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LIMIT POINT AND LIMIT CIRCLE TRICHOTOMY FOR STURM-LIOUVILLE PROBLEMS WITH COMPLEX POTENTIALS

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Abstract. The limit point and limit circle classification of real Sturm-Liouville problems by H. Weyl more than 100 years ago was extended by A.R. Sims around 60 years ago to the case when the coefficients are complex. Here the main result is a collection of various criteria which allow us to decide to which class of Sims' scheme a given Sturm-Liouville problem with complex coefficients belongs. This is subsequently applied to a second order differential equation defined on a ray in $\mathbb{C}$ which is motivated by the recent intensive research connected with $\mathcal{PT}$-symmetric Hamiltonians.

1. Introduction

The search for criteria that guarantee that a Sturm-Liouville equation with real coefficients is in the limit point or the limit circle case has a long tradition since the seminal paper of H. Weyl [18] in 1910 where he classifies Sturm-Liouville problems into two classes: Either all solutions of the eigenvalue problem are square integrable (limit circle) or there exists at least one solution without this property (limit point). This behaviour is independent of the chosen eigenvalue parameter.

Sturm-Liouville problems with *complex-valued* coefficients were investigated in the (also seminal) paper by A.R. Sims in 1957 [17] with further refinements in [6] and [15, 16]. The classification proposed by A.R. Sims contains three different cases, where one takes into account also the behaviour of the derivative of the solutions (for details we refer to Section 2).

Our interest in the classification proposed by A.R. Sims arises from a second order differential equation defined on a ray in $\mathbb{C}$. This is motivated by the recent intensive research connected with $\mathcal{PT}$-symmetric Hamiltonians, cf. [5]. In the seminal paper by C.M. Bender and S. Boettcher [5] a new view at Quantum Mechanics was proposed which adopts all its axioms except the one that restricts the Hamiltonian to be Hermitian, relaxing it to the assumption that the Hamiltonian is $\mathcal{PT}$-symmetric. Here, $\mathcal{P}$ is parity and $\mathcal{T}$ is time reversal. Since then, $\mathcal{PT}$-symmetric Hamiltonians have been analyzed intensively by many authors. In [12] $\mathcal{PT}$-symmetry was embedded into a more general mathematical framework: pseudo-Hermiticity or, what is the same, self-adjoint operators in Krein spaces, [1, 9, 10, 11]. For a general introduction into $\mathcal{PT}$-symmetric Quantum Mechanics we refer to [13] and [3].

Let Γ be a ray in the complex plane with angle $\phi \in (-\pi/2, \pi/2)$,

$$\Gamma := \{z \in \mathbb{C} : z = xe^{i\phi}, x \in [a, \infty)\}$$

for some $a \geq 0$. Our main interest is to obtain a Weyl criterion for the differential equation

$$-y(z)'' + \mathbf{q}(z)y(z) = \lambda y(z), \qquad z \in \Gamma, \tag{1.1}$$

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where $\mathbf{q} : \Gamma \to \mathbb{C}$ is locally integrable.

A prominent class of potentials consists of the $\mathcal{PT}$-symmetric potentials

$$\mathbf{q}(z) := -(iz)^{N+2}$$

where N is a positive integer [5, 4]. Other Hamilitonians can be found in [7, 13].

Via a parametrization (1.1) can be mapped back to the real line leading to a Sturm-Liouville problem on an interval of the form

$$-\big(py'\big)' + qy = w\lambda y. \tag{1.2}$$

We give an asymptotic approximation for its solutions in Section 3 via an approach based on [8]. A careful analysis of these asymptotic approximations leads to new criteria for limit point/limit circle cases in the sense of A.R. Sims for the equation in (1.2) (see Section 4) and then, via the parametrization, also for (1.1), see Section 5.

Notations. For $-\infty < a < b \le \infty$ we denote by $\mathrm{AC}_{\mathrm{loc}}(a, b)$ the set of locally absolutely continuous functions on each compact subinterval of (a, b). For a locally integrable function $w : (a, b) \to \mathbb{C}$ we set $L^2_w(a, b) := \{f : (a, b) \to \mathbb{C} : f \text{ measurable}, \int_a^b |f(x)|^2 |w(x)|\,\mathrm{d}x < \infty\}$. If $w = 1$, then we write $L^2(a, b)$. Recall that the normed space of uniformly locally integrable functions $L^1_{\mathrm{u}}(a, b)$ is defined as

$$L^1_{\mathrm{u}}(a, b) = \left\{ f \in L^1_{\mathrm{loc}}(a, b) : \sup_{n\in\mathbb{Z}} \int_{[n,n+1]\cap(a,b)} |f(t)|\,\mathrm{d}t < \infty \right\}.$$

2. Weyl's alternative for complex Sturm-Liouville problems

Consider the Sturm-Liouville problem

$$-\big(p(x)y'(x)\big)' + q(x)y(x) = w(x)\lambda y(x), \quad x \in [a, b) \tag{2.1}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{\infty\}$, $\lambda \in \mathbb{C}$ and w, $1/p$, $q : [a, b) \to \mathbb{C}$ are locally integrable in $[a, b)$ and satisfy $w(x) > 0$, $p(x) \neq 0$ for a.a. $x \in [a, b)$. Here we always assume that the end point a is regular and b is singular (which is indicated by writing $x \in [a, b)$ or $L^1_{\mathrm{loc}}[a, b)$). A **solution** for (2.1) is a function y such that y, $py' \in \mathrm{AC}_{\mathrm{loc}}(a, b)$ and y satisfies (2.1) for a.a. $x \in [a, b)$.

