Existence of Ground States for Infrared-Critical Models of Quantum Field Theory

Dissertation
zur Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat
der Fakultät für Mathematik und Informatik
der Friedrich-Schiller-Universität Jena

von M. Sc. Benjamin Hinrichs

geboren am 29.07.1991 in Jena

Gutachter:

1. **Prof. Dr. David Hasler** (Friedrich-Schiller-Universität Jena)
2. **Prof. Dr. Ira Herbst** (University of Virginia)
3. **Prof. Dr. Oliver Matte** (Aalborg Universitet)

Tag der öffentlichen Verteidigung: 7. Januar 2022

To Ida, Rasmus and Jonathan.

Zusammenfassung

In der Betrachtung von Modellen, in welchen nichtrelativistische quantenmechanische Teilchen mit einem Feld relativistischer masseloser Bosonen wechselwirken, begegnet man einem Infrarot-Problem. Anschaulich geht dies darauf zurück, dass bereits kleine Energie-Schwankungen zur Erzeugung unendlich vieler niederenergetischer Bosonen führen können. Dies führt zur Infrarot-Divergenz des Systems. Aus mathematischer Sicht erwartet man für solche Systeme, dass kein Grundzustand, also kein stationärer Zustand niedrigster Energie, existieren kann. In dieser Dissertation behandeln wir zwei Arten von infrarot-kritischen Modellen.

Das translationsinvariante Nelson-Modell beschreibt ein nichtrelativistisches spinloses quantenmechanisches Teilchen, welches linear an das Bosonenfeld gekoppelt ist. Wir betrachten das System bei festgehaltenem Gesamtimpuls und verallgemeinern vorangegangene Beweise für die Abwesenheit von Grundzuständen auf das ultraviolett-renormierte Modell. Hierbei behandeln wir die Abhängigkeit des renormierten Operators vom Gesamtimpuls und zeigen insbesondere, dass die Definitionsbereiche von renormierten Operatoren für unterschiedliche Gesamtimpulse im physikalischen Fall lediglich den trivialen Vektorraum gemein haben.

Als zweites Modell behandeln wir das Spin-Boson-Modell. Dieses beschreibt ein quantenmechanisches System mit zwei Zuständen – Spin genannt – welches ebenfalls linear an das Bosonenfeld gekoppelt ist. In der Vergangenheit wurde mit Hilfe störungstheoretischer Methoden gezeigt, dass selbst im physikalischen infrarot-kritischen Fall ein Grundzustand existiert. Dies wird darauf zurückgeführt, dass Symmetrien des Modells zur gegenseitigen Aufhebung von Divergenzen führen. Wir führen in dieser Arbeit einen neuen nicht-störungstheoretischen Beweis für die Existenz von Grundzuständen, der es uns erlaubt das Resultat auf singulärere Fälle auszuweiten. Weiterhin können wir eine explizite Kopplungskonstante angeben, bis zu welcher der Grundzustand existiert. Es wird die Vermutung aufgestellt, dass es eine kritische Kopplung gibt, oberhalb welcher die Existenz des Grundzustands endet.

Abstract

When considering models of nonrelativistic quantum mechanical particles interacting with a field of massless relativistic bosons, one encounters an infrared problem. Heuristically, this is due to the fact that small energy fluctuations can create an infinite number of low-energy bosons, which causes infrared-divergences. From a mathematical perspective, this leads to the expectation that such systems cannot have a ground state, i.e., a stationary state at the lowest possible energy. In this thesis, we study two types of infrared-critical models.

In case of the translation-invariant Nelson model, which describes the interaction of a nonrelativistic spinless quantum mechanical particle linearly coupled to the boson field, we extend previous proofs for the absence of ground states at fixed total momentum to the ultraviolet-renormalized model. Along the way, we study the dependence of the renormalized operators on the total momentum. Especially, we prove that in the physical case the domains of the renormalized operators with different total momentum have only the trivial vector space as intersection.

The second model we study is the spin boson model. It describes a two-state quantum mechanical system, called spin, again linearly coupled to a boson field. It was previously proven by perturbative methods that the model has a ground state even in the physical infrared-critical case. This is attributed to a cancellation of divergences due to an underlying symmetry of the model. We provide a new non-perturbative proof for this fact, which allows us to extend the result to more singular models. Further, we can give an explicit coupling constant, below which the ground state exists. It is conjectured that there exists a critical coupling constant, above which the ground state ceases to exist.

Acknowledgements

First of all, I want to thank my advisor David Hasler for his scientific and academic input, which both provided a solid foundation for my PhD time. Especially, I want to mention our many hours spent in video conferences discussing over mathematics as well as reading and editing drafts of our manuscripts during the Corona pandemic.

I also want to thank Oliver Siebert for the fruitful joint work and for his enthusiastic participation in many of those mentioned online meetings.

For a joyful Danish-German friendship highlighting in a collaboration which found its way into this thesis and from which I learned quite a lot, I am very grateful to Thomas Norman Dam.

In this context, I also want to mention Jacob Schach Møller, who invited me for a one-week visit to Aarhus Universitet in 2018 on the occasion of Thomas' PhD thesis defense. He was also a great host as organizer of the QMath14 in August 2019. On both occasions, he provided us with the surroundings to work on parts of the joint work.

As a basis for my teaching of the lecture series "Analysis für Lehramt Gymnasium" from autumn 2019 to spring 2021, I want to express my gratitude to Dorothee Haroske, Aicke Hinrichs and Marcel Schmidt for providing their lecture notes as an orientation and for a variety of helpful advice.

For sacrificing their time and carefully proofreading parts of this thesis, I am very thankful to Glenn Byrenheid, David Hasler, Aicke Hinrichs and Oliver Siebert. The future readers profited a lot from your helpful comments.

I do not want to leave unmentioned all the friendly people at our institute of mathematics in Jena, starting with my (former) office mates and (at the time) fellow PhD students Markus, Oliver and Jannis. Apart from spending quite an amount of time together in the office, I especially want to highlight our joint visits to conferences and workshops in Stuttgart, Venice, Rome, Tucson, Montreal and Toronto, all of which I enjoyed very much. I gratefully acknowledge funding for parts of these trips by the research training group GRK 1523 "Quantum and Gravitational Fields".

A part of the good atmosphere in our institute is due to the PhD seminar. I want to give my thanks to Maike and Daniel for taking me in as an organizer of the seminar and to Aljoscha and Julian for taking over the seminar later on.

For being as enthusiastic about a coffee break at "Rossi" as I am, I want to mention Glenn, Henning, Markus, Mathias and Oliver, without claiming completeness in this list.

I want to express my deepest gratitude for their reliable and loving care for our children, whenever there might be the need, to my parents Maren and Aicke, my parents-in-law Angelika and Peter, and my grandparents-in-law Waltraut and Hermann.

Last but far from least, I thank my wife Helena for sharing and caring about all the small (and also not so small) moments in life, for being the best imaginable every-evening companion and for the occasional supporting kick in the behind, when it is required.

Contents

Zusammenfassung/Abstract v

Acknowledgements vii

1 Introduction 1

Organization of the Thesis . . . 3

1.1 The Translation-Invariant Nelson Model . . . 4

1.1.1 Definition and Results . . . 4

1.1.2 Relation to the Literature . . . 7

1.1.3 Relation to [DH21] . . . 8

1.2 The Spin Boson Model . . . 8

1.2.1 Definition and Results . . . 9

1.2.2 Relation to the Literature . . . 13

1.2.3 Relation to [HHS21, HHS22a, HHS22b] . . . 14

2 Renormalization of the Translation-Invariant Nelson Hamiltonian 15

2.1 The Gross-Transformed Operators . . . 16

2.1.1 Mapping Properties of Weyl Operators I . . . 16

2.1.2 Explicit Calculation of the Transformed Operators . . . 17

2.1.3 Continuity Properties at Finite Cutoff . . . 18

2.1.4 Form Bounds . . . 20

2.1.5 The Renormalized Operators in the Gross Regime . . . 23

2.2 Construction of the Renormalized Operators . . . 24

2.2.1 Mapping Properties of Weyl Operators II . . . 25

2.2.2 Regularity and Domain of the Renormalized Operators . . . 29

3 Absence of Ground States in the Nelson Model 33

3.1 Energy Minimum at $P = 0$ . . . 34

3.2 An HVZ Theorem . . . 35

3.3 Rotation Invariance and Resolvent Bounds . . . 38

3.4 Pull-Through Formula . . . 45

3.5 Proof of Absence of Ground States . . . 47

4 Correlation Bounds in 1D Ising Models 49

4.1 The Ising Model on $\mathbb{Z}$ . . . 49

4.1.1 Griffiths' Inequalities . . . 51

4.1.2 Correlation Bound for Finite Chains . . . 55

4.1.3 Proof of the Correlation Bound on $\mathbb{Z}$ . . . 56

4.1.4 Ising Model with Nearest Neighbor Coupling . . . 58

4.2 The Ising Model on $\mathbb{R}$ . . . 60

4.2.1 Weak Convergence of Probability Measures 63
4.2.2 The Skorokhod Space 64
4.2.3 The Continuum Limit of the Discrete Ising Model 67
4.2.4 Proof of the Correlation Bound 72

5 FKN Formula for the Spin Boson Model with External Magnetic Field 75
5.1 The FKN Formula 76
5.2 Ground State Energy of the Spin Boson Model 83

6 Existence of Ground States in the Spin Boson Model 89
6.1 Uniqueness of Ground States 89
6.2 Ground States in the Massive Spin Boson Model 90
6.3 Ground State Properties 94
6.3.1 Minimizing Sequence 96
6.3.2 Infrared Bounds 96
6.3.3 The Compactness Argument 99
6.4 Proof of Existence 101

A Operators on Hilbert Spaces 105

B Fock Space Analysis 125

Nomenclature 135

List of Figures 137

Bibliography 139

1. Introduction

The key developments in theoretical physics of the twentieth century can be summarized in two categories: the theory of relativity describing large-scale physics and quantum theories describing physics at small scales. Our interest is focused on the latter. One way to formulate the mathematical description of such a quantum theory goes back to von Neumann [Neu32] and uses the methods of functional analysis. The state of the physical system is described by an element of a Hilbert space $\mathcal{H}$ and the energy of this system is given by a selfadjoint operator H acting on this space and which, for simplicity, we assume to be time-independent. The time-evolution $\psi : \mathbb{R} \to \mathcal{H}$ is then described by a Schrödinger equation, which (setting $\hbar = c = 1$) is the initial value problem

$$\mathsf{i}\dot{\psi} = H\psi, \quad \psi(0) = \psi_0.$$

An essential ingredient to the investigation of above equation is the spectral analysis of the so-called *Hamilton operator* (or *Hamiltonian*) H. Especially, it is important to ask for the existence of eigenvalues, which in view of the Schrödinger equation correspond to the stationary states of the system.

In this thesis, we treat quantum systems for which the possible energy – or mathematically speaking the spectrum of H – is bounded from below. Our main object of investigation is the question whether there exists a stationary state corresponding to the lowest possible energy – the infimum of the spectrum. We call such a state a *ground state*. The systems we investigate are models describing the interaction of a rather simple quantum-mechanical system, from now on referred to as *particle*, with a field of bosons (cf. Fig. 1). If such a system has a ground state, it is expected that the particle is "dressed" by a cloud of bosons. This state of the system is also called a *quasi-particle*. If one assumes the bosons to be massless, this causes infrared-divergences. The physical interpretation of this phenomenon is sometimes referred to as the soft photon catastrophe:

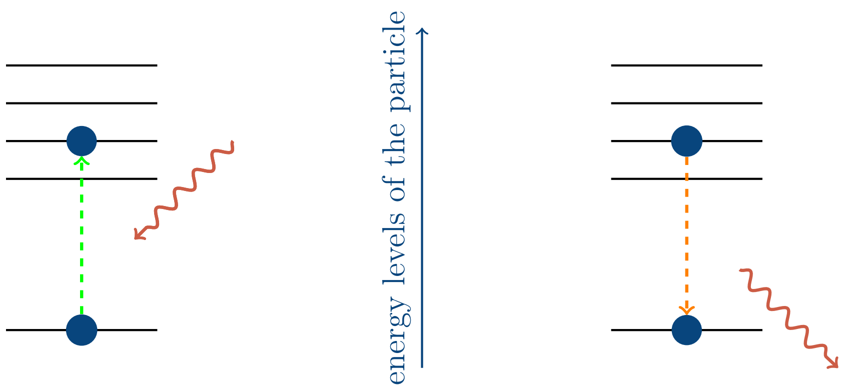

Figure 1: Heuristic concept of the considered models: a quantum mechanical particle (blue), which can absorb (left) or emit (right) bosons (red).

even an infinite number of low-energy ("soft") bosons can have finite combined energy. Since the uncertainty principle allows for small energy fluctuations even in the vacuum state of the boson field, this leads to the creation of infinitely many soft bosons, which causes an infrared-divergence of the system. For a first treatment of this problem in the physics literature, we refer to [BN37].

Let us discuss the setup a little further:
The central information for our setting are the *dispersion relation* $\omega : \mathbb{R}^d \to [0,\infty)$, describing the free energy of a boson, and the *form factor* $v \in L^2(\mathbb{R}^d)$, describing the interaction of particle and field. Here and throughout the thesis, $d \in \mathbb{N}$ denotes the dimension of the underlying space. The usual physical example, which we have in mind, is given by

$$d = 3, \qquad \omega(k) = \sqrt{\mu^2 + |k|^2} \qquad \text{and} \qquad v(k) = \lambda \eta(k) |k|^{-1/2}, \tag{1.1}$$

where the parameter $\mu \geq 0$ is the *mass* of the bosons, $\lambda \in \mathbb{R}$ is a *coupling constant*, describing the strength of the particle field interaction, and $\eta : \mathbb{R}^d \to [0,1]$ is an ultraviolet cutoff function chosen such that $v \in L^2(\mathbb{R}^d)$. One typical choice, which we will use for this introductory discussion, is the characteristic function χ_K of some compact set $K \subset \mathbb{R}^d$. Often, one chooses K to be a closed ball centered at zero with a large but finite radius.

In the case $\mu > 0$ or in presence of an infrared cutoff, i.e., in case $0 \notin K$, there is a lower bound on the energy of bosons which can interact with the particle and one expects that there exists a ground state, being a quasi-particle with a cloud of bosons, which is square-integrable. This phenomenon has been extensively studied in the mathematical literature and, apart from discussing results more thoroughly for the models explicitly studied below, we refer to the exemplary results in [Gro72, Frö74, AH97, BFS98a, BFS98b, Gér00]. In fact, it turns out that a usually sufficient infrared regularity condition is obtained if one replaces the exponent $-1/2$ of v in (1.1) by $-1/2 + \varepsilon$ for some $\varepsilon > 0$, independent of any further cutoff assumptions.

We are, on the contrary, studying the infrared-critical case, which in the situation (1.1) with $\eta = \chi_K$ means that both $\mu = 0$ and K contains some ball centered at 0. In the translation-invariant Nelson model, we will prove that the infrared-divergence indeed causes absence of ground states. Especially, our proof includes the case in which the ultraviolet cutoff η has been removed. Then, we study the spin boson model and prove that a ground state still exists in the infrared-critical case. This can be interpreted as a cancellation of divergences due to an underlying symmetry of the model, since the expectation of the interaction of soft bosons with respect to the free ground state of the particle vanishes, cf. [LMS02] for a more detailed discussion. Results of this type in the model of non-relativistic quantum electrodynamics can be found in [BFS99, GLL01, HH11a, HH11c, BCFS07, HS20].

Before we come to a more rigorous introduction of the considered models and our results on infrared-criticality, let us fix the following objects throughout the thesis:

- the dimension $d \in \mathbb{N}$,
- a measurable function $\omega : \mathbb{R}^d \to [0,\infty)$,
- a locally square-integrable function $v : \mathbb{R}^d \to \mathbb{C}$.

We will later on strengthen the assumptions from above appropriately in the places where it is necessary. Especially, the infrared-criticality assumption then translates into

$$\int_{|k|\le K} \frac{|v(k)|^2}{\omega(k)^2} \mathsf{d}k = \infty \qquad \text{for any } K > 0.$$

Organization of the Thesis

This thesis is organized as follows. In the next two introductory sections, we give rigorous definitions of the models considered in this thesis and state the main results on infrared-criticality in the sense introduced above. We further compare our results to the literature. The two models studied are the translation-invariant Nelson model in Section 1.1 and the spin boson model in Section 1.2, respectively.

