Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 23/12

The rank-one perturbation problem for linear relations

Baragaña, Itziar; Martínez Pería, Francisco; Roca, Alicia; Trunk, Carsten

Dezember 2023

URN: urn:nbn:de:gbv:ilm1-2023200305

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
https://www.tu-ilmenau.de/mathematik/

ilmedia

The rank-one perturbation problem for linear relations

Itziar Baragaña[a,1], Francisco Martínez Pería[b,c,2,3], Alicia Roca [d,1,3,*], Carsten Trunk[e,3]

[a] *Departamento de Ciencia de la Computación e I.A., Universidad del País Vasco UPV/EHU, Apartado 649, 20080 Donostia-San Sebastián, Spain*
[b] *Instituto Argentino de Matemática "Alberto P. Calderón" Saavedra 15, Piso 3 (1083) Buenos Aires, Argentina*
[c] *Centro de Matemática de La Plata – Facultad de Ciencias Exactas Universidad Nacional de La Plata, Calles 50 y 115 (1900) La Plata, Argentina*
[d] *Departamento de Matemática Aplicada, IMM, Universitat Politècnica de València, 46022 Valencia, Spain*
[e] *Department of Mathematics, Technische Universität Ilmenau, Postfach 100565, 98648 Ilmenau, Germany*

Abstract

We use the recently introduced Weyr characteristic of linear relations in $\mathbb{C}^n$ and its relation with the Kronecker canonical form of matrix pencils to describe their dimension. Then, this is applied to study one-dimensional perturbations of linear relations.

Keywords: linear relations, perturbation, Weyr characteristic, Kronecker canonical form
2020 MSC: 15A21, 15A22, 47A06

---

1. Introduction

Linear relations are a natural generalization of linear operators and they can be traced back to [1], see also [7]. Linear relations in $\mathbb{C}^n$ are nothing else than (linear) subspaces of $\mathbb{C}^n \times \mathbb{C}^n$, but there is a well developed spectral theory behind them which is mainly expressed in terms of (proper) eigenvalues, Jordan and singular chains, and multishifts, see [1, 7, 4, 5, 6, 11].

Recently, the notion of the Weyr characteristics for a linear relation were introduced, both as a tool for developing a canonical form for linear relations in

*Corresponding author
Email addresses: `itziar.baragana@ehu.eus` (Itziar Baragaña), `francisco@mate.unlp.edu.ar` (Francisco Martínez Pería), `aroca@mat.upv.es` (Alicia Roca), `carsten.trunk@tu-ilmenau.de` (Carsten Trunk)

[1] Partially supported by grant PID2021-124827NB-I00 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe" by the "European Union".
[2] Partially supported by CONICET PIP 11220200102127CO and UNLP 11X974.
[3] Partially supported by Deutsche Forschungsgemeinschaft DFG TR903/24-1.

finite dimensional vector spaces [6] and also to relate the Kronecker invariants of a matrix pencil with the invariants of its kernel and range representations [11]. Given a matrix pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$, its kernel representation is defined by $E^{-1}F := N(\begin{bmatrix} F & -E \end{bmatrix}) \subseteq \mathbb{C}^m \times \mathbb{C}^m$, where $N(X)$ stands for the kernel of a matrix X. Analogously, its range representation is given by $FE^{-1} = R\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) \subseteq \mathbb{C}^n \times \mathbb{C}^n$, where $R(X)$ stands for the range of a matrix X.

We show that the Weyr characteristics of the kernel and range representations of a pencil $P(s)$ can be recovered from the Weyr characteristic of $P(s)$. Conversely, given a linear relation S, it is possible to find matrix pencils of different sizes whose range or kernel representations are S. We also show that the dimension $d = \dim S$ is the minimal number of columns of a pencil necessary to describe S as the range representation of it. Also, the minimal number of rows necessary to describe S as the kernel representation of a matrix pencil is $2n - d$. Hence, from the Weyr characteristic of the linear relation we are able to recover the Weyr characteristic of the pencil, which depends on the size of the chosen pencil.

Moreover, in [11] new perturbation results for the Kronecker form of matrix pencils under rank-one perturbations were derived from perturbation results for the Weyr characteristic of linear relations given in [12].

The main goal of this paper is to use this relationship one more time, but in the opposite direction. Now, given linear relations S and T in $\mathbb{C}^n$, we would like to determine if there exist linear relations $\tilde{S}, \tilde{T} \in \mathbb{C}^n \times \mathbb{C}^n$ which are strictly equivalent to S and T, respectively, and such that $\tilde{S}$ and $\tilde{T}$ are one dimensional perturbations of each other in the sense of [11]. To do so, we make use of recent results from [2, 3] and [9].

The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3 we present some properties of linear relations, mainly of the kernel and range representations of matrix pencils. In Section 4 we state the problem to be studied, and we relate it to a matrix pencil completion problem. Section 5 contains some known results about matrix pencil completion problems which are used later, and we show that they can be stated in terms of the Weyr characteristics of the pencils involved. In Section 6 we obtain necessary conditions to solve the problem, and in Section 4 we solve the problem completely. Finally, in the Appendix we include some technical results.

2. Preliminaries

Let $\mathbb{C}$ be the field of complex numbers, and $\bar{\mathbb{C}}$ the extended field $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. $\mathbb{C}[s]$ denotes the ring of polynomials in the indeterminate s with coefficients in $\mathbb{C}$, $\mathbb{C}^{n\times m}$ is the vector spaces of $n \times m$ matrices over $\mathbb{C}$, and $\mathbb{C}[s]^{n\times m}$ is the bimodule of $n \times m$ matrices over $\mathbb{C}[s]$, respectively. $\mathrm{Gl}_n(\mathbb{C})$ is the general linear group of invertible matrices in $\mathbb{C}^{n\times n}$.

Given a matrix $X \in \mathbb{C}^{n\times m}$, $R(X) \subseteq \mathbb{C}^n$ is the subspace spanned by the columns of S and $N(X) \subseteq \mathbb{C}^m$ is the kernel of X.

We call partition to a finite or infinite sequence of nonnegative integers $a = (a_1, a_2, \ldots)$, almost all being zero, such that $a_1 \geq a_2 \geq \ldots$. The number of nonzero components of a is the length of a (denoted $\ell(a)$) and $|a|$ is the sum of the components of a, i.e., $|a| = \sum_{i=1}^{\ell(a)} a_i$. Given a finite partition $a = (a_1, a_2, \ldots, a_n)$, if necessary, we take $a_i = 0$ if $i > n$. We identify two partitions that differ only in the number of zero components. The conjugate of a partition $a = (a_1, a_2, \ldots)$ is the partition $\bar{a} = (\bar{a}_1, \bar{a}_2, \ldots)$, where $\bar{a}_k := \#\{i : a_i \geq k\},\ k \geq 1$.

We will also work with finite sequences of integers $\mathbf{c} = (c_1, c_2, \ldots, c_m)$ such that $c_1 \geq c_2 \geq \ldots \geq c_m$. When necessary, we take $c_i = +\infty$ if $i < 1$ and $c_i = -\infty$ if $i > m$. Observe that a finite sequence has a fixed number of components. As before, $|\mathbf{c}| = \sum_{i=1}^{m} c_i$. All along this paper, the sequences of integers involved have nonnegative components. The conjugate of a finite sequence of nonnegative integers $\mathbf{c} = (c_1, \ldots, c_m)$ is the conjugate partition of the partition $c = (c_1, \ldots, c_m, 0, \ldots)$. When necessary, we define the term $\bar{c}_0 = \#\{i : c_i \geq 0\} = m$.

A polynomial matrix of the form $P(s) = sE - F,\ E, F \in \mathbb{C}^{n\times m}$, is a matrix pencil. For basic notions on matrix pencils we refer to [10, Chapter XII]. Two matrix pencils $P_1(s) = sE_1 - F_1$ and $P_2(s) = sE_2 - F_2$ in $\mathbb{C}[s]^{n\times m}$ are strictly equivalent, denoted $P_1(s) \overset{s.e.}{\sim} P_2(s)$, if there exist invertible matrices $U \in \mathrm{Gl}_n(\mathbb{C})$, $V \in \mathrm{Gl}_m(\mathbb{C})$, such that $P_2(s) = UP_1(s)V$.

Given a matrix pencil $P(s) \in \mathbb{C}[s]^{n\times m}$, the normal rank of $P(s)$, denoted $\mathrm{rank}(P(s))$, is the order of the largest nonidentically zero minor of $P(s)$, i.e., it is the rank of $P(s)$ considered as a matrix on the field of fractions of $\mathbb{C}[s]$. The spectrum of the pencil $P(s) = sE - F$, denoted $\Lambda(P(s))$, is defined as $\Lambda(P(s)) = \{\lambda \in \bar{\mathbb{C}} : \mathrm{rank}(P(\lambda)) < \mathrm{rank}(P(s))\}$, where we agree that $P(\infty) = E$. The elements $\lambda \in \Lambda(P(s))$ are the eigenvalues of $P(s)$. If $\mathrm{rank}(P(s)) = r$ and $\Lambda(P(s)) = \{\lambda_1, \ldots, \lambda_\ell\}$, then the Kronecker invariants of $P(s)$ are ℓ partitions $n(\lambda_i) = (n_1(\lambda_i), n_2(\lambda_i), \ldots)$, where $n(\lambda_i)$ is the Segre characteristic at λ_i of $P(s)$ and $\ell(n(\lambda_i)) \leq r$, and two sequences of nonnegative integers, $\epsilon = (\epsilon_1, \ldots, \epsilon_{m-r})$, $\eta = (\eta_1, \ldots, \eta_{n-r})$, called column minimal indices and row minimal indices of $P(s)$, respectively. They satisfy:

$$\sum_{i=1}^{\ell} |n(\lambda_i)| + |\epsilon| + |\eta| = r. \tag{1}$$

For $\lambda \in \bar{\mathbb{C}} \setminus \Lambda(P(s))$ we define $n(\lambda) = (0, 0, \ldots)$.

The homogeneous invariant factors of $P(s)$ are homogeneous polynomials in the indeterminate s and t, $\phi_1(s,t) \mid \cdots \mid \phi_r(s,t)$, that collect the information of the finite and infinite eigenvalues. They are defined as

$$\phi_j(s,t) = t^{n_{r-j+1}(\infty)} \prod_{\lambda \in \Lambda(P(s))\setminus\{\infty\}} (s - \lambda t)^{n_{r-j+1}(\lambda)}, \quad 1 \leq j \leq r.$$

As usual, we take $\phi_j(s,t) = 1$ for $j < 1$, and $\phi_j(s,t) = 0$ for $j > r$.

Following the notation of [11], we define the sequences of nonnegative integers α, β, γ as

$$\alpha = n(\infty), \quad \beta = (\epsilon_1 + 1, \ldots, \epsilon_{m-r} + 1), \quad \gamma = (\eta_1 + 1, \ldots, \eta_{n-r} + 1).$$

The Weyr characteristic of $P(s)$ is (w, b, c), where $w = (w(\lambda_1), \ldots, w(\lambda_\ell))$ and $w(\lambda_i)$ is the conjugate partition of $n(\lambda_i)$, $1 \leq i \leq \ell$, $b = (b_1, b_2, \ldots)$ is the conjugate partition of β, and $c = (c_1, c_2, \ldots)$ is the conjugate partition of γ (see [11, Definition 4.1]). Notice that (1) is equivalent to

$$\sum_{i=1}^{\ell} |w(\lambda_i)| + (|b| - b_1) + (|c| - c_1) = r. \tag{2}$$

We denote $|w| = \sum_{i=1}^{\ell} w(\lambda_i)$, and for $\lambda \in \bar{\mathbb{C}} \setminus \Lambda(P(s))$ we define $w(\lambda) = (0, 0, \ldots)$.

Two matrix pencils $P_1(s)$ and $P_2(s)$ are strictly equivalent if and only if their Weyr characteristics (equivalently, their Kronecker invariants) coincide ([10, Chapter XII, Theorem 5]). A canonical form for the strict equivalence of matrix pencils is the Kronecker canonical form. It is a matrix pencil of the form

$$P_c(s) = sE_c - F_c = \begin{bmatrix} sI_{n_0} - J_0 & O & O & O \\ O & sN_\alpha - I_{|\alpha|} & O & O \\ O & O & sK_\beta - L_\beta & O \\ & & & \\ O & O & O & sK_\gamma^T - L_\gamma^T \end{bmatrix}, \tag{3}$$

where $n_0 = \sum_{\lambda \in \Lambda(P(s)) \cap \mathbb{C}} |n(\lambda)|$ and J_0 is a diagonal of Jordan blocks, $N_\alpha = \operatorname{diag}(N_{\alpha_1}, \ldots, N_{\alpha_{w_1(\infty)}})$ (observe that $\alpha_1 \geq \cdots \geq \alpha_{w_1(\infty)} > 0 = \alpha_{w_1(\infty)+1} = \ldots$) and

$$N_k = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{C}^{k \times k},$$

$$K_\beta = \begin{bmatrix} \operatorname{diag}(K_{\beta_1}, \ldots, K_{\beta_{b_2}}) & O_{(|\beta| - b_1) \times (b_1 - b_2)} \end{bmatrix}, \quad K_\gamma^T = \begin{bmatrix} \operatorname{diag}(K_{\gamma_1}^T, \ldots, K_{\gamma_{c_2}}^T) \\ O_{(c_1 - c_2) \times (|\gamma| - c_1)} \end{bmatrix},$$

$$L_\beta = \begin{bmatrix} \operatorname{diag}(L_{\beta_1}, \ldots, L_{\beta_{b_2}}) & O_{(|\beta| - b_1) \times (b_1 - b_2)} \end{bmatrix}, \quad L_\gamma^T = \begin{bmatrix} \operatorname{diag}(L_{\gamma_1}^T, \ldots, L_{\gamma_{c_2}}^T) \\ O_{(c_1 - c_2) \times (|\gamma| - c_1)} \end{bmatrix},$$

and for $k > 1$,

$$K_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}, \quad L_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \end{bmatrix} \in \mathbb{C}^{(k-1) \times k}.$$

Observe that b_1 is the number of column minimal indices, i.e., $b_1 = m - r$, and $b_1 - b_2$ is the number of column minimal indices equal to 0, i.e., $b_1 - b_2 = \#\{i : \beta_i = 1\}$. Analogously, $c_1 = n - r$, and $c_1 - c_2 = \#\{i : \gamma_i = 1\}$.