Given $A \subset \mathbb{C}$, we define $\overline{\mathrm{co}}(A)$ as the closed convex hull of A. We impose

$$Q := \overline{\mathrm{co}} \left\{ \frac{q(x)}{w(x)} + rp(x) \ : \ 0 < r < \infty,\ x \in [a, b) \right\} \neq \mathbb{C}. \tag{2.2}$$

Let $\lambda \notin Q$. Then there exists a unique point $K \in Q$ which minimizes the distance between λ and Q. Moreover, there exists an angle θ such that

$$\mathrm{Re}[e^{i\theta}\,(z - K)] \ge 0 \quad \text{for all } z \in Q. \tag{2.3}$$

In fact, let $Q - K := \{z \in \mathbb{C} : z = w - K, w \in Q\}$. Since the set $\mathrm{Int}(Q - K) = \mathrm{Int}(Q) - K$ is open and convex and does not intersect the trivial subspace $\{0\}$, the geometric form of the Hahn-Banach Theorem (see e.g. [14, Theorem 7.7.4]) shows that there is a linear functional $f : \mathbb{C} \to \mathbb{C}$ such that $\mathrm{Re}[f(z)] > 0$ for all $z \in \mathrm{Int}(Q) - K$. Without restriction we may assume that f is normalized. Hence there exists a real number θ such that $f(z) = e^{i\theta}z$ for all $z \in \mathbb{C}$ and $\mathrm{Re}[e^{i\theta}v] > 0$ for all $v \in \mathrm{Int}(Q) - K$.

For $K \in \mathbb{C}$ and $\theta \in \mathbb{R}$ we define the open half-plane

$$\Lambda_{K,\theta} := \left\{ z \in \mathbb{C} : \mathrm{Re}[e^{i\theta}\,(z - K)] < 0 \right\},$$

and the set

$$S := \{(\theta, K) : (2.3) \text{ is satisfied}\}.$$

In [6] it is proved that the Sturm-Liouville problem (2.1) falls in exactly one of the cases of the next definition.

Definition 2.1. Given $(\theta, K) \in S$ and $\lambda \in \Lambda_{K,\theta}$, we have the following cases:

(1) There is, up to a multiplicative constant, only one solution y of the equation (2.1) such that

$$\int_a^b \operatorname{Re}(e^{i\theta} p)|y'|^2 dt + \int_a^b \operatorname{Re}[e^{i\theta}(q - Kw)]|y|^2 dt + \int_a^b w|y|^2 dt < \infty \tag{2.4}$$

and this is the only solution belonging to $L^2_w(a,b)$. In this case we say that (2.1) is in the **limit point I case**.

(2) All solutions of (2.1) are in $L^2_w(a,b)$ but, up to a multiplicative constant, there is only one solution that satisfies (2.4). In this case we say that (2.1) is in the **limit point II case**.

(3) All solutions of (2.1) are in $L^2_w(a,b)$ and all solutions satisfy (2.4). In this case we say that (2.1) is in the **limit circle case**.

Remark 2.2. In the situation of Definition 2.1 (3) we have

$$\operatorname{Re}(e^{i\theta}(q - Kw)) = \operatorname{Re}(we^{i\theta}(\frac{q}{w} - K)) = w \operatorname{Re}(e^{i\theta}(\frac{q}{w} - K)) \geq 0.$$

Hence the three summands on the left hand side of (2.4) are nonnegative and therefore, if a solution of (2.1) satisfies (2.4), then it is automatically in $L^2_w(a,b)$.

Remark 2.3. In [6, Remark 2.2] the method of variation of parameters is used to deduce that the classification is independent of λ, that is

- If all solutions satisfy (2.4) for some $\lambda_0 \in \Lambda_{\theta,K}$, then the same is true for all $\lambda \in \mathbb{C}$.
- If all solutions are in $L^2_w(a,b)$ for some $\lambda_0 \in \mathbb{C}$, then the same is true for all $\lambda \in \mathbb{C}$.

Note that $\theta = \frac{\pi}{2}$ in the case of real coefficients. Hence the first two terms in (2.4) are zero and therefore the limit point case II is not possible for real Sturm-Liouville equations.

3. Asymptotic approximation of second order differential equations

In this section we find an asymptotic approximation of solutions of (3.1). Since (1.1) can be transformed into such an equation, this will allow us to establish limit point/limit circle criteria in Section 5. Our approximations of the solutions are primarily based on [8, Theorem 1.3.1]. Consider the following differential equation:

$$(py')'(x) = s(x)y(x), \quad x \in [a,b) \tag{3.1}$$

where $s, p : [a,b) \to \mathbb{C}$ are functions such that $\frac{1}{p}, s \in L^1_{\text{loc}}[a,b)$. In what follows we define the nth root of a complex number $z = re^{i\arg(z)}$ with $-\pi < \arg(z) \leq \pi$ as $z^{1/n} = r^{1/n}e^{i\arg(z)/n}$. Our first theorem is a variation of Theorem 2.5.1 in [8]. It leads to slightly different assertions which form the basis for the subsequent section.

Theorem 3.1. Assume that $p(x) \neq 0$, $s(x) \neq 0$, $\arg\frac{s(x)}{p(x)} \neq \pi$, $\arg p(x)s(x) \neq \pi$ for all $x \in [a,b)$ and let $u := (ps)^{-1/4}$. Assume that $u, pu' \in \text{AC}_{\text{loc}}(a,b)$ and $u(pu')' \in L^1(a,b)$. Then there exist

$F_j, \widehat{F}_j : [a,b) \to \mathbb{C}$ and a fundamental system $\{y, \widehat{y}\}$ for (3.1) with

$$\begin{aligned} y(x) &= (p(x)s(x))^{-1/4} e^{-\int_a^x \sqrt{s(t)/p(t)}\,\mathrm{d}t}(F_1(x)+1), \quad x \in [a,b), \\ ((ps)^{1/4}y)'(x) &= (s(x)/p(x))^{1/2} e^{-\int_a^x \sqrt{s(t)/p(t)}\,\mathrm{d}t}(F_2(x)-1), \quad x \in [a,b), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \widehat{y}(x) &= (p(x)s(x))^{-1/4} e^{\int_a^x \sqrt{s(t)/p(t)}\,\mathrm{d}t}(\widehat{F}_1(x)+1), \qquad x \in [a,b), \\ ((ps)^{1/4}\widehat{y})'(x) &= (s(x)/p(x))^{1/2} e^{\int_a^x \sqrt{s(t)/p(t)}\,\mathrm{d}t}(\widehat{F}_2(x)-1), \qquad x \in [a,b), \end{aligned} \tag{3.3}$$

and, for $j = 1,2$,

$$\|F_j\|_\infty,\ \|\widehat{F}_j\|_\infty \le 2e^{2M} - 2, \quad \text{and} \quad F_j(x),\ \widehat{F}_j(x) \to 0, \quad \text{when } x \to b,$$

where $\|\cdot\|_\infty$ is the supremum norm and $M := \|u(pu')'\|_{L^1}$.

