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Spectral inclusion property for a class of block operator matrices

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Abstract

The numerical range and the quadratic numerical range is used to study the spectrum of a class of block operator matrices. We show that the approximate point spectrum is contained in the closure of the quadratic numerical range. In particular, the spectral enclosures yield a spectral gap. It is shown that these spectral bounds are tighter than classical numerical range bounds.

Keywords — Block operator matrices; Numerical range; Quadratic numerical range; Spectrum; Spectral Enclosures

1 Introduction

Block operator matrices play an important role in system theory and theoretical physics (e.g. see [13]). The spectral enclosure and the spectral properties of a block operator matrices are of major interest and have received a great deal of attention in recent years.

In the present paper, we consider the following framework. Let H_1, H_2 be complex Hilbert spaces and let

$$\mathcal{A}_{\pm D} = \begin{bmatrix} 0 & B \\ -B^* & \pm D \end{bmatrix} : \mathcal{D}(-B^*) \oplus \mathcal{D}(B) \subset H_1 \oplus H_2 \to H_1 \oplus H_2, \tag{1.1}$$

be an off-diagonally dominant operator of the form where B is a densely defined closed operator, D is accretive in H_2, *i.e.* $\mathrm{Re}(Dz, z) \geq 0, z \in \mathcal{D}(B)$. We use $\mathcal{A}_{+D}$ or $\mathcal{A}_{-D}$ to denote the case that the element $\pm D$ in (1.1) is $+D$ or $-D$, respectively. In particular, the operator $\mathcal{A}_{-D}$ is dissipative, and generates a C^0-semigroup of contractions if there exists a $\lambda_0 > 0$ such that $\lambda_0 I - \mathcal{A}_{-D}$ is surjective. Moreover, if D is a self-adjoint operator, then the operator $\mathcal{A}_{\pm D}$ is $\mathcal{J}$-symmetric with $\mathcal{J} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Furthermore, if D is self-adjoint and B-bound< 1, the operator $\mathcal{A}_{\pm D}$ is $\mathcal{J}$-self-adjoint, *i.e.* $(\mathcal{J}\mathcal{A}_{\pm D})^* = \mathcal{J}\mathcal{A}_{\pm D}$ [4].

The aim of this paper is to establish a new enclosure for the spectrum of the operator $\mathcal{A}_{\pm D}$ in (1.1) by using the numerical range and the quadratic numerical range. In 1918, the numerical range of a linear operator in a Hilbert space was first studied by O. Teoplitz in [12]. Under some mild assumptions, it gives a localization of the spectrum. However, it generally does not capture finer structures of the spectrum. To circumvent the limitedness, a new concept, the quadratic numerical range, was introduced in 1998 in [7], which came with subsequent development in [5] and [6]. Unlike the numerical range, the quadratic numerical range consists of at most two connected components which need not be convex. In addition, since the quadratic numerical range is always contained in the numerical range, it potentially gives a tighter spectral enclosure.

C. Tretter proved in [14] that the quadratic numerical range of the unbounded block operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ has the spectral inclusion property if $\mathcal{A}$ is diagonally dominant or off-diagonally dominant of order 0. We show in this article that the spectral inclusion property of off-diagonally dominant matrices of arbitrary order under the assumption that they have the special form given in (1.1).

In 2017, M. Langer and M. Strauss studied certain upper dominant and certain diagonally dominant unbounded $\mathcal{J}$-self-adjoint block operator matrices. They used the Schur complement and the quadratic numerical range to obtain enclosures for the spectrum and to derive variational principles for real eigenvalues even in the presence of non-real spectrum in [8].

B. Jacob, C. Tretter, C. Trunk and H. Vogt used the quadratic numerical range to establish a new method for obtaining non-convex spectral enclosures for operator matrices as the form $\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}$, associated with second order differential equations in a Hilbert space in [3]. For unbounded Hamiltonian operator matrices ($B = B^*$, $C = C^*$, and $D = -A^*$), the quadratic numerical range is shown to be symmetric with respect to the imaginary axis and has the spectral inclusion property under certain assumptions [2]. Further studies about the enclosures of spectrum, the numerical range and the quadratic numerical range of linear operators can be found in [15, 10, 11, 9].

The present paper is organized as follows. In Section 2 and Section 3, we introduce the fundamental definitions and properties of $\mathcal{A}_{\pm D}$. Section 4 is devoted to the spectral properties of bounded block operator matrices $\mathcal{A}_{\pm D}$. We prove that the spectral enclosures of $\mathcal{A}_{\pm D}$ yield a vertical strip free of spectrum under rather weak assumptions. The precise statement is summarized in Theorem 4.2. In Section 5, we show that the approximate point spectrum of the unbounded $\mathcal{A}_{\pm D}$ is contained in the closure of the quadratic numerical range and investigate the location of spectrum of $\mathcal{A}_{\pm D}$ by using the quadratic numerical range.

2 Preliminaries

Let H be a complex Hilbert space with scalar product $(\cdot, \cdot)$. The domain and range of a densely defined closable linear operator T in H are denoted by $\mathcal{D}(T)$ and $\mathbb{R}(T)$, respectively. We use T^* to denote the adjoint of T. The resolvent set of T is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} |\ T - \lambda \text{ is bijection and } (T-\lambda)^{-1} \text{ is bounded in } H\},$$

and the set

$$\sigma(T) := \mathbb{C} \setminus \rho(T)$$

is called the spectrum of T. The point spectrum is denoted by $\sigma_p(T)$ and the approximate point spectrum of T is defined as

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} |\ \text{there exists } (x_n) \text{ in } \mathcal{D}(T),\ \|x_n\| = 1,\ \text{such that } (T-\lambda)x_n \to 0,\ n \to \infty\}.$$

Definition 2.1 ([4]). *Let T, S be operators in H, S is called relatively bounded with respect to T or T-bounded if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and there exist constants $a_S,\ b_S \geq 0$ such that*

$$\|Sx\| \leq a_S\|x\| + b_S\|Tx\|,\ x \in \mathcal{D}(T).$$

The infimum δ_S of all b_S so that the above equality holds for some $a_S \geq 0$ is called relative bound of S with respect to T or T-bound of S.