The subsequent chapters are then devoted to the proofs of the statements. In Chapter 2, we study the ultraviolet renormalization procedure for the Nelson model and prove new regularity results for the renormalized fiber operators. In Chapter 3, we then prove absence of ground states in the Nelson model. Chapter 4 takes an intermediate role between the study of the two quantum field theoretic models. Therein, we prove correlation bounds for one-dimensional Ising models, which are interesting by themselves, as we motivate in the beginning of that chapter. Our motivation to study Ising models becomes apparent in Chapter 5, where we derive a Feynman-Kac-Nelson (FKN) type formula for the spin boson model with external magnetic field. This provides a direct connection between the studied Ising correlation functions and the derivative of the ground state energy in the spin boson model with respect to the strength of the external magnetic field. Finally, in Chapter 6, we prove the existence of ground states in the spin boson model even in the infrared-critical case. The correlation bound from Chapter 4 and its analogue for the derivatives of the ground state energy obtained using the FKN formula from Chapter 5 will provide a key ingredient of the proof.

The results presented are contained in [DH21, HHS22a, HHS22b, HHS21]. However, this thesis is written with an aim to be as self-contained as possible. Hence, proofs of results which are not contained in the corresponding articles are added. For the convenience of the reader, Sections 1.1.3 and 1.2.3 contain a comparison of the articles with the corresponding chapters in this thesis.

Also with the purpose to be self-contained, the thesis is complemented by a collection of appendices. These contain basic definitions and statements which can be found in the usual textbooks, but go beyond the knowledge of standard functional analysis and measure theory courses. In Appendix A, we give an overview of the theory of operators on Hilbert spaces, while we study the notions of Fock space analysis in Appendix B.

We will assume basic definitions from the appendices in the following, which especially include: selfadjointness (Definition A.32), semiboundedness (Definition A.39), and Fock space basics such as the second quantization operators $\mathsf{d}\Gamma(\cdot)$ and $\Gamma(\cdot)$ (Definitions B.11 and B.14), the field operators $\varphi(\cdot)$ (Definition B.21) and the Weyl operators $W(\cdot)$ (Definition B.24).

Furthermore, we fix some notation. We will work under the convention $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The characteristic function of a set $M \subset \mathbb{R}^d$ is denoted by χ_M and, for $R > 0$, we denote the open ball of radius R centered at $k \in \mathbb{R}^d$ by $B_R(k)$. The identity operator on a Hilbert space $\mathcal{H}$ is denoted by $\mathbb{1}_{\mathcal{H}}$. If the Hilbert space is clear from the context, we drop the index $\mathcal{H}$. Further, we also sometimes denote the operator $\alpha\mathbb{1}$ with some $\alpha \in \mathbb{C}$ only by the symbol α. Inner products and sesquilinear forms are assumed to be anti-linear in the first and linear in the second argument. If the Hilbert space is apparent from the context, then we will not use it as an index for inner products or norms.

Throughout the thesis, $\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^d))$ denotes the Fock space over the one-particle space $L^2(\mathbb{R}^d)$. Similarly, we denote by $\mathcal{F}^{(n)} = \mathcal{F}^{(n)}(L^2(\mathbb{R}^d)) \cong L^2_{\mathsf{s}}(\mathbb{R}^{nd})$ the corresponding n-particle subspaces, which are unitarily equivalent to the space of L^2-functions in n d-dimensional variables and symmetric under permutation of these variables. We will use this equivalence without further mention, cf. Remark B.3 for more details.

A more extensive list of the standard symbols used in this thesis can be found in the Nomenclature.

Next, we want to introduce our models explicitly and before that fix the notion of a ground state.

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space and let H be a selfadjoint lower-semibounded operator on $\mathcal{H}$ and let $E = \inf \sigma(H)$. We say $\psi \in \mathcal{D}(H)$ is a *ground state* of H if $H\psi = E\psi$. We say that the ground state is *unique* or *nondegenerate* if $\dim \ker(H - E) = 1$.

1.1. The Translation-Invariant Nelson Model

The Nelson model was originally treated by Edward Nelson [Nel64]. It describes a particle minimally coupled to a boson field. Here, the particle is spinless and uncharged and is described by the Schrödinger operator $-\Delta + V$ acting on $L^2(\mathbb{R}^d)$, where $-\Delta$ denotes the positive Laplacian in $\mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is an appropriately chosen potential. The Nelson model has been extensively studied throughout the literature, since it exhibits two characteristic behaviors: First, although directly only being well-defined in the presence of an ultraviolet cutoff similar to the one in (1.1), there exists an explicit procedure to remove the cutoff and obtain a selfadjoint lower-semibounded operator [Nel64]. Second, the model displays a serious infrared criticality and serves as an important platform for the study of scattering theory in presence of infrared criticalities. This was first studied by Fröhlich [Frö73, Frö74] and has inspired a variety of subsequent papers, cf. the discussion in Section 1.1.2.

In the case $V = 0$ the model is translation-invariant, which means the Hamiltonian strongly commutes with the total momentum operator $\mathsf{P} = -\mathsf{i}\nabla + \mathsf{d}\Gamma(\mathbf{m})$, where $\mathbf{m}$ is the momentum operator on $L^2(\mathbb{R}^d)$, see (1.3). Hence, it decomposes with respect to the spectrum of P, i.e., it is unitarily equivalent to the direct integral over so-called fiber operators describing the system at fixed total momentum. In case of the Nelson model, the fiber operators can explicitly be defined as operators on Fock space [LLP53], which is the starting point of our investigation.

1.1.1. Definition and Results

Our study of the translation-invariant Nelson model is concerned with the fiber operators at fixed total momentum $P \in \mathbb{R}^d$, which are defined on the Fock space $\mathcal{F}$.

For the form factor v, we define the ultraviolet cutoff version

$$v_\Lambda = v\chi_{\{|\cdot| \le \Lambda\}} \qquad \text{for } \Lambda \ge 0. \tag{1.2}$$

For well-definedness, we introduce the following assumptions.

Hypothesis N0.

(i) $\omega > 0$ almost everywhere.

(ii) $v_\Lambda \in \mathcal{D}(\omega^{-1/2})$ for all $\Lambda \geq 0$.

Remark 1.2. In (ii) and from now on, we understand ω as selfadjoint multiplication operator. Note that, in this sense, Hypothesis N0 (i) implies that ω is positive and injective.

For the definition of the operators we are investigating, we define the vector of multiplication operators $\mathbf{m} = (m_1, \ldots, m_d)$ on $L^2(\mathbb{R}^d)$ given by

$$m_i(k) = k_i \qquad \text{for } k = (k_1, \ldots, k_d) \in \mathbb{R}^d. \tag{1.3}$$

Definition 1.3 (Fiber Operators of the Translation-Invariant Nelson Model)**.**
For $P \in \mathbb{R}^d$ and $\Lambda \geq 0$, we define

$$H_{\mathsf{N},\Lambda}(P) = |P - \mathsf{d}\Gamma(\mathbf{m})|^2 + \mathsf{d}\Gamma(\omega) + \varphi(v_\Lambda)$$

as operator on $\mathcal{F}$.

Remark 1.4. Since $\mathbf{m}$ is a vector of selfadjoint multiplication operators, $\mathsf{d}\Gamma(\mathbf{m})$ is a vector of strongly commuting selfadjoint operators, by Lemma B.15 (ii). Hence, the operator $|P - \mathsf{d}\Gamma(\mathbf{m})|^2$ is well-defined by the functional calculus, cf. Definition A.73.

Lemma 1.5. *Assume Hypothesis N0. Then, for all $\Lambda \geq 0$ and $P \in \mathbb{R}^d$, the operator $H_{\mathsf{N},\Lambda}(P)$ is selfadjoint on the domain $\mathcal{D}(H_{\mathsf{N},\Lambda}(P)) = \mathcal{D}(|\mathsf{d}\Gamma(\mathbf{m})|^2) \cap \mathcal{D}(\mathsf{d}\Gamma(\omega))$ and has form domain $\mathcal{Q}(H_{\mathsf{N},\Lambda}(P)) = \mathcal{D}(|\mathsf{d}\Gamma(\mathbf{m})|) \cap \mathcal{D}(\mathsf{d}\Gamma(\omega)^{1/2})$. Further, it is lower-semibounded uniformly in P.*

Proof. As a sum of strongly commuting and positive selfadjoint operators, $H_{\mathsf{N},0}(P)$ is positive and selfadjoint, cf. Lemmas A.61, A.69 and B.15. Further, $\varphi(v_\Lambda)$ is infinitesimally $\mathsf{d}\Gamma(\omega)$-bounded, cf. Lemma B.22. Hence, the Kato-Rellich theorem (Theorem A.45) proves selfadjointness and semiboundedness. The uniform lower bound can be seen from the standard bounds in Lemma B.20 (vii). The explicit calculation of domain and form domain follows from Lemmas B.17 and A.87, respectively. □

Now and henceforth, we denote the domain and form domain of the ultraviolet regular Nelson operators $H_{\mathsf{N},\Lambda}(P)$ as

$$\mathscr{D}_{\mathsf{N}} = \mathcal{D}(|\mathsf{d}\Gamma(\mathbf{m})|^2) \cap \mathcal{D}(\mathsf{d}\Gamma(\omega)) \qquad \text{and} \qquad \mathfrak{Q}_{\mathsf{N}} = \mathcal{D}(|\mathsf{d}\Gamma(\mathbf{m})|) \cap \mathcal{D}(\mathsf{d}\Gamma(\omega)^{1/2}). \tag{1.4}$$

Using the ideas of Nelson, one can renormalize the fiber operators. To give a precise statement, we need the following assumptions.

Hypothesis NR. We assume Hypothesis N0 and the following:

(i) There exists $\sigma > 0$ such that $\inf\{\omega(k) : |k| \geq \sigma\} > 0$,

$$\int_{|k|>\sigma} \frac{|v(k)|^2}{\omega(k)^{1/2}(1+|k|^2)} \mathsf{d}k < \infty, \quad \text{and} \quad \int_{|k|>\sigma} \frac{|v(k)|^2\omega(k)}{(1+|k|^2)^2} \mathsf{d}k < \infty.$$

(ii) $v(k) = v(-k)$ and $\omega(k) = \omega(-k)$ for all $k \in \mathbb{R}^d$.

(iii) $k \mapsto \omega(k)(1 + |k|^2)^{-1}$ is bounded.

We define the self-energy of the Nelson model as

$$E_\Lambda = \int_{|k|\leq\Lambda} \frac{|v(k)|^2}{\omega(k) + |k|^2} \mathsf{d}k, \tag{1.5}$$

which is finite by assumption Hypothesis N0 (ii).

Now, we can state the self-energy renormalization of the Nelson model.

Lemma 1.6. *Assume Hypothesis NR holds. Then, for all $P \in \mathbb{R}^d$, there exists a unique selfadjoint lower-semibounded operator $H_{\mathsf{N},\infty}(P)$ such that $H_{\mathsf{N},\Lambda}(P) + E_\Lambda$ converges to $H_{\mathsf{N},\infty}(P)$ in the norm resolvent sense as $\Lambda \to \infty$.*

Proof. The statement is contained in that of Theorem 2.1. □

Our main theorem now contains the infrared criticality statement both for the ultraviolet-regular as well as the renormalized Nelson model. We need the following assumptions.

Hypothesis NA.

(i) $d \geq 2$.

(ii) ω is continuous and $|k_1| > |k_2|$ implies $\omega(k_1) > \omega(k_2)$.

(iii) ω and v are rotation invariant.

(iv) $v_\Lambda \notin \mathcal{D}(\omega^{-1})$ for one and hence all $\Lambda > 0$.

(v) The limit $C_\omega = \lim\limits_{k\to 0} \dfrac{|k|}{\omega(k)} \in (0, \infty)$ exists.

Theorem 1.7. *Assume $\Lambda \in (0, \infty)$ and Hypothesis N0 or $\Lambda = \infty$ and Hypothesis NR hold. Further, assume Hypothesis NA holds. Then $H_{\mathsf{N},\Lambda}(P)$ does not have a ground state for any choice of $P \in \mathbb{R}^d$.*

Our emphasis lies on the proof for the renormalized case. Therein, it will be especially important to treat differences of fiber operators at different total momentum. For the case $\Lambda < \infty$ and for $P_1, P_2 \in \mathbb{R}^d$, a straightforward calculation from Definition 1.3 leads to the transformation law

$$H_{\mathsf{N},\Lambda}(P_2) = H_{\mathsf{N},\Lambda}(P_1) - 2\mathsf{d}\Gamma((P_2 - P_1) \cdot \mathbf{m}) + 2(P_2 - P_1) \cdot P_1 + |P_2 - P_1|^2. \tag{1.6}$$

However, in the case $\Lambda = \infty$, this statement heavily depends on the domains of the involved operators. Explicitly, we prove the following statement in Chapter 2, which holds independent of any infrared-regularity conditions. For the definitions of the Weyl operators $W(\cdot)$ used therein, we again refer to Appendix B.

Theorem 1.8. *Assume Hypothesis NR holds and let σ be as in Hypothesis NR (i). For $K \geq \sigma$, we define $B_K \in L^2(\mathbb{R}^d)$ as*

$$B_K(k) = \chi_{\{K\leq|k|\}}(k) \frac{v(k)}{\omega(k) + |k|^2}.$$

The following domain and transformation statements hold:

(i) *The form domain of* $H_{\mathsf{N},\infty}(P)$ *is independent of* $P \in \mathbb{R}^d$. *Explicitly,*

$$\mathcal{Q}(H_{\mathsf{N},\infty}(P)) = W(B_K)^* \mathfrak{Q}_{\mathsf{N}} \qquad \text{for any } K \geq \sigma.$$

Further, $\mathcal{Q}(H_{\mathsf{N},\infty}(P)) \subset \mathcal{D}(\mathsf{d}\Gamma(\omega)^{1/2}) \cap \mathcal{D}(|\mathsf{d}\Gamma(\mathbf{m})|^{2/3})$ *and the transformation rule* (1.6) *with* $\Lambda = \infty$ *holds in the sense of sesquilinear forms for all* $P_1, P_2 \in \mathbb{R}^d$.

(ii) *For* $P_1, P_2 \in \mathbb{R}^d$, *the operator domains satisfy* $\mathcal{D}(H_{\mathsf{N},\infty}(P_1)) = \mathcal{D}(H_{\mathsf{N},\infty}(P_2))$ *if and only if* $k \mapsto (P_2 - P_1) \cdot kB_K(k)$ *is square-integrable. In this case, the transformation rule* (1.6) *with* $\Lambda = \infty$ *holds. Otherwise,* $\mathcal{D}(H_{\mathsf{N},\infty}(P_1)) \cap \mathcal{D}(H_{\mathsf{N},\infty}(P_2)) = \{0\}$.

Let us finish with a discussion of the physical example.

Example 1.9. Similar to (1.1), we discuss

$$d = 3, \quad \omega(k) = \sqrt{\mu^2 + k^2} \quad \text{and} \quad v = \lambda\omega^{-1/2}$$

with boson mass $\mu \geq 0$ and coupling constant $\lambda \neq 0$.

First, note that Hypothesis NR is satisfied for any choice of these constants. By integration in polar coordinates, we observe that $(P_2 - P_1) \cdot kB_K(k)$ is not square-integrable for any choice of $P_1 \neq P_2$ and $K > 0$. Hence, Theorem 1.8 gives us that $\mathcal{D}(H_{\mathsf{N},\infty}(P_1)) \cap \mathcal{D}(H_{\mathsf{N},\infty}(P_2)) = \{0\}$ if $P_1 \neq P_2$.

As we already discussed, the physical model is infrared divergent in the massless case $\mu = 0$, i.e., it then satisfies Hypothesis NA. For any choice of $\Lambda \in (0, \infty]$ and $P \in \mathbb{R}^d$, Theorem 1.7 now implies that there is no ground state.

1.1.2. Relation to the Literature

In [Nel64], Nelson introduced the full Nelson operator $H_{\mathsf{N},\Lambda}$, which is unitarily equivalent to the direct integral $\int_{\mathbb{R}^d}^{\oplus} H_{\mathsf{N},\Lambda}(P)\mathsf{d}P$ [LLP53]. In fact, Nelson considered several non-relativistic particles interacting with a bosonic field. He proved that $H_{\mathsf{N},\Lambda} + E_\Lambda$ converges to a selfadjoint lower-semibounded operator $H_{\mathsf{N},\infty}$ in the *strong* resolvent sense. The first proof for norm resolvent convergence of the full operator is due to Ammari [Amm00]. A similar proof to that in [Nel64] for the fiber operators investigated here was first given by Cannon in [Can71]. Further constructions of the norm resolvent limit of the fiber operators can be found in [MM18, Lam21]

The domain of the full renormalized operator has independently been studied in [GW18, LS19, Sch21]. Whereas Griesemer and Wünsch in [GW18] use the original renormalization procedure due to Nelson [Nel64] and its later improvement by Ammari [Amm00], Lampart and Schmidt in [LS19, Sch21] use a novel direct description of the renormalized Nelson Hamiltonian by interior boundary conditions. Our approach in Chapter 2 is closely related to that of [GW18].