Proof. Note that y is a solution for (3.1) if and only if Y is a solution for $Y'(x) = A(x)Y(x)$ where

$$Y(x) := \begin{pmatrix} y(x) \\ p(x)y'(x) \end{pmatrix} \quad \text{and} \quad A(x) := \begin{pmatrix} 0 & 1/p(x) \\ s(x) & 0 \end{pmatrix} \quad x \in [a,b). \tag{3.4}$$

We call Y a solution for $Y' = AY$ if Y is locally absolutely continuous in $[a,b)$ and satisfies the differential equation (3.4) for a.a. $x \in [a,b)$.

For $z(x) := u(x)^{-1}y(x)$ and

$$Z(x) := \begin{pmatrix} z(x) \\ p(x)u(x)^2 z'(x) \end{pmatrix}, \tag{3.5}$$

we obtain the relations

$$Y(x) = \begin{pmatrix} u(x) & 0 \\ p(x)u'(x) & u^{-1}(x) \end{pmatrix} Z(x) \tag{3.6}$$

and

$$Z(x) = \begin{pmatrix} u^{-1}(x) & 0 \\ -p(x)u'(x) & u(x) \end{pmatrix} Y(x).$$

The transformation (3.6) takes (3.4) into

$$Z'(x) = \begin{pmatrix} 0 & p(x)^{-1}u(x)^{-2} \\ s(x)u(x)^2 - u(x)\,(p(x)u'(x))' & 0 \end{pmatrix} Z(x), \qquad x \in [a,b). \tag{3.7}$$

By the transformation

$$W(x) := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} Z(x) \tag{3.8}$$

with inverse transformation

$$Z(x) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} W(x)$$

we obtain the following system, using that $p^{-1}u^{-2} = su^2 = \sqrt{s/p} \in L^1_{\mathrm{loc}}[a,b)$,

$$W'(x) = \begin{pmatrix} \left(\frac{s(x)}{p(x)}\right)^{1/2} & 0 \\ 0 & -\left(\frac{s(x)}{p(x)}\right)^{1/2} \end{pmatrix} W(x) - \frac{1}{2}(u(x)(pu')'(x)) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} W(x). \tag{3.9}$$

If we set

$$S(x) := \frac{-u(x)(pu')'(x)}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

then (3.9) can be written as

$$W'(x) = \left(\frac{s(x)}{p(x)}\right)^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W(x) + S(x)W(x). \tag{3.10}$$

Note that by hypothesis $\int_a^b \|S(t)\|_{\mathbb{C}^2}\,\mathrm{d}t = \int_a^b \|u(pu')'(t)\|\,\mathrm{d}t = M < \infty$ and

$$M(x) := \int_x^b \|S(t)\|_{\mathbb{C}^2}\,\mathrm{d}t \le M < \infty, \qquad x \in [a,b)$$

where $\|\cdot\|_{\mathbb{C}^2}$ denotes the operator norm of a 2×2 matrix. Now we will construct two linearly independent solutions of (3.9) using a fixed point argument. For our first solution, we set $V(x) := W(x)e^{\int_a^x \sqrt{s(t)/p(t)}\,\mathrm{d}t}$ which solves the differential equation

$$V'(x) = \begin{pmatrix} 2\left(\frac{s(x)}{p(x)}\right)^{1/2} & 0 \\ 0 & 0 \end{pmatrix} V(x) + S(x)V(x), \qquad x \in [a,b). \tag{3.11}$$

A fundamental system for the homogeneous differential equation

$$V_0'(x) = \begin{pmatrix} 2\left(\frac{s(x)}{p(x)}\right)^{1/2} & 0 \\ 0 & 0 \end{pmatrix} V_0(x) \tag{3.12}$$

is given by the matrix

$$\Phi(x) = \begin{pmatrix} e^{2\int_a^x \sqrt{s(t)/p(t)}\,\mathrm{d}t} & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\operatorname{Re}(p(x)s(x))^{1/2} \ge 0$ in $[a,b)$, we obtain

$$\|\Phi(x)\Phi(\widehat{x})^{-1}\|_2 = \left\| \begin{pmatrix} e^{-2\int_x^{\widehat{x}} \left(\frac{s(t)}{p(t)}\right)^{1/2}\mathrm{d}t} & 0 \\ 0 & 1 \end{pmatrix} \right\|_{\mathbb{C}^2} = 1, \qquad a \le x < \widehat{x}. \tag{3.13}$$

Let $C([a,b),\mathbb{C}^2)$ denote the Banach space of continuous bounded functions $f : [a,b) \to \mathbb{C}^2$ equipped with the supremum norm $\|\cdot\|_\infty$. We define the continuous operator $F : C([a,b),\mathbb{C}^2) \to C([a,b),\mathbb{C}^2)$ by

$$(Ff)(x) := \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \Phi(x)\int_x^b \Phi(t)^{-1} S(t) f(t)\,\mathrm{d}t. \tag{3.14}$$

Next we construct a sequence of functions $\{h_k\}_{k\in\mathbb{N}}$ by

$$h_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad h_{k+1}(x) = (Fh_k)(x), \quad k \ge 1.$$

We proceed by induction to prove that for all $x \ge a$

$$\|h_{k+1}(x) - h_k(x)\| \le \frac{1}{k!}\left(\int_x^b \|S(t)\|_{\mathbb{C}^2}\,\mathrm{d}t\right)^k = \frac{(M(x))^k}{k!} \le \frac{M^k}{k!}. \tag{3.15}$$

To this end, note that for all $x \ge a$

$$\|h_2(x) - h_1(x)\| = \left\| -\Phi(x)\int_x^b \Phi(t)^{-1} S(t) h_1(t)\,\mathrm{d}t \right\| \le \int_x^b \|S(t)\|_{\mathbb{C}^2}\,\mathrm{d}t = M(x) \le M.$$

