |T_G f(x)|. \tag{3.25}$$

Then, it has been proven that the sequence of iterations w_n is bounded from above by $|T_G f(x)|^{1/(2-p)}$. This bound depends only on p and f.
Additionally, it can be shown that the sequence $|w_n(x)|$ is increasing for all $x \in G$. Multiplying the inequality $|w_{n-1}(x)|^{2-p} \leq |T_G f(x)|$ by $|w_{n-1}(x)|^{2-p}$ results in

$$|w_{n-1}(x)|^{2-p} |w_{n-1}(x)|^{2-p} \leq |w_{n-1}(x)|^{2-p} |T_G f(x)| = |w_n(x)|,$$

or

$$|w_{n-1}(x)|^{4-2p} \leq |w_n(x)|.$$

If $p = \frac{3}{2}$, it means that

$$|w_{n-1}(x)|^{4-2p} = |w_{n-1}(x)| \,,$$

and consequently, one obtains

$$|w_{n-1}(x)| \leq |w_n(x)| \,. \tag{3.26}$$

Taking the L_p norm in Inequality (3.26), we find that the sequence w_n is an increasing sequence in L_p norm.

$$\|w_{n-1}\|_{0,p} \leq \|w_n\|_{0,p} \tag{3.27}$$

It has to be proved that the sequence w_n is bounded in the L_p norm.
It is already known that $|w_n|^{2-p} \in W^{1,\frac{p}{2-p}}(G)$. If $p = \frac{3}{2}$ then $\frac{p}{2-p} = 3$, and $|w_n|^{2-p} = |w_n|^{1/2} \in W^{1,3}(G)$. Furthermore, $|T_G f| \in W^{1,\tilde{p}}(G)$, where $\tilde{p} = \frac{p}{p-1}$. The case $p = \frac{3}{2}$ leads to $\tilde{p} = 3$. Now, we can take the L_3 norm in Inequality (3.25), resulting in

$$\left\| |w_n|^{2-p} \right\|_{0,3} \leq \|T_G f\|_{0,3} \,.$$

From Remark 5 we know $|T_G f|^2 \in W^{1,p}(G)$ and $|w_n| \in W^{1,p}(G)$. Taking L_p norm in Inequality (3.24), it results in

$$\|w_n\|_{0,p} \leq \left\| |T_G f|^2 \right\|_{0,p} \,.$$

These properties will be essential for the application of Banach's fixed-point theorem.

3.3.2.2 Norm Conditions for $p = 3/2$

In this case, we will start with the norm condition. This condition is

$$\|w_0\|_{0,p} \leq \left(\|T_G f\|_{1,\tilde{p}} \right)^{\frac{1}{2-p}} \,.$$

We will prove that this condition remains true for the whole iterated sequence. Let us assume that the condition $\|w_k\|_{0,p} \leq \left(\|T_G f\|_{1,\tilde{p}} \right)^{\frac{1}{2-p}}$ is true for $k = 0, \dots, n-1$ and we prove that it holds also for $k = n$.
Recalling Inequality (3.10)

$$\|w_k\|_{0,p} \leq \left(\|w_{k-1}\|_{0,p} \right)^{2-p} \|T_G f\|_{1,\tilde{p}} \,,$$

and substituting $\|w_{k-1}\|_{0,p} \leq \left(\|T_G f\|_{1,\tilde{p}}\right)^{\frac{1}{2-p}}$ in the inequality above, we obtain

$$\|w_k\|_{0,p} \leq \left(\left(\|T_G f\|_{1,\tilde{p}}\right)^{\frac{1}{2-p}}\right)^{2-p} \|T_G f\|_{1,\tilde{p}} = \left(\|T_G f\|_{1,\tilde{p}}\right)^2 .$$

Indeed, if $p = \frac{3}{2}$ we have proved

$$\|w_n\|_{0,p} \leq \left(\|T_G f\|_{1,\tilde{p}}\right)^2 = \left(\|T_G f\|_{1,\tilde{p}}\right)^{\frac{1}{2-p}} .$$

Then, it have been proven that the norm of the sequence of iterations w_n is bounded from above by $\left(\|T_G f\|_{1,\tilde{p}}\right)^{\frac{1}{2-p}}$. This bound depends only on p and f.

Actually, it can not be shown that the norm of the sequence $w_n(x)$ is increasing for all $x \in G$ by using the norm condition. For that, we need the point-wise condition as proven in the previous case (see Inequality (3.26)). Finally, we have proven that the sequence w_n is increasing in the L_p norm (see Inequality (3.27)).

Remark 9. *It is necessary to study the norm conditions, but it has been observed that point-wise conditions are stronger than norm conditions and by considering only point-wise conditions, we are able to prove all necessary assumptions for apply Banach's fixed-point theorem.*

Collecting all obtained estimates for $p = 3/2$ and $\tilde{p} = 3$, we have

$$w_n(x) = |w_{n-1}(x)|^{2-p}\, T_G f(x), \quad |w_{n-1}(x)| \leq |w_n(x)|\,, \quad |w_n(x)| \leq |T_G f(x)|^2\,,$$

$$\|w_{n-1}\|_{0,p} \leq \|w_n\|_{0,p}\,, \quad \|w_n\|_{0,p} \leq \left(\|T_G f\|_{1,\tilde{p}}\right)^2 .$$

We can conclude the absolute difference of $w_n(x)$ and $w_{n-1}(x)$, we get

$$|w_n(x) - w_{n-1}(x)| = |T_G f(x)| \left| |w_{n-1}(x)|^{1/2} - |w_{n-2}(x)|^{1/2} \right| \frac{|w_{n-1}(x)|^{1/2} + |w_{n-2}(x)|^{1/2}}{|w_{n-1}(x)|^{1/2} + |w_{n-2}(x)|^{1/2}}$$

$$= |T_G f(x)| \frac{\left||w_{n-1}(x)| - |w_{n-2}(x)|\right|}{|w_{n-1}(x)|^{1/2} + |w_{n-2}(x)|^{1/2}}. \tag{3.28}$$

Estimating the denominator from below by applying Remark 4 (b) to the denominator leads to

$$|w_{n-1}(x)|^{1/2} + |w_{n-2}(x)|^{1/2} \geq 2\sqrt{|w_{n-1}(x)|^{1/2}\, |w_{n-2}(x)|^{1/2}} = 2\, |w_{n-1}(x)|^{1/4}\, |w_{n-2}(x)|^{1/4} .$$

Using

$$\frac{1}{|w_{n-1}(x)|^{1/2} + |w_{n-2}(x)|^{1/2}} \leq \frac{1}{2\,|w_{n-1}(x)|^{1/4}\,|w_{n-2}(x)|^{1/4}}$$

in Equation (3.28), yields

$$|w_n(x) - w_{n-1}(x)| \leq \frac{|T_G f(x)|\,||w_{n-1}(x)| - |w_{n-2}(x)||}{2\,|w_{n-1}(x)|^{1/4}\,|w_{n-2}(x)|^{1/4}}. \tag{3.29}$$

From the inequality $|w_{n-1}(x)| \leq |w_n(x)|$ we obtain that

$$|w_0(x)|^{1/4} \leq |w_{n-1}(x)|^{1/4},\ |w_0(x)|^{1/4} \leq |w_{n-2}(x)|^{1/4}$$

and consequently, one can obtain

$$2\,|w_{n-1}(x)|^{1/4}\,|w_{n-2}(x)|^{1/4} \geq 2\,|w_0(x)|^{1/2}. \tag{3.30}$$

Substituting Inequality (3.30) in Inequality (3.29) results in

$$|w_n(x) - w_{n-1}(x)| \leq \frac{|T_G f(x)|\,||w_{n-1}(x)| - |w_{n-2}(x)||}{2\,|w_0(x)|^{1/2}}. \tag{3.31}$$

The condition for the initial guess for $p = \frac{3}{2}$ reads as $|w_0(x)| \geq k_1\,|T_G f(x)|^2$. By substituting this inequality in (3.31), it gives

$$|w_n(x) - w_{n-1}(x)| \leq \frac{|T_G f(x)|\,||w_{n-1}(x)| - |w_{n-2}(x)||}{2\sqrt{k_1}\,|T_G f(x)|}$$

$$= \frac{||w_{n-1}(x)| - |w_{n-2}(x)||}{2\sqrt{k_1}} \leq \frac{|w_{n-1}(x) - w_{n-2}(x)|}{2\sqrt{k_1}}.$$

With the constant $c_1 = \frac{1}{2\sqrt{k_1}}$ it has been shown that

$$|w_n(x) - w_{n-1}(x)| \leq c_1\,|w_{n-1}(x) - w_{n-2}(x)|. \tag{3.32}$$

The constant $c_1 = \frac{1}{2\sqrt{k_1}}$ is called the contractivity constant if $c_1 < 1$. It can be proved that $\frac{1}{2\sqrt{k_1}} < 1$ if $k_1 > \frac{1}{4}$ and we know that $k_1 < 1$. Then, we have proven that $c_1 < 1$ for $\frac{1}{4} < k_1 < 1$, or we can say $k_1 \in (1/4, 1)$.

Finally, by taking the L_p norm in Inequality (3.32), one obtains

$$\|w_n - w_{n-1}\|_{0,p} \leq c_1\,\|w_{n-1} - w_{n-2}\|_{0,p}.$$

Now all the necessary assumptions for the application of Banach's fixed-point theorem are fulfilled.

Theorem 12. *Let G be a bounded domain with smooth boundary $\partial G = \Gamma$ in $\mathbb{R}^N$, $N \geq 2$. Let $f \in L_{\tilde{p}}(G) \cap L_q^{loc}(G)$, with $\tilde{p} = \frac{p}{p-1}$, $q > N$, $tr_\Gamma T_G f = 0$ and $p = \frac{3}{2}$. Then the nonlinear boundary value problem*

$$D\left(|w|^{p-2}\, w\right) = f \quad in \quad G \tag{3.33}$$

$$w = 0 \quad on \quad \partial G = \Gamma, \tag{3.34}$$

has a unique solution $w \in L_p(G)$. Under the condition

$$k_1 \left|T_G f\right|^{1/(2-p)} \leq |w_0(x)| \leq \left|T_G f\right|^{1/(2-p)}, \ \frac{1}{4} < k_1 < 1,$$

the sequence defined by $w_n = |w_{n-1}|^{2-p} T_G f$ for $n \in \mathbb{N}$ converges in $L_p(G)$ to the unique solution.

Remark 10. *The special importance of Theorem 12 lies in the fact that for $p = 3/2$ no upper bound for $T_G f$ is necessary. Accepting such bounds for the size of the right hand side an existence and uniqueness result could be proved in Theorem 11 that holds for the whole interval $p \in (1, 2)$. In this case the bound and the contraction constant depend in a natural way on p and provides us with a two-sided norm estimate for the solution of the boundary value problem.*

Remark 11. *Obviously from all these point-wise estimates we can derive also a point-wise version of this fixed point result.*

3.3.3 Better Regularity

In this subsection, we will use all results presented up to now to prove that the solution w has a better regularity. We have proved in Theorems 11 and 12 that there is a unique solution in $L_p(G)$ for the main problem

$$D\left(|w|^{p-2}\, w\right) = f \quad \text{in } G, \tag{3.35}$$

$$w = 0 \quad \text{on } \Gamma. \tag{3.36}$$

Now, we will prove that under the original regularity assumptions for the right-hand side f, the solution w belongs not only to L_p space but has a better regularity.

Theorem 13. *Let G be a bounded domain in $\mathbb{R}^3$, $f \in L_{\tilde{p}}(G)$, where $\tilde{p} = \frac{p}{p-1}$. Then the solution w of the main problem belongs to the space $L_r(G)$*
(I) if $p = 1.5$, such that $w \in L_r(G)$ for all $r < \infty$,
(II) for each $p \in (1, 1.5)$, it exists $r > 3$ such that $w \in L_r(G)$,
(III) for each $p \in (1.5, 2)$, it exists $r > 6$ such that $w \in L_r(G)$.

Proof. For proving this theorem, we consider the solution $w \in L_p(G)$, where $1 < p < 2$ and solve the equation

$$w = |w|^{2-p} T_G f.$$

To calculate r successively we start with a number r_1 that we obtain as follows: We know that

$$w \in L_p(G) \text{ for } 1 < p < 2,\ |w|^{2-p} \in L_{\frac{p}{2-p}}(G),$$

and we can obtain

$$T_G f \in W^{1,\tilde{p}}(G).$$

Using Theorem 5, we conclude that

$$|w|^{2-p}\, T_G f \in L_{r_1}(G),$$

where r_1 is

$$\frac{2-p}{p} + \frac{p-1}{p} - \frac{1}{3} < \frac{1}{r_1}.$$

We find $p < r_1$ and we have $p < r_1 < 3p/(3-p)$.
From the regularity results of the problem, we get

$$w \in L_{r_1}(G) \text{ for all } p < r_1 < \frac{3p}{3-p}.$$

Now, to prove part (I), fixing $p = 3/2$ and assuming

$$w \in L_{r_1}(G), \text{ where } 3/2 < r_1 < 3,$$

we find, by using a new study of the same problem that $|w|^{1/2}$ belongs to $L_{2r_1}(G)$ and by using Theorem 5 again, we conclude that:

$$|w|^{1/2}\, T_G f \in L_{r_2}(G), \text{ for } r_1 < r_2 \text{ and } r_2 \text{ is } \frac{1}{2r_1} < \frac{1}{r_2}.$$

Finally, it has proved that $r_1 < r_2 < 2r_1$.
Repeating this consideration by assuming $w \in L_{r_2}(G)$, where $r_1 < r_2 < 2r_1$ we find by

using a new study of the same problem that

$$|w|^{1/2} \in L_{2r_2}(G),$$

and by using Theorem 5, we conclude that

$$|w|^{1/2} T_G f \in L_{r_3}(G), \text{ for } r_2 < r_3 \text{ and } r_3 \text{ is } \frac{1}{2r_2} < \frac{1}{r_3}.$$

Finally, we have proved $r_2 < r_3 < 2r_2$.
Repeating this consideration over and over, for $p = 3/2$, we obtain the inequalities $r_{i-1} < r_i < 2r_{i-1}$ and we can arrive at any $r < \infty$.
For the proof of (II), we calculate r_1 as above by

$$p < r_1 < 3p/(3-p) \text{ for } p \in (1, 1.5).$$

If we repeat the same process from the discussion earlier, we got that

$$|w|^{2-p} T_G f \in L_{r_2}(G), \text{ for } r_1 < r_2 \text{ and } r_2 \text{ is } \frac{2-p}{r_1} + \frac{p-1}{p} - \frac{1}{3} < \frac{1}{r_2}.$$

Finally, we have proved

$$r_1 < r_2 < \frac{3r_1 p}{3p(2-p) + r_1(2p-3)}.$$

Repeating this consideration over and over, for $p \in (1, 1.5)$, we get the inequalities

$$r_{i-1} < r_i < \frac{3r_{i-1} p}{3p(2-p) + r_{i-1}(2p-3)}.$$

We observe that after some steps the iteration stops because the upper bound of the interval becomes smaller than the lower bound. There is a condition for r_{i-1} and this condition is

$$r_{i-1} \leq \frac{-3p(2-p)}{2p-3}.$$

Consequently, r is bounded from below and greater than 3 if $p \in (1, 1.5)$.
Finally, we can prove (III) if $p \in (1.5, 2)$ with the same method and find that r is bounded from below by 6.

□

Remark 12. *In Theorem 13, it has been proven that the solution w belongs not only* *to L_p space but has a better regularity and it belongs to L_r space and for each $p \in (1,2)$ one can get different r dependant on p.*

Remark 13. *After studying the behavior of the solution w, under the original regularity assumptions for f, we observed that the solution w belongs not only to L_p space but has a better regularity. We noticed that the solution w belongs to L_r and we obtained a general formula which is valid to calculate r for each $p \in (1,2)$ and $N \geq 2$.*
The general formula to conclude $r = r(N,p)$ is given by:

$$r \begin{cases} p < r_1 < \dfrac{Np}{N-p}, & N = 2,3,... \\ r_{i-1} < r_i < \dfrac{Npr_{i-1}}{Np(2-p) + r_{i-1}((N-1)p - N)} & , i = 2,3,4,... \ , \ N = 3,4,.... \end{cases}$$

Figure 3.1 shows the behavior of the space L_r for each $p \in (1,2)$ and $N = 3$.

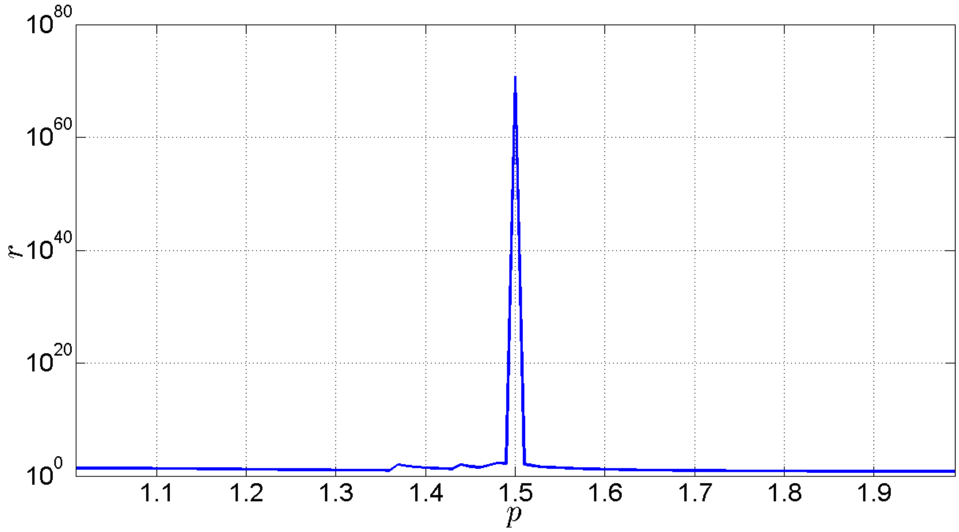


Figure 3.1: The better regularity of the solution w in $\mathbb{R}^3$

Looking at the graph in Figure 3.1, we see that it goes to infinity at $p = \frac{3}{2}$, which illustrates part(I) in Theorem 13. Also, we see that the graph in Figure 3.2 goes to infinity if p tends to 2 in the two dimensional case.

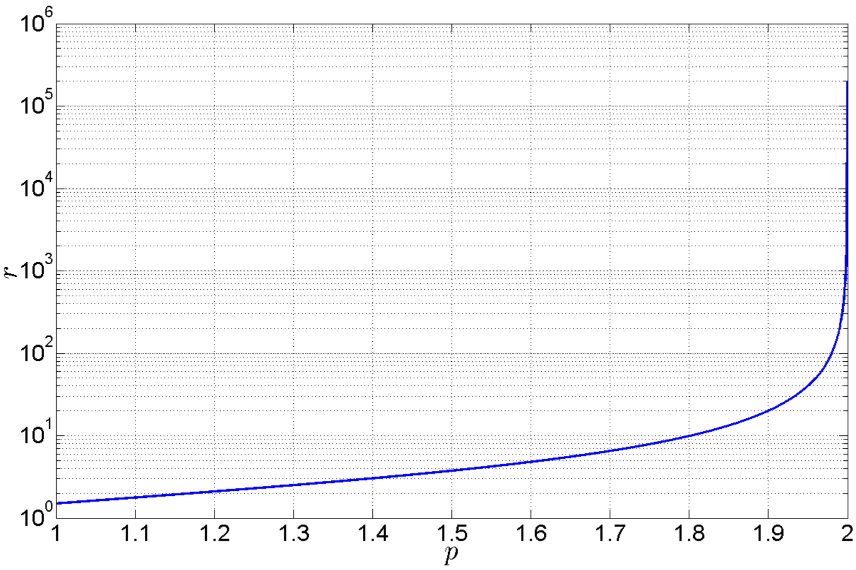


Figure 3.2: The better regularity for the solution w in $\mathbb{R}^2$

In the next section, we will consider the nonlinear p-Dirac problem for $2 < p < 3$; a big part of Subsection 3.4.1 has been published in article [Al-Yasiri and Gürlebeck, 2015].

3.4 Boundary Value Problem for $2 < p < 3$

3.4.1 The Higher Dimensional Case

Let G be a bounded domain with smooth boundary $\partial G = \Gamma$ in $\mathbb{R}^N$, $N \geq 3$. We will deal with the case $N = 2$ separately later on, the situation is then different between condition $2 < p < 3$ means then the p is greater than the space dimension. We search for $w \in L_p(G)$, $2 < p < 3$, satisfying

$$\begin{aligned} D\left(|w|^{p-2}\, w\right) &= f \qquad \text{in } G && (3.37) \\ w &= 0 \qquad \text{on } \Gamma. && (3.38) \end{aligned}$$

In addition to the right-hand side f, we assume some necessary conditions:

$$f \in L_{\tilde{p}}(G) \cap L_q^{loc}(G), \text{ with } \tilde{p} = \frac{p}{p-1},\ q > N \text{ and } \mathrm{tr}_\Gamma T_G f = 0.$$

The condition $q > N$ is needed for technical reasons because it guarantees that $T_G f \in$

$C(G)$.
First, we will apply the operator T_G to Equation (3.37), resulting in

$$T_G(D(|w|^{p-2}\,w)) = T_G f.$$

By applying the Borel-Pompeiu formula, we obtain

$$T_G(D(|w|^{p-2}\,w)) = |w|^{p-2}\,w - F_\Gamma(|w|^{p-2}\,w) = T_G f.$$

From the boundary condition $w|_\Gamma = 0$, one obtains

$$|w|^{p-2}\,w|_\Gamma = 0 \Rightarrow F_\Gamma(|w|^{p-2}\,w) = 0.$$

Finally, we get

$$|w|^{p-2}\,w = T_G f.$$

From now on, we assume that $|w|^{p-2} \neq 0$ in G. Dividing the previous equation by $|w|^{p-2}$, we obtain the equation

$$w = \frac{T_G f}{|w|^{p-2}}. \tag{3.39}$$

This equation will be iterated. Starting from an initial guess w_0, we define for $n = 1, 2, ...$

$$w_n = \frac{T_G f}{|w_{n-1}|^{p-2}}. \tag{3.40}$$

We consider the subsequence with even indices and introduce a new sequence in terms of g_n, where

$$g_0 = w_0,\ g_1 = w_2,\ \ldots, \text{ and } g_n = w_{2n}.$$

Finally, we get the equations

$$g_n = \frac{T_G f}{|T_G f|^{p-2}}\,|g_{n-1}|^{(p-2)^2}, n = 1, 2, \ldots. \tag{3.41}$$

Taking the modulus on both sides of Equation (3.41), one obtains

$$|g_n| = |T_G f|^{3-p}\,|g_{n-1}|^{(p-2)^2}. \tag{3.42}$$

We can express all g_n by using only g_0 as follows:

$$g_n = T_G f\,|T_G f|^{\sum_{k=0}^{n-2}(1-(p-2))(p-2)^{2(k+1)}-(p-2)}\,|g_0|^{(p-2)^{2n}}.$$

It can be seen from the properties of $T_G f$ that $g_n = w_{2n}$ satisfies the boundary condition if g_0 satisfies it.

At first, we will study the mapping properties of the mapping defined by Equation (3.42). Because of our basic assumption for the right hand side $f \in L_{\tilde{p}}(G)$, where $\tilde{p} = \frac{p}{p-1}$, using the property $T_G : L_{\tilde{p}}(G) \longrightarrow W^{1,\tilde{p}}(G)$, we find that $T_G f \in W^{1,\tilde{p}}(G)$ and

$$|T_G f|^{3-p} \in W^{1,\frac{\tilde{p}}{3-p}}(G).$$

We know that $g_n \in L_p(G)$ and using the same idea of Lemma 1, we can obtain that $|g_{n-1}|^{(p-2)^2} \in L_{\frac{p}{(p-2)^2}}(G)$. Using a theorem on products of functions belonging to certain Sobolev spaces (see [Reed and Simon, 1978]), we conclude that:

$$|T_G f|^{3-p} |g|^{(p-2)^2} \in L_r(G), \text{ for } p < r < \frac{Np}{N-p}.$$

By using embedding theorems, we have $L_r(G) \subset L_p(G)$. All together means

$$g_n \in L_p(G), \text{ where } 2 < p < 3.$$

Now, we are ready to study the iteration procedure. To avoid zeros of the iterates in G let us assume that the initial guess g_0 satisfies the condition

$$k_2 |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}} \leq |g_0(x)| \leq |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}},$$

for all $x \in G$ with $0 < k_2 < 1$.

Supposing that $|g_k(x)| \leq |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}}$ for $k = 0, \ldots, n-1$, we prove that it holds also for $k = n$.
Recalling Equation (3.42) for $k = n$ and substituting $|g_{k-1}(x)| \leq |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}}$, we obtain

$$|g_k(x)| \leq |T_G f(x)|^{3-p} \left(|T_G f(x)|^{\frac{3-p}{1-(p-2)^2}} \right)^{(p-2)^2} = |T_G f(x)|^{3-p} |T_G f(x)|^{\frac{(3-p)(p-2)^2}{1-(p-2)^2}}.$$

This proves that for all n

$$|g_n(x)| \leq |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}}. \tag{3.43}$$

Finally, the L_p norm in (3.43) leads to

$$\|g_n\|_{0,p} \leq \left\| |T_G f|^{\frac{3-p}{1-(p-2)^2}} \right\|_{0,p}. \tag{3.44}$$

Then, we have proved that the sequence of iterations g_n or w_{2n} is bounded from above by $|T_G f|^{\frac{3-p}{1-(p-2)^2}}$. This bound depends only on p and f.

Additionally, it can be shown that the sequence $|g_n(x)|$ is increasing for all $x \in G$. Recalling Inequality (3.43) and taking the $(1-(p-2)^2)$-th power on both sides, we get

$$|g_{n-1}|^{1-(p-2)^2} \leq |T_G f|^{3-p}.$$

Multiplying both sides of our inequality by $|g_{n-1}|^{(p-2)^2}$ results in

$$|g_{n-1}|^{1-(p-2)^2} |g_{n-1}|^{(p-2)^2} \leq |T_G f|^{3-p} |g_{n-1}|^{(p-2)^2}.$$

This means for all n and $2 < p < 3$, we have proved that

$$|g_{n-1}| \leq |g_n|. \tag{3.45}$$

By taking the L_p norm in Inequality (3.45) we discover that the sequence of the norms of g_n or w_{2n} is also an increasing sequence

$$\|g_{n-1}\|_{0,p} \leq \|g_n\|_{0,p}. \tag{3.46}$$

Now, we will study the contractivity of the mapping. We will collect all obtained estimates for $2 < p < 3$. We have

$$g_n = \frac{T_G f}{|T_G f|^{p-2}} |g_{n-1}|^{(p-2)^2}, \ |g_{n-1}| \leq |g_n| \leq |T_G f|^{\frac{3-p}{1-(p-2)^2}},$$

$$\|g_{n-1}\|_{0,p} \leq \|g_n\|_{0,p} \leq \left\| |T_G f|^{\frac{3-p}{1-(p-2)^2}} \right\|_{0,p}.$$

Now, we can conclude that

$$|g_n - g_{n-1}| = \left| \frac{T_G f}{|T_G f|^{p-2}} \left(|g_{n-1}|^{(p-2)^2} - |g_{n-2}|^{(p-2)^2} \right) \right|$$

$$\leq |T_G f|^{3-p} \left| |g_{n-1}|^{(p-2)^2} - |g_{n-2}|^{(p-2)^2} \right|. \tag{3.47}$$

Considering the function $y(s) = s^{(p-2)^2}$ for $s > 0$. Fixing $x \in G$ and applying the mean value theorem to

$$y(|g_{n-1}(x)|) = |g_{n-1}(x)|^{(p-2)^2} \text{ and } y(|g_{n-2}(x)|) = |g_{n-2}(x)|^{(p-2)^2}.$$

With $|\zeta| \in [|g_{n-2}(x)|, |g_{n-1}(x)|]$ and $|y'(|\zeta|)| = \frac{(p-2)^2}{|\zeta|^{1-(p-2)^2}}$ we have

$$\left||g_{n-1}(x)|^{(p-2)^2} - |g_{n-2}(x)|^{(p-2)^2}\right| = \frac{(p-2)^2}{|\zeta|^{1-(p-2)^2}} \left||g_{n-1}(x)| - |g_{n-2}(x)|\right|.$$

We use $|\zeta| \in [|g_{n-2}(x)|,\ |g_{n-1}(x)|]$, we know that:

$$|g_{n-2}(x)| \leq |g_{n-1}(x)| \text{ and } |g_0| \geq k_2 |T_G f|^{\frac{3-p}{1-(p-2)^2}}.$$

Then, we have

$$|\zeta|^{1-(p-2)^2} \geq |g_{n-2}|^{1-(p-2)^2} \geq |g_0|^{1-(p-2)^2} \geq k_2{}^{1-(p-2)^2} |T_G f|^{3-p}$$

and

$$\frac{1}{|\zeta|^{1-(p-2)^2}} \leq \frac{1}{|g_0|^{1-(p-2)^2}} \leq \frac{1}{k_2{}^{1-(p-2)^2} |T_G f|^{3-p}}.$$

By substituting the latter estimates in Equation (3.47), one obtains

$$|g_n - g_{n-1}| \leq \frac{(p-2)^2 |T_G f|^{3-p}}{k_2{}^{1-(p-2)^2} |T_G f|^{3-p}} \left||g_{n-1}| - |g_{n-2}|\right| \leq \frac{(p-2)^2}{k_2{}^{1-(p-2)^2}} |g_{n-1} - g_{n-2}|. \quad (3.48)$$

It has been shown that

$$|g_n - g_{n-1}| \leq c_3 |g_{n-1} - g_{n-2}|, \quad (3.49)$$

with the constant c_3

$$c_3 = \frac{(p-2)^2}{k_2{}^{1-(p-2)^2}}. \quad (3.50)$$

The last step is to take the L_p norm in the Inequality (3.49). We obtain

$$\|g_n - g_{n-1}\|_{0,p} \leq c_3 \|g_{n-1} - g_{n-2}\|_{0,p}.$$

That means we have proved for

$$k_2 > (p-2)^{\frac{2}{1-(p-2)^2}},$$

dependent on $p \in (2,3)$ there is contractivity coefficient $c_3 < 1$. All conditions for the application of Banach's fixed-point theorem are now fulfilled.

Before stating the next theorem, remember that the formula for g_n is given by $g_0 = w_0$, $g_1 = w_2$, ..., $g_n = w_{2n}$. That means we state the theorem in terms of w_{2n}.

Theorem 14. *Let G be a bounded domain with smooth boundary $\partial G = \Gamma$ in $\mathbb{R}^N$, $N \geq 3$. Let $f \in L_{\tilde{p}}(G) \cap L_q^{loc}(G)$, with $\tilde{p} = \frac{p}{p-1}$, $q > N$, $tr_\Gamma T_G f = 0$ and $2 < p < 3$. Then the boundary value problem*

$$D\left(|w|^{p-2} w\right) = f \quad in \quad G \tag{3.51}$$

$$w = 0 \quad on \quad \partial G = \Gamma, \tag{3.52}$$

has a unique solution $w \in L_p(G)$. Under the condition

$$k_2 \left|T_G f\right|^{\frac{3-p}{1-(p-2)^2}} \leq |w_0(x)| \leq |T_G f|^{\frac{3-p}{1-(p-2)^2}},$$

with $(p-2)^{\frac{2}{1-(p-2)^2}} < k_2 < 1$.
The sequence defined by

$$w_{2n} = \frac{T_G f}{|T_G f|^{p-2}} \left|w_{2(n-1)}\right|^{(p-2)^2}$$

for $n \in \mathbb{N} \setminus \{0\}$ converges in $L_p(G)$ to the unique solution.

We have proved in Theorem 14 that there is a unique solution in $L_p(G)$ for the main problem. Our question is to know if the solution w belongs to another space? We will prove now, that under the original regularity assumptions for the right-hand side f, the solution w belongs not only to L_p but has a much better regularity.

To demonstrate that we are able to answer this question, we will formulate the next theorem for $2 < p < 3$.

Theorem 15. *Let G be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, $f \in L_{\tilde{p}}(G)$, where $\tilde{p} = \frac{p}{p-1}$. Then, the solution w of the main problem belongs to the space $L_r(G)$ with $p < r < \frac{Np}{N-p}$ if $2 < p < 3$.*

Proof. The proof of this theorem is completely analogous to the proof in Theorem 13.

We start our proof by taking $w \in L_p(G)$, where $2 < p < 3$ and solve the equation

$$w = |T_G f|^{3-p} |w|^{(p-2)^2}.$$

We have already proved that

$$|T_G f|^{3-p} \in W^{1,\frac{\tilde{p}}{3-p}}(G) \text{ and } |w|^{(p-2)^2} \in L_{\frac{p}{(p-2)^2}}(G).$$

Using Theorem 5, we conclude that:

$$|T_G f|^{3-p} |w|^{(p-2)^2} \in L_r(G), \text{ for } p < r < \frac{Np}{N-p}.$$

From the regularity results of the problem, we get

$$w \in L_r(G), \text{ for } p < r < \frac{Np}{N-p}.$$

By trying a new study of the same problem $w = |T_G f|^{3-p} |w|^{(p-2)^2}$, we observe that after one step, the iteration will stop. □

Remark 14.
- *In the theoretical investigation of solving the p-Dirac problem for $2 < p < 3$, we are interacting to study the problem in higher dimensional case $(\mathbb{R}^N,\ N \geq 3)$.*
- *But also, one should study the two dimensional case because in this case for $2 < p < 3$ and $N = 2$, we have $p > N$ and the embedding theorem will be different in this case.*
- *The two dimensional case $\mathbb{R}^2$ is important in the numerical investigation because all applications of p-Dirac problem will be in $\mathbb{R}^2$.*

For these reasons, we will consider the problem in $\mathbb{R}^2$ for $2 < p < 3$.

3.4.2 The Two Dimensional Case

Let G be a bounded domain with smooth boundary $\partial G = \Gamma$ in $\mathbb{R}^2$. We search for $w \in L_p(G)$, $2 < p < 3$, satisfies Equation (3.37) and (3.38). With the right-hand side f satisfy the conditions:

$$f \in L_{\tilde{p}}(G) \cap L_q^{loc}(G), \text{ with } \tilde{p} = \frac{p}{p-1},\ q > 2 \text{ and } \mathrm{tr}_\Gamma T_G f = 0.$$

Most of our considerations in Subsection 3.4.1 can be repeated analogously. We will only study the mapping properties of the mapping defined in Equation (3.42) by a different way.

We know that the right hand side $f \in L_{\tilde{p}}(G)$, where $\tilde{p} = \frac{p}{p-1}$ and $2 < p < 3$. Using the property $T_G : L_{\tilde{p}}(G) \longrightarrow W^{1,\tilde{p}}(G)$, we obtain that $T_G f \in W^{1,\tilde{p}}(G)$ and using embedding theorem, we can get $T_G f \in L_{\tilde{p}}(G)$ and similarly to Lemma 1, we can prove that

$$|T_G f|^{3-p} \in L_{\frac{\tilde{p}}{3-p}}(G).$$

We know that $|g_{n-1}|^{(p-2)^2} \in L_{\frac{p}{(p-2)^2}}(G)$. Let $p^\star = \frac{\tilde{p}}{3-p}$ and $p^{\star\star} = \frac{p}{(p-2)^2}$, using the generalized Hölder's inequality, it can be proven that

$$|T_G f|^{3-p} |g_{n-1}|^{(p-2)^2} \in L_r(G),$$

where $\frac{1}{r} = \frac{1}{p^\star} + \frac{1}{p^{\star\star}}$. Calculating r we obtain

$$\begin{aligned} \frac{1}{r} &= \frac{1}{p^\star} + \frac{1}{p^{\star\star}} = \frac{3-p}{\tilde{p}} + \frac{(p-2)^2}{p} \\ &= \frac{(3-p)(p-1) + (p-2)^2}{p} = \frac{1}{p}. \end{aligned}$$

All together means, we have proved that g_n and $|T_G f|^{3-p} |g_{n-1}|^{(p-2)^2}$ belong to the same space $L_p(G)$, where $G \in \mathbb{R}^2$ and $2 < p < 3$. Analogously, all the necessary assumption to apply Banach's fixed-point theorem can be proven as it is mentioned in Subsection 3.4.1.

3.5 Generalized Normalized p-Laplace Equations with ∞-Laplacian and 1-Laplacian

This section is organized in two parts. The first part is dependent on the ∞-Laplacian and ∞-Dirac operators, and the connection between these two operators. Also, we will consider the 1- Laplacian and 1-Dirac operators and the connection between these two operators. In the second part, we will introduce a new operator called normalized generalized p-Laplacian Δ_p^c and the connection between the generalized normalized ∞-Laplacian Δ_∞^c and the generalized normalized 1-Laplacian Δ_1^c. These operators are essential in the applications. In addition to that, the existence and the uniqueness to the solutions of the ∞-Laplace equation and 1-Laplace equation are established.