Definition 2.2 ([13]). *Let H_1, H_2 be Hilbert spaces. The block operator matrix*

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix},\ \mathcal{D}(\mathcal{A}) := (\mathcal{D}(A) \cap \mathcal{D}(C)) \oplus (\mathcal{D}(B) \cap \mathcal{D}(D)) \subset H_1 \oplus H_2 \to H_1 \oplus H_2, \tag{2.1}$$

is called off-diagonally dominant if A is C-bounded and D is B-bounded, and

$$\begin{array}{ll} A : \mathcal{D}(A) \to H_1 & B : \mathcal{D}(B) \to H_1 \\ C : \mathcal{D}(C) \to H_2 & D : \mathcal{D}(D) \to H_2 \end{array}$$

are closable operators with dense domains $\mathcal{D}(A), \mathcal{D}(C) \subset H_1$ and $\mathcal{D}(B), \mathcal{D}(D) \subset H_2$. In particular, this implies that the operator $\mathcal{A}$ in (2.1) is densely defined.

Definition 2.3 ([12]). *Let T be a linear operator in Hilbert space H. The numerical range $W(T)$ of T is defined by*

$$W(T) := \{(Tx, x) |\ x \in \mathcal{D}(T),\ \|x\| = 1\}.$$

Definition 2.4 ([7]). *Let $\mathcal{A}$ be a block operator matrix of the form in (2.1), $(f, g)^\top \in \mathcal{D}(\mathcal{A})$ such that $\|f\| = \|g\| = 1$ and let*

$$\mathcal{A}_{f,g} := \begin{bmatrix} (Af, f) & (Bg, f) \\ (Cf, g) & (Dg, g) \end{bmatrix} \in M_2(\mathbb{C})$$

be a 2×2 matrix associated to f, g and the operator $\mathcal{A}$. Then, the set

$$W^2(\mathcal{A}) := \bigcup \{\sigma_p(\mathcal{A}_{f,g}) |\ (f, g)^T \in \mathcal{D}(\mathcal{A}),\ \|f\| = \|g\| = 1\}$$

is called the quadratic numerical range of $\mathcal{A}$.

3 Fundamental properties

In this section, we introduce some fundamental properties of an off-diagonally dominant operator matrix

$$\mathcal{A}_{\pm D} = \begin{bmatrix} 0 & B \\ -B^* & \pm D \end{bmatrix} : \mathcal{D}(-B^*) \oplus \mathcal{D}(B) \subset H_1 \oplus H_2 \to H_1 \oplus H_2 \tag{3.1}$$

where H_1, H_2 are Hilbert spaces, B is a densely defined closed operator, D is B-bounded and accretive in H_2, *i.e.* $\mathrm{Re}(Dz, z) \geq 0, z \in \mathcal{D}(B)$. Then, by [13, Corollary 2.2.9 (ii)], $\mathcal{A}_{\pm D}$ is a closed operator. Moreover, from [13, Theorems 2.5.3, 2.5.4, and 2.5.9] we obtain the following.

Proposition 3.1. For the block operator matrix $\mathcal{A}_{\pm D}$ in (3.1) we have

$$\sigma_p(\mathcal{A}_{\pm D}) \subset W^2(\mathcal{A}_{\pm D}) \subset W(\mathcal{A}_{\pm D}).$$

If, in addition, $\dim H_1, \dim H_2 > 1$, then

$$W(\pm D) \cup \{0\} \subset W^2(\mathcal{A}_{\pm D}). \tag{3.2}$$

For the block operator matrix $\mathcal{A}_{\pm D}$ in (3.1) define β and γ as

$$\beta := \inf_{z\in\mathcal{D}(B)\setminus\{0\}} \frac{\operatorname{Re}(Dz,z)}{\|z\|^2} \in [0,\infty) \qquad \gamma := \sup_{z\in\mathcal{D}(B)\setminus\{0\}} \frac{\operatorname{Re}(Dz,z)}{\|z\|^2} \in [0,\infty]. \tag{3.3}$$

Note that $\beta > 0$ means that $\gamma > 0$ and the operator D is uniformly accretive. We have the following equivalent formulation.

Remark 3.2. We have $\lambda \in W(\mathcal{A}_{\pm D})$ if and only if there is $(f,g)^\top \in \mathcal{D}(\mathcal{A}_{\pm D})$, $\|f\|^2+\|g\|^2 = 1$, with

$$\begin{aligned}\lambda = \left(\begin{bmatrix} 0 & B \\ -B^* & \pm D\end{bmatrix}\begin{bmatrix} f \\ g\end{bmatrix}, \begin{bmatrix} f \\ g\end{bmatrix}\right) &= \left(\begin{bmatrix} Bg \\ -B^*f \pm Dg\end{bmatrix}\begin{bmatrix} f \\ g\end{bmatrix}\right) \\ &= (Bg,f) - (B^*f,g) \pm (Dg,g) = -2\mathrm{i}\operatorname{Im}(f,Bg) \pm (Dg,g).\end{aligned}$$

Together with (3.3), the real part $\operatorname{Re} W(\mathcal{A}_{\pm D})$ satisfies

$$\begin{aligned} &\min(\operatorname{Re} W(\mathcal{A}_{+D})) = 0, && \sup(\operatorname{Re} W(\mathcal{A}_{+D})) = \gamma \\ &\min(\operatorname{Re} W(\mathcal{A}_{-D})) = -\gamma, && \sup(\operatorname{Re} W(\mathcal{A}_{-D})) = 0. \end{aligned} \tag{3.4}$$

Remark 3.3. For $(f,g)^\top \in \mathcal{D}(\mathcal{A}_{\pm D})$, with $\|f\| = \|g\| = 1$, set

$$\triangle_\pm(f,g;\lambda) := \lambda^2 \mp \lambda(Dg,g) + |(f,Bg)|^2,$$

then

$$W^2(\mathcal{A}_{\pm D}) = \Big\{\lambda \in \mathbb{C} | \text{ there exists } (f,g)^\top \in \mathcal{D}(\mathcal{A}_{\pm D}),\ \|f\| = \|g\| = 1 \text{ with } \triangle_\pm(f,g;\lambda) = 0\Big\}$$

and the real part $\operatorname{Re}\left(W^2(\mathcal{A}_{\pm D})\right)$ satisfies the same relations as in (3.4), where one has to replace $W(\mathcal{A}_{+D})$ by $W^2(\mathcal{A}_{+D})$.