By the induction hypothesis we obtain

$$\begin{aligned}
|h_{k+1}(x) - h_k(x)| &\leq \int_x^b \|S(t)\|_{\mathbb{C}^2} |h_k(t) - h_{k-1}(t)| \,\mathrm{d}t \\
&\leq \int_x^b \|S(t)\|_{\mathbb{C}^2} \frac{\left(\int_t^b \|S(u)\|_{\mathbb{C}^2} \,\mathrm{d}u\right)^{k-1}}{(k-1)!} \,\mathrm{d}t \\
&\leq \left| - \int_x^b \frac{\mathrm{d}}{\mathrm{d}t} \frac{\left(\int_t^b \|S(u)\|_{\mathbb{C}^2} \,\mathrm{d}u\right)^{k}}{k!} \,\mathrm{d}t \right| \\
&= \frac{1}{k!} \left(\int_x^b \|S(u)\|_{\mathbb{C}^2} \,\mathrm{d}u \right)^k = \frac{(M(x))^k}{k!} \leq \frac{M^k}{k!}.
\end{aligned}$$

For all $n \geq m \geq 1$ we have that

$$\|h_n(x) - h_m(x)\| = \left\| \sum_{k=m}^{n-1} h_{k+1}(x) - h_k(x) \right\| \leq \sum_{k=m}^{n-1} \frac{(M(x))^k}{k!} \leq e^{M(x)}$$

and consequently $\|h_n - h_m\|_\infty \leq e^M$, so $\{h_n\}_{k\in\mathbb{N}}$ converges uniformly to a bounded and continuous function $h \in C([a,b), \mathbb{C}^2)$. Moreover, h is a fixed point of F because F is continuous. Let us show that h is a solution of (3.11).

$$\begin{aligned}
h'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(F(h))(x) \\
&= \begin{pmatrix} 2\sqrt{s(x)/p(x)} & 0 \\ 0 & 0 \end{pmatrix} h_1 - \Phi'(x) \int_x^b \Phi(t)^{-1} S(t) h(t) \,\mathrm{d}t + S(x)h(x) \\
&= \begin{pmatrix} 2\sqrt{s(x)/p(x)} & 0 \\ 0 & 0 \end{pmatrix} \left(h_1 - \Phi(x) \int_x^b \Phi(t)^{-1} S(t) h(t) \,\mathrm{d}t \right) + S(x)h(x) \\
&= \begin{pmatrix} 2\sqrt{s(x)/p(x)} & 0 \\ 0 & 0 \end{pmatrix} F(h)(x) + S(x)h(x) = \begin{pmatrix} 2\sqrt{s(x)/p(x)} & 0 \\ 0 & 0 \end{pmatrix} h(x) + S(x)h(x).
\end{aligned}$$

From (3.15) we obtain

$$\|h - h_1\|_\infty =: \left\| \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \right\|_\infty \leq \sum_{k=1}^{\infty} \frac{M^k}{k!} = e^M - 1. \tag{3.16}$$

Finally, by hypothesis, for any $\epsilon > 0$, there exists $C \in \mathbb{R}$ such that $M(C) < \ln(1+\epsilon)$ and therefore, by (3.15),

$$|h(x) - h_1(x)| \leq e^{M(x)} - 1 < \epsilon$$

for all $x \geq C$. It follows that

$$G_i(x) \to 0, \quad \text{when} \quad x \to b, \quad \text{for } i = 1, 2,$$

hence by (3.16)

$$h(x) = \begin{pmatrix} G_1(x) \\ G_2(x) + 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{for } x \to b.$$

Since h is a solution for (3.11), a solution Z_0 for (3.7) is given by

$$Z_0(x) = \begin{pmatrix} z_0(x) \\ p(x)u^2 z_0'(x) \end{pmatrix} = e^{-\int_a^x \sqrt{s(t)/p(t)}dt} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} G_1(x) \\ G_2(x)+1 \end{pmatrix}. \tag{3.17}$$

Recall that $z = uy$. Setting $F_1 := G_1 + G_2$ and $F_2 := G_1 - G_2$ we obtain

$$y(x) = (p(x)s(x))^{-1/4} e^{-\int_a^x \sqrt{s(t)/p(t)}dt}(1 + F_1(x))$$

and

$$((ps)^{1/4}y)'(x) = (s/p)^{1/2}(x) e^{-\int_a^x \sqrt{s(t)/p(t)}dt}(F_2(x) - 1)$$

where we used that $(ps)^{1/4}y = z_0$ and z_0' as given in (3.17).

In order to obtain a second solution of (3.9), we set $\widehat{V}(x) := W(x)e^{-\int_a^x \sqrt{s(t)/p(t)}\,\mathrm{d}t}$. It satisfies

$$\widehat{V}'(x) = \begin{pmatrix} 0 & 0 \\ 0 & -2\sqrt{s(x)/p(x)} \end{pmatrix} \widehat{V}(x) + \frac{1}{2}(-u(x)(pu')'(x)) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \widehat{V}(x). \tag{3.18}$$

As before, we note that the constant $\widehat{h}_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution of the homogeneous equation

$$\widehat{V}'(x) = \begin{pmatrix} 0 & 0 \\ 0 & -2\sqrt{s(x)/p(x)} \end{pmatrix} \widehat{V}(x) \tag{3.19}$$

and that

$$\widehat{\Phi}(x) := \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\int_a^x \sqrt{s(t)/p(t)}\,\mathrm{d}t} \end{pmatrix}$$

is a fundamental system for (3.19). Again, for $a \le \widehat{x} < x$

$$\|\widehat{\Phi}(x)\widehat{\Phi}^{-1}(\widehat{x})\|_2 = \left\| \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\int_{\widehat{x}}^x \sqrt{s(t)/p(t))}\,\mathrm{d}t} \end{pmatrix} \right\|_2 = 1$$

and arguing as before, the operator $\widehat{F}$ defined by

$$(\widehat{F}f)(x) := \widehat{h}_1 + \widehat{\Phi}(x) \int_a^x \widehat{\Phi}^{-1}(t) S(t) f(t)\,\mathrm{d}t$$

yields a solution $\widehat{h} := \lim_{n\to\infty} F^n(\widehat{h}_1)$ for (3.18) which in turn leads to a solution $\widehat{y}$ of (3.1) which satisfies (3.3).