3.5.1 The ∞-Laplacian

The ∞-Laplace operator appears naturally in several applications, for example in image processing and in optimal transportation. The solution to ∞-Laplace equation is called infinity harmonic function.
In [Juutinen et al., 1999], [Juutinen and Lindqvist, 2005] and [Charro and Peral, 2007] they have been studied the eigenvalues for the ∞-Laplace problem. The Tug-of-War games and the ∞-Laplacian were studied in [Peres et al., 2009], [Gómez and Rossi, 2013].

By taking the limit $p \to \infty$ in the p-Laplace operator, we will get the ∞-Laplacian. The ∞-Laplacian is a nonlinear operator and it is usually denoted by Δ_∞.
Starting with the generalized p-Laplace equation

$$\Delta_p u = D\left(|Du|^{p-2} Du\right) = 0, \tag{3.53}$$

where u is a vector-valued function $u \in W^{1,p}(G)$, $1 \le p < \infty$. Note that $D^2 = -\Delta$ is the Laplacian in $\mathbb{R}^N$.
One can write the p-Laplacian of u as

$$\begin{aligned}\Delta_p u = D\left(|Du|^{p-2} Du\right) &= D\left(|Du|^{p-2}\right) Du + |Du|^{p-2} DDu \\ &= \frac{(p-2)}{2} |Du|^{p-4}\, 2\left(\sum_{j=1}^{N} Du_j DDu_j\right) Du + |Du|^{p-2} DDu \\ &= (p-2)\,|Du|^{p-4}\left\{\sum_{i,j=1}^{N} Du_j DDu_j Du_i + \frac{1}{p-2}|Du|^2 \Delta u\right\} = 0.\end{aligned}$$

Dividing by $(p-2)\,|Du|^{p-4}$ and sending $p \to \infty$, passing to limit in the equation $\Delta_p u = 0$, to obtain

$$\Delta_\infty u = \sum_{i,j=1}^{N} Du_j DDu_j Du_i = DuD^2u\overline{Du} = 0.$$

The nonlinear operator Δ_∞ is called the ∞-Laplacian.

P. Juutinen and B. Kawohl studied the evolution governed by the ∞-Laplacian, they studied the non-linear degenerate equation

$$u_t = \Delta_\infty u$$

which is a parabolic version of the increasingly popular ∞-Laplace equation, and they proved the existence and uniqueness results in [Juutinen and Kawohl, 2006].
After defining the Δ_∞ operator, we can rewrite the p-Laplace equation in form of:

$$\Delta_p u = (p-2)\,|Du|^{p-4}\left\{\Delta_\infty u + \frac{1}{p-2}\,|Du|^2\,\Delta u\right\}, \tag{3.54}$$

or

$$\Delta_p u = |Du|^{p-2}\left\{\frac{p-2}{|Du|^2}\Delta_\infty u + \Delta u\right\}, \tag{3.55}$$

and we will use these formulas later.

3.5.2 The ∞-Dirac Operator

There are some links between the first order nonlinear p-Dirac equation and the second order nonlinear p-Laplace equation when p tends to infinity. T. Bieske and J. Ryan have introduced a first order non-linear infinite Dirac operator associated with the non-linear infinite Laplacian in $\mathbb{R}^N$ by using fundamental solution of p-Dirac equation as $p \to \infty$.

After they define the ∞-Dirac operator on the sphere S^n, they can also formally define the ∞-Laplace operator on the sphere S^n and they found a link between these operators on the sphere S^n. (see [Bieske and Ryan, 2010]).

By sending p to infinity, we will define the ∞-Dirac operator. Let $\mathrm{D}_p w(x)$ the p-Dirac operator and $w = Du$, where w is a vector-valued function. We define the ∞-Dirac operator acting on a function w as $\mathrm{D}_\infty w$ also by taking the limit of $\mathrm{D}_p w$ as $p \to \infty$. Recalling p-Dirac equation

$$\mathrm{D}_p w = D\left(|w|^{p-2}\,w\right) = (p-2)\,|w|^{p-4}\sum_{i,j=1}^{N} w_j D w_j w_i + |w|^{p-2}\,Dw$$

$$= (p-2)\,|w|^{p-4}\left\{\sum_{i,j=1}^{N} w_j D w_j w_i + \frac{1}{p-2}\,|w|^2\,Dw\right\} = 0.$$

Dividing by $(p-2)\,|w|^{p-4}$ and sending $p\to\infty$, to obtain

$$\mathrm{D}_\infty w = wDw\overline{w} = 0.$$

That means we have proved

$$\mathrm{D}_\infty w(x) = \mathrm{D}_\infty Du(x).$$

Finally, the p-Dirac equation can be written in a different form, by using ∞-Dirac, as follows

$$\begin{aligned}\mathrm{D}_p w &= (p-2)\,|w|^{p-4}\left\{\mathrm{D}_\infty w + \frac{1}{p-2}\,|w|^2\,\mathrm{D}w\right\}\\ &= |w|^{p-2}\left\{\frac{p-2}{|w|^2}\mathrm{D}_\infty w + \mathrm{D}w\right\} = 0.\end{aligned}$$

3.5.3 The 1-Laplacian and 1-Dirac Operator

Let us consider G be an open set of $\mathbb{R}^N$, $N\geq 1$. For a function u belonging to Sobolev space and $Du\neq 0$ in G, we can define the generalized 1-Poisson (1-Laplacian) equation by substituting $p=1$ in the generalized p-Laplace equation

$$D\left(\frac{Du}{|Du|}\right) = f. \tag{3.56}$$

If u is a scalar part, then $Du=\nabla u$, where $|\nabla u|$ is the length of gradient ∇u of u and div $\left(\frac{\nabla u}{|\nabla u|}\right)$ is said to be the 1-tension field of u. The $1-$Laplace operator is denoted as Δ_1 and defined as

$$\Delta_1 u = D\left(\frac{Du}{|Du|}\right) = \mathrm{H},$$

where H is the Mean Curvature.

Investigations regarding 1-Laplace type operators could be, initially, found in image restoration studies, as well as, the theory of torsion, specifically in some optimal design problems ([Kawohl, 1991], [Sapiro, 2001], [Andreu-Vaillo et al., 2004]). Furthermore, in [Andreu et al., 2001a] and [Andreu et al., 2001b] a systematic mathematical study of problems involving 1-Laplace type operators started.

In fact, as first observed in Osher's paper [Osher and Sethian, 1988], also Osher's monograph [Osher and Fedkiw, 2002], the 1-Laplacian operator is closely connected to the mean curvature operator as follows: Considering the surface given by the level surface $u(x) = k$; then its normal unit vector is formally given by $\mathrm{n}(x) = \frac{Du}{|Du|}$ operator. Therefore, the mean curvature of the surface at the point x is formally given by

$$\mathrm{H}(x) = \mathrm{div}(\mathrm{n})(x) = \mathrm{div}\left(\frac{Du}{|Du|}\right)(x);$$

As shown by this relationship, the behavior at the boundary ∂G of the solutions to (3.56) can depend on the boundary's mean curvature, and more specifically on the mean curvature's boundedness.

By substituting $Du = w$, we will have the 1-Dirac operator.
We also study the relation between the 1-Laplacian and ∞-Laplacian as follows

$$\Delta_1 u = D\left(\frac{Du}{|Du|}\right) = \frac{D\,(Du)}{|Du|} - \frac{Du\left(\sum_{j=1}^{N} Du_j DDu_j\right)}{|Du|^3}$$

$$\Delta_1 u = \frac{\Delta u}{|Du|} - \frac{\Delta_\infty u}{|Du|^3}.$$

The linear Laplacian can be written in terms of Δ_1 and Δ_∞

$$\Delta u = |Du|\Delta_1 u + \frac{\Delta_\infty u}{|Du|^2}.$$

3.5.4 Generalized Normalized p-Laplace Equation

Recent studies have suggested a variant of the p-Laplacian. That variant is known as the "normalized p-Laplacian". It has been introduced in conjunction with the stochastic game, "Tug-of-War" with noise [Peres and Sheffield, 2008]. Also, a similar interpretation of a p-harmonic extension $u(x)$ as the limit of the values of certain stochastic games has been developed.

The normalized p-Laplacian is formulated based on the generalize p-Laplacian as follows

$$\Delta_p^c u = \frac{1}{p}|\nabla u|^{2-p}\Delta_p u.$$

The normalized p-Laplacian is also called game-theoretic p-Laplacian.

By substituting ∇ by D, we can define the normalized generalized p-Laplacian

$$\Delta_p^c u = \frac{1}{p}|Du|^{2-p}\Delta_p u = \frac{1}{p}|Du|^{2-p}D\left(Du|Du|^{p-2}\right), \quad 1 \leq p < \infty. \tag{3.57}$$

By substituting (3.55) in Equation (3.57), we obtain

$$\begin{aligned}\Delta_p^c u &= \frac{1}{p}|Du|^{2-p}\left\{|Du|^{p-2}\left\{\frac{p-2}{|Du|^2}\Delta_\infty u + \Delta u\right\}\right\}\\ &= \frac{p-2}{p}\,|Du|^{-2}\,\Delta_\infty u + \frac{1}{p}\Delta u.\end{aligned}$$

Sending p to ∞ leads

$$\Delta_\infty^c u = |Du|^{-2}\,\Delta_\infty u. \tag{3.58}$$

For $p = 1$, it has the following relationships

$$\Delta_1^c u = |Du|\Delta_1 u. \tag{3.59}$$

It has been proven in the previous subsection that $\Delta u = |Du|\Delta_1 u + \frac{\Delta_\infty u}{|Du|^2}$, by substituting (3.58) and (3.59), we get

$$\Delta u = \Delta_1^c u + \Delta_\infty^c u.$$

Another relationship between the normalized generalized p-Laplace for any $p > 1$, and $p = 1$ and $p = \infty$ can be considered.

Starting with the generalized p-Laplacian

$$\Delta_p u = D\left(|Du|^{p-2}\,Du\right) = |Du|^{p-2}\left\{\frac{p-2}{|Du|^2}\Delta_\infty u + \Delta u\right\},$$

by using Equation (3.58), we obtain

$$\begin{aligned}\Delta_p u &= |Du|^{p-2}\left\{(p-2)\Delta_\infty^c u + \Delta u\right\}\\ &= |Du|^{p-2}\left\{(p-1)\Delta_\infty^c u + (\Delta u - \Delta_\infty^c u)\right\}.\end{aligned}$$

By substituting $\Delta_1^c u = \Delta u - \Delta_\infty^c u$ in the last equation, it results in

$$D\left(|Du|^{p-2}\,Du\right) = |Du|^{p-2}\left\{(p-1)\Delta_\infty^c u + \Delta_1^c u\right\}.$$

Multiplying the above equation by $|Du|^{2-p}$, one gets

$$|Du|^{2-p} D\left(|Du|^{p-2} Du\right) = (p-1)\Delta_\infty^c u + \Delta_1^c u$$

and dividing it by p, results in

$$\frac{1}{p}|Du|^{2-p} D\left(|Du|^{p-2} Du\right) = \frac{p-1}{p}\Delta_\infty^c u + \frac{1}{p}\Delta_1^c u.$$

Finally, we get

$$\Delta_p^c u = \frac{1}{p}\Delta_1^c u + \frac{1}{q}\Delta_\infty^c u, \tag{3.60}$$

where q is the Hölder conjugate of p and $\frac{1}{p}+\frac{1}{q}=1$. For $p=2 \Rightarrow q=2$ we get

$$\Delta_2^c u = \frac{1}{2}\Delta u.$$

Finally, we consider the normalized p-Dirac equation by substituting $w = Du$ in Equation (3.57)

$$\mathrm{D}_p^c u = \frac{1}{p}|w|^{2-p}\mathrm{D}_p u = \frac{1}{p}|w|^{2-p} D\left(w|w|^{p-2}\right) = 0 \qquad 1 \le p < \infty.$$

Analogously, all the other relationships of the normalized p-Dirac, the normalized 1-Dirac and the normalized ∞-Dirac can be obtained.

Concluding

- It could be shown in the present chapter that the p-Poisson equation with certain Neumann-type boundary conditions can be transferred to a Dirichlet boundary value problem for the p-Dirac equation.
- The main tool was an operator calculus taken from hypercomplex function theory.
- The obtained p-Dirac equation could be solved iteratively by a fixed-point iteration.
- This was possible for all $1 < p < 2$ and as an estimate for the contractivity constant, the value $c_2 = (2-p)$ was obtained.
- The case $p = 3/2$ plays a special role because, in this case, no bound for the right-hand side was necessary.

- Also, this was possible for all $p \in (2, 3)$ and as an estimate for the contractivity constant, the value $c_3 = \frac{(p-2)^2}{{k_2}^{1-(p-2)^2}}$, for $(p-2)^{\frac{2}{1-(p-2)^2}} < k_2 < 1$ was obtained.

- These results support the idea to reduce, as in the linear case, the study of the second-order differential equation to two equations of first order, where we can apply function theoretic methods. For the first step, discussed in this chapter, it is visible.

- This work is extending the applicability to the use of hypercomplex operator calculus to nonlinear equations with nonlinearities other than products of values and partial derivatives only. In particular, the results of this work allow to extend the results for the Navier–Stokes equations from [Gürlebeck and Sprößig, 1990] to the case of non-Newtonian fluids.

- Another benefit of the proposed approach is that we obtain strong solutions of the studied operator equations.

- Finally, the normalized generalized p-Laplace and p-Dirac problems were considered and the relation of the ∞-Laplace, ∞-Dirac, 1-Laplace and 1-Dirac operators were discussed.

4 Numerical Analysis and Discrete Function Theory

This chapter is dedicated to the consideration of a very special discrete operator: the theory of difference potentials is an important technique for describing the numerical solution of interior and exterior boundary value problems. The aim of this work consists in solving discrete boundary value problems by the method of difference potentials operating on a uniform grid of step size h.

Here, we are dealing with the plane case. The idea of studying the numerical approximations is not new. There are similar problems in the complex case. These problems have been under consideration for a long time. They are based on the concept of translating the whole function theory into the language of discretization methods. One of the approaches used is the language of finite difference approximations, and this way ends with a discrete function theory. This work started in the 1940's and 1950's (see [Isaacs, 1941], [Ferrand, 1944], [Isaacs, 1952], [Duffin, 1956], [Hayabara, 1966] and [Deeter and Lord, 1969]). To start with, we will present only a short summary of the method of difference potentials which is published in the following references [Duffin, 1953], [Ryabenkij, 1987] and [Hommel, 1998].

In this chapter, we will perform the following steps:

1. We will start with an overview of the discrete fundamental solutions of a discrete Laplace operator, a discrete Cauchy-Riemann operator and its adjoint, and their properties will be given. Then, we will calculate the discrete fundamental solution of a discrete Laplace operator in the plane with different methods and make a comparison between them, dependent on the time of simulation, the region and the accuracy of the absolute error.

2. We will calculate the discrete fundamental solutions of the Cauchy-Riemann operator and its adjoint in the plane by using the fundamental solution of a

discrete Laplacian.

3. By using the discrete fundamental solution of the Cauchy-Riemann operator and its adjoint, we define generalized discrete T_h^1 and T_h^2 operators (Teodorescu transform), which are right-inverse to the discrete Cauchy-Riemann operator and its adjoint in $2D$. Also, we define the discrete orthoprojections operators $\mathbf{P}_h$ and $\mathbf{Q}_h$. In conjunction, we prove explicit representations for the orthoprojections decomposition.

4. We consider a generalized discrete boundary operator F_h (a discrete Cauchy integral). Using discrete T_h and F_h operators, we formulate in a theorem a discrete Borel-Pompeiu formula and we prove it.

5. After introducing all important discrete operators, we will consider examples to construct discrete holomorphic complex-valued functions. By applying a generalized boundary operator F_h to these functions, we will verify the accuracy of that operator.

6. Finally, we construct a discrete complex-valued function from $\operatorname{Im} \mathbf{Q}_h^2$ and apply a T_h^1-operator to that function.

A computer program that defines the fundamental solution of a discrete Laplacian and Cauchy-Riemann operator was developed. The numerical computation for the functional theory, which is able to compute the discrete difference operators $\mathbf{D}_{h,M}^1$ and its adjoint $\mathbf{D}_{h,M}^2$, Teodorescu operators (T_h^1 and T_h^2), discrete Cauchy integral operators (F_h^1 and F_h^2) and a discrete Borel-Pompeiu formula in the domain $\mathbb{R}_h^2(G_h)$ was built. The presented numerical results of discrete function theory have applications in many areas of physics and engineering, in particular they can be used to solve the discrete p-Dirac problem in the next chapter.

For doing all these steps, we have to define the uniform lattice in the plane and provide some definitions related to the discrete norm.

4.1 The Uniform Lattice

Let G be a bounded domain in $\mathbb{R}^2$, $\mathbb{R}^2$ be the two-dimensional Euclidean space where $b_1 = (1, 0)$ and $b_2 = (0, 1)$ are the canonical basis vectors in $\mathbb{R}^2$, and let $x = (x_1, x_2)$ be an arbitrary element in $\mathbb{R}^2$.

We define the uniform lattice over this space by

$$\mathbb{R}_h^2 = \{mh = (m_1h, m_2h)\}$$

and we work on it, with the mesh width $h > 0$ and $m_1, m_2 \in \mathbb{Z}$.
Each discrete complex-valued function $f(mh)$ in this chapter is defined by

$$f_h(mh) = \mathrm{Re} f_h(mh) + i \mathrm{Im} f_h(mh) = (f_0(mh), f_1(mh)),$$

where i is the imaginary unit, satisfying the equation $i^2 = -1$.
The definition of the inner product in uniform lattice is given here:

Definition 13. *Let f_h and g_h be defined on $\mathbb{R}_h^2$. Then we introduce the inner product*

$$\langle f_h, g_h \rangle_h = \sum_{x \in \mathbb{R}_h^2} \overline{f_h(x)} g_h(x) h^2. \tag{4.1}$$

Using this discrete inner product we define the $L_p(\mathbb{R}_h^2)$.

Definition 14. *Let $f_h \in L_p(\mathbb{R}_h^2)$ be a discrete function with $1 < p < \infty$. The discrete norm is defined by*

$$||f_h||_{L_p} = \left(\sum_{x \in \mathbb{R}_h^2} |f_h(x)|^p \, h^2 \right)^{1/p} < \infty,$$

where h is the mesh width.

The forward and backward differences can be introduced by

$$D_h^j f_k(mh) = h^{-1}(f_k(mh + hb_j) - f_k(mh))$$

and

$$D_h^{-j} f_k(mh) = h^{-1}(f_k(mh) - f_k(mh - hb_j)),$$

respectively, where $j = 1, 2$ and $k = 0, 1$.
We introduce the norm in the space $W^{1,2}(\mathbb{R}_h^2)$ by

$$||f_h||_{W^{1,2}(G_h)} = \left(||f_h||_{L_2}^2 + \sum_{j=1}^{2} ||D_h^{\pm j} f_h(mh)||_{L_p}^2 \right)^{1/2}.$$

A discrete Laplace operator in $\mathbb{R}_h^2$ is given by:

$$-\Delta_h u_h\,(mh) = -\sum_{j=1}^{2} D_h^{-j} D_h^{j} u_h\,(mh)\,.$$

The Cauchy-Riemann operators can be introduced approximately with the difference operators

$$\mathbf{D}^1_{h,M} = \begin{pmatrix} D_h^{-2} & D_h^1 \\ -D_h^{-1} & D_h^2 \end{pmatrix} \text{ and } \mathbf{D}^2_{h,M} = \begin{pmatrix} D_h^{2} & -D_h^1 \\ D_h^{-1} & D_h^{-2} \end{pmatrix}.$$

The discrete Cauchy-Riemann operators have the following properties

$$\mathbf{D}^1_{h,M} f(mh) = \begin{pmatrix} D_h^{-2} & D_h^1 \\ -D_h^{-1} & D_h^2 \end{pmatrix} \begin{pmatrix} f_0(mh) \\ f_1(mh) \end{pmatrix} = \begin{pmatrix} D_h^{-2} f_0(mh) + D_h^1 f_1(mh) \\ -D_h^{-1} f_0(mh) + D_h^2 f_1(mh) \end{pmatrix}$$

$$= (D_h^{-2} f_0(mh) + D_h^1 f_1(mh), -D_h^{-1} f_0(mh) + D_h^2 f_1(mh))$$
$$= (-i)\left[\left(D_h^{-1} f_0(mh) + iD_h^1 f_1(mh)\right) + i\left(D_h^{-2} f_0(mh) + iD_h^2 f_1(mh)\right)\right]$$

and

$$\mathbf{D}^2_{h,M} f(mh) = \begin{pmatrix} D_h^{2} & -D_h^1 \\ D_h^{-1} & D_h^{-2} \end{pmatrix} \begin{pmatrix} f_0(mh) \\ f_1(mh) \end{pmatrix} = \begin{pmatrix} D_h^{2} f_0(mh) - D_h^1 f_1(mh) \\ D_h^{-1} f_0(mh) + D_h^{-2} f_1(mh) \end{pmatrix}$$

$$= (D_h^{2} f_0(mh) - D_h^1 f_1(mh), D_h^{-1} f_0(mh) + D_h^{-2} f_1(mh))$$
$$= i\left[\left(D_h^{-1} f_0(mh) + iD_h^1 f_1(mh)\right) - i\left(D_h^{2} f_0(mh) + iD_h^{-2} f_1(mh)\right)\right],$$

where the right-hand side of the above two equations is converging to the continuous Cauchy-Riemann operators

$$\mathbf{D}^1 f(x_1, x_2) = (-i)\left(\frac{\partial f}{\partial x_1} + i\frac{\partial f}{\partial x_2}\right) \text{ and } \mathbf{D}^2 f(x_1, x_2) = i\left(\frac{\partial f}{\partial x_1} - i\frac{\partial f}{\partial x_2}\right),$$

respectively, if h tends to zero. That indicates that these difference operators $\mathbf{D}^1_{h,M}$ and $\mathbf{D}^2_{h,M}$ can approximate the Cauchy-Riemann operators $\mathbf{D}^1$ and $\mathbf{D}^2$. The Cauchy-Riemann operators can be approximated in different ways (for more details see [Isaacs, 1941], [Ferrand, 1944], [Isaacs, 1952] and [Duffin, 1956]). Also, there are different possibilities to define a discrete Laplacian. Here, we used the difference operators $\mathbf{D}^1_{h,M}$ and $\mathbf{D}^2_{h,M}$ and we can factorize a discrete Laplace operator by the discrete Cauchy-Riemann operator and its adjoint. A discrete Laplace operator is defined by:

$$\mathbf{D}^1_{h,M}\mathbf{D}^2_{h,M} = \mathbf{D}^2_{h,M}\mathbf{D}^1_{h,M} = I_2\Delta_h,$$

where I is the 2×2 identical operator and Δ_h is the discrete Laplacian.
There are many known complex function theory tools that are not applicable in the discrete function theory, because they have no analytic power function. We can say that the set of discrete analytic functions is not a commutative algebra. For example, if $f \in \ker \mathbf{D}^1_{h,M}$ and $g \in \ker \mathbf{D}^1_{h,M}$ in general it does not follow that $fg \in \ker \mathbf{D}^1_{h,M}$.

Now, it is possible to define the fundamental solution of a discrete Laplacian in the plane.

4.2 The Discrete Fundamental Solution of a Laplacian in the Plane

A discrete Laplace operator in $\mathbb{R}^2_h$ is given by:

$$-\Delta_h u_h\,(mh) = -\sum_{j=1}^{2} D_h^{-j} D_h^{j} u_h\,(mh)\,, \tag{4.2}$$

by using finite difference approximations. We calculate $E_h\,(mh)$ which is the discrete fundamental solution of a Laplace operator. Each solution of the equation

$$-\Delta_h E_h(mh) = \delta_h(mh) = \begin{cases} h^{-2} & \text{for } mh = 0 \\ 0 & \text{for } mh \neq 0 \end{cases} \tag{4.3}$$

is called the discrete fundamental solution of the discrete Laplacian, for all mh, where $mh = (m_1 h, m_2 h)$ with $m_j \in \mathbb{Z}$, $j = 1, 2$ and $h > 0$ is the mesh width. In the plane case where the dimension is equal to 2, it refers to any solution of Equation (4.3), which does not grow faster than $\ln |x|$ at infinity as a fundamental solution. It is not necessary here to prove existence, uniqueness and representations of these fundamental solutions. Furthermore, we refer to the literature:
[Stummel, 1967], [Thomée, 1968], [Boor et al., 1989] and [Stern, 1991] these references contain general results for existence. [Duffin, 1953], [Gürlebeck, 1988], [Boor et al., 1989] and [Gürlebeck and Hommel, 1994] discussed special fundamental solutions with uniqueness results.

In the next section, we will investigate three methods to calculate the discrete fundamental solutions of Laplacian Δ_h in more details. We will consider three algorithms for the computation of this fundamental solution. The discrete fundamental solution of Laplacian in $\mathbb{R}_h^2$ will be calculated by using the finite difference methods. This is a difficult case, because at infinity it should not grow faster than the logarithm. Numerically, it is a challenging task to get such a behavior.

This fact creates a reason to calculate the fundamental solution of a discrete Laplacian by three different methods to find the best way to get the fundamental solution. It is necessary to have an explicit expression of the fundamental solution of a discrete Laplacian because it is an important part of numerical applications, and for the constructive analytical considerations.

In this part, we will present these methods and make a comparison between the methods dependent on the time of simulation, the region and the absolute error between the fundamental solution of a discrete Laplacian Δ_h of each method and the fundamental solution of the proper Laplacian Δ. All these calculations will be done by using finite difference methods. MATLAB R2013a and Maple 2015 programmes will be used for these calculations.

The convergence of the discrete fundamental solution $E_h(mh)$ of Δ_h is an important information in the theory of difference potentials. The proof of the convergence of the difference potentials was considered in [Hommel, 1998]. It has been proved that

$$|E_h\,(mh) + K - E\,(mh)| \leq C_1 h |mh|^{-1}$$

for $h > 0$ and a special constant K. This constant is $K = \frac{1}{2\pi}\left(\frac{5}{2}\ln 2 - \ln h\right)$ which was proved theoretically in [Hommel, 1998]. Numerically, for $h = 1$, we found that $K = 0.257303603247324$ gives a better accuracy, where C_1 is a constant and $E\,(mh) = \frac{-1}{2\pi}\left(C_\gamma - \ln 2 + \ln|mh|\right)$ is the continuous fundamental solution and C_γ is the Euler-Mascheroni constant.

A. Hommel provided a formula to estimate the approximation error of the discrete fundamental solution of Laplacian in the space $L_p(G_h)$ with $1 \leq p < \infty$ for each bounded domain $G_h = (G \cap \mathbb{R}_h^2) \subset \mathbb{R}_h^2$ and $h \leq e^{-1}$. This form can also be used to estimate the approximation error in L_p. The reported form is:

$$\|E_h(mh) + K - E_h^*(mh)\|_{L_p(G_h)} \leq \begin{cases} Ch & 1 \leq p < 2 \\ Ch\sqrt{|\ln h|} & p = 2 \\ Ch^{2/p} & 2 < p < \infty \end{cases},$$

where

$$E_h^*(mh) = \begin{cases} E(mh) & \text{for} \quad mh \neq 0, \\ 0 & \text{for} \quad mh = 0. \end{cases}$$

The discrete fundamental solution is an important part of the potential theoretical investigations. Therefore, it should be studied in particular numerical details.

4.2.1 Three Methods to Calculate the Discrete Fundamental Solution of the Laplacian in the Lattice Cell

In this subsection, we present three numerical methods to calculate the fundamental solution in the first quadrants. Using the symmetrical behavior of the discrete fundamental solution, we will complete the whole lattice region. Using these results, we construct a new numerical technique which is obtained by incorporating the best result of each method in a specific part of the domain and we cover the whole plane.

This fundamental solution is defined on the whole lattice $\mathbb{R}_h^2$. Finally, we will summarize the results of each numerical way to calculate the fundamental solution of Δ_h and calculate the difference between the fundamental solution of discrete Laplacian and the fundamental solution of continuous Laplacian individually. Besides, there are some important questions that will be considered later with respect to this fundamental solution.

We investigate the discrete fundamental solution $E_h(mh)$ only for $h = 1$. Because of symmetry properties, it is sufficient to evaluate

$$E_1(m_1, m_2) = E_1(m_2, m_1) = E_1(-m_1, m_2) = E_1(m_1, -m_2),$$

only at the points (m_1, m_2) with $m_1, m_2 \in \mathbb{N}$ and $0 \leq m_2 \leq m_1$.
In addition, we have

$$E_h(m_1 h, m_2 h) = E_1(m_1, m_2) \qquad \forall h > 0.$$

Therefore, it is sufficient to make all computations for only one mesh width.

The fundamental solution which will be calculated from the first method is $E_h^1(mh)$. $E_h^2(mh)$ and $E_h^3(mh)$ are the fundamental solution of Δ_h which are calculated from the second method and the third method, respectively.

Note that to get high accuracy, we will use the fixed-decimal format with 16 digits after the decimal point for the whole numerical calculation.

1^{st} Method: **Normalized Discrete Fourier Transform**

In this part, the integral representation for the discrete fundamental solution of the Laplacian which was obtained by discrete Fourier transform is discussed. The most familiar and elementary method for calculating the discrete fundamental solution of Laplacian is to consider the integral below. In particular, using the discrete Fourier transform, we obtain:

$$E_h^1(mh) = \frac{1}{(2\pi)^2} \int_{-\pi/h}^{\pi/h} \int_{-\pi/h}^{\pi/h} \frac{e^{-i(m_1 h\xi_1 + m_2 h\xi_2)} - 1}{d^2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \tag{4.4}$$

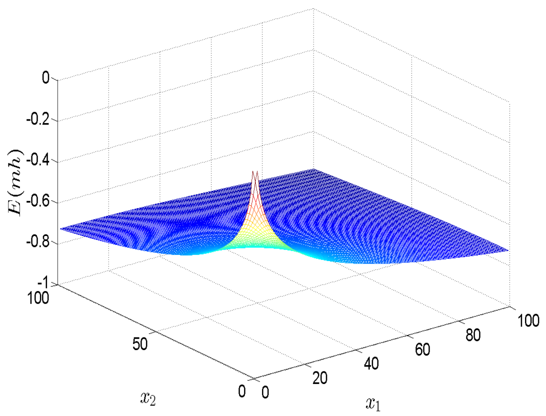
with $d^2 = \frac{4}{h^2}(\sin^2(h\xi_1/2) + \sin^2(h\xi_2/2))$, where $mh = (m_1 h, m_2 h)$ and $h > 0$ is the mesh width. (See [Thomée, 1968].)
This integral representation is obtained from the ideas of Stummel [Stummel, 1967]. This integration has also been studied in [Hommel, 1998].
Analytically, this fundamental solution is known by the normalized Fourier integral (see, [Stummel, 1967], [Thomée, 1968]).
The continuous fundamental solution of Laplacian at lattice points is given by

$$E(mh) = \frac{-\log|mh|}{2\pi}.$$

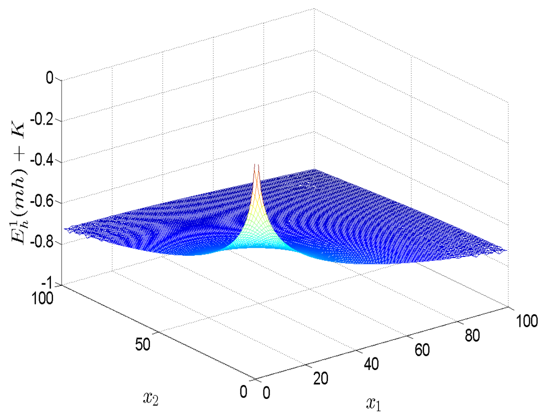
There are many different methods to calculate the discrete fundamental solution $E_h^1(mh)$. Each method has different strengths, but to solve the Integral (4.4) numerically, the tiled approach is the fastest way with more accurate results. Using generalized Gaussian quadrature rule, which is defined in MATLAB by the code "*quad2d*" using the tiled method, we will calculate the fundamental solution $E_h^1(mh)$.

Hint: This approach divides the region into quadrants approximating the integral over each quadrant by a two dimensional quadrature rule. If the error condition on the rectangle has not been achieved, the rectangle is divided again into quadrants and so forth.

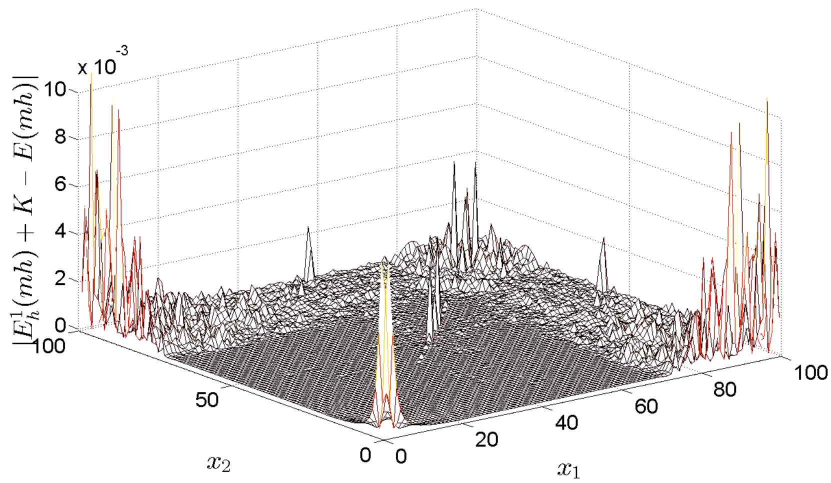
The plots of the continuous fundamental solution $E(mh)$ and the discrete fundamental solution $E_h^1(mh) + K$ can be seen in Figure 4.1a and 4.1b, respectively. Figure 4.1c shows the absolute error between the fundamental solution $E_h^1(mh) + K$ and the fundamental solution $E(mh)$. Close to the center of the graph it represents an optimal result. But as the value points diverge from the center, greater instability is the result, increasing in all parameters.



(a) The F.S of Laplacian Δ



(b) The F.S of Laplacian Δ_h



(c) The absolute error

Figure 4.1: The absolute difference between the fundamental solutions of the Laplacian Δ and the discrete Laplacian Δ_h.

The numerical studies of the fundamental solution $E_h^1(mh)$ which is given in (4.4), communicate that this method to calculate the fundamental solution of a discrete Laplacian Δ_h is unstable if the lattice domain becomes big. For that reason, we do not need to calculate the fundamental solution $E_h^1(mh)$ in a big domain. The simulation process to calculate the fundamental solution $E_h^1(mh)$ was not fast enough. Also, the absolute difference between the fundamental solution $E_h^1(mh)+K$ and the fundamental solution $E(mh)$ is too big for the values which are not near the origin, as it is clear in the above Figure 4.1c.
Now, it is necessary to study other approaches to calculate the discrete fundamental solution of a discrete Laplacian with a more accurate and stable method.

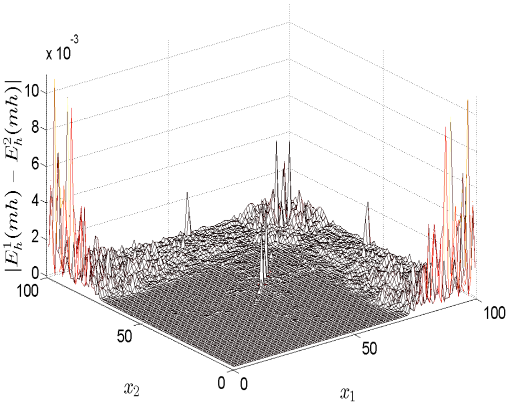
2^{nd} Method: **Simple Numerical Implementation**
Another representation formula to the fundamental solution (4.4) for the mesh width $h = 1$ is proved by S. Sobolev. [Sobolev, 1952], proved the following formula at the main diagonal of the lattice:

$$E_1^2(n,n) = -\frac{1}{\pi}\left(1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right), \qquad n \geq 1, \quad n \in \mathbb{N}. \tag{4.5}$$

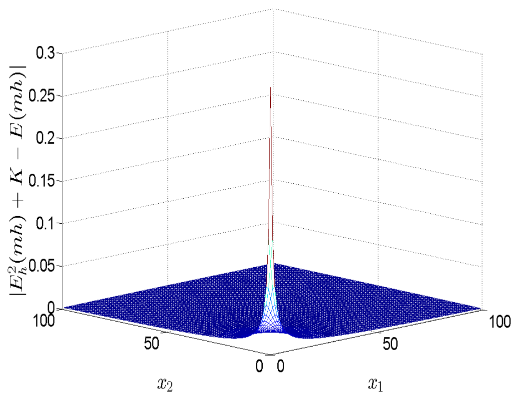
Using the Integral (4.4), we can calculate 3 primary points $E_1^2(0,0) = 0$ and $E_1^2(1,0) = E_1^2(0,1) = -1/4$. Starting from these known values (evaluated by a finite number of operations at each point), the values at all the other points of the whole lattice $\mathbb{R}_h^2$ can be computed using the difference equation.