The infrared behavior of the Nelson model has been intensively investigated. Different types of infrared regularity conditions, such as massive bosons and infrared cutoffs, were introduced and existence of ground states proven. Results of this type for the full Nelson Hamiltonian and different types of potentials can be found in [Spo98, Gér00, BHL+02, HHS05] for the ultraviolet regular case and in [HM21] for the ultraviolet renormalized model. Similar results for the fiber operators were proven in [Can71, Frö74]. Vice versa, the infrared-critical case has also been topic of ongoing research. In [Frö73], Fröhlich considered a non-equivalent representation of the fiber operators and constructed ground states therein. This approach was continued in [Piz03, BDP12]. As far as rigorous proofs for absence of ground states are concerned, the full operator in the ultraviolet regular case was treated in [LMS02, DG04], while a proof for the full ultraviolet renormalized operator was only recently given in [HM21]. The ultraviolet regular part of Theorem 1.7 is contained in the result of Dam [Dam20]. He used a technique developed by Hasler and Herbst in [HH08] for the Pauli-Fierz model. In case of the Pauli-Fierz model absence of ground states is proven for all total momenta at which the derivative of the ground state energy does not vanish. This result is optimal, since a ground state exists at total momentum $P = 0$ [BCFS07, HS20], where the derivative vanishes due to the global energy minimum. In case of the Nelson model, differentiability is only proven for small total momentum [AH12]. Furthermore, our results imply that no ground state exists even if the derivative of the ground state energy vanishes. In the proof of Dam, the differentiability condition is replaced by rotation invariance and a non-degeneracy assumption for the potential ground state. A major novelty of the proof presented in Chapter 3 is that this non-degeneracy condition is removed from the proof. Whereas it is well-known for the operators with cutoff [Gro72, Frö73, Møl05] and also recently proven for the renormalized fiber operators [Miy21, Lam21], our adaptions come with two major advantages: First, proofs for the non-degeneracy of ground states usually employ a Perron-Frobenius-Faris argument (cf. [Far72] or Theorem A.112) and hence employ an additional technical approach. Second, there are cases in which one expects the degeneracy of the ground state if it exists, e.g., in models involving spin. Therefore, we expect our extended method to have further applications in the future.

1.1.3. Relation to [DH21]

Chapters 2 and 3 are based on Sections 3 and 4 in [DH21], respectively. The proof technique in Chapter 2, however, differs from the one in the article, since we use a different approach to the renormalized fiber operators $H_{\mathsf{N},\infty}(P)$. In the article, for simplicity reasons, we rely on the results in [GW18] and the direct integral decomposition of the full Nelson operator to derive properties of the fibers. In this thesis, we elaborate those arguments in more detail and provide a self-contained construction of the renormalized fiber operators, which is independent of the full operator. The proof in Chapter 3 is mostly the same as in the article, but arguments which were deferred to the literature are presented more rigorously. This especially includes the full proofs of Lemmas 3.1, 3.4 and 3.12 and more details on the proof of Proposition 3.2. Further, material which has been presented in the appendices of [DH21] is now included in the main body. This concerns Sections 2.1.1, 2.2.1 and 3.4.

1.2. The Spin Boson Model

In case of the spin boson model, the "particle" interacting with a boson field is a two-level quantum mechanical system, shortly referred to as spin. This can, for example, be seen as a coarse aproximation of an atom. Although it is one of the simplest models for the interaction of matter with radiation, it has been extensively studied both in the physics and the mathematics literature. Here, we just refer to [LCD$^+$87] for a review from the physics perspective and to [HS95a] for one from the mathematical side. More detailed references will be given in Section 1.2.2.

The main interest for our study of the spin boson model is that the absence of diagonal entries in the interaction causes infrared divergences to cancel. This was first observed in [HH11b] and we aim for an extension of the results therein.

1.2.1. Definition and Results

As described in the introduction, the spin is a quantum mechanical system with two degrees of freedom. We represent this on the state space $\mathbb{C}^2$, choose the standard basis as the eigenvectors of the free energy of the system and set the energy eigenvalues to ± 1. In our formulation, we need the usual 2×2 Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.7}$$

For well-definedness of the spin boson Hamiltonian, we introduce the following assumptions.

Hypothesis SB0.

(i) $\omega > 0$ almost everywhere.

(ii) $v \in \mathcal{D}(\omega^{-1/2})$.

Remark 1.10. Note that Hypothesis SB0 is stronger than Hypothesis N0. This represents that we do not strive for a renormalization of the ultraviolet divergence in the spin boson model. In fact, it was proven in [DM20a] that a physically non-trivial, i.e., with non-vanishing interaction between spin and boson field, self-energy renormalization of the spin boson Hamiltonian does not exist.

We now define the considered operators.

Definition 1.11 (Spin Boson Hamiltonian)**.** For $\lambda \in \mathbb{R}$, we define

$$H_{\mathsf{SB}}(\lambda) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}\Gamma(\omega) + \lambda \sigma_x \otimes \varphi(v).$$

Remark 1.12. The constant λ describes the coupling strength between spin and field. As our main result below depends on the coupling strength, we introduce it here as a parameter of the model, whereas it was hidden in the form factor v in our definition of the Nelson model (Definition 1.3).

Lemma 1.13. *For all $\lambda \in \mathbb{R}$, the operator $H_{\mathsf{SB}}(\lambda)$ is selfadjoint and lower-semibounded on the domain $\mathcal{D}(H_{\mathsf{SB}}(\lambda)) = \mathcal{D}(\mathbb{1} \otimes \mathsf{d}\Gamma(\omega))$.*

Proof. Since σ_x and σ_z are bounded, this follows directly from $\mathsf{d}\Gamma(\omega)$ being positive and selfadjoint (Lemma B.15), $\varphi(v)$ being infinitesimally $\mathsf{d}\Gamma(\omega)$-bounded (Lemma B.22) and the Kato-Rellich theorem (Theorem A.45). □

Our main result on the spin boson Hamiltonian is the existence of ground states for the spin boson model, even if the form factor is infrared-critical – under some mild integrability conditions and the assumption that the coupling constant is below a critical value. Let us first state the necessary assumptions.

Hypothesis SBE. We assume Hypothesis SB0 and the following:

(i) There exists $\alpha_1 > 0$ such that ω is locally α_1-Hölder continuous.

(ii) $\omega(k) = \omega(-k)$ for all $k \in \mathbb{R}^d$.

(iii) $\omega(k) \xrightarrow{|k|\to\infty} \infty$.

(iv) There exists $\epsilon > 0$ such that $\omega^{-1/2}v \in L^{2+\epsilon}(\mathbb{R}^d)$.

(v) v has real Fourier transform, i.e., $v(k) = \overline{v(-k)}$ for all $k \in \mathbb{R}^d$.

(vi) There exists $\alpha_2 > 0$ such that $\sup_{|p|\le 1} \int_{\mathbb{R}^d} \frac{|v(k+p) - v(k)|}{\sqrt{\omega(k)}|p|^{\alpha_2}} \mathsf{d}k < \infty$.

(vii) $\sup_{|p|\le 1} \int_{\mathbb{R}^d} \frac{|v(k)|}{\sqrt{\omega(k)}\omega(k+p)} \mathsf{d}k < \infty$.

Theorem 1.14. *Assume Hypothesis SBE holds. Then $H_{\mathsf{SB}}(\lambda)$ has a unique ground state for all $\lambda \in \mathbb{R}$ with $|\lambda| < \|\omega^{-1/2}v\|/\sqrt{5}$.*

Let us again discuss the physical example.

Example 1.15. We consider the choice

$$\omega(k) = |k|^\alpha \qquad \text{and} \qquad v(k) = \eta(k)|k|^\beta,$$

where η is a suitably chosen cutoff function, e.g., $\eta = \chi_{B_\Lambda(0)}$ for some $\Lambda \in (0,\infty)$ or $\eta(k) = e^{-ck^2}$ for some $c > 0$. Then Hypothesis SBE holds if

$$d > \max\left\{\alpha - 2\beta, \frac{3}{2}\alpha - \beta, \frac{1}{2}\alpha - 2\beta\right\}.$$

Especially, in the case $d = 3$ and $\alpha = 1$ this translates to $\beta > -1$ which includes the physical case (1.1) with vanishing boson mass and even more infrared-singular couplings.

Remark 1.16. For the case $\alpha = 1$ in the previous example our result is optimal in the sense that the interaction is not bounded with respect to the free energy anymore when $\beta \le -1$, i.e., Hypothesis SB0 ceases to hold.

We believe our result is close to optimal in another sense. Let us state this as a conjecture for now, which we will illustrate below.

Conjecture 1.17. As before, we assume $\omega > 0$ almost everywhere. Further, we assume $v \in \mathcal{D}(\omega^{-1/2}) \setminus \mathcal{D}(\omega^{-1})$. Then there exists a critical coupling constant $\lambda_{\mathsf{c}} \in (0,\infty)$ such that $H_{\mathsf{SB}}(\lambda)$ has no ground state for $\lambda \in \mathbb{R}$ with $|\lambda| > \lambda_{\mathsf{c}}$.

Our proof for Theorem 1.14 in Chapter 6 uses a compactness argument. Explicitly, we replace ω by an infrared-regularized dispersion relation, e.g., $\omega_m(k) = \sqrt{m^2 + |k|^2}$ with $m > 0$ in the case (1.1). The parameter m can physically be interpreted as an artificial boson mass, which is why we will refer to the infrared regularized model as *massive* spin boson model in the following. In this case there is no infrared-singularity and the existence of ground states for the corresponding spin boson Hamiltonian can be inferred by well-known arguments. We prove that the ground states of the massive spin boson model lie in a compact set independent of the parameter m. Hence, there exists a sequence $(m_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ of parameters such that $m_n \xrightarrow{n\to\infty} 0$ and the sequence of corresponding ground states converges strongly. We identify the limit of this sequence as the ground state of $H_{\mathsf{SB}}(\lambda)$.

The key argument is the construction of the compact set. Therein, we use a resolvent bound. To prove this bound, we need to investigate the spin boson model in presence of an external magnetic field.

Definition 1.18 (Spin Boson Model with External Magnetic Field)**.** For $\lambda, \mu \in \mathbb{R}$, we define

$$H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda,\mu) = \sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}\Gamma(\omega) + \sigma_x \otimes (\lambda\varphi(v) + \mu\mathbb{1})$$

as an operator on $\mathbb{C}^2 \otimes \mathcal{F}$.

Lemma 1.19. *For all $\lambda, \mu \in \mathbb{R}$, the operator $H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda,\mu)$ is selfadjoint and lower-semibounded on the domain $\mathcal{D}(\mathbb{1} \otimes \mathsf{d}\Gamma(\omega))$.*

Proof. Since $H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda,\mu) = H_{\mathsf{SB}}(\lambda) + \mu\sigma_x \otimes \mathbb{1}$ and σ_x is bounded, this again follows from the Kato-Rellich theorem (Theorem A.45). □

Using second order analytic perturbation theory in μ, we can show that a sufficient condition for the resolvent bound to hold is a bound on the magnetic susceptibility of the massive spin boson model with external magnetic field, which is proportional to the second derivative of the ground state energy of $H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda,\mu)$ with respect to μ.

We investigate this bound through a functional integration point of view. To that end, let X be a jump process taking values in $\{\pm 1\}$ with independent increments and jump times being Poisson distributed with parameter 1, for a rigorous definition see Definition 4.16 in the beginning of Chapter 4. Expectation values w.r.t. the probability measure associated to X are denoted by $\mathbb{E}_X$. In Chapter 5, we prove that the semigroup of the spin boson model with external magnetic field can be treated in terms of this stochastic process through a Feynman-Kac-Nelson (FKN) formula.

For a precise statement, we define

$$\Omega_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \Omega, \tag{1.8}$$

where Ω denotes the Fock space vacuum (cf. Definition B.7) and

$$W(t) = \frac{1}{2}\int_{\mathbb{R}^d} |v(k)|^2 e^{-|t|\omega(k)}\mathsf{d}k. \tag{1.9}$$

For the FKN formula to hold, we need the following assumptions.

Hypothesis SBF. We assume Hypothesis SB0 and

(i) $\omega(k) = \omega(-k)$ for all $k \in \mathbb{R}^d$,

(ii) $v(k) = \overline{v(-k)}$ for all $k \in \mathbb{R}^d$.

Theorem 1.20. *Assume Hypothesis SBF holds. Then, for all $\lambda, \mu \in \mathbb{R}$ and $T > 0$,*

$$e^{-T}\left\langle \Omega_\downarrow, e^{-TH_{\mathsf{SB}}^{(\mathsf{m})}(\lambda,\mu)}\Omega_\downarrow \right\rangle = \mathbb{E}_X\left[\exp\left(\lambda^2\int_0^T\int_0^T W(t-s)X_tX_s\mathsf{d}t\mathsf{d}s - \mu\int_0^T X_t\mathsf{d}t\right)\right].$$

Remark 1.21. The integrals on the right hand side are Riemann integrals. Assume X is realized as a random variable on a probability space Ω. Then, for almost every $\nu \in \Omega$, the function $t \mapsto X_t(\nu)$ has only finitely many discontinuities in the interval $[0, T]$, cf. Lemma 4.15 (ii). Since W can easily be seen to be continuous by definition, this implies that the functions $t \mapsto W(t-s)X_t(\nu)X_s(\nu)$ and $t \mapsto X_t(\nu)$ are Riemann-integrable for almost every $\nu \in \Omega$ and the right hand side is well-defined.

Remark 1.22. The formula in the above theorem is in fact not the FKN formula, but a corollary of it obtained by integrating out the degrees of freedom of the boson field in the vacuum state. The full FKN formula is the statement of Theorem 5.3.

Remark 1.23. The right hand side of the formula in Theorem 1.20 is the partition function of an Ising model on $\mathbb{R}$ with pair interaction function W and external magnetic field μ. In our study of Ising models in Chapter 4, we prove that this model can be seen as the limit $\delta \to 0$ of a discrete Ising model on a lattice with Ising spin distance δ.

Originating from the previous remark, let us define the partition function of the Ising model given by W as

$$\mathfrak{Z}_T(\lambda, \mu) = \mathbb{E}_X \left[\exp \left(\lambda^2 \int_0^T \int_0^T W(t-s) X_t X_s \mathsf{d}t \mathsf{d}s - \mu \int_0^T X_t \mathsf{d}t \right) \right] \quad \text{for } T > 0. \tag{1.10}$$

It is a well-known result that the expectation values of the semigroup can be used to calculate the ground state energy. In our case, this amounts to the following.

Corollary 1.24. *Assume Hypothesis SBF holds. Then, for all $\lambda, \mu \in \mathbb{R}$,*

$$\inf \sigma(H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda, \mu)) = - \lim_{T \to \infty} \left(\frac{1}{T} \log \mathfrak{Z}_T(\lambda, \mu) + 1 \right).$$

We want to generalize this to the derivatives of the ground state energy. To that end, we define expectation values in the above continuous Ising model. For a real-valued random variable Y defined on the same probability space as X, we set

$$\langle\!\langle Y \rangle\!\rangle_{T,\lambda,\mu} = \frac{1}{\mathfrak{Z}_T(\lambda, \mu)} \mathbb{E}_X \left[Y \exp \left(\lambda^2 \int_0^T \int_0^T W(t-s) X_t X_s \mathsf{d}t \mathsf{d}s - \mu \int_0^T X_t \mathsf{d}t \right) \right]. \tag{1.11}$$

To give a connection between correlation functions in our continuous Ising model and the derivative of the ground state energy, we say a selfadjoint lower-semibounded operator A has a *spectral gap* if $\operatorname{dist}(\{\inf \sigma(A)\}, \sigma(A) \setminus \{\inf \sigma(A)\}) > 0$, where dist denotes the usual distance of sets in metric spaces (cf. (A.2)). Further, we denote by Π_n the set of all partitions of the set $\{1, \dots, n\}$ and by $|M|$ the cardinality of a finite set M.

Theorem 1.25. *Assume Hypothesis SBF holds. Let $\lambda, \mu \in \mathbb{R}$ and suppose $H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda, \mu)$ has a spectral gap. Then, for all $n \in \mathbb{N}$, the following derivatives exist and satisfy*

$$\partial_\mu^n \inf \sigma(H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda, \mu)) = \lim_{T \to \infty} \frac{1}{T} \sum_{\mathfrak{P} \in \Pi_n} (-1)^{|\mathfrak{P}|+n} (|\mathfrak{P}| - 1)! \prod_{B \in \mathfrak{P}} \left\langle\!\!\left\langle \left(\int_0^T X_t \mathsf{d}t \right)^{|B|} \right\rangle\!\!\right\rangle_{T,\lambda,\mu}.$$

Especially, in the case of the massive spin boson model the spectral gap assumption is satisfied. Hence, we can calculate the desired second derivative by means of the pair correlation functions of the continuous Ising model. We prove an upper bound on these in Chapter 4, by taking the continuum limit of a discrete Ising model.