We prove that y and $\widehat{y}$ are linearly independent. We calculate the following Wronskian

$$\begin{aligned} W((sp)^{1/4}y, (sp)^{1/4}\widehat{y})(x) &= (sp)^{1/4} \Big(((sp)^{1/4}y)'(x)\widehat{y}(x) - y(x)((sp)^{1/4}\widehat{y})'(x) \Big) \\ &= \left(\frac{s(x)}{p(x)}\right)^{1/2} \Big((F_2(x)-1)(1+\widehat{F}_1(x)) - (1+\widehat{F}_2(x))(1+F_1(x)) \Big). \end{aligned}$$

As $F_j(x), \widehat{F}_j(x) \to 0$ for $x \to b$ and $s(x) \neq 0$ for $x \in [a,b)$ the Wronskian is non-zero for all x. ☐

4. Summability properties of solutions

Now we discuss properties of the solutions of (3.1) for the case $p = 1$ and $b = \infty$. In this section we will always assume that the conditions of the Theorem 3.1 hold. Let y and $\widehat{y}$ be the fundamental system of (3.1) as in (3.2) and (3.3).

Remark 4.1. Note that $\mathrm{Re}[s(x)^{1/2}] > 0$ for all $x \in [a, \infty)$. Since both F_1 and $\widehat{F}_1$ tend to zero for $x \to \infty$, it follows that $|y(x)| \le |\widehat{y}(x)|$ for large x. In particular, $\widehat{y} \in L^2[a, \infty)$ implies $y \in L^2[a, \infty)$.

We start with a helpful comparison.

Lemma 4.2. Assume $p = 1$ and $b = \infty$ and that the conditions of the Theorem 3.1 are fulfilled. Let y and $\widehat{y}$ be the fundamental system in (3.2) and (3.3). Let $\psi \ge 0$ be a measurable function on $[a, \infty)$.

(a) If $\psi \in L^1(\alpha, \infty)$ for some $\alpha \ge a$ and
$$\limsup_{x\to\infty} \frac{1}{\psi(x)} \frac{e^{2\int_a^x \mathrm{Re}[s(t)^{1/2}]\,\mathrm{d}t}}{|s(x)|^{1/2}} < \infty,$$
then $\widehat{y} \in L^2(a, \infty)$.

(b) If $\psi \notin L^1(\alpha, \infty)$ for any $\alpha \ge a$ and
$$\liminf_{x\to\infty} \frac{1}{\psi(x)} \frac{e^{2\int_a^x \mathrm{Re}[s(t)^{1/2}]\,\mathrm{d}t}}{|s(x)|^{1/2}} > 0,$$
then $\widehat{y} \notin L^2(a, \infty)$.

(c) All solutions of (3.1) are in $L^2[a, \infty)$ if and only if the following function is in $L^2[a, \infty)$:
$$x \mapsto s(x)^{-1/4} e^{\int_a^x \sqrt{s(t)}\,\mathrm{d}t}$$

Proof. Note that for any nonnegative function ψ we have that
$$|\widehat{y}(x)|^2 = |s(x)|^{-1/2} |1 + \widehat{F}_1(x)|^2 e^{2\int_a^x \mathrm{Re}[s(t)^{1/2}]\,\mathrm{d}t} = \psi(x) \frac{e^{2\int_a^x \mathrm{Re}[s(t)^{1/2}]\,\mathrm{d}t}}{\psi(x)|s(x)|^{1/2}} |1 + \widehat{F}_1(x)|^2.$$
Since $\lim_{x\to\infty} \widehat{F}_1(x) = 0$, we have for large enough x
$$\frac{1}{2}\psi(x) \liminf_{\eta\to\infty} \frac{e^{2\int_a^\eta \mathrm{Re}[s(t)^{1/2}]\,\mathrm{d}t}}{\psi(\eta)|s(\eta)|^{1/2}} \le |\widehat{y}(x)|^2 \le 2\psi(x) \limsup_{\eta\to\infty} \frac{e^{2\int_a^\eta \mathrm{Re}[s(t)^{1/2}]\,\mathrm{d}t}}{\psi(\eta)|s(\eta)|^{1/2}}$$
and (a) and (b) are proved. Assertion (c) follows from the equality in (3.3) and from Remark 4.1. □

The next corollary follows from Lemma 4.2 (a), Remark 4.1 and setting $\psi(x) := x^{-\rho}$.

Corollary 4.3. If for some $\rho > 1$
$$\limsup_{x\to\infty} \frac{x^\rho e^{2\int_a^x \mathrm{Re}[s(t)^{1/2}]\,\mathrm{d}t}}{|s(x)|^{1/2}} < \infty,$$
then all solutions of (3.1) are in $L^2[a, \infty)$.

Theorem 4.4. Assume $p = 1$ and $b = \infty$ and that the conditions of the Theorem 3.1 are fulfilled. Let y and $\widehat{y}$ be the fundamental system as in (3.2) and (3.3). Then
$$\widehat{y} \notin L^2[a, \infty)$$
if one (or more) of the following conditions is satisfied.

(a) $|s|^{-1/2} \notin L^1[a,\infty)$.
(b) The function s is bounded, i.e., $\|s\|_\infty < \infty$.
(c) We have

$$\int_a^\infty \operatorname{Re}[s(x)^{1/2}]\,\mathrm{d}t < \infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{\operatorname{Re}[s(x)^{1/2}]}{|s(x)|^{1/2}} > 0.$$

(d) We have

$$\int_a^\infty \operatorname{Re}(s(x)^{1/2})\,\mathrm{d}t = \infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{\big(\operatorname{Re}[s(x)^{1/2}]\big)^N}{|s(x)^{1/2}|} > 0 \quad \text{for some } N \in \mathbb{N}.$$

(e) $|\arg s| \le \pi - \epsilon_0$ for some $\epsilon_0 > 0$.

Proof. (a) By assumption, $\arg(s(t)) \neq \pi$, hence $\operatorname{Re}[s(t)^{1/2}] > 0$ for all $t \in [a,\infty)$. Moreover $\widehat{F}_1$ is a bounded function with $\lim_{x\to\infty} \widehat{F}_1(x) = 0$. Therefore, for any $c \in (0,1)$ we can take $\alpha > a$ such that $|1 + \widehat{F}_1(x)| > c$ and hence

$$|\widehat{y}(x)| = |s(x)|^{-1/4} |1 + \widehat{F}_1(x)| e^{\int_a^x \operatorname{Re}[s(t)^{1/2}]\,\mathrm{d}t} \ge c|s(x)|^{-1/4}, \qquad x > \alpha. \tag{4.1}$$

Recall that $s^{-1/4}$ is absolutely continuous by assumption in Theorem 3.1, hence the right hand side in (4.1) is not square integrable on $[\alpha,\infty)$ and therefore also $\widehat{y} \notin L^2[a,\infty)$.

