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Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
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On the essential spectrum of operator pencils

Hassen Khlif[1], Carsten Trunk[2], and Mitsuru Wilson[2]

[1] University of Sfax, Department of Mathematics BP1171, Sfax 3000 Tunisia, hassen.khelif7@gmail.com
[2] Ilmenau University of Technology, PF 10 05 65, D–98684 Ilmenau, Germany, carsten.trunk@tu-ilmenau.de, mitsuru.wilson@tu-ilmenau.de

Abstract

For a closed densely defined linear operator A and a bounded linear operator B on a Banach space X whose essential spectrums are contained in disjoint sectors, we show that the essential spectrum of the associated operator pencil $\lambda A + B$ is contained in a sector of the complex plane whose boundaries are determined purely by the angles that define the two sectors, which contain the essential spectrums of A and B.

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1 Introduction

It is a well-known fact that the essential spectrum of a linear operator is invariant under compact perturbations. Here we understand the essential spectrum as the complement of the Fredholm domain. In many applications, e.g. in mathematical physics or in transport theory, one is interested in the (essential) spectrum of operator pencils, see, e.g., [3, 4]. There are also applications in the theory of differential-algebraic equations, see [12].

A linear operator pencil is a first order polynomial with operators as coefficients, that is, it is of the form

$$P(\lambda) = \lambda A + B,$$

where $\lambda \in \mathbb{C}$, A is a closed and B a bounded operator acting in a Banach space. By definition (see, e.g., [6, 10]) a complex number λ is in the spectrum of the pencil P if zero is in the spectrum of the operator $\lambda A + B$. In the same way the essential spectrum of P is defined as the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda A+B$ is no Fredholm operator. Obviously, it follows immediately from this definition that the essential spectrum of such an operator pencils remains unchanged if the coefficients A and B are perturbed by a compact operator. For a somehow different class of perturbation results which are based on the theory of linear relations we refer to [2].

Here, we investigate the question what can be said of the essential spectrum of the pencil P if the essential spectra of its coefficients A and of B lie in two disjoint sectors of the complex plane. If in addition, A and B fulfill some mild commutation property (i.e., the commutator is compact) then it is the main result of this paper to determine a sector in the complex plane which contains the essential spectrum of P, cf. Theorem 3.2.

2 Preliminaries

Here we recall some standard terminology connected with Fredholm operators and the essential spectrum of operators and operator pencils. For this let X be a Banach space and $\mathcal{L}(X)$ be the set of bounded operators on X. We denote by $\mathcal{K}(X) \subset \mathcal{L}(X)$ the set of compact operators and by $\mathcal{C}(X)$ the closed densely defined operators. An operator $F \in \mathcal{C}(X)$ is called a Fredholm operator if the dimensions of its kernel and cokernel is finite and the set of Fredholm operators in X is denoted by $\Phi(X) \subset \mathcal{C}(X)$. Consequently, its Fredholm index

$$\mathrm{ind}(F) = \dim(\ker(F)) - \dim(\mathrm{coker}(F))$$

is well defined and its value is finite, see, e.g., [5]. For some operator $A \in \mathcal{C}(X)$ we call

$$\Phi_A = \{\lambda \in \mathbb{C} : \ \lambda I - A \in \Phi(X)\} \subset \mathbb{C}$$

the Fredholm domain of $A \in \mathcal{C}(X)$ and set

$$\Phi_{A,0} = \{\lambda \in \mathbb{C} : \ \mathrm{ind}(\lambda I - A) = 0\} \subset \Phi_A.$$

The set $\sigma_e(A) := \mathbb{C}\backslash\Phi_A$ is called the essential spectrum. Note that the boundary $\partial\Phi_{A,0}$ of $\Phi_{A,0}$ belongs to $\sigma_e(A)$ [5, Chapter IV 5.4].

In what follows, we will drop the letter I in $\lambda I - A$ and write for simplicity $\lambda - A$ instead of $\lambda I - A$. The spectrum $\sigma(A)$ of $A \in \mathcal{C}(X)$ is defined as

$$\sigma(A) := \{\lambda \in \mathbb{C} : (\lambda - A)^{-1} \text{ has a bounded inverse}\}$$

and the resolvent as $\rho(A) := \mathbb{C} \setminus \sigma(A)$. For $\lambda \in \rho(A)$, the resolvent operator is defined as

$$R_\lambda(A) := (\lambda - A)^{-1} \in \mathcal{L}(X)$$

Let $A \in \mathcal{C}(X)$ and $B \in \mathcal{L}(X)$. An operator pencil P associated to A and B is the map

$$\begin{aligned} P : \mathbb{C} &\to \mathcal{C}(X) \\ \lambda &\mapsto \lambda A + B, \end{aligned}$$

The essential spectrum of the operator pencil P is defined by

$$\sigma_e(P) = \{\lambda \in \mathbb{C} \mid 0 \in \sigma_e(\lambda A + B)\} .$$

If not otherwise stated, the branch of the argument $\arg z$ of a complex number $z \in \mathbb{C}$ is $[0, 2\pi)$. Sometimes it is convenient, for the sake of brevity, to make an exception from this rule. Whenever we do such an exception, it is stated explicitly.