After calculating the discrete fundamental solution $E_1^2(mh)$ using Maple programme, the results will be stored. Using MATLAB, we plot the absolute difference between the fundamental solution $E_h^2(mh)$ and the fundamental solution $E_h^1(mh)$ as it is shown in Figure 4.2a. One can see that the absolute error is big through the diagonal and when the distance increases from the origin.

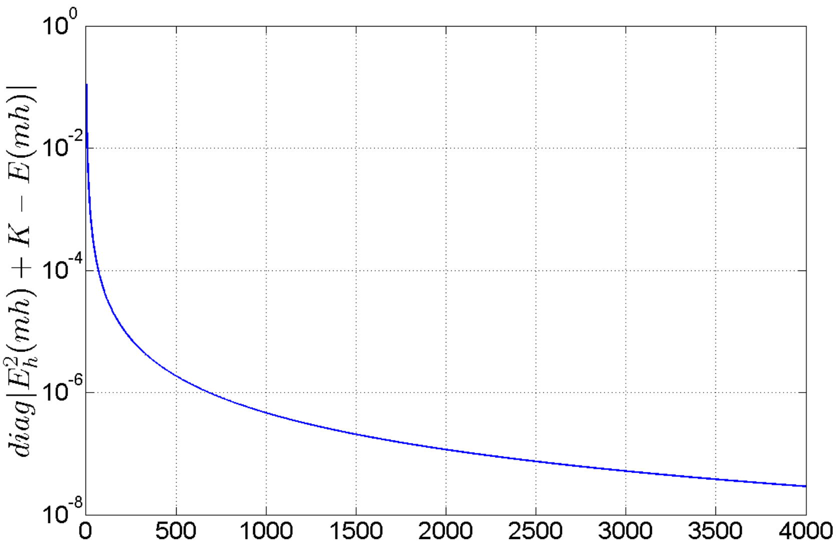
Figure 4.2b and 4.2c show the absolute error between the discrete fundamental solution $E_h^2(mh)+K$ and the fundamental solution $E(mh)$ of Laplacian. In order to achieve minimum absolute error, which converges to zero, the region must be enlarged. In addition, the simulation process was extremely fast. One can observe that the method used to calculate the fundamental solution of a discrete Laplacian is successful in the large area and the absolute error converges to zero as it is seen in 4.2c.



(a) The absolute difference of $E_h^1(mh)$ and $E_h^2(mh)$



(b) The absolute error



(c) The diagonal plot of the absolute error

Figure 4.2: Plot of the absolute difference between the fundamental solution of a discrete Laplace operator which are calculated by the 1^{st} and 2^{nd} method. The second and third plot is the difference between the fundamental solution of the Laplacian Δ and the discrete Laplacian Δ_h calculated by the second method.

The final method to calculate the fundamental solution of a discrete Laplacian is to solve Poisson's equation.

3^{rd} Method: **The Boundary Value Problem on a Grid**

By solving the second-order linear elliptic (Poisson's equation) in a rectangular region

$$-\Delta_h E^3(x,y) = \begin{cases} \dfrac{1}{h^2} & \text{for } (x,y) = 0 \\ 0 & \text{for } (x,y) \neq 0 \end{cases} \tag{4.6}$$

with the boundary condition (this boundary is equal to the fundamental solution of the continuous Laplacian)

$$E^3(x,y) = -\frac{log\,(x^2+y^2)}{(4\pi)} \qquad , \text{on } \Gamma_h,$$

the fundamental solution of a discrete Laplacian will be calculated.
As before, we are interested in numerical methods. Therefore, we use the finite difference discretization of the partial derivatives. i.e.,

$$E^3_{xx}(x,y) = \frac{1}{h^2}\left[E^3(x+h,y) - 2E^3(x,y) + E^3(x-h,y)\right]$$

and

$$E^3_{yy}(x,y) = \frac{1}{h^2}\left[E^3(x,y+h) - 2E^3(x,y) + E^3(x,y-h)\right].$$

Using the lattice notations $E^3(x,y) = E^3_h(m_1h, m_2h) = E^3_{h,i,j}$ and $E^3(x+h,y) = E^3_h(m_1h+h, m_2h) = E^3_{h,i+1,j}$, Equation (4.6) turns into the difference equations

$$\tfrac{-1}{h^2}\left[4E^3_{h,i,j} - E^3_{h,i-1,j} - E^3_{h,i+1,j} - E^3_{h,i,j+1} - E^3_{h,i,j-1}\right] = \begin{cases} \frac{1}{h^2}, \\ 0. \end{cases}$$

This equation can be rewritten as

$$4E^3_{h,i,j} - E^3_{h,i-1,j} - E^3_{h,i+1,j} - E^3_{h,i,j+1} - E^3_{h,i,j-1} = \begin{cases} -1 \\ 0 \end{cases}$$

or

$$E^3_{h,i,j} = \begin{cases} \frac{1}{4}\left[E^3_{h,i-1,j} + E^3_{h,i+1,j} + E^3_{h,i,j+1} + E^3_{h,i,j-1}\right] - \frac{1}{4}, & (m_1h, m_2h) = 0 \\ \frac{1}{4}\left[E^3_{h,i-1,j} + E^3_{h,i+1,j} + E^3_{h,i,j+1} + E^3_{h,i,j-1}\right], & (m_1h, m_2h) \neq 0. \end{cases} \tag{4.7}$$

We obtain the discrete fundamental solution of Laplacian $E^3_h(mh)$ by solving the discrete boundary value problem (4.7) iteratively (using the Jacobi iterative method).
When we study this approach and compare the result with the exact fundamental solution $E(mh)$ we find that this method has a better result as it is shown in

Figure 4.3, but if we need to calculate the fundamental solution of a Δ_h in a very large domain, the second method will be more suitable.

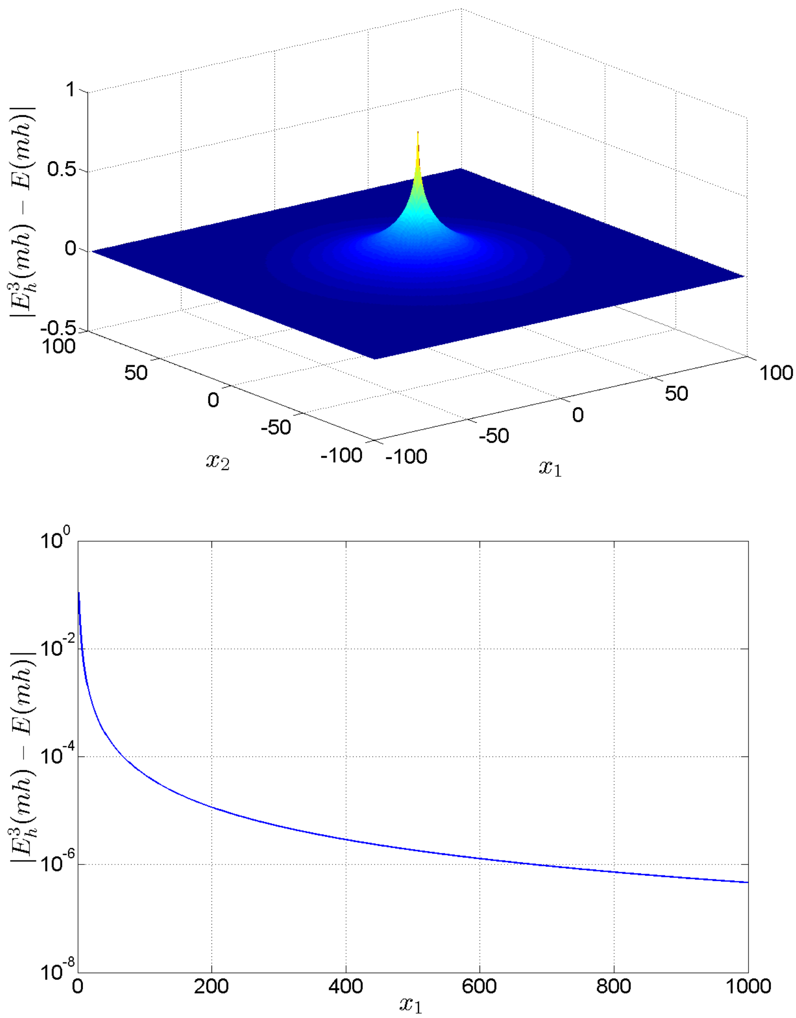


Figure 4.3: The absolute error of $E_h^3(mh)$ and the continuous fundamental solution $E(mh)$, is represented in a $3D$ plot and the $2D$ plot, using the logarithm scale in the y- axis.

In the previous subsection we have calculated the discrete fundamental solution of Laplacian by using three different approaches. We can summarize the formulas for one step of each of our methods, by focusing on the advantages of each approach and find the most effective method to calculate the fundamental solution of a discrete Laplacian.

4.2.2 Summary of Methods Behavior

To review what we have seen, all three methods can be used to calculate the discrete fundamental solution in $\mathbb{R}^2_h$ seen above: The stability of these methods can depend differently on the following factors; the regions, time of simulation, accuracy and precision of absolute error. The absolute error is the absolute difference between the fundamental solution of a discrete Laplacian which are calculated by three methods and the fundamental solution of Laplacian.

Using the first method to calculate the fundamental solution $E^1_h(mh)$ is recommendable if we near the origin, but we will only need to replace the points through the diagonal by calculating them using the second method.
We are often interested in approximation of the discrete fundamental solution by solving the Poisson problem. The results from the third method are favorable and consistent. However, to achieve the best results to calculate the discrete fundamental solution of discrete Laplacian in regard to computing time and higher accuracy, the second method appears to be more effective.

The second method to calculate the fundamental solution of a discrete Laplacian is more effective in a large region and the absolute error between the fundamental solution $E^2_h(mh)$ with the fundamental solution $E(mh)$ is very small. After the achievement of the minimum absolute error, one can use the fundamental solution of Laplacian.

Now, $E_h(mh)$ is the discrete fundamental solution of Laplacian which is calculated by using the best part of each method. We introduce a basic graph to explain the concept of combining the previously used methods and the symmetry properties, resulting in all quadrants having points as shown in Figure 4.4.

Remark 15. *The graduation on the axis in Figure 4.4 is not drawn to scale.*

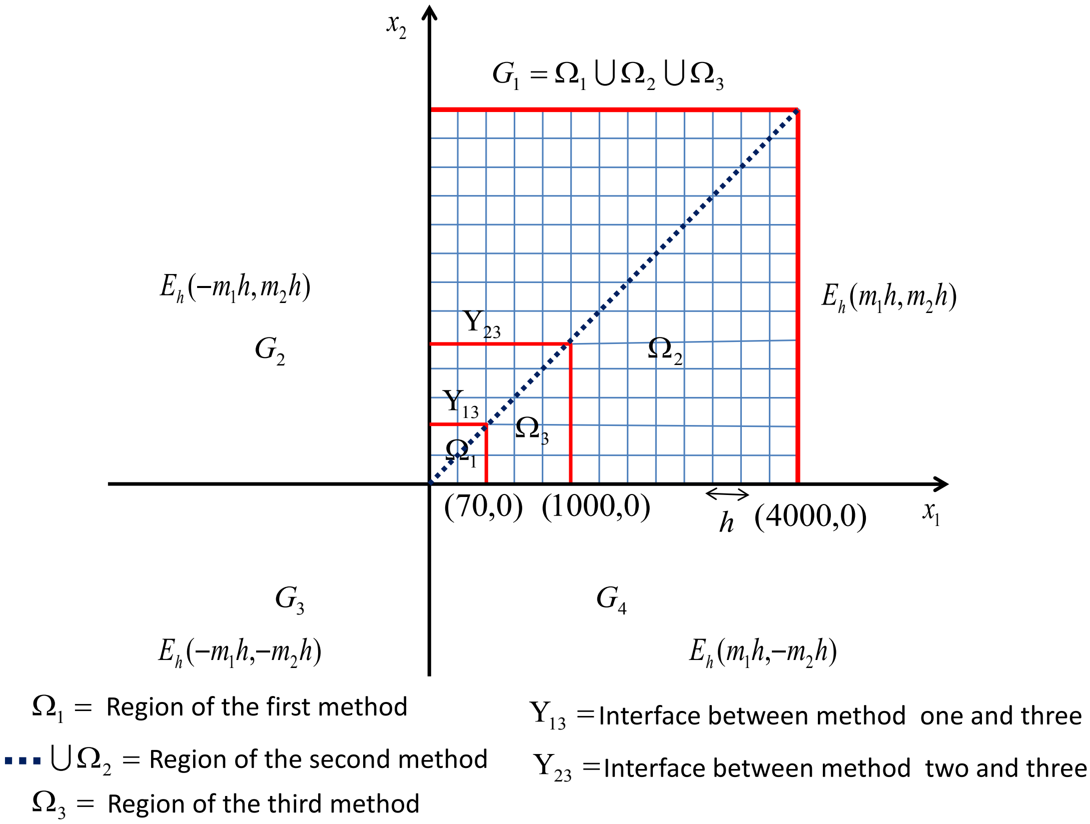


Figure 4.4: Combined results

For the purpose of this work, we have calculated the discrete fundamental solution for 8001×8001. Finally, we have checked that the discrete fundamental solution $E_h(mh)$ satisfies Equation (4.3).

There is still an important question we should ask, "What happens when we combine the methods?" To answer this question, we will apply a discrete Laplacian to the fundamental solution $E_h(mh)$ and plot the result of this calculation. In Figure 4.5 we plot the region of the interface Υ_{13} after applying Δ_h to the fundamental solution $E_h(mh)$. Figure 4.5 displays a very small margin of error, which is insignificant in the final calculation.

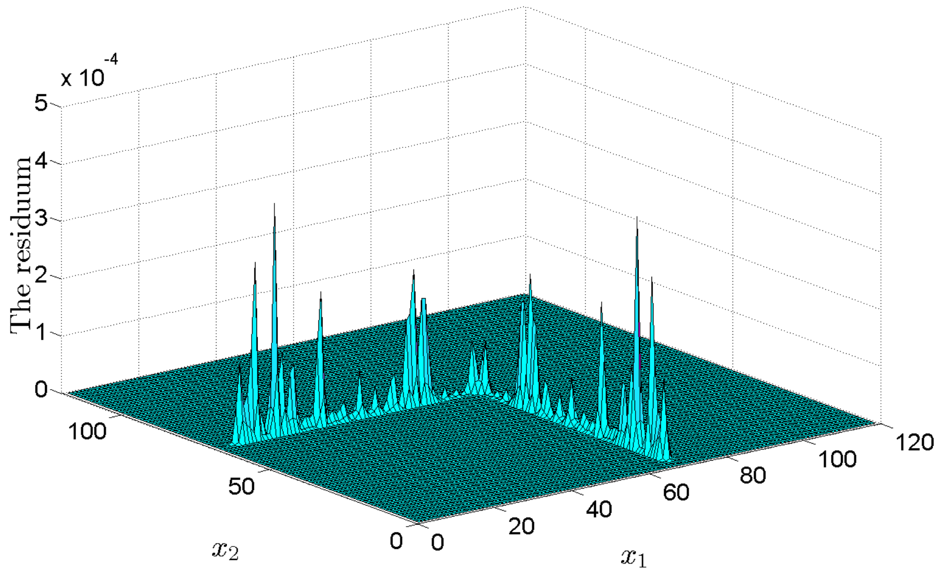


Figure 4.5: Influence of the interface Y_{13} on the error.

It is common for researchers to test methods by comparing their solutions with those obtained by other methods as we do in some parts of the present work. This kind of test is not an optimal test and can even be misleading. The differences between the various solutions are often simply errors differences; consequently, it is difficult to evaluate which is the best method to calculate the fundamental solution of a discrete Laplacian. One can wonder: "Why this method?" and other approaches can be suggested. To answer this question, we will compare the discrete fundamental solution $E_h(mh)$ with the exact fundamental solution $E(mh)$. The comparison with the exact solution (i.e. accurate error) is a more significant measure of quality.

Figure 4.6 shows the diagonal plot of the absolute difference between the $E_h(mh)$ and the exact one.

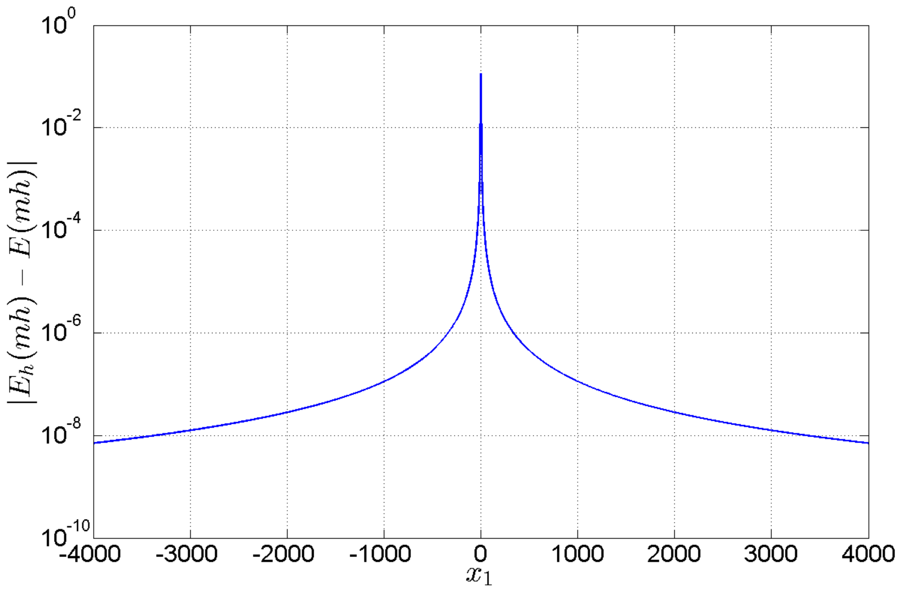


Figure 4.6: The diagonal plot of the absolute error between the fundamental solution $E_h(mh)$ and the exact fundamental solution using the logarithmic scale in y- axis.

One can recognize that the error between the fundamental solution $E_h(mh)$ of Δ_h and the fundamental solution of Laplacian is 10^{-8} if we are near 4000. One can use the exact fundamental solution to complete the value beyond this region. Also, if we need more accuracy (less than 10^{-8}), we can calculate the fundamental solution of a discrete Laplacian in a larger region. Here, we find that the error margin is stable with respect to the study.

Conclusion and Remarks

- The absolute error of the discrete fundamental solution of Δ_h and the fundamental solution of Laplacian has been investigated.
- The algorithm to calculate the fundamental solution of a discrete Laplacian using three methods is comparable in speed and storage requirements. The absolute error for the discrete fundamental solution $E_h(mh)$ and the exact fundamental solution $E(mh)$ of Laplacian is determined.

Now, we will use the fundamental solution $E_h(mh)$ of a discrete Laplacian to calculate the discrete fundamental solution of the Cauchy-Riemann operator and its adjoint in the plane.

4.3 The Discrete Fundamental Solution of the Cauchy-Riemann Operator and its Adjoint

The discrete fundamental solution of the discrete Cauchy-Riemann operator $\mathbf{D}^1_{h,M}$ is $\mathbf{e}^1_h(mh)$, which is a solution of the system

$$\begin{pmatrix} D_h^{-2} & D_h^1 \\ -D_h^{-1} & D_h^2 \end{pmatrix} \begin{pmatrix} e^1_{h11}(mh) & e^1_{h12}(mh) \\ e^1_{h21}(mh) & e^1_{h22}(mh) \end{pmatrix} = \begin{pmatrix} \delta_h(mh) & 0 \\ 0 & \delta_h(mh) \end{pmatrix} \tag{4.8}$$

and the $\mathbf{e}^2_h(mh)$ is the discrete fundamental solution of the adjoint operator $\mathbf{D}^2_{h,M}$ which is a solution of the system

$$\begin{pmatrix} D_h^2 & -D_h^1 \\ D_h^{-1} & D_h^{-2} \end{pmatrix} \begin{pmatrix} e^2_{h11}(mh) & e^2_{h12}(mh) \\ e^2_{h21}(mh) & e^2_{h22}(mh) \end{pmatrix} = \begin{pmatrix} \delta_h(mh) & 0 \\ 0 & \delta_h(mh) \end{pmatrix},$$

where the discrete δ-function is defined in Equation (4.3).

Using for each fundamental solution $\mathbf{e}^1_h(mh)$ and $\mathbf{e}^2_h(mh)$ entry the discrete Fourier transform, see [Stummel, 1967]

$$(F_h u_h)(\xi) = \begin{cases} \frac{h^2}{2\pi} \sum\limits_{mh \in \mathbb{R}^2_h} u_h(mh) e^{ih\langle m,\xi\rangle} & \xi \in Q_h \\ 0 & \xi \in \mathbb{R}^2 \setminus Q_h, \end{cases}$$

with $Q_h = \left\{ \xi \in \mathbb{R}^2 : -\frac{\pi}{h} < \xi_j < \frac{\pi}{h}, \quad j = 1,2 \right\}$, the scalar product $\langle m, \xi\rangle = m_1\xi_1 + m_2\xi_2$ and the properties:

$$\begin{aligned} F_h(D_h^j u_h) &= -\xi^h_{-j} F_h u_h, \quad \xi^h_{-j} = h^{-1}(1 - e^{-ih\xi_j}) \ , \quad j = 1,2 \\ F_h(D_h^{-j} u_h) &= \xi^h_j F_h u_h, \quad \xi^h_j = h^{-1}(1 - e^{ih\xi_j}) \ , \quad j = 1,2. \end{aligned} \tag{4.9}$$

Integral representation for the matrix elements of $\mathbf{e}^1_h(mh)$ and $\mathbf{e}^2_h(mh)$ is obtained by using the inverse discrete Fourier transform $(F_h)^{-1} = R_h F$, where $R_h u$ denotes the restriction of the function u to the lattice $\mathbb{R}^2_h$ and the symbol F stands for the classical Fourier transform. The integral representation is defined by the formulas:

$$\mathbf{e}^1_h(mh) = \frac{1}{2\pi} \begin{pmatrix} R_h F(\xi^h_{-2}/d^2) & R_h F(-\xi^h_{-1}/d^2) \\ R_h F(-\xi^h_1/d^2) & R_h F(-\xi^h_2/d^2) \end{pmatrix}$$

and

$$\mathbf{e}_h^2(mh) = \frac{1}{2\pi}\begin{pmatrix} R_hF(-\xi_2^h/d^2) & R_hF(\xi_{-1}^h/d^2) \\ R_hF(\xi_1^h/d^2) & R_hF(\xi_{-2}^h/d^2) \end{pmatrix},$$

where $d^2 = \frac{4}{h^2}(\sin^2(h\xi_1/2) + \sin^2(h\xi_2/2))$.

For calculating the discrete fundamental solution of Cauchy-Riemann operator, we will use the discrete fundamental solution of Laplacian, for this purpose, we will write the fundamental solution of the discrete Laplacian $E_h(mh)$ as a matrix. By applying one time the adjoint operator $\mathbf{D}_{h,M}^2$ to the discrete fundamental solution $E_h(mh)$ and next time applying the operator $\mathbf{D}_{h,M}^1$ to the discrete fundamental solution $E_h(mh)$, we will calculate the discrete fundamental solution of Cauchy-Riemann operator $\mathbf{e}_h^1(mh)$ and its adjoint $\mathbf{e}_h^2(mh)$ respectively

$$\begin{pmatrix} D_h^2 & -D_h^1 \\ D_h^{-1} & D_h^{-2} \end{pmatrix}\begin{pmatrix} E_h(mh) & 0 \\ 0 & E_h(mh) \end{pmatrix} = \begin{pmatrix} e_{h11}^1(mh) & e_{h12}^1(mh) \\ e_{h21}^1(mh) & e_{h22}^1(mh) \end{pmatrix},$$

and

$$\begin{pmatrix} D_h^{-2} & D_h^1 \\ -D_h^{-1} & D_h^2 \end{pmatrix}\begin{pmatrix} E_h(mh) & 0 \\ 0 & E_h(mh) \end{pmatrix} = \begin{pmatrix} e_{h11}^2(mh) & e_{h12}^2(mh) \\ e_{h21}^2(mh) & e_{h22}^2(mh) \end{pmatrix}.$$

We have stored the discrete fundamental solution $E_h(mh)$ in the 1^{st} quadrant and this is enough to calculate the discrete fundamental solutions of Cauchy-Riemann operator and its adjoint in the full domain.

One should check the error estimate between the discrete and the continuous fundamental solution of Cauchy-Riemann operator and its adjoint in the plane. The continuous fundamental solutions of Cauchy-Riemann operator and its adjoint are $\mathbf{e}^1(mh)$ and $\mathbf{e}^2(mh)$, where

$$\mathbf{e}^1(mh) = \frac{i}{2\pi(m_1h + im_2h)}$$

and

$$\mathbf{e}^2(mh) = \frac{i}{2\pi(m_1h - im_2h)}.$$

We calculate the continuous fundamental solution by using MATLAB and then compute the absolute difference between the continuous and the discrete fundamental solution of Cauchy-Riemann operator and its adjoint. In Figure 4.7, can be seen the curve of the absolute difference one time between $e^1_{h11}(mh)$ and $\operatorname{Re}\mathbf{e}^1(mh)$ and second time between $e^2_{h21}(mh)$ and $\operatorname{Im}\mathbf{e}^2(mh)$.

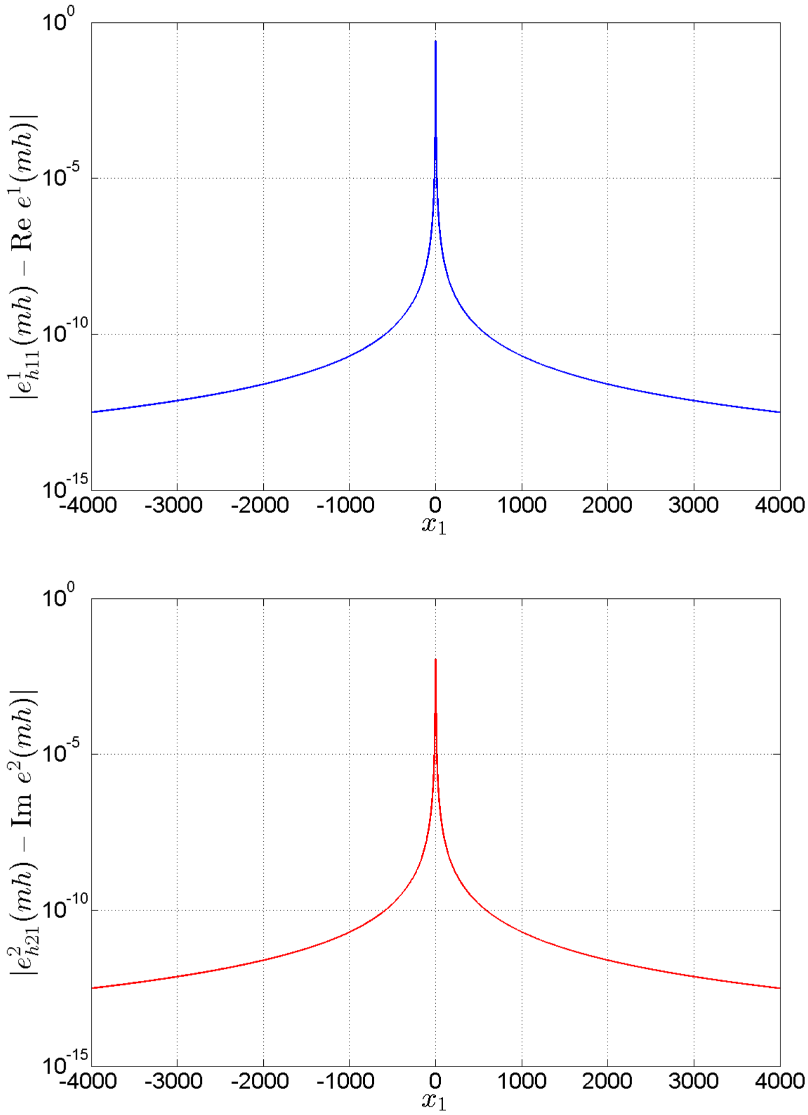


Figure 4.7: The absolute difference between the exact and the discrete fundamental solution of Cauchy-Riemann operator.

From these figures one can see that the difference between the discrete and the continuous fundamental solution of Cauchy-Riemann operator and its adjoint is miniscule below 10^{-12} near ± 4000. There is an option here to use the continuous fundamental solution of Cauchy-Riemann if required outside this domain.

This numerical calculation achieves the best obtainable result and now one can use the discrete fundamental solutions of Cauchy-Riemann operator and its adjoint to define discrete T_h^1 and T_h^2 operators, which are a right-inverse to the discrete Cauchy-Riemann operator and its adjoint in the plane.

4.4 A Discrete Teodorescu Transform (T_h -Operator)

In this section, we will generalize discrete T_h^1 and T_h^2 operators which are right inverse operators to the discrete Cauchy-Riemann operator and its adjoint. Let us consider a bounded domain $G \subseteq \mathbb{R}^2$ and denote $G_h = (G \cap \mathbb{R}_h^2)$ as the discrete bounded domain in $\mathbb{R}_h^2$. To define the discrete boundary γ_h^-, we use the canonical basis vectors $b_1 = (1,0)$ and $b_2 = (0,1)$. The discrete boundary γ_h^- can be defined by

$$\gamma_h^- = \{rh \in \mathbb{R}_h^2 \setminus G_h : \exists b_i \text{ with } (r \pm b_i)h \in G_h,\, i = 1,2.\}$$

Frequently the boundary γ_h^- is split into four parts:

$$\gamma_{hj}^- = \begin{cases} rh \in \gamma_h^- : (r + b_i)h \in G_h & \text{for } j = 1,2\ ,\ i = 1,2 \\ rh \in \gamma_h^- : (r - b_i)h \in G_h & \text{for } j = 3,4\ ,\ i = 1,2. \end{cases}$$

The first layer in the interior points β_h can be defined by

$$\beta_h = \{rh \in G_h : \exists b_i \text{ with } (r \pm b_i)h \in \gamma_h^-,\, i = 1,2.\}$$

Frequently the first layer in the interior points β_h is split into four parts:

$$\beta_{hj} = \begin{cases} rh \in \beta_h : (r - b_i)h \in \gamma_{hj}^- & \text{for } j = 1,2\ ,\ i = 1,2 \\ rh \in \beta_h : (r + b_i)h \in \gamma_{hj}^- & \text{for } j = 3,4\ ,\ i = 1,2. \end{cases}$$

Moreover, let us define the corner points

$$\Gamma_{sz} = \{lh \in \mathbb{R}_h^2 \setminus (G_h \cup \gamma_h^-) : (l \pm b_i)h \in \gamma_{hs}^- \quad \text{and} \quad (l \pm b_i)h \in \gamma_{hz}^-\},$$

with $i = 1,2$ and $s \neq z \in \{1,\dots,4\}$. These corners consist of inner corners and outer corners. The inner corners are defined by:

$$\Gamma_{sz}^* = \{lh \in \mathbb{R}_h^2 \setminus (G_h \cup \gamma_h^-) : (l \pm b_i)h \in \gamma_{hs}^- \text{ or } G_h \quad \text{and} \quad (l \pm b_i)h \in \gamma_{hz}^- \text{ or } G_h\}.$$

The set $G_h \pm hb_i$ is defined by:

$$G_h \pm hb_i = \begin{cases} G_h + hb_i = \left\{lh \in (G_h \setminus \beta_{hi}) \cup \gamma^-_{h(i+2)}\right\} \\ G_h - hb_i = \left\{lh \in (G_h \setminus \beta_{h(i+2)}) \cup \gamma^-_{hi}\right\} \end{cases}, \qquad \text{for } i = 1, 2.$$

In [Gürlebeck and Hommel, 2003], the boundary values are set to be zero on Γ_{sz} were considered, because these corners are not important for solving the discrete Laplace equation. They have mentioned if one consider an arbitrary complex-valued function $f_h(mh) = \mathrm{Re} f_h(mh) + i\mathrm{Im} f_h(mh) = (f_0(mh), f_1(mh))$, then the function $f_0(mh) = f_1(mh) = 0$ for all $mh \in \Gamma_{14}$ and $mh \in \Gamma_{23}$ which are special outer corners.

We need to extend the definition of a discrete T_h-operator because it will be applied to the function from the image of discrete orthoprojections operators and the value of this function can be different from zero on the corners.

Definition 15. *The discrete orthoprojections operators are denoted by* $\mathbf{P}^i_h$ *and* $\mathbf{Q}^i_h$ *with* $\mathbf{Q}^i_h = I - \mathbf{P}^i_h$ *where* $i = 1, 2$. *They are defined by*

$$\mathbf{P}^i_h : \;\; L_{2,h}(G_h) \;\; \overset{onto}{\longrightarrow} \;\; \ker \boldsymbol{D}^i_{h,M}(G_h)$$

$$\mathbf{Q}^1_h : \;\; L_{2,h}(G_h) \;\; \overset{onto}{\longrightarrow} \;\; \boldsymbol{D}^2_{h,M}(W^{1,2}_0(G_h)), \qquad \mathbf{Q}^2_h : \;\; L_{2,h}(G_h) \;\; \overset{onto}{\longrightarrow} \;\; \boldsymbol{D}^1_{h,M}(W^{1,2}_0(G_h)).$$

These ortoprojectors are defined by the orthogonal decomposition of the space $L_{2,h}$ as it is considered in the next theorem. The statement of the next theorem is taken from [Gürlebeck and Sprößig, 1990] and [Gürlebeck and Sprößig, 1997]. We will repeat here the proof of the next theorem to inform the reader that when we change on the definition of the boundary and include the corner points, for any function values on the corners without any restriction, this will not have changes in the orthogonal decomposition.

Theorem 16. *The discrete space* $L_{2,h}(G_h)$ *allows the orthogonal decomposition*

$$L_{2,h}(G_h) = \ker \boldsymbol{D}^1_{h,M}(G_h) \oplus_{\langle,\rangle_h} \boldsymbol{D}^2_{h,M}(W^{1,2}_0(G_h)),$$

with respect to the discrete inner product (4.1).

The $\oplus_{\langle,\rangle_h}$ stands for the topological sum relative to the discrete inner product which is given in (4.1).

Proof. We follow the proof which is given in [Gürlebeck and Sprößig, 1997]. We will prove first

$$\mathbf{D}^2_{h,M}(W_0^{1,2}(G_h))^\perp \subset \ker \mathbf{D}^1_{h,M}(G_h).$$

Let $f_h(x) \in W_0^{1,2}(G_h)$ and $g_h(x) \in \mathbf{D}^2_{h,M}(W_0^{1,2}(G_h))$. Assume that

$$\langle f_h, g_h \rangle_h = 0, \ \forall g_h(x) \in \mathbf{D}^2_{h,M}(W_0^{1,2}(G_h)),$$

in other words, we have

$$\langle f_h, g_h \rangle_h = \sum_{x \in \mathbb{R}^2_h} \overline{f_h(x)} g_h(x) h^2 = 0.$$

Then, for all $s_h(x) \in W_0^{1,2}(G_h)$ we get $g_h(x) = \mathbf{D}^2_{h,M} s_h(x)$ and

$$\langle f_h, g_h \rangle_h = \langle f_h, \mathbf{D}^2_{h,M} s_h \rangle_h = \sum_{x \in \mathbb{R}^2_h} \overline{f_h(x)} \mathbf{D}^2_{h,M} s_h(x) h^2 = 0$$

$$\Rightarrow - \sum_{x \in \mathbb{R}^2_h} \overline{\mathbf{D}^1_{h,M} f_h(x)} s_h(x) h^2 = -\langle \mathbf{D}^1_{h,M} f_h, s_h \rangle_h = 0.$$

Because the equality should be achieved for all $s_h \in W_0^{1,2}(G_h)$, it yields that $f_h(x) \in \ker \mathbf{D}^1_{h,M}(G_h)$. It has now been proved that

$$\ker \mathbf{D}^1_{h,M}(G_h) \subset \mathbf{D}^2_{h,M}(W_0^{1,2}(G_h))^\perp.$$

For $f_h(x) \in \ker \mathbf{D}^1_{h,M}(G_h)$, we get analogously

$$\langle f_h, g_h \rangle_h = \langle f_h, \mathbf{D}^2_{h,M} s_h \rangle_h = -\langle \mathbf{D}^1_{h,M} f_h, s_h \rangle_h = 0.$$

Finally we obtain that:

$$\mathbf{D}^2_{h,M}(W_0^{1,2}(G_h))^\perp = \ker \mathbf{D}^1_{h,M}(G_h),$$

now the proof is complete. □

Analogously to Theorem 16, we can prove orthogonal decomposition with respect to the discrete Cauchy-Riemann operator and its adjoint operator, respectively.