(b) If s is bounded, then clearly $|s|^{-1/2}$ is not square integrable on $[a,\infty)$ and the claim follows from (a).

(c) By assumption,

$$\liminf_{x\to\infty} \frac{\operatorname{Re}[s(x)^{1/2}]}{|s(x)|^{1/2}} := c > 0 \quad \text{and} \quad 0 < \int_a^\infty \operatorname{Re}[s(t)^{1/2}]\,\mathrm{d}t =: R < \infty.$$

Hence the Lebesgue measure of the set $A := \{t \in [a,\infty) : \operatorname{Re}[s(t)^{1/2}] \le 1\}$ is infinite. Note that the function $x \mapsto \int_a^x \operatorname{Re}\sqrt{s(t)}\,\mathrm{d}t$ is non-decreasing and for large enough x, we have that

$$\int_a^x \operatorname{Re}\sqrt{s(t)}\,\mathrm{d}t \ge \frac{R}{2}.$$

We conclude

$$|\widehat{y}(x)|^2 = |1 + F_1(x)|^2 \frac{e^{2\int_a^x \operatorname{Re}\sqrt{s(t)}\,\mathrm{d}t}}{|s(x)|^{1/2}} \ge \frac{ce^R}{2} \frac{1}{\operatorname{Re}[s(x)^{1/2}]} \ge \frac{ce^R}{2} \chi_A(x)$$

where χ_A is the characteristic function of A. Since $R > 0$ it follows that $\widehat{y} \notin L^2[a,\infty)$.

(d) We define

$$\psi(x) := \frac{e^{2\int_a^x \operatorname{Re}[s(t)^{1/2}]\,\mathrm{d}t}}{(\operatorname{Re} s(x)^{1/2})^N}, \qquad x \in [a,\infty).$$

Then

$$\psi(x)^{-1} \frac{e^{2\int_a^x \operatorname{Re}[s(t)^{1/2}]\,\mathrm{d}t}}{|s(x)|^{1/2}} = \frac{(\operatorname{Re} s(x)^{1/2})^N}{|s(x)|^{1/2}}$$

and by Lemma 4.2 (b) it suffices to show that $\psi \notin L^1[a,\infty)$. If we set

$$g(x) := \int_a^x \operatorname{Re}[s(t)^{\frac{1}{2}}]\,\mathrm{d}t,$$

then g satisfies the differential equation

$$\frac{e^{2g(x)}}{(g'(x))^N} = \psi(x).$$

For

$$G(x) := \int_a^x (\psi(t))^{-1/N}\,\mathrm{d}t$$

we obtain

$$e^{-\frac{2}{N}g(x)} = 1 - \frac{2}{N}G(x).$$

By assumption, $g(x) \to \infty$ for $x \to \infty$, so $G(x) \to N/2$ and $\int_a^\infty \psi^{-1/N}(t)\,\mathrm{d}t < \infty$. Similarly as above, the Lebesgue measure of the set $\{t \in [a,\infty) : \psi^{-1/N}(t) \le 1\}$ is infinite. Hence

$$\int_a^\infty \psi^{1/N}(t)\,\mathrm{d}t = \infty,$$

which implies that $\int_a^\infty \psi(t)\,\mathrm{d}t$ diverges.

(e) The assumption on s implies that $|\arg(s(x))^{1/2}| < (\pi - \epsilon_0)/2$. Setting $c := \cos\left(\frac{\pi-\epsilon_0}{2}\right)$, we have that $\operatorname{Re} s(x)^{1/2} = |s(x)|^{1/2}\cos\arg(s(x))^{1/2}$ and

$$\frac{\operatorname{Re} s(x)^{1/2}}{|s(x)|^{1/2}} \ge c, \qquad x \in [a,\infty).$$

If $\int_a^\infty \operatorname{Re}[s(t)^{1/2}]\,\mathrm{d}t = \infty$, then the claim follows from (d) with $N = 1$. If on the other hand $\int_a^\infty \operatorname{Re}[s(t)^{1/2}]\,\mathrm{d}t =: R < \infty$, the claim follows from (c).

□

Remark 4.5. If, in addition to the assumptions in Theorem 4.4 (b), we have $\operatorname{Re}[s^{1/2}] \in L^1[a,\infty)$, then there exists $K > 0$ such that

$$|y(x)| \ge K|s(x)|^{-1/4}|1 + F_1(x)| \ge K\|s\|_\infty^{-1/4}|1 + F_1(x)|,$$

which implies that $y \notin L^2[a,\infty)$.

5. Sturm-Liouville equation on a ray

In this section the results of the previous section are used to investigate a second order differential equation defined on a ray in $\mathbb{C}$. This is motivated by the recent intensive research connected with $\mathcal{PT}$-symmetric Hamiltonians, cf. [5]. Let Γ be a ray with angle $\phi \in (-\pi/2, \pi/2)$,

$$\Gamma := \{z \in \mathbb{C} : z = xe^{i\phi}, x \in [a,\infty)\} \tag{5.1}$$

for some $a \ge 0$. Our main interest is to obtain a Weyl criterion for the differential equation

$$-y(z)'' + \mathbf{q}(z)y(z) = \lambda y(z), \qquad z \in \Gamma$$

where $\mathbf{q} : \Gamma \to \mathbb{C}$ is locally integrable. Setting

$$v(x) := y(z(x)) \text{ and } q(x) := \mathbf{q}(z(x)) \text{ with } z(x) := xe^{i\phi}, \quad x \in [a,\infty), \tag{5.2}$$

we obtain the Sturm-Liouville problem

$$-e^{-2i\phi}v''(x) + q(x)v(x) = \lambda v(x), \qquad x \in [a,\infty). \tag{5.3}$$

In order to describe the solutions of (5.3), we re-write it in the form

$$v''(x) = e^{2i\phi}\,(q(x) - \lambda)\, v(x), \qquad x \in [a, \infty). \tag{5.4}$$

If we set

$$s(x) := e^{2i\phi}\,(q(x) - \lambda)\,, \qquad x \in [a, \infty), \tag{5.5}$$

then (5.4) is in the form of (3.1) with $p = 1$ and solutions for (5.3) are obtained from Theorem 3.1.