3 Main result

In what follow, we wish to determine the sector in the complex plane which contain the essential spectrum of the operator pencil $P(\lambda) = \lambda A + B$. In the present article, we base our results on the following assumption.

Assumption (A).

(i) The operators A and B satisfy

$$A \in \mathcal{C}(X) \quad \text{and} \quad B \in \mathcal{L}(X).$$

(ii) Let S_1 and S_2 be two sectors,

$$S_1 = \{\lambda \mid \varphi_1 \leqslant \arg \lambda \leqslant \varphi_2\} \quad \text{and} \quad S_2 = \{\lambda \mid \theta_1 \leqslant \arg \lambda \leqslant \theta_2\},$$

with $\varphi_1, \varphi_2, \theta_1, \theta_2 \in [0, 2\pi)$ such that

(a) $\varphi_1 \leqslant \varphi_2 < \theta_1 \leqslant \theta_2$, that is, $S_1 \cap S_2 = \{0\}$,

(b) $\sigma_e(A) \subset S_1$ and $\sigma_e(B) \subset S_2$,

(c) $\mathbb{C} \setminus \sigma_e(A) \subset \Phi_{A,0}$ and assume that $\Phi_{A,0}$ is connected,

(d) there exists $\lambda_0 \in \rho(A)$ such that $\lambda_0 \in \mathbb{C} \setminus S_1$.

Remark 3.1. Obviously, by imposing Assumption (A), the essential spectra of the operators A and B are contained in two different sectors such that the essential spectrum of A is in a sector with the smaller arguments, i.e., $\varphi_1 \leqslant \varphi_2 < \theta_1 \leqslant \theta_2$. Moreover item (ii) in Assumption (A) implies that the essential spectra of A and of B are not contained in a neighbourhood of the positive real axis. These two conditions (essential spectrum of A is contained in the sector with the smaller arguments and the essential spectra of A and of B are not contained in a neighbourhood of $\mathbb{R}^+$) are no restrictions. Multiplying both operators A and B by $\exp(i\theta)$ for an appropriate $\theta \in (0, 2\pi)$ leads to the operators $\exp(i\theta)A$ and $\exp(i\theta)B$. These operators satisfy Assumption (A) for an appropriate angle $\theta \in (0, 2\pi)$. The essential spectrum of A and B are rotated proportionally to θ. Likewise if the essential spectrum of A or B is contained in a neighbourhood of the positive x-axis, it can be dilated such that the positive x-axis does not intersect the essential spectrum.

Our main result is the following theorem, which characterizes the sector W that contains the essential spectrum of the operator pencil. The angles of this sector is calculated as $-\pi < \theta_1 - \varphi_2 - \pi \leqslant \theta_2 - \varphi_1 - \pi < \pi$ (oriented counterclockwise) and Σ may contain the positive real line. Note, that here, for the sake of simplicity, we allow the arguments to be between $-\pi$ and $+\pi$. The key tool for proving this result is Lemma 4.2 below.

Theorem 3.2. *Suppose that Assumption (A) holds and that there exist a natural number n and a compact operator $K \in \mathcal{K}(X)$ such that $B : D(A^n) \to D(A)$ and*

$$ABx = BAx + Kx \text{ for all } x \in D(A^n).$$

Then, the essential spectrum $\sigma_e(P)$ of the operator pencil

$$P(\lambda) = \lambda A + B$$

is contained in the sector Σ defined by the angles between $\theta_1 - \varphi_2 - \pi$ and $\theta_2 - \varphi_1 - \pi$ oriented counterclockwise, where these angles lie between $-\pi$ and π.

Some auxilliary statements are necessary in order to prove Theorem 3.2 and this proof is presented at the end of Section 4.

4 Proof of Theorem 3.2

The following theorem is essentially contained in [8]. Our statement and proof are adopted to the present setup, and we include the proof for completeness.