4 Spectral inclusion. The case of bounded entries

In this section, we establish new spectral enclosures for bounded block operator matrices

$$\mathcal{A}_{\pm D} = \begin{bmatrix} 0 & B \\ -B^* & \pm D\end{bmatrix} : H_1 \oplus H_2 \to H_1 \oplus H_2, \tag{4.1}$$

where H_1, H_2 are Hilbert spaces, D is accretive , *i.e.* $\operatorname{Re}(Dz,z) \geq 0$. By Theorem 2.3 in [5], for bounded operator matrices, we have

$$\sigma(\mathcal{A}_{\pm D}) \subset \overline{W^2(\mathcal{A}_{\pm D})} \subset \overline{W(\mathcal{A}_{\pm D})}.$$

For bounded operator $\mathcal{A}_{\pm D}$ we set

$$b := \sup_{\substack{f\in H_1,\ g\in H_2 \\ \|f\|^2+\|g\|^2=1}} |\operatorname{Im}(f,Bg)|, \qquad c := \sup_{\substack{g\in H_2 \\ \|g\|=1}} |\operatorname{Im}(Dg,g)|. \tag{4.2}$$

Suppose $f \in H_1$ and $g \in H_2$ such that $\|f\|^2 + \|g\|^2 = 1$. Then, we have $|(f,Bg)|^2 \leq \|f\|^2\|g\|^2\|B\|^2 = (1-\|g\|^2)\|g\|^2\|B\|^2$, which takes its maximum at $\|g\|^2 = 1/2$ and we obtain

$$b \in [0, \|B\|/2] \quad \text{and} \quad c \in [0, \|D\|].$$

The following proposition is an immediate consequence of Remark 3.2.

Proposition 4.1. Suppose $\mathcal{A}_{\pm D}$ in (4.1) is bounded. Then

$$\sigma(\mathcal{A}_{\pm D}) \subset \overline{W(\mathcal{A}_{\pm D})} \subset \{\lambda \in \mathbb{C} |\ 0 \le \pm \operatorname{Re} \lambda \le \gamma,\ |\operatorname{Im} \lambda| \le 2b + c\}.$$

If, in addition, there exists $k \ge 0$ such that $|\operatorname{Im}(Dz, z)| \le k \operatorname{Re}(Dz, z)$, $z \in H_2$ then

$$\sigma(\mathcal{A}_{\pm D}) \subset \overline{W(\mathcal{A}_{\pm D})} \subset \{\lambda \in \mathbb{C} |\ 0 \le \pm \operatorname{Re} \lambda \le \gamma,\ |\operatorname{Im} \lambda| \le 2b + k|\operatorname{Re} \lambda|\}.$$

The above propositions are proved by the numerical range of $\mathcal{A}_{\pm D}$. In the following, we investigate the spectral enclosures of $\mathcal{A}_{\pm D}$ in terms of the quadratic numerical range.

Theorem 4.2. Let $\mathcal{A}_{\pm D}$ in (4.1) be bounded. Assume $B \ne 0$ and $\beta > 0$. We define

$$t_1 = \beta \left(1 - \frac{1}{1 + \left(\frac{2b+c}{\|B\|}\right)^2}\right),\ t_2 = \frac{\beta}{2}\left(1 - \sqrt{1 - \frac{4\|B\|^2}{\beta^2}}\right) \text{ and } t_3 = \frac{\beta}{2}\left(1 + \sqrt{1 - \frac{4\|B\|^2}{\beta^2}}\right),$$

and for $x \in [0, \beta)$

$$g(x) := f^2(x) - x^2 \quad \text{and} \quad f(x) := \|B\| \sqrt{\frac{x}{\beta - x}}.$$

Then the following statements hold.

(I) If $\beta < 2\|B\|$, then

$$\begin{aligned}\sigma(\mathcal{A}_{\pm D}) \setminus \{0\} \subset &\{\lambda \in \mathbb{C} |\ \pm \operatorname{Re} \lambda \in (0, t_1],\ |\operatorname{Im} \lambda| \le \sqrt{g(|\operatorname{Re} \lambda|)}\} \\ &\cup \{\lambda \in \mathbb{C} |\ \pm \operatorname{Re} \lambda \in (t_1, \gamma], |\operatorname{Im} \lambda| \le 2b + c\}.\end{aligned}$$

(II) If $\beta \ge 2\|B\|$, and

(1) $t_1 < t_2$, then

$$\begin{aligned}\sigma(\mathcal{A}_{\pm D}) \setminus \{0\} \subset &\{\lambda \in \mathbb{C} |\ \pm \operatorname{Re} \lambda \in (0, t_1],\ |\operatorname{Im} \lambda| \le \sqrt{g(|\operatorname{Re} \lambda|)}\} \\ &\cup \{\lambda \in \mathbb{C} |\ \pm \operatorname{Re} \lambda \in (t_1, \gamma] \setminus (t_2, t_3),\ |\operatorname{Im} \lambda| \le 2b + c\};\end{aligned}$$

(2) $t_2 \le t_1 \le t_3$, then

$$\begin{aligned}\sigma(\mathcal{A}_{\pm D}) \setminus \{0\} \subset &\{\lambda \in \mathbb{C} |\ \pm \operatorname{Re} \lambda \in (0, t_2],\ |\operatorname{Im} \lambda| \le \sqrt{g(|\operatorname{Re} \lambda|)}\} \\ &\cup \{\lambda \in \mathbb{C} |\ \pm \operatorname{Re} \lambda \in (t_3, \gamma], |\operatorname{Im} \lambda| \le 2b + c\};\end{aligned}$$

(3) $t_1 > t_3$, then

$$\begin{aligned}\sigma(\mathcal{A}_{\pm D}) \setminus \{0\} \subset &\{\lambda \in \mathbb{C} |\ \pm \operatorname{Re} \lambda \in (0, t_1] \setminus (t_2, t_3),\ |\operatorname{Im} \lambda| \le \sqrt{g(|\operatorname{Re} \lambda|)}\} \\ &\cup \{\lambda \in \mathbb{C} |\ \pm \operatorname{Re} \lambda \in (t_1, \gamma],\ |\operatorname{Im} \lambda| \le 2b + c\}.\end{aligned}$$

There is a spectral gap in the case $\beta > 2\|B\|$ around $\beta/2$ of length $t_3 - t_2 = \sqrt{\beta^2 - 4\|B\|^2}$.