A linear relation S in $\mathbb{C}^n$ is a vector subspace of $\mathbb{C}^n \times \mathbb{C}^n$. A matrix $X \in \mathbb{C}^{n\times n}$ can be identified with a linear relation in $\mathbb{C}^n$ via its graph $\Gamma(X) := \{(x, Xx) : x \in \mathbb{C}^n\}$. For basic notions and properties of linear relations we refer to [1, 15]. Two linear relations S and T are strictly equivalent, denoted $S \overset{s.e.}{\sim} T$, if there exists $P \in \mathrm{Gl}_n(\mathbb{C})$ such that

$$T = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} S.$$

The set of proper eigenvalues of a linear relation S (see [11, Section 2] for the definition) will be denoted by $\Lambda(S)$. Let $\Lambda(S) \cap \mathbb{C} = \{\lambda_1, \ldots, \lambda_\ell\}$, then the Weyr characteristic of S consists of $\ell + 3$ partitions $W(\lambda_1), \ldots, W(\lambda_\ell)$, A, B, and C (for details see [11, Definitions 3.1 and 3.2]). In this work we put $A = W(\infty)$, thus, the Weyr characteristics of S will be denoted by (W, B, C). We denote $\mid W \mid = \sum_{i=1}^{\ell} \mid W(\lambda_i) \mid$, and for $\lambda \in \bar{\mathbb{C}} \setminus \Lambda(S)$ we define $W(\lambda) = (0, 0, \ldots)$.

Two linear relations S and T are strictly equivalent if and only if their Weyr characteristics coincide ([11, Theorem 5.4]).

3. Linear relations and their dimension

In this section we show that there exists a close relationship between linear relations and matrix pencils, and we analyze how the corresponding Weyr characteristics are related. We must introduce some notions about linear relations.

The product of two linear relations S_1 and S_2 in $\mathbb{C}^n$ is the linear relation in $\mathbb{C}^n$ defined by

$$S_1 S_2 = \{(x, z) : \exists\, y \in \mathbb{C}^n \text{ such that } (y, z) \in S_1 \text{ and } (x, y) \in S_2\}.$$

The inverse of a linear relation S in $\mathbb{C}^n$ is the linear relation S^{-1} in $\mathbb{C}^n$ defined by

$$S^{-1} = \{(y, x) \in \mathbb{C}^n \times \mathbb{C}^n : \; (x, y) \in S\}.$$

Also, the adjoint of S is the linear relation S^* in $\mathbb{C}^n$ given by

$$S^* = \{(u, v) \in \mathbb{C}^n \times \mathbb{C}^n : \; \langle\, x \,,\, v \,\rangle = \langle\, y \,,\, u \,\rangle \text{ for every } (x, y) \in S\}.$$

Geometrically, the adjoint of S can be described as $S^* = -S^\perp = (-S)^\perp$, where $-S = \{(x, -y) : \; (x, y) \in S\}$, see [14].

Given a matrix pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$, the range and the kernel representations of $P(s)$ are the linear relations

$$FE^{-1} = R\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) \subseteq \mathbb{C}^n \times \mathbb{C}^n \quad \text{and} \quad E^{-1}F = N\left(\begin{bmatrix} F & -E \end{bmatrix}\right) \subseteq \mathbb{C}^m \times \mathbb{C}^m,$$

respectively, see [11]. Then,

$$\dim FE^{-1} = \operatorname{rank}\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) \quad \text{and} \quad \dim E^{-1}F = 2m - \operatorname{rank}\left(\begin{bmatrix} F & -E \end{bmatrix}\right).$$

Remark 3.1. *Given $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$, define $P^*(s) := sE^* - F^* \in \mathbb{C}[s]^{m\times n}$. Then,*

$$(FE^{-1})^* = (F^*)^{-1}E^* \quad \textit{and} \quad (E^{-1}F)^* = F^*(E^*)^{-1},$$

i.e. the adjoint of the range representation of $P(s)$ is the kernel representation of $P^*(s)$ and the adjoint of the kernel representation of $P(s)$ is the range representation of $P^*(s)$.

In the next remark we analyze when two pencils have the same range or the same kernel representation. Also, we show that given a pencil, there exists another one having the same range (kernel) representation and minimal number of columns (rows).

Remark 3.2. *Given two pencils $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times m}$, then $FE^{-1} = \bar{F}\bar{E}^{-1}$ if and only if there exists an invertible matrix $V \in \mathrm{Gl}_m(\mathbb{C})$ such that $\begin{bmatrix} E \\ F \end{bmatrix} V = \begin{bmatrix} \bar{E} \\ \bar{F} \end{bmatrix}$, equivalently, $\bar{P}(s) = P(s)V$. Moreover, if $\dim FE^{-1} = d$, $V \in \mathrm{Gl}_m(\mathbb{C})$ can be chosen such that $\begin{bmatrix} E \\ F \end{bmatrix} V = \begin{bmatrix} E_1 & O \\ F_1 & O \end{bmatrix}$, where $\begin{bmatrix} E_1 \\ F_1 \end{bmatrix} \in \mathbb{C}^{(n+n)\times d}$ has full (column) rank. Hence, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times d}$ has the same range representation as $P(s)$, i.e. $F_1E_1^{-1} = FE^{-1}$, and minimal number of columns.*

Analogously, $E^{-1}F = \bar{E}^{-1}\bar{F}$ if and only if there exists $U \in \mathrm{Gl}_n(\mathbb{C})$ such that $U\begin{bmatrix} F & -E \end{bmatrix} = \begin{bmatrix} \bar{F} & -\bar{E} \end{bmatrix}$, equivalently, $\bar{P}(s) = UP(s)$. If $\dim FE^{-1} = 2m - r$ then $U \in \mathrm{Gl}_n(\mathbb{C})$ can be chosen such that $U\begin{bmatrix} F & -E \end{bmatrix} = \begin{bmatrix} F_1 & -E_1 \\ O & O \end{bmatrix}$, where $\begin{bmatrix} F_1 & -E_1 \end{bmatrix} \in \mathbb{C}^{r\times(m+m)}$ has full (row) rank. Hence, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{r\times m}$ has the same kernel representation as $P(s)$, i.e. $E_1^{-1}F_1 = E^{-1}F$.

Lemma 3.3 ([4, Theorem 3.3])**.** *Let S be a linear relation in $\mathbb{C}^n$ with $\dim S = d$. Then there exists a pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$ with $\operatorname{rank}\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) = d$ such that $S = FE^{-1}$.*

Moreover, for $r = 2n - d$ there exists a pencil $Q(s) = sG - H \in \mathbb{C}[s]^{r\times n}$ with $\operatorname{rank}(\begin{bmatrix} H & -G \end{bmatrix}) = r$ such that $S = G^{-1}H$.

Lemma 3.4. *Given matrix pencils $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times m}$, let $S = N(\begin{bmatrix} F & -E \end{bmatrix})$, $\bar{S} = N(\begin{bmatrix} \bar{F} & -\bar{E} \end{bmatrix})$ be their kernel representations. Then,*

$$P(s) \overset{s.e.}{\sim} \bar{P}(s) \Leftrightarrow S \overset{s.e.}{\sim} \bar{S}.$$

Proof. If the matrix pencils are strictly equivalent, the strict equivalence of their kernel representations follows from [11, Proposition 4.3].

Conversely, if $S \overset{s.e.}{\sim} \bar{S}$ then there exists $T \in \mathrm{Gl}_m(\mathbb{C})$ such that $\bar{S} = \begin{bmatrix} T & O \\ O & T \end{bmatrix} \cdot S$. Therefore,

$$\begin{aligned}\bar{S} &= \{(Tx_1, Ty_1) \, : \, (x_1, y_1) \in S\} = \{(Tx_1, Ty_1) \, : \, Fx_1 = Ey_1\} \\ &= \{(x_2, y_2) \, : \, FT^{-1}x_2 = ET^{-1}y_2\} = N\left(\begin{bmatrix} FT^{-1} & -ET^{-1} \end{bmatrix}\right) \\ &= (ET^{-1})^{-1}(FT^{-1}).\end{aligned}$$

Now, let $P'(s) = P(s)T^{-1} = sET^{-1} - FT^{-1}$. Since $\bar{E}^{-1}\bar{F} = (ET^{-1})^{-1}(FT^{-1})$, by Remark 3.2 there exists $U \in \mathrm{Gl}_n(\mathbb{C})$ such that $P'(s) = U\bar{P}(s)$. Hence, $P(s) = U\bar{P}(s)T$. □

The following result for range representations can be proved similarly.

Lemma 3.5. *Given matrix pencils $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times m}$, let $S = FE^{-1}$, $\bar{S} = \bar{F}\bar{E}^{-1}$ be their range representations. Then,*

$$P(s) \overset{s.e.}{\sim} \bar{P}(s) \Leftrightarrow S \overset{s.e.}{\sim} \bar{S}.$$

In the next lemma we calculate the dimensions of the range and kernel representations of a given pencil.

Lemma 3.6. *Let $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ be a matrix pencil with Weyr characteristic (w, b, c). Then,*

$$\dim FE^{-1} = m - b_1 + b_2, \quad \dim E^{-1}F = 2m - n + c_1 - c_2.$$

Proof. Since $P(s) \overset{s.e.}{\sim} P_c(s)$, where $P_c(s) = sE_c - F_c$ is its Kronecker canonical form (3), by Lemmas 3.4 and 3.5, we have that $E^{-1}F \overset{s.e.}{\sim} E_c^{-1}F_c$ and $FE^{-1} \overset{s.e.}{\sim} F_cE_c^{-1}$. Then, $\dim E^{-1}F = \dim E_c^{-1}F_c$ and $\dim FE^{-1} = \dim F_cE_c^{-1}$.

It is easy to see that $\operatorname{rank}\begin{bmatrix} K_\beta \\ L_\beta \end{bmatrix} = |\beta| - b_1 + b_2$ and $\operatorname{rank}\begin{bmatrix} K_\gamma^T \\ L_\gamma^T \end{bmatrix} = |\gamma| - c_1$. Hence,

$$\begin{aligned}\dim FE^{-1} &= \operatorname{rank}\begin{bmatrix} E_c \\ F_c \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I_{n_0} \\ J_0 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} N_\alpha \\ I_{|\alpha|} \end{bmatrix} + \operatorname{rank}\begin{bmatrix} K_\beta \\ L_\beta \end{bmatrix} + \operatorname{rank}\begin{bmatrix} K_\gamma^T \\ L_\gamma^T \end{bmatrix} \\ &= n_0 + |\alpha| + |\beta| + |\gamma| - c_1 - b_1 + b_2 = m - b_1 + b_2.\end{aligned}$$

Analogously, $\operatorname{rank}\begin{bmatrix} L_\beta & -K_\beta \end{bmatrix} = |\beta| - b_1$ and $\operatorname{rank}\begin{bmatrix} L_\gamma^T & -K_\gamma^T \end{bmatrix} = |\gamma| - c_1 + c_2$, meanwhile $\begin{bmatrix} J_0 & -I_{n_0} \end{bmatrix}$ and $\begin{bmatrix} I_{|\alpha|} & -N_\alpha \end{bmatrix}$ have full rank. Therefore,

$$\operatorname{rank}\begin{bmatrix} F_c & -E_c \end{bmatrix} = n_0 + |\alpha| + |\beta| - b_1 + |\gamma| - c_1 + c_2,$$

and

$$\begin{aligned}\dim E^{-1}F &= 2m - \operatorname{rank}\begin{bmatrix} F_c & -E_c \end{bmatrix} = 2m - (n_0 + |\alpha| + |\beta| - b_1 + |\gamma| - c_1 + c_2) \\ &= 2m - n + c_1 - c_2.\end{aligned}$$ □

As an immediate consequence of Lemma 3.6, we obtain

$$\dim FE^{-1} = m \quad \Leftrightarrow \quad b_1 = b_2,$$

and

$$\dim E^{-1}F = 2m - n \quad \Leftrightarrow \quad c_1 = c_2.$$

In [11] the relationship between the eigenvalues and the Weyr characteristic of a matrix pencil and those of its kernel and range representations was obtained. We state those results adapting the notation to the one used in this work.