The analogy to the continuous Ising model also gives an argument towards Conjecture 1.17, which we here only discuss for the physical case $\omega(k) = |k|$ and $v(k) = \kappa(k)|k|^{-1/2}$. In this situation, the interaction function of the Ising model satisfies $W(t) \sim t^{-2}$ for $t \to \infty$. One-dimensional long-range Ising models decaying quadratically are known to have a phase transition [AN86, IN88]. Hence, we expect that at large coupling the correlation functions of the Ising model cease to be uniformly bounded and a ground state ceases to exist. In fact, a proof for absence of ground states in the spin boson model at large coupling would provide a new proof for this phase transition in the Ising model.

1.2.2. Relation to the Literature

Spectral aspects of the spin boson model have been intensively studied in the past, see for example [FNV88, Ama91, HS95b] and references therein, and are still an active area of research up to now, e.g., [BDH19, Rek20, DM20b] and references therein.

Especially, existence of ground states in presence of an infrared regularity assumption is well-known [Spo89, AH95, Gér00] and the analytic dependence of the ground state and the ground state energy on the coupling constants has been proven [GH09]. As far as the infrared singular case goes, Arai, Hirokawa and Hiroshima [AHH99] proved a vanishing expectation value condition in their study of generalized spin boson models. However, in our case this amounts to $\langle \psi, (\sigma_x \otimes \mathbb{1})\psi \rangle = 0$ for a ground state ψ of $H_{\mathsf{SB}}(\lambda)$, which can be seen to hold in any case due to a simple symmetry argument.

Hasler and Herbst [HH11b] gave the first proof for existence of ground states even in the infrared singular case if the coupling constant is sufficiently small. Their assumptions especially include the physical situation (1.1). The result was obtained using operator theoretic renormalization and especially includes analyticity of the ground state and the ground state energy in the coupling constant. A further proof using iterated perturbation theory was recently given by Bach et al. [BBKM17]. Our result goes beyond these results in two ways. First, we use a non-perturbative method and in this sense the upper bound for the coupling constant in Theorem 1.20 is not of perturbative nature, but necessary for the resolvent bound to hold. In fact, as discussed above, we believe that the ground states cease to exist for large coupling constants. Second, our result holds for more severe divergences than the ones covered in [HH11b, BBKM17], see also Example 1.15. The method employed in our proof is built upon that used in [GLL01, HS20] for the Pauli-Fierz model.

We note that finite temperature KMS states of the spin boson Hamiltonian for the case $W(t) \sim t^{-2}$ for large t were investigated in [Spo89]. Using results about the one-dimensional continuous Ising model, it was established that the weak limit of the KMS states exists as the temperature tends to zero. In particular, it was shown that there exists a critical coupling such that the expectation of the number of bosons is finite below and infinite at and above the critical coupling strength, which is somewhat analogous to our Conjecture 1.17.

The duality between the spin boson model and a one-dimensional Ising model has been first discussed by Emery and Luther [EL74] and was further used for the study of the spin boson model in [SD85, FN88, Abd11], see also references therein. Especially, in [Abd11], the formula from Corollary 1.24 for the case $\mu = 0$ was used to prove analyticity of the ground state energy in the parameter λ. Note that the infimum of the spectrum can be analytic although there does not exist a ground state, cf. [AH12] for the case of the Nelson model. Also see [HHL14] for a recent article on the spin boson model using functional integration methods. For more references on correlation bounds in the Ising model and FKN type formulas in quantum field theory, we refer to the introductions of Chapters 4 and 5, respectively.

1.2.3. Relation to [HHS21, HHS22a, HHS22b]

Chapters 4, 5 and 6 are based on [HHS22a], [HHS22b] and [HHS21], respectively. The structure of Chapter 4 has been slightly altered, to separate the discussions of the discrete and the continuous Ising model. In the proof of the correlation bound for the discrete Ising model, the proofs for the well-known Griffiths' inequalities in Section 4.1.1 have been added. In the discussion of the continuous Ising model, the definition of the jump process in the article is similar to Remark 4.18, whereas in this thesis the construction of X is done as a continuous-time Markov process. Further, weak convergence of measures and the Skorokhod topology are more extensively discussed in Sections 4.2.1 and 4.2.2, respectively. Also, a proof for Lemma 4.40 has been added. Since [HHS22b] is fairly self-contained by itself, the alterations in Chapter 5 are only of notational nature. Finally, in Chapter 6 the proof for the existence of a ground state in the massive spin boson model (Section 6.2) has been added. It is an adaption of a part of an earlier draft of [HHS21]. Further, the concluding Section 6.4 is partly based on the last section in [HHS22b].

2. Renormalization of the Translation-Invariant Nelson Hamiltonian

In this chapter, we discuss the renormalization procedure of the translation-invariant Nelson operators, which leads to Lemma 1.6. From there, we extract a variety of regularity properties of the renormalized fibers, which we then use to prove Theorem 1.8. The results obtained in this chapter are further an essential ingredient in the proof of Theorem 1.7 in the next chapter.

The main result of this chapter is Theorem 2.1. For the statement, we need to introduce some notation. First of all, for $P \in \mathbb{R}^d$, $\Lambda \in [0,\infty)$ and $z \notin \sigma(H_{\mathsf{N},\Lambda}(P) + E_\Lambda)$, we write

$$R_{P,\Lambda}(z) = \left(H_{\mathsf{N},\Lambda}(P) + E_\Lambda - z\right)^{-1}. \tag{2.1}$$

We will extend this definition to the case $\Lambda = \infty$, by setting $E_\infty = 0$. Further, for $0 \le K \le \Lambda \le \infty$, we define

$$B_{K,\Lambda}(k) = \chi_{\{K \le |\cdot| \le \Lambda\}}(k) \frac{v(k)}{\omega(k) + |k|^2}. \tag{2.2}$$

Finally, we introduce the operators

$$A_s := 1 + \mathsf{d}\Gamma(\omega)^{1/2} + |\mathsf{d}\Gamma(\mathbf{m})|^s \qquad \text{for } s \in [0,1]. \tag{2.3}$$

As a sum of positive and strongly commuting selfadjoint operators, they are positive and selfadjoint on $\mathcal{D}(A_s) = \mathcal{D}(|\mathsf{d}\Gamma(\mathbf{m})|^s) \cap \mathcal{D}(\mathsf{d}\Gamma(\omega)^{1/2})$, cf. Lemmas A.69 and B.17.

Theorem 2.1. *Assume Hypothesis NR holds.*
Let $s \in [0,1]$ and let $\sigma > 0$ as in Hypothesis NR (i).

(i) $H_{\mathsf{N},\Lambda}(P) + E_\Lambda$ *is uniformly bounded below in* $P \in \mathbb{R}^d$ *and* $\Lambda \in [0,\infty)$ *and converges to a selfadjoint lower-semibounded operator* $H_{\mathsf{N},\infty}(P)$ *in the norm resolvent sense as* $\Lambda \to \infty$.

(ii) $\mathcal{Q}(H_{\mathsf{N},\infty}(P)) = W(B_{K,\infty})^* \mathfrak{Q}_{\mathsf{N}}$ *for all* $P \in \mathbb{R}^d$, $K \ge \sigma$.

(iii) *If* $B_{K,\infty} \in \mathcal{D}(|\mathbf{m}|^s)$ *for some* $K \ge \sigma$, *then* $\mathcal{D}(H_{\mathsf{N},\infty}(P)) \subset \mathcal{Q}(H_{\mathsf{N},\infty}(P)) \subset \mathcal{D}(A_s)$ *and*

$$\operatorname*{s-lim}_{\Lambda\to\infty} A_s R_{P,\Lambda}(\lambda)^{1/2} = A_s R_{P,\infty}(\lambda)^{1/2} =: C_\lambda(P) \quad \textit{for any } \lambda < \inf \sigma(H_{\mathsf{N},\infty}(P)).$$

For λ *sufficiently small, the maps* $P \mapsto C_\lambda(P) C_\lambda(P)^*$ *and* $P \mapsto \|C_\lambda(P)\|$ *are continuous.*

Remark 2.2. In Lemma 2.5, we discuss regularity properties of $B_{K,\infty}$ similar to the condition in Theorem 2.1 (iii). Especially, the assumption $B_{K,\infty} \in \mathcal{D}(|\mathbf{m}|^s)$ for all $s \leq \frac{2}{3}$ follows directly from Hypothesis NR.

For the remainder of this chapter, we drop the lower index N of $H_{\mathsf{N},\Lambda}$, assume Hypothesis NR holds and fix $\sigma > 0$ as in Hypothesis NR (i).

This chapter is structured as follows. In Section 2.1, we will introduce the Gross transformed Nelson operators and prove they converge in the norm resolvent sense. This approach was already used in [Nel64, Can71]. We further derive similar regularity properties to the above ones for the Gross transformed operators, utilizing methods close to the ones recently applied to the full model in [HM21]. In Section 2.2, we then use convergence considerations similar to [GW18] to prove Theorem 2.1 as well as Theorem 1.8.

2.1. The Gross-Transformed Operators

The central idea of Nelson [Nel64] was to apply a dressing transformation introduced by Gross [Gro62] to improve the ultraviolet properties of the operator. Explicitly, the transformed operators are defined as

$$\widetilde{H}_{K,\Lambda}(P) = W(B_{K,\Lambda})H_\Lambda(P)W(B_{K,\Lambda})^* + E_\Lambda \quad \text{for } \sigma \leq K \leq \Lambda < \infty \text{ and } P \in \mathbb{R}^d. \tag{2.4}$$

Note that $B_{K,\Lambda} \in L^2(\mathbb{R}^d)$ by Hypothesis N0 for any choice of K and Λ, so the definition makes sense.

2.1.1. Mapping Properties of Weyl Operators I

To explicitly study the transformed operators, we need to discuss some mapping properties of Weyl operators. This is done in the next two abstract lemmas.

This first statement, in a slightly less general setting, can be found in [GW18].

Lemma 2.3. *Assume $\nu : \mathbb{R}^d \to \mathbb{R}$ is measurable, satisfies $\nu > 0$ almost everywhere and, for some $p \in \mathbb{N}$, $h : \mathbb{R}^d \to \mathbb{R}^p$ is measurable. Let $\mathscr{D} = \mathcal{D}(\mathsf{d}\Gamma(\nu)^{1/2}) \cap \mathcal{D}(|\mathsf{d}\Gamma(h)|)$. If $g \in \mathcal{D}(|h|) \cap \mathcal{D}(\nu^{-1/2}|h|)$, then $W(f)\mathscr{D} = \mathscr{D}$ and for all $i \in \{1, \dots, p\}$*

$$W(f)\mathsf{d}\Gamma(h_i)W(f)^* = \mathsf{d}\Gamma(h_i) - \varphi(h_i f) + \langle f, h_i f\rangle \quad \textit{holds on } \mathscr{D}. \tag{2.5}$$

Proof. Similar to (2.3), we define the selfadjoint operator $A = 1 + \mathsf{d}\Gamma(\nu)^{1/2} + |\mathsf{d}\Gamma(h)|$ on the domain $\mathscr{D}$.

Now, let $\mathscr{E} = \operatorname{span} \mathcal{E}(\mathcal{D}(\nu + |h|))$. By Lemma B.10, $\mathscr{E}$ is dense in $\mathcal{F}(\mathfrak{h})$. Further, by Lemmas A.64 and B.15, it is left invariant by $\Gamma(e^{it(\nu+|h|)}) = e^{it\mathsf{d}\Gamma(\nu+|h|)}$. Hence, using Lemma A.65, $\mathscr{E}$ is a core for $\mathsf{d}\Gamma(\nu + |h|)$. Since A strongly commutes with $\mathsf{d}\Gamma(\nu + |h|)$ (Lemma B.15) and is $\mathsf{d}\Gamma(\nu+|h|)$-bounded (Lemma A.44), this implies $\mathscr{E}$ is also a core for A, cf. Lemma A.70. Now, the left hand side of (2.5) is a closed operator and the right hand side is A-bounded, so it suffices to show (2.5) holds on $\mathscr{E}$ (cf. Lemma A.33).

Let $g_1, g_2 \in \mathcal{D}(\nu+|h|)$. Using Definitions B.8 and B.24 and Lemmas B.15 (iv), B.20 (iii) and B.26 (i), we have

$$\begin{aligned}
\langle \epsilon(g_2), W(f)\mathsf{d}\Gamma(h_i)W(f)^*\epsilon(g_1)\rangle &= e^{-\|f\|^2 + \langle f, g_1\rangle + \langle g_2, f\rangle} \langle \epsilon(g_2 - f), \mathsf{d}\Gamma(h_i)\epsilon(g_1 - f)\rangle \\
&= \langle g_2 - f, h_i(g_1 - f)\rangle\, e^{\langle g_2, g_1\rangle} \\
&= \langle \epsilon(g_2), (\mathsf{d}\Gamma(h_i) - \varphi(h_i f) + \langle f, h_i f\rangle)\epsilon(g_1)\rangle .
\end{aligned}$$

As $\mathcal{E}(\mathcal{D}(\nu+|h|))$ is total (Lemma B.10), this proves (2.5) holds on $\mathscr{E}$. □

The next lemma is standard and can, e.g., be found in [BR97].

Lemma 2.4. *For all $f,g \in L^2(\mathbb{R}^d)$, we have $W(f)\mathcal{D}(\varphi(g)) = \mathcal{D}(\varphi(g))$ and*

$$W(f)\varphi(g)W(f)^* = \varphi(g) - 2\operatorname{Re}\langle f,g\rangle \qquad \textit{holds on } \mathcal{D}(\varphi(g)).$$

Proof. By Lemma B.22, $\mathcal{E}(L^2(\mathbb{R}^d))$ spans a core for $\varphi(g)$, so it suffices to prove the statement on $\mathcal{E}(L^2(\mathbb{R}^d))$. To that end, let $h_1,h_2 \in L^2(\mathbb{R}^d)$. Then, again using Definition B.24 and Lemmas B.20 (iii) and B.26 (i), it follows that

$$\begin{aligned}\langle \epsilon(h_2), W(f)\varphi(g)W(f)^*\epsilon(h_1)\rangle &= e^{-\|f\|^2+\langle f,h_1\rangle+\langle h_2,f\rangle}\langle \epsilon(h_2-f),\varphi(g)\epsilon(h_1-f)\rangle \\ &= e^{\langle h_2,h_1\rangle}\left(\langle h_2-f,g\rangle+\langle g,h_1-f\rangle\right) \\ &= \langle \epsilon(h_2),(\varphi(g)-2\operatorname{Re}\langle f,g\rangle)\epsilon(h_1)\rangle .\end{aligned}$$

Since $\mathcal{E}(L^2(\mathbb{R}^d))$ is total (Lemma B.10), this proves the statement. □

2.1.2. Explicit Calculation of the Transformed Operators

To obtain an explicit formula for the transformed operators defined in (2.4), we need to study the regularity properties of the functions $B_{K,\Lambda}$ with respect to ω and m, as Lemma 2.3 and the definition (2.4) show. We include the case $\Lambda=\infty$ for later reference.

Lemma 2.5. *Let $\sigma \le K < \Lambda < \infty$. Then the following holds:*

(i) $B_{K,\Lambda} \in \mathcal{D}(\omega^a)\cap\mathcal{D}(|\mathbf{m}|^b)\cap\mathcal{D}(\omega^a|\mathbf{m}|^b)$ *for all $a,b\in\mathbb{R}$,*

(ii) $B_{K,\infty} \in \mathcal{D}(\omega^{1/2})\cap\mathcal{D}(\omega^{-r}|\mathbf{m}|)\cap\mathcal{D}(|\mathbf{m}|^s)$ *for all $s\le\frac{2}{3}$ and $r\ge\frac{1}{4}$.*

Proof. (i) holds, since $B_{K,\Lambda}$ is compactly supported and $\omega \in L^\infty_{\mathsf{loc}}(\mathbb{R}^d)$ by Hypothesis NR (iii). Now, (ii) follows due to the inequality

$$|\mathbf{m}|^{2/3} \le \omega^{1/2}\chi_{\{|\mathbf{m}|^{2/3}\le\omega^{1/2}\}} + |\mathbf{m}|\omega^{-1/4}\chi_{\{|\mathbf{m}|^{2/3}>\omega^{1/2}\}},$$

the integrability conditions in Hypothesis NR (i) and Hypothesis NR (iii). □

We now apply above lemmas to explicitly calculate the transformed Nelson operators.