Theorem 5.1. The differential equation (5.3) with $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\lambda \in \mathbb{C}$ such that

(i) $q, q' \in \mathrm{AC}_{\mathrm{loc}}[a, \infty)$,

(ii) $\lambda \notin \overline{\mathrm{co}}\{e^{-2i\phi} r + q(x) \,:\, 0 < r < \infty,\, x \in [a, \infty)\}$ and

(iii) $M := \displaystyle\int_a^\infty \left| \frac{5(e^{2i\phi} q'(x))^2}{16(e^{2i\phi}(q(x) - \lambda))^{5/2}} - \frac{e^{2i\phi} q''(x)}{4(e^{2i\phi}(q(x) - \lambda))^{3/2}} \right| dx < \infty$

has a fundamental system $\{y, \widehat{y}\}$ of the form

$$y(x) = s(x)^{-1/4} e^{-\int_a^x \sqrt{s(t)}\,\mathrm{d}t}(1 + F_1(x)), \qquad x \in [a, \infty), \tag{5.6}$$

$$\widehat{y}(x) = s(x)^{-1/4} e^{\int_a^x \sqrt{s(t)}\,\mathrm{d}t}(1 + \widehat{F}_1(x)), \qquad x \in [a, \infty), \tag{5.7}$$

with $|\widehat{F}_1(x)|, |F_1(x)| \to 0$ when $x \to +\infty$, $\|F_1\|_\infty \leq 2e^M - 2$ and $\|\widehat{F}_1\|_\infty \leq 2e^M - 2$.

Proof. We show that the assumptions of Theorem 3.1 for the function s in (5.5) are fulfilled.

We have $p = 1$ and it follows from (ii) (by sending r to zero) that $q(x) \neq \lambda$ for all $x \in [a, \infty]$, hence $s(x) \neq 0$. Assume that there exists $x_0 \in [a, \infty]$ with $s(x_0) \in (-\infty, 0]$. Therefore, by definition of s in (5.5),

$$\lambda = q(x_0) - s(x_0) e^{-2i\phi},$$

a contradiction to (ii). Hence $\arg s(x) \neq \pi$ for all $x \in [a, \infty)$. Moreover, by (i), $s^{-1/4}$ and $(s^{-1/4})'$ are in $\mathrm{AC}_{\mathrm{loc}}[a, \infty)$ and we have

$$s^{-1/4}(s^{-1/4})'' = s^{-1/4}\left(\frac{5}{16}(s')^2 s^{-9/4} - \frac{1}{4} s'' s^{-5/4}\right).$$

Now, (iii) implies that

$$s^{-1/4}(s^{-1/4})'' \in L^1(a, \infty)$$

and we obtain the desired solutions from Theorem 3.1. □

We prove a limit point/limit circle criteria for the differential expression (5.3). This is our main result. Note that, if $\overline{\mathrm{co}}\{e^{-2i\phi} r + q(x) : 0 < r < \infty, x \in [a, \infty)\} \neq \mathbb{C}$, then the equation (5.3) is in the Weyl trichotomy described in the Definition 2.1.

Theorem 5.2. Assume that the assumptions (i)–(iii) from Theorem 5.1 are satisfied for some $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\lambda \in \mathbb{C}$. Then the following is true.

(a) The equation (5.3) is in the limit point I case if one of the following conditions is fulfilled:

(α) We have $|q - \lambda|^{-1/2} \notin L^1[a, \infty)$ or

(β) we have $\|q\|_\infty < \infty$ or

(γ) we have

$$\int_a^\infty \mathrm{Re}\left(e^{i\phi}\sqrt{q(x) - \lambda}\right) dx < \infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{\mathrm{Re}\left(e^{i\phi}\sqrt{q(x) - \lambda}\right)}{\left|\sqrt{q(x) - \lambda}\right|} > 0 \quad \text{or}$$

(δ) for some $N \in \mathbb{N}$ we have

$$\int_a^\infty \operatorname{Re}\left(e^{i\phi}\sqrt{q(x)-\lambda}\right)dx = \infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{\left(\operatorname{Re} e^{i\phi}\sqrt{q(x)-\lambda}\right)^N}{\left|\sqrt{q(x)-\lambda}\right|} > 0 \quad \text{or}$$

(ϵ) we have for some $\epsilon_0 > 0$

$$|\arg e^{i\phi}\sqrt{q(x)-\lambda}| \le \pi - \epsilon_0.$$

In the cases (α)–(δ) this classification is independent of the choice of λ, cf. Remark 2.3.

(b) If for some $\rho > 1$

$$q \in L^1_{\mathrm{u}}(a,b) \quad \text{and} \quad \limsup_{x\to\infty} \frac{x^\rho e^{2\int_a^x \operatorname{Re}\left(e^{i\phi}\sqrt{q(x)-\lambda}\right)\,\mathrm{d}t}}{|\sqrt{q(x)-\lambda}|} < \infty,$$

then the equation (5.3) is in the limit circle case. This classification is independent of the choice of λ, cf. Remark 2.3.

Proof. By Theorem 5.1 we know that there exist two linearly independent solutions y and $\widehat{y}$. Cases (α)–(ϵ) follow directly from Theorem 4.4. It remains to show item (b). By Corollary 4.3 all solutions of (3.1) are in $L^2[a,\infty)$. It is easy to see that the statements in the appendix of [2] also hold true for non-real (i.e. complex-valued) potentials. Then multiplying (5.3) by $e^{2i\phi}$ and applying [2, Lemma A.2 (i)] one obtains that all solutions fulfill (2.4). □

Example 5.3. Consider Γ as in (5.1) with $\phi = 0$. Moreover we set $\lambda = 0$ and $q(x) := -f(x) + i$ with $f(x) > 0$ in $[a,\infty)$ and $f, f' \in \mathrm{AC}_{\mathrm{loc}}[a,\infty)$ such that item (iii) of Theorem 5.1 is satisfied.

(i) If $\|f\|_\infty < \infty$, then (5.3) is in the limit point I.