Lemma 3.7 ([11, Proposition 4.2])**.** *Let $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ be a matrix pencil, then*

$$\Lambda(P(s)) = \Lambda(E^{-1}F) = \Lambda(FE^{-1}).$$

Lemma 3.8 ([11, Theorem 5.1])**.** *Let $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ be a matrix pencil with Weyr characteristic (w, b, c). If (W, B, C) is the Weyr characteristic of the kernel representation $E^{-1}F$, then $W = w$, $B = b$, and if $c = (c_1, c_2, \dots,)$, then $C = (c_3, c_4, \dots)$.*

Lemma 3.9 ([11, Proposition 5.2])**.** *Let S be a linear relation in $\mathbb{C}^m$ with Weyr characteristic (W, B, C), where $C = (C_1, C_2, \dots)$. If S is the kernel representation of a pencil $P(s) = sE-F \in \mathbb{C}[s]^{n\times m}$, then the Weyr characteristic (w, b, c) of $P(s)$ is given by $w = W$, $b = B$, and $c = (n - m + B_1, m - |W| - |B| - |C|, C_1, C_2, \dots)$.*

Lemma 3.10 ([11, Theorem 6.1])**.** *Let $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ be a matrix pencil with Weyr characteristic (w, b, c). If (W, B, C) is the Weyr characteristic of the range representation FE^{-1}, then $W = w$, and if $b = (b_1, b_2, \dots,)$ and $c = (c_1, c_2, \dots,)$, then $B = (b_2, b_3, \dots)$ and $C = (c_2, c_3, \dots)$.*

Lemma 3.11 ([11, Proposition 6.2])**.** *Let S be a linear relation in $\mathbb{C}^m$ with Weyr characteristic (W, B, C), where $B = (B_1, B_2, \dots)$ and $C = (C_1, C_2, \dots)$. If S is the range representation of a pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$, then the Weyr characteristic (w, b, c) of $P(s)$ is given by $w = W$, $b = (m - |W| - |B| - |C|, B_1, B_2, \dots)$ and $c = (n - |W| - |B| - |C|, C_1, C_2, \dots)$.*

Notice that given a matrix pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times m}$ with Weyr characteristic (w, b, c), and (W_k, B_k, C_k) and (W_r, B_r, C_r) as Weyr characteristics of its kernel $E^{-1}F \subseteq \mathbb{C}^m \times \mathbb{C}^m$ and range representations $FE^{-1} \subseteq \mathbb{C}^n \times \mathbb{C}^n$, respectively, by (2) and Lemmas 3.6, 3.9 and 3.11 we obtain

$$\begin{aligned}
\operatorname{rank}(P(s)) &= |W_k| + |B_k| - B_{k,1} + |C_k| + c_2 = |W_r| + |B_r| + |C_r|, \qquad (4)\\
\dim E^{-1}F &= 2m - n + c_1 - c_2 = |W_k| + |B_k| + |C_k| + B_{k,1} \quad \text{and}\\
\dim FE^{-1} &= m - b_1 + b_2 = |W_r| + |B_r| + |C_r| + B_{r,1}.
\end{aligned}$$

Lemma 3.12. *Given an integer $n \geq 0$, a finite subset $\{\lambda_1, \dots, \lambda_\ell\} \subset \bar{\mathbb{C}}$, two partitions $B = (B_1, B_2, \dots)$, $C = (C_1, C_2, \dots)$, and a collection of partitions $W = (W(\lambda_1), \dots, W(\lambda_\ell))$, where $W(\lambda_i) = (W_1(\lambda_i), W_2(\lambda_i), \dots)$, $1 \leq i \leq \ell$,* *there exists a linear relation $S \subseteq \mathbb{C}^n \times \mathbb{C}^n$ with Weyr characteristic (W, B, C) if and only if*

$$n - (|W| + |B| + |C|) \geq C_1.$$

Proof. Assume that (W, B, C) is the Weyr characteristic of a linear relation $S \subseteq \mathbb{C}^n \times \mathbb{C}^n$ and denote $d = \dim S$. By Lemma 3.3, there exists a pencil $P(s) = sE - F \in \mathbb{F}[s]^{n\times d}$ such that $S = FE^{-1}$. Let (w, b, c) the Weyr characteristic of $P(s)$. Then by Lemma 3.10, $w = W$, $b = (b_1, B_1, B_2, \dots)$, $c = (c_1, C_1, C_2, \dots)$, and by (4) we have $\operatorname{rank}(P(s)) = |W| + |B| + |C|$ and

$$C_1 = c_2 \leq c_1 = n - \operatorname{rank}(P(s)) = n - (|W| + |B| + |C|).$$

Conversely, assume that $n - (|W| + |B| + |C|) \geq C_1$. Define $w = W$, $b = (B_1, B_1, B_2, \dots)$, $c = (n-(|W|+|B|+|C|), C_1, C_2, \dots)$ and let $P(s) = sE-F$ be a pencil with Weyr characteristic (w, b, c) (for instance, the Kronecker canonical form with Weyr characteristic (w, b, c)). Note that

$$\operatorname{rank}(P(s)) = |w| + (|b| - b_1) + (|c| - c_1) = |W| + |B| + |C|.$$

The number of rows of $P(s)$ is given by $c_1 + \operatorname{rank}(P(s)) = n - (|W| + |B| + |C|) + \operatorname{rank}(P(s)) = n$. Hence, $S := FE^{-1}$ is a linear relation in $\mathbb{C}^n$ and, by Lemma 3.10, its Weyr characteristic is (W, B, C). □

Remark 3.13. *The above lemma can also be proved using the kernel representation of a matrix pencil and Lemma 3.8. We omit the details.*

4. Rank one perturbation of linear relations

Given two linear relations S and T in a vector space X, we define

$$r(S, T) = \max\left\{\dim \frac{S}{S \cap T}, \dim \frac{T}{S \cap T}\right\}.$$

Notice that $r(S, T) \geq 0$, and $r(S, T) = 0$ if and only if $S = T$. Also, $r(S, T)$ can be alternatively calculated as

$$r(S, T) = \max\left\{\dim \frac{S + T}{T}, \dim \frac{S + T}{S}\right\}.$$

Using this notation we can state the low rank perturbation problem for linear relations in $\mathbb{C}^n$ as follows:

Problem 4.1. *(low rank perturbation for linear relations). Given two linear relations $S, T \subseteq \mathbb{C}^n \times \mathbb{C}^n$ and a nonnegative integer r, find necessary and sufficient conditions for the existence of linear relations $\bar{S}, \bar{T} \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that*

$$\bar{S} \overset{s.e.}{\sim} S, \quad \bar{T} \overset{s.e.}{\sim} T \quad \text{and} \quad r(\bar{S}, \bar{T}) \leq r.$$

Remark 4.2. *If $S \overset{s.e.}{\sim} T$, then Problem 4.1 is trivial. Taking $\bar{S} = \bar{T} = T$ we have $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = r(T, T) = 0 \leq r$.*

Given a pair of linear relations S and T in $\mathbb{C}^n$, to solve Problem 4.1 we will represent them as range or kernel representations of a pair of suitable matrix pencils with n rows or n columns, respectively. We start analyzing the case when one of the linear relations contains the other one and the difference of their dimensions is r.

Lemma 4.3. *Given two linear relations S and U in $\mathbb{C}^n$, assume that $U \subseteq S$. Denote $d = \dim S$, $g = \dim U$ and $m = 2n - g = \dim U^\perp$. Let $r \geq 1$ be an integer. The following statements are equivalent:*

(i) $\dim \frac{S}{U} = r$.

(ii) There exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n \times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n \times (d-r)}$ and $P_2(s) = sE_2 - F_2 \in \mathbb{C}[s]^{n \times r}$ such that

$$P(s) = \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix}, \quad FE^{-1} = S \;\; \textit{and} \;\; F_1 E_1^{-1} = U.$$

(iii) There exist pencils $Q(s) = sG - H \in \mathbb{C}[s]^{m \times n}$, $Q_1(s) = sG_1 - H_1 \in \mathbb{C}[s]^{(m-r) \times n}$ and $Q_2(s) = sG_2 - H_2 \in \mathbb{C}[s]^{r \times n}$ such that

$$Q(s) = \begin{bmatrix} Q_1(s) \\ Q_2(s) \end{bmatrix}, \quad G_1^{-1} H_1 = S \;\; \textit{and} \;\; G^{-1} H = U.$$

Proof. Assume that (i) holds. Then $g = \dim U = d - r$. By Lemma 3.3 there exists a pencil $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n \times (d-r)}$ such that rank $\begin{bmatrix} E_1 \\ F_1 \end{bmatrix} = d - r$ and $F_1 E_1^{-1} = U$. Note that the columns of $\begin{bmatrix} E_1 \\ F_1 \end{bmatrix}$ form a basis of $U \subset S$. Let $\{(e_1, f_1), \ldots, (e_r, f_r)\}$ be a basis for a subspace $V \subset S$ such that $S = U \dot{+} V$, and let $E_2, F_2 \in \mathbb{C}^{n \times r}$ be the matrices whose columns are $\{e_1, \ldots, e_r\}$ and $\{f_1, \ldots, f_r\}$, respectively. Then, defining $P_2(s) := sE_2 - F_2$ and $P(s) := \begin{bmatrix} P_1(s) & P_2(s) \end{bmatrix} = sE - F$, it is immediate that $FE^{-1} = R\left(\begin{bmatrix} E_1 & E_2 \\ F_1 & F_2 \end{bmatrix}\right) = U \dot{+} V = S$, which proves (ii).

Conversely, assume that (ii) holds. Then, $d = \dim S = \dim R\left(\begin{bmatrix} E \\ F \end{bmatrix}\right) = \dim R\left(\begin{bmatrix} E_1 & E_2 \\ F_1 & F_2 \end{bmatrix}\right)$. Hence, both $\begin{bmatrix} E \\ F \end{bmatrix}$ and $\begin{bmatrix} E_1 \\ F_1 \end{bmatrix}$ have full (column) rank. Therefore, $g = \dim U = \dim R\left(\begin{bmatrix} E_1 \\ F_1 \end{bmatrix}\right) = d - r$, or equivalently, $\dim \frac{S}{U} = r$. This completes the proof of the equivalence $(i) \Leftrightarrow (ii)$.

The equivalence $(i) \Leftrightarrow (iii)$ is obtained applying the above case to the inclusion $S^\perp \subset U^\perp$, i.e., $\dim \frac{U^\perp}{S^\perp} = r$ if and only if there exist pencils $Q^*(s) = sG^* - H^* \in \mathbb{C}[s]^{n \times m}$, $Q_1^*(s) = sG_1^* - H_1^* \in \mathbb{C}[s]^{n \times (m-r)}$ and $Q_2^*(s) = sG_2^* - H_2^* \in$

$\mathbb{C}[s]^{n\times r}$ such that $Q^*(s) = \begin{bmatrix} Q_1^*(s) & Q_2^*(s) \end{bmatrix}$, $H^*(G^*)^{-1} = U^\perp$ and $H_1^*(G_1^*)^{-1} = S^\perp$. Moreover, $H^*(G^*)^{-1} = U^\perp$ if and only if $G^{-1}H = U$, and $H_1^*(G_1^*)^{-1} = S^\perp$ if and only if $G_1^{-1}H_1 = S$. □

The main objective of this work is to characterize the Weyr characteristics of two linear relations S and T such that $r(S,T) \leq 1$, i.e. to give a solution to Problem 4.1 for $r = 1$. As mentioned, we deal with linear relations as kernel or range representations of appropriate pencils. Note that every rank one matrix pencil in $\mathbb{C}[s]^{n\times m}$ can be written in one of the following ways:

$$(su - v)w^*, \quad (0,0) \neq (u,v) \in \mathbb{C}^n \times \mathbb{C}^n,\ 0 \neq w \in \mathbb{C}^m, \tag{5}$$

or

$$w(su^* - v^*), \quad (0,0) \neq (u,v) \in \mathbb{C}^m \times \mathbb{C}^m,\ 0 \neq w \in \mathbb{C}^n. \tag{6}$$

In [11] we have the following result.

Lemma 4.4 ([11, Lemma 7.3])**.** *Let $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times m}$,*

(a) If $\bar{P}(s) - P(s)$ is a rank one matrix as in (5), then $r(FE^{-1}, \bar{F}\bar{E}^{-1}) \leq 1$.

(b) If $\bar{P}(s) - P(s)$ is a rank one matrix as in (6), then $r(E^{-1}F, \bar{E}^{-1}\bar{F}) \leq 1$.

The next two corollaries follow straightforward from Lemma 4.3.

Corollary 4.5. *Given two linear relations S and T in $\mathbb{C}^n$, denote $d = \dim S$, $g = \dim(S \cap T)$ and $m = 2n - g = (S\cap T)^\perp$. Then, the following statements are equivalent:*

(i) $\dim \frac{S}{S\cap T} = 1$ and $\dim \frac{T}{S\cap T} = 0$.

(ii) There exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$ and $u(s) = se - f \in \mathbb{C}[s]^{n\times 1}$ such that $P(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$, $FE^{-1} = S$ and $F_1E_1^{-1} = T$.

(iii) There exist pencils $Q_1(s) = sG_1 - H_1 \in \mathbb{C}[s]^{(m-1)\times n}$, $Q(s) = sG - H \in \mathbb{C}[s]^{m\times n}$ and $v(s) = sg - h \in \mathbb{C}[s]^{n\times 1}$ such that $Q(s) = \begin{bmatrix} Q_1(s) \\ v(s)^* \end{bmatrix}$, $G_1^{-1}H_1 = S$ and $G^{-1}H = T$.

Corollary 4.6. *Given two linear relations S and T in $\mathbb{C}^n$ denote $d = \dim S$, $g = \dim(S\cap T)$ and $m = 2n - g = \dim(S\cap T)^\perp$. Then, the following statements are equivalent:*

(i) $\dim \frac{S}{S\cap T} = \dim \frac{T}{S\cap T} = 1$.

(ii) $\dim S = \dim T$ and there exist pencils $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$, and $u(s) = se - f, \bar{u}(s) = s\bar{e} - \bar{f} \in \mathbb{C}[s]^{n\times 1}$ such that $P(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$, $\bar{P}(s) = \begin{bmatrix} P_1(s) & \bar{u}(s) \end{bmatrix}$, $FE^{-1} = S$, $\bar{F}\bar{E}^{-1} = T$ and $F_1E_1^{-1} = S\cap T$.

(iii) *There exist pencils* $Q_1(s) = sG_1 - H_1, \bar{Q}_1(s) = s\bar{G}_1 - \bar{H}_1 \in \mathbb{C}[s]^{(m-1)\times n}$, $Q(s) = sG - H, \bar{Q}(s) = s\bar{G} - \bar{H} \in \mathbb{C}[s]^{m\times n}$, *and* $v(s) = sg - h, \bar{v}(s) = s\bar{g} - \bar{h}, \in \mathbb{C}[s]^{n\times 1}$ *such that* $Q(s) = \begin{bmatrix} Q_1(s) \\ v(s)^* \end{bmatrix}$, $\bar{Q}(s) = \begin{bmatrix} \bar{Q}_1(s) \\ \bar{v}(s)^* \end{bmatrix}$, $G_1^{-1}H_1 = S$, $\bar{G}_1^{-1}\bar{H}_1 = T$ *and* $G^{-1}H = \bar{G}^{-1}\bar{H} = S \cap T$.