Lemma 2.6. *Let $P\in\mathbb{R}^d$.*

(i) *The operator $\widetilde{H}_{K,\Lambda}(P)$ is selfadjoint on $\mathscr{D}_{\mathsf{N}}$.*

(ii) *On $\mathscr{D}_{\mathsf{N}}$, we have*

$$\begin{aligned}\widetilde{H}_{K,\Lambda}(P) = H_K(P) &+ E_K \\ &+ a^\dagger(\mathbf{m}B_{K,\Lambda})^2 + a(\mathbf{m}B_{K,\Lambda})^2 + 2a^\dagger(\mathbf{m}B_{K,\Lambda})a(\mathbf{m}B_{K,\Lambda}) \\ &- 2a^\dagger(\mathbf{m}B_{K,\Lambda})\cdot(P-\mathsf{d}\Gamma(\mathbf{m})) - 2(P-\mathsf{d}\Gamma(\mathbf{m}))\cdot a(\mathbf{m}B_{K,\Lambda}).\end{aligned}$$

Remark 2.7. In the formula above, we use vector notation for arguments of the creation and annihilation operators, similar to Definition B.16.

Proof. By the definition (2.4), we have $\mathcal{D}(\widetilde{H}_{K,\Lambda}(P)) = W(B_{K,\Lambda})\mathcal{D}(H_\Lambda(P))$. Hence, the domain statement (i) follows by combining Lemmas 1.5, 2.4 and 2.5. Now, by Lemmas 2.3, 2.4 and 2.5 and using that $\langle v_\Lambda, B_{K,\Lambda}\rangle = E_\Lambda - E_K \in \mathbb{R}$ by (1.5) and (2.2), we have

$$W(B_{K,\Lambda})\mathsf{d}\Gamma(\omega)W(B_{K,\Lambda})^* = \mathsf{d}\Gamma(\omega) - \varphi(\omega B_{K,\Lambda}) + \langle B_{K,\Lambda}, \omega B_{K,\Lambda}\rangle \quad \text{on } \mathscr{D}_{\mathsf{N}} \tag{2.6}$$
$$W(B_{K,\Lambda})\varphi(v_\Lambda)W(B_{K,\Lambda})^* = \varphi(v_\Lambda) - 2(E_\Lambda - E_K). \tag{2.7}$$

Further, by Lemmas 2.5 and B.20 (viii), $\varphi(m_i B_{K,\Lambda})\mathscr{D}_{\mathsf{N}} \subset \mathfrak{Q}_{\mathsf{N}}$. Hence, we can twice apply Lemma 2.3 and, on $\mathscr{D}_{\mathsf{N}}$, obtain

$$W(B_{K,\Lambda})(P - \mathsf{d}\Gamma(\mathbf{m}))^2 W(B_{K,\Lambda})^* = (P - \mathsf{d}\Gamma(\mathbf{m}) - \varphi(\mathbf{m}B_{K,\Lambda}) + \langle B_{K,\Lambda}, \mathbf{m}B_{K,\Lambda}\rangle)^2.$$

A simple symmetry argument yields $\langle B_{K,\Lambda}, m_i B_{K,\Lambda}\rangle = 0$ for $i = 1, \dots, d$. Then, applying the commutation relations from Lemmas B.20 (iv) and (viii), we obtain the formula

$$\begin{aligned} &W(B_{K,\Lambda})(P - \mathsf{d}\Gamma(\mathbf{m}))^2 W(B_{K,\Lambda})^* \\ &\quad = (P - \mathsf{d}\Gamma(\mathbf{m}))^2 - \varphi(|\mathbf{m}|^2 B_{K,\Lambda}) \\ &\qquad + 2a(\mathbf{m}B_{K,\Lambda}) \cdot (P - \mathsf{d}\Gamma(\mathbf{m})) + 2(P - \mathsf{d}\Gamma(\mathbf{m})) \cdot a^\dagger(\mathbf{m}B_{K,\Lambda}) \\ &\qquad + a^\dagger(\mathbf{m}B_{K,\Lambda})^2 + a(\mathbf{m}B_{K,\Lambda})^2 + 2a^\dagger(\mathbf{m}B_{K,\Lambda}) \cdot a(\mathbf{m}B_{K,\Lambda})^2 \\ &\qquad - \||\mathbf{m}|^2 B_{K,\Lambda}\|^2 \qquad\qquad \text{holds on } \mathscr{D}_{\mathsf{N}}. \end{aligned} \tag{2.8}$$

Now, we observe $v_\Lambda - (\omega + |\mathbf{m}|^2)B_{K,\Lambda} = v_K$, by the definition (2.2). Hence, using the additivity of the field operator, the $\varphi(\cdot)$ terms in (2.6) – (2.8) sum op to $\varphi(v_K)$. By the same argument

$$\langle B_{K,\Lambda}, \omega B_{K,\Lambda}\rangle + \||\mathbf{m}|^2 B_{K,\Lambda}\| = \langle v_\Lambda - v_K, B_{K,\Lambda}\rangle = E_\Lambda - E_K.$$

Therefore, summing up (2.6), (2.7) and (2.8) proves the statement. □

2.1.3. Continuity Properties at Finite Cutoff

Before we renormalize the transformed Nelson operators, let us discuss some continuity properties of the Nelson fibers – both in the usual representation as well as the Gross transformed version.

Since we will need to treat fiber operators with different total momentum throughout our considerations on the Nelson model, we introduce the operators

$$D_P(k) = 2k \cdot (P - \mathsf{d}\Gamma(\mathbf{m})) + |k|^2 = -2\mathsf{d}\Gamma(k \cdot \mathbf{m}) + 2k \cdot P + |k|^2, \tag{2.9}$$

which by (1.6) satisfy

$$H_\Lambda(P + k) = H_\Lambda(P) + D_P(k) \qquad \text{for all } \Lambda \in [0, \infty),\ P, k \in \mathbb{R}^d. \tag{2.10}$$

The next lemma collects some statements on the regularity of $D_P(k)$. Here and henceforth, we denote

$$B_P = (H_0(P) + 1)^{1/2} \qquad \text{for } P \in \mathbb{R}^d. \tag{2.11}$$

We note that $\mathcal{D}(B_P) = \mathfrak{Q}_{\mathsf{N}}$ for all $P \in \mathbb{R}^d$.

Lemma 2.8. *Let $a \geq 0$ and $c \in \mathbb{R}$. The operator $|D_P(k) - c|^a$ is B_P^a-bounded for all $P, k \in \mathbb{R}^d$. Further, $\|D_P(k)B_P^{-1}\| \leq 4|k|$ holds for all $k, P \in \mathbb{R}^d$ with $|k| \leq 1$ and*

$$B_{P+k}^{-a} = B_P^{-a}(1 + D_P(k)B_P^{-2})^{-a/2} \qquad \text{for } |k| < \tfrac{1}{4}\ , \tag{2.12}$$

$$\lim_{k\to 0}(1 + D_P(k)B_P^{-2})^{-a/2} = 1. \tag{2.13}$$

Proof. We observe that the operator $|D_P(k) - c|^a B_P^{-a}$ acts on $\mathcal{F}^{(n)}$ as multiplication by the function

$$f_n(k_1, \ldots, k_n) = \frac{\left|2k \cdot (P - k_1 - \cdots - k_n) + |k|^2 - c\right|^a}{(1 + \omega(k_1) + \cdots + \omega(k_n) + |P - k_1 - \cdots - k_n|^2)^a}.$$

Since $|f_n| \leq 2^a(|k|^a + ||k|^2 - c|^a)$, the operator $|D_P(k) - c|^a B_P^a$ is bounded. Hence, $|D_P(k) - c|^a$ is B_P^a-bounded and the bound $\|D_P(k)B_P^{-1}\| \leq 4|k|$ holds for $|k| \leq 1$. For $|k| < \frac{1}{4}$ we have $\|D_P(k)B_P^{-2}\| < 4|k| < 1$ as $\|B_P\| \leq 1$ and hence $1 + D_P(k)B_P^{-2}$ is invertible. (2.12) follows using (2.10) and the fact that $H_0(P)$ and $D_P(k)$ commute. We now see

$$\left\|(1 + D_P(k)B_P^{-2})^{-a/2} - 1\right\| \leq \sup_{|x|\leq|k|} ((1 + 4x)^{-a/2} - 1) \xrightarrow{k\to 0} 0. \qquad \square$$

Similar to (2.1), we define the resolvents of the Gross transformed operators

$$\widetilde{R}_{P,K,\Lambda}(z) = (\widetilde{H}_{K,\Lambda}(P) - z)^{-1} \qquad \text{for } \sigma \leq K \leq \Lambda < \infty \text{ and } P \in \mathbb{R}^d. \tag{2.14}$$

Further, we denote the infima of the respective spectra by

$$\Sigma_\Lambda(P) = \inf \sigma(H_\Lambda(P)) \qquad \text{and} \qquad \widetilde{\Sigma}_{K,\Lambda}(P) = \inf \sigma(\widetilde{H}_{K,\Lambda}(P)). \tag{2.15}$$

In the next lemma, we now give a variety of continuity statements.

Lemma 2.9. *Let $k, P \in \mathbb{R}^d$.*

(i) *Fix $\Lambda \in [0, \infty)$. Then $H_\Lambda(P)$ is uniformly bounded below in P and for arbitrary $\lambda < \inf_{P\in\mathbb{R}^d}(\Sigma_\Lambda(P) + E_\Lambda)$ the map $P \mapsto R_{P,\Lambda}(\lambda)$ is continuous.*

(ii) *Fix $\sigma \leq K < \Lambda < \infty$. Then $\widetilde{H}_{K,\Lambda}(P)$ is uniformly bounded below in P and $\mathfrak{Q}_\mathsf{N} \subset \mathcal{Q}(\widetilde{H}_{K,\Lambda}(P))$. If $\lambda < \inf_{P\in\mathbb{R}^d} \widetilde{\Sigma}_{K,\Lambda}(P)$ and $a \in [0, 1]$, then the map*

$$r_{a,\lambda} : P \mapsto B_P^a \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2}(B_P^a \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2})^*$$

is continuous in norm.

Proof. To prove (i), we first note that the uniform lower bound was already proved in Lemma 1.5. Now, we fix Λ, λ as stated in (i) and $P \in \mathbb{R}^d$. From Lemmas 1.5, 2.8 and A.87, we know $\mathfrak{Q}_\mathsf{N} = \mathcal{Q}(H_\Lambda(P)) = \mathcal{D}(B_P) \subset \mathcal{D}(D_P(k))$ and

$$\|D_P(k)R_{P,\Lambda}(\lambda)^{1/2}\| \leq \|D_P(k)B_P^{-1}\|\|B_P R_{P,\Lambda}(\lambda)^{1/2}\| \xrightarrow{k\to 0} 0.$$

Recalling (2.10) and Lemma 2.8 and using the resolvent identity (Lemma A.29), we obtain

$$R_{P+k,\Lambda}(\lambda) = R_{P,\Lambda}(\lambda)(1 + D_P(k)R_{P,\Lambda}(\lambda))^{-1} \qquad \text{for } |k| \text{ sufficiently small.}$$

Taking the limit $k \to 0$ proves the claim.

The uniform lower bound in (ii) follows directly from Lemma 2.9 (i) and (2.4), while $\mathfrak{Q}_{\mathsf{N}} \subset \mathcal{Q}(\widetilde{H}_{K,\Lambda}(P))$ follows from Lemmas 2.6 (i) and A.87. Fix K, Λ, λ as stated in (ii) and $P \in \mathbb{R}^d$. We define $\widetilde{D}_P(k) = W(B_{K,\Lambda}) D_P(k) W(B_{K,\Lambda})^*$ and note that

$$\widetilde{D}_P(k) \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2} = W(B_{K,\Lambda}) D_P(k) R_{P,\Lambda}(\lambda + E_\Lambda)^{1/2} W(B_{K,\Lambda})^* \xrightarrow{k\to 0} 0 \quad \text{in norm.}$$

In particular, the operator

$$Z(k) = \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2} \widetilde{D}_P(k) \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2}$$

is bounded and goes to 0 for $k \to 0$. We easily deduce $\widetilde{H}_{K,\Lambda}(P+h) = \widetilde{H}_{K,\Lambda}(P) + \widetilde{D}_P(h)$ from (2.10) an, therefore obtain

$$\widetilde{R}_{P+k,K,\Lambda}(\lambda) = \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2} (1 + Z(k))^{-1} \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2} \qquad \text{for } |k| \text{ sufficiently small.}$$

Setting $C = B_P^a \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2}$, this yields

$$r_{a,\lambda}(P+k) = (1 + D_P(k) B_P^{-2})^{a/2} C (1 + Z(k))^{-1} C^* (1 + D_P(k) B_P^{-2})^{a/2},$$

so $r_{a,\lambda}(P+k)$ converges to $CC^* = r_a(P)$ in norm as $k \to 0$, by Lemma 2.8. □

2.1.4. Form Bounds

To derive the convergence of the transformed operators, Nelson studied the quadratic form associated to $\widetilde{H}_{K,\Lambda} - H_0$. Similar derivations are used, e.g., in [Can71, Amm00, GW18]. Following an idea from [HM21], we use a slightly different approach, fix $P \in \mathbb{R}^d$, $K \geq \sigma$ and some sufficiently large $L \geq K$ and study $\widetilde{H}_{K,\Lambda}(P) - \widetilde{H}_{K,L}(P)$ in the limit $\Lambda \to \infty$.

Explictly, we define the quadratic form

$$\mathfrak{Q}^{(P)}_{K,L,\Lambda}(\psi) = \mathfrak{q}_{\widetilde{H}_{K,\Lambda}(P)}(\psi) - \mathfrak{q}_{\widetilde{H}_{K,L}(P)}(\psi) \qquad \text{for } \sigma \leq K \leq L \leq \Lambda,\ P \in \mathbb{R}^d,\ \psi \in \mathfrak{Q}_{\mathsf{N}}. \quad (2.16)$$

We now obtain the following theorem, by an appropriate adaption of the ideas of Nelson [Nel64], see also [Amm00, GW18, HM21] for later refinements.

Theorem 2.10. *Fix some $K \geq \sigma$.*

(i) *For all $\varepsilon > 0$, there is $b > 0$ such that, for all $P \in \mathbb{R}^d$, there exists $L_P \geq K$ with*

$$|\mathfrak{Q}^{(P)}_{K,L,\Lambda}(\psi)| \leq \varepsilon \mathfrak{q}_{\widetilde{H}_{K,L}(P)}(\psi) + b\|\psi\|^2 \qquad \text{for all } \psi \in \mathfrak{Q}_{\mathsf{N}},\ \Lambda \geq L \geq L_P.$$

(ii) *For all $L \geq K$ and $\varepsilon > 0$, there exist $\Lambda_0 \geq L$ and $c \geq 0$ such that, for all $P \in \mathbb{R}^d$, we have*

$$|\mathfrak{Q}^{(P)}_{K,L,\Lambda}(\psi) - \mathfrak{Q}^{(P)}_{K,L,\Lambda'}(\psi)| \leq \varepsilon(\mathfrak{q}_{\widetilde{H}_{K,L}(P)}(\psi) + c\|\psi\|^2) \ \text{ for all } \psi \in \mathfrak{Q}_{\mathsf{N}},\ \Lambda, \Lambda' \geq \Lambda_0.$$

Except for the standard estimates from Lemma B.20 (vii), the key ingredient to the proof of the above theorem is the following lemma.

Lemma 2.11. *For all* $f, g \in \mathcal{D}(\omega^{-1/4}) \cap \mathcal{D}(\omega^{-1/2})$ *and* $\psi \in \mathcal{Q}(\mathsf{d}\Gamma(\omega))$*, we have*

$$|\langle a^\dagger(f)\psi, a(g)\psi\rangle| \\ \leq 3\|\max(\omega^{-1/4}, \omega^{-1/2})f\|\|\max(\omega^{-1/4}, \omega^{-1/2})g\|\|(1+\mathsf{d}\Gamma(\omega))^{1/2}\psi\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|.$$

Proof. Similar proofs are presented in [Nel64, Lemma 5] or [HM21, Lemma C.2].