Theorem 17. *Let $\boldsymbol{D}^1_{h,M}$ be the discrete Cauchy-Riemann operator and $\boldsymbol{D}^2_{h,M}$ its adjoint* *operator. Then, we have the discrete orthogonal decompositions*

$$L_{2,h}(G_h) = \ker \boldsymbol{D}^2_{h,M}(G_h) \oplus_{\langle,\rangle_h} \boldsymbol{D}^1_{h,M}(W_0^{1,2}(G_h)),$$

with respect to the discrete inner product (4.1).

Proof. The theorem can be proven analogously. □

Lemma 6. *The orthoprojectors $\mathbf{P}^i_h$ and $\mathbf{Q}^i_h$ have the following properties*

$$\boldsymbol{D}^i_{h,M}(\mathbf{P}^i_h(f_h))(mh) = 0, \qquad \forall mh \in G_h,$$

and

$$\begin{aligned}\boldsymbol{D}^i_{h,M}(\mathbf{Q}^i_h(f_h))(mh) &= \boldsymbol{D}^i_{h,M}((I - \mathbf{P}^i_h)(f_h))(mh)\\ &= \boldsymbol{D}^i_{h,M} f_h(mh),\end{aligned}$$

where I is identity operator and $i = 1, 2$.

Now, we will generalize the construction of a right inverse operator and it will include the corner points Γ_{sz} to overcome the above mentioned restrictions for the values of the functions at these points in the summands of the structure of T_h-operator. The discrete complex T^1_h-operator is defined by

$$(T^1_h(f_h))(mh) = \left((T^1_{h1}(f_h))(mh), (T^1_{h2}(f_h))(mh)\right).$$

The coordinates of a operator T^1_h are represented by

$$(T^1_{hk}(f_h))(mh) = (T^{1,G}_{hk}(f_h))(mh) + (T^{1,\gamma^-_h}_{hk}(f_h))(mh)$$

with

$$(T^{1,G}_{hk}(f_h))(mh) = \sum_{lh\in G_h} h^2 \begin{pmatrix} e^1_{hk1}(mh-lh)\\ e^1_{hk2}(mh-lh)\end{pmatrix}^T \begin{pmatrix} f_0(lh)\\ f_1(lh)\end{pmatrix} \tag{4.10}$$

and

$$\begin{aligned}(T^{1,\gamma^-_h}_{hk}(f_h))(mh) = &\sum_{lh\in\gamma^-_{h1}\cup\gamma^-_{h4}\cup\Gamma_{14}\cup\Gamma^*_{14}\cup\Gamma^*_{34}\cup\Gamma^*_{12}} h^2 \begin{pmatrix} e^1_{hk1}(mh-lh)\\ e^1_{hk2}(mh-lh)\end{pmatrix}^T \begin{pmatrix} f_0(lh)\\ 0\end{pmatrix}\\ &+ \sum_{lh\in\gamma^-_{h2}\cup\gamma^-_{h3}\cup\Gamma_{23}\cup\Gamma^*_{23}\cup\Gamma^*_{34}\cup\Gamma^*_{12}} h^2 \begin{pmatrix} e^1_{hk1}(mh-lh)\\ e^1_{hk2}(mh-lh)\end{pmatrix}^T \begin{pmatrix} 0\\ f_1(lh)\end{pmatrix},\end{aligned} \tag{4.11}$$

where $k = 1, 2$ and the inner corners are counted if they exist. In some shapes some corners will be a union of other corners.
These definitions of discrete T_h^1 and T_h^2 operators are applicable to apply them to any complex valued-function which is defined on any shape, in some shapes some inner corners will be missing.

Theorem 18. *For an arbitrary function $f_h(mh) = (f_0(mh), f_1(mh))$ which is defined on G_h, it holds that*

$$\boldsymbol{D}^1_{h,M}(T_h^1(f_h))(mh) = f_h(mh) \qquad\qquad \forall\ mh \in G_h. \tag{4.12}$$

Proof. The idea is taken from [Gürlebeck and Hommel, 2003], let us start from the left hand side of (4.12)

$$\mathbf{D}^1_{h,M}(T_h^1(f_h))(mh) = \begin{pmatrix} D_h^{-2} & D_h^1 \\ -D_h^{-1} & D_h^2 \end{pmatrix} \begin{pmatrix} (T^1_{h1}(f_h))(mh) \\ (T^1_{h2}(f_h))(mh) \end{pmatrix}$$

$$= \begin{pmatrix} D_h^{-2}(T^1_{h1}(f_h))(mh) + D_h^1(T^1_{h2}(f_h))(mh) \\ -D_h^{-1}(T^1_{h1}(f_h))(mh) + D_h^2(T^1_{h2}(f_h))(mh) \end{pmatrix}$$

$$= \begin{pmatrix} D_h^{-2}(T^{1,G}_{h1}(f_h))(mh) + D_h^1(T^{1,G}_{h2}(f_h))(mh) \\ -D_h^{-1}(T^{1,G}_{h1}(f_h))(mh) + D_h^2(T^{1,G}_{h2}(f_h))(mh) \end{pmatrix}$$

$$+ \begin{pmatrix} D_h^{-2}(T^{1,\gamma_h^-}_{h1}(f_h))(mh) + D_h^1(T^{1,\gamma_h^-}_{h2}(f_h))(mh) \\ -D_h^{-1}(T^{1,\gamma_h^-}_{h1}(f_h))(mh) + D_h^2(T^{1,\gamma_h^-}_{h2}(f_h))(mh) \end{pmatrix}$$

First we simplify $M1$, where

$$M1 = D_h^{-2}(T^{1,G}_{h1}(f_h))(mh) + D_h^1(T^{1,G}_{h2}(f_h))(mh)$$

$$= D_h^{-2}\left\{\sum_{lh\in G_h} h^2 \begin{pmatrix} e^1_{h11}(mh-lh) \\ e^1_{h12}(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix}\right\} + D_h^1\left\{\sum_{lh\in G_h} h^2 \begin{pmatrix} e^1_{h21}(mh-lh) \\ e^1_{h22}(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix}\right\},$$

where D_h^{-2} is acting on the variable mh

$$M1 = \sum_{lh\in G_h} h^2 \begin{pmatrix} D_h^{-2} \\ D_h^1 \end{pmatrix}^T \left[\begin{pmatrix} e^1_{h11}(mh-lh) & e^1_{h12}(mh-lh) \\ e^1_{h21}(mh-lh) & e^1_{h22}(mh-lh) \end{pmatrix} \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix}\right].$$

Based on the property of the fundamental solution $\mathbf{e}^1_h(mh)$ which is given in System

(4.8), we get

$$M1 = \sum_{lh \in G_h} h^2 \left[\begin{pmatrix} D_h^{-2} \\ D_h^1 \end{pmatrix}^T \begin{pmatrix} e^1_{h11}(mh-lh) & e^1_{h12}(mh-lh) \\ e^1_{h21}(mh-lh) & e^1_{h22}(mh-lh) \end{pmatrix} \right] \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix}$$

$$= \sum_{lh \in G_h} h^2 \begin{pmatrix} \delta_h(mh-lh) \\ 0 \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix} = \begin{cases} f_0(mh) & \forall mh \in G_h \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta_h(mh-lh) = \begin{cases} h^{-2} & \text{for } mh = lh \\ 0 & \text{for } mh \neq lh. \end{cases}$

In a comparable way, one can get

$$M2 = D_h^{-2}(T_{h1}^{1,\gamma_h^-}(f_h))(mh) + D_h^1(T_{h2}^{1,\gamma_h^-}(f_h))(mh)$$

$$= \sum_{lh \in \gamma_{h1}^- \cup \gamma_{h4}^- \cup \Gamma_{14} \cup \Gamma_{14}^* \cup \Gamma_{34}^* \cup \Gamma_{12}^*} h^2 \begin{pmatrix} \delta_h(mh-lh) \\ 0 \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ 0 \end{pmatrix}$$

$$+ \sum_{lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23} \cup \Gamma_{23}^* \cup \Gamma_{34}^* \cup \Gamma_{12}^*} h^2 \begin{pmatrix} \delta_h(mh-lh) \\ 0 \end{pmatrix}^T \begin{pmatrix} 0 \\ f_1(lh) \end{pmatrix}$$

$$= \begin{cases} f_0(mh) & \forall mh \in \gamma_{h1}^- \cup \gamma_{h4}^- \cup \Gamma_{14} \cup \Gamma_{14}^* \cup \Gamma_{34}^* \cup \Gamma_{12}^* \\ 0 & \text{otherwise.} \end{cases}$$

Utilizing the same idea we get

$$M3 + M4 = (-D_h^{-1}(T_{h1}^{1,G}(f_h))(mh) + D_h^2(T_{h2}^{1,G}(f_h))(mh))$$

$$+(-D_h^{-1}(T_{h1}^{1,\gamma_h^-}(f_h))(mh) + D_h^2(T_{h2}^{1,\gamma_h^-}(f_h))(mh))$$

$$= \sum_{lh \in G_h} h^2 \begin{pmatrix} 0 \\ \delta_h(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix} + \sum_{lh \in \gamma_{h1}^- \cup \gamma_{h4}^- \cup \Gamma_{14} \cup \Gamma_{14}^* \cup \Gamma_{34}^* \cup \Gamma_{12}^*} h^2 \begin{pmatrix} 0 \\ \delta_h(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ 0 \end{pmatrix}$$

$$+ \sum_{lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23} \cup \Gamma_{23}^* \cup \Gamma_{34}^* \cup \Gamma_{12}^*} h^2 \begin{pmatrix} 0 \\ \delta_h(mh-lh) \end{pmatrix}^T \begin{pmatrix} 0 \\ f_1(lh) \end{pmatrix}$$

$$= \begin{cases} f_1(mh) & \forall mh \in G_h \cup \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23} \cup \Gamma_{23}^* \cup \Gamma_{34}^* \cup \Gamma_{12}^* \\ 0 & \text{otherwise.} \end{cases}$$

From $M1 + M2$ and $M3 + M4$ we get the right hand side of (4.12), which provides the result. $\square$

Remark 16. *In the proof of Theorem 18, we showed that* $\boldsymbol{D}^1_{h,M}(T^1_h(f_h))(mh) = f_h(mh)$ *is true for all the interior points and it is also true for other points belonging to some part of the boundary and some corner points. This is more than in the continuous case.*

Using a comparable structure for the adjoint discrete operator $\mathbf{D}^2_{h,M}$ such that

$$\mathbf{D}^2_{h,M}(T^2_h(f_h))(mh) = f_h(mh) \qquad \forall\ mh \in G_h, \tag{4.13}$$

where a right inverse operator T^2_h is defined by

$$(T^2_h(f_h))(mh) = \Big((T^2_{h1}(f_h))(mh), (T^2_{h2}(f_h))(mh)\Big),$$

with the coordinates

$$(T^2_{hk}(f_h))(mh) = (T^{2,G}_{hk}(f_h))(mh) + (T^{2,\gamma^-_h}_{hk}(f_h))(mh).$$

The above summands have the structure

$$(T^{2,G}_{hk}(f_h))(mh) = \sum_{lh \in G_h} h^2 \begin{pmatrix} e^2_{hk1}(mh-lh) \\ e^2_{hk2}(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix}$$

and

$$\begin{aligned}(T^{2,\gamma^-_h}_{hk}(f_h))(mh) = &\sum_{lh \in \gamma^-_{h1} \cup \gamma^-_{h2} \cup \Gamma_{12} \cup \Gamma^*_{12} \cup \Gamma^*_{14} \cup \Gamma^*_{23}} h^2 \begin{pmatrix} e^2_{hk1}(mh-lh) \\ e^2_{hk2}(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ 0 \end{pmatrix} \\ &+ \sum_{lh \in \gamma^-_{h3} \cup \gamma^-_{h4} \cup \Gamma_{34} \cup \Gamma^*_{34} \cup \Gamma^*_{14} \cup \Gamma^*_{23}} h^2 \begin{pmatrix} e^2_{hk1}(mh-lh) \\ e^2_{hk2}(mh-lh) \end{pmatrix}^T \begin{pmatrix} 0 \\ f_1(lh) \end{pmatrix}.\end{aligned}$$

We have been developing a program to calculate $(T^1_h(f_h))(mh)$ and $(T^2_h(f_h))(mh)$ for any arbitrary complex-valued function $f_h(mh) = (f_0(mh), f_1(mh))$ which is defined on the whole domain including all corners. The more important task is to check the accuracy and the application of this operator. Finally, we will verify that these discrete operators are a right inverse of Cauchy-Riemann operator and satisfy Equation (4.12) and (4.13).

In the last part of this chapter, we consider an example to construct discrete complex-valued functions from $\operatorname{Im}\mathbf{Q}_h^2$, which is necessary in the numerical study of the discrete p-Dirac problem. The discrete complex-valued functions from $\operatorname{Im}\mathbf{Q}_h^2$ will be the nonhomogeneous part of the p-Dirac problem. Now, we visualize the discretization and the different categories of lattice points in the rectangular domain. Ultimately, this will explain the points that belong to the boundary, interior points $mh \in G_h$ and outer corner points Γ_{sz} as it is shown in Figure 4.8.

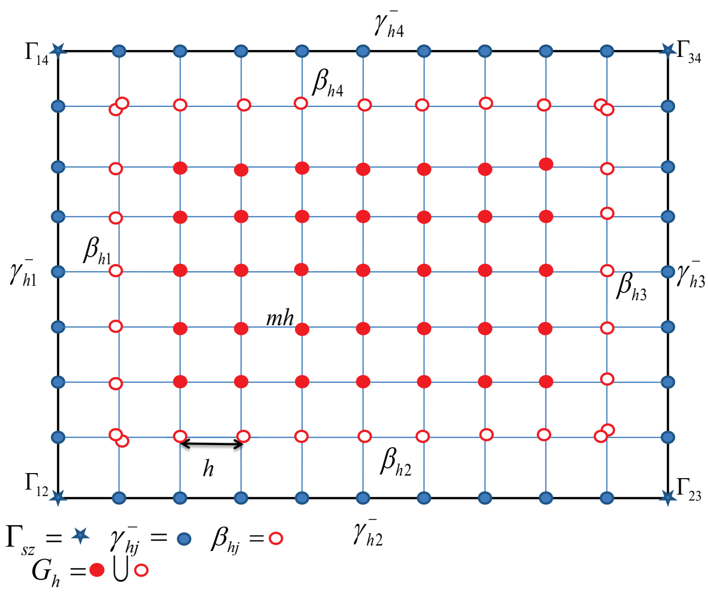


Figure 4.8: Rectangular lattice region.

In Figure 4.8, the ∘ are the points which belong to the set β_{hj} and the •⋃∘ are the interior points $mh \in G_h$. It can be seen that • are the boundary points which belong to γ_{hj}^-. Finally, the ⋆ are the points which are on the outer corner Γ_{sz} and the h is the step size. In Figure 4.9, the L-shaped lattice is visualized. The corner point Γ_{34} is the union of two corners ($\Gamma_{34} = \Gamma_{34}^1 \cup \Gamma_{34}^2$) and we can also see the inner corner Γ_{34}^*. By rotating the L-shaped three times around 90° clockwise, the union will result in a Cross shape +.

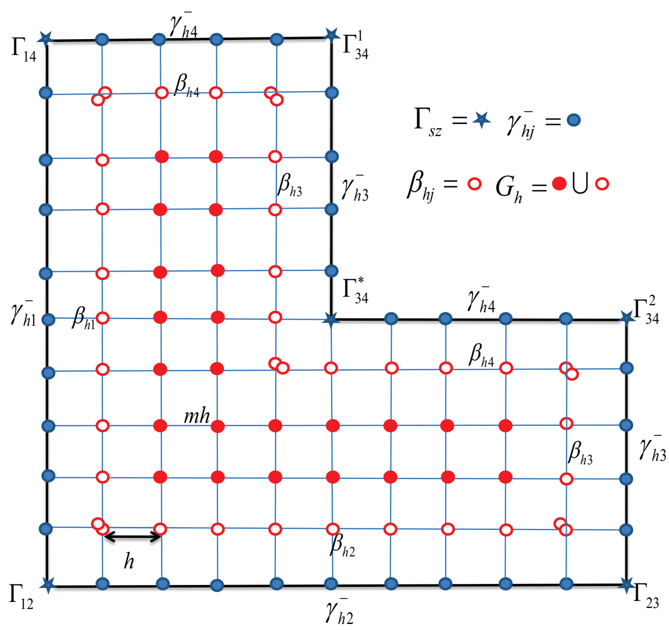


Figure 4.9: L-shaped lattice domain.

After describing a right inverse operator T_h^1 and T_h^2, it is possible to consider a discrete Borel-Pompeiu formula.

4.5 Discrete Borel-Pompeiu Formula and a Boundary Operator F_h

Before considering a discrete Borel-Pompeiu formula, we introduce a generalized boundary operator F_h^1 (Cauchy-Bitsadze operator), which is defined by:

$$(F_h^1(f_h))(mh) = \Big((F_{h1}^1(f_h))(mh), (F_{h2}^1(f_h))(mh)\Big),$$

with

$$(F_{hk}^1(f_h))(mh) = (F_{hk}^{1,\gamma_h^-}(f_h))(mh) + (F_{hk}^{1,\Gamma}(f_h))(mh),$$

where

$$
\begin{aligned}
(F_{hk}^{1,\gamma_h^-}(f_h))(mh) = &\sum_{lh\in\gamma_{h1}^-} h \begin{pmatrix} -e_{hk2}^1(mh-lh) \\ e_{hk1}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix} \\
&+ \sum_{lh\in\gamma_{h2}^-} h \begin{pmatrix} e_{hk1}^1(mh-lh) \\ e_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix} \\
&+ \sum_{lh\in\gamma_{h3}^-} h \begin{pmatrix} e_{hk2}^1(mh-lh) \\ -e_{hk1}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix} \\
&+ \sum_{lh\in\gamma_{h4}^-} h \begin{pmatrix} -e_{hk1}^1(mh-lh) \\ -e_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix},
\end{aligned}
\tag{4.14}
$$

and

$$
\begin{aligned}
&(F_{hk}^{1,\Gamma}(f_h))(mh) = \\
&h\Bigg\{\sum_{lh\in\Gamma_{12}\cup\Gamma_{23}^*} e_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in\Gamma_{14}\cup\Gamma_{14}^*\cup\Gamma_{34}^*} e_{hk1}^1(mh-lh)f_0(lh) \\
&+ \sum_{lh\in\Gamma_{14}\cup\Gamma_{14}^*\cup\Gamma_{12}^*} e_{hk1}^1(mh-lh)f_1(lh) - \sum_{lh\in\Gamma_{34}\cup\Gamma_{23}^*} e_{hk1}^1(mh-lh)f_1(lh) \\
&+ \sum_{lh\in\Gamma_{23}\cup\Gamma_{23}^*\cup\Gamma_{34}^*} e_{hk2}^1(mh-lh)f_0(lh) - \sum_{lh\in\Gamma_{12}\cup\Gamma_{14}^*} e_{hk2}^1(mh-lh)f_0(lh) \\
&+ \sum_{lh\in\Gamma_{23}\cup\Gamma_{23}^*\cup\Gamma_{12}^*} e_{hk2}^1(mh-lh)f_1(lh) - \sum_{lh\in\Gamma_{34}\cup\Gamma_{14}^*} e_{hk2}^1(mh-lh)f_1(lh)\Bigg\}
\end{aligned}
\tag{4.15}
$$

for $k = 1, 2$. Now we are able to consider the most important theorem of this chapter.

Theorem 19. *The discrete Borel-Pompeiu formula is given by*

$$(T_{hk}^1(\boldsymbol{D}_{h,M}^1(f_h)))(mh) + (F_{hk}^1(f_h))(mh) = f_{k-1}(mh)\chi_{k-1}, \quad k = 1, 2,$$

with the characteristic function

$$\chi_0 = \begin{cases} 1 & \forall mh \in G_h \cup \gamma_{h1}^- \cup \gamma_{h2}^- \cup \Gamma_{12} \cup \Gamma_{12}^* \\ 0 & else \end{cases}$$

and

$$\chi_1 = \begin{cases} 1 & \forall mh \in G_h \cup \gamma_{h3}^- \cup \gamma_{h4}^- \cup \Gamma_{34} \cup \Gamma_{34}^* \\ 0 & else. \end{cases}$$

Proof. At first we will calculate $(T_{hk}^1(\mathbf{D}_{h,M}^1(f_h)))(mh)$ where the coordinates of the operator T_{hk}^1 are represented by:

$$(T_{hk}^1(D_h^{-2}f_0 + D_h^1 f_1, -D_h^{-1}f_0 + D_h^2 f_1))(mh) =$$

$$T_{hk}^{1,G}(D_h^{-2}f_0+D_h^1 f_1, -D_h^{-1}f_0+D_h^2 f_1)(mh)+T_{hk}^{1,\gamma_h^-}(D_h^{-2}f_0+D_h^1 f_1, -D_h^{-1}f_0+D_h^2 f_1)(mh).$$

We start by applying $T_{hk}^{1,G}(D_h^{-2}f_0 + D_h^1 f_1, -D_h^{-1}f_0 + D_h^2 f_1)(mh)$, using Formula (4.10)

$$T_{hk}^{1,G}(D_h^{-2}f_0 + D_h^1 f_1, -D_h^{-1}f_0 + D_h^2 f_1)(mh)$$

$$= \sum_{lh\in G_h} h^2 \begin{pmatrix} e_{hk1}^1(mh-lh) \\ e_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} D_h^{-2}f_0(lh) + D_h^1 f_1(lh) \\ -D_h^{-1}f_0(lh) + D_h^2 f_1(lh) \end{pmatrix}.$$

We split the above sum into four parts

$$= M_1^{G_h} + M_2^{G_h} + M_3^{G_h} + M_4^{G_h},$$

with respect to the four difference quotients:

$$M_1^{G_h} = \sum_{lh\in G_h} h^2 e_{hk1}^1(mh-lh) D_h^{-2} f_0(lh), \qquad M_2^{G_h} = \sum_{lh\in G_h} h^2 e_{hk1}^1(mh-lh) D_h^1 f_1(lh),$$

$$M_3^{G_h} = -\sum_{lh\in G_h} h^2 e_{hk2}^1(mh-lh) D_h^{-1} f_0(lh) \quad \text{and} \quad M_4^{G_h} = \sum_{lh\in G_h} h^2 e_{hk2}^1(mh-lh) D_h^2 f_1(lh).$$

Next we apply $T_{hk}^{1,\gamma_h^-}(D_h^{-2}f_0+D_h^1 f_1, -D_h^{-1}f_0+D_h^2 f_1)(mh)$ by using Formula (4.11)

$$T_{hk}^{1,\gamma_h^-}(D_h^{-2}f_0 + D_h^1 f_1, -D_h^{-1}f_0 + D_h^2 f_1)(mh)$$

$$= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{14}^*\cup\Gamma_{34}^*\cup\Gamma_{12}^*} h^2 \begin{pmatrix} e_{hk1}^1(mh-lh) \\ e_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} D_h^{-2}f_0(lh) + D_h^1 f_1(lh) \\ 0 \end{pmatrix}$$

$$+\sum_{lh\in\gamma_{h2}^{-}\cup\gamma_{h3}^{-}\cup\Gamma_{23}\cup\Gamma_{23}^{*}\cup\Gamma_{34}^{*}\cup\Gamma_{12}^{*}} h^2\begin{pmatrix} e_{hk1}^1(mh-lh)\\ e_{hk2}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}0\\ -D_h^{-1}f_0(lh)+D_h^2 f_1(lh)\end{pmatrix}.$$

Once again, we split the above sum into four parts

$$=M_1^{\gamma_h^-}+M_2^{\gamma_h^-}+M_3^{\gamma_h^-}+M_4^{\gamma_h^-},$$

with respect to the four difference quotients:

$$M_1^{\gamma_h^-}=\sum_{lh\in\gamma_{h1}^{-}\cup\gamma_{h4}^{-}\cup\Gamma_{14}\cup\Gamma_{14}^{*}\cup\Gamma_{34}^{*}\cup\Gamma_{12}^{*}} h^2 e_{hk1}^1(mh-lh)D_h^{-2}f_0(lh),$$

$$M_2^{\gamma_h^-}=\sum_{lh\in\gamma_{h1}^{-}\cup\gamma_{h4}^{-}\cup\Gamma_{14}\cup\Gamma_{14}^{*}\cup\Gamma_{34}^{*}\cup\Gamma_{12}^{*}} h^2 e_{hk1}^1(mh-lh)D_h^{1}f_1(lh),$$

$$M_3^{\gamma_h^-}=-\sum_{lh\in\gamma_{h2}^{-}\cup\gamma_{h3}^{-}\cup\Gamma_{23}\cup\Gamma_{23}^{*}\cup\Gamma_{34}^{*}\cup\Gamma_{12}^{*}} h^2 e_{hk2}^1(mh-lh)D_h^{-1}f_0(lh),$$

$$M_4^{\gamma_h^-}=\sum_{lh\in\gamma_{h2}^{-}\cup\gamma_{h3}^{-}\cup\Gamma_{23}\cup\Gamma_{23}^{*}\cup\Gamma_{34}^{*}\cup\Gamma_{12}^{*}} h^2 e_{hk2}^1(mh-lh)D_h^{2}f_1(lh).$$

By rewriting each of these eight parts: first, when the complex-valued function $f_h(mh)$ is defined on the uniform rectangular lattice domain; second when it is defined on the L-shaped lattice domain, in conjunction with 3-times 90° clockwise rotations. Finally, when the complex-valued function $f_h(mh)$ is defined on the Cross shape domain and on the irregular domain. We will prove the Borel-Pompeiu formula in each domain.

Using the uniform rectangular lattice domain

Part 1: Technically the first summand will result in

$$
\begin{aligned}
& M_1^{G_h} \\
& = \sum_{lh \in G_h} h^2 e_{hk1}^1(mh - lh) D_h^{-2} f_0(lh) \\
& = \sum_{lh \in G_h} h^2 e_{hk1}^1(mh - lh) \left\{ h^{-1}(f_0(lh) - f_0(lh - hb_2)) \right\} \\
& = \sum_{lh \in G_h} h e_{hk1}^1(mh - lh) f_0(lh) - \sum_{lh \in G_h} h e_{hk1}^1(mh - lh) f_0(lh - hb_2) \\
& = \sum_{lh \in G_h} h e_{hk1}^1(mh - lh) f_0(lh) - \sum_{lh \in G_h - hb_2} h e_{hk1}^1(mh - (lh + hb_2)) f_0(lh) \\
& = \sum_{lh \in G_h} h e_{hk1}^1(mh - lh) f_0(lh) - \sum_{lh \in (G_h / \beta_{h4}) \cup \gamma_{h2}^-} h e_{hk1}^1((mh - lh) - hb_2) f_0(lh)
\end{aligned}
$$

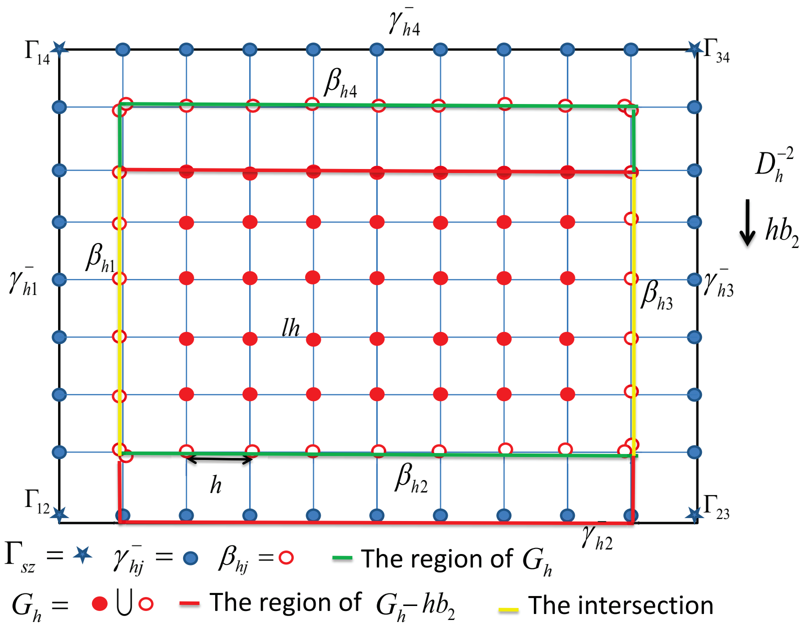


Figure 4.10: The graph of the points on the G_h which are moved down one step by using shift $-hb_2$, $(b_2 = (0, 1))$.

Looking at the graph of $lh \in G_h - hb_2$ in Figure 4.10, we see that it equals $lh \in (G_h \setminus \beta_{h4}) \cup \gamma_{h2}^-$.

For the summand $M_1^{\gamma_h^-}$ the index of the sum will be $\gamma_{h1}^- \cup \gamma_{h4}^- \cup \Gamma_{14}$ and the other parts are $\emptyset$ in the rectangular domain, $(\Gamma_{14}^* = \Gamma_{34}^* = \Gamma_{12}^* = \emptyset$, there are no inner corners in the rectangular domain).

$$
\begin{aligned}
M_1^{\gamma_h^-}
&= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{14}^*\cup\Gamma_{34}^*\cup\Gamma_{12}^*} h^2 e_{hk1}^1(mh-lh)D_h^{-2}f_0(lh)\\
&= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}} h^2 e_{hk1}^1(mh-lh)\left\{h^{-1}(f_0(lh)-f_0(lh-hb_2))\right\}\\
&= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_0(lh-hb_2)\\
&= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}-hb_2} he_{hk1}^1(mh-(lh+hb_2))f_0(lh)\\
&= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h1}^-\cup\beta_{h4}\cup\Gamma_{12}} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)
\end{aligned}
$$

In Figure 4.11, $\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}-hb_2$ is equal to $\gamma_{h1}^-\cup\beta_{h4}\cup\Gamma_{12}$.

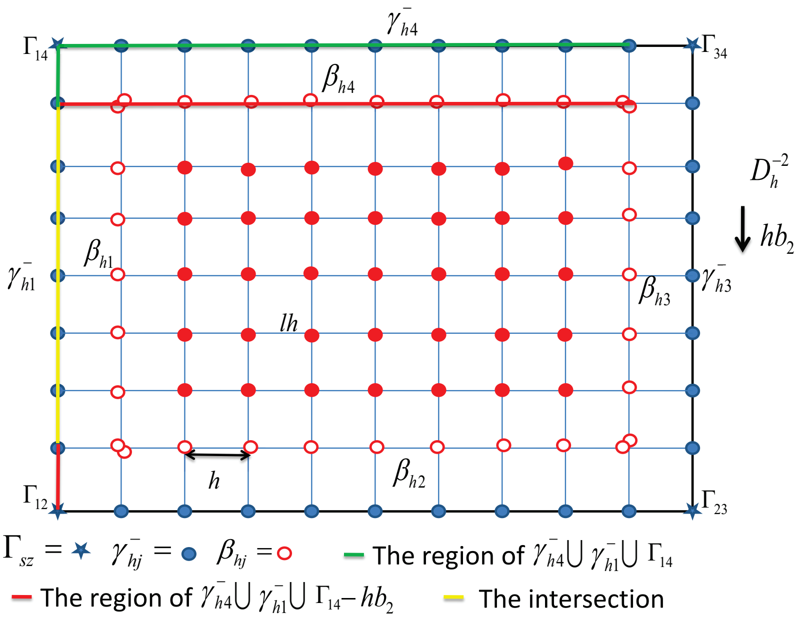


Figure 4.11: The graph of the points on the $\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}$ which are moved down one step by using shift $-hb_2$.

In the next step we will add to $M_1^{G_h}$ the expression $M_1^{\gamma_h^-}$

$$
M_1^{G_h}+M_1^{\gamma_h^-} = \sum_{lh\in G_h} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in(G_h\backslash\beta_{h4})\cup\gamma_{h2}^-} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)
$$

$$+\sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h1}^-\cup\beta_{h4}\cup\Gamma_{12}} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)$$

$$=\sum_{lh\in G_h\cup\gamma_{h1}^-} he_{hk1}^1(mh-lh)f_0(lh)-\sum_{lh\in(G_h\backslash\beta_{h4})\cup\gamma_{h2}^-\cup\gamma_{h1}^-\cup\beta_{h4}\cup\Gamma_{12}} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h2}^-} he_{hk1}^1(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h2}^-} he_{hk1}^1(mh-lh)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h4}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_0(lh)$$

$$+\sum_{lh\in\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh)-\sum_{lh\in\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh)$$

$$=\sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh)-\sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)$$

$$-\sum_{lh\in\gamma_{h2}^-} he_{hk1}^1(mh-lh)f_0(lh)+\sum_{lh\in\gamma_{h4}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_0(lh)$$

$$-\sum_{lh\in\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh).$$

Finally we get:

$$M_1^{G_h}+M_1^{\gamma_h^-}=\sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} h^2D_h^{-2}e_{hk1}^1(mh-lh)_hf_0(lh)$$

$$-\sum_{lh\in\gamma_{h2}^-\cup\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh)+\sum_{lh\in\gamma_{h4}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_0(lh).$$

Part 2: Technically, we simplify $M_3^{G_h}$ and $M_3^{\gamma_h^-}$:

$$M_3^{G_h}$$

$$=-\sum_{lh\in G_h} h^2e_{hk2}^1(mh-lh)D_h^{-1}f_0(lh)$$

$$=-\sum_{lh\in G_h} h^2e_{hk2}^1(mh-lh)\left\{h^{-1}(f_0(lh)-f_0(lh-hb_1))\right\}$$

$$= -\left\{ \sum_{lh \in G_h} h e^1_{hk2}(mh - lh) f_0(lh) - \sum_{lh \in G_h - hb_1} h e^1_{hk2}(mh - (lh + hb_1)) f_0(lh) \right\}$$

$$= -\left\{ \sum_{lh \in G_h} h e^1_{hk2}(mh - lh) f_0(lh) - \sum_{lh \in (G_h \setminus \beta_{h3}) \cup \gamma^-_{h1}} h e^1_{hk2}((mh - lh) - hb_1) f_0(lh) \right\},$$

similarly we set $\Gamma^*_{23} = \Gamma^*_{34} = \Gamma^*_{12} = \emptyset$ on the index of the sum in the $M_3^{\gamma^-_h}$:
$M_3^{\gamma^-_h}$

$$= - \sum_{lh \in \gamma^-_{h2} \cup \gamma^-_{h3} \cup \Gamma_{23}} h^2 e^1_{hk2}(mh - lh) D_h^{-1} f_0(lh)$$

$$= - \sum_{lh \in \gamma^-_{h2} \cup \gamma^-_{h3} \cup \Gamma_{23}} h^2 e^1_{hk2}(mh - lh) \{h^{-1}(f_0(lh) - f_0(lh - hb_1))\}$$

$$= -\left\{ \sum_{lh \in \gamma^-_{h2} \cup \gamma^-_{h3} \cup \Gamma_{23}} h e^1_{hk2}(mh - lh) f_0(lh) - \sum_{lh \in \gamma^-_{h2} \cup \gamma^-_{h3} \cup \Gamma_{23} - hb_1} h e^1_{hk2}(mh - (lh + hb_1)) f_0(lh) \right\}$$

$$= -\left\{ \sum_{lh \in \gamma^-_{h2} \cup \gamma^-_{h3} \cup \Gamma_{23}} h e^1_{hk2}(mh - lh) f_0(lh) - \sum_{lh \in \gamma^-_{h2} \cup \beta_{h3} \cup \Gamma_{12}} h e^1_{hk2}((mh - lh) - hb_1) f_0(lh) \right\}.$$

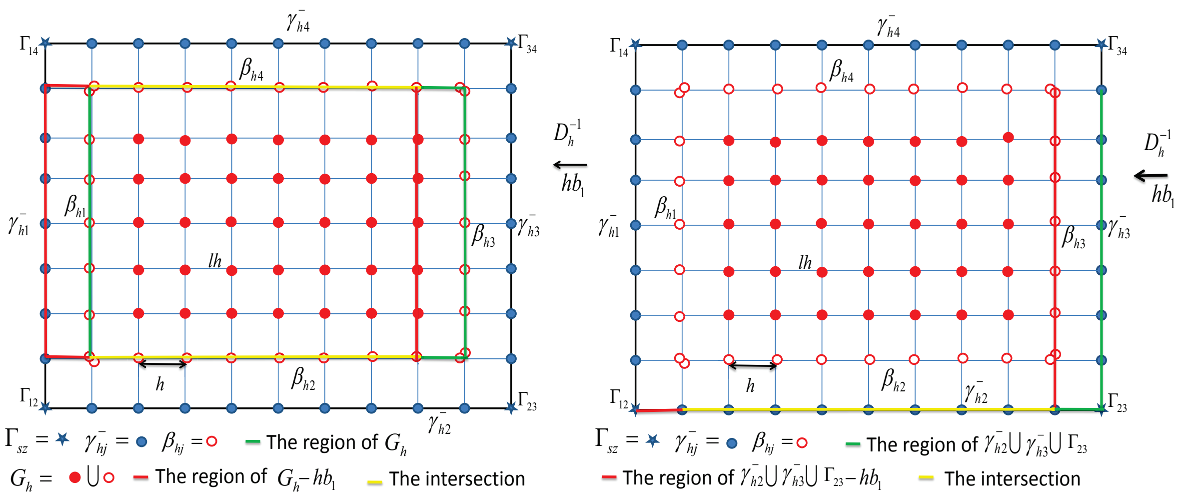


Figure 4.12: The graph of the points which are moved to the left one step by using shift $-hb_1$.