Now, we can prove the converse of Lemma 4.4.

Theorem 4.7. *Given two linear relations* S *and* T *in* $\mathbb{C}^n$, *denote* $d = \dim S$, $g = \dim T$, *and* $m = \dim T^\perp = 2n - g$. *Then, the following statements are equivalent:*

(i) $r(S, T) \leq 1$.

(ii) *There exist pencils* $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times d}$ *such that* $S = FE^{-1}$, $T = \bar{F}\bar{E}^{-1}$ *and*

$$\bar{P}(s) - P(s) = (su - v)w^*, \quad (u, v) \in \mathbb{C}^n \times \mathbb{C}^n, \quad w \in \mathbb{C}^d.$$

(iii) *There exist pencils* $Q(s) = sG - H, \bar{Q}(s) = s\bar{G} - \bar{H} \in \mathbb{C}[s]^{m\times n}$ *such that* $S = G^{-1}H$, $T = \bar{G}^{-1}\bar{H}$ *and*

$$\bar{Q}(s) - Q(s) = w(su^* - v^*), \quad (u, v) \in \mathbb{C}^n \times \mathbb{C}^n, \quad w \in \mathbb{C}^m.$$

Proof. The implications $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ are immediate consequences of Lemma 4.4.

Conversely, assume that (i) holds.

If $r(S, T) = 0$, then $S = T$ and $g = d$. By Lemma 3.3 there exists a pencil $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$ such that $FE^{-1} = S = T$. Taking $\bar{P}(s) = P(s)$, (ii) follows.

If $\dim \frac{S}{S\cap T} = 1$ and $\dim \frac{T}{S\cap T} = 0$, then $g = \dim(S \cap T) = d - 1$. By Corollary 4.5 there exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$ and $u(s) = se - f \in \mathbb{C}[s]^{n\times 1}$ such that $P(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$, $FE^{-1} = S$ and $F_1E_1^{-1} = T$. Let $\bar{P}(s) = s\bar{E} - \bar{F} = \begin{bmatrix} sE_1 - F_1 & O \end{bmatrix} \in \mathbb{C}[s]^{n\times d}$. Then $\bar{F}\bar{E}^{-1} = F_1E_1^{-1} = T$ and $\bar{P}(s) - P(s) = \begin{bmatrix} O & -u(s) \end{bmatrix} = u(s)w^*$, where $w^* = \begin{bmatrix} O & -1 \end{bmatrix} \in \mathbb{C}[s]^{1\times((d-1)+1)}$.

If $\dim \frac{S}{S\cap T} = 0$ and $\dim \frac{T}{S\cap T} = 1$, the proof is analogous.

If $\dim \frac{S}{S\cap T} = \dim \frac{T}{S\cap T} = 1$, then $g = d$. By Corollary 4.6 there exist pencils $P(s) = sE - F, \bar{P}(s) = s\bar{E} - \bar{F} \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$, $u(s) = se - f, \bar{u}(s) = s\bar{e} - \bar{f} \in \mathbb{C}[s]^{n\times 1}$ such that $P(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$, $\bar{P}(s) = \begin{bmatrix} P_1(s) & \bar{u}(s) \end{bmatrix}$, $FE^{-1} = S$, $\bar{F}\bar{E}^{-1} = T$ and $F_1E_1^{-1} = S \cap T$. Hence,

$$\bar{P}(s) - P(s) = \begin{bmatrix} O & \bar{u}(s) - u(s) \end{bmatrix} = (\bar{u}(s) - u(s))w^*,$$

where $w^* = \begin{bmatrix} O & 1 \end{bmatrix} \in \mathbb{C}[s]^{1\times((d-1)+1)}$.

We can prove $(i) \Rightarrow (iii)$ in a similar way, or it can be derived from case (ii) applied to the adjoint pencils (see Remark 3.1).

□

5. Matrix pencil completion theorems

As we have seen in Section 4, the rank perturbation problem of linear relations is related to a matrix pencil completion problem. In this section we introduce, in Lemmas 5.2 and 5.4, some known results about the latter problem. Although they are valid over arbitrary fields, here we state them over $\mathbb{C}$. First, we need to define the 1step-majorization, which is a particular case of generalized majorization [8, Definition 2].

Definition 5.1. *Given two finite sequences of integers* $\mathbf{g} = (g_1, \dots, g_m)$ *and* $\mathbf{d} = (d_1, \dots, d_{m+1})$*, we say that* $\mathbf{d}$ *is 1step-majorized by* $\mathbf{g}$ *(denoted by* $\mathbf{d} \prec' \mathbf{g}$*) if*

$$g_i = d_{i+1}, \quad h \leq i \leq m,$$

where $h = \min\{i = 1, \dots, m : \ g_i < d_i\}$.

Lemmas 5.2 and 5.4 are particular cases of [9, Theorem 4.3] and they can also be seen in [2, Lemmas 4.3 and 4.4].

Lemma 5.2. *Given two matrix pencils* $H_1(s) \in \mathbb{C}[s]^{(n+p)\times(n+m)}$*,* $H(s) \in \mathbb{C}[s]^{(n+p+1)\times(n+m)}$ *of* $\operatorname{rank}(H_1(s)) = \operatorname{rank}(H(s)) = n$*, let* $\pi_1^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$*,* $g_1 \geq \cdots \geq g_m \geq 0$ *and* $t_1 \geq \cdots \geq t_p \geq 0$ *be the homogeneous invariant factors, the column and the row minimal indices of* $H_1(s)$*, respectively, and let* $\pi_1(s,t) \mid \cdots \mid \pi_n(s,t)$*,* $k_1 \geq \cdots \geq k_m \geq 0$ *and* $u_1 \geq \cdots \geq u_{p+1} \geq 0$ *be the homogeneous invariant factors, the column and the row minimal indices indices of* $H(s)$*, respectively.*

Let $\mathbf{g} = (g_1, \dots, g_m)$*,* $\mathbf{t} = (t_1, \dots, t_p)$*,* $\mathbf{k} = (k_1, \dots, k_m)$*,* $\mathbf{u} = (u_1, \dots, u_{p+1})$*. There exists a pencil* $h(s) \in \mathbb{C}[s]^{1\times(n+m)}$ *such that* $H(s) \overset{s.e.}{\sim} \begin{bmatrix} h(s) \\ H_1(s) \end{bmatrix}$ *if and only if*

$$\pi_i(s,t) \mid \pi_i^1(s,t) \mid \pi_{i+1}(s,t), \quad 1 \leq i \leq n, \tag{7}$$

$$\mathbf{u} \prec' \mathbf{t}, \tag{8}$$

$$\mathbf{g} = \mathbf{k}. \tag{9}$$

Remark 5.3. *Let* $\theta = \#\{i \ : \ t_i > 0\}$ *and* $\bar{\theta} = \#\{i \ : \ u_i > 0\}$*. Lemma 4.3 in [2] also contains the condition*

$$\bar{\theta} \geq \theta. \tag{10}$$

But we show that (7)-(9) implies (10): we have $\operatorname{rank}(H(s)) = n = \sum_{i=1}^n \deg(\pi_i) + \sum_{i=1}^m k_i + \sum_{i=1}^{p+1} u_i$ *and* $\operatorname{rank}(H_1(s)) = n = \sum_{i=1}^n \deg(\pi_i^1) + \sum_{i=1}^m g_i + \sum_{i=1}^p t_i$*. Therefore* $\sum_{i=1}^p t_i = \sum_{i=1}^{p+1} u_i + \sum_{i=1}^n (\deg(\pi_i) - \deg(\pi_i^1)) + \sum_{i=1}^m (k_i - g_i)$*. From (7) and (9) we obtain* $\sum_{i=1}^p t_i \leq \sum_{i=1}^{p+1} u_i$*. Then, by (8) and Lemma 5.10 of [2] we derive (10).*

Lemma 5.4. *Given matrix pencils* $H_1(s) \in \mathbb{C}[s]^{(n+p)\times(n+m)}$ *with* $\operatorname{rank}(H_1(s)) = n$*, and* $H(s) \in \mathbb{C}[s]^{(n+p+1)\times(n+m)}$ *with* $\operatorname{rank}(H(s)) = n+1$*, let* $\pi_1^1(s,t) \mid \cdots \mid \pi_n^1(s,t)$*,* $g_1 \geq \cdots \geq g_m \geq 0$ *and* $t_1 \geq \cdots \geq t_p \geq 0$ *be the homogeneous invariant factors, the column and the row minimal indices of* $H_1(s)$*, respectively, and let*

$\pi_1(s,t) \mid \cdots \mid \pi_{n+1}(s,t)$, $k_1 \geq \cdots \geq k_{m-1} \geq 0$ and $u_1 \geq \cdots \geq u_p \geq 0$ be the homogeneous invariant factors, the column and the row minimal indices of $H(s)$, respectively.

Let $\mathbf{g} = (g_1, \dots, g_m)$, $\mathbf{t} = (t_1, \dots, t_p)$, $\mathbf{k} = (k_1, \dots, k_{m-1})$, and $\mathbf{u} = (u_1, \dots, u_p)$. There exists a pencil $h(s) \in \mathbb{C}[s]^{1\times(n+m)}$ such that $H(s) \stackrel{s.e.}{\sim} \begin{bmatrix} h(s) \\ H_1(s) \end{bmatrix}$ if and only if (7),

$$\mathbf{g} \prec' \mathbf{k}, \tag{11}$$

$$\mathbf{t} = \mathbf{u}. \tag{12}$$

To solve Problem 4.1, we must express these results in terms of the Weyr characteristics of the pencils involved.

Lemma 5.5 ([13, Lemma 3.2], see also [3, Lemma 4.3])**.** *Let $(a_1, \dots)$ and $(b_1, \dots)$ be partitions. Let $(p_1, \dots) = \overline{(a_1, \dots)}$ and $(q_1, \dots) = \overline{(b_1, \dots)}$ be the conjugate partitions. Let $k \geq 0$ be an integer. Then,*

$$a_j \geq b_{j+k}, \quad j \geq 1,$$

if and only if

$$p_j \geq q_j - k, \quad j \geq 1.$$

Lemma 5.6. *For $i = 1, 2$ let $P^i(s) \in \mathbb{C}[s]^{n^i \times m^i}$ be matrix pencils such that* $\operatorname{rank}(P^i(s)) = \rho_i$. *Let $\phi^i_1(s,t) \mid \cdots \mid \phi^i_{\rho_i}(s,t)$ be the homogeneous invariant factors of $P^i(s)$.*

For $\lambda \in \bar{\mathbb{C}}$, let $n^i(\lambda) = (n^i_1(\lambda), n^i_2(\lambda) \dots)$ be the Segre characteristic at λ of $P^i(s)$, and let $(w^i_1(\lambda), w^i_2(\lambda), \dots) = \overline{(n^i(\lambda))}$, be the conjugate partition of $n^i(\lambda)$.

Let $x \geq \rho_2 - \rho_1$ be an integer. Then, the following statements are equivalent

(i) $\phi^1_j(s,t) \mid \phi^2_{j+x}(s,t), \quad j \geq 1,$

(ii) $n^1_{j+\rho_1-\rho_2+x}(\lambda) \leq n^2_j(\lambda), \quad \lambda \in \bar{\mathbb{C}}, \quad j \geq 1,$

(iii) $w^1_j(\lambda) + \rho_2 - \rho_1 - x \leq w^2_j(\lambda), \quad \lambda \in \bar{\mathbb{C}}, \quad j \geq 1.$

Proof. The equivalence between (i) and (ii) is immediate, it is enough to take into account that

$$\phi^i_j(s,t) = t^{n^i_{\rho_i - j+1}(\infty)} \prod_{\lambda \in \Lambda(P^i(s)) \setminus \{\infty\}} (s - \lambda t)^{n^i_{\rho_i-j+1}(\lambda)}, \quad 1 \leq j \leq \rho_i.$$

The equivalence between (ii) and (iii) follows from Lemma 5.5. ☐

Definition 5.7 ([3, Definition 4.1])**.** *Given two partitions $r = (r_0, r_1, \dots)$ and $s = (s_0, s_1, \dots)$, we say that s is conjugate majorized by r (denoted by $s \angle r$) if $r_0 = s_0 + 1$ and*

$$r_i = s_i + 1, \quad 0 \leq i \leq g,$$

where $g = \max\{i : r_i > s_i\}$.

Notation. Given two partitions $p=(p_1,p_2,\dots)$ and $q=(q_1,q_2,\dots)$, we write $p\le q$ if $p_j\le q_j$ for $j\ge 1$.

Remark 5.8. *Notice that, if* $(s_0,s_1,\dots)\angle(r_0,r_1,\dots)$ *and* $k\ge 1$ *is an integer, then* $(r_k,r_{k+1},\dots)\le(s_k,s_{k+1},\dots)$ *or* $(s_k,s_{k+1},\dots)\angle(r_k,r_{k+1},\dots)$.

Lemma 5.9 ([3, Proposition 4.5]). *Given two finite sequences of nonnegative integers* $\mathbf{k}=(k_1,\dots,k_{m+1})$ *and* $\mathbf{d}=(d_1,\dots,d_m)$, *let* $(r_1,\dots)=\overline{(k_1,\dots,k_{m+1})}$, $(s_1,\dots)=\overline{(d_1,\dots,d_m)}$ *be the conjugate partitions,* $r_0=m+1=s_0+1$, *and* $r=(r_0,r_1,\dots)$, $s=(s_0,s_1,\dots)$. *Then* $\mathbf{k}\prec'\mathbf{d}$ *if and only if* $s\angle r$.