Let $\Omega = \{\omega \geq 1\}$ and set $f_> = f\chi_\Omega$, $f_< = f - f_>$, $g_> = g\chi_\Omega$, and $g_< = g - g_>$. From the standard estimates in Lemma B.20 (vii), we have

$$|\langle a^\dagger(f_<)\psi, a(g)\psi\rangle| \leq \|\omega^{-1/2}f_<\|\|\omega^{-1/2}g\|\|(1+\mathsf{d}\Gamma(\omega))^{1/2}\psi\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|. \tag{2.17}$$

Similarly, by combining Lemma B.20 (vii) with the canonical commutation relations (Lemma B.20 (iv)), we obtain

$$\begin{aligned} |\langle a^\dagger(f_>)\psi, a(g_<)\psi\rangle| &= |\langle a^\dagger(g_<)\psi, a(f_>)\psi\rangle| \\ &\leq \|\omega^{-1/2}g_<\|\|\omega^{-1/2}f\|\|(1+\mathsf{d}\Gamma(\omega))^{1/2}\psi\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|. \end{aligned} \tag{2.18}$$

By the Cauchy-Schwarz inequality, we observe that, for any $\varepsilon > 0$,

$$\begin{aligned} &\frac{\left|\int_{\mathbb{R}^{2d}} \overline{f_>(k)g_>(k')}\psi^{(n+2)}(k,k',k_1,\ldots,k_n)\mathsf{d}(k,k')\right|^2}{1+\varepsilon+\frac{1}{2}\sum_{j=1}^n \chi_\Omega(k_j)} \\ &\leq \|\omega^{-1/4}f_>\|^2\|\omega^{-1/4}g_>\|^2 \int_{\Omega^2} \frac{\omega(k)^{1/2}\omega(k')^{1/2}|\psi^{(n+2)}(k,k',k_1,\ldots,k_n)|^2\mathsf{d}(k,k')}{\varepsilon+\frac{1}{2}\Big(\chi_\Omega(k)+\chi_\Omega(k')+\sum_{j=1}^n\chi_\Omega(k_j)\Big)}. \end{aligned}$$

By the definition of the Fock space norm, the creation and annihilation operators, the permutation symmetry of $\psi^{(n+2)}$ and the inequality $\sqrt{\omega(k)\omega(k')} \leq \frac{1}{2}(\omega(k)+\omega(k'))$, this yields

$$\begin{aligned} &\left\|(1+\mathsf{d}\Gamma(\chi_\Omega/2))^{-1/2}a(f_>)a(g_>)\psi\right\| \\ &\qquad \leq \|\omega^{-1/4}f_>\|\|\omega^{-1/4}g_>\| \sup_{\varepsilon>0}\left\|(\varepsilon+\mathsf{d}\Gamma(\chi_\Omega))^{-1/2}\mathsf{d}\Gamma(\omega^{1/2}\chi_\Omega)\psi\right\|. \end{aligned}$$

Now, we observe

$$\sum_{j=1}^n \chi_\Omega(k_j)\omega^{1/2}(k_j) \leq \left(\sum_{j=1}^n \chi_\Omega(k_j)\right)^{1/2}\left(\sum_{j=1}^n \omega(k_j)\right)^{1/2},$$

so $\left\|(\varepsilon+\mathsf{d}\Gamma(\chi_\Omega))^{-1/2}\mathsf{d}\Gamma(\omega^{1/2}\chi_\Omega)\psi\right\| \leq \|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|$ for all $\varepsilon > 0$. Combined, we obtain

$$\begin{aligned} |\langle a^\dagger(f_>)\psi, a(g_>)\psi\rangle| &\leq \left\|(1+\mathsf{d}\Gamma(\chi_\Omega))^{1/2}\psi\right\|\left\|(1+\mathsf{d}\Gamma(\chi_\Omega/2))^{-1/2}a(f_>)a(g_>)\psi\right\| \\ &\leq \|\omega^{-1/4}f_>\|\|\omega^{-1/4}g_>\|\|(1+\mathsf{d}\Gamma(\omega))^{1/2}\psi\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|. \end{aligned} \tag{2.19}$$

Putting together (2.17), (2.18) and (2.19) proves the statement. □

We use the above lemma to prove Theorem 2.10.

Proof of Theorem 2.10 (i). Throughout this proof, if we have two vectors of operators $\mathbf{A} = (A_1, \dots, A_d)$ and $\mathbf{B} = (B_1, \dots, B_d)$, we will write $\langle \mathbf{A}\phi, \mathbf{B}\psi \rangle = \sum_{i=1}^{d} \langle A_i\phi, B_i\psi \rangle$ and similar for the norm. Further, we assume $\psi \in \mathfrak{Q}_{\mathsf{N}}$ and fix $P \in \mathbb{R}^d$ without further mention.

First, observe that $B_{K,L}$ and $B_{L,\Lambda}$ have disjoint support (up to a set of measure zero). Hence, $a^\dagger(\mathbf{m}B_{K,L})$ and $a(\mathbf{m}B_{L,\Lambda})$ weakly commute, by Lemma B.20 (iv). Now, using Lemma 2.6 and the additivity of creators and annihilators, we obtain

$$\begin{aligned}\mathfrak{Q}^{(P)}_{K,L,\Lambda}(\psi) = \operatorname{Re}\Big[4\big\langle a(\mathbf{m}B_{L,\Lambda})\psi, (\mathsf{d}\Gamma(\mathbf{m}) - P + \varphi(\mathbf{m}B_{K,L}))\psi\big\rangle \\ + 2\big\langle a^\dagger(\mathbf{m}B_{L,\Lambda})\psi, a(\mathbf{m}B_{L,\Lambda})\psi\big\rangle\Big] + 2\big\|a(\mathbf{m}B_{L,\Lambda})\psi\big\|^2.\end{aligned} \tag{2.20}$$

We estimate the terms one by another. To begin with, we observe that Lemma B.20 (vii) and the Cauchy-Schwarz inequality imply that, for all $\varepsilon > 0$,

$$\begin{aligned}&\big|\big\langle a(\mathbf{m}B_{L,\Lambda})\psi, (\mathsf{d}\Gamma(\mathbf{m}) - P + \varphi(\mathbf{m}B_{K,L}))\psi\big\rangle\big| \\ &\quad\leq \|\omega^{-1/2}|\mathbf{m}|B_{L,\Lambda}\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|\|(\mathsf{d}\Gamma(\mathbf{m}) - P + \varphi(\mathbf{m}B_{K,L}))\psi\| \\ &\quad\leq \varepsilon\|(\mathsf{d}\Gamma(\mathbf{m}) - P + \varphi(\mathbf{m}B_{K,L}))\psi\|^2 + \frac{4}{\varepsilon}\|\omega^{-1/2}|\mathbf{m}|B_{L,\infty}\|^2\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|^2.\end{aligned} \tag{2.21}$$

Similarly, we have

$$\big\|a(\mathbf{m}B_{L,\Lambda})\psi\big\|^2 \leq \|\omega^{-1/2}|\mathbf{m}|B_{L,\infty}\psi\|^2\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|^2. \tag{2.22}$$

Using Lemma 2.11, we further have

$$\begin{aligned}&\big|\big\langle a^\dagger(\mathbf{m}B_{L,\Lambda})\psi, a(\mathbf{m}B_{L,\Lambda})\psi\big\rangle\big| \\ &\quad\leq 3\|\max(\omega^{-1/4}, \omega^{-1/2})|\mathbf{m}|B_{L,\infty}\|^2\|(1 + \mathsf{d}\Gamma(\omega))^{1/2}\psi\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|\end{aligned} \tag{2.23}$$

From Lemma 2.6, we recall there exists a constant $C_{K,L} > 0$ (independent of P) such that

$$\begin{aligned}\mathfrak{q}_{\widetilde{H}_{K,L}(P)}(\psi) = \|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|^2 + \|(\mathsf{d}\Gamma(\mathbf{m}) - P + \varphi(\mathbf{m}B_{K,L}))\psi\|^2 \\ + \langle\psi, \varphi(v_K + |\mathbf{m}|^2B_{K,L})\psi\rangle + C_{K,L}\|\psi\|^2.\end{aligned}$$

Hence, again applying Lemmas 2.5 and B.20 (vii), we find there is a constant $D_{K,L} > 0$ such that

$$\|(\mathsf{d}\Gamma(\mathbf{m}) - P + \varphi(\mathbf{m}B_{K,L}))\psi\|^2 + \|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|^2 \leq \mathfrak{q}_{\widetilde{H}_{K,L}(P)}(\psi) + D_{K,L}\|\psi\|^2. \tag{2.24}$$

Combining (2.21) – (2.24) and inserting them into (2.20) proves the statement. □

Proof of Theorem 2.10 (ii). This proof is very similar to the previous one and we use the same notation. From (2.20), we directly obtain

$$\begin{aligned}\mathfrak{Q}^{(P)}_{K,L,\Lambda}(\psi) - \mathfrak{Q}^{(P)}_{K,L,\Lambda'}(\psi) = \operatorname{Re}\Big[4\big\langle a(\mathbf{m}B_{\Lambda,\Lambda'})\psi, (\mathsf{d}\Gamma(\mathbf{m}) - P + \varphi(\mathbf{m}B_{K,L}))\psi\big\rangle \\ + 2\big\langle a^\dagger(\mathbf{m}B_{\Lambda,\Lambda'})\psi, a(\mathbf{m}B_{L,\Lambda})\psi\big\rangle + 2\big\langle a^\dagger(\mathbf{m}B_{L,\Lambda'})\psi, a(\mathbf{m}B_{\Lambda,\Lambda'})\psi\big\rangle\Big] \\ + 2\,\langle a(\mathbf{m}B_{\Lambda,\Lambda'})\psi, a(\mathbf{m}B_{L,\Lambda})\psi\rangle - 2\,\langle a(\mathbf{m}B_{L,\Lambda'})\psi, a(\mathbf{m}B_{\Lambda,\Lambda'})\psi\rangle\,.\end{aligned}$$

Applying the same bounds as before, this results in

$$\begin{aligned}
&\left|\mathfrak{Q}^{(P)}_{K,L,\Lambda}(\psi) - \mathfrak{Q}^{(P)}_{K,L,\Lambda'}(\psi)\right| \\
&\qquad \leq 4\|\omega^{-1/2}|\mathbf{m}|B_{\Lambda,\Lambda'}\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|\|(\mathsf{d}\Gamma(\mathbf{m}) - P + \varphi(\mathbf{m}B_{K,L}))\psi\| \\
&\qquad\qquad + 4\|\omega^{-1/2}|\mathbf{m}|B_{L,\infty}\|\|\omega^{-1/2}|\mathbf{m}|B_{\Lambda,\Lambda'}\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|^2 \\
&\qquad\qquad + 12\|\max(\omega^{-1/4},\omega^{-1/2})|\mathbf{m}|B_{L,\infty}\|\|\max(\omega^{-1/4},\omega^{-1/2})|\mathbf{m}|B_{\Lambda,\Lambda'}\| \\
&\qquad\qquad\qquad \times \|(1+\mathsf{d}\Gamma(\omega))^{1/2}\psi\|\|\mathsf{d}\Gamma(\omega)^{1/2}\psi\|.
\end{aligned}$$

Using (2.24) again proves the statement. □

The following statement now is a direct consequence of Theorem 2.10.

Corollary 2.12. *For all $L \geq K \geq \sigma$ and $P \in \mathbb{R}^d$, there exists a symmetric sesquilinear form $\mathfrak{Q}^{(P)}_{K,L,\infty}$ with form domain $\mathfrak{Q}_{\mathsf{N}}$ such that*

$$\mathfrak{Q}^{(P)}_{K,L,\infty}(\psi,\phi) = \lim_{\Lambda\to\infty} \mathfrak{Q}^{(P)}_{K,L,\Lambda}(\psi,\phi) \qquad \text{for all } \psi,\phi \in \mathfrak{Q}_{\mathsf{N}}.$$

Further, the bounds in Theorem 2.10 (i) and (ii) are also satisfied for $\Lambda = \infty$.

Since this implies that $\mathfrak{Q}^{(P)}_{K,L,\infty}$ is a small form perturbation of $\widetilde{H}_{K,L}(P)$ for L sufficiently large, we can now obtain the renormalized Nelson operators in the Gross regime by the KLMN theorem (Theorem A.90). In the next section, we prove that these operators are in fact the norm resolvent limit of the Gross transformed Nelson operators.

2.1.5. The Renormalized Operators in the Gross Regime

In the next theorem, we construct the norm resolvent limit of the Gross transformed fiber operators and collect similar regularity results to the ones stated in Theorem 2.1. Therein, we extend the definition (2.14) to the case $\Lambda = \infty$.

Theorem 2.13. *Fix some $K \geq \sigma$.*

(i) *The operators $\widetilde{H}_{K,\Lambda}(P)$ are bounded below uniformly in $\Lambda \geq K$ and $P \in \mathbb{R}^d$.*

(ii) *For all $P \in \mathbb{R}^d$, the operators $\widetilde{H}_{K,\Lambda}(P)$ converge to a selfadjoint lower-semibounded operator $\widetilde{H}_{K,\infty}(P)$ in the norm resolvent sense as $\Lambda \to \infty$. They have that form domain $\mathcal{Q}(\widetilde{H}_{K,\infty}(P)) = \mathfrak{Q}_{\mathsf{N}}$.*

(iii) *For $P \in \mathbb{R}^d$ and $\lambda < \inf\sigma(\widetilde{H}_{K,\Lambda}(P))$, we write $\widetilde{C}_{P,K,\Lambda}(\lambda) = B_P\widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2}$. Then, for all $P \in \mathbb{R}^d$ and $\lambda < \inf\sigma(\widetilde{H}_{K,\infty}(P))$,*

$$\begin{aligned}
&\lim_{\Lambda\to\infty} \widetilde{C}_{P,K,\Lambda}(\lambda)\widetilde{C}_{P,K,\Lambda}(\lambda)^* = \widetilde{C}_{P,K,\infty}(\lambda)\widetilde{C}_{P,K,\infty}(\lambda)^*, \\
&\operatorname*{s-lim}_{\Lambda\to\infty} \widetilde{C}_{P,K,\Lambda}(\lambda) = \widetilde{C}_{P,K,\infty}(\lambda).
\end{aligned}$$

Further, for λ sufficiently small, the map $P \mapsto \widetilde{C}_{P,K,\infty}(\lambda)\widetilde{C}_{P,K,\infty}(\lambda)^*$ is continuous.

Proof. The uniform lower bound in (i) directly follows from the uniform lower bound in Lemma 2.9 (ii), Theorem 2.10 (i) and the KLMN theorem (Theorem A.90).

Now, fix $P \in \mathbb{R}^d$ and choose L_P corresponding to $\varepsilon = \frac{1}{2}$ in Theorem 2.10 (i). Then, by Corollary 2.12, the symmetric form $\mathfrak{Q}^{(P)}_{K,L_P,\infty} = \lim_{\Lambda\to\infty} \mathfrak{Q}^{(P)}_{K,L_P,\Lambda}$ is $\mathfrak{q}_{\widetilde{H}_{K,L_P}(P)}$-form bounded with relative bound $\frac{1}{2}$. Let $\widetilde{H}_{K,\infty}(P)$ be the selfadjoint lower-semibounded operator corresponding to $\mathfrak{q}_{\widetilde{H}_{K,L_P}(P)} + \mathfrak{Q}^{(P)}_{K,L_P,\infty}$ by the KLMN theorem (Theorem A.90). We note that the domain statement in (ii) directly follows from the KLMN theorem.

Since Theorem 2.10 (i) also holds for $\Lambda = \infty$, by Corollary 2.12, we can choose the spectral parameter $\lambda < \inf_{\Lambda\in[0,\infty]} \inf\sigma(\widetilde{H}_{K,\Lambda}(P))$. Put $Z = \|H_0(P)^{1/2}\widetilde{R}_{P,K,\infty}(\lambda)^{1/2}\|$. For any $\varepsilon > 0$, we can choose δ_ε such that

$$Z^2\delta_\varepsilon < 1 \qquad \text{and} \qquad \|\widetilde{C}_{P,K,\infty}(\lambda)\|^2 \frac{Z^2\delta_\varepsilon}{1 - Z^2\delta_\varepsilon} < \varepsilon.$$

Further, by Theorem 2.10 (ii) and Corollary 2.12 and the observation $H_0(P) = \widetilde{H}_{K,K}(P)$, there exists Λ_ε such that the form $\mathfrak{v}_{K,\Lambda} := \mathfrak{q}_{\widetilde{H}_{K,\Lambda}(P)} - \mathfrak{q}_{\widetilde{H}_{K,\infty}(P)}$ satisfies

$$|\mathfrak{v}_{K,\Lambda}(\psi)| = |\mathfrak{Q}^{(P)}_{K,K,\Lambda}(\psi) - \mathfrak{Q}^{(P)}_{K,K,\infty}(\psi)| \le \delta_\varepsilon \mathfrak{q}_{H_0(P)} + c\delta_\varepsilon\|\psi\|^2 \qquad \text{for all } \psi \in \mathfrak{Q}_{\mathsf{N}} \text{ and } \Lambda \ge \Lambda_\varepsilon.$$

Hence, by Lemma A.91, there exists a bounded operator D_ε with $\|D_\varepsilon\| \le Z^2\delta_\varepsilon$ corresponding to the form $\mathfrak{v}_{K,\Lambda}(\widetilde{R}_{P,K,\infty}(\lambda)^{1/2}\cdot)$. Further,

$$\widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2} = \widetilde{R}_{P,K,\infty}(\lambda)^{1/2}(1 + D_\varepsilon)^{-1}\widetilde{R}_{P,K,\infty}(\lambda)^{1/2}$$

and hence by Lemma A.14

$$\begin{aligned}\|\widetilde{C}_{P,K,\Lambda}(\lambda)\widetilde{C}_{P,K,\Lambda}(\lambda)^* - \widetilde{C}_{P,K,\infty}(\lambda)\widetilde{C}_{P,K,\infty}(\lambda)^*\| &= \|\widetilde{C}_{P,K,\infty}(\lambda)((1 + D_\varepsilon)^{-1} - 1)\widetilde{C}_{P,K,\infty}(\lambda)^*\| \\ &\le \|\widetilde{C}_{P,K,\infty}(\lambda)\|^2 \frac{Z^2\delta_\varepsilon}{1 - Z^2\delta_\varepsilon} < \varepsilon.\end{aligned}$$

This proves the convergence statements in (iii) for λ sufficiently small, by Lemma A.78. Now, since

$$\widetilde{R}_{P,K,\Lambda}(\lambda) = B_P^{-1}\widetilde{C}_{P,K,\Lambda}(\lambda)\widetilde{C}_{P,K,\Lambda}(\lambda)^* B_P^{-1}$$

this implies (ii), by Lemma A.76 (i). The norm resolvent convergence in turn yields that for any $\lambda < \inf\sigma(\widetilde{H}_{K,\infty}(P))$ also $\lambda < \inf\sigma(\widetilde{H}_{K,\Lambda}(P))$ for Λ sufficiently large, so above argument still holds.