Proof. By Proposition 4.1 we have

$$\sigma(\mathcal{A}_{\pm D}) \subset \{\lambda \in \mathbb{C} |\ 0 \le \pm \operatorname{Re}\lambda \le \gamma,\ |\operatorname{Im}\lambda| \le 2b + c\}. \tag{4.3}$$

If $\lambda \in \sigma(\mathcal{A}_{\pm D})\setminus\{0\}$, then $\lambda \in W^2(\mathcal{A}_{\pm D})\setminus\{0\}$. By Remark 3.3, there exists $(f,g)^\top \in \mathcal{D}(\mathcal{A}_{\pm D})$, $\|f\| = \|g\| = 1$ with

$$\lambda^2 \mp \lambda(Dg,g) + |(f,Bg)|^2 = 0,$$

and hence

$$\operatorname{Re}(Dg,g) = |\operatorname{Re}\lambda| \left(1 + \frac{|(f,Bg)|^2}{|\lambda|^2}\right). \tag{4.4}$$

If $0 \le \pm\operatorname{Re}\lambda < \beta$ and using $|(f,Bg)\ |^2 \le \|B\|^2$, $0 < \beta \le \operatorname{Re}(Dg,g)$ and equality (4.4), we infer

$$|\lambda|^2 \le \frac{|\operatorname{Re}\lambda|\ \|B\|^2}{\beta - |\operatorname{Re}\lambda|}, \tag{4.5}$$

which implies

$$|\operatorname{Re}\lambda| \le \|B\| \sqrt{\frac{|\operatorname{Re}\lambda|}{\beta - |\operatorname{Re}\lambda|}},$$

$$|\operatorname{Im}\lambda| \le \|B\| \sqrt{\frac{|\operatorname{Re}\lambda|}{\beta - |\operatorname{Re}\lambda|}} = f(|\operatorname{Re}\lambda|), \tag{4.6}$$

and

$$|\operatorname{Im}\lambda|^2 \le f^2(|\operatorname{Re}\lambda|) - |\operatorname{Re}\lambda|^2 = g(|\operatorname{Re}\lambda|),$$

while

$$g(|\operatorname{Re}\lambda|) = |\operatorname{Re}\lambda| \frac{|\operatorname{Re}\lambda|^2 - \beta|\operatorname{Re}\lambda| + \|B\|^2}{\beta - |\operatorname{Re}\lambda|} = |\operatorname{Re}\lambda| \frac{\big(|\operatorname{Re}\lambda| - t_2\big)\big(|\operatorname{Re}\lambda| - t_3\big)}{\beta - |\operatorname{Re}\lambda|}.$$

By constructions, we have

(i) $g(x) \le f^2(x)$ for $x \in (0,\beta)$;

(ii) if $\beta < 2\|B\|$, then $g(x)$ has no zeros in $(0,\beta)$ and is always positive;

(iii) if $\beta \ge 2\|B\|$, then $g(0) = g(t_2) = g(t_3) = 0$ and $g(x) < 0$ for all $x \in (t_2,t_3) \subset (0,\beta)$.

In what follows we compute when $f(x)$ (and hence $g(x)$) is smaller than $2b + c$. In this case, the estimate in (4.6) is better than (4.3). Note that $f(x)$ is monotone increasing, and we have

$$f(x) = 2b + c$$

if and only if $x = t_1$. Now the statement in (I) follows from (i), (ii) and (4.6), the statements in (II) follow from (i), (iii) and (4.6). □

In the following figures, Theorem 4.2 and subsequent improvements in the quadratic numerical range (red for colour online version) over the numerical range (light grey) are depicted. In particular, if the interval $I = (t_2,t_3)$ is non-empty, then there is a spectral free strip for $\operatorname{Re}\lambda \in I$. We suppose that

$$\|B\| = 2, \quad \gamma = 7 \quad \text{and} \quad b = 1/6. \tag{4.7}$$

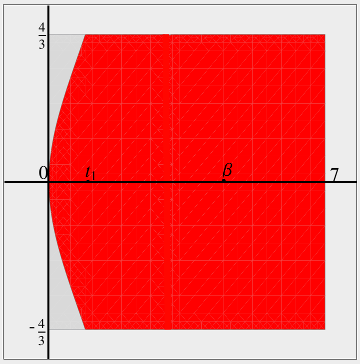

Figure 1: Case I of Theorem 4.2 for $\beta = 3, b = \frac{1}{6}, c = 1$ and $t_1 = \frac{12}{13}$.

The following three figures illustrate Case II of Theorem 4.2. In addition to (4.7), we assume $\beta = 5$, which implies $t_2 = 1$ and $t_3 = 4$. Consequently, there is a spectral gap in $(1, 4)$.

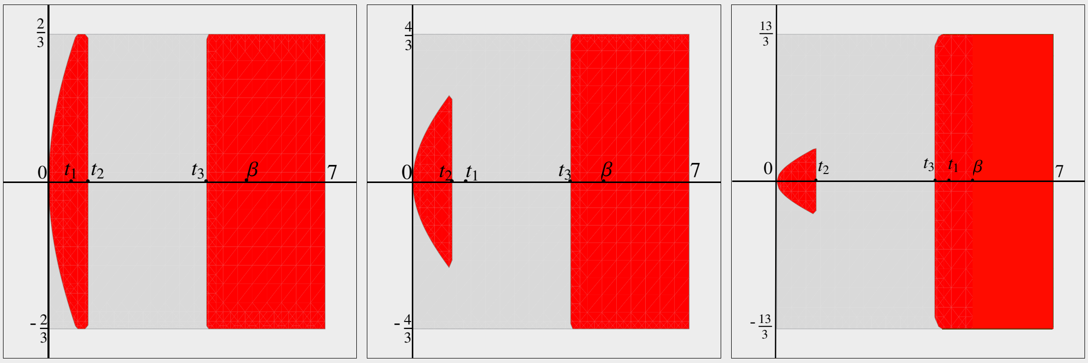

Figure 2: $c = \frac{1}{3}$ and $t_1 = \frac{1}{2}$ (left). $c = 1$ and $t_1 = \frac{20}{13}$ (center). $c = 4$ and $t_1 = \frac{169}{41}$ (right).