Theorem 4.1. *Suppose that Assumption (A) hold, there exist a natural number n and a compact operator $K \in \mathcal{K}(X)$ such that $B : D(A^n) \to D(A)$ and*

$$ABx = BAx + Kx \text{ for all } x \in D(A^n).$$

Then, for $\lambda \in \rho(A_1) \cap \rho(B)$, we have

$$\mathcal{R}_\lambda(A_1)\mathcal{R}_\lambda(B) = \mathcal{R}_\lambda(B)\mathcal{R}_\lambda(A_1) + K' \tag{4.1}$$

where $A_1 = \gamma - A$ for some $\gamma \in \mathbb{C}$ and $K' \in \mathcal{K}(X)$. Moreover, we have

$$\sigma_e(A + B) \subseteq \sigma_e(A) + \sigma_e(B). \tag{4.2}$$

Proof. We prove (4.1). Indeed, let $x \in D(A^n)$ then we have

$$\begin{aligned}
(\lambda - A_1)(\lambda - B)x =& (\lambda - \gamma + A)(\lambda - B)x \\
=& (\lambda^2 - \lambda B - \lambda\gamma + \gamma B + \lambda A - AB)x \\
=& (\lambda^2 - \lambda B - \lambda\gamma + \gamma B + \lambda A - BA)x - Kx \\
=& (\lambda - B)(\lambda - A_1)x - Kx.
\end{aligned} \tag{4.3}$$

We multiply on the left and on the right by $(\lambda - B)^{-1}$ and obtain for $x \in D(A^n)$

$$(\lambda - B)^{-1}(\lambda - A_1)x = (\lambda - A_1)(\lambda - B)^{-1}x - K_1 x \tag{4.4}$$

where $K_1 \in \mathcal{K}(X)$. By multiplying Equation (4.4) on the right by $(\lambda - A_1)^{-1}$ one obtains

$$(\lambda - B)^{-1}x = (\lambda - A_1)(\lambda - B)^{-1}(\lambda - A_1)^{-1}x - K_1(\lambda - A_1)^{-1}x,$$

which equal to

$$(\lambda - A_1)^{-1}(\lambda - B)^{-1}x = (\lambda - B)^{-1}(\lambda - A_1)^{-1}x + K'x$$

for $K'x = -(\lambda - A_1)^{-1}K_1(\lambda - A_1)^{-1}$.

Since $(\lambda - A_1)^{-1}(\lambda - B)^{-1}$ and $(\lambda - B)^{-1}(\lambda - A_1)^{-1} + K'$ are bounded and $D(A^n) = D((\gamma - A)^n) = D((A_1^n) = D((\lambda - A_1)^n)$ is dense by [9, Theorem 2.5], (4.1) follows by continuity.

If $\sigma_e(A) + \sigma_e(B)$ is the entire complex plane, the theorem holds trivially. Therefore, we assume $\sigma_e(A) + \sigma_e(B) \neq \mathbb{C}$. Let $\gamma \notin \sigma_e(A) + \sigma_e(B)$. We show that $\gamma \in \Phi_{A+B}$. If $\lambda \in \sigma_e(B)$ then $(\gamma - \lambda) \in \Phi_A$. Now, let $A_1 = \gamma - A$. We have

$$\begin{aligned}
\gamma - \lambda \in \Phi_A &\Leftrightarrow (\gamma - \lambda) - A \in \Phi(X) \\
&\Leftrightarrow A_1 - \lambda \in \Phi(X) \\
&\Leftrightarrow \lambda - A_1 \in \Phi(X) \\
&\Leftrightarrow \lambda \in \Phi_{A_1}.
\end{aligned}$$

The same arguments imply $\gamma - \lambda \in \Phi_{A,0} \Leftrightarrow \lambda \in \Phi_{A_1,0}$. This and item (ii) (c) in Assumption (A) imply

$$\sigma_e(B) \subset \Phi_{A_1,0}. \tag{4.5}$$

As B is bounded, Assumption (A) and (4.5) imply that there exists an open, bounded set $U \subset \mathbb{C}$ with

$\sigma_e(B) \subset U \subset \Phi_{A_1,0}$,

∂U consists of finite many rectifiable arcs.

As B is bounded, the unbounded component of Φ_B has a non-empty intersection with $\rho(B)$. Hence, it consists only of Fredholm operators of index zero. As $\partial\Phi_{B,0} \subset \sigma_e(B)$, we obtain $\mathbb{C} \setminus U \subset \Phi_{B,0}$. By the punctured neighbourhood theorem, in this component the spectrum of B consists only of isolated normal eigenvalues [5, Chapter VI , 5.4].