Applying Lemmas 5.6 and 5.9, Lemmas 5.2 and 5.4 can be expressed in terms of the Weyr characteristics of the pencils $H(s)$ and $H_1(s)$. By transposition the results also apply for column completion instead of row completion. For convenience, we present next the second option.

Lemma 5.10. *Given two matrix pencils* $H_1(s)\in\mathbb{C}[s]^{(n+p)\times(n+m)}$, $H(s)\in\mathbb{C}[s]^{(n+p)\times(n+m+1)}$ *of* $\operatorname{rank}(H_1(s))=\operatorname{rank}(H(s))=n$, *let* (w,b,c) *and* (w^1,b^1,c^1) *be the Weyr characteristics of* $H(s)$ *and* $H_1(s)$, *respectively.*

There exists a pencil $h(s)\in\mathbb{C}[s]^{(n+p)\times 1}$ *such that* $H(s)\overset{s.e.}{\sim}\begin{bmatrix}h(s) & H_1(s)\end{bmatrix}$ *if and only*

$$w_j(\lambda)\le w^1_j(\lambda)\le w_j(\lambda)+1,\quad \lambda\in\bar{\mathbb{C}},\quad j\ge 1, \tag{13}$$

$$b^1\angle b. \tag{14}$$

$$c=c^1, \tag{15}$$

Lemma 5.11. *Given two matrix pencils* $H_1(s)\in\mathbb{C}[s]^{(n+p)\times(n+m)}$, $H(s)\in\mathbb{C}[s]^{(n+p)\times(n+m+1)}$ *of* $\operatorname{rank}(H_1(s))=n$ *and* $\operatorname{rank}(H(s))=n+1$, *let* (w,b,c) *and* (w^1,b^1,c^1) *be the Weyr characteristics of* $H(s)$ *and* $H_1(s)$, *respectively.*

There exists a pencil $h(s)\in\mathbb{C}[s]^{(n+p)\times 1}$ *such that* $H(s)\overset{s.e.}{\sim}\begin{bmatrix}h(s) & H_1(s)\end{bmatrix}$ *if and only if*

$$w_j(\lambda)-1\le w^1_j(\lambda)\le w_j(\lambda),\quad \lambda\in\bar{\mathbb{C}},\quad j\ge 1, \tag{16}$$

$$b=b^1, \tag{17}$$

$$c\angle c^1. \tag{18}$$

6. Necessary conditions for Problem 4.1 with $r=1$

To find necessary conditions for solving problem 4.1 we distinguish two cases: when $\dim S>\dim T$ or when $\dim S=\dim T$. We start analyzing the case when one of the linear relations is included in the other one.

Theorem 6.1. *Let* S,U *be two linear relations in* $\mathbb{C}^n$ *such that* $U\subseteq S$, $\dim S=d$ *and* $\dim\frac{S}{U}=1$.

Let (W,B,C) *and* (W^1,B^1,C^1) *be the Weyr characteristics of* S *and* U, *respectively. Then one of the two following conditions holds:*

(a)

$$W_j(\lambda) \le W_j^1(\lambda) \le W_j(\lambda) + 1, \quad j \ge 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{19}$$

$$B^1 \angle B, \tag{20}$$

$$C = C^1. \tag{21}$$

(b)

$$W_j(\lambda) - 1 \le W_j^1(\lambda) \le W_j(\lambda), \quad j \ge 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{22}$$

$$B = B^1, \tag{23}$$

$$c \angle c^1, \tag{24}$$

where $c = (n - d + B_1, C_1, C_2, \dots)$ *and* $c^1 = (n - d + B_1 + 1, C_1^1, C_2^1 \dots)$.

Remark 6.2. *Condition (24) is equivalent to*

$$C \angle C^1 \text{ or } C^1 \le C.$$

Proof. By Lemma 4.3, there exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n\times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n\times(d-1)}$, $u(s) = se - f \in \mathbb{C}[s]^{n\times 1}$ such that $P(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$, $FE^{-1} = S$ and $F_1E_1^{-1} = U$.

Let (w, b, c) and (w^1, b^1, c^1) be the Weyr characteristics of $P(s)$ and $P_1(s)$, respectively. Then by Lemma 3.10 $w = W$, $w^1 = W^1$,

$$b_j = B_{j-1}, \quad b_j^1 = B_{j-1}^1, \quad c_j = C_{j-1}, \quad c_j^1 = C_{j-1}^1, \quad j \ge 2.$$

As $\dim S = \dim FE^{-1} = d$ and $\dim U = \dim F_1E_1^{-1} = d - 1$, we have $b_1 = b_2 = B_1$ and $b_1^1 = b_2^1 = B_1^1$. Then $\operatorname{rank}(P(s)) = d - b_1 = d - B_1$ and $\operatorname{rank}(P_1(s)) = d - 1 - b_1^1 = d - 1 - B_1^1$; hence $c_1 = n - d + B_1$ and $c_1^1 = n - d + 1 + B_1^1$.

We have $\operatorname{rank}(P_1(s)) \le \operatorname{rank}(P(s)) \le \operatorname{rank}(P_1(s)) + 1$.

(a) If $\operatorname{rank}(P(s)) = \operatorname{rank}(P_1(s))$, then, by Lemma 5.10, conditions (13), (14) and (15) hold. From (13) and (15) we obtain inmediatly (19) and (21).

As $b_1 = b_2$ and $b_1^1 = b_2^1$, from (14) we obtain $b_2 = b_2^1 + 1$. By Remark 5.8, from (14) we derive $(b_2^1, \dots) \angle (b_2, \dots)$, equivalently (20).

(b) If $\operatorname{rank}(P(s)) = \operatorname{rank}(P_1(s)) + 1$, then, by Lemma 5.11, we obtain (16), (17) and (18), which are equivalent to (22)-(24).

□

Remark 6.3. *In the previous proof we have applied the equivalence between (i) and (ii) of Lemma 4.3. Analogously, the proof could be made by applying the equivalence between (i) and (iii)*

As an immediate consequence of Theorem 6.1 we obtain in the next theorem necessary conditions for Problem 4.1, with $r = 1$.

Theorem 6.4. *Let S, T be two linear relations in $\mathbb{C}^n$ such that $r(S,T) = 1$ and $\dim S = d \geq \dim T$.*

Let (W, B, C) and $(\bar{W}, \bar{B}, \bar{C})$ be the Weyr characteristics of S and T, respectively.

1. *If $\dim \frac{S}{S\cap T} = 1$ and $\dim \frac{T}{S\cap T} = 0$ then one of the two following conditions holds:*

   (a)
$$W_i(\lambda) \leq \bar{W}_i(\lambda) \leq W_i(\lambda) + 1, \quad i \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{25}$$
$$\bar{B} \angle B, \tag{26}$$
$$C = \bar{C}. \tag{27}$$

   (b)
$$W_i(\lambda) - 1 \leq \bar{W}_i(\lambda) \leq W_i(\lambda), \quad i \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{28}$$
$$B = \bar{B}, \tag{29}$$
$$c \angle \bar{c}, \tag{30}$$

   where $c = (n-d+B_1, C_1, C_2 \ldots)$ and $\bar{c} = (n-d+B_1+1, \bar{C}_1, \bar{C}_2, \ldots)$.

2. *If $\dim \frac{S}{S\cap T} = \dim \frac{T}{S\cap T} = 1$, let (W^1, B^1, C^1) be the Weyr characteristic of $S \cap T$. Then one of the four following conditions holds:*

   (c)
$$\max\{W_i(\lambda), \bar{W}_i(\lambda)\} \leq W_i^1(\lambda) \leq \min\{W_i(\lambda), \bar{W}_i(\lambda)\}+1, \ i \geq 1, \ \lambda \in \bar{\mathbb{C}}, \tag{31}$$
$$B^1 \angle B, \quad B^1 \angle \bar{B} \tag{32}$$
$$C = \bar{C} = C^1, \tag{33}$$

   (d)
$$\max\{W_i(\lambda), \bar{W}_i(\lambda)-1\} \leq W_i^1(\lambda) \leq \min\{W_i(\lambda)+1, \bar{W}_i(\lambda)\}, \ i \geq 1, \ \lambda \in \bar{\mathbb{C}}, \tag{34}$$
$$\bar{B} = B^1 \angle B \tag{35}$$
$$C = C^1 \text{ and } \bar{c} \angle c, \tag{36}$$

   where $c = (n-d+B_1, C_1, C_2, \ldots)$ and $\bar{c} = (n-d+B_1-1, \bar{C}_1, \bar{C}_2, \ldots,)$.

   (e)
$$\max\{W_i(\lambda)-1, \bar{W}_i(\lambda)\} \leq W_i^1(\lambda) \leq \min\{W_i(\lambda), \bar{W}_i(\lambda)+1\}, \ i \geq 1, \ \lambda \in \bar{\mathbb{C}}, \tag{37}$$
$$B = B^1 \angle \bar{B} \tag{38}$$
$$\bar{C} = C^1 \text{ and } c \angle \bar{c}, \tag{39}$$

   where $c = (n-d+B_1, C_1, C_2, \ldots)$ and $\bar{c} = (n-d+B_1+1, \bar{C}_1, \bar{C}_2, \ldots)$.

(f)

$$\max\{W_i(\lambda), \bar{W}_i(\lambda)\}-1 \leq W_i^1(\lambda) \leq \min\{W_i(\lambda), \bar{W}_i(\lambda)\},\ i \geq 1,\ \lambda \in \bar{\mathbb{C}}, \tag{40}$$

$$B = \bar{B} = B^1, \tag{41}$$

$$c \angle c^1 \text{ and } \bar{c} \angle c^1, \tag{42}$$

where $c = (n-d+B_1, C_1, C_2, \dots)$, $\bar{c} = (n-d+B_1, \bar{C}_1, \bar{C}_2, \dots)$, *and* $c^1 = (n-d+B_1+1, C_1^1, C_2^1, \dots)$.

7. Solution to Problem 4.1 with $r = 1$

Lemma 7.1. *Let S, U be two linear relations in $\mathbb{C}^n$. Then, there exists a linear relation $\bar{S}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$ and $U \subseteq \bar{S}$ if and only if there exists a linear relation $\bar{U}$ in $\mathbb{C}^n$ such that $\bar{U} \overset{s.e.}{\sim} U$ and $\bar{U} \subseteq S$.*

Proof. Let us assume that $\bar{S} \overset{s.e.}{\sim} S$ and $U \subseteq \bar{S}$. Then, there exists $T \in \mathrm{Gl}_n(\mathbb{C})$ such that $\bar{S} = \begin{bmatrix} T & O \\ O & T \end{bmatrix} S$. Let $\bar{U} = \begin{bmatrix} T^{-1} & O \\ O & T^{-1} \end{bmatrix} U$. Then $\bar{U} \overset{s.e.}{\sim} U$ and $\bar{U} \subseteq \begin{bmatrix} T^{-1} & O \\ O & T^{-1} \end{bmatrix} \bar{S} = S$.

Conversely, let us assume that $\bar{U} \overset{s.e.}{\sim} U$ and $\bar{U} \subseteq S$. Then, there exists $V \in Gl_n(\mathbb{C})$ such that $\bar{U} = \begin{bmatrix} V & O \\ O & V \end{bmatrix} U$. Let $\bar{S} = \begin{bmatrix} V^{-1} & O \\ O & V^{-1} \end{bmatrix} S$. Then $\bar{S} \overset{s.e.}{\sim} S$ and $U = \begin{bmatrix} V^{-1} & O \\ O & V^{-1} \end{bmatrix} \bar{U} \subseteq \begin{bmatrix} V^{-1} & O \\ O & V^{-1} \end{bmatrix} S = \bar{S}$.

□

Theorem 7.2. *Let S, U be two linear relations in $\mathbb{C}^n$ such that $\dim S = d = \dim U + 1$ and let (W, B, C) and (W^1, B^1, C^1) be the Weyr characteristics of S and U, respectively. Then there exists a linear relation $\bar{S}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$ and $U \subset \bar{S}$ (equivalently, there exists a linear relation $\bar{U}$ in $\mathbb{C}^n$ such that $\bar{U} \overset{s.e.}{\sim} U$ and $\bar{U} \subset S$) if and only if one of the conditions (a) or (b) of Theorem 6.1 holds.*

Proof. Assume that there exists $\bar{S}$ such that $\bar{S} \overset{s.e.}{\sim} S$, $U \subset \bar{S}$. Then $\dim \frac{\bar{S}}{U} = \dim S - \dim U = 1$. By Theorem 5.4 of [11], (W, B, C) is the Weyr characteristic of $\bar{S}$. By Theorem 6.1, (a) or (b) holds.

By Lemma 3.3 there exist pencils $P(s) = sE - F \in \mathbb{C}[s]^{n \times d}$, $P_1(s) = sE_1 - F_1 \in \mathbb{C}[s]^{n \times (d-1)}$ such that $\operatorname{rank} \begin{bmatrix} E \\ F \end{bmatrix} = d$, $\operatorname{rank} \begin{bmatrix} E_1 \\ F_1 \end{bmatrix} = d-1$, $FE^{-1} = S$. and $F_1 E_1^{-1} = U$.