Corollary 4.3. Suppose $\mathcal{A}_{\pm D}$ in (4.1) is bounded and D is self-adjoint. We define for $x \in [0, \|B\|]$

$$t_4 = \sqrt{\|B\|^2 - 4b^2} \quad \text{and} \quad h(x) = \sqrt{\|B\|^2 - x^2}.$$

(I) If $\|D\|/2 \leq \|B\|$, and

(i) $\|D\|/2 \geq t_4$, then

$$\begin{aligned}\sigma(\mathcal{A}_{\pm D}) \subset &\{\lambda \in \mathbb{C} |\ 0 \leq \pm \operatorname{Re} \lambda \leq t_4, |\operatorname{Im} \lambda| \leq 2b\} \\ &\cup \{\lambda \in \mathbb{C} |\ t_4 < \pm \operatorname{Re} \lambda \leq \|D\|/2, |\operatorname{Im} \lambda| \leq h(|\operatorname{Re} \lambda|)\} \\ &\cup \{\lambda \in \mathbb{R} |\ \|D\|/2 < \pm\lambda \leq \|D\|\};\end{aligned}$$

(ii) $\|D\|/2 < t_4$, then

$$\sigma(\mathcal{A}_{\pm D}) \subset \{\lambda \in \mathbb{C} |\ 0 \leq \pm \operatorname{Re} \lambda \leq \|D\|/2,\ |\operatorname{Im} \lambda| \leq 2b\} \cup \{\lambda \in \mathbb{R} |\ \|D\|/2 < \pm\lambda \leq \|D\|\}.$$

(II) If $\|D\|/2 > \|B\|$, then

$$\begin{aligned}\sigma(\mathcal{A}_{\pm D}) \subset &\{\lambda \in \mathbb{C} |\ 0 \leq \pm \operatorname{Re} \lambda \leq t_4, |\operatorname{Im} \lambda| \leq 2b\} \\ &\cup \{\lambda \in \mathbb{C} |\ t_4 < \pm \operatorname{Re} \lambda \leq \|B\|, |\operatorname{Im} \lambda| \leq h(|\operatorname{Re} \lambda|)\} \\ &\cup \{\lambda \in \mathbb{C} |\ \|B\| < \pm \operatorname{Re} \lambda \leq \|D\|/2,\ |\operatorname{Im} \lambda| \leq 2b\} \\ &\cup \{\lambda \in \mathbb{R} |\ \|D\|/2 < \pm\lambda \leq \|D\|\}.\end{aligned}$$

Proof. Let $\lambda \in W^2(\mathcal{A}_{\pm D})$, Remark 3.3 implies that $0 \leq \pm \operatorname{Re} \lambda \leq \|D\|$ and there exists $(f,g)^\top \in \mathcal{D}(\mathcal{A}_{\pm D})$ with $\|f\| = \|g\| = 1$, such that

$$\lambda^2 \mp \lambda(Dg,g) + |(f,Bg)|^2 = 0. \tag{4.8}$$

Since B is bounded, by Remark 3.2, we have

$$\sigma(\mathcal{A}_{\pm D}) \subset \overline{W(\mathcal{A}_{\pm D})} \subset \{\lambda \in \mathbb{C} |\ 0 \leq \pm \operatorname{Re} \lambda \leq \|D\|,\ |\operatorname{Im} \lambda| \leq 2b\}. \tag{4.9}$$

In addition, from (4.8), it is easy to see that

$$\pm \operatorname{Im}(Dg,g) = \operatorname{Im} \lambda (1 - |(f,Bg)|^2 |\lambda|^{-2}) = 0 \tag{4.10}$$

and

$$|\lambda|^2 = |(f,Bg)|^2 \leq \|B\|^2. \tag{4.11}$$

Hence

$$|\operatorname{Im} \lambda| \leq \sqrt{\|B\|^2 - |\operatorname{Re} \lambda|^2} = h(|\operatorname{Re} \lambda|), \quad \text{for } |\operatorname{Re} \lambda| \in [0, \|B\|]. \tag{4.12}$$

In the following, we will compute when $h(x)$ is smaller than $2b$. Note that $h(x) = \sqrt{\|B\|^2 - x^2}$ is monotone decreasing, and we have

$$h(x) = 2b$$

if and only if $x = t_4$. Moreover, (4.8) implies $\operatorname{Re}(Dg,g) = |\operatorname{Re} \lambda|(1 + |(f,Bg)|^2 |\lambda|^{-2})$. If $\operatorname{Im} \lambda \neq 0$, then by (4.10) we have $|(f,Bg)|^2|\lambda|^{-2} = 1$ and it follows that $0 \leq \pm \operatorname{Re} \lambda \leq \frac{\|D\|}{2}$. Thus, the statements in (I), (II) follow from (4.9) and (4.12). □

The following figures illustrate an improvement for the quadratic numerical range (red for coloured online version) over the numerical range (light grey) for $\|B\| = 2$ and $b = 1/2$.

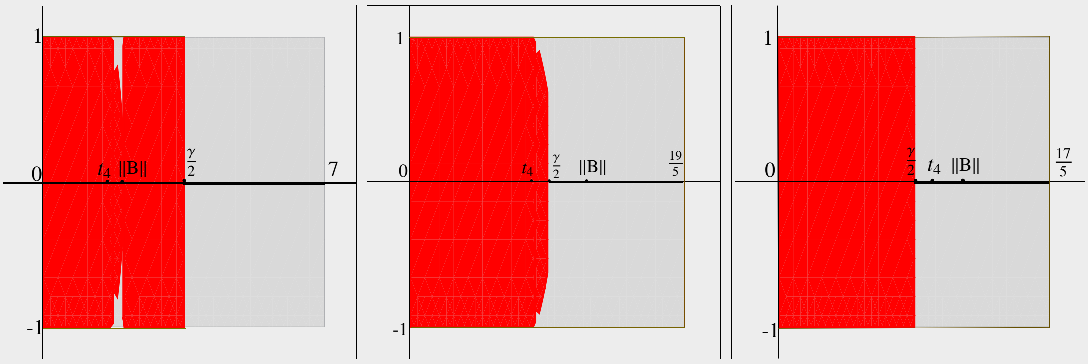

Figure 3: $\|D\| = 7, t_4 = \sqrt{3}$ and $t_4 \leq \|B\| \leq \|D\|/2$ (left). $\|D\| = \frac{19}{5}, t_4 = \sqrt{3}$ and $t_4 \leq \|D\|/2 \leq \|B\|$ (center). $\|D\| = \frac{17}{5}, t_4 = \sqrt{3}$ and $\|D\|/2 \leq t_4 \leq \|B\|$ (right).