The same argument applies to the operator A_1. Item (c) in Assumption (A) implies that $\Phi_{A,0}$ is connected and, hence, $\Phi_{A_1,0}$ is connected, By item (d) the component $\Phi_{A_1,0}$ has non-empty intersection with the resolvent set of A_1. Therefore, by the same arguments as above, there are at most finitely many (isolated) eigenvalues of A_1 and of B on ∂U. After changing U slightly, we find a neighbourhood $\widetilde{U}$ with $\partial\widetilde{U}$ consists of finite many rectifiable arcs which satisfies $\partial\widetilde{U} \subset \rho(A_1) \cap \rho(B)$ and $\sigma_e(B) \subset \widetilde{U} \subset \Phi_{A_1,0}$.

Define the two operators S_1 and S_2 by

$$S_1 = \frac{-1}{2i\pi} \int_{\partial\widetilde{U}} \mathcal{R}_\lambda(A_1)\mathcal{R}_\lambda(B)d\lambda,$$
$$S_2 = \frac{-1}{2i\pi} \int_{\partial\widetilde{U}} \mathcal{R}_\lambda(B)\mathcal{R}_\lambda(A_1)d\lambda.$$

We will show that there is a compact operator $\widetilde{K}$ such that $(\gamma-B-A)S_1 = I+\widetilde{K}$. Indeed, we have

$$\gamma - B - A = (\gamma - \lambda - A) + (\lambda - B) = -(\lambda - A_1) + (\lambda - B).$$

Thus,

$$(\gamma - B - A)S_1 = \frac{1}{2i\pi} \int_{\partial\widetilde{U}} \mathcal{R}_\lambda(B)d\lambda - \frac{1}{2i\pi} \int_{\partial\widetilde{U}} (\lambda - B)\mathcal{R}_\lambda(A_1)\mathcal{R}_\lambda(B)d\lambda.$$

The first integral in the right hand side equals the spectral projection $P_{\widetilde{U}}(B)$ onto the spectral set $\widetilde{U}$ of B, see, e.g., [1, VII.3]. By the above construction, we see that $\widetilde{U}$ contains the spectrum of B except for at most finitely many isolated eigenvalues of B. Hence, $P_{\widetilde{U}}(B) = I - K,\ K \in \mathcal{K}(X)$. On the other hand, applying Equation (4.1) on the second integral in the right hand side we obtain

$$\frac{-1}{2i\pi} \int_{\partial\widetilde{U}} (\lambda - B)\mathcal{R}_\lambda(A_1)\mathcal{R}_\lambda(B)d\lambda = \frac{-1}{2i\pi} \int_{\partial\widetilde{U}} \mathcal{R}_\lambda(A_1)d\lambda - \frac{1}{2i\pi} \int_{\partial\widetilde{U}} (\lambda - B)K'd\lambda$$
$$\in \mathcal{K}(X)$$

since $\frac{1}{2i\pi} \int_{\partial\widetilde{U}} \mathcal{R}_\lambda(A_1)d\lambda$ is the spectral projection onto the spectral set $\widetilde{U}$ of A which contains only finitely many isolated eigenvalues of A_1.

By the same reasoning, we deduce that

$$\begin{aligned} S_2(\gamma - B - A) &= \frac{1}{2i\pi} \int_{\partial\widetilde{U}} \mathcal{R}_\lambda(B)d\lambda - \frac{1}{2i\pi} \int_{\partial\widetilde{U}} \mathcal{R}_\lambda(B)\mathcal{R}_\lambda(A_1)(\lambda - B)d\lambda \\ &= I + \widehat{K}, \end{aligned}$$

where $\widehat{K} \in \mathcal{K}(X)$. Consequently, using [9, Lemma 2.4], we can deduce that $(\gamma - B - A) \in \Phi(X)$. □

Lemma 4.2. *Suppose that assumptions in Theorem 4.1 are fulfilled. Let* $\alpha \in \mathbb{C}$. *Then,* $\alpha\sigma_e(A) \subset \mathbb{C} \setminus (-S_2)$ *implies that*

$$\alpha \notin \sigma_e(P).$$

Proof. Suppose that $0 \in \alpha\sigma_e(A) + \sigma_e(B)$. Let $\alpha_1 \in \alpha\sigma_e(A)$ and $\alpha_2 \in \sigma_e(B)$ such that $0 = \alpha_1 + \alpha_2$.