(ii) If $\lim_{x\to\infty} f(x) = \infty$, then (5.3) is in the limit point I if and only if $f^{-1/2} \in L^1[a,\infty)$.

Proof. Note that in both cases

$$\overline{\mathrm{co}}\{r + q(x) : x \in [a,\infty),\, 0 < r < \infty\} \subseteq \{t + i : t \in \mathbb{R}\},$$

with equality in case (ii), and that

$$|q(x)| = \sqrt{f(x)^2+1}, \qquad \arg q(x) = \pi - \arctan\left(\frac{1}{f(x)}\right).$$

The claim in (i) follows directly from Theorem 4.4(b).

Now assume that $\lim_{x\to\infty} f(x) = \infty$. Theorem 5.1 shows that $\widehat{y} \in L^2[a,\infty)$ if and only if the function

$$x \mapsto e^{2\int_a^x \left(f(t)^2+1\right)^{1/4}\cos\left(\frac{\pi-\arctan(1/f(t))}{2}\right)\mathrm{d}t}\left(f(x)^2+1\right)^{-1/4}$$

is in $\in L^1[a,\infty)$. Note that

$$\cos\left(\frac{\pi - \arctan(1/f(x))}{2}\right) = \sin\left(\frac{\arctan(1/f(x))}{2}\right).$$

Given that

$$\lim_{x\to\infty} \frac{\sin\left(\frac{\arctan(1/f(x))}{2}\right)}{\frac{1}{2f(x)}} = \lim_{x\to\infty} \frac{\sin\left(\frac{\arctan(1/f(x))}{2}\right)}{\frac{\arctan(1/f(x))}{2}} \times \frac{\arctan(1/f(x))}{\frac{1}{f(x)}} = 1$$

and

$$\lim_{x\to\infty} \frac{f(x)^{-1/2}}{(f^2(x)+1)^{-1/4}} = 1,$$

it follows that $\widehat{y} \in L^2[a,\infty)$ if and only if the function

$$x \mapsto e^{\int_a^x f^{-1/2}(t)\,\mathrm{d}t} f^{-1/2}(x)$$

is in $L^1[a,\infty)$. Note that

$$\int_a^l e^{\int_a^x f^{-1/2}(t)\,\mathrm{d}t} f^{-1/2}(x)dx = \int_0^\alpha e^{y(x)} y'(x)dx = e^{\int_a^l f^{-1/2}(t)\,\mathrm{d}t} - 1,$$

with $\alpha := \int_a^l f^{-1/2}(t)\,\mathrm{d}t$. Hence (5.3) is in the limit point I case if and only if $f^{-1/2} \in L^1[a,\infty)$. $\square$

Example 5.4. We review an example of more relevance from theoretical physics. In [5] the authors consider for $N \geq 1$, $N \in \mathbb{N}$,

$$\mathbf{q}(z) := -(iz)^{N+2}, \quad \text{with} \quad z \in \Gamma,$$

where Γ is as in (5.1). Then the function q in (5.2) is given by

$$q(x) = -i^{N+2} x^{N+2} e^{(N+2)i\phi}.$$

We apply the results from Section 4. Equation (5.4) equals now

$$v''(x) = e^{2i\phi}(-i^{N+2}x^{N+2}e^{(N+2)i\phi} - \lambda)v(x), \quad x \in [a,\infty). \tag{5.8}$$

Here we choose for simplicity $a = 1$ because then we can choose $\lambda = 0$, see below. This is without loss of generality, as the classification into limit point/limit circle just describes the behaviour of the solutions at the singular endpoint, which is here ∞. If

$$\phi \neq -\frac{(N+2)}{2(N+4)}\pi + \frac{2k}{N+4}\pi, \quad \text{for } k = 0,\dots,N+3, \tag{5.9}$$

then

$$0 \notin \overline{\mathrm{co}}\{e^{-2i\phi}r - i^{N+2}x^{N+2}e^{(N+2)i\phi} \,:\, 0<r<\infty,\, x\in[1,\infty)\}.$$

That is, assumptions (i) and (ii) from Theorem 5.1 are satisfied for $\lambda = 0$. Moreover,

$$\int_1^\infty \left| \frac{5(e^{2i\phi}q'(x))^2}{16(e^{2i\phi}q(x))^{5/2}} - \frac{e^{2i\phi}q''(x)}{4(e^{2i\phi}q(x))^{3/2}} \right| dx \leq \int_1^\infty \frac{5(N+2)^2}{16x^{3+N/2}} + \frac{(N+2)(N+1)}{4x^{3+N/2}} dx < \infty$$

and (iii) from Theorem 5.1 is satisfied. Finally, note that

$$\operatorname{Re} e^{i\phi} q(x)^{1/2} = x^{\frac{N+2}{2}} \cos\theta, \qquad \operatorname{Im} e^{i\phi} q^{1/2}(x) = x^{\frac{N+2}{2}} \sin\theta,$$

where

$$\theta := \left(\frac{N+4}{2}\right)\left(\phi + \frac{\pi}{2}\right)$$

A straightforward calculation shows that $\cos\theta \neq 0$ if (5.9) holds. In this case,

$$\int_1^\infty \operatorname{Re}\left(e^{i\phi}\sqrt{q(x)-\lambda}\right) dx = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{|\operatorname{Im} e^{i\phi}q(x)^{1/2}|}{|\operatorname{Re} e^{i\phi}q(x)^{1/2}|} = \frac{|\sin\theta|}{|\cos\theta|} = |\tan\theta| < \infty.$$

Therefore, for large x, we have $|\operatorname{Im} e^{i\phi}q(x)^{1/2}| \leq K|\operatorname{Re} e^{i\phi}q(x)^{1/2}|$ for some $K>0$. Then

$$\liminf_{x\to\infty} \frac{\left(\operatorname{Re} e^{i\phi}\sqrt{q(x)-\lambda}\right)^2}{\left|\sqrt{q(x)-\lambda}\right|} = \liminf_{x\to\infty} \frac{x^{N+2}\cos^2\theta}{x^{\frac{N+2}{2}}} = \liminf_{x\to\infty} x^{\frac{N+2}{2}} \cos^2\theta > 0$$

and applying Theorem 5.2 (γ) we obtain that (5.8) is in the limit point I case as long as ϕ satisfies (5.9). This was already proven in [11] with a somehow different argument.

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