Looking at the left part of Figure 4.12, one can see the graph of the $lh \in G_h - hb_1$. On the right side of this figure we see that $lh \in \gamma^-_{h2} \cup \gamma^-_{h3} \cup \Gamma_{23} - hb_1 = \gamma^-_{h2} \cup \beta_{h3} \cup \Gamma_{12}$.

Now we will add $M_3^{G_h}$ to $M_3^{\gamma_h^-}$

$$M_3^{G_h}+M_3^{\gamma_h^-}=-\sum_{lh\in G_h} he^1_{hk2}(mh-lh)f_0(lh)+\sum_{lh\in(G_h\setminus\beta_{h3})\cup\gamma_{h1}^-} he^1_{hk2}((mh-lh)-hb_1)f_0(lh)$$

$$-\sum_{lh\in\gamma_{h2}^-\cup\gamma_{h3}^-\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_0(lh)+\sum_{lh\in\gamma_{h2}^-\cup\beta_{h3}\cup\Gamma_{12}} he^1_{hk2}((mh-lh)-hb_1)f_0(lh)$$

$$=-\sum_{lh\in G_h\cup\gamma_{h2}^-} he^1_{hk2}(mh-lh)f_0(lh)+\sum_{lh\in(G_h\setminus\beta_{h3})\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\beta_{h3}\cup\Gamma_{12}} he^1_{hk2}((mh-lh)-hb_1)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h1}^-} he^1_{hk2}(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h1}^-} he^1_{hk2}(mh-lh)f_0(lh)$$

$$-\sum_{lh\in\gamma_{h3}^-\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_0(lh)$$

$$+\sum_{lh\in\Gamma_{12}} he^1_{hk2}(mh-lh)f_0(lh)-\sum_{lh\in\Gamma_{12}} he^1_{hk2}(mh-lh)f_0(lh)$$

$$=-\sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} he^1_{hk2}(mh-lh)f_0(lh)+\sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} he^1_{hk2}((mh-lh)-hb_1)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h1}^-\cup\Gamma_{12}} he^1_{hk2}(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h3}^-\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_0(lh).$$

Finally we have:

$$M_3^{G_h}+M_3^{\gamma_h^-}=-\sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} h^2D_h^{-1}e^1_{hk2}(mh-lh)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h1}^-\cup\Gamma_{12}} he^1_{hk2}(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h3}^-\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_0(lh).$$

Part 3: We will use a similar method with $M_2^{G_h}$ and $M_2^{\gamma_h^-}$

$M_2^{G_h}$

$$= \sum_{lh\in G_h} h^2 e^1_{hk1}(mh-lh)D^1_h f_1(lh)$$

$$= \sum_{lh\in G_h} h^2 e^1_{hk1}(mh-lh)\left\{h^{-1}(f_1(lh+hb_1)-f_1(lh))\right\}$$

$$= \sum_{lh\in G_h} he^1_{hk1}(mh-lh)f_1(lh+hb_1) - \sum_{lh\in G_h} he^1_{hk1}(mh-lh)f_1(lh)$$

$$= \sum_{lh\in G_h+hb_1} he^1_{hk1}(mh-(lh-hb_1))f_1(lh) - \sum_{lh\in G_h} he^1_{hk1}(mh-lh)f_1(lh)$$

$$= \sum_{lh\in(G_h\setminus\beta_{h1})\cup\gamma^-_{h3}} he^1_{hk1}((mh-lh)+hb_1)f_1(lh) - \sum_{lh\in G_h} he^1_{hk1}(mh-lh)f_1(lh),$$

here also the inner corners will be an empty set and the summand $M_2^{\gamma_h^-}$ will result:
$M_2^{\gamma_h^-}$

$$= \sum_{lh\in\gamma^-_{h1}\cup\gamma^-_{h4}\cup\Gamma_{14}} h^2 e^1_{hk1}(mh-lh)D^1_h f_1(lh)$$

$$= \sum_{lh\in\gamma^-_{h1}\cup\gamma^-_{h4}\cup\Gamma_{14}} he^1_{hk1}(mh-lh)f_1(lh+hb_1) - \sum_{lh\in\gamma^-_{h1}\cup\gamma^-_{h4}\cup\Gamma_{14}} he^1_{hk1}(mh-lh)f_1(lh)$$

$$= \sum_{lh\in\gamma^-_{h1}\cup\gamma^-_{h4}\cup\Gamma_{14}+hb_1} he^1_{hk1}(mh-(lh-hb_1))f_1(lh) - \sum_{lh\in\gamma^-_{h1}\cup\gamma^-_{h4}\cup\Gamma_{14}} he^1_{hk1}(mh-lh)f_1(lh)$$

$$= \sum_{lh\in\gamma^-_{h4}\cup\beta_{h1}\cup\Gamma_{34}} he^1_{hk1}((mh-lh)+hb_1)f_1(lh) - \sum_{lh\in\gamma^-_{h1}\cup\gamma^-_{h4}\cup\Gamma_{14}} he^1_{hk1}(mh-lh)f_1(lh).$$

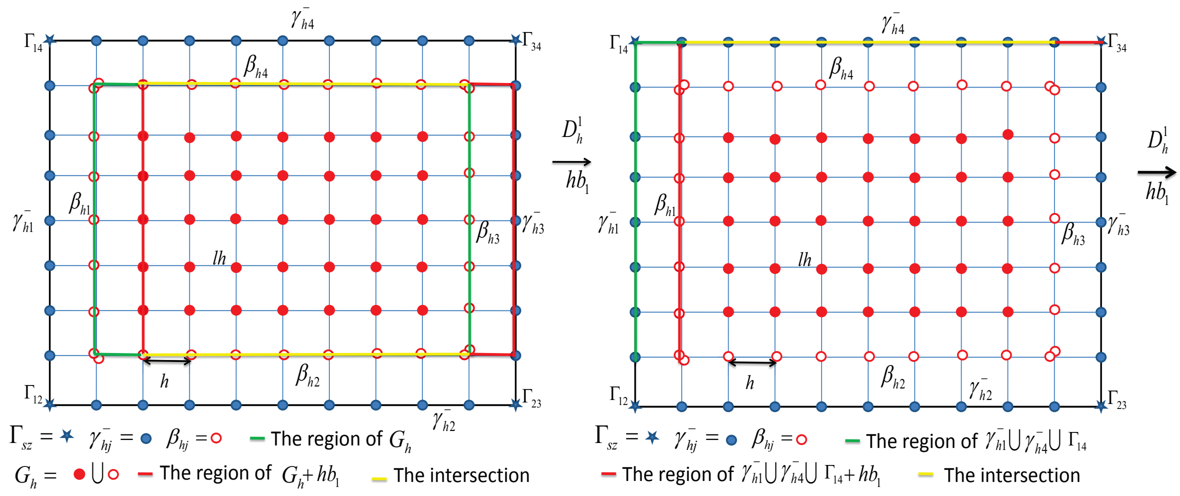


Figure 4.13: The graph of the points which are moved to the right one step by using shift hb_1.

Looking at the left part of Figure 4.13, one can see that the graph of the $lh \in G_h + hb_1$ is equal to $(G_h \setminus \beta_{h1}) \cup \gamma_{h3}^-$. On the right side of this figure it is seen that $lh \in \gamma_{h1}^- \cup \gamma_{h4}^- \cup \Gamma_{14} + hb_1 = \gamma_{h4}^- \cup \beta_{h1} \cup \Gamma_{34}$.
Adding $M_2^{G_h}$ to $M_2^{\gamma_h^-}$ we get

$$
\begin{aligned}
M_2^{G_h} + M_2^{\gamma_h^-} = & \sum_{lh \in (G_h \setminus \beta_{h1}) \cup \gamma_{h3}^- \cup \gamma_{h4}^- \cup \beta_{h1} \cup \Gamma_{34}} h e_{hk1}^1((mh - lh) + hb_1) f_1(lh) \\
& - \sum_{lh \in G_h \cup \gamma_{h4}^-} h e_{hk1}^1(mh - lh) f_1(lh) - \sum_{lh \in \gamma_{h1}^- \cup \Gamma_{14}} h e_{hk1}^1(mh - lh) f_1(lh) \\
& + \sum_{lh \in \gamma_{h3}^- \cup \Gamma_{34}} h e_{hk1}^1(mh - lh) f_1(lh) - \sum_{lh \in \gamma_{h3}^- \cup \Gamma_{34}} h e_{hk1}^1(mh - lh) f_1(lh)
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{lh \in G_h \cup \gamma_{h3}^- \cup \gamma_{h4}^- \cup \Gamma_{34}} h e_{hk1}^1((mh - lh) + hb_1) f_1(lh) - \sum_{lh \in G_h \cup \gamma_{h3}^- \cup \gamma_{h4}^- \cup \Gamma_{34}} h e_{hk1}^1(mh - lh) f_1(lh) \\
& - \sum_{lh \in \gamma_{h1}^- \cup \Gamma_{14}} h e_{hk1}^1(mh - lh) f_1(lh) + \sum_{lh \in \gamma_{h3}^- \cup \Gamma_{34}} h e_{hk1}^1(mh - lh) f_1(lh).
\end{aligned}
$$

Finally

$$
\begin{aligned}
M_2^{G_h} + M_2^{\gamma_h^-} = & \sum_{lh \in G_h \cup \gamma_{h3}^- \cup \gamma_{h4}^- \cup \Gamma_{34}} h^2 D_h^1 e_{hk1}^1(mh - lh) f_1(lh) \\
& - \sum_{lh \in \gamma_{h1}^- \cup \Gamma_{14}} h e_{hk1}^1(mh - lh) f_1(lh) + \sum_{lh \in \gamma_{h3}^- \cup \Gamma_{34}} h e_{hk1}^1(mh - lh) f_1(lh).
\end{aligned}
$$

Part 4: The technical details displayed in the last two summands $M_4^{G_h}$ and $M_4^{\gamma_h^-}$:
$M_4^{G_h}$

$$= \sum_{lh \in G_h} h^2 e^1_{hk2}(mh - lh) D_h^2 f_1(lh)$$

$$= \sum_{lh \in G_h} h^2 e^1_{hk2}(mh - lh) \{h^{-1}(f_1(lh + hb_2) - f_1(lh))\}$$

$$= \sum_{lh \in G_h} he^1_{hk2}(mh - lh) f_1(lh + hb_2) - \sum_{lh \in G_h} he^1_{hk2}(mh - lh) f_1(lh)$$

$$= \sum_{lh \in G_h + hb_2} he^1_{hk2}(mh - (lh - hb_2)) f_1(lh) - \sum_{lh \in G_h} he^1_{hk2}(mh - lh) f_1(lh)$$

$$= \sum_{lh \in (G_h \setminus \beta_{h2}) \cup \gamma_{h4}^-} he^1_{hk2}((mh - lh) + hb_2) f_1(lh) - \sum_{lh \in G_h} he^1_{hk2}(mh - lh) f_1(lh),$$

and
$M_4^{\gamma_h^-}$

$$= \sum_{lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23}} h^2 e^1_{hk2}(mh - lh) D_h^2 f_1(lh)$$

$$= \sum_{lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23}} he^1_{hk2}(mh - lh) f_1(lh + hb_2) - \sum_{lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23}} he^1_{hk2}(mh - lh) f_1(lh)$$

$$= \sum_{lh \in lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23} + hb_2} he^1_{hk2}(mh - (lh - hb_2)) f_1(lh) - \sum_{lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23}} he^1_{hk2}(mh - lh) f_1(lh)$$

$$= \sum_{lh \in \gamma_{h3}^- \cup \beta_{h2} \cup \Gamma_{34}} he^1_{hk2}((mh - lh) + hb_2) f_1(lh) - \sum_{lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23}} he^1_{hk2}(mh - lh) f_1(lh).$$

Looking at the left part of Figure 4.14, one can see that the graph of the $lh \in G_h + hb_2$ is equal to $(G_h \setminus \beta_{h2}) \cup \gamma_{h4}^-$. On the right side of this figure it is seen that $lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23} + hb_2 = \gamma_{h3}^- \cup \beta_{h2} \cup \Gamma_{34}$.

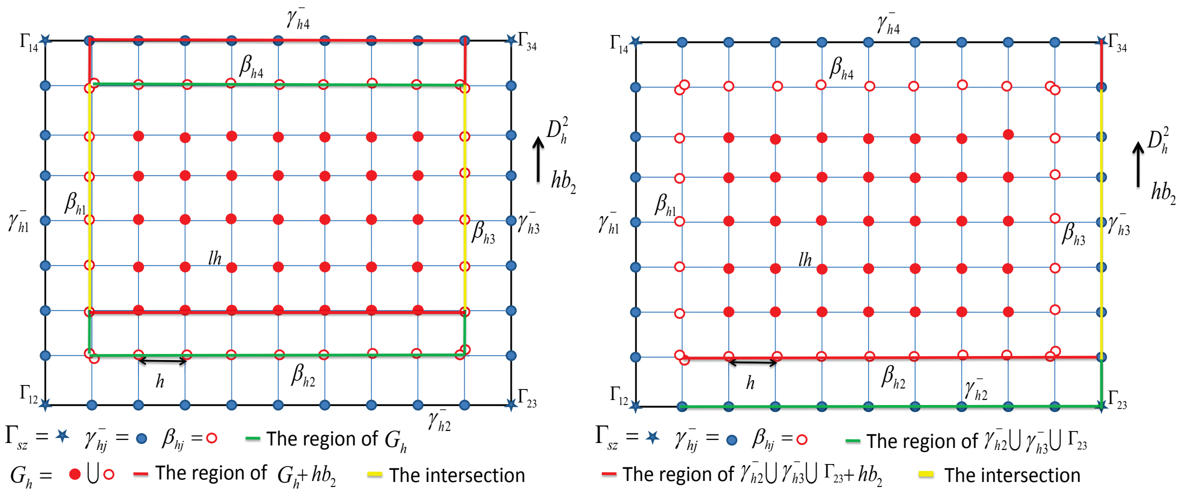


Figure 4.14: The graph of the points which are moved up one step by using shift hb_2.

Adding $M_4^{G_h}$ to $M_4^{\gamma_h^-}$ we get

$$
\begin{aligned}
M_4^{G_h} + M_4^{\gamma_h^-} = & \sum_{lh \in (G_h \setminus \beta_{h2}) \cup \gamma_{h4}^- \cup \gamma_{h3}^- \cup \beta_{h2} \cup \Gamma_{34}} h e^1_{hk2}((mh-lh)+hb_2) f_1(lh) \\
& - \sum_{lh \in G_h \cup \gamma_{h3}^-} h e^1_{hk2}(mh-lh) f_1(lh) - \sum_{lh \in \gamma_{h2}^- \cup \Gamma_{23}} h e^1_{hk2}(mh-lh) f_1(lh) \\
& + \sum_{lh \in \gamma_{h4}^- \cup \Gamma_{34}} h e^1_{hk2}(mh-lh) f_1(lh) - \sum_{lh \in \gamma_{h4}^- \cup \Gamma_{34}} h e^1_{hk2}(mh-lh) f_1(lh).
\end{aligned}
$$

$$
\begin{aligned}
M_4^{G_h} + M_4^{\gamma_h^-} = & \sum_{lh \in G_h \cup \gamma_{h3}^- \cup \gamma_{h4}^- \cup \Gamma_{34}} h^2 D_h^2 e^1_{hk2}(mh-lh) f_1(lh) \\
& - \sum_{lh \in \gamma_{h2}^- \cup \Gamma_{23}} h e^1_{hk2}(mh-lh) f_1(lh) + \sum_{lh \in \gamma_{h4}^- \cup \Gamma_{34}} h e^1_{hk2}(mh-lh) f_1(lh).
\end{aligned}
$$

Part 5: Finally by adding the eight summands we obtain

$$
\begin{aligned}
(T^1_{hk}(\mathbf{D}^1_{h,M}(f_h)))(mh) = & \sum_{lh \in G_h \cup \gamma_{h1}^- \cup \gamma_{h2}^- \cup \Gamma_{12}} h^2 D_h^{-2} e^1_{hk1}(mh-lh) f_0(lh) \\
& - \sum_{lh \in \gamma_{h2}^- \cup \Gamma_{12}} h e^1_{hk1}(mh-lh) f_0(lh) + \sum_{lh \in \gamma_{h4}^- \cup \Gamma_{14}} h e^1_{hk1}(mh-lh) f_0(lh) \\
& - \sum_{lh \in G_h \cup \gamma_{h1}^- \cup \gamma_{h2}^- \cup \Gamma_{12}} h^2 D_h^{-1} e^1_{hk2}(mh-lh) f_0(lh)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{12}} he^1_{hk2}(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_0(lh)\\
&\qquad + \sum_{lh\in G_h\cup\gamma_{h3}^{-}\cup\gamma_{h4}^{-}\cup\Gamma_{34}} h^2 D_h^1 e^1_{hk1}(mh-lh)f_1(lh)\\
&- \sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{14}} he^1_{hk1}(mh-lh)f_1(lh) + \sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{34}} he^1_{hk1}(mh-lh)f_1(lh)\\
&\qquad + \sum_{lh\in G_h\cup\gamma_{h3}^{-}\cup\gamma_{h4}^{-}\cup\Gamma_{34}} h^2 D_h^2 e^1_{hk2}(mh-lh)f_1(lh)\\
&- \sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_1(lh) + \sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{34}} he^1_{hk2}(mh-lh)f_1(lh)
\end{aligned}
$$

$$
\begin{aligned}
= &\sum_{lh\in G_h\cup\gamma_{h1}^{-}\cup\gamma_{h2}^{-}\cup\Gamma_{12}} h^2 \left\{D_h^{-2} e^1_{hk1}(mh-lh) - D_h^{-1} e^1_{hk2}(mh-lh)\right\} f_0(lh)\\
+ &\sum_{lh\in G_h\cup\gamma_{h3}^{-}\cup\gamma_{h4}^{-}\cup\Gamma_{34}} h^2 \left\{D_h^{1} e^1_{hk1}(mh-lh) + D_h^{2} e^1_{hk2}(mh-lh)\right\} f_1(lh)\\
&- \sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{12}} he^1_{hk1}(mh-lh)f_0(lh) + \sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{14}} he^1_{hk1}(mh-lh)f_0(lh)\\
&+ \sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{12}} he^1_{hk2}(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_0(lh)\\
&- \sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{14}} he^1_{hk1}(mh-lh)f_1(lh) + \sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{34}} he^1_{hk1}(mh-lh)f_1(lh)\\
&- \sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_1(lh) + \sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{34}} he^1_{hk2}(mh-lh)f_1(lh).
\end{aligned}
$$

Using the properties of the discrete Fourier transform which is given in (4.9) we can show that:

$$F_h D_h^{-1} e^1_{h12}(mh) = \xi_1^h F_h e^1_{h12}(mh) = \frac{-\xi_1^h \xi_{-1}^h}{2\pi\, d^2} = \xi_{-1}^h F_h e^1_{h21}(mh) = -F_h D_h^1 e^1_{h21}(mh),$$

also we can get using the same properties

$$F_h D_h^{1} e^1_{h11}(mh) = -\xi_{-1}^h F_h e^1_{h11}(mh) = \frac{-\xi_{-1}^h \xi_{-2}^h}{2\pi\, d^2} = \xi_{-2}^h F_h e^1_{h12}(mh) = -F_h D_h^2 e^1_{h12}(mh),$$

with $\xi_j^h = h^{-1}(1-e^{ih\xi_j})$ and $\xi_{-j}^h = h^{-1}(1-e^{-ih\xi_j})$ for $j=1,2$.

By using the inverse Fourier transform we get:

$$-D_h^{-1}e^1_{h12}(mh) = D_h^1 e^1_{h21}(mh) \quad \text{and} \quad D_h^1 e^1_{h11}(mh) = -D_h^2 e^1_{h12}(mh). \tag{4.16}$$

For $k=1$ we obtain that $(T^1_{h1}(\mathbf{D}^1_{h,M}(f_h)))(mh)$ is

$$\begin{aligned}
= & \sum_{lh\in G_h\cup\gamma^-_{h1}\cup\gamma^-_{h2}\cup\Gamma_{12}} h^2\left\{D_h^{-2}e^1_{h11}(mh-lh) - D_h^{-1}e^1_{h12}(mh-lh)\right\} f_0(lh)\\
+ & \sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}} h^2\left\{D_h^{1}e^1_{h11}(mh-lh) + D_h^{2}e^1_{h12}(mh-lh)\right\} f_1(lh)\\
& - \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{12}} he^1_{h11}(mh-lh)f_0(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{14}} he^1_{h11}(mh-lh)f_0(lh)\\
& + \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{12}} he^1_{h12}(mh-lh)f_0(lh) - \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{23}} he^1_{h12}(mh-lh)f_0(lh)\\
& - \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{14}} he^1_{h11}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{34}} he^1_{h11}(mh-lh)f_1(lh)\\
& - \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{23}} he^1_{h12}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{34}} he^1_{h12}(mh-lh)f_1(lh),
\end{aligned}$$

using the properties which are given in (4.16) we get that $(T^1_{h1}(\mathbf{D}^1_{h,M}(f_h)))(mh)$ is

$$\begin{aligned}
= & \sum_{lh\in G_h\cup\gamma^-_{h1}\cup\gamma^-_{h2}\cup\Gamma_{12}} h^2\left\{D_h^{-2}e^1_{h11}(mh-lh) + D_h^{1}e^1_{h21}(mh-lh)\right\} f_0(lh)\\
+ & \sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}} h^2\left\{-D_h^{2}e^1_{h12}(mh-lh) + D_h^{2}e^1_{h12}(mh-lh)\right\} f_1(lh)\\
& - \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{12}} he^1_{h11}(mh-lh)f_0(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{14}} he^1_{h11}(mh-lh)f_0(lh)\\
& + \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{12}} he^1_{h12}(mh-lh)f_0(lh) - \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{23}} he^1_{h12}(mh-lh)f_0(lh)\\
& - \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{14}} he^1_{h11}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{34}} he^1_{h11}(mh-lh)f_1(lh)\\
& - \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{23}} he^1_{h12}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{34}} he^1_{h12}(mh-lh)f_1(lh).
\end{aligned}$$

Based on the properties of the discrete fundamental solution we get

$$\sum_{lh\in G_h\cup\gamma_{h1}^{-}\cup\gamma_{h2}^{-}\cup\Gamma_{12}} h^2\left\{D_h^{-2}e_{h11}^1(mh-lh)+D_h^1e_{h21}^1(mh-lh)\right\}f_0(lh)$$

$$=\sum_{lh\in G_h\cup\gamma_{h1}^{-}\cup\gamma_{h2}^{-}\cup\Gamma_{12}} h^2\,\delta_h(mh-lh)f_0(lh)$$

$$=\begin{cases} f_0(mh) & \forall mh\in G_h\cup\gamma_{h1}^{-}\cup\gamma_{h2}^{-}\cup\Gamma_{12}\\ 0 & \text{otherwise,}\end{cases}$$

$$=f_0(mh)\chi_0.$$

Finally, for $k=1$ we obtain:

$$\begin{aligned}
(T_{h1}^1(\mathbf{D}_{h,M}^1(f_h)))(mh) &= f_0(mh)\chi_0\\
&-\sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{12}} he_{h11}^1(mh-lh)f_0(lh)+\sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{14}} he_{h11}^1(mh-lh)f_0(lh)\\
&+\sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{12}} he_{h12}^1(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{23}} he_{h12}^1(mh-lh)f_0(lh)\\
&-\sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{14}} he_{h11}^1(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{34}} he_{h11}^1(mh-lh)f_1(lh)\\
&-\sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{23}} he_{h12}^1(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{34}} he_{h12}^1(mh-lh)f_1(lh).
\end{aligned}\tag{4.17}$$

Using Equation (4.14) and (4.15), the boundary operator $(F_{hk}^1(f_h))(mh)$ for $k=1$ on the uniform rectangular lattice after setting all the inner corners as an empty set, will result:

$$(F_{h1}^1(f_h))(mh)=(F_{h1}^{1,\gamma_h^-}(f_h))(mh)+(F_{h1}^{1,\Gamma}(f_h))(mh)$$

$$=\sum_{lh\in\gamma_{h1}^{-}} h\begin{pmatrix}-e_{h12}^1(mh-lh)\\ e_{h11}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}f_0(lh)\\ f_1(lh)\end{pmatrix}+\sum_{lh\in\gamma_{h2}^{-}} h\begin{pmatrix}e_{h11}^1(mh-lh)\\ e_{h12}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}f_0(lh)\\ f_1(lh)\end{pmatrix}$$

$$+\sum_{lh\in\gamma_{h3}^{-}} h\begin{pmatrix}e_{h12}^1(mh-lh)\\ -e_{h11}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}f_0(lh)\\ f_1(lh)\end{pmatrix}+\sum_{lh\in\gamma_{h4}^{-}} h\begin{pmatrix}-e_{h11}^1(mh-lh)\\ -e_{h12}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}f_0(lh)\\ f_1(lh)\end{pmatrix}$$

$$
\begin{aligned}
&+h\left\{\sum_{lh\in\Gamma_{12}} e^1_{h11}(mh-lh)f_0(lh) - \sum_{lh\in\Gamma_{14}} e^1_{h11}(mh-lh)f_0(lh)\right.\\
&+\sum_{lh\in\Gamma_{14}} e^1_{h11}(mh-lh)f_1(lh) - \sum_{lh\in\Gamma_{34}} e^1_{h11}(mh-lh)f_1(lh)\\
&+\sum_{lh\in\Gamma_{23}} e^1_{h12}(mh-lh)f_0(lh) - \sum_{lh\in\Gamma_{12}} e^1_{h12}(mh-lh)f_0(lh)\\
&\left.+\sum_{lh\in\Gamma_{23}} e^1_{h12}(mh-lh)f_1(lh) - \sum_{lh\in\Gamma_{34}} e^1_{h12}(mh-lh)f_1(lh)\right\}.
\end{aligned}
\tag{4.18}
$$

By adding $(T^1_{h1}(\mathbf{D}^1_{h,M}(f_h)))(mh)$ to $(F^1_{h1}(f_h))(mh)$ which are given in Equation (4.17) and (4.18) we get:

$$
(T^1_{h1}(\mathbf{D}^1_{h,M}(f_h)))(mh) + (F^1_{h1}(f_h))(mh) = f_0(mh)\chi_0. \tag{4.19}
$$

For $k=2$ we obtain that $(T^1_{h2}(\mathbf{D}^1_{h,M}(f_h)))(mh)$ is

$$
\begin{aligned}
= &\sum_{lh\in G_h\cup\gamma^-_{h1}\cup\gamma^-_{h2}\cup\Gamma_{12}} h^2\left\{D_h^{-2}e^1_{h21}(mh-lh) - D_h^{-1}e^1_{h22}(mh-lh)\right\}f_0(lh)\\
+ &\sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}} h^2\left\{D_h^{1}e^1_{h21}(mh-lh) + D_h^{2}e^1_{h22}(mh-lh)\right\}f_1(lh)\\
&- \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{12}} he^1_{h21}(mh-lh)f_0(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{14}} he^1_{h21}(mh-lh)f_0(lh)\\
&+ \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{12}} he^1_{h22}(mh-lh)f_0(lh) - \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{23}} he^1_{h22}(mh-lh)f_0(lh)\\
&- \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{14}} he^1_{h21}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{34}} he^1_{h21}(mh-lh)f_1(lh)\\
&- \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{23}} he^1_{h22}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{34}} he^1_{h22}(mh-lh)f_1(lh).
\end{aligned}
$$

From the properties of the discrete Fourier transform which is given in (4.9) we get

$$
F_hD_h^{-2}e^1_{h21}(mh) = F_hD_h^{-1}e^1_{h22}(mh) \text{ and } F_hD_h^{2}e^1_{h22}(mh) = F_hD_h^{-2}e^1_{h11}(mh).
$$

Based on the inverse Fourier transform we obtain

$$
D_h^{-2}e^1_{h21}(mh) = D_h^{-1}e^1_{h22}(mh) \text{ and } D_h^{2}e^1_{h22}(mh) = D_h^{-2}e^1_{h11}(mh). \tag{4.20}
$$

By substituting (4.20) in the last summand the result is

$$
\begin{aligned}
&(T^1_{h2}(\mathbf{D}^1_{h,M}(f_h)))(mh)\\
&= \sum_{lh\in G_h\cup\gamma^-_{h1}\cup\gamma^-_{h2}\cup\Gamma_{12}} h^2\left\{D_h^{-1}e^1_{h22}(mh-lh) - D_h^{-1}e^1_{h22}(mh-lh)\right\} f_0(lh)\\
&+ \sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}} h^2\left\{D_h^{1}e^1_{h21}(mh-lh) + D_h^{-2}e^1_{h11}(mh-lh)\right\} f_1(lh)\\
&- \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{12}} he^1_{h21}(mh-lh)f_0(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{14}} he^1_{h21}(mh-lh)f_0(lh)\\
&+ \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{12}} he^1_{h22}(mh-lh)f_0(lh) - \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{23}} he^1_{h22}(mh-lh)f_0(lh)\\
&- \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{14}} he^1_{h21}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{34}} he^1_{h21}(mh-lh)f_1(lh)\\
&- \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{23}} he^1_{h22}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{34}} he^1_{h22}(mh-lh)f_1(lh).
\end{aligned}
$$

Using the properties of the discrete fundamental solution we get

$$
\begin{aligned}
&\sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}} h^2\left\{D_h^{1}e^1_{h21}(mh-lh) + D_h^{-2}e^1_{h11}(mh-lh)\right\} f_1(lh)\\
&= \sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}} h^2\,\delta_h(mh-lh)f_1(lh)\\
&= \begin{cases} f_1(mh) & \forall mh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}\\ 0 & \text{otherwise,}\end{cases}\\
&= f_1(mh)\chi_1.
\end{aligned}
$$

Finally, for $k=2$ we obtain:

$$(T^1_{h2}(\mathbf{D}^1_{h,M}(f_h)))(mh) = f_1(mh)\chi_1$$

$$
\begin{aligned}
&-\sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{12}} he_{h21}^{1}(mh-lh)f_0(lh)+\sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{14}} he_{h21}^{1}(mh-lh)f_0(lh)\\
&+\sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{12}} he_{h22}^{1}(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{23}} he_{h22}^{1}(mh-lh)f_0(lh)\\
&-\sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{14}} he_{h21}^{1}(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{34}} he_{h21}^{1}(mh-lh)f_1(lh)\\
&-\sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{23}} he_{h22}^{1}(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{34}} he_{h22}^{1}(mh-lh)f_1(lh).
\end{aligned}
\tag{4.21}
$$

Using Equation (4.14) and (4.15), the boundary operator $(F_{hk}^{1}(f_h))(mh)$ for $k=2$ on the uniform rectangular lattice with $\Gamma_{sz}^{*}=\emptyset$ will result:

$$
\begin{aligned}
&(F_{h2}^{1}(f_h))(mh)=(F_{h2}^{1,\gamma_h^{-}}(f_h))(mh)+(F_{h2}^{1,\Gamma}(f_h))(mh)\\
&=\sum_{lh\in\gamma_{h1}^{-}} h\begin{pmatrix}-e_{h22}^{1}(mh-lh)\\ e_{h21}^{1}(mh-lh)\end{pmatrix}^{T}\begin{pmatrix}f_0(lh)\\ f_1(lh)\end{pmatrix}+\sum_{lh\in\gamma_{h2}^{-}} h\begin{pmatrix}e_{h21}^{1}(mh-lh)\\ e_{h22}^{1}(mh-lh)\end{pmatrix}^{T}\begin{pmatrix}f_0(lh)\\ f_1(lh)\end{pmatrix}\\
&+\sum_{lh\in\gamma_{h3}^{-}} h\begin{pmatrix}e_{h22}^{1}(mh-lh)\\ -e_{h21}^{1}(mh-lh)\end{pmatrix}^{T}\begin{pmatrix}f_0(lh)\\ f_1(lh)\end{pmatrix}+\sum_{lh\in\gamma_{h4}^{-}} h\begin{pmatrix}-e_{h21}^{1}(mh-lh)\\ -e_{h22}^{1}(mh-lh)\end{pmatrix}^{T}\begin{pmatrix}f_0(lh)\\ f_1(lh)\end{pmatrix}\\
&\qquad +h\Bigg\{\sum_{lh\in\Gamma_{12}} e_{h21}^{1}(mh-lh)f_0(lh)-\sum_{lh\in\Gamma_{14}} e_{h21}^{1}(mh-lh)f_0(lh)\\
&\qquad +\sum_{lh\in\Gamma_{14}} e_{h21}^{1}(mh-lh)f_1(lh)-\sum_{lh\in\Gamma_{34}} e_{h21}^{1}(mh-lh)f_1(lh)\\
&\qquad +\sum_{lh\in\Gamma_{23}} e_{h22}^{1}(mh-lh)f_0(lh)-\sum_{lh\in\Gamma_{12}} e_{h22}^{1}(mh-lh)f_0(lh)\\
&\qquad +\sum_{lh\in\Gamma_{23}} e_{h22}^{1}(mh-lh)f_1(lh)-\sum_{lh\in\Gamma_{34}} e_{h22}^{1}(mh-lh)f_1(lh)\Bigg\}.
\end{aligned}
\tag{4.22}
$$

By adding $(T_{h2}^{1}(\mathbf{D}_{h,M}^{1}(f_h)))(mh)$ to $(F_{h2}^{1}(f_h))(mh)$ which are given in Equation (4.21) and (4.22) we get:

$$
(T_{h2}^{1}(\mathbf{D}_{h,M}^{1}(f_h)))(mh)+(F_{h2}^{1}(f_h))(mh)=f_1(mh)\chi_1. \tag{4.23}
$$

Finally, from (4.19) and (4.23) the Borel-Pompeiu formula is proved for the discrete complex-valued function which is defined on the uniform rectangular lattice domain. The idea to prove the Borel-Pompeiu formula for a function defined on L-shaped lattice domain will be similar to the rectangular lattice domain but more corners will appear in the definition of a discrete T_h^1-operator.