Since $\alpha_2 \in \sigma_e(B) \subset S_2$, we obtain that

$$\alpha_1 = -\alpha_2 \in -S_2,$$

which contradicts $\alpha_1 \in \alpha\sigma_e(A) \in \mathbb{C} \setminus (-S_2)$. Subsequently,

$$0 \notin \alpha\sigma_e(A) + \sigma_e(B). \tag{4.6}$$

Thus, using Theorem 4.1, we obtain $0 \notin \sigma_e(\alpha A + B)$. Therefore,

$$\alpha \notin \sigma_e(P).$$

□

In order to make use of Lemma 4.2 in the proof of Theorem 3.2, one needs the defining angles of $-S_2$. These defining angles could be expressed as $\theta_1 - \pi$ and $\theta_2 - \pi$. This is an easy description but now the values of $\theta_1 - \pi$ and $\theta_2 - \pi$ are no longer in $[0, 2\pi]$.

Proof of Theorem 3.2. Let

$$\tilde{\theta} \in [0, \theta_1 - \varphi_2) \cup (\theta_2 - \varphi_1, 2\pi) \quad \text{and} \quad z \in \sigma_e(A). \tag{4.7}$$

Note that $\arg z \in [\varphi_1, \varphi_2]$. We see that

$$\tilde{\theta} + \arg z \in [\varphi_1, \theta_1) \cup (\theta_2, 2\pi + \varphi_2)$$

Hence, for some $r > 0$ we set

$$\tilde{\alpha} := re^{i\tilde{\theta}}$$

and we have

$$\arg \tilde{\alpha}z \in [0, \theta_1) \cup (\theta_2, 2\pi)$$

as $\varphi_1 \leqslant \varphi_2 < \theta_1$ by Assumption (A), that is

$$\tilde{\alpha}z \in \mathbb{C} \setminus S_2. \tag{4.8}$$

Set

$$\theta := \tilde{\theta} - \pi \quad \text{and} \quad \alpha := re^{i\theta}.$$

Then together with (4.8)

$$\alpha z = re^{i\theta}z = -re^{\tilde{\theta}i}z = -\tilde{\alpha}z \in \mathbb{C} \setminus \{-S_2\}$$

for all $z \in \sigma_e(A)$. Hence, by Lemma 4.2,

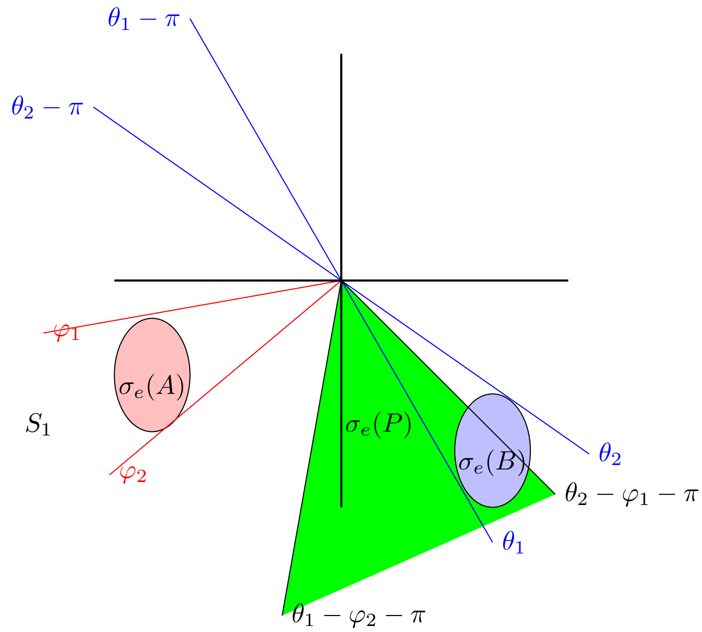
$$\alpha \notin \sigma_e(P).$$

Note that $\theta = \tilde{\theta} - \pi$ in the above proof takes values in $[-\pi, \pi)$, more precisely, by (4.7),

$$\theta \in [-\pi, \theta_1 - \varphi_2 - \pi) \cup (\theta_2 - \varphi_1 - \pi, \pi)$$

and the statement of the theorem follows. □

Figure 1: The essential spectrum $\sigma_e(P)$ of the operator pencil is contained in the wedge between $\theta_1 - \varphi_2 - \pi = 260°$ mod 2π and $\theta_2 - \varphi_1 - \pi = 315°$ mod 2π (shaded in green) for $\varphi_1 = 190°, \varphi_2 = 220°, \theta_1 = 300°, \theta_2 = 325°$ in view of Theorem 3.2.



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