Let (w, b, c) and (w^1, b^1, c^1) be the Weyr characteristics of $P(s)$ and $P_1(s)$, respectively. As in the proof of Theorem 6.1, $\operatorname{rank}(P(s)) = d - B_1$, $\operatorname{rank}(P_1(s)) = d - 1 - B_1^1$,

$$w = W, \quad b = (B_1, B_1, B_2, \dots), \quad c = (n-d+B_1, C_1, C_2, \dots),$$

$$w^1 = W^1, \quad b^1 = (B_1^1, B_1^1, B_2^1, \dots), \quad c^1 = (n-d+1+B_1^1, C_1^1, C_2^1, \dots).$$

- Assume that (a) holds. Condition (19) is equivalent to (13). From (20) we derive $B_1 = B_1^1 + 1$; hence $b_1 = b_1^1 + 1$, $c_1 = c_1^1$ and $\operatorname{rank}(P(s)) = \operatorname{rank}(P_1(s))$. Thus, from (20) and (21) we obtain (14) and (15). By Lemma 5.10 here exists a pencil $u(s) = se - f \in \mathbb{F}[s]^{n\times 1}$ such that $P(s) \overset{s.e.}{\sim} \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$.
- Assume that (b) holds. Condition (22) is equivalent to (16). From (23) we derive $B_1 = B_1^1$; hence $b_1 = b_1^1$, $c_1 = c_1^1+1$ and $\operatorname{rank}(P(s)) = \operatorname{rank}(P_1(s))+1$. Thus, from (23) and (24) we obtain (17) and (18). By Lemma 5.11 here exists a pencil $u(s) = se - f \in \mathbb{F}[s]^{n\times 1}$ such that $P(s) \overset{s.e.}{\sim} \begin{bmatrix} P_1(s) & u(s) \end{bmatrix}$.

In both cases, let $\bar{P}(s) = \begin{bmatrix} P_1(s) & u(s) \end{bmatrix} = s\begin{bmatrix} E_1 & e \end{bmatrix} - \begin{bmatrix} F_1 & f \end{bmatrix}$ and $\bar{S} = \begin{bmatrix} F_1 & f \end{bmatrix}\begin{bmatrix} E_1 & e \end{bmatrix}^{-1} = R\left(\begin{bmatrix} E_1 & e \\ F_1 & f \end{bmatrix}\right)$. It is clear that $U \subset \bar{S}$. By Lemma 3.5, $\bar{S} \overset{s.e.}{\sim} S$.

□

Remark 7.3. *As in Theorem 6.1, a proof of Theorem 7.2 can be made using pencils such that their kernel representations are the relations S and U (instead of the range representations).*

As an immediate consequence of Theorem 7.2 we obtain a solution to Problem 4.1 with $r = 1$ when $\dim S = \dim T + 1$.

Theorem 7.4. *Let S, T be two linear relations in $\mathbb{C}^n$ such that $\dim S = d = \dim T + 1$. Let (W, B, C) and $(\bar{W}, \bar{B}, \bar{C})$ be the Weyr characteristics of S and T, respectively. Then there exists a linear relation $\bar{S}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$ and $r(\bar{S}, T) = 1$ (equivalently, there exists a linear relation $\bar{T}$ in $\mathbb{C}^n$ such that $\bar{T} \overset{s.e.}{\sim} T$ and $r(S, \bar{T}) = 1$) if and only if one of the conditions (a) or (b) of Theorem 6.4 holds.*

Proof. There exists $\bar{S}$ such that $\bar{S} \overset{s.e.}{\sim} S$ and $r(\bar{S}, T) = 1$ (there exists $\bar{T}$ such that $\bar{T} \overset{s.e.}{\sim} T$ and $r(S, \bar{T}) = 1$) if and only if there exists $\bar{S}$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\dim \frac{\bar{S}}{\bar{S}\cap T} = 1$ and $\dim \frac{T}{\bar{S}\cap T} = 0$ (there exists $\bar{T}$ such that $\bar{T} \overset{s.e.}{\sim} T$, $\dim \frac{S}{S\cap\bar{T}} = 1$ and $\dim \frac{\bar{T}}{S\cap\bar{T}} = 0$) if and only if there exists $\bar{S}$ such that $\bar{S} \overset{s.e.}{\sim} S$, and $T \subset \bar{S}$ (there exists $\bar{T}$ such that $\bar{T} \overset{s.e.}{\sim} S$, and $\bar{T} \subset S$). By Theorem 7.2 this occurs if and only if one of the conditions (a) or (b) of Theorem 6.4 holds.

□

The solution to the case $\dim S = \dim T$ is given in the next theorem. The proof follows the ideas of [2, Theorem 5.1]. We need some technical lemmas from [2] and [3], which, for the reader's convenience, we include in Appendix Appendix A.

Theorem 7.5. *Let S, T be two linear relations in $\mathbb{C}^n$ such that $\dim S = \dim T = d$ and $S \overset{s.e.}{\not\sim} T$. Let (W, B, C) and $(\bar{W}, \bar{B}, \bar{C})$ be the Weyr characteristics of S and T, respectively, and let $\Lambda(S) \cup \Lambda(T) = \{\lambda_1, \ldots, \lambda_\ell\}$.*

1. *If $B = \bar{B}$ and $C = \bar{C}$, then there exist linear relations $\bar{S}, \bar{T} \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = 1$ if and only if*

$$W_i(\lambda) - 1 \leq \bar{W}_i(\lambda) \leq W_i(\lambda) + 1, \quad i \geq 1, \quad \lambda \in \bar{\mathbb{C}}. \tag{43}$$

2. *If $B = \bar{B}$ and $C \neq \bar{C}$, let $x = \min\{i \ : \ C_i \neq \bar{C}_i\}$,*

$$e = \min\{i \geq x-1 : \ \bar{C}_{i+1} \geq C_{i+1}\}, \quad e' = \min\{i \in \{i \geq x-1 : \ C_{i+1} \geq \bar{C}_{i+1}\},$$

$$X =| W | + | C | - \sum_{i=1}^{\ell} \sum_{j \geq 1} \min\{W_j(\lambda_i), \bar{W}_j(\lambda_i)\} - 1.$$

Then there exist linear relations $\bar{S}, \bar{T} \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = 1$ if and only if (43) and

$$X \leq \sum_{i \geq 1} \min\{C_i, \bar{C}_i\} + \max\{e, e'\}. \tag{44}$$

3. *If $B \neq \bar{B}$ and $B_1 = \bar{B}_1$, let $\bar{x} = \min\{i \ : \ B_i \neq \bar{B}_i\}$,*

$$\bar{e} = \min\{i \geq \bar{x}-1 : \ \bar{B}_{i+1} \geq B_{i+1}\}, \quad \bar{e}' = \min\{i \in \{i \geq \bar{x}-1 : \ B_{i+1} \geq \bar{B}_{i+1}\},$$

$$Y =| W | + | B | - \sum_{i=1}^{\ell} \sum_{j \geq 1} \max\{W_j(\lambda_i), \bar{W}_j(\lambda_i)\}.$$

Then there exist linear relations $\bar{S}, \bar{T} \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = 1$ if and only if (43),

$$C = \bar{C}, \tag{45}$$

and

$$Y \geq \sum_{i \geq 1} \max\{B_i, \bar{B}_i\} - \max\{\bar{e}, \bar{e}'\}. \tag{46}$$

4. *If $B_1 \neq \bar{B}_1$, then there exist linear relations $\bar{S}, \bar{T} \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = 1$ if and only if one of the two following conditions hold:*
   (a)

$$\bar{W}_j(\lambda) - 2 \leq W_j(\lambda) \leq \bar{W}_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{47}$$

$$\bar{B} \angle B, \quad \bar{c} \angle c, \tag{48}$$

where $c = (n - d + B_1, C_1, C_2, \dots)$, $\bar{c} = (n - d + \bar{B}_1, \bar{C}_1, \bar{C}_2, \dots)$,

$$\sum_{i=1}^{\ell} \sum_{j \geq 1} \max\{W_j(\lambda_i), \bar{W}_j(\lambda_i) - 1\} \leq x \leq \sum_{i=1}^{\ell} \sum_{j \geq 1} \min\{W_j(\lambda_i) + 1, \bar{W}_j(\lambda_i)\}, \tag{49}$$

where $x =| W | + | B | - | \bar{B} |$.

(b)

$$W_j(\lambda) - 2 \leq \bar{W}_j(\lambda) \leq W_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \tag{50}$$

$$B \angle \bar{B}, \quad c \angle \bar{c}, \tag{51}$$

where $c = (n - d + B_1, C_1, C_2, \dots)$, $\bar{c} = (n - d + \bar{B}_1, \bar{C}_1, \bar{C}_2, \dots)$,

$$\sum_{i=1}^{\ell} \sum_{j \geq 1} \max\{W_j(\lambda_i) - 1, \bar{W}_j(\lambda_i)\} \leq y \leq \sum_{i=1}^{\ell} \sum_{j \geq 1} \min\{W_j(\lambda_i), \bar{W}_j(\lambda_i) + 1\}, \tag{52}$$

where $y = \mid \bar{W} \mid + \mid \bar{B} \mid - \mid B \mid$.

Remark 7.6. *If* $\lambda \notin \{\lambda_1, \dots, \lambda_\ell\}$, *then* $\min\{W_j(\lambda), \bar{W}_j(\lambda)\} = 0$. *Therefore, in item 2. we can define*

$$X = \mid W \mid + \mid C \mid - \sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \min\{W_j(\lambda), \bar{W}_j(\lambda)\} - 1.$$

Analogously, in item 3.,

$$Y = \mid W \mid + \mid B \mid - \sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \max\{W_j(\lambda), \bar{W}_j(\lambda)\}.$$

and, in item 4. conditions (49) and (52) can be written, respectively, as

$$\sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \max\{W_j(\lambda), \bar{W}_j(\lambda) - 1\} \leq x \leq \sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \min\{W_j(\lambda) + 1, \bar{W}_j(\lambda)\},$$

and

$$\sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \max\{W_j(\lambda) - 1, \bar{W}_j(\lambda)\} \leq y \leq \sum_{\lambda \in \bar{\mathbb{C}}} \sum_{j \geq 1} \min\{W_j(\lambda), \bar{W}_j(\lambda) + 1\}.$$

Proof. Necessity. Let us assume that there exist linear relations $\bar{S}, \bar{T} \subseteq \mathbb{C}^n \times \mathbb{C}^n$, such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $r(\bar{S}, \bar{T}) = 1$. As $\dim \bar{S} = \dim \bar{T}$ and $\bar{S} \neq \bar{T}$, we have $\dim \frac{\bar{S}}{\bar{S} \cap \bar{T}} = \dim \frac{\bar{T}}{\bar{S} \cap \bar{T}} = 1$. Let (W^1, B^1, C^1) be the Weyr characteristic of $\bar{S} \cap \bar{T}$. Then one of the four conditions (c), (d), (e) or (f) of Theorem 6.4 hold.

1. If $B = \bar{B}$ and $C = \bar{C}$, then (c) or (f) holds. Condition (43) is derived from (31) if (c) holds, and from (40) if (f) holds.
2. If $B = \bar{B}$ and $C \neq \bar{C}$, then (f) holds. From (40) we derive (43). By Lemma Appendix A.8, from (42) we have that

$$\mid C^1 \mid \leq \sum_{i \geq 1} \min\{C_i, \bar{C}_i\} + \max\{e, e'\}. \tag{53}$$

   We have $\mid W^1 \mid + \mid B^1 \mid + \mid C^1 \mid + B^1_1 = \dim(\bar{S} \cap \bar{T}) = d - 1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1$. From (41) we obtain $\mid C^1 \mid = \mid W \mid + \mid C \mid - \mid W^1 \mid - 1$. From (40), $X \leq \mid C^1 \mid$. Therefore, from (53) we obtain (44).

3. If $B \neq \bar{B}$ and $B_1 = \bar{B}_1$, then (c) holds. From (31) we derive (43) and from (33), condition (45) is immediate. From (32), for any integer $Z \geq B_1 = \bar{B}_1$, $(Z-1, B^1_1, B^1_2 \dots) \angle (Z, B_1, B_2, \dots,)$ and $(Z-1, B^1_1, B^1_2 \dots) \angle (Z, \bar{B}_1, \bar{B}_2, \dots,)$. By Lemma Appendix A.8,
$$\mid B^1 \mid \geq \sum_{i \geq 1} \max\{B_i, \bar{B}_i\} - \max\{\bar{e}, \bar{e}'\}. \tag{54}$$
We have $\mid W^1 \mid + \mid B^1 \mid + \mid C^1 \mid + B^1_1 = \dim(\bar{S} \cap \bar{T}) = d - 1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1$. From (32) and (33) we obtain $\mid B^1 \mid = \mid W \mid + \mid B \mid - \mid W^1 \mid$. From (31), $Y \geq \mid B^1 \mid$. Therefore, from (54) we obtain (46).
4. If $B_1 \neq \bar{B}_1$, then (d) or (e) holds.
   Assume that (d) holds. From (34) we derive (47) and from (35) and (36), condition (48) is immediate. We have $\mid W^1 \mid + \mid B^1 \mid + \mid C^1 \mid + B^1_1 = \dim(\bar{S} \cap \bar{T}) = d - 1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1$. From (35) and (36) we obtain $\mid W^1 \mid = \mid W \mid + \mid B \mid - \mid \bar{B} \mid$. From (34) we derive $\Lambda(\bar{S} \cap \bar{T}) \subseteq \Lambda(\bar{T})$; hence $\mid W^1 \mid = \sum_{i=1}^{\ell} \sum_{j \geq 1} W^1_j(\lambda_i)$ and from (34) we obtain (49).
   Analogously, if (e) is satisfied, then we obtain (50)-(52).

Sufficiency.