5 Spectral inclusion. The case of unbounded entries

In this section, we give some results of an off-diagonally dominant unbounded block operator matrices

$$\mathcal{A}_{\pm D} = \begin{bmatrix} 0 & B \\ -B^* & \pm D \end{bmatrix} : \mathcal{D}(-B^*) \oplus \mathcal{D}(B) \subset H_1 \oplus H_2 \to H_1 \oplus H_2 \tag{5.1}$$

where H_1, H_2 are Hilbert spaces, B is a densely defined closed operator, D is B-bounded and accretive in H_2, *i.e.* $\operatorname{Re}(Dz,z) \geq 0, z \in \mathcal{D}(B)$.

Theorem 5.1. For the unbounded block operator $\mathcal{A}_{\pm D}$ in (5.1), we have

$$\sigma_{ap}(\mathcal{A}_{\pm D}) \subset \overline{W^2(\mathcal{A}_{\pm D})}.$$

Proof. Let $\lambda \in \sigma_{ap}(\mathcal{A}_{\pm D})$, if $\lambda = 0$, then $\lambda \in \overline{W^2(\mathcal{A}_{\pm D})}$ by Proposition 3.1. Thus, in the following, we only have to show the case for $\lambda \neq 0$. By the definition of $\sigma_{ap}(\mathcal{A}_{\pm D})$, here exists a sequence $(f_n, g_n)^\top \in \mathcal{D}(\mathcal{A}_{\pm D})$, $\|f_n\|^2 + \|g_n\|^2 = 1$, with

$$\lim_{n\to\infty} (\mathcal{A}_{\pm D} - \lambda) \begin{bmatrix} f_n \\ g_n \end{bmatrix} = 0.$$

i.e.

$$\lim_{n\to\infty} (Bg_n - \lambda f_n) = 0, \tag{5.2}$$

$$\lim_{n\to\infty} (\pm D g_n - B^* f_n - \lambda g_n) = 0. \tag{5.3}$$

By taking inner products on both sides of (5.2) and (5.3), we obtain

$$\lim_{n\to\infty} (Bg_n - \lambda f_n, f_n) = 0 = \lim_{n\to\infty} (\pm D g_n - B^* f_n - \lambda g_n, g_n). \tag{5.4}$$

Since $\|f_n\|$ and $\|g_n\|$ are bounded, there exist a convergent subsequence $(\|f_{n_k}\|)_k$ of $(\|f_n\|)_n$ and a convergent subsequence $(\|g_{n_k}\|)_k$ of $(\|g_n\|)_n$ with

$$\lim_{k\to\infty} \|f_{n_k}\| =: \mu \quad \text{and} \quad \lim_{k\to\infty} \|g_{n_k}\| =: \nu \tag{5.5}$$

for some $\mu, \nu \in [0, 1]$. Hence, (5.4) implies

$$\lim_{k\to\infty} (Bg_{n_k}, f_{n_k}) = \lambda\mu^2 \quad \text{and} \quad \lim_{k\to\infty} (B^* f_{n_k}, g_{n_k}) = \lim_{k\to\infty} (f_{n_k}, Bg_{n_k}) = \overline{\lambda}\mu^2, \tag{5.6}$$

Similarly, now (5.4) and (5.6) imply

$$\begin{aligned} \lim_{k\to\infty} (\pm D g_{n_k}, g_{n_k}) &= \lim_{n\to\infty} ((\pm D g_{n_k} - B^* f_{n_k} - \lambda g_{n_k}) + (B^* f_{n_k} + \lambda g_{n_k}), g_{n_k}) \\ &= \overline{\lambda}\mu^2 + \lambda\nu^2. \end{aligned} \tag{5.7}$$

If $\mu = 0$, then $\lambda \in \overline{W(\pm D)} \subset \overline{W^2(\mathcal{A}_{\pm D})}$ due to equality (5.7) and Proposition 3.1. If $\mu > 0$, we consider the sequence of polynomials

$$\triangle(f_{n_k}, g_{n_k}; z) := \det \begin{bmatrix} -z(f_{n_k}, f_{n_k}) & (Bg_{n_k}, f_{n_k}) \\ -(B^* f_{n_k}, g_{n_k}) & \pm(D g_{n_k}, g_{n_k}) - z(g_{n_k}, g_{n_k}) \end{bmatrix}.$$

By (5.5), (5.6) and (5.7), this sequence of polynomials converges locally uniformly to the polynomial $\triangle(z)$ where $\triangle(z)$ is given by

$$\triangle(z) := \det \begin{bmatrix} -z\mu^2 & \lambda\mu^2 \\ -\overline{\lambda}\mu^2 & \overline{\lambda}\mu^2 + \lambda\nu^2 - z\nu^2 \end{bmatrix}.$$

It is easy to see that $\triangle(\lambda) = 0$ and $\triangle \not\equiv 0$. Hence, by Hurwitz's theorem [1, Theorem VII.2.5] for every $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that, for $k \geq K$, the quadratic polynomial $\triangle(f_{n_k}, g_{n_k}; \cdot)$ has a null point $z_{n_k} \in \mathbb{C}$ with $|z_{n_k} - \lambda| < \varepsilon$. Since $z_{n_k} \in W^2(\mathcal{A}_{\pm D})$, it follows that $\lambda \in \overline{W^2(\mathcal{A}_{\pm D})}$. $\square$