Using the uniform L-shaped lattice: again we will simplify each of the eight summands for this case. We set that $\Gamma_{12}^* = \Gamma_{23}^* = \Gamma_{14}^* = \emptyset$ and Γ_{34}^* will be active here.
Part 1^*: The first summand will result in

$$\begin{aligned}
&M_1^{G_h}\\
&= \sum_{lh\in G_h} h^2 e_{hk1}^1(mh-lh) D_h^{-2} f_0(lh)\\
&= \sum_{lh\in G_h} h^2 e_{hk1}^1(mh-lh)\left\{h^{-1}(f_0(lh)-f_0(lh-hb_2))\right\}\\
&= \sum_{lh\in G_h} h e_{hk1}^1(mh-lh) f_0(lh) - \sum_{lh\in(G_h/\beta_{h4})\cup\gamma_{h2}^-} h e_{hk1}^1((mh-lh)-hb_2) f_0(lh).
\end{aligned}$$

For the second summand we obtain

$$\begin{aligned}
&M_1^{\gamma_h^-}\\
&= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} h^2 e_{hk1}^1(mh-lh) D_h^{-2} f_0(lh)\\
&= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} h e_{hk1}^1(mh-lh) f_0(lh)\\
&\qquad - \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*-hb_2} h e_{hk1}^1(mh-(lh+hb_2)) f_0(lh)\\
&= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} h e_{hk1}^1(mh-lh) f_0(lh) - \sum_{lh\in\gamma_{h1}^-\cup\beta_{h4}\cup\Gamma_{12}} h e_{hk1}^1((mh-lh)-hb_2) f_0(lh).
\end{aligned}$$

In the next step, we will add to $M_1^{G_h}$ the expression $M_1^{\gamma_h^-}$

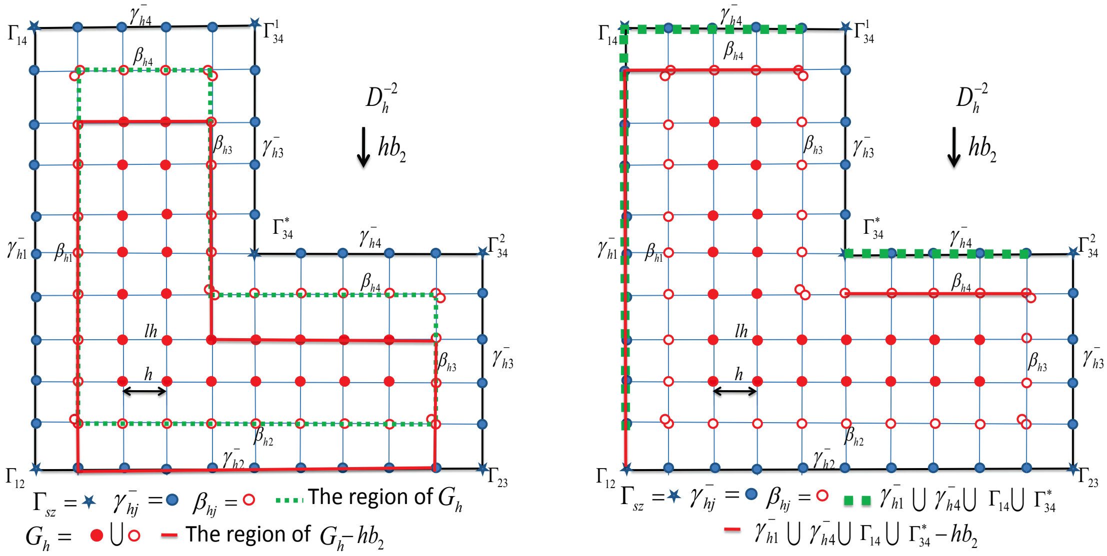


Figure 4.15: The graph of the points which are moved down one step by using shift $-hb_2$.

$$M_1^{G_h} + M_1^{\gamma_h^-} = \sum_{lh\in G_h} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in (G_h\setminus\beta_{h4})\cup\gamma_{h2}^-} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)$$

$$+ \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h1}^-\cup\beta_{h4}\cup\Gamma_{12}} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)$$

$$= \sum_{lh\in G_h\cup\gamma_{h1}^-} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in (G_h\setminus\beta_{h4})\cup\gamma_{h2}^-\cup\gamma_{h1}^-\cup\beta_{h4}\cup\Gamma_{12}} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)$$

$$+ \sum_{lh\in\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} he_{hk1}^1(mh-lh)f_0(lh)$$

$$+ \sum_{lh\in\gamma_{h2}^-\cup\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h2}^-\cup\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh)$$

$$= \sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh) - \sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} he_{hk1}^1((mh-lh)-hb_2)f_0(lh)$$

$$- \sum_{lh\in\gamma_{h2}^-\cup\Gamma_{12}} he_{hk1}^1(mh-lh)f_0(lh) + \sum_{lh\in\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} he_{hk1}^1(mh-lh)f_0(lh).$$

Finally, we get:

$$M_1^{G_h} + M_1^{\gamma_h^-} = \sum_{lh \in G_h \cup \gamma_{h1}^- \cup \gamma_{h2}^- \cup \Gamma_{12}} h^2 D_h^{-2} e_{hk1}^1 (mh - lh)_h f_0(lh)$$

$$- \sum_{lh \in \gamma_{h2}^- \cup \Gamma_{12}} h e_{hk1}^1 (mh - lh) f_0(lh) + \sum_{lh \in \gamma_{h4}^- \cup \Gamma_{14} \cup \Gamma_{34}^*} h e_{hk1}^1 (mh - lh) f_0(lh).$$

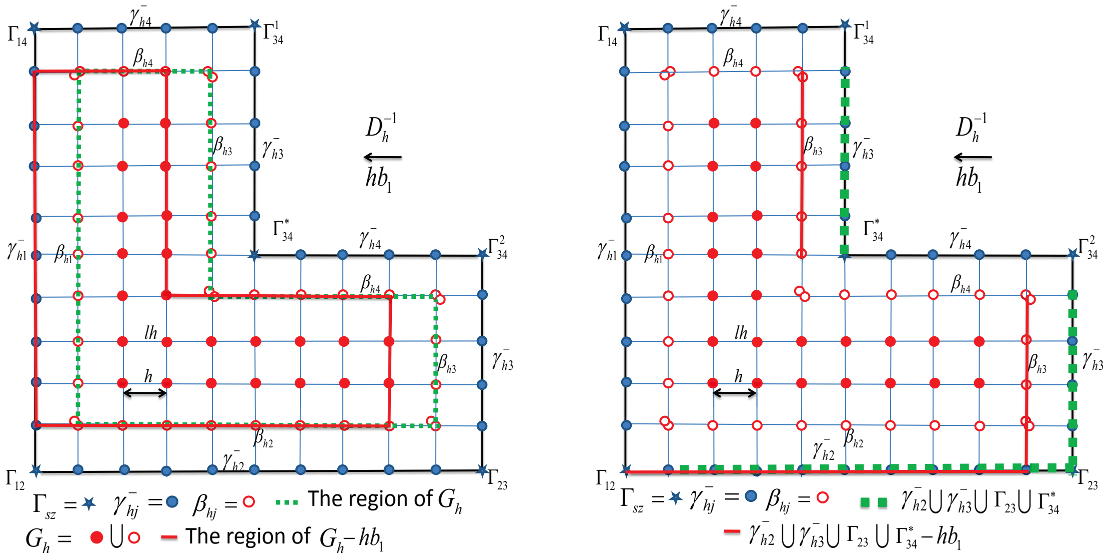


Figure 4.16: The graph of the points which are moved to the left one step by using shift $-hb_1$.

Part 2*: By adding $M_3^{G_h}$ to $M_3^{\gamma_h^-}$ it results

$$M_3^{G_h} + M_3^{\gamma_h^-} = -\sum_{lh \in G_h} h e_{hk2}^1 (mh - lh) f_0(lh) + \sum_{lh \in (G_h \setminus \beta_{h3}) \cup \gamma_{h1}^-} h e_{hk2}^1 ((mh - lh) - hb_1) f_0(lh)$$

$$- \sum_{lh \in \gamma_{h2}^- \cup \gamma_{h3}^- \cup \Gamma_{23} \cup \Gamma_{34}^*} h e_{hk2}^1 (mh - lh) f_0(lh) + \sum_{lh \in \gamma_{h2}^- \cup \beta_{h3} \cup \Gamma_{12}} h e_{hk2}^1 ((mh - lh) - hb_1) f_0(lh)$$

$$= -\sum_{lh \in G_h \cup \gamma_{h2}^-} h e_{hk2}^1 (mh - lh) f_0(lh) + \sum_{lh \in (G_h \setminus \beta_{h3}) \cup \gamma_{h1}^- \cup \gamma_{h2}^- \cup \beta_{h3} \cup \Gamma_{12}} h e_{hk2}^1 ((mh - lh) - hb_1) f_0(lh)$$

$$+ \sum_{lh \in \gamma_{h1}^- \cup \Gamma_{12}} h e_{hk2}^1 (mh - lh) f_0(lh) - \sum_{lh \in \gamma_{h1}^- \cup \Gamma_{12}} h e_{hk2}^1 (mh - lh) f_0(lh)$$

$$- \sum_{lh \in \gamma_{h3}^- \cup \Gamma_{23} \cup \Gamma_{34}^*} h e_{hk2}^1 (mh - lh) f_0(lh)$$

$$= -\sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} he_{hk2}^1(mh-lh)f_0(lh) + \sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} he_{hk2}^1((mh-lh)-hb_1)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h1}^-\cup\Gamma_{12}} he_{hk2}^1(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h3}^-\cup\Gamma_{23}\cup\Gamma_{34}^*} he_{hk2}^1(mh-lh)f_0(lh).$$

Finally it results in:

$$M_3^{G_h} + M_3^{\gamma_h^-} = -\sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} h^2 D_h^{-1} e_{hk2}^1(mh-lh)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h1}^-\cup\Gamma_{12}} he_{hk2}^1(mh-lh)f_0(lh) - \sum_{lh\in\gamma_{h3}^-\cup\Gamma_{23}\cup\Gamma_{34}^*} he_{hk2}^1(mh-lh)f_0(lh).$$

Part 3*: We will use a similar method with $M_2^{G_h}$ and $M_2^{\gamma_h^-}$
$M_2^{G_h}$

$$= \sum_{lh\in G_h} h^2 e_{hk1}^1(mh-lh)D_h^1 f_1(lh)$$

$$= \sum_{lh\in G_h} h^2 e_{hk1}^1(mh-lh)\left\{h^{-1}(f_1(lh+hb_1)-f_1(lh))\right\}$$

$$= \sum_{lh\in G_h+hb_1} he_{hk1}^1(mh-(lh-hb_1))f_1(lh) - \sum_{lh\in G_h} he_{hk1}^1(mh-lh)f_1(lh)$$

$$= \sum_{lh\in(G_h\setminus\beta_{h1})\cup\gamma_{h3}^-\cup\Gamma_{34}^*} he_{hk1}^1((mh-lh)+hb_1)f_1(lh) - \sum_{lh\in G_h} he_{hk1}^1(mh-lh)f_1(lh),$$

and
$M_2^{\gamma_h^-}$

$$= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} h^2 e_{hk1}^1(mh-lh)D_h^1 f_1(lh)$$

$$= \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*+hb_1} he_{hk1}^1(mh-(lh-hb_1))f_1(lh)$$

$$-\sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} he_{hk1}^1(mh-lh)f_1(lh)$$

$$= \sum_{lh\in\gamma_{h4}^-\cup\beta_{h1}\cup\Gamma_{34}^1\cup\Gamma_{34}^2} he_{hk1}^1((mh-lh)+hb_1)f_1(lh) - \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} he_{hk1}^1(mh-lh)f_1(lh).$$

Adding $M_2^{G_h}$ to $M_2^{\gamma_h^-}$ we get

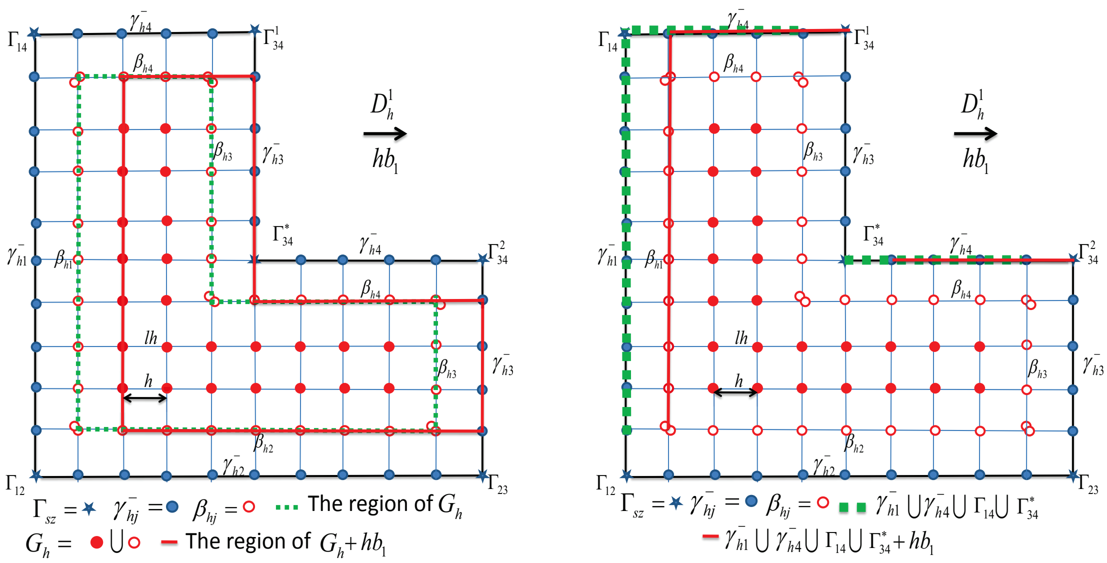


Figure 4.17: The graph of the points which are moved to the right one step by using shift hb_1.

$$
\begin{aligned}
&M_2^{G_h}+M_2^{\gamma_h^-}\\
&=\sum_{lh\in(G_h\setminus\beta_{h1})\cup\gamma_{h3}^-\cup\gamma_{h4}^-\cup\beta_{h1}\cup\Gamma_{34}^1\cup\Gamma_{34}^2\cup\Gamma_{34}^*} he_{hk1}^1((mh-lh)+hb_1)f_1(lh)\\
&-\sum_{lh\in G_h\cup\gamma_{h4}^-\cup\Gamma_{34}^*} he_{hk1}^1(mh-lh)f_1(lh)-\sum_{lh\in\gamma_{h1}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_1(lh)\\
&+\sum_{lh\in\gamma_{h3}^-\cup\Gamma_{34}^1\cup\Gamma_{34}^2} he_{hk1}^1(mh-lh)f_1(lh)-\sum_{lh\in\gamma_{h3}^-\cup\Gamma_{34}^1\cup\Gamma_{34}^2} he_{hk1}^1(mh-lh)f_1(lh)\\
&=\sum_{lh\in G_h\cup\gamma_{h3}^-\cup\gamma_{h4}^-\cup\Gamma_{34}^1\cup\Gamma_{34}^2\cup\Gamma_{34}^*} h\left\{e_{hk1}^1((mh-lh)+hb_1)-e_{hk1}^1(mh-lh)\right\}f_1(lh)\\
&-\sum_{lh\in\gamma_{h1}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h3}^-\cup\Gamma_{34}} he_{hk1}^1(mh-lh)f_1(lh).
\end{aligned}
$$

Finally

$$
\begin{aligned}
M_2^{G_h}+M_2^{\gamma_h^-}&=\sum_{lh\in G_h\cup\gamma_{h3}^-\cup\gamma_{h4}^-\cup\Gamma_{34}\cup\Gamma_{34}^*} h^2D_h^1e_{hk1}^1(mh-lh)f_1(lh)\\
&-\sum_{lh\in\gamma_{h1}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h3}^-\cup\Gamma_{34}} he_{hk1}^1(mh-lh)f_1(lh).
\end{aligned}
$$

Part 4^*: The technical details shown on the remaining two summands $M_4^{G_h}$ and $M_4^{\gamma_h^-}$ are:

$$
\begin{aligned}
M_4^{G_h} &= \sum_{lh\in G_h} h^2 e^1_{hk2}(mh-lh) D_h^2 f_1(lh) \\
&= \sum_{lh\in G_h} h^2 e^1_{hk2}(mh-lh)\left\{h^{-1}(f_1(lh+hb_2)-f_1(lh))\right\} \\
&= \sum_{lh\in G_h+hb_2} he^1_{hk2}(mh-(lh-hb_2))f_1(lh) - \sum_{lh\in G_h} he^1_{hk2}(mh-lh)f_1(lh) \\
&= \sum_{lh\in (G_h\setminus\beta_{h2})\cup\gamma_{h4}^-\cup\Gamma_{34}^*} he^1_{hk2}((mh-lh)+hb_2)f_1(lh) - \sum_{lh\in G_h} he^1_{hk2}(mh-lh)f_1(lh),
\end{aligned}
$$

and

$$
\begin{aligned}
M_4^{\gamma_h^-} &= \sum_{lh\in\gamma_{h2}^-\cup\gamma_{h3}^-\cup\Gamma_{23}\cup\Gamma_{34}^*} h^2 e^1_{hk2}(mh-lh)D_h^2 f_1(lh) \\
&= \sum_{lh\in lh\in\gamma_{h2}^-\cup\gamma_{h3}^-\cup\Gamma_{23}\cup\Gamma_{34}^*+hb_2} he^1_{hk2}(mh-(lh-hb_2))f_1(lh) \\
&\qquad - \sum_{lh\in\gamma_{h2}^-\cup\gamma_{h3}^-\cup\Gamma_{23}\cup\Gamma_{34}^*} he^1_{hk2}(mh-lh)f_1(lh) \\
&= \sum_{lh\in\gamma_{h3}^-\cup\beta_{h2}\cup\Gamma_{34}^1\cup\Gamma_{34}^2} he^1_{hk2}((mh-lh)+hb_2)f_1(lh) - \sum_{lh\in\gamma_{h2}^-\cup\gamma_{h3}^-\cup\Gamma_{23}\cup\Gamma_{34}^*} he^1_{hk2}(mh-lh)f_1(lh).
\end{aligned}
$$

Adding $M_4^{G_h}$ to $M_4^{\gamma_h^-}$ we get

$$
\begin{aligned}
M_4^{G_h}+M_4^{\gamma_h^-} &= \sum_{lh\in(G_h\setminus\beta_{h2})\cup\gamma_{h4}^-\cup\Gamma_{34}^*\cup\gamma_{h3}^-\cup\beta_{h2}\cup\Gamma_{34}^1\cup\Gamma_{34}^2} he^1_{hk2}((mh-lh)+hb_2)f_1(lh) \\
&\quad - \sum_{lh\in G_h\cup\gamma_{h3}^-\cup\Gamma_{34}^*} he^1_{hk2}(mh-lh)f_1(lh) - \sum_{lh\in\gamma_{h2}^-\cup\Gamma_{23}} he^1_{hk2}(mh-lh)f_1(lh) \\
&\quad + \sum_{lh\in\gamma_{h4}^-\cup\Gamma_{34}^1\cup\Gamma_{34}^2} he^1_{hk2}(mh-lh)f_1(lh) - \sum_{lh\in\gamma_{h4}^-\cup\Gamma_{34}^1\cup\Gamma_{34}^2} he^1_{hk2}(mh-lh)f_1(lh).
\end{aligned}
$$

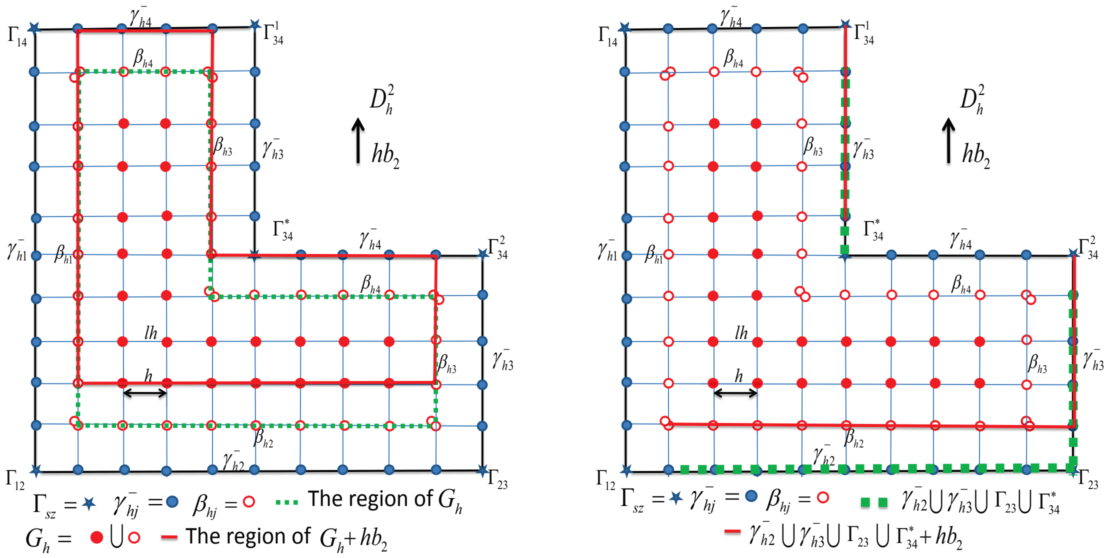


Figure 4.18: The graph of the points which are moved up one step by using shift hb_2.

$$M_4^{G_h}+M_4^{\gamma_h^-} = \sum_{lh\in G_h\cup\gamma_{h3}^-\cup\gamma_{h4}^-\cup\Gamma_{34}\cup\Gamma_{34}^*} h^2 D_h^2 e_{hk2}^1(mh-lh)f_1(lh)$$

$$- \sum_{lh\in\gamma_{h2}^-\cup\Gamma_{23}} h e^1_{hk2}(mh-lh)f_1(lh) + \sum_{lh\in\gamma_{h4}^-\cup\Gamma_{34}} h e^1_{hk2}(mh-lh)f_1(lh).$$

Part 5*: Finally, by collecting all eight summands we obtain

$$(T_{hk}^1(\mathbf{D}_{h,M}^1(f_h)))(mh)$$

$$= \sum_{lh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^-\cup\Gamma_{12}} h^2\left\{D_h^{-2}e_{hk1}^1(mh-lh) - D_h^{-1}e_{hk2}^1(mh-lh)\right\} f_0(lh)$$

$$+ \sum_{lh\in G_h\cup\gamma_{h3}^-\cup\gamma_{h4}^-\cup\Gamma_{34}\cup\Gamma_{34}^*} h^2\left\{D_h^{1}e_{hk1}^1(mh-lh) + D_h^{2}e_{hk2}^1(mh-lh)\right\} f_1(lh)$$

$$- \sum_{lh\in\gamma_{h2}^-\cup\Gamma_{12}} h e_{hk1}^1(mh-lh)f_0(lh) + \sum_{lh\in\gamma_{h4}^-\cup\Gamma_{14}\cup\Gamma_{34}^*} h e_{hk1}^1(mh-lh)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{12}} he_{hk2}^{1}(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{23}\cup\Gamma_{34}^{*}} he_{hk2}^{1}(mh-lh)f_0(lh)$$

$$-\sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{14}} he_{hk1}^{1}(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{34}} he_{hk1}^{1}(mh-lh)f_1(lh)$$

$$-\sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{23}} he_{hk2}^{1}(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{34}} he_{hk2}^{1}(mh-lh)f_1(lh).$$

For $k=1$, using the properties which are given in (4.16) we obtain that a discrete $(T_{h1}^{1}(\mathbf{D}_{h,M}^{1}(f_h)))(mh)$ is

$$=\sum_{lh\in G_h\cup\gamma_{h1}^{-}\cup\gamma_{h2}^{-}\cup\Gamma_{12}} h^2\left\{D_h^{-2}e_{h11}^{1}(mh-lh)+D_h^{1}e_{h21}^{1}(mh-lh)\right\}f_0(lh)$$

$$+\sum_{lh\in G_h\cup\gamma_{h3}^{-}\cup\gamma_{h4}^{-}\cup\Gamma_{34}\cup\Gamma_{34}^{*}} h^2\left\{-D_h^{2}e_{h12}^{1}(mh-lh)+D_h^{2}e_{h12}^{1}(mh-lh)\right\}f_1(lh)$$

$$-\sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{12}} he_{h11}^{1}(mh-lh)f_0(lh)+\sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{14}\cup\Gamma_{34}^{*}} he_{h11}^{1}(mh-lh)f_0(lh)$$

$$+\sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{12}} he_{h12}^{1}(mh-lh)f_0(lh)-\sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{23}\cup\Gamma_{34}^{*}} he_{h12}^{1}(mh-lh)f_0(lh)$$

$$-\sum_{lh\in\gamma_{h1}^{-}\cup\Gamma_{14}} he_{h11}^{1}(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h3}^{-}\cup\Gamma_{34}} he_{h11}^{1}(mh-lh)f_1(lh)$$

$$-\sum_{lh\in\gamma_{h2}^{-}\cup\Gamma_{23}} he_{h12}^{1}(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h4}^{-}\cup\Gamma_{34}} he_{h12}^{1}(mh-lh)f_1(lh).$$

Based on the properties of the discrete fundamental solution, we get

$$\begin{aligned}\sum_{lh\in G_h\cup\gamma_{h1}^{-}\cup\gamma_{h2}^{-}\cup\Gamma_{12}} & h^2\left\{D_h^{-2}e_{h11}^{1}(mh-lh)+D_h^{1}e_{h21}^{1}(mh-lh)\right\}f_0(lh)\\ &=\sum_{lh\in G_h\cup\gamma_{h1}^{-}\cup\gamma_{h2}^{-}\cup\Gamma_{12}} h^2\,\delta_h(mh-lh)f_0(lh)\\ &=\begin{cases} f_0(mh) & \forall mh\in G_h\cup\gamma_{h1}^{-}\cup\gamma_{h2}^{-}\cup\Gamma_{12}\\ 0 & \text{otherwise,}\end{cases}\\ &=f_0(mh)\chi_0.\end{aligned}$$

Similarly for $k = 2$ we obtain based on the properties which given in (4.20) that $(T^1_{h2}(\mathbf{D}^1_{h,M}(f_h)))(mh)$ is

$$= \sum_{lh\in G_h\cup\gamma^-_{h1}\cup\gamma^-_{h2}\cup\Gamma_{12}} h^2\left\{D^{-1}_h e^1_{h22}(mh-lh) - D^{-1}_h e^1_{h22}(mh-lh)\right\} f_0(lh)$$

$$+ \sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}\cup\Gamma^*_{34}} h^2\left\{D^1_h e^1_{h21}(mh-lh) + D^{-2}_h e^1_{h11}(mh-lh)\right\} f_1(lh)$$

$$- \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{12}} he^1_{h21}(mh-lh)f_0(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{14}\cup\Gamma^*_{34}} he^1_{h21}(mh-lh)f_0(lh)$$

$$+ \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{12}} he^1_{h22}(mh-lh)f_0(lh) - \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{23}\cup\Gamma^*_{34}} he^1_{h22}(mh-lh)f_0(lh)$$

$$- \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{14}} he^1_{h21}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{34}} he^1_{h21}(mh-lh)f_1(lh)$$

$$- \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{23}} he^1_{h22}(mh-lh)f_1(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{34}} he^1_{h22}(mh-lh)f_1(lh).$$

Using the properties of the discrete fundamental solution, we get

$$\sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}\cup\Gamma^*_{34}} h^2\left\{D^1_h e^1_{h21}(mh-lh) + D^{-2}_h e^1_{h11}(mh-lh)\right\} f_1(lh)$$

$$= \sum_{lh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}\cup\Gamma^*_{34}} h^2\,\delta_h(mh-lh)f_1(lh)$$

$$= \begin{cases} f_1(mh) & \forall mh\in G_h\cup\gamma^-_{h3}\cup\gamma^-_{h4}\cup\Gamma_{34}\cup\Gamma^*_{34} \\ 0 & \text{otherwise,}\end{cases}$$

$$= f_1(mh)\chi_1.$$

Finally, for $k = 1, 2$, we obtain:

$$(T^1_{hk}(\mathbf{D}^1_{h,M}(f_h)))(mh) = f_{k-1}(mh)\chi_{k-1}$$

$$- \sum_{lh\in\gamma^-_{h2}\cup\Gamma_{12}} he^1_{hk1}(mh-lh)f_0(lh) + \sum_{lh\in\gamma^-_{h4}\cup\Gamma_{14}\cup\Gamma^*_{34}} he^1_{hk1}(mh-lh)f_0(lh)$$

$$+ \sum_{lh\in\gamma^-_{h1}\cup\Gamma_{12}} he^1_{hk2}(mh-lh)f_0(lh) - \sum_{lh\in\gamma^-_{h3}\cup\Gamma_{23}\cup\Gamma^*_{34}} he^1_{hk2}(mh-lh)f_0(lh)$$

$$\begin{aligned}&-\sum_{lh\in\gamma_{h1}^-\cup\Gamma_{14}} he_{hk1}^1(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h3}^-\cup\Gamma_{34}} he_{hk1}^1(mh-lh)f_1(lh)\\&-\sum_{lh\in\gamma_{h2}^-\cup\Gamma_{23}} he_{hk2}^1(mh-lh)f_1(lh)+\sum_{lh\in\gamma_{h4}^-\cup\Gamma_{34}} he_{hk2}^1(mh-lh)f_1(lh).\end{aligned}\tag{4.24}$$

Using Equation (4.14) and (4.15), the boundary operator $(F_{hk}^1(f_h))(mh)$ for $k=1,2$ on the uniform L-shaped lattice domain with $\Gamma_{12}^*=\Gamma_{23}^*=\Gamma_{14}^*=\emptyset$, will result in:

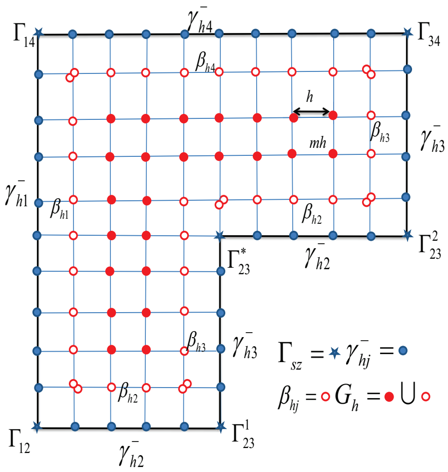
$$\begin{aligned}&(F_{hk}^1(f_h))(mh)=(F_{hk}^{1,\gamma_h^-}(f_h))(mh)+(F_{hk}^{1,\Gamma}(f_h))(mh)\\
&=\sum_{lh\in\gamma_{h1}^-} h\begin{pmatrix}-e_{hk2}^1(mh-lh)\\ e_{hk1}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}f_0(lh)\\f_1(lh)\end{pmatrix}+\sum_{lh\in\gamma_{h2}^-} h\begin{pmatrix}e_{hk1}^1(mh-lh)\\ e_{hk2}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}f_0(lh)\\f_1(lh)\end{pmatrix}\\
&+\sum_{lh\in\gamma_{h3}^-} h\begin{pmatrix}e_{hk2}^1(mh-lh)\\ -e_{hk1}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}f_0(lh)\\f_1(lh)\end{pmatrix}+\sum_{lh\in\gamma_{h4}^-} h\begin{pmatrix}-e_{hk1}^1(mh-lh)\\ -e_{hk2}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix}f_0(lh)\\f_1(lh)\end{pmatrix}\\
&\quad+h\Bigg\{\sum_{lh\in\Gamma_{12}} e_{hk1}^1(mh-lh)f_0(lh)-\sum_{lh\in\Gamma_{14}\cup\Gamma_{34}^*} e_{hk1}^1(mh-lh)f_0(lh)\\
&\quad+\sum_{lh\in\Gamma_{14}} e_{hk1}^1(mh-lh)f_1(lh)-\sum_{lh\in\Gamma_{34}} e_{hk1}^1(mh-lh)f_1(lh)\\
&\quad+\sum_{lh\in\Gamma_{23}\cup\Gamma_{34}^*} e_{hk2}^1(mh-lh)f_0(lh)-\sum_{lh\in\Gamma_{12}} e_{hk2}^1(mh-lh)f_0(lh)\\
&\quad+\sum_{lh\in\Gamma_{23}} e_{hk2}^1(mh-lh)f_1(lh)-\sum_{lh\in\Gamma_{34}} e_{hk2}^1(mh-lh)f_1(lh)\Bigg\}.\end{aligned}\tag{4.25}$$

By adding $(T_{hk}^1(\mathbf{D}_{h,M}^1(f_h)))(mh)$ to $(F_{hk}^1(f_h))(mh)$ which are given in Equation (4.24) and (4.25) we get:

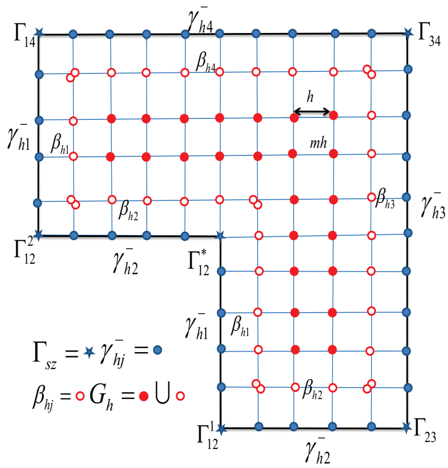
$$(T_{hk}^1(\mathbf{D}_{h,M}^1(f_h)))(mh)+(F_{hk}^1(f_h))(mh)=f_{k-1}(mh)\chi_{k-1}.\tag{4.26}$$

Finally, from (4.26) a Borel-Pompeiu formula is proved for a discrete complex-valued function which is defined on the L-shaped lattice domain.

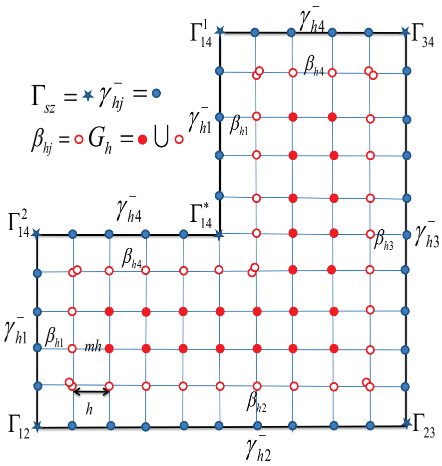
The methodology for proving the other cases (after rotating L-shape 3-time 90° clockwise) is similar to the rectangular and L-shaped proof. But some concepts will be different dependent on the location of the inner corners.



(a) 90° rotation



(b) 180° rotation



(c) 270° rotation

The proof of the other cases is similar and requires no new ideas. Therefore, it is sufficient here to explain each case by illustrating the idea. From the figures above, we see that after rotating the L-shaped 90° clockwise, the inner corner Γ_{23}^{*} will be active and the other inner corner will be empty and when rotating the L-shaped 180° and 270° clockwise the inner corners Γ_{12}^{*} and Γ_{14}^{*} respectively will be active. After studying each case, we find that a discrete Borel-Pompeiu formula is true for any discrete complex-valued function which is defined on irregular domain. We should note that some inner corners will be a union from other inner corners for example $\Gamma_{34}^{*} = \Gamma_{34}^{*1} \cup \Gamma_{34}^{*2}$. Also, some out corners can be a union from other out corners. For a better understanding, we plot Figure 4.20. $\square$

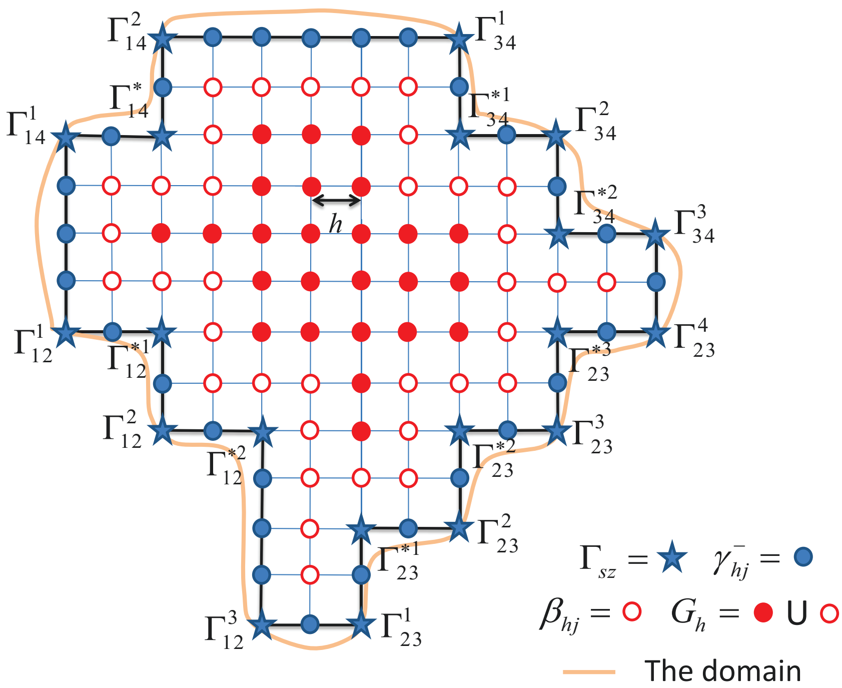


Figure 4.20: General domain

The next theorem is another discrete Borel-Pompeiu formula which is analogous to the classical theory (to the continuous case):

Theorem 20. *For all points $mh \in G_h$, we have a discrete Borel-Pompeiu formula:*

$$(T_h^1(\boldsymbol{D}^1_{h,M}(f_h)))(mh) + (F_h^1(f_h))(mh) = f_h(mh).$$

Remark 17. *A discrete Borel-Pompeiu formula with respect to $\boldsymbol{D}^2_{h,M}$, T_h^2 and F_h^2 can be obtained in a similar way.*

Next step: we restrict our considerations to construct discrete holomorphic functions. Obtaining such a function is a non-trivial task. The advantage of discrete holomorphic functions is that they are uniquely determined already by their boundary values and the extension is given by the discrete boundary operator. After constructing a discrete holomorphic function, the generalized boundary operator F_h^1 will be applied.

Example 3. *Considering an arbitrary function $g_h(x_1, x_2)$ as a boundary condition of a discrete Laplace equation (4.2), where $x_1 \in (-L, L)$ and $x_2 \in (-L, L)$ and letting*

$$g_h(x_1, x_2) = \sin\left(\frac{x_1^2 + x_2^2}{(Lh+1)^2}\right) + \cos\left(\frac{x_1 x_2}{(Lh+1)^2}\right),$$

where h is mesh step size.
By solving the boundary value problem of discrete Laplace equation with this boundary condition we will get a real-valued function $U_h(x_1, x_2)$.