1., 2. Case $B = \bar{B}$. Assume that (43) holds, and that, if $C \neq \bar{C}$, (44) also holds. As $d = \mid W \mid + \mid B \mid + \mid C \mid + B_1 = \mid \bar{W} \mid + \mid \bar{B} \mid + \mid \bar{C} \mid + \bar{B}_1$, we obtain $\mid W \mid + \mid C \mid = \mid \bar{W} \mid + \mid \bar{C} \mid$.
Define
$$\hat{B} = B = \bar{B}, \tag{55}$$
and
$$\hat{W}_j(\lambda) = \min\{W_j(\lambda), \bar{W}_j(\lambda)\}, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}.$$
Then $\hat{W}_j(\lambda) \geq \hat{W}_{j+1}(\lambda)$, for $j \geq 1$ and $\lambda \in \bar{\mathbb{C}}$, and from (43) we derive
$$\begin{array}{ll} W_j(\lambda) - 1 \leq \hat{W}_j(\lambda) \leq W_j(\lambda), & j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \\ \bar{W}_j(\lambda) - 1 \leq \hat{W}_j(\lambda) \leq \bar{W}_j(\lambda), & j \geq 1, \quad \lambda \in \bar{\mathbb{C}}. \end{array} \tag{56}$$
Define
$$\hat{W}(\lambda_i) = (\hat{W}_1(\lambda_i), \hat{W}_2(\lambda_i), \dots), \quad 1 \leq i \leq \ell,$$
$$\hat{W} = (\hat{W}(\lambda_1), \dots, \hat{W}(\lambda_\ell)).$$
We have $\mid \hat{W} \mid \leq \mid W \mid$ and $\mid \hat{W} \mid \leq \mid \bar{W} \mid$. Let $c = (n - d + B_1, C_1, C_2, \dots)$, $\bar{c} = (n - d + B_1, \bar{C}_1, \bar{C}_2, \dots)$ and let $X = \mid W \mid + \mid C \mid - \mid \hat{W} \mid -1 = \mid \bar{W} \mid + \mid \bar{C} \mid - \mid \hat{W} \mid -1$. Then $X \geq \mid C \mid -1 \geq -1$ and $X \geq \mid \bar{C} \mid -1$.
Let us see that $X \geq 0$. If $X = -1$, then $C = \bar{C} = 0$ and $\mid W \mid = \mid \bar{W} \mid = \mid \hat{W} \mid$; i.e., $\sum_{i=1}^{\ell} \sum_{j \geq 1} (\hat{W}_j(\lambda_i) - W_j(\lambda_i)) = \sum_{i=1}^{\ell} \sum_{j \geq 1} (\hat{W}_j(\lambda_i) - \bar{W}_j(\lambda_i)) = 0$, from where $\hat{W}_j(\lambda_i) = W_j(\lambda_i) = \bar{W}_j(\lambda_i))$ for $1 \leq i \leq \ell$ and $j \geq 1$; hence, $\hat{W} = W = \bar{W}$. Then $(W, B, C) = (\bar{W}, \bar{B}, \bar{C})$, which contradicts $S \overset{s.e.}{\not\sim} T$. Therefore $X \geq 0$.

- If $C \neq \bar{C}$, by Lemma Appendix A.8, from (44), there exists a partition of nonnegative integers $\hat{C} = (\hat{C}_1, \hat{C}_2, \dots)$ such that $\mid \hat{C} \mid= X$ and

$$c \angle (n-d+B_1+1, \hat{C}_1, \hat{C}_2, \dots), \quad \bar{c} \angle (n-d+B_1+1, \hat{C}_1, \hat{C}_2, \dots). \tag{57}$$

where $c = (n-d+B_1, C_1, C_2, \dots)$ and $\bar{c} = (n-d+B_1, \bar{C}_1, \bar{C}_2, \dots)$.

- If $C = \bar{C}$, by Lemma Appendix A.2 there exists a partition of nonnegative integers $\hat{C} = (\hat{C}_1, \hat{C}_2, \dots)$ such that $\mid \hat{C} \mid= X$ and (57) holds.

From (57), $\hat{C}_1 \leq C_1 + 1$ and $\mid \hat{W} \mid + \mid \hat{B} \mid + \mid \hat{C} \mid +\hat{C}_1 =\mid \hat{W} \mid + \mid B \mid +X + \hat{C}_1 =\mid W \mid + \mid B \mid + \mid C \mid -1 + \hat{C}_1 \leq\mid W \mid + \mid B \mid + \mid C \mid +C_1$. By Lemma 3.12, $\mid W \mid + \mid B \mid + \mid C \mid +C_1 \leq n$. By the same Lemma, there exists a linear relation U in $\mathbb{C}^n$ such that the Weyr characteristic of U is $(\hat{W}, \hat{B}, \hat{C})$. Then $\dim U =\mid \hat{W} \mid + \mid \hat{B} \mid + \mid \hat{C} \mid +\hat{B}_1 =\mid W \mid + \mid B \mid + \mid C \mid +B_1 - 1 = d-1$. From (55)-(57), by Theorem 7.2 there exists linear relations $\bar{S}$, $\bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $U \subseteq \bar{S} \cap \bar{T}$. As $\dim U = d-1 \leq \dim(\bar{S} \cap \bar{T}) < \dim \bar{S} = d$, we have that $U = \bar{S} \cap \bar{T}$; hence $\dim \frac{\bar{S}}{\bar{S}\cap\bar{T}} = \dim \frac{\bar{T}}{\bar{S}\cap\bar{T}} = 1$.

3. Case $B \neq \bar{B}$ and $B_1 = \bar{B}_1$. Assume that (43), (45) and (46) hold. Let $Y' = \sum_{i\geq 1} \max\{B_i, \bar{B}_i\} - \max\{\bar{e}, \bar{e}'\}$ and $Z = B_1 = \bar{B}_1$. By Lemma Appendix A.8, there exists a partition of nonnonnegative integers $\hat{B}$ such that $(Z-1, \hat{B}_1, \dots) \angle (Z, B_1, \dots)$, $(Z-1, \hat{B}_1, \dots) \angle (Z, \bar{B}_1, \dots)$ and $\mid \hat{B} \mid= Y'$. As $\hat{B}_1 \leq Z-1 = B_1 - 1 < B_1$, we have $B_1 = \bar{B}_1 = \hat{B}_1 + 1$; hence
$$\hat{B} \angle B, \quad \hat{B} \angle \bar{B}. \tag{58}$$
Define
$$\hat{C} = C = \bar{C}, \tag{59}$$
and $y = Y - Y'$. From (46), $y \geq 0$.
Fix $\lambda_0 \notin \{\lambda_1, \dots, \lambda_\ell\}$ and define
$$\begin{aligned} &\hat{W}_j(\lambda) = \max\{W_j(\lambda), \bar{W}_j(\lambda)\}, \quad j \geq 1, \quad \lambda_0 \neq \lambda \in \bar{\mathbb{C}}, \\ &\hat{W}_j(\lambda_0) = 1, \quad 1 \leq j \leq y, \\ &\hat{W}_j(\lambda_0) = 0, \quad j > y. \end{aligned}$$
Then $\hat{W}_j(\lambda) \geq \hat{W}_{j+1}(\lambda)$, for $j \geq 1$ and $\lambda \in \bar{\mathbb{C}}$, and from (43) we derive
$$\begin{aligned} W_j(\lambda) \leq \hat{W}_j(\lambda) \leq W_j(\lambda) + 1, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \\ \bar{W}_j(\lambda) \leq \hat{W}_j(\lambda) \leq \bar{W}_j(\lambda) + 1, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}. \end{aligned} \tag{60}$$
Let
$$\hat{W}(\lambda) = (\hat{W}_1(\lambda), \hat{W}_2(\lambda), \dots), \quad \lambda \in \bar{\mathbb{C}},$$
and $\hat{W} = (\hat{W}(\lambda_0), \hat{W}(\lambda_1), \dots, \hat{W}(\lambda_\ell))$.

We have $\mid \hat{W} \mid = \sum_{i=1}^{\ell} \sum_{j\geq 1} \max\{W_j(\lambda_i), \bar{W}_j(\lambda_i)\} + y = \mid W \mid + \mid B \mid - Y'$ and $\mid \hat{W} \mid + \mid \hat{B} \mid + \mid \hat{C} \mid + \hat{C}_1 = \mid W \mid + \mid B \mid + \mid C \mid + C_1$. By Lemma 3.12, $\mid W \mid + \mid B \mid + \mid C \mid + C_1 \leq n$. By the same Lemma, there exists a linear relation U in $\mathbb{C}^n$ such that the Weyr characteristic of U is $(\hat{W}, \hat{B}, \hat{C})$. Then $\dim U = \mid \hat{W} \mid + \mid \hat{B} \mid + \mid \hat{C} \mid + \hat{B}_1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1 = d - 1$. From (58)-(60), by Theorem 7.2 there exists linear relations $\bar{S}$, $\bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $U \subseteq \bar{S} \cap \bar{T}$. As $\dim U = d - 1 \leq \dim(\bar{S} \cap \bar{T}) < \dim \bar{S} = d$, we have that $U = \bar{S} \cap \bar{T}$; hence $\dim \frac{\bar{S}}{\bar{S}\cap\bar{T}} = \dim \frac{\bar{T}}{\bar{S}\cap\bar{T}} = 1$.

4. Case $B_1 \neq \bar{B}_1$.
Assume that (47)-(49) hold. Define

$$\hat{B} = \bar{B}, \tag{61}$$

$$\hat{C} = C. \tag{62}$$

From (48) we obtain

$$\hat{B} \angle B, \tag{63}$$

and

$$\bar{c} \angle \hat{c}, \tag{64}$$

where $\bar{c} = (n - d + \bar{B}_1, \bar{C}_1, \bar{C}_2, \dots)$ and $\hat{c} = (n - d + \bar{B}_1 + 1, \hat{C}_1, \hat{C}_2 \dots)$. For $\lambda \in \bar{\mathbb{C}}$ and $j \geq 1$,

$$m_j(\lambda) = \max\{W_j(\lambda), \bar{W}_j(\lambda) - 1\}, \quad M_j(\lambda) = \min\{W_j(\lambda) + 1, \bar{W}_j(\lambda)\}.$$

Then $m_j(\lambda) \geq m_{j+1}(\lambda)$ and $M_j(\lambda) \geq M_{j+1}(\lambda)$ for $j \geq 1$ and $\lambda \in \bar{\mathbb{C}}$. Let $m(\lambda) = (m_1(\lambda), \dots,)$ and $M(\lambda) = (M_1(\lambda), \dots,)$ for $\lambda \in \bar{\mathbb{C}}$. Con this notation, condition (49) becomes

$$\sum_{i=1}^{\ell} \mid m(\lambda_i) \mid \leq x \leq \sum_{i=1}^{\ell} \mid M(\lambda_i) \mid .$$

From (47), we have

$$m_j(\lambda) \leq M_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}};$$

hence $\mid m(\lambda_i) \mid \leq \mid M(\lambda_i) \mid$ for $1 \leq i \leq \ell$. From Lemma Appendix A.9, there exist integers $x(\lambda_1), \dots, x(\lambda_\ell)$ such that

$$\sum_{i=1}^{\ell} x(\lambda_i) = x \text{ and } \mid m(\lambda_i) \mid \leq x(\lambda_i) \leq \mid M(\lambda_i) \mid, \quad 1 \leq i \leq \ell. \tag{65}$$

From (47) we have $\Lambda(S) \subseteq \Lambda(T)$; hence $\Lambda(T) = \{\lambda_1, \dots, \lambda_\ell\}$. For $1 \leq i \leq \ell$, let $\bar{n}_i = \max\{j : : \bar{W}_j(\lambda_i) > 0\}$. Then $m_j(\lambda_i) = M_j(\lambda_i) = 0$ for $j > \bar{n}_i$ and

$$\mid m(\lambda_i) \mid = \sum_{j=1}^{\bar{n}_i} m_j(\lambda_i), \quad \mid M(\lambda_i) \mid = \sum_{j=1}^{\bar{n}_i} M_j(\lambda_i).$$

Again by Lemma Appendix A.9, from (65), for $1 \leq i \leq \ell$, there exist integers $\hat{W}_1(\lambda_i) \geq \cdots \geq \hat{W}_{\bar{n}_i}(\lambda_i)$ such that

$$\sum_{j=1}^{\bar{n}_i} \hat{W}_j(\lambda_i) = x(\lambda_i) \text{ and } m_j(\lambda_i) \leq \hat{W}_j(\lambda_i) \leq M_j(\lambda_i), \quad 1 \leq j \leq \bar{n}_1. \quad (66)$$

Define $\hat{W}(\lambda_i) = (\hat{W}_1(\lambda_i), \ldots)$, for $1 \leq i \leq \ell$, $\hat{W} = (\hat{W}(\lambda_1), \ldots, \hat{W}(\lambda_\ell))$ and $\hat{W}(\lambda) = (0, \ldots)$ if $\lambda \notin \{\lambda_1, \ldots, \lambda_\ell\}$. From (66) we have

$$W_j(\lambda) \leq \hat{W}_j(\lambda) \leq W_j(\lambda) + 1, \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}, \quad (67)$$

and

$$\bar{W}_j(\lambda) - 1 \leq \hat{W}_j(\lambda) \leq \bar{W}_j(\lambda), \quad j \geq 1, \quad \lambda \in \bar{\mathbb{C}}. \quad (68)$$

From (66) and (65), $\mid \hat{W} \mid = \sum_{i=1}^{\ell} \mid \hat{W}(\lambda_i) \mid = \sum_{i=1}^{\ell} x(\lambda_i) = x$; hence $\mid \hat{W} \mid + \mid \hat{B} \mid + \mid \hat{C} \mid + \hat{C}_1 = \mid W \mid + \mid B \mid + \mid C \mid + C_1$. As in the case 3., by Lemma 3.12, $\mid W \mid + \mid B \mid + \mid C \mid + C_1 \leq n$. By the same Lemma, there exists a linear relation U in $\mathbb{C}^n$ such that the Weyr characteristic of U is $(\hat{W}, \hat{B}, \hat{C})$. Then $\dim U = \mid \hat{W} \mid + \mid \hat{B} \mid + \mid \hat{C} \mid + \hat{B}_1 = \mid W \mid + \mid B \mid + \mid C \mid + B_1 - 1 = d - 1$. On one hand, from (67), (63) and (62) and on the other hand, from (68), (61) and (64), by Theorem 7.2 there exists linear relations $\bar{S}$, $\bar{T}$ in $\mathbb{C}^n$ such that $\bar{S} \overset{s.e.}{\sim} S$, $\bar{T} \overset{s.e.}{\sim} T$ and $U \subseteq \bar{S} \cap \bar{T}$. As $\dim U = d - 1 \leq \dim(\bar{S} \cap \bar{T}) < \dim \bar{S} = d$, we have that $U = \bar{S} \cap \bar{T}$; hence $\dim \frac{\bar{S}}{\bar{S} \cap \bar{T}} = \dim \frac{\bar{T}}{\bar{S} \cap \bar{T}} = 1$.
If (50)-(52) hold, the proof is analogous.