Theorem 5.2. Let $\mathcal{A}_{\pm D}$ be as in (5.1). If a component Ω of $\mathbb{C} \setminus \overline{W^2(\mathcal{A}_{\pm D})}$ contains a point $\mu \in \rho(\mathcal{A}_{\pm D})$, then $\Omega \subset \rho(\mathcal{A}_{\pm D})$; in particular if every component of $\mathbb{C} \setminus \overline{W^2(\mathcal{A}_{\pm D})}$ contains a point $\mu \in \rho(\mathcal{A}_{\pm D})$, then

$$\sigma(\mathcal{A}_{\pm D}) \subset \overline{W^2(\mathcal{A}_{\pm D})}.$$

Proof. See the proof of [4, Theorem V.3.2]. □

Theorem 5.3. Let $\mathcal{A}_{\pm D}$ be as in (5.1). Assume there exists $k \geq 0$ such that

$$|\operatorname{Im}(Dg, g)| \leq k \operatorname{Re}(Dg, g), \;\; g \in \mathcal{D}(B)$$

and $\beta > 0, \;\gamma < \infty$. If there are $\pm\lambda_1, \pm\lambda_2 \in \rho(\mathcal{A}_{\pm D})$ with $\pm\lambda_1 < 0$ and $\pm\lambda_2 > \gamma$, then

$$\sigma(\mathcal{A}_{\pm D}) \subset \left\{ \lambda \in \mathbb{C} |\; 0 \leq \pm \operatorname{Re}\lambda \leq \gamma, \;|\operatorname{Im}\lambda| \leq \begin{cases} \frac{k|\operatorname{Re}\lambda|}{1-\frac{2}{\beta}|\operatorname{Re}\lambda|}, & \pm\operatorname{Re}\lambda \in [0, \frac{\beta}{2}) \\ \infty, & \pm\operatorname{Re}\lambda \in [\frac{\beta}{2}, \frac{\gamma}{2}] \\ \frac{k|\operatorname{Re}\lambda|}{\frac{2}{\gamma}|\operatorname{Re}\lambda|-1}, & \pm\operatorname{Re}\lambda \in (\frac{\gamma}{2}, \gamma]. \end{cases} \right\}. \tag{5.8}$$

Proof. Let $\lambda \in W^2(\mathcal{A}_{\pm D}) \setminus \{0\}$, Remark 3.3 implies that $0 \leq \pm\operatorname{Re}\lambda \leq \gamma$ and there exists $(f, g)^\top \in \mathcal{D}(\mathcal{A}_{\pm D})$ with $\|f\| = \|g\| = 1$, such that the equality

$$\lambda^2 \mp \lambda(Dg, g) + |(f, Bg)|^2 = 0$$

holds and it follows that

$$\operatorname{Re}(Dg, g) = \left(1 + \frac{|(f, Bg)|^2}{|\lambda|^2}\right) |\operatorname{Re}\lambda|, \tag{5.9}$$

$$\pm\operatorname{Im}(Dg, g) = \left(1 - \frac{|(f, Bg)|^2}{|\lambda|^2}\right) \operatorname{Im}\lambda. \tag{5.10}$$

As $\beta > 0$ we have $\operatorname{Re}(Dg, g) > 0$, thus (5.9) implies that $|\operatorname{Re}\lambda| \neq 0$. Moreover, by Theorem 5.2 and Remark 3.3 one sees that the real part of the quadratic numerical range of $\mathcal{A}_{+D}$ (resp. $\mathcal{A}_{-D}$) lies in the interval $[0, \gamma]$ (resp. $[-\gamma, 0]$). The interval $[0, \gamma]$ (resp. $[-\gamma, 0]$) is a subset of the right hand side of (5.8). Hence, we assume that $\operatorname{Im}\lambda \neq 0$. By (5.9) and (5.10), we infer that

$$\frac{\operatorname{Re}(Dg, g)}{|\operatorname{Re}\lambda|} \pm \frac{\operatorname{Im}(Dg, g)}{\operatorname{Im}\lambda} = 2.$$

Multiplying this identity by $\frac{|\operatorname{Re}\lambda| \;\operatorname{Im}\lambda}{\operatorname{Re}(Dg,g)}$, we get

$$\operatorname{Im}\lambda \pm \frac{\operatorname{Im}(Dg, g)}{\operatorname{Re}(Dg, g)}|\operatorname{Re}\lambda| = \frac{2|\operatorname{Re}\lambda| \operatorname{Im}\lambda}{\operatorname{Re}(Dg, g)}. \tag{5.11}$$

For $0 < \pm\operatorname{Re}\lambda < \frac{\beta}{2}$, we have

$$|\operatorname{Im}\lambda| \leq \frac{k|\operatorname{Re}\lambda|}{1 - 2\frac{|\operatorname{Re}\lambda|}{\beta}}.$$

By $\operatorname{Re}(Dg, g) \geq \beta$, it follows that

$$W^2(\mathcal{A}_{\pm D}) \subset \left\{ \lambda \in \mathbb{C} |\; 0 < \pm\operatorname{Re}\lambda < \frac{\beta}{2}, \;|\operatorname{Im}\lambda| \leq \frac{k|\operatorname{Re}\lambda|}{1 - 2\frac{|\operatorname{Re}\lambda|}{\beta}} \right\}.$$

For $\frac{\gamma}{2} < \pm \operatorname{Re} \lambda \leq \gamma$, by the definition of γ and (5.11), we deduce that

$$|\operatorname{Im} \lambda| \leq \frac{k|\operatorname{Re} \lambda|}{2^{\frac{|\operatorname{Re} \lambda|}{\gamma}} - 1}.$$

i.e.

$$W^2(\mathcal{A}_{\pm D}) \subset \left\{ \lambda \in \mathbb{C} | \ \frac{\gamma}{2} < \pm \operatorname{Re} \lambda \leq \gamma, \ |\operatorname{Im} \lambda| \leq \frac{k|\operatorname{Re} \lambda|}{2^{\frac{|\operatorname{Re} \lambda|}{\gamma}} - 1} \right\}.$$

By Theorem 5.2 and the assumption that there exist $\pm\lambda_1 < 0$ and $\pm\lambda_2 > \gamma$ such that $\pm\lambda_1, \pm\lambda_2 \in \rho(\mathcal{A}_{\pm D})$, we have $\sigma(\mathcal{A}_{\pm D}) \subset \overline{W^2(\mathcal{A}_{\pm D})}$ and the statement is proved. □