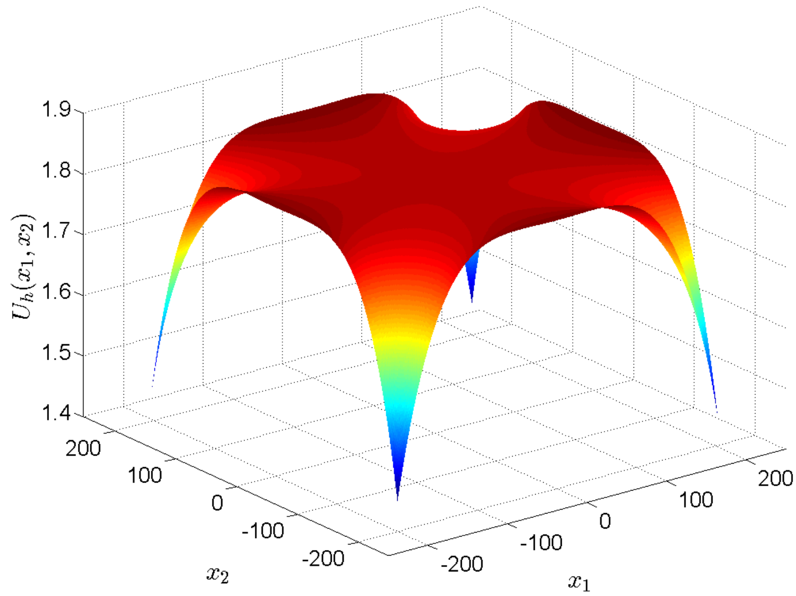


Figure 4.21: The solution of a discrete value problem (Laplace equation) with the boundary condition $g_h(x_1, x_2)$.

Applying the adjoint discrete Cauchy-Riemann operator $\boldsymbol{D}^2_{h,M}$ to the discrete real-valued function $U_h(x_1, x_2)$ will result in

$$\boldsymbol{D}^2_{h,M} U_h(x_1, x_2) = \begin{pmatrix} D_h^2 & -D_h^1 \\ D_h^{-1} & D_h^{-2} \end{pmatrix} \begin{pmatrix} U_h(x_1, x_2) \\ 0 \end{pmatrix} = \begin{pmatrix} D_h^2 U_h(x_1, x_2) \\ D_h^{-1} U_h(x_1, x_2) \end{pmatrix}.$$

The result is a discrete holomorphic complex-valued function $f_h(x_1, x_2)$, where

$$f_0(x_1, x_2) = D_h^2 U_h(x_1, x_2) \text{ and } f_1(x_1, x_2) = D_h^{-1} U_h(x_1, x_2).$$

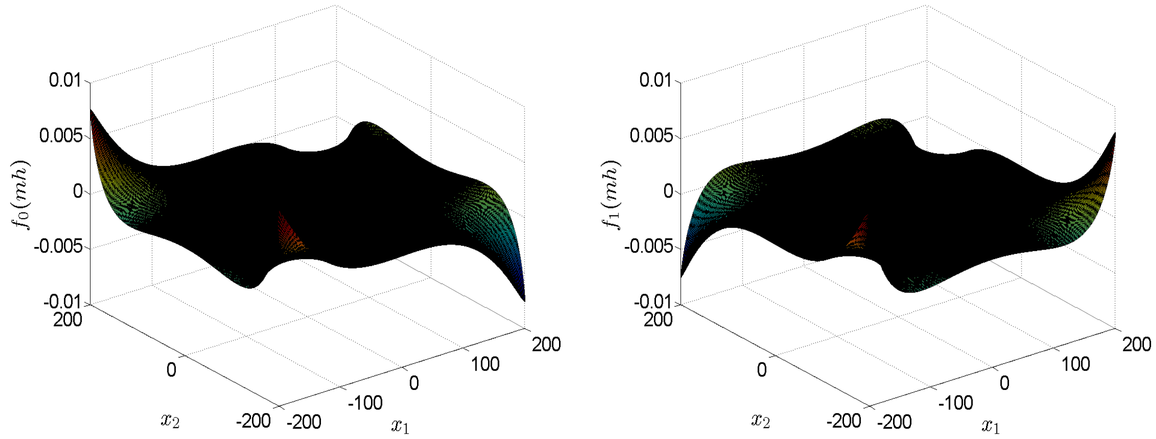


Figure 4.22: Discrete holomorphic complex-valued function $f_h(x_1, x_2)$.

By applying the generalized boundary operator F_h^1 *to the boundary values of the discrete holomorphic complex-valued function* $f_h(x_1, x_2)$*, it is possible to calculate the absolute error between* $(F_h^1(f_h))(mh)$ *and the discrete holomorphic complex-valued function. Then, we plot the absolute error as shown in Figure 4.23*

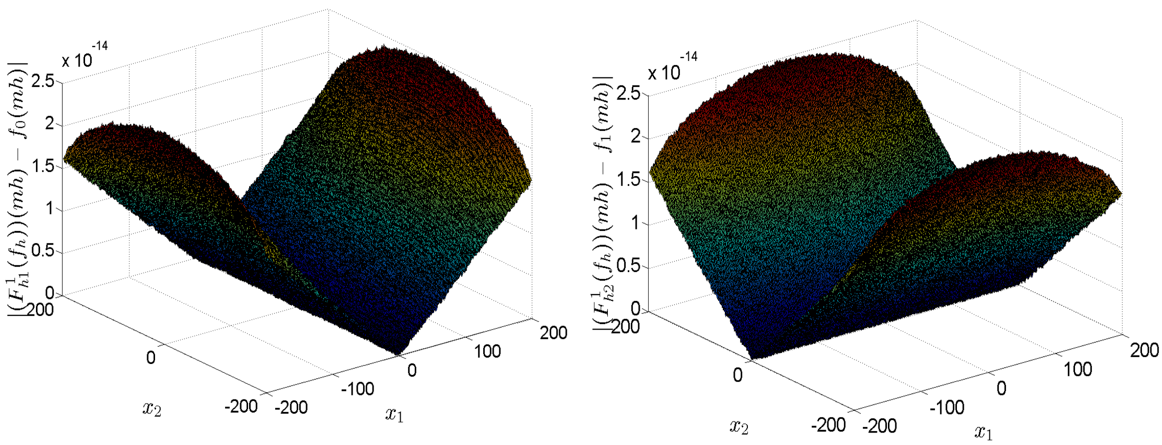


Figure 4.23: The absolute difference between $(F_h^1(f_h))(mh)$ and the discrete holomorphic function $f_h(mh)$.

From this investigation of the boundary operator after applying it to the discrete holomorphic function $(F_h^1(f_h))(mh)$*, we can say that the accuracy of this operator is perfect and the small error comes from the numerical calculation of the discrete fundamental solution of Cauchy-Riemann operator.*

In the next example, we will construct a different discrete holomorphic function by changing the boundary condition of a discrete Laplace equation and the same steps will be used as in Example 3.

Example 4. *Consider another function $g_h(x_1, x_2)$ as a boundary condition of a discrete Laplace equation (4.2), where $x_1 \in (-L, L)$ and $x_2 \in (-L, L)$ and let*

$$g_h(x_1, x_2) = \log(x_1^2 + x_2^2).$$

By solving the boundary value problem of a discrete Laplace equation with this boundary conditions we will get a real-valued function $V_h(x_1, x_2)$.

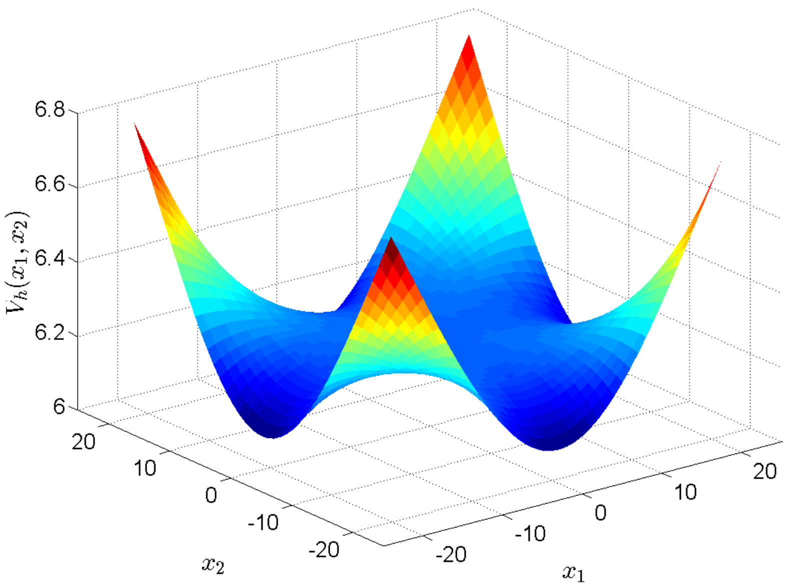


Figure 4.24: The solution of a discrete value problem (Laplace equation) with the boundary condition $g_h(x_1, x_2)$.

Applying the adjoint discrete Cauchy-Riemann operator $\boldsymbol{D}_{h,M}^2$ to the discrete real-valued function $V_h(x_1, x_2)$, it will result in a discrete holomorphic complex-valued function $q_h(x_1, x_2)$.

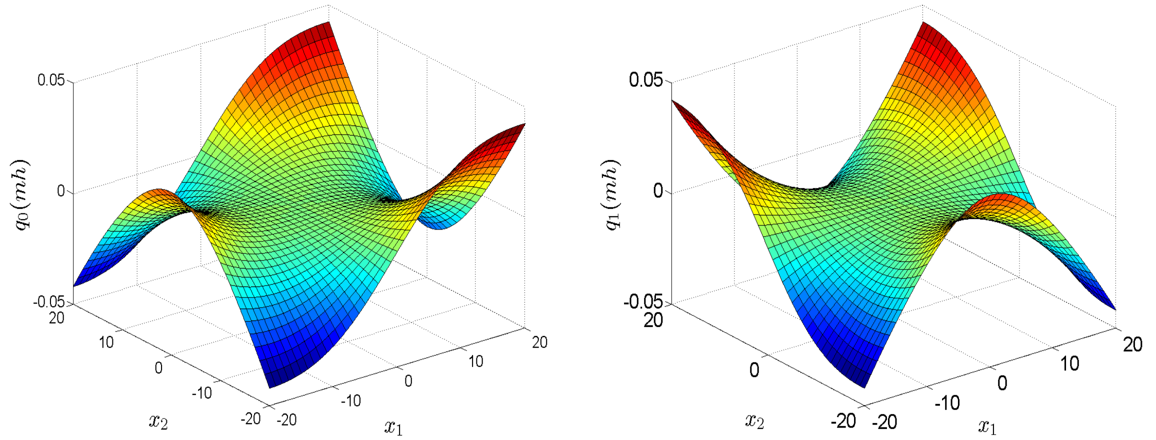


Figure 4.25: Discrete holomorphic complex-valued function $q_h(x_1, x_2)$.

Finally, we calculate the absolute difference between $(F_h^1(q_h))(mh)$ *and the discrete holomorphic complex-valued functions* $q_h(mh)$*. As shown in Figure 4.26 the absolute error in this example is too small.*

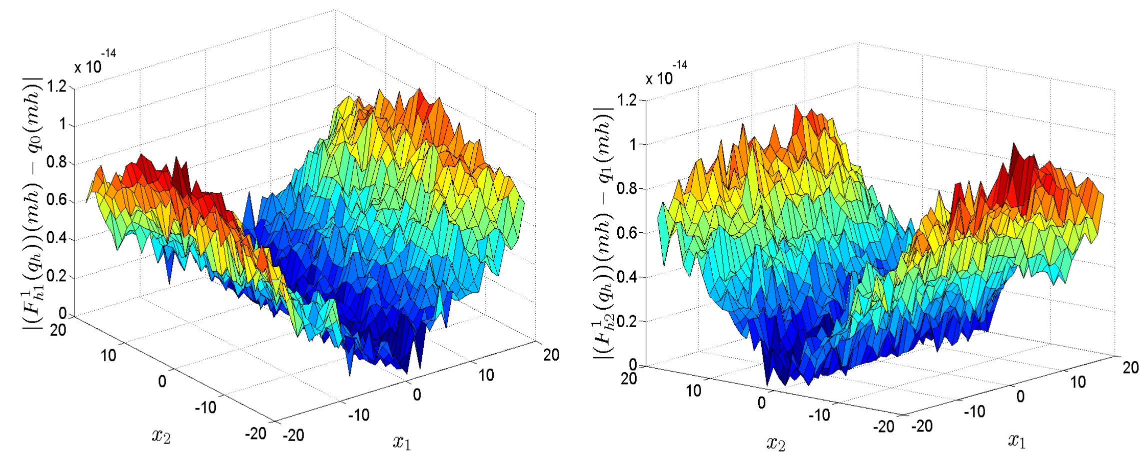


Figure 4.26: The absolute difference between $(F_h^1(q_h))(mh)$ and the discrete holomorphic function $q_h(mh)$.

The next example is to apply the generalized boundary operator F_h^1 to the constant complex-valued function which has taken place on the L-shaped lattice domain, which is the square $[-L, L]^2$ with the upper right quadrant removed.

Example 5. *Consider a constant complex function* $f_h(x_1, x_2)$ *on the L-shaped lattice* *domain* $G_h \in \mathbb{R}^2$ *where* $G_h = [-L, L] \times [-L, L] \setminus (L, 0) \times (L, 0)$ *and let*

$$f_h(x_1, x_2) = 1 + i.$$

Apply the generalized boundary operator F_h^1 *to the boundary values of the discrete constant complex-valued function* $f_h(x_1, x_2)$. *Next, calculate the absolute error between the boundary operator* $(F_h^1(f_h))(mh)$ *and the constant function* $f_h(mh)$ *several times dependent on the number of nodes point in the L-shaped lattice. The result is seen in Table 4.1.*

Number of nodes	$\lvert(F_h^1(f_h))(mh) - f_h(mh)\rvert \leq \varepsilon$ $\varepsilon =$
341	10^{-16}
7701	10^{-15}
30401	10^{-14}
120801	10^{-14}
752001	10^{-14}

Table 4.1

Example 6. *In this example, we want to see the accuracy of the generalized boundary operator* F_h^1 *when it is applied. First, when it is applied to a discrete holomorphic constant complex function* $q_h(x_1, x_2)$. *Next, when it is applied to the discrete holomorphic function* $f_h(x_1, x_2)$ *which is constructed in Example 3. Both functions will be considered in the same domain* $G_h \in \mathbb{R}_h^2$ *where* G_h *is the square domain* $[-L, L]^2$. *We will use the same number of nodes in this comparison and calculate the absolute errors* ε_1 *and* ε_2 *with*

$$\lvert(F_h^1(q_h))(mh) - q_h(mh)\rvert \leq \varepsilon_1$$

and

$$\lvert(F_h^1(f_h))(mh) - f_h(mh)\rvert \leq \varepsilon_2.$$

It is shown in Table 4.2 that the absolute error ε_1 *is smaller than* ε_2 *when the number of nodes increases. The reason for this difference is coming from the absolute values of holomorphic functions* $f_h(x_1, x_2)$ *on the boundary.*

Now, we intend to clarify the structure of a function from $\operatorname{Im} \mathbf{Q}_h^i$ where $i = 1, 2$. First, we should know how to construct such a function. The answer will not be difficult and we will explain how one can construct a function from $\operatorname{Im} \mathbf{Q}_h^2$ in the last example of the present chapter. One more question arises, how one can verify if a function is from $\operatorname{Im} \mathbf{Q}_h^i$ or not? To answer that question, we have to consider the following theorem.

Number of nodes	The absolute error ε_1	The absolute error ε_2
1681	10^{-16}	10^{-15}
10201	10^{-15}	10^{-14}
40401	10^{-15}	10^{-14}
160801	10^{-14}	10^{-14}
1002001	10^{-14}	10^{-9}

Table 4.2

Theorem 21. *The function f_h belongs to $Im\,\mathbf{Q}_h^2$ if and only if the trace of $T_h^1 f_h$ is equal to zero. (i.e $f_h \in Im\,\mathbf{Q}_h^2$ iff $tr_\Gamma T_h^1 f_h = 0$.)*

Proof. Let $f_h \in \operatorname{Im}\mathbf{Q}_h^2$, then there is a function g_h such that $f_h = \mathbf{D}_{h,M}^1 g_h$, where $g_h \in W_0^{1,2}(G_h)$. Applying T_h^1-operator to the function f_h, we obtain

$$T_h^1 f_h = T_h^1 \mathbf{D}_{h,M}^1 g_h.$$

With the help of discrete Borel-Pompeiu's formula, we obtain that

$$T_h^1 \mathbf{D}_{h,M}^1 g_h = g_h - F_h^1 g_h = g_h.$$

This leads to $\mathrm{tr}_\Gamma T_h^1 f_h = \mathrm{tr}_\Gamma T_h^1 \mathbf{D}_{h,M}^1 g_h = \mathrm{tr}_\Gamma g_h = 0$, because $g_h \in W_0^{1,2}(G_h)$.
Now, let us assume that $\mathrm{tr}_\Gamma T_h^1 f_h = 0$. Then, the function f_h can be decomposed into the sum of two functions

$$f_h = u_h + v_h,$$

with $u_h \in \ker \mathbf{D}_{h,M}^2(G_h)$ and $v_h \in \operatorname{Im}\mathbf{Q}_h^2$. We obtain

$$0 = \mathrm{tr}_\Gamma T_h^1 u_h + \mathrm{tr}_\Gamma T_h^1 v_h.$$

The first term belongs to $\ker \mathbf{D}_{h,M}^2(G_h)$. That means $\mathbf{D}_{h,M}^2 \mathbf{D}_{h,M}^1 u_h = 0$. Therefore, $\mathbf{D}_{h,M}^1 u_h = 0$ and from the discrete Borel-Pompeiu formula, we obtain $T_h^1 \mathbf{D}_{h,M}^1 u_h = u_h = 0$. We get the value of $T_h^1 u_h$ on the interior and on the boundary is equal to zero and $u_h = 0$. For the second term we get the representation

$$v_h = \mathbf{D}_{h,M}^1 g_h, \text{ where } g_h \in W_0^{1,2}(G_h).$$

Hence $\mathrm{tr}_\Gamma T_h^1 \mathbf{D}_{h,M}^1 g_h = \mathrm{tr}_\Gamma g_h = 0$. Therefore, we proved that $f_h = v_h \in \operatorname{Im}\mathbf{Q}_h^2$. □

The property of a function from $\in \operatorname{Im}\mathbf{Q}_h^2$ will be essential for the investigation of boundary value problems. Now, it is necessary to construct a discrete complex-valued function from $\operatorname{Im}\mathbf{Q}_h^2$ in the rectangular domain. We will use this function when we study the discrete non-linear p-Dirac problem. The discussion will involve one example of constructing a function from $\in \operatorname{Im}\mathbf{Q}_h^2$ and we will apply the T_h^1-operator to this function:

Example 7. *Considering a discrete complex-valued function which is defined on $G_h \in \mathbb{R}_h^2$ where $G_h = [-L, L]\times[-L, L]$, where $x_1 \in (-L, L)$ and $x_2 \in (-L, L)$. The boundary values of this function are equal to zero. Let us consider*

$$g_h(x_1, x_2) = g_0(x_1, x_2) + ig_1(x_1, x_2) = (g_0(x_1, x_2), g_1(x_1, x_2)),$$

where

$$g_0(x_1, x_2) = \left(\frac{\sin(x_1 - Lh)}{Lh}\right)^3 \left(1 - \exp\left(\frac{x_1 + Lh}{Lh}\right)\right)$$
$$\left(\frac{\sin(x_2 + Lh)}{Lh}\right)^3 \left(1 - \exp\left(\frac{x_2 - Lh}{Lh}\right)\right)$$

and

$$g_1(x_1, x_2) = (x_1 - Lh)\,(x_1 + Lh)\,(x_2 - Lh)\,(x_2 + Lh)\,/\,(\,Lh)^4\,.$$

By applying the discrete Cauchy-Riemann operator $\boldsymbol{D}_{h,M}^1$ to the function $g_h(x_1, x_2)$,

$$\boldsymbol{D}_{h,M}^1 g_h(x_1, x_2) = \begin{pmatrix} D_h^{-2} & D_h^1 \\ -D_h^{-1} & D_h^2 \end{pmatrix} \begin{pmatrix} g_0(x_1, x_2) \\ g_1(x_1, x_2) \end{pmatrix} = \begin{pmatrix} D_h^{-2} g_0(x_1, x_2) + D_h^1 g_1(x_1, x_2) \\ -D_h^{-1} g_0(x_1, x_2) + D_h^2 g_1(x_1, x_2) \end{pmatrix}$$

the result is a discrete complex-valued function $f_h(x_1, x_2)$,

$$\begin{aligned} D_h^{-2} g_0(x_1, x_2) + D_h^1 g_1(x_1, x_2) &= f_0(x_1, x_2) \\ -D_h^{-1} g_0(x_1, x_2) + D_h^2 g_1(x_1, x_2) &= f_1(x_1, x_2) \end{aligned} \tag{4.27}$$

which is from $Im\,\mathbf{Q}_h^2$. By fixing the points $L = 20$ and $h = 1$, we calculate numerically the complex-valued function f_h which is defined on the lattice and then, we plot the result in Figure 4.27. Figure 4.27 provides a view of the complex-valued function f_h. That figure contains the real part $f_0(x_1, x_2)$ and the imaginary part $f_1(x_1, x_2)$ of the discrete complex-valued function $f_h(x_1, x_2)$ which is from $Im\,\mathbf{Q}_h^2$. This program is convertible and adaptive for any arbitrary n and one can consider other complex-valued functions $g_h(x_1, x_2)$.

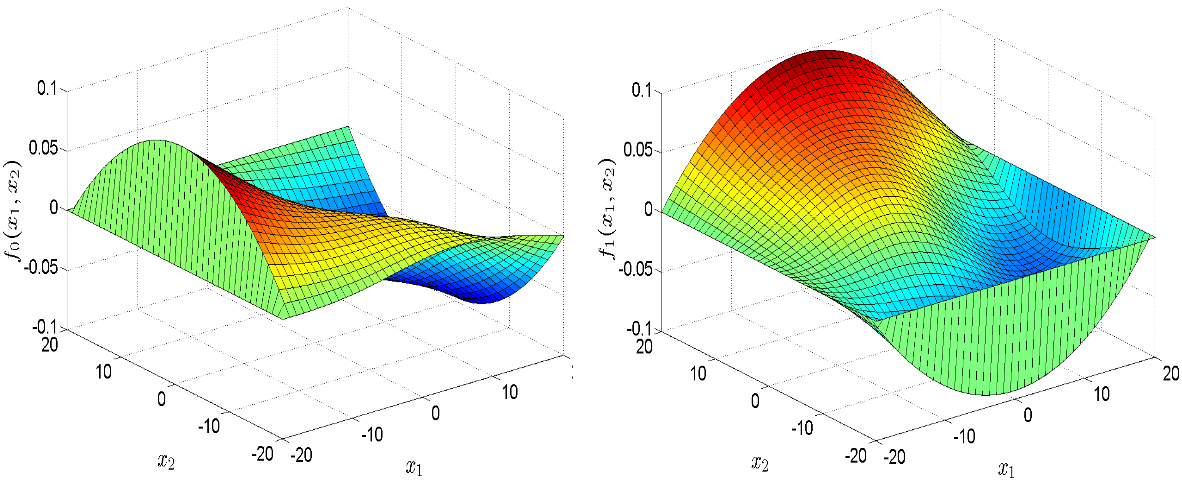


Figure 4.27: Discrete complex-valued function $f_h(x_1, x_2)$ from $\operatorname{Im} \mathbf{Q}_h^2$.

Later, the discrete complex-valued functions $f_h(x_1, x_2)$ *from* $Im\, \mathbf{Q}_h^2$ *will be used as the right hand side of the discrete p-Dirac equation.*
Here, we will use the fundamental solution $\boldsymbol{e}_h^1(mh)$ *of the discrete Cauchy-Riemann operator to calculate a discrete* T_h^1*-operator. Then, we apply the discrete* T_h^1*-operator to the discrete* $f_h(x_1, x_2)$*. The components of the operator* T_h^1 *are calculated by combining the output of the two formulas which are given in (4.10) and (4.11). The result of this calculation will be* $(T_h^1(f_h))(mh) = ((T_{h1}^1(f_h))(mh), (T_{h2}^1(f_h))(mh))$ *and we can see the graph of* $(T_h^1(f_h))(mh)$ *in Figure 4.28*

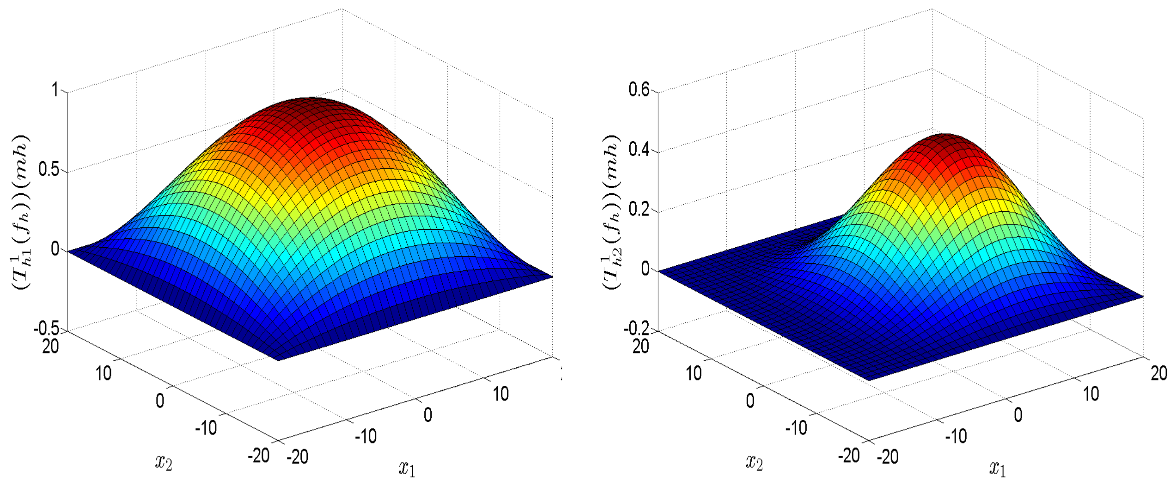


Figure 4.28: The graph of $f_h(x_1, x_2)$ from $\operatorname{Im} \mathbf{Q}_h^2$ after applying a discrete T_h^1-operator.

One can see in Figure 4.28, that the boundary values of $(T_h^1(f_h))(mh)$ *are equal* *to zero. They are satisfying the properties for any function* $f_h(mh) \in Im\,\mathbf{Q}_h^2 \iff tr_\Gamma T_h^1 f_h = 0$.

Finally, we will check the accuracy and stability of a discrete T_h^1*-operator by applying the discrete Cauchy-Riemann operator* $\boldsymbol{D}_{h,M}^1$ *to* $(T_h^1(f_h)(mh))$*, which is given by:*

$$\begin{pmatrix} D_h^{-2} & D_h^1 \\ -D_h^{-1} & D_h^2 \end{pmatrix} \begin{pmatrix} (T_{h1}^1(f_h))(mh) \\ (T_{h2}^1(f_h))(mh) \end{pmatrix} = \begin{pmatrix} D_h^{-2}(T_{h1}^1(f_h))(mh) + D_h^1(T_{h2}^1(f_h))(mh) \\ -D_h^{-1}(T_{h1}^1(f_h))(mh) + D_h^2(T_{h2}^1(f_h))(mh) \end{pmatrix}.$$

To achieve that result, the following equations are calculated:

$$D_h^{-2}(T_{h1}^1(f_h))(mh) + D_h^1(T_{h2}^1(f_h))(mh) = f_0(mh)$$

and

$$-D_h^{-1}(T_{h1}^1(f_h))(mh) + D_h^2(T_{h2}^1(f_h))(mh) = f_1(mh).$$

Or, we should check whether the result of the two equations below are equal to zero

$$f_0(mh) - \Big(D_h^{-2}(T_{h1}^1(f_h))(mh) + D_h^1(T_{h2}^1(f_h))(mh)\Big) = 0,$$

$$f_1(mh) - \Big(-D_h^{-1}(T_{h1}^1(f_h))(mh) + D_h^2(T_{h2}^1(f_h))(mh)\Big) = 0$$

In Figure 4.29, it is clear that the difference of $f_h(mh) - \boldsymbol{D}_{h,M}^1(T_h^1(f_h))(mh)$ *is equal to zero for all* $mh \in G_h$.

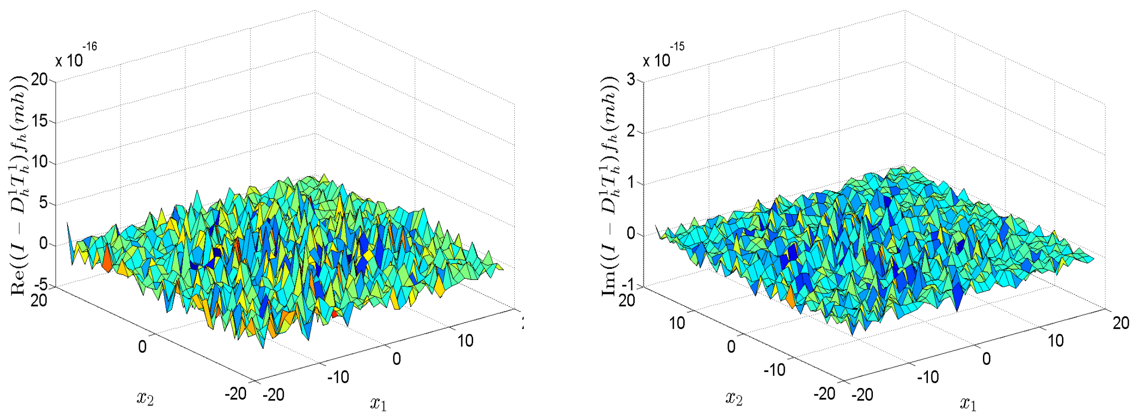


Figure 4.29: Plot to verify that T_h^1 is a right inverse of $\mathbf{D}_{h,M}^1$.

Concluding Remarks:

- When the recommended algorithms were implemented, it was discovered that not all results from the literature about discrete T_h^1 and T_h^2 are applicable to our work. Therefore, it was necessary to generalize discrete T_h^1 and T_h^2 operators. The properties of these operators should be theoretically and numerically checked.
- New formulas for the generalized boundary operators F_h^1 and F_h^2 were considered.
- A discrete Borel-Pompeiu formula was proved using our discrete T_h^1 operator and F_h^1 operator.
- For an implementation of programs, a suitable software was created and some tests were applied to check the correctness of our programs and to check the accuracy and the stability of it. For example, the theoretical problem of the integral representation of the fundamental solutions of the Laplace operator is already solved and all the properties were proved in the literature. While, on the computer the formula is not working well (except near to the origin). So, we found an applicable solutions to solve the problem numerically.
- Then, some examples of the discrete holomorphic complex-valued functions were constructed and the generalized boundary operator F_h^1 was applied to these functions.
- Finally, a discrete complex-valued function from $\operatorname{Im}\mathbf{Q}_h^2$ was constructed and a generalized discrete operator T_h^1 was applied to this discrete complex-valued function.

5 Numerical Results of Discrete Nonlinear Problems

The solutions of the p-Laplace equation and p-Dirac equation are the main tasks of this thesis. After introducing the discrete operators, we can define the discrete p-Dirac equation, using an iteration procedure to describe the solution of the discrete p-Dirac equation in the plane.

In Chapter 3, the existence and uniqueness of the solution of the nonlinear p-Dirac equation for $1 < p < 2$ in Theorem 11, for $p = 3/2$ in Theorem 12 and for $2 < p < 3$ in Theorem 14 has been proven. In the last chapter, we defined all important operators and constructed discrete complex-valued functions from $\operatorname{Im}\mathbf{Q}_h^2$, and now, we are able to solve the discrete p-Dirac equation in the uniform lattice by using the fixed-point iteration.

Under the same conditions as it is aforementioned in Chapter 3, we will find the approximate solutions of nonlinear discrete p-Dirac equations and give the error estimates related to the solutions where several boundary value problems are mainly considered dependent on p.
We discuss the nonlinear discrete p-Dirac equation in $\mathbb{R}_h^2(G_h(mh))$, i.e.

$$\begin{aligned} \mathbf{D}_{h,M}^1\left(|w_h(mh)|^{p-2}\, w_h(mh)\right) &= f_h(mh) \qquad \forall mh \in G_h(mh) && (5.1)\\ w_h(rh) &= 0 \qquad \forall rh \in \partial G_h = \Gamma_h, && (5.2) \end{aligned}$$

where $f_h(mh)$ is the complex-valued function and $f_h(mh) \in \operatorname{Im}\mathbf{Q}_h^2 \Longleftrightarrow \operatorname{tr}_\Gamma T_h^1 f_h = 0$. Using the result of Chapter 3, the problem (5.1) and (5.2) has a unique solution $w_h(mh) \in L_p(G_h)$ under some conditions, where $w_h(mh)$ is a complex-valued function

$w_h(mh) = (w_h^1(mh), w_h^2(mh))$ and

$$(T_h^1(f_h))(mh) = \left((T_{h1}^1(f_h))(mh), (T_{h2}^1(f_h))(mh)\right).$$

The sequence defined by

$$w_{h,n}(mh) = |w_{h,n-1}(mh)|^{2-p}\left(T_h^1(f_h)\right)(mh) \tag{5.3}$$

for $n \in \mathbb{N}$ converges in $L_p(G_h)$ to the unique solution.

Our main objective in the next part is to introduce a new theory and discuss the convergence of the iteration procedure and study the error estimate. The iteration number of convergence and the accuracy of the error will be investigated. We will consider several cases dependent on p and study each case.

5.1 The Discrete Nonlinear p-Dirac Problem for $p = 3/2$

Considering the discrete problem given by

$$w_{h,n}(mh) = |w_{h,n-1}(mh)|^{2-p}\left(T_h^1(f_h)\right)(mh),$$

where $w_h(mh)$ and $f_h(mh)$ are complex-valued functions, with the initial-value $w_{h,0}(mh) = (w_{h,0}^1(mh), w_{h,0}^2(mh))$ satisfying the condition in the Theorem 12

$$|w_{h,0}(mh)| \leq \left|\left(T_h^1(f_h)\right)(mh)\right|^{1/(2-p)}.$$

Let us consider

$$w_{h,0}^1(mh) = c\left|(T_h^1(f_h))(mh)\right|(T_{h1}^1(f_h))(mh)$$

and

$$w_{h,0}^2(mh) = c\left|(T_h^1(f_h))(mh)\right|(T_{h2}^1(f_h))(mh),$$

where c is a constant and the nonhomogeneous part $f_h(mh) = (f_0(mh),\ f_1(mh))$ is from $\operatorname{Im}\mathbf{Q}_h^2$. We will use the same discrete complex-valued function $f_h(mh)$ as in Example 7 Equation (4.27).

In general, the estimate of the difference between the n-th and the $(n-1)$-th iteration is not enough to stop the iteration. But in Banach fixed-point iteration, one can relate this difference to the true error by the a-posteriori estimates.

The iteration will be stopped at step n when

$$\frac{C}{1-C}\left\|w_{h,n}(mh)-w_{h,n-1}(mh)\right\|_{L_p(G_h)} \leq \varepsilon_1,$$

with the contractivity constant C. Then the iteration approximates the solution $w_{h,n}$ with accuracy ε_1. That means, we need to stop the iterative process at the first step n for which the discrete L_p norm of the difference between two consecutive iterates is at most $(1-C)\varepsilon_1/C$.

To do this task, the L_p norm of the difference of Equation (5.3) for $w_{h,n}$ and $w_{h,n-1}$ should be small enough, i.e.

$$\left\|w_{h,n}(mh)-w_{h,n-1}(mh)\right\|_{L_p(G_h)} \leq \varepsilon,$$

where $0<\varepsilon<1$, the minimum $\varepsilon = 10^{-16}$.

To calculate the initial starting point $w_{h,0}(mh) = (w^1_{h,0}(mh), w^2_{h,0}(mh))$, we should calculate

$$|(T^1_h(f_h))(mh)| = |(T^1_{h1}(f_h))(mh) + (T^1_{h2}(f_h))(mh)|.$$

Multiplying the result one time with $c\,(T^1_{h1}(f_h))(mh)$ and second time with $c\,(T^1_{h2}(f_h))(mh)$ to compute $w^1_{h,0}(mh)$ and $w^2_{h,0}(mh)$ respectively.