To conclude, we note that δ_ε and Λ_ε are independent of P. Hence, the convergence in (iii) is in fact uniform in P and the continuity statement follows, by Lemma 2.9. □

2.2. Construction of the Renormalized Operators

After having renormalized the Gross transformed operator, we move back to the usual operators $H_\Lambda(P)$. It is a direct consequence of the strong continuity of Weyl operators (Lemma B.26 (iii)) and Theorem 2.13 that, for any $P \in \mathbb{R}^d$, the operator $H_\Lambda(P)$ converges to

$$H_\infty(P) = W(B_{K,\infty})^*\widetilde{H}_{K,\infty}(P)W(B_{K,\infty}) \tag{2.25}$$

in the *strong* resolvent sense for an arbitrary choice of $K \geq \sigma$. This was the implication used in [Nel64, Can71]. However, using further properties of the Weyl operators, it is possible to improve the convergence statement to norm resolvent convergence and deduce further regularity properties of the operators $H_\infty(P)$.

2.2.1. Mapping Properties of Weyl Operators II

Before we can give proofs of Theorems 1.8 and 2.1, we need to further study mapping properties of Weyl operators.

In this section, we assume $\nu : \mathbb{R}^d \to [0,\infty)$ is measurable and $\nu > 0$ almost everywhere and $h : \mathbb{R}^d \to \mathbb{R}^p$ for some $p \in \mathbb{N}$ is measurable. For $\varepsilon > 0$ and $s \in [0,1]$, we define

$$T_\varepsilon = (1+\varepsilon \mathsf{d}\Gamma(\nu))^{1/2}, \quad S_s = 1+\mathsf{d}\Gamma(\nu)^{1/2}+|\mathsf{d}\Gamma(h)|^s \quad \text{and} \quad \mathscr{D}_s = \mathcal{D}(\mathsf{d}\Gamma(\nu)^{1/2})\cap\mathcal{D}(|\mathsf{d}\Gamma(h)|^s).$$

Note that $\mathcal{D}(T_\varepsilon) = \mathscr{D}_0$ and $\mathcal{D}(S_s) = \mathscr{D}_s$.

We prove the following statement.

Theorem 2.14. *Let $f \in \mathcal{D}(\nu^{1/2})$, $s \in [0,1]$ and $i \in \{1,\dots,p\}$.*

(i) *$W(f)\mathscr{D}_0 = \mathscr{D}_0$ and $\|(\mathsf{d}\Gamma(\nu)+1)^{1/2}W(f)(\mathsf{d}\Gamma(\nu)+1)^{-1/2}\| \leq 1 + \|\nu^{1/2}f\|$.*

(ii) *If $f \in \mathcal{D}(|h|^{1/2}) \cap \mathcal{D}(|h|\nu^{-1/2}) \setminus \mathcal{D}(h_i)$, then $\mathcal{D}(\mathsf{d}\Gamma(h_i)) \cap W(f)^*\mathscr{D}_1 = \{0\}$.*

(iii) *If $f \in \mathcal{D}(|h|^s) \cap \mathcal{D}(\nu^{-1/2}|h|^s)$, then $W(f)\mathscr{D}_s = \mathscr{D}_s$.*
Furthermore, if $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(|h|^s) \cap \mathcal{D}(\nu^{-1/2}|h|^s)$ converges to f simultaneously in $\nu^{1/2}$-, $|h|^s$- and $\nu^{-1/2}|h|^s$-norm, then

$$\operatorname*{s-lim}_{n\to\infty} S_s W(f_n) S_s^{-1} = S_s W(f_n) S_s^{-1}. \tag{2.26}$$

Remark 2.15. This theorem is an extension of [GW18, Lemmas C.3, C.4 & Cor. C.5] and [HM21, Lemma A.4].

We prove the theorem in several lemmas. To that end, for $f \in L^2(\mathbb{R}^d)$ satisfying $W(f)\mathscr{D}_0 \subset \mathscr{D}_0$ and $\varepsilon > 0$, we write

$$Q_{f,\varepsilon} = T_\varepsilon W(f) T_\varepsilon^{-1}.$$

Lemma 2.16. *Let $\varepsilon > 0$ and $(f_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ and assume $\lim_{n\to\infty} f_n = f$ in the L^2-sense. If $W(f_n)\mathscr{D}_0 \subset \mathscr{D}_0$ for all $n \in \mathbb{N}$ and both $Q^*_{f_n,\varepsilon}Q_{f_n,\varepsilon}$ and $Q^*_{-f_n,\varepsilon}Q_{-f_n,\varepsilon}$ converge strongly, then $W(f)\mathscr{D}_0 \subset \mathscr{D}_0$ and $\operatorname*{s-lim}_{n\to\infty} Q_{f_n,\varepsilon} = Q_{f,\varepsilon}$, $\operatorname*{s-lim}_{n\to\infty} Q^*_{f_n,\varepsilon} = Q^*_{f,\varepsilon}$.*

Proof. Since $\|Q^*_{\pm f_n,\varepsilon}Q_{\pm f_n,\varepsilon}\| = \|Q^*_{\pm f_n,\varepsilon}\|^2$ (cf. Lemma A.14), we see $Q^*_{\pm f_n,\varepsilon}$ is uniformly bounded. As $\operatorname*{s-lim}_{n\to\infty} T_\varepsilon^{-1}W(f_n) = T_\varepsilon^{-1}W(f)$ by Lemma B.26 (iii), Lemma A.18 implies $W(f)\mathscr{D}_0 \subset \mathscr{D}_0$ and $\operatorname*{s-lim}_{n\to\infty} Q^*_{\pm f_n,\varepsilon} = Q^*_{\pm f,\varepsilon}$. Taking adjoints, this yields $\operatorname*{w-lim}_{n\to\infty} Q_{f_n,\varepsilon} = Q_{f,\varepsilon}$. Now, the equality $Q_{f_n,\varepsilon} = Q^*_{-f_n,\varepsilon}(Q^*_{f_n,\varepsilon}Q_{f_n,\varepsilon})$ shows the convergence is actually strong. □

Lemma 2.17. *Let $f,g : \mathbb{R}^d \to \mathbb{C}$ satisfy $\nu^{-1/2}f, \nu^{-1/2}g \in L^2(\mathbb{R}^d)$ and assume $\varepsilon > 0$.*

(i) *There is $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\nu^{-1/2})$ such that $\lim_{n\to\infty} \nu^{-1/2}f_n = \nu^{-1/2}f$ in $L^2(\mathbb{R}^d)$.*

(ii) *If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence as in (i), then there is a bounded operator* $\widetilde{\varphi}_\varepsilon(f)$ *independent of the sequence such that* $\widetilde{\varphi}_\varepsilon(f) = \lim_{n\to\infty} T_\varepsilon^{-1}\varphi(f_n)T_\varepsilon^{-1}$.

(iii) $\|\widetilde{\varphi}_\varepsilon(f)\| \le 2\varepsilon^{-1/2}\|\nu^{-1/2}f\|$ *and* $\widetilde{\varphi}_\varepsilon(f) - \widetilde{\varphi}_\varepsilon(g) = \widetilde{\varphi}_\varepsilon(f-g)$.

Proof. A possible choice in (i) is

$$f_n(k) = f(k)\chi_{\{|f|<n, n^{-1}<\nu, |k|<n\}}(k).$$

Lemma B.20 (vii) yields $\|a(h)T_\varepsilon^{-1}\| \le \varepsilon^{-1/2}\|\nu^{-1/2}h\|$ for $h \in \mathcal{D}(\nu^{-1/2})$. Hence, using $\|T_\varepsilon^{-1}a^\dagger(h)\| = \|(a(h)T_\varepsilon^{-1})^*\|$ (Lemmas A.14 and B.20 (i)) and $T_\varepsilon \ge 1$, we find

$$\|T_\varepsilon^{-1}\varphi(h)T_\varepsilon^{-1}\| \le 2\varepsilon^{-1/2}\|\nu^{-1/2}h\|.$$

This inequality, closedness of the bounded operators and the fact that

$$T_\varepsilon^{-1}\varphi(h_1)T_\varepsilon^{-1} - T_\varepsilon^{-1}\varphi(h_2)T_\varepsilon^{-1} = T_\varepsilon^{-1}\varphi(h_1-h_2)T_\varepsilon^{-1} \qquad \text{for } h_1, h_2 \in \mathcal{D}(\nu^{-1/2})$$

then finish the proof. □

Lemma 2.18. *Let* $\varepsilon > 0$ *and* $f \in \mathcal{D}(\nu^{1/2})$. *Then* $W(f)\mathscr{D}_0 \subset \mathscr{D}_0$.
Further, if $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\nu^{1/2})$ *converges to* f *in* $\nu^{1/2}$*-norm, then* $\operatorname{s-lim}_{n\to\infty} Q_{f_n,\varepsilon} = Q_{f,\varepsilon}$, $\operatorname{s-lim}_{n\to\infty} Q^*_{f_n,\varepsilon} = Q^*_{f,\varepsilon}$ *and* $\|Q_f\| \le 1 + \|\nu^{1/2}f\|$.

Proof. We set $g_n = f\chi_{\{\nu<n\}} \in \mathcal{D}(\nu)$. We apply Lemma 2.3 with $p = 1$ and $h = \nu$. This yields $W(g_n)\mathcal{D}(\mathsf{d}\Gamma(\nu)) \subset \mathcal{D}(\mathsf{d}\Gamma(\nu))$, which implies $W(g_n)\mathscr{D}_0 \subset \mathscr{D}_0$ by Lemma A.87. Further, on $\mathscr{D}_0$,

$$Q^*_{g_n,\varepsilon}Q_{g_n,\varepsilon} = T_\varepsilon^{-1}W(g_n)^*T_\varepsilon^2W(g_n)T_\varepsilon^{-1} = 1 - T_\varepsilon^{-1}\varphi(\nu f_n)T_\varepsilon^{-1} + \|\nu^{1/2}g_n\|^2T_\varepsilon^{-2}.$$

By Lemma 2.17, the right hand side converges in norm as $n\to\infty$, so Lemma 2.16 shows $W(f)\mathscr{D}_0 \subset \mathscr{D}_0$ and, for any $f \in \mathcal{D}(\nu^{1/2})$,

$$Q^*_{f,\varepsilon}Q_{f,\varepsilon} = 1 - \widetilde{\varphi}_1(\nu f) + \|\nu^{1/2}f\|^2T_\varepsilon^{-2}. \tag{2.27}$$

Another application of Lemma 2.17 shows $Q^*_{f_n,\varepsilon}Q_{f_n,\varepsilon}$ is convergent if $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\nu^{1/2})$ converges to f in $\nu^{1/2}$-norm. Hence, Lemma 2.16 shows that $\operatorname{s-lim}_{n\to\infty} Q_{f_n,\varepsilon} = Q_{f,\varepsilon}$ and $\operatorname{s-lim}_{n\to\infty} Q^*_{f_n,\varepsilon} = Q^*_{f,\varepsilon}$. Lemma 2.17 and (2.27) now imply

$$\|Q_{f,1}\|^2 = \|Q^*_{f,1}Q_{f,1}\| \le 1 + 2\|\nu^{1/2}f\| + \|\nu^{1/2}f\|^2.$$ □

Lemma 2.19. *Let* $i \in \{1,..,p\}$ *and* $f \in \mathcal{D}(|h_i|^{1/2}) \cap \mathcal{D}(\nu^{1/2}) \cap \mathcal{D}(|h_i|\nu^{-1/2})$.
Set $f_\Lambda = f\chi_{\{|h|<\Lambda, \nu<\Lambda\}}$. *If* $\psi \in \mathscr{D}_0 \cap \mathcal{D}(\mathsf{d}\Gamma(h_i))$ *and* $W(f)\psi \in \mathcal{D}(\mathsf{d}\Gamma(h_i))$, *then*

$$\limsup_{\varepsilon\to 0}\limsup_{\Lambda\to\infty}\|T_\varepsilon^{-1}a^\dagger(h_if_\Lambda)\psi\| < \infty.$$

Proof. By Lemma 2.3, we have

$$T_\varepsilon^{-1}W(f_\Lambda)\mathsf{d}\Gamma(h_i)^{-1}W(f_\Lambda)^*\psi = \mathsf{d}\Gamma(h_i)T_\varepsilon^{-1}\psi - \widetilde{\varphi}(h_if_\Lambda)T_\varepsilon\psi + \langle f_\Lambda, h_if_\Lambda\rangle T_\varepsilon^{-1}\psi. \tag{2.28}$$

Setting $C = \|\mathsf{d}\Gamma(h_i)\psi\| + \||h_i|\nu^{1/2}f\|\|\mathsf{d}\Gamma(\nu)^{1/2}\psi\| + \||h_i|^{1/2}f\|^2\|\psi\|$, Lemma B.20 (vii) and (2.28) imply

$$\|T_\varepsilon^{-1}a^\dagger(h_if_\Lambda)\psi\| \leq \|T_\varepsilon^{-1}W(f_\Lambda)\mathsf{d}\Gamma(h_i)W(f_\Lambda)\psi\| + C.$$

Note that Lemma 2.17 implies the right hand side of (2.28) converges as $\Lambda \to \infty$. Hence,

$$T_\varepsilon^{-1}\mathsf{d}\Gamma(h_i)W(f_\Lambda)^*\psi = Q^*_{f_\Lambda,\varepsilon}T_\varepsilon^{-1}W(f_\Lambda)\mathsf{d}\Gamma(h_i)^{-1}W(f_\Lambda)^*\psi$$

also converges, by Lemma 2.18. As $T_\varepsilon^{-1}\mathsf{d}\Gamma(h_i)$ is closed, the limit is $T_\varepsilon^{-1}\mathsf{d}\Gamma(h_i)W(f)^*\psi$. Hence, we obtain

$$T_\varepsilon^{-1}W(f_\Lambda)\mathsf{d}\Gamma(h_i)^{-1}W(f_\Lambda)^*\psi = Q^*_{-f_\Lambda,\varepsilon}T_\varepsilon^{-1}\mathsf{d}\Gamma(h_i)W(f_\Lambda)^*\psi$$

converges to $T_\varepsilon^{-1}W(f)\mathsf{d}\Gamma(h_i)^{-1}W(f)^*\psi$ as $\Lambda \to \infty$. We obtain

$$\limsup_{\varepsilon\to 0}\limsup_{\Lambda\to\infty}\|T_\varepsilon^{-1}W(f_\Lambda)\mathsf{d}\Gamma(h_i)W(f_\Lambda)\psi\| = \|\mathsf{d}\Gamma(h_i)W(f_\Lambda)\psi\| < \infty. \qquad \square$$

Lemma 2.20. *Let $i \in \{1, .., p\}$ and $f \in \mathcal{D}(h_i\nu^{-1/2}) \setminus \mathcal{D}(h_i)$. Set $f_\Lambda = f\chi_{\{|h|<\Lambda,\nu<\Lambda\}}$. If $\psi \in \mathscr{D}_0$ and $\limsup_{\varepsilon\to 0}\limsup_{\Lambda\to\infty}\|T_\varepsilon^{-1}a^\dagger(h_if_\Lambda)\psi\| < \infty$, then $\psi = 0$.*