□

Appendix A. Auxiliary results to prove Theorem 7.5

Lemma Appendix A.1 ([2, Lemma 5.5]). *Let $X \geq 0$ be a nonnegative integer and let $\mathbf{a} = (a_1, \ldots, a_m)$ be a finite sequence of nonnegative integers. Then there exists a finite sequence of nonnegative integers $\mathbf{g} = (g_1, \ldots, g_{m+1})$ such that $\mid \mathbf{g} \mid = X$ and $\mathbf{g} \prec' \mathbf{a}$.*

From Lemmas Appendix A.1 and 5.9, we obtain Lemma Appendix A.2.

Lemma Appendix A.2. *Let $X \geq 0$ be a nonnegative integer and let $A = (A_1, A_2, \ldots,)$ be a partition. Then there exists a partition $G = (G_1, G_2, \ldots)$ such that $\mid G \mid = X$ and $A \angle (A_1 + 1, G_1, G_2, \ldots)$.*

Lemma Appendix A.3 ([2, Lemma 5.8]). *Let $X, Y \geq 0$ be nonnegative integers and let $\mathbf{c} = (c_1, \ldots, c_m)$, $\mathbf{d} = (d_1, \ldots, d_m)$ be finite sequences of nonnegative integers such that $\mathbf{c} \neq \mathbf{d}$. Let $\ell = \max\{i : c_i \neq d_i\}$, $f = \max\{i \in \{1, \ldots, \ell\} : c_i < d_{i-1}\}$ and $f' = \max\{i \in \{1, \ldots, \ell\} : d_i < c_{i-1}\}$.*

1. *There exists a finite sequence of nonnegative integers* $\mathbf{g} = (g_1, \dots, g_{m+1})$ *such that* $|\mathbf{g}| = X$, $\mathbf{g} \prec' \mathbf{c}$ *and* $\mathbf{g} \prec' \mathbf{d}$ *if and only if*
$$X \leq \sum_{i=1}^{m} \min\{c_i, d_i\} + \max\{c_f, d_{f'}\}.$$
2. *If* $f > 1$ *and* $f' > 1$, *there exists a finite sequence of nonnegative integers* $\mathbf{e} = (e_1, \dots, e_{m-1})$ *such that* $|\mathbf{e}| = Y$, $\mathbf{c} \prec' \mathbf{e}$ *and* $\mathbf{d} \prec' \mathbf{e}$ *if and only if*
$$Y \geq \sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_f, d_{f'}\}.$$
3. *If* $f = 1$ *or* $f' = 1$, *there exists a finite sequence of nonnegative integers* $\mathbf{e} = (e_1, \dots, e_{m-1})$ *such that* $|\mathbf{e}| = Y$, $\mathbf{c} \prec' \mathbf{e}$ *and* $\mathbf{d} \prec' \mathbf{e}$ *if and only if*
$$Y = \textstyle\sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_f, d_{f'}\},$$
or
$$Y \geq \textstyle\sum_{i=1}^{m} \max\{c_i, d_i\} - \max\{c_{f+1}, d_{f'+1}\}.$$
Equivalently,
$$Y = \sum_{i=2}^{m} \max\{c_i, d_i\} \text{ or } Y \geq \max\{c_1, d_1\} + \sum_{i=3}^{m} \max\{c_i, d_i\}.$$

Lemma Appendix A.4 ([3, Lemma 4.7]). *Given two finite sequence of nonnegative integers* $\mathbf{a} = (a_1, \dots, a_m)$ *and* $\mathbf{b} = (b_1, \dots, b_m)$, *let* $x_i = \min\{a_i, b_i\}$, $1 \leq i \leq m$. *Let* $(r_1, r_2 \dots) = \overline{(a_1, \dots, a_m)}$, $(s_1, s_2 \dots) = \overline{(b_1, \dots, b_m)}$, *and* $y_i = \min\{r_i, s_i\}$, $i \geq 1$. *Then*
$$(y_1, \dots) = \overline{(x_1, \dots, x_m)}.$$

Analogously we can prove Lemma Appendix A.5.

Lemma Appendix A.5. *Given two finite sequence of nonnegative integers* $\mathbf{a} = (a_1, \dots, a_m)$ *and* $\mathbf{b} = (b_1, \dots, b_m)$, *let* $X_i = \max\{a_i, b_i\}$, $1 \leq i \leq m$. *Let* $(r_1, r_2, \dots) = \overline{(a_1, \dots, a_m)}$, $(s_1, s_2, \dots) = \overline{(b_1, \dots, b_m)}$, *and* $Y_i = \max\{r_i, s_i\}$, $i \geq 1$. *Then*
$$(Y_1, \dots) = \overline{(X_1, \dots, X_m)}.$$

Lemma Appendix A.6 ([3, Lemma 4.6]). *Given two sequences of nonnegative integers* $\mathbf{a} = (a_1, \dots, a_m)$ *and* $\mathbf{b} = (b_1, \dots, b_m)$ *such that* $\mathbf{a} \neq \mathbf{b}$, *let*
$$\ell = \max\{i \ : \ a_i \neq b_i\},$$
$$f = \max\{i \in \{1, \dots, \ell\} \ : \ a_i < b_{i-1}\}, \quad f' = \max\{i \in \{1, \dots, \ell\} \ : \ b_i < a_{i-1}\}.$$
Let $(r_1, r_2, \dots) = \overline{(a_1, \dots, a_m)}$, $(s_1, s_2, \dots) = \overline{(b_1, \dots, b_m)}$, $r_0 = s_0 = m$,
$$x = \min\{i \ : \ r_i \neq s_i\},$$
$$e = \min\{i \geq x - 1 : \ s_{i+1} \geq r_{i+1}\}, \quad e' = \min\{i \geq x - 1 : \ r_{i+1} \geq s_{i+1}\}.$$
Then
$$e = a_f, \quad e' = b_{f'}.$$

Remark Appendix A.7. *From Lemmas Appendix A.5 and Appendix A.6, we have* $\sum_{i\geq 1}\max\{r_i,s_i\}-\max\{e,e'\}=\sum_{i=1}^{m}\max\{a_i,b_i\}-\max\{a_f,b_{f'}\}$. *If* $f\leq f'$, *then* $\max\{a_f,b_f\}\geq\max\{a_f,b_{f'}\}$ *and, if* $f\geq f'$, *then* $\max\{a_{f'},b_{f'}\}\geq\max\{a_f,b_{f'}\}$. *Therefore*

$$\sum_{i\geq 1}\max\{r_i,s_i\}-\max\{e,e'\}=\sum_{i=1}^{m}\max\{a_i,b_i\}-\max\{a_f,b_{f'}\}\geq 0.$$

From Lemmas Appendix A.3-Appendix A.6 and 5.9, we obtain Lemma Appendix A.8.

Lemma Appendix A.8. *Let* $X,Y\geq 0$ *be nonnegative integers and let* $A=(A_1,A_2,\dots)$, $B=(B_1,B_2,\dots)$ *be partitions such that* $A\neq B$.

Let $x=\min\{i\ :\ A_i\neq B_i\}$,

$$e=\min\{i\geq x-1:\ B_{i+1}\geq A_{i+1}\},\quad e'=\min\{i\in\{i\geq x-1:\ A_{i+1}\geq B_{i+1}\}.$$

Let Z *be an integer such that* $Z\geq\max\{A_1,B_1\}$.

1. *There exists a partition* $G=(G_1,G_2,\dots)$ *such that* $\mid G\mid=X$, $(Z,A_1,\dots)\angle(Z+1,G_1,\dots)$ *and* $(Z,B_1,\dots)\angle(Z+1,G_1,\dots)$ *if and only if*
$$X\leq\sum_{i\geq 1}\min\{A_i,B_i\}+\max\{e,e'\}.$$
2. *If there exists a partition* $E=(E_1,E_2,\dots)$ *such that* $\mid E\mid=Y$, $(Z-1,E_1,\dots)\angle(Z,A_1,\dots)$ *and* $(Z-1,E_1,\dots)\angle(Z,B_1,\dots)$ *then*
$$Y\geq\sum_{i\geq 1}\max\{A_i,B_i\}-\max\{e,e'\}.$$
3. *If*
$$Y=\sum_{i\geq 1}\max\{A_i,B_i\}-\max\{e,e'\}.$$
then there exists a partition $E=(E_1,E_2,\dots)$ *such that* $\mid E\mid=Y$, $(Z-1,E_1,\dots)\angle(Z,A_1,\dots)$ *and* $(Z-1,E_1,\dots)\angle(Z,B_1,\dots)$.

Lemma Appendix A.9. *Let* $m_1,\dots,m_n$, $M_1,\dots,M_n$ *and* x *be integers such that*

$$\sum_{i=1}^{n}m_i\leq x\leq\sum_{i=1}^{n}M_i\ \text{and}\ m_i\leq M_i,\quad 1\leq i\leq n.$$

Then, there exist integers $x_1,\dots,x_n$ *such that*

$$\sum_{i=1}^{n}x_i=x\ \text{and}\ m_i\leq x_i\leq M_i,\quad 1\leq i\leq n.$$

And, if $m_1\geq\cdots\geq m_n$ *and* $M_1\geq\cdots\geq M_n$, *then* $x_1\geq\cdots\geq x_n$.

Proof. Let $k = \min\{j \geq 0 \;:\; x \leq \sum_{i=1}^{j} M_i + \sum_{i=j+1}^{n} m_i\}$. Then $k \leq n$.

If $k = 0$, then $x = \sum_{i=1}^{n} m_i$. Define $x_i = m_i$, $1 \leq i \leq n$. Then $x_1, \dots, x_n$ satisfy the conditions.

If $k > 0$, then

$$\sum_{i=1}^{k-1} M_i + \sum_{i=k}^{n} m_i < x \leq \sum_{i=1}^{k} M_i + \sum_{i=k+1}^{n} m_i. \tag{A.1}$$

Define

$$\begin{aligned} x_i &= M_i, \quad 1 \leq i \leq k-1, \\ x_k &= x - \textstyle\sum_{i=1}^{k-1} M_i - \sum_{i=k+1}^{n} m_i, \\ x_i &= m_i, \quad k+1 \leq i \leq n. \end{aligned}$$

It is clear that $\sum_{i=1}^{n} x_i = x$ and that $m_i \leq x_i \leq M_i$ for $1 \leq i \leq k-1$ and $k+1 \leq i \leq n$. From (A.1), we obtain $m_k < x_k \leq M_k$.

If $m_1 \geq \cdots \geq m_n$ and $M_1 \geq \cdots \geq M_n$, then $x_1 \geq \cdots \geq x_{k-1}$ and $x_{k+1} \geq \cdots \geq x_n$. Moreover, $x_{k-1} = M_{k-1} \geq M_k \geq x_k > m_k \geq m_{k+1} = x_{k+1}$.

□

Acknowledgements

The first and third authors acknowledge the support of the grant PID2021-124827NB-I00 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe" by the "European Union". The second author acknowledges the support from CONICET PIP 11220200102127CO, UNLP 11X974. The second, third and fourth authors acknowledge the support from the Deutsche Forschungsgemeinschaft DFG TR903/24-1.

References

[1] R. Arens, Operational calculus of linear relations, Pacific J. Math. (1961), 9–23.

[2] I. Baragaña and A. Roca, Rank-one perturbations of matrix pencils, Linear Alg. Appl. 606 (2020), 170-–191.

[3] I. Baragaña and A. Roca, On the Change of the Weyr Characteristics of Matrix Pencils After Rank-One Perturbations, SIAM J. Matrix Anal. Appl. 43 (2022), 981–1002.

[4] T. Berger, C. Trunk, and H. Winkler, Linear relations and the Kronecker canonical form, Linear Algebra Appl. 488 (2016), 13–44.

[5] T. Berger, H. de Snoo, C. Trunk, and H. Winkler, Linear relations and their singular chains, Methods Funct. Anal. Topology 27 (2021), 287–301.

[6] T. Berger, H. de Snoo, C. Trunk, and H. Winkler, A Jordan-like Decomposition for Linear Relations in Finite-Dimensional Spaces, https://arxiv.org/abs/2209.14234, 2022.

[7] R. Cross, Multivalued Linear Operators, Monographs and Textbooks in Pure and Applied Mathematics 213, Marcel Dekker, Inc., New York, 1998.

[8] M. Dodig and M. Stošić, On convexity of polynomial paths and generalized majorizations, The Electronic Journal of Combinatorics 17 (2010) #61.

[9] M. Dodig and M. Stošić, The general matrix pencil completion problem: A minimal case, SIAM J. Matrix Anal. Appl. 40 (2019), 347-–369.

[10] F. Gantmacher, The theory of Matrices, Chelsea, New York, 1959.

[11] H. Gernandt, F. Martínez Pería, F. Philipp, and C. Trunk, On characteristic invariant of matrix pencils and linear relations, SIAM J. Matrix Anal. Appl. 44 (2023), 1510–1539.

[12] L. Leben, F. Martínez Pería, F. Philipp, C. Trunk, and H. Winkler, Finite rank perturbations of linear relations and singular matrix pencils, Complex Anal. Oper. Theory 15 (2021), 37.

[13] R. A. Lippert and G. Strang, The Jordan forms of AB and BA, Electronic Journal of the International Linear Algebra Society, 18 (2009), 281–288.

[14] A. Sandovici, On the Adjoint of Linear Relations in Hilbert Spaces, Mediterr. J. Math. (2020) 17:68.

[15] A. Sandovici, H. de Snoo, and H. Winkler, The structure of linear relations in Euclidean spaces, Linear Algebra Appl. 397 (2005), 141–169.