Next, we calculate $|w_{h,0}(mh)|^{2-p} = |w^1_{h,0}(mh)+w^2_{h,0}(mh)|^{2-p}$ and compute $w_{h,1}(mh) = (w^1_{h,1}(mh), w^2_{h,1}(mh))$ by solving the following equations

$$w^1_{h,n}(mh) = |w_{h,n-1}(mh)|^{2-p}\left(T^1_{h1}(f_h)\right)(mh)$$

$$w^2_{h,n}(mh) = |w_{h,n-1}(mh)|^{2-p}\left(T^1_{h2}(f_h)\right)(mh).$$

If the discrete L_p norm of the difference of Equation (5.3) for $w_{h,1}$ and $w_{h,0}$ is less than ε, i.e.

$$\left\|w_{h,1}(mh)-w_{h,0}(mh)\right\|_{L_p(G_h)} \leq \varepsilon$$

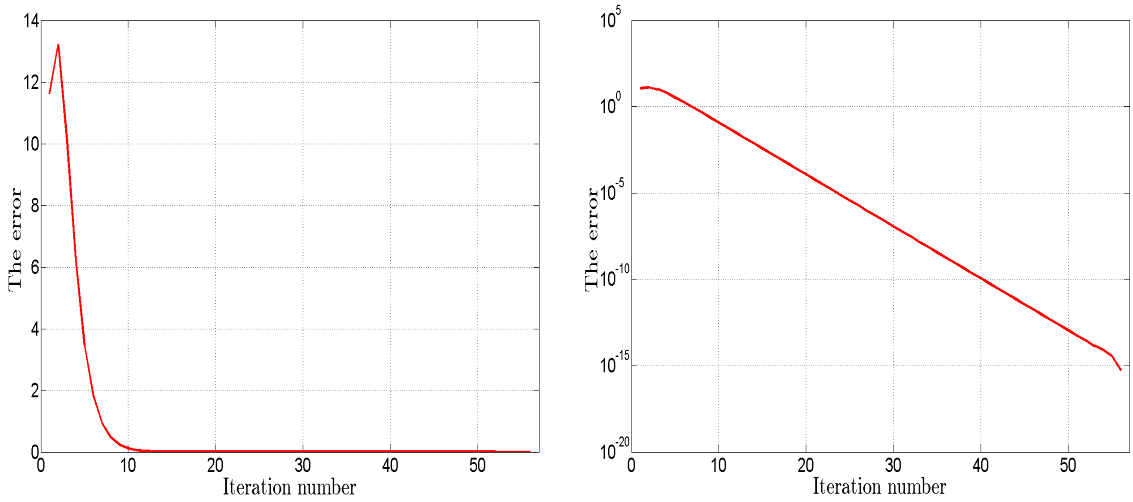
then $w_{h,1}(mh)$ is a sufficiently good approximation solution of the discrete equation, if not, then, similarly we repeat the procedure to compute $w_{h,2}(mh)$ and so on, until we get a sufficiently good approximation solution.

In Table 5.1, one can see if c is fixed and ε is changed, then the number of iterations will also change to achieve the approximation solution

$\|w_{h,n}(mh) - w_{h,n-1}(mh)\|_{L_p(G_h)} \leq \varepsilon$ $\varepsilon =$	Iteration number	Time of simulation
10^{-6}	27	50.4 seconds
10^{-9}	37	50.6 seconds
10^{-12}	47	52.1 seconds
10^{-16}	57	52.8 seconds

Table 5.1: Fix $c = 0.1$ and $p = 3/2$.

The plot of the convergence with the minimum ε can be seen in Figure 5.1.



(a) (b)

Figure 5.1: The convergence of the error for $p = 3/2$ and $c = 0.1$. Using the linear scale one time and the logarithm scale, we plot the iteration number versus the error (ε).

For a complete understanding of the behavior of the convergence, one should discuss the case if ε is fixed ($\varepsilon = 10^{-16}$) and the initial starting point is changed. By changing the value of the constant c, we will recognize that the number of iteration will change but it is still bounded as in Table 5.2

Initial value dependent on c $c=$	Iteration number	Time of simulation
0.0001	59	53.75 seconds
0.001	59	53.33 seconds
0.01	58	53.19 seconds
0.1	57	52.8 seconds
0.2	57	52.85 seconds
0.5	56	52.71 seconds
0.9	53	52.3 seconds
2	55	52.7 seconds
5	56	52.73 seconds
20	57	52.76 seconds
50	58	52.83 seconds
100	58	52.76 seconds
1000	58	52.83 seconds
10000	59	53.19 seconds

Table 5.2: Fix $\varepsilon = 10^{-16}$ and $p = 3/2$.

Even with high degree of accuracy for ε, the maximum iteration number of convergence is 59. From this point forward, we will ignore the time of calculation due to time redundancy.
Finally, the contractivity constant c_1 in this case, for $p = 3/2$, can be obtained by the equation below

$$c_1 = \frac{1}{2\sqrt{k_1}}, \text{ for } \frac{1}{4} < k_1 < 1.$$

We have considered the case $p = 3/2$ and studied the numerical results of the discrete p-Dirac problem. We now address the case $1 < p < 2$ in the next section.

Remark 18. *In the continuous p-Dirac problem, it has been shown that the contractivity constant of the Banach fixed-point theorem depends on p as it is given in Chapter 3 Theorem 11. When there are changes in the value of* $p \in (1,2)$ *and changes in the right hand side* f*, one can consider hundreds of cases, but this is not the purpose of the numerical study of the discrete p-Dirac problem for* $1 < p < 2$*.*
The main point here is if p is in proximity to 1 *or* 2 *because of contractivity, the number of iteration will be dependent on p* ($c_2 = 2 - p$)*. One can ask what will happen if p is near* 1 *or* 2*. Is the iteration number of the convergence to the approximation solution small or too large? To accomplish this, we consider the next section.*

5.2 The Discrete Nonlinear p-Dirac Problem for $1 < p < 2$

Considering the discrete nonlinear p-Dirac Equation (5.1) and (5.2) for $1 < p < 2$, we will solve the discrete system (5.3) iteratively

$$w^1_{h,n}(mh) = |w_{h,n-1}(mh)|^{2-p}\left(T^1_{h1}(f_h)\right)(mh)$$

$$w^2_{h,n}(mh) = |w_{h,n-1}(mh)|^{2-p}\left(T^1_{h2}(f_h)\right)(mh).$$

Let us consider $w_{h,0}(mh) = (w^1_{h,0}(mh), w^2_{h,0}(mh))$ be the initial-value which satisfies the condition in the Theorem 11

$$\left|\left(T^1_h(f_h)\right)(mh)\right|^{\frac{1}{p-1}} \leq |w_{h,0}(mh)| \leq \left|\left(T^1_h(f_h)\right)(mh)\right| \leq 1$$

where $f_h(mh)$ is a complex-valued function from $\operatorname{Im}\mathbf{Q}^2_h$, which we discussed in Example 7. Therefore, Theorem 11 is applicable and we can conclude the approximation solution.

We fix the constant c in the initial value $w_{h,0}(mh)$ and fix the value of p and compute $w_{h,1}(mh) = (w^1_{h,1}(mh), w^2_{h,1}(mh))$ from system (5.3). Next we compute the discrete L_p norm of the difference of Equation (5.3) for $w_{h,1}$ and $w_{h,0}$, if it is less than ε, i.e.

$$\|w_{h,1}(mh) - w_{h,0}(mh)\|_{L_p(G_h)} \leq \varepsilon,$$

then $w_{h,1}(mh)$ is a sufficiently good approximation solution of the discrete equation. If not, then similarly we repeat the procedure to compute $w_{h,2}(mh)$ and so on, until we get the approximation solution. In this calculation, ε should be small enough.

Firstly, we fix $p = 1.005$ which is close to 1 and start the iteration procedure with initial value satisfying the condition above with $c = 0.1$. After 6421 iteration we will get the best approximation solution of the discrete nonlinear problem with $\varepsilon = 10^{-14}$. The time of simulation is still less than one minute even if the number of iteration is 6421. Due to this fact, we will ignore the time of simulation.
In Figure 5.2, we plot the curve of the error estimate versus the number of iteration using the logarithm scale on the y-axis to recognize the large range of quantities and the continuation of values near 0 as it is seen

We are interested in seeing what will happen if the nonlinear p-Dirac problem is

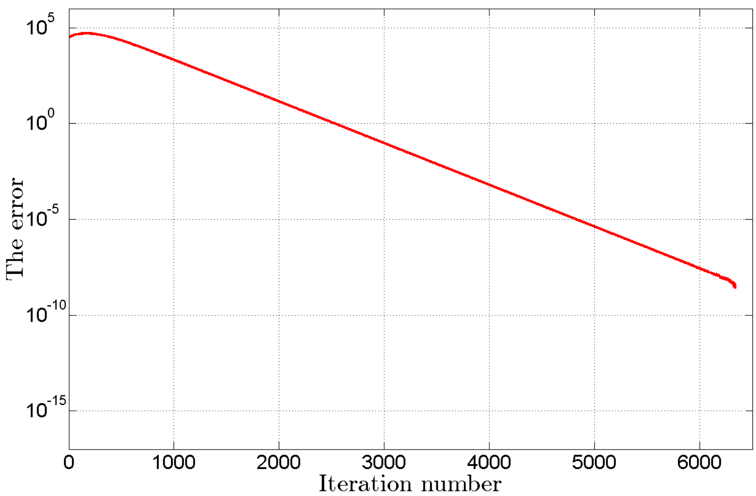


Figure 5.2: The convergence of the error for $p = 1.005$ and $c = 0.1$. Plot of the iteration number versus ε using the logarithm scale.

near the linear case. Let us consider $p = 1.999$, we will start the iteration procedure with the same right hand side $f_h(mh)$ and with the initial value satisfying the condition in Theorem 11. We recognize that we are able to reach the sufficiently good approximation solution after a few iteration sequences. In Figure 5.3, one can see the plot of the error versus the number of iteration, after fixing the constant $c = 0.1$

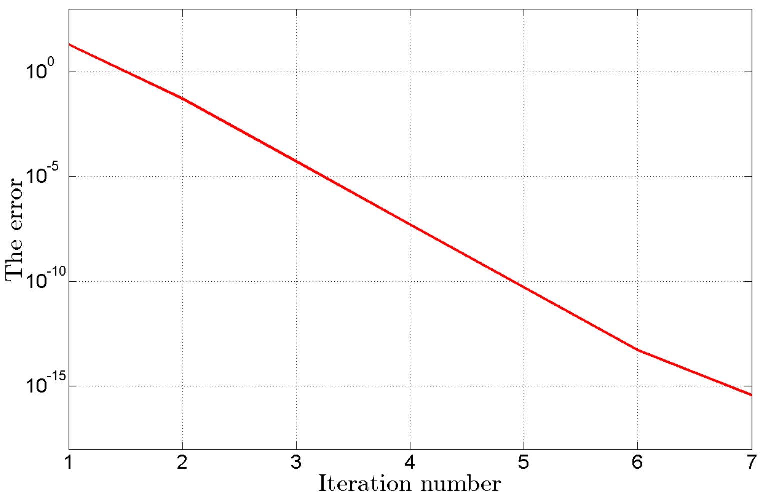


Figure 5.3: The convergence of the error for $p = 1.999$ and $c = 0.1$. Plot of the iteration number versus ε using the logarithm scale.

In Figure 5.3, one can see that the absolute error is less than $\varepsilon = 10^{-15}$.

We still have interest in solving the discrete nonlinear p-Dirac equation with different value of p, but also those values are near 1 and 2. The result of these investigation can be seen in Table 5.3 below

p	Initial value $c =$	$\|w_{h,n}(mh) - w_{h,n-1}(mh)\|_{L_p(G_h)} \leq \varepsilon$ $\varepsilon =$	Iteration number
1.005	0.1	10^{-14}	6421
1.01	0.1	10^{-14}	3272
1.1	0.1	10^{-16}	345
1.1	0.9	10^{-16}	316
1.9	0.1	10^{-16}	19
1.9	0.5	10^{-16}	18
1.99	0.1	10^{-16}	9
1.999	0.1	10^{-16}	7

Table 5.3

One can recognize from Table 5.3 that when p has reached lateral limit 1 the iteration number of convergence to the solution is too great and when p is too proximate to 2, as in the linear case, the number of iteration is at its lowest relevance for our study's purpose, as it is seen in Figure 5.4

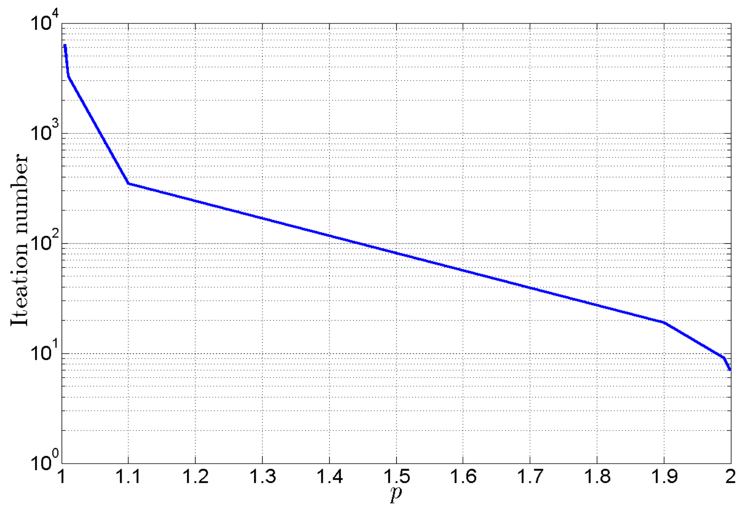


Figure 5.4: Plot of the iteration number versus p.

There are many applications of the p-Dirac problem for $p = 1$, this gives a reason to study the limit case as p tends to 1. We explain what will happen for a p-Dirac equation if p tends to 1. To gain a better understanding of this approximation, it is required to study a few numbers which are close to 1 with the same starting value

$w_{h,0}(mh)$ and $\varepsilon = 10^{-6}$ for all p^s. In Figure 5.5, one can see all convergence's curves for the different p^s in one plot.

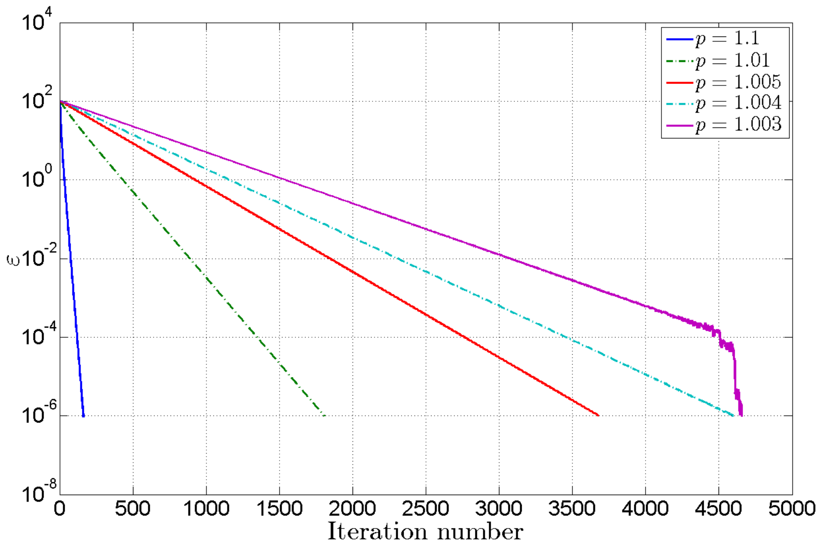


Figure 5.5: The convergence of p'^s in proximity to 1.

We calculate the discrete L_1 norm of the difference of the solution for each two values

$$\left\| w_{h,n(p_i)}(mh) - w_{h,n(p_j)}(mh) \right\|_{L_1(G_h)},$$

where $n(p_i) \neq n(p_j)$, for example $n(p_i)$ is the number of iteration to reach the solution for $p = 1.1$ and $n(p_j)$ is the number of iteration to reach the solution for $p = 1.01$, we compute the result of the all values and it converges to zero, as shown in Table 5.4

$\left\| w_{h,n(1.1)}(mh) - w_{h,n(1.01)}(mh) \right\|_{L_1(G_h)}$	$4.943063072158984e+11$
$\left\| w_{h,n(1.01)}(mh) - w_{h,n(1.005)}(mh) \right\|_{L_1(G_h)}$	$1.029152235963072e+09$
$\left\| w_{h,n(1.005)}(mh) - w_{h,n(1.004)}(mh) \right\|_{L_1(G_h)}$	$2.773382799636123e+07$
$\left\| w_{h,n(1.004)}(mh) - w_{h,n(1.003)}(mh) \right\|_{L_1(G_h)}$	$2.444842519761996e+04$

Table 5.4

In the next section, we are considering the discrete nonlinear p-Dirac problem for $2 < p < 3$, which is being studied by many people. We want to mention that the functional theory is successful to solve the p-Dirac problem for $2 < p < 3$ as we have proved in Chapter 3 Theorem 14. It has been proven that there is a unique solution

$w_h(mh) \in L_p(G_h)$ for the main problem (5.1) and (5.2). Under the condition

$$k_2 \left|\left(T_h(f_h)\right)(mh)\right|^{\frac{3-p}{1-(p-2)^2}} \leq |w_{h,0}(mh)| \leq \left|\left(T_h(f_h)\right)(mh)\right|^{\frac{3-p}{1-(p-2)^2}}, \tag{5.4}$$

with $(p-2)^{\frac{2}{1-(p-2)^2}} < k_2 < 1$. The sequence defined by

$$w_{h,2n} = \frac{\left(T_h(f_h)\right)(mh)}{\left|\left(T_h(f_h)\right)(mh)\right|^{p-2}} \left|w_{h,2(n-1)}\right|^{(p-2)^2} \tag{5.5}$$

for $n \in \mathbb{N} \setminus \{0\}$ converges in $L_p(G_h)$ to the unique solution.

5.3 The Discrete Nonlinear p-Dirac Problem for $2 < p < 3$

Considering the discrete p-Dirac equation (5.1) and (5.2) for $2 < p < 3$, we will solve the discrete system (5.5) iteratively

$$w^1_{h,2n} = \frac{\left(T^1_{h1}(f_h)\right)(mh)}{\left|\left(T^1_h(f_h)\right)(mh)\right|^{p-2}} \left|w_{h,2(n-1)}\right|^{(p-2)^2}$$

$$w^2_{h,2n} = \frac{\left(T^1_{h2}(f_h)\right)(mh)}{\left|\left(T^1_h(f_h)\right)(mh)\right|^{p-2}} \left|w_{h,2(n-1)}\right|^{(p-2)^2}.$$

We stop the iteration procedure when the L_p norm of the difference of Equation (5.5) for $2n$ and $2(n-1)$ is small enough, i.e.

$$\left\|w_{h,2n}(mh) - w_{h,2(n-1)}(mh)\right\|_{L_p(G_h)} \leq \varepsilon$$

Let us start with the initial-value $w_{h,0}(mh) = (w^1_{h,0}(mh), w^2_{h,0}(mh))$ which is satisfying the condition (5.4). Let us consider

$$w^1_{h,0}(mh) = \left((p-2)^{\frac{2}{1-(p-2)^2}} + c\right) \left|(T^1_h(f_h))(mh)\right|^{\frac{(p-3)(p-2)}{1-(p-2)^2}} (T^1_{h1}(f_h))(mh)$$

and

$$w^2_{h,0}(mh) = \left((p-2)^{\frac{2}{1-(p-2)^2}} + c\right) \left|(T^1_h(f_h))(mh)\right|^{\frac{(p-3)(p-2)}{1-(p-2)^2}} (T^1_{h2}(f_h))(mh),$$

the right hand side $f_h(mh) \in \operatorname{Im} \mathbf{Q}^2_h$, which we discussed in Example 7.
We fix the constant c in the initial value $w_{h,0}(mh)$ and fix the value of p and compute

$w_{h,2}(mh) = (w^1_{h,2}(mh), w^2_{h,2}(mh))$ from the system (5.5). Then, we compute the discrete L_p norm of the difference of Equation (5.5) for $w_{h,2}$ and $w_{h,0}$, if it is less than ε, i.e.

$$\|w_{h,2}(mh) - w_{h,0}(mh)\|_{L_p(G_h)} \leq \varepsilon$$

then $w_{h,2}(mh)$ is a sufficiently good approximation solution of the discrete p-Dirac problem. If not, then similarly, we repeat the procedure to compute $w_{h,4}(mh)$ and so on, until we get the approximation solution. In this calculation, ε should be 10^{-16} at its lowest value.

We solve the discrete nonlinear p-Dirac problem with different value of $2 < p < 3$ and different initial value. The result of this investigation can be seen in Table 5.5.

p	Initial value $c =$	$\|w_{h,n}(mh) - w_{h,n-1}(mh)\|_{L_p(G_h)} \leq \varepsilon$ $\varepsilon =$	Iteration number
2.0001	0.1	10^{-16}	4
2.001	0.1	10^{-16}	5
2.01	0.1	10^{-16}	6
2.1	0.1	10^{-16}	11
2.1	0.5	10^{-16}	10
2.5	0.1	10^{-16}	30
2.9	0.5	10^{-16}	172
2.9	0.1	10^{-16}	178
2.99	0.1	10^{-16}	1705
2.999	0.1	10^{-16}	15748
2.9999	0.1	10^{-16}	144703*

* Simulation time = 126.89 seconds.

Table 5.5

In Figure 5.6, one can see that the iteration number to reach the solution is small when the value of p is close to 2 (which is the linear case) and too large when p is close to 3.

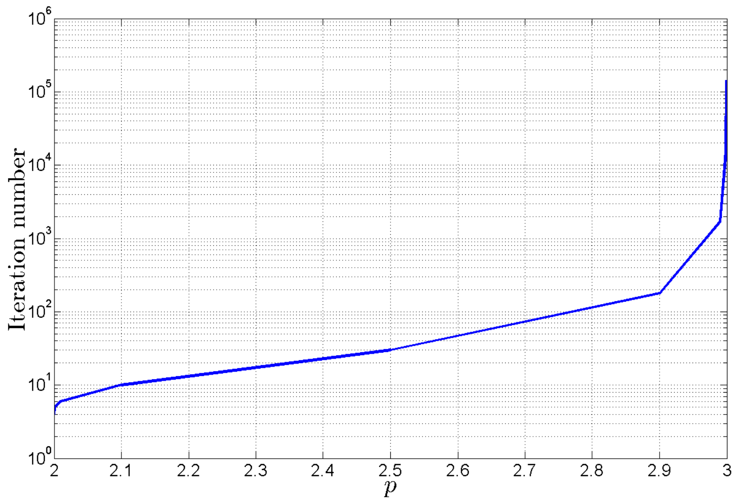
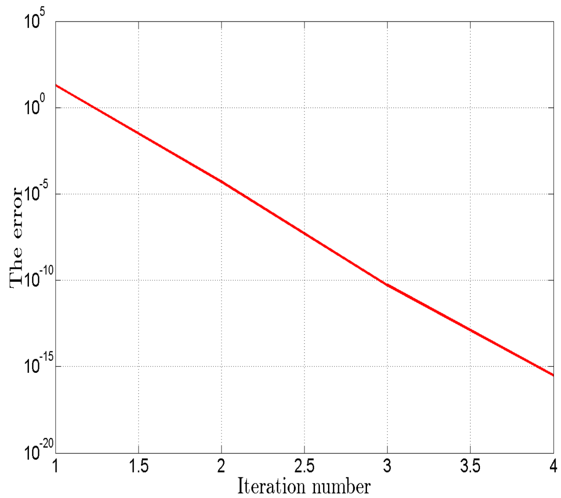
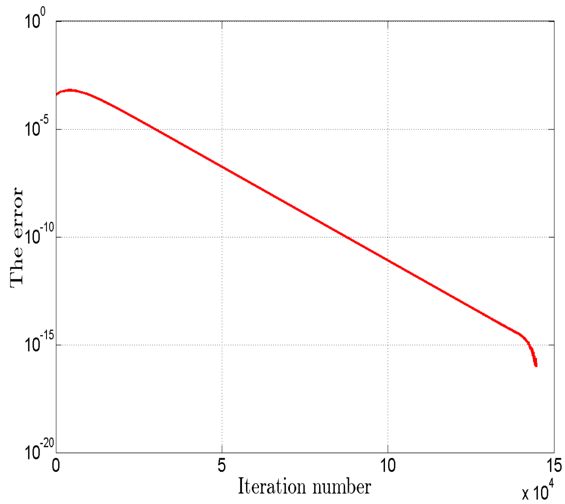


Figure 5.6: Plot of the iteration number versus p.

From Table 5.5, we plot the convergence of the error for the discrete p-Dirac problem, one time by using the result of the first row of the table, which is an important case near the linear problem, where $p = 2.0001$ as it is seen in 5.7a, and the second part of Figure 5.7b is of the last row of the table, in which the number of iterations is large when p is close to 3.



(a) $p = 2.0001$



(b) $p = 2.9999$

Figure 5.7: The convergence of the error.

Remark 19. *One can recognize from the numerical calculation of solving the discrete p-Dirac problem iteratively that when the non-linear boundary value problem is near* *the linear problem from both sides to $p = 2$ the necessary number of iterations is small. Theoretically it is dependent on the contractivety constant, which is depending on p. Numerically it is clear in Section 5.2 and Section 5.3.*

It is necessary here to discuss the a-priori and a-posteriori error estimates in the L_p norm. In our analysis, we need a-priori estimates, which is a-priori upper bound for the difference between the exact solution and the numerical solution.

5.4 A-priori and A-posteriori Error Estimates

Usually, a-priori error and a-posteriori error estimates are important in assessing the accuracy of the simulation and in helping to determine an adaptive strategy to improve the accuracy where needed. So, the a-posteriori estimation gives us a direct stopping criterion for the iterative approximation of fixed points by Banach fixed-point iteration, while the a-priori estimation indirectly offers a stopping criterion. The a-priori estimate suffices to show that the system has a unique solution. This estimate gives a certain presentation on how the error decreases. A-priori estimate is used in order to obtain upper bounds on the number of iteration steps. We have the a-priori error estimate, which is described by

$$\|w_{exact}(mh) - w_{h,n}(mh)\|_{L_p(G_h)} \leq \frac{C^n}{1-C} \|w_{h,1}(mh) - w_{h,0}(mh)\|_{L_p(G_h)} \tag{5.6}$$

and the a-posteriori error estimate

$$\|w_{exact}(mh) - w_{h,n}(mh)\|_{L_p(G_h)} \leq \frac{C}{1-C} \|w_{h,n}(mh) - w_{h,n-1}(mh)\|_{L_p(G_h)} \tag{5.7}$$

for all $n \in \mathbb{N}$, with the contractivity constant C.

One of the most important questions is when to stop the iteration procedure. One can stop the iterations too early when the norm of the error is larger than the tolerance, or too late to obtain the required accuracy. An important aspect of the Banach fixed-point iterative method is to determine when the iterations should be stopped. The a-priori estimate is used in order to obtain upper bounds on the number of iteration steps, which is necessary to achieve a desired accuracy error.

For a given ε, by the a-priori estimate one can calculate M

$$M \geq \frac{\log\left((1-C)\varepsilon / \left\| w_{h,1}(mh) - w_{h,0}(mh) \right\|_{L_p(G_h)}\right)}{\log(C)}, \tag{5.8}$$

which is the maximum number of necessary iteration steps.

We know the exact solution of the p-Dirac problem for $2 < p < 3$; we will discuss the a-priori and a-posteriori error estimates for this case in the next example.

Example 8. *Considering the discrete p-Dirac problem for $2 < p < 3$, as it is in Section 5.3, with the initial-value $w_{h,0}(mh) = (w^1_{h,0}(mh), w^2_{h,0}(mh))$ which is satisfying the condition (5.4) and the right hand side $f_h(mh) \in Im\,\mathbf{Q}^2_h$, which we discussed in Example 7.*

We can find $w_{h,2}(mh)$ from the system (5.5) which is the first iteration step. (Remember that in Subsection 3.4.1 we study the p-Dirac problem for $2 < p < 3$ in term of g_n and g_n is given by $g_0 = w_0$, $g_1 = w_2$, ..., $g_n = w_{2n}$.)

The error estimate is useful for figuring out how many iterations we need. For this, we need to know the contraction constant c_3 (typically we get this constant from Section 3.4.1 Equation (3.50)), which is given by

$$c_3 = \frac{(p-2)^2}{{k_2}^{1-(p-2)^2}},$$

where the constant k_2 should satisfy the condition

$$(p-2)^{\frac{2}{1-(p-2)^2}} < k_2 < 1. \tag{5.9}$$

To calculate the contractivity constant c_3, we choose constant k_2 where

$$k_2 = (p-2)^{\frac{2}{1-(p-2)^2}} + a,$$

with $a > 0$. For a different value of $p \in (2,3)$, we will choose $a = 0.1$, then the ${k_2}^s$ satisfies the condition (5.9). Now, we can calculate the contractivity constant c_3 for these p^s. By using Inequality (5.8), we will compute the maximum number of necessary iteration steps (M) for the different p.

In Table 5.6, we can see the necessary maximum number of iteration steps and the true iteration number of our calculation.

p	Maximum number of iteration $M =$	Iteration number
2.99	7895	1705
2.9	776	178
2.8	366	87
2.7	226	56
2.6	152	40
2.5	104	30
2.4	71	23
2.3	47	18
2.2	29	14
2.1	16	11

Table 5.6

The expression $\frac{{c_3}^n}{1-c_3}\left\|w_{h,2}(mh) - w_{h,0}(mh)\right\|_{L_p(G_h)}$ *in (5.6) can be determined for every* n *before starting the iteration procedure (only* $w_{h,2}(mh)$ *is required), thus giving an a-priori error estimate for the exact error.*
Let us consider three values of p*,* $p = 2.1$*,* $p = 2.5$ *and* $p = 2.9$ *and let us choose the constant* k_2

$$k_2 = (p-2)^{\frac{2}{1-(p-2)^2}} + a,$$

where $a > 0$*; for these three values of* p *we will choose* $a = 0.1$*.*

Calculating the exact error for each p*, which is the* L_p *norm of the difference between the exact and the approximation solutions*

$$exact\ error = \left\|w_{exact}(mh) - w_{h,n}(mh)\right\|_{L_p(G_h)}.$$

Computing the a-priori error estimate for each p *and plot the graph of the exact error and a-priori error estimate in Figure 5.8.*

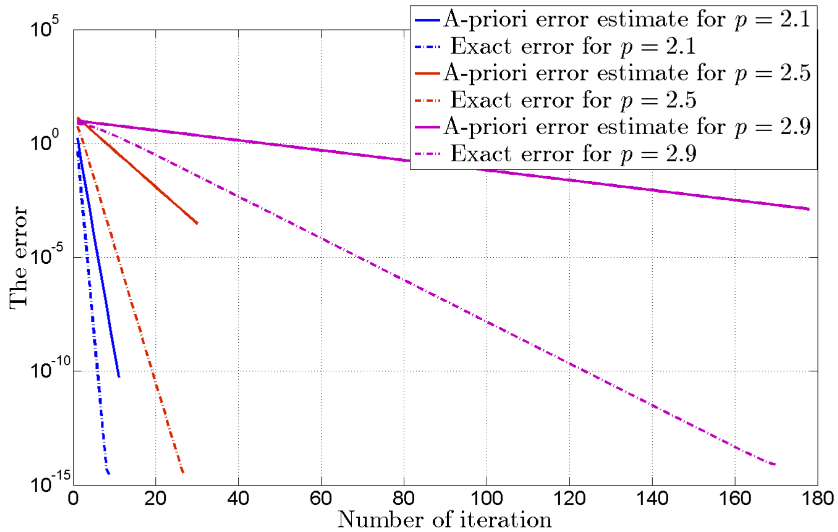


Figure 5.8: A-priori error estimate and exact error.

Choosing other p^s, $p = 2.9$ and $p = 2.99$ and computing the a-posteriori error estimate for each p. One can see the exact error for these p^s and the a-posteriori error estimate in Figure 5.9, one can see that the a-posteriori estimation is better than the a-priori estimation.

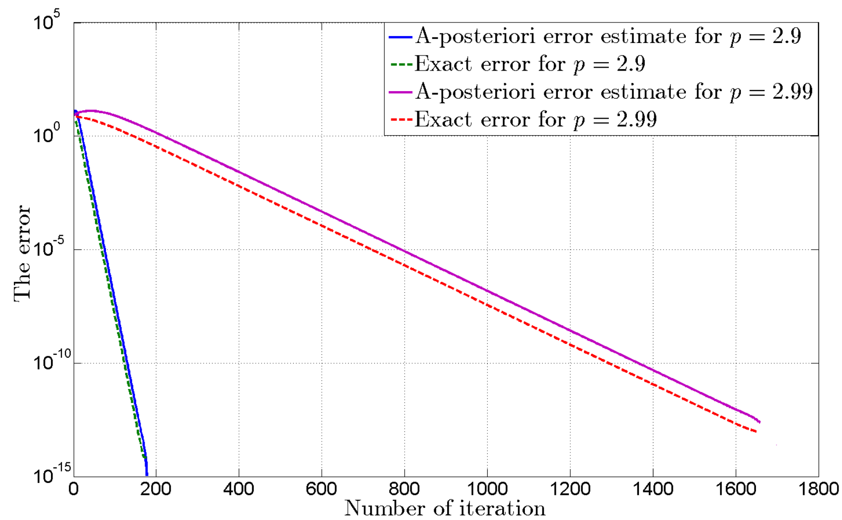


Figure 5.9: A-posteriori error estimate and exact error.

These calculations are done with highest accuracy where $\varepsilon = 10^{-16}$. Our estimates given have better results comparatively to the a-priori error estimate. From Figure 5.8 our desired result for each p can be clearly seen.

6 Summary, Conclusions, Results and Future Work

In the present thesis, it is shown that the p-Poisson equation with certain Neumann-type boundary conditions can be transferred to a Dirichlet boundary value problem for the p-Dirac equation. The main tool was an operator calculus taken from hypercomplex function theory. The obtained p-Dirac equation could be solved iteratively by a Banach's fixed-point iteration. We proved the existence and uniqueness of the solution of the main boundary problem. This was possible for all $p \in (1,2)$ and as an estimate, for the contractivity constant, the value $c_2 = 2 - p$ was obtained. The case $p = 3/2$ plays a special role because, in this case, no bound for the right hand side $T_G f$ was necessary. For $p \in (1,2)$, we obtained a two-sided norm estimate for the solution formulated as $\left\| |T_G f|^{\frac{1}{p-1}} \right\|_{0,p} \leq \|w\|_{0,p} \leq \|T_G f\|_{0,p}$. This was also possible for all $p \in (2,3)$ and as an estimate for the contractivity constant, the value $c_3 = \frac{(p-2)^2}{{k_2}^{1-(p-2)^2}}$, for $(p-2)^{\frac{2}{1-(p-2)^2}} < k_2 < 1$ was obtained. These results support the idea to reduce, as in the linear case, the study of the second order differential equation to two equations of first order, where we can apply function theoretic methods. For the first step, discussed in this work, this is visible.

The main goal of this work is to apply the theory of discrete functions to the solution of the p-Dirac problem. The sub-goals include approximating the solutions to the boundary value problems by adapting finite difference schemes. The discrete fundamental solution of the Cauchy-Riemann operator and its adjoint in the plane based on the discrete fundamental solution of Laplacian were calculated. It was discovered that not all results from the literature about discrete T_h^1 and T_h^2 operators (which are the right inverse of the Cauchy-Riemann operator and its adjoint) are applicable to our work. Therefore, it was necessary to generalize discrete T_h^1 and T_h^2 operators. The properties of these operators should be theoretically and numerically checked. New formulas for the generalized boundary operators F_h^1 and F_h^2 were considered.

A generalized discrete Borel-Pompeiu formula was proved using discrete T_h^1 operator and F_h^1 operator. Furthermore, a discrete complex-valued function from $\operatorname{Im}\mathbf{Q}_h^2$ was constructed and a generalized discrete T_h^1-operator was applied to this discrete complex-valued function.

Finally, the nonlinear discrete p-Dirac equation in the uniform lattice was solved using Banach's fixed-point iteration and the convergence of the iteration procedure and the error estimate were discussed. Several cases of the discrete nonlinear p-Dirac problem dependent on p were studied.

This work is extending the applicability to the use of hypercomplex operator calculus to nonlinear equations with other nonlinearities than only products of values and partial derivatives. Specifically, the results of this work allow to extend the results for the nonlinear boundary value problems like Navier-Stokes equations from [Gürlebeck and Sprößig, 1997] and [Gürlebeck et al., 2016] to the case of non-Newtonian fluids. The discrete operator calculus (Teodorescu transform, boundary operator and Borel-Pompeiu formula) plays an essential role and one can use them to solve many discrete nonlinear problems in the complex plane. Another benefit of the proposed approach is that we get strong solutions of the studied operator equations. The numerical studies for the p-Dirac equation were performed for simplicity only for the two dimensional case. A generalization of the numerical part to higher dimensions is a natural task. There are different ways to develop this work: The next step with respect to calculating u from $Du = w$ is yet to be solved. Another task is to consider p-Poisson equation for the case $p \geq 3$ and then solve the problem.

Finally, one can extend this work by considering some real nonlinear problems related to p-Laplacian and solve them numerically using the results of this thesis. For that purpose, we are interested in studying rockfill dam problem and image processing in future works.

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