Theorem 5.4. Let $\mathcal{A}_{\pm D}$ be as in (5.1) and assume that D is a bounded and self-adjoint operator. If there are $\pm\lambda_1, \pm\lambda_2 \in \rho(\mathcal{A}_{\pm D})$ with $\pm\lambda_1 < 0$ and $\pm\lambda_2 > \|D\|$, then

$$\sigma(\mathcal{A}_{\pm D}) \subset \{\lambda \in \mathbb{C} | \ 0 \leq \pm \operatorname{Re} \lambda \leq \|D\|/2\} \cup [\|D\|/2, \|D\|]. \tag{5.12}$$

Proof. Let $\lambda \in W^2(\mathcal{A}_{\pm D})$, Remark 3.3 implies that $0 \leq \pm \operatorname{Re} \lambda \leq \|D\|$ and there exists $(f, g)^\top \in \mathcal{D}(\mathcal{A}_{\pm D})$ with $\|f\| = \|g\| = 1$ such that

$$\lambda^2 \mp \lambda(Dg, g) + |(f, Bg)|^2 = 0.$$

By Proposition 3.1, in the following we only need to consider $\lambda \neq 0$, and we have

$$\operatorname{Re}(Dg, g) = |\operatorname{Re} \lambda| \left(1 + \frac{|(f, Bg)|^2}{|\lambda|^2}\right), \tag{5.13}$$

$$\pm \operatorname{Im}(Dg, g) = \operatorname{Im} \lambda \left(1 - \frac{|(f, Bg)|^2}{|\lambda|^2}\right) = 0. \tag{5.14}$$

If $\operatorname{Im} \lambda \neq 0$, then $\frac{|(f,Bg)|^2}{|\lambda|^2} = 1$. From (5.13) and (5.14), it follows that $0 \leq \pm \operatorname{Re} \lambda \leq \frac{\|D\|}{2}$, therefore the set $W^2(\mathcal{A}_{\pm D})$ is contained in the right hand side of (5.12). By Theorem 5.2 and the assumption that there exist $\pm\lambda_1 < 0$ and $\pm\lambda_2 > \|D\|$ such that $\pm\lambda_1, \pm\lambda_2 \in \rho(\mathcal{A}_{\pm D})$, we have $\sigma(\mathcal{A}_{\pm D}) \subset \overline{W^2(\mathcal{A}_{\pm D})}$. □

The following figures illustrate improvements for the quadratic numerical range (red for colour online version) over the numerical range (light grey).

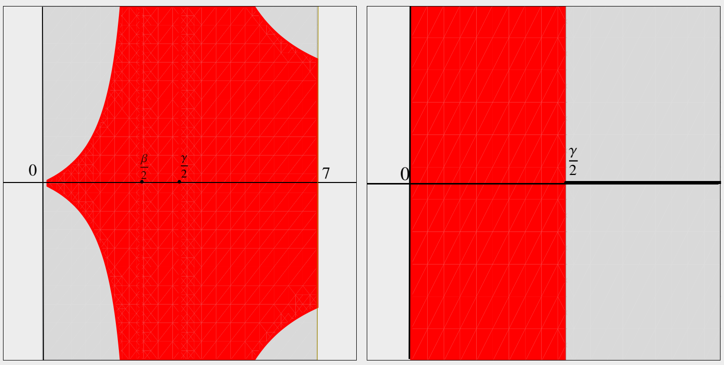

Figure 4: Theorem 5.3 for $\gamma = 7$, $\beta = 5$ and $k = \frac{1}{2}$ (left). Theorem 5.4 for $\gamma = \|D\| = 7$ (right).

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References

[1] J. B. Conway. *Functions of one complex variable.* Springer, 1996.

[2] J. Huang, J. Liu, and A. Chen. Symmetry of the quadratic numerical range and spectral inclusion properties of hamiltonian operator matrices. *Math. Notes*, 103(5):1007–1013, 2018.

[3] B. Jacob, C. Tretter, C. Trunk, and H. Vogt. Systems with strong damping and their spectra. *Math. Methods Appl. Sci.*, 41(16):6546–6573, 2018.

[4] T. Kato. *Perturbation theory for linear operators.* Springer, 2013.

[5] H. Langer, A. Markus, V. Matsaev, and C. Tretter. A new concept for block operator matrices: the quadratic numerical range. *Linear Algebra Appl.*, 330(1-3):89–112, 2001.

[6] H. Langer, A. Markus, and C. Tretter. Corners of numerical ranges. In A. Dijksma, M. A. Kaashoek, and A. C. M. Ran, editors, *Recent Advances in Operator Theory*, pages 385–400. Birkhäuser Basel, 2001.

[7] H. Langer and C. Tretter. Spectral decomposition of some nonselfadjoint block operator matrices. *J. Operator Theory*, 39(2):339–359, 1998.

[8] M. Langer and M. Strauss. Spectral properties of unbounded j-self-adjoint block operator matrices. *J. Spectr. Theory*, 7(1):137–190, 2017.

[9] H. Linden. The quadratic numerical range and the location of zeros of polynomials. *SIAM J. Matrix Anal. Appl.*, 25(1):266–284, 2003.

[10] A. Muhammad and M. Marletta. *Approximation of quadratic numerical range of block operator matrices.* PhD thesis, Cardiff University, 2012.

[11] A. Muhammad and M. Marletta. A numerical investigation of the quadratic numerical range of Hain-Lüst operators. *Int. J. Comput. Math.*, 90(11):2431–2451, 2013.

[12] O. Toeplitz. Das algebraische Analogon zu einem Satze von Fejér. *Math. Zeitschrift*, 2:187–197, 1918.

[13] C. Tretter. *Spectral theory of block operator matrices and applications.* Imperial College Press, 2008.

[14] C. Tretter. Spectral inclusion for unbounded block operator matrices. *J. Funct. Anal.*, 256(11):3806–3829, 2009.

[15] C. Tretter. The quadratic numerical range of an analytic operator function. *Complex Anal. Oper. Theory*, 4(2):449–469, 2010.