Proof. We use Definitions B.11 and B.18 and obtain

$$\begin{aligned}(a(h_if_\Lambda)T_\varepsilon^{-2}a^\dagger(h_if_\Lambda)\psi^{(n)})(k_1, ..., k_n) &= \int_{\mathbb{R}^d} \frac{|h_i(k)f_\Lambda(k)|^2\psi(k_1, ..., k_n)}{1 + \varepsilon\,(\nu(k) + \nu(k_1) + \cdots + \nu(k_n))}\mathsf{d}k \\ &+ \sum_{j=1}^n h_i(k_j)f(k_j)\int_{\mathbb{R}^d} \frac{\overline{h_i(k)f(k)}\psi^{(n)}(k, k_1, ..., \hat{k}_i, ..., k_n)}{1 + \varepsilon\,(\nu(k) + \nu(k_1) + \cdots + \nu(k_n))}\mathsf{d}k.\end{aligned}$$

The second term is bounded by $a^\dagger(|h_if_\Lambda|)a(|h_if_\Lambda|)|\psi^{(n)}|(k_1, \ldots, k_n)$, since $\nu \geq 0$. Hence, we obtain

$$\begin{aligned}&\|T_\varepsilon^{-1}a^\dagger(h_if_\Lambda)\psi^{(n)}\|^2 \\ &\quad\geq \int_{\mathbb{R}^d}|h_i(k)f_\Lambda(k)|^2\|(1 + \varepsilon(\nu(k) + \mathsf{d}\Gamma(\nu)))^{1/2}\psi^{(n)}\|^2\mathsf{d}k - \|a(|h_if_\Lambda|)|\psi^{(n)}|\|^2.\end{aligned}$$

By Lemma B.20 (vii), $\|a(|h_if_\Lambda|)|\psi^{(n)}|\|^2 \leq \|h_i\nu^{-1/2}f\|\|T_1\psi^{(n)}\|$, so summing over n and using monotone convergence in the limits $\Lambda \to \infty$ and $\varepsilon \to \infty$ we get

$$\infty > \limsup_{\Lambda\to\infty}\int_k |h_i(k)f_\Lambda(k)|^2\mathsf{d}k\|\psi\|^2 - \|h_i\nu^{-1/2}f\|^2\|T_1\psi\|^2.$$

Since h_if is not square-integrable, this implies $\|\psi\| = 0$. $\square$

Lemma 2.21. *Let $s \in [0, 1]$ and $f \in \mathcal{D}(\nu^{1/2}) \cap \mathcal{D}(|h|^s) \cap (\nu^{-1/2}|h|^s)$.*
Then there is a unique bounded operator $D_{f,s}$ such that (cf. Definition A.92)

$$\mathfrak{c}_{|\mathsf{d}\Gamma(h)|^s,\varphi(f)}(\psi, T_1^{-1}\psi) = \langle\psi, D_{f,s}\psi\rangle \quad \text{for } \psi \in \mathscr{D}_s.$$

Further, $\|D_{f,s}\| \leq 2\|(1 + \nu^{-1/2})|h|^sf\|$.
If f_n converges to f in $\nu^{-1/2}, \nu^{-1/2}|h|^s$ and $|h|^s$ norm then $\lim_{n\to\infty} D_{f_n,s} = D_{f,s}$.

Proof. Let $g \in \mathcal{D}(|\mathsf{d}\Gamma^{(n)}(h)|^s)$ (cf. Definition B.11). We use $||x+y|^s - |x|^s| \le |y|^s$ for all $x, y \in \mathbb{R}^\nu$ and the definition of the annihilation operator (cf. Definition B.18) and see the inequality

$$\left|a(f)|\mathsf{d}\Gamma^{(n)}(h)|^s g - |\mathsf{d}\Gamma^{(n-1)}(h)|^s a(f) g\right| \le a(|h|^s|f|)|g|$$

holds pointwise. Now let $\phi, \psi \in \mathscr{D}_s$ and define $|\psi| = \{|\psi^{(n)}|\}, |\phi| = \{|\phi^{(n)}|\} \in \mathscr{D}_s$. We now write $\mathfrak{q}_{f,s} = \mathfrak{c}_{|\mathsf{d}\Gamma(h)|^s, \varphi(f)}$ and use the above inequality to obtain

$$|\mathfrak{q}_{f,s}(\psi,\phi)| \le \langle |\psi|, a(|h|^s|f|)|\phi|\rangle + \langle a(|h|^s|f|)|\psi|, |\phi|\rangle = \langle |\psi|, \varphi(|h|^s|f|)|\phi|\rangle.$$

Since $\nu \ge 0$, we have $|T_1^{-1}\psi| = T_1^{-1}|\psi|$. Inserting $\phi = T_1^{-1}\psi$ and combining with Lemma B.20 (vii) and the Cauchy-Schwarz inequality, then yield

$$|\mathfrak{q}_{f,s}(\psi, T_1^{-1}\psi)| \le 2\|(1+\nu^{-1/2})|h|^s f\| \|\psi\|^2.$$

This proves existence and upper bound of $D_{f,s}$.

We observe $\mathfrak{q}_{f,s} - \mathfrak{q}_{f_n,s} = \mathfrak{q}_{f-f_n,s}$, which yields $D_{f,s} - D_{f_n,s} = D_{f-f_n,s}$ by the bound above. The convergence statement directly follows. □

Proof of Theorem 2.14. Recalling $W(f)^* = W(-f)$ from Lemma B.26, the statement of (i) follows from Lemma 2.18. Further, Lemmas 2.19 and 2.20 yield (ii).

Hence, it remains to prove (iii). Therefore, let $f_\Lambda(k) = \chi_{\{|k|\le\Lambda\}} f(k)$ and recall that, by Lemma 2.3, $W(tf_\Lambda)$ maps $\mathscr{D}_1$ onto itself for all $t \in \mathbb{R}$. Now, we assume $\psi, \phi \in \mathscr{D}_1$ and define

$$g_{\Lambda,\psi,\phi}(t) = \langle W(tf_\Lambda)\psi, |\mathsf{d}\Gamma(h)|^s W(tf_\Lambda)\phi\rangle \qquad \text{for } t \in \mathbb{R}. \tag{2.29}$$

For all $i \in \{1, \dots, p\}$, the map

$$t \mapsto \mathsf{d}\Gamma(h_i)W(tf_\Lambda)\psi = W(tf_\Lambda)(\mathsf{d}\Gamma(h_i)\psi - t\varphi(f_\Lambda)\psi + t^2\langle h_i f_\Lambda, f_\Lambda\rangle\psi)$$

is continuous by Lemmas B.20 (vii) and B.26 (iii), so $W(tf_\Lambda)\psi$ is continuous in $\mathsf{d}\Gamma(h_i)$-norm. Since

$$\||\mathsf{d}\Gamma(h)|^s\eta\| \le \|\eta\| + \sum_{i=1}^{n} \|\mathsf{d}\Gamma(h_i)\eta\| \qquad \text{for all } \eta \in \mathcal{D}(\mathsf{d}\Gamma(h))$$

by the spectral theorem (cf. Lemma A.61 (i)), $t \mapsto |\mathsf{d}\Gamma(h)|^s W(tf_\Lambda)\psi$ is continuous, so we can apply Lemma A.93 to (2.29). Hence, $g_{\Lambda,\psi,\phi}$ is continuously differentiable with derivative

$$g'_{\Lambda,\psi,\phi}(t) = -\mathsf{i}\mathfrak{c}_{|\mathsf{d}\Gamma(h)|^s,\varphi(f_\Lambda)}(W(tf_\Lambda)\psi, W(tf_\Lambda)\phi).$$

By Lemma 2.21, the form $\mathfrak{c}_{|\mathsf{d}\Gamma(h)|^s,\varphi(f_\Lambda)}(\psi, T_1^{-1}\psi)$ corresponds to an operator $D_\Lambda \in \mathcal{B}(\mathcal{F})$ bounded uniformly in Λ and satisfying $\lim\limits_{\Lambda\to\infty} D_\Lambda = D_\infty$. Therefore, we have

$$\begin{aligned}
\langle \psi, |\mathsf{d}\Gamma(h)|^s W(f_\Lambda)\phi\rangle &= g_{\Lambda, W(-f_\Lambda)\psi,\phi}(1) \\
&= g_{\Lambda, W(-f_\Lambda)\psi,\phi}(0) + \int_0^1 g'_{\Lambda, W(-f_\Lambda)\psi,\phi}(t)\mathsf{d}t \\
&= \langle \psi, W(f_\Lambda)|\mathsf{d}\Gamma(h)|^s\phi\rangle - \mathsf{i}\int_0^1 \langle \psi, W((1-t)f_\Lambda)D_\Lambda T_1 W(tf_\Lambda)\phi\rangle \, \mathsf{d}t.
\end{aligned}$$

Since $\psi \in \mathscr{D}_1$ was arbitrary and $\mathscr{D}_1$ is dense, this yields

$$|\mathsf{d}\Gamma(h)|^s W(f_\Lambda)\phi = W(f_\Lambda)|\mathsf{d}\Gamma(h)|^s\phi - \mathsf{i}\int_0^1 W((1-t)f_\Lambda)D_\Lambda T_1 W(tf_\Lambda)\phi\,\mathsf{d}t,$$

where we use the Bochner integral on the right hand side. By the dominated convergence theorem and Lemmas 2.18 and B.26 (iii), we can take the limit $\Lambda \to \infty$ and obtain $W(f)\phi \in \mathcal{D}(|\mathsf{d}\Gamma(h)|^s)$ as well as

$$|\mathsf{d}\Gamma(h)|^s W(f)\phi = W(f)|\mathsf{d}\Gamma(h)|^s\phi - \mathsf{i}\int_0^1 W((1-t)f)D_\infty T_1 W(tf)\phi\,\mathsf{d}t. \tag{2.30}$$

Again $W(-f) = W(f)^*$ directly yields $W(f)\mathscr{D}_s = \mathscr{D}_s$. Finally, we deduce (2.26) from Lemma 2.18 and a dominated convergence type argument applied to (2.30), as we did for finite Λ above. □

2.2.2. Regularity and Domain of the Renormalized Operators

After having provided all technical ingredients, we can now move to the renormalization of the fiber operators. To that end, we recall the definition of $H_\infty(P)$ in (2.25).

Proof of Theorem 2.1. First, we observe that the uniform lower bound directly follows from Theorem 2.13 (i) and (2.2). Further, we observe that for λ sufficiently small

$$\begin{aligned} R_{P,\Lambda}(\lambda) - R_{P,\infty}(\lambda) &= W(B_{K,\Lambda})^*(B_P\widetilde{R}_{P,K,\Lambda}(\lambda))^* B_P^{-1}(W(B_{K,\Lambda}) - W(B_{K,\infty})) \\ &\quad + W(B_{K,\Lambda})^*\left(\widetilde{R}_{P,K,\Lambda}(\lambda) - \widetilde{R}_{P,K,\infty}(\lambda)\right)W(B_{K,\infty}) \\ &\quad + (W(B_{K,\Lambda})^* - W(B_{K,\infty})^*)B_P^{-1}B\widetilde{R}_{P,K,\Lambda}(\lambda)W(B_{K,\infty}). \end{aligned}$$

This converges to zero in norm by Theorems 2.1 and 2.14, so we have proven (i)

Now, (ii) follows directly from (2.25).

To prove (iii), we first notice the domain statement follows directly from (ii), Lemma 2.5 and Theorem 2.14 (iii). For Λ large enough that $\lambda < \Sigma_\Lambda(\xi) + E_\Lambda$ (cf. Lemma A.76), we can calculate similar to above and obtain

$$A_s R_{P,\Lambda}(\lambda)^{1/2} = A_s W(B_{K,\Lambda})^* B_P^{-1} B_P \widetilde{R}_{P,K,\Lambda}(\lambda)^{1/2} W(B_{K,\Lambda}).$$

Convergence now follows from Theorems 2.13 (iii) and 2.14. To prove the continuity statements, we first observe $\|g_\lambda(P)g_\lambda(P)^*\| = \|g_\lambda(P)\|^2$ (Lemma A.14), so it is enough to see that $P \mapsto g_\lambda(P)g_\lambda(P)^*$ is continuous in norm. Writing $\widetilde{C}$ as in Theorem 2.13 (iii), we have

$$g_\lambda(P)g_\lambda(P)^* = A_s W(B_{K,\infty})^* B_P^{-1}\widetilde{C}_{P,K,\Lambda}(\lambda)C_{P,K,\Lambda}(\lambda)^* B_P^{-1} W(B_{K,\infty})A_s.$$

By Lemma 2.8, the map $P \mapsto A_s W(B_{K,\infty})^* B_P^{-1}$ is continuous, so the statement follows from Theorem 2.13 (iii). □

The next lemma shows that we can replace $A_{\frac{2}{3}}$ by $B_P^{1/2}$ in the convergence statements of Theorem 2.1 (iii). It especially provides a key ingredient in the proof of Theorem 1.8.

Lemma 2.22. *Let $P \in \mathbb{R}^d$. Then $B_P^{1/2}$ is infinitesimally $A_{\frac{2}{3}}$ bounded.*

Proof. Let $\varepsilon > 0$. Pick C such that

$$x^{1/4} \le \frac{1}{2}\varepsilon x^{1/2} + C \qquad \text{and} \qquad x^{1/2} \le \frac{1}{2^{5/4}}\varepsilon x^{2/3} + C \qquad \text{for all } x \ge 0.$$

Then the sub-additivity of $x \mapsto x^{1/4}$ and $|P - k_1 - ... - k_n|^2 \le 2|P|^2 + 2|k_1 + ... + k_n|^2$ lead to

$$\frac{(1 + \omega(k_1) + \cdots + \omega(k_n) + |P - k_1 - ... - k_n|^2)^{1/4}}{1 + (\omega(k_1) + \cdots + \omega(k_n))^{1/2} + |P - k_1 - ... - k_n|^{2/3}}$$
$$\le \varepsilon + (1 + 2^{1/4}|P|^{1/2} + 3C)(1 + (\omega(k_1) + \cdots + \omega(k_n))^{1/2} + |k_1 + ... + k_n|^{2/3})^{-1}.$$

As this holds uniformly for all $n \in \mathbb{N}$, we obtain $\mathcal{D}(A_{\frac{2}{3}}) \subset \mathcal{D}(B_P)$. To finish the proof we observe

$$\|B_P\psi\| \le \varepsilon\|A_{\frac{2}{3}}\psi\| + (1 + 2^{1/4}|P|^{1/2} + 3C)\|\psi\| \qquad \text{for all } \psi \in \mathcal{D}(A_{\frac{2}{3}}). \qquad \square$$

We conclude this chapter with the proof of Theorem 1.8. To that end, we recall the definition of $D_P(k)$ in (2.9).

Proof of Theorem 1.8 (i). Fix $P, k \in \mathbb{R}^d$. We note that by Theorem 2.1 (ii) and Lemmas 2.8 and 2.22, we already know $\mathfrak{q}_{D_P(k)}$ is infinitesimally $\mathfrak{q}_{H_\infty(P)}$ bounded. Hence, let H_k denote the selfadjoint operator corresponding to $\mathfrak{q}_{H_\infty(P)} + \mathfrak{q}_{D_P(k)}$ by the KLMN theorem (Theorem A.90). It only remains to prove that $H_k = H_\infty(P + k)$. To do this, it suffices to prove

$$(H_k - \lambda)^{-1} = R_{P+k,\infty}(\lambda) \qquad \text{for some (and hence all) } \lambda \in \sigma(H_\infty(P + k))^c.$$

Therefore, we pick λ_0 and Λ_0 such that $\lambda_0 < \min\{\inf \sigma(H_k), \Sigma_\Lambda(P) + E_\Lambda\}$ for $\Lambda > \Lambda_0$. By the uniform boundedness principle, Theorem 2.1 (iii) and Lemma 2.5, we can set

$$a := \sup_{\Lambda \in [\Lambda_0, \infty]} \|A_{\frac{2}{3}} R_{P,\Lambda}(\lambda_0)^{1/2}\|.$$

Since $\|R_{P,\Lambda}(\lambda_0)^{-1/2} R_{P,\Lambda}(\lambda)^{1/2}\| < 1$ by the spectral theorem, this yields

$$\|A_{\frac{2}{3}} R_{P,\Lambda}(\lambda)^{1/2}\| \le a \qquad \text{for all } \lambda < \lambda_0,\ \Lambda > \Lambda_0.$$

By Lemmas 2.8 and 2.22, we can choose $C > 0$ such that

$$\|B_P^{1/2}\psi\| \le \frac{1}{\||D_P(k)|^{1/2} B_P^{-1/2}\|}\left(\frac{1}{4a}\|A_{\frac{2}{3}}\psi\| + C\|\psi\|\right) \qquad \text{for all } \psi \in \mathcal{D}(A_{\frac{2}{3}}).$$

We now fix $\lambda < \lambda_0$ small enough that $C\|R_{P,\Lambda}(\lambda)^{1/2}\| \le \frac{1}{4}$. For $\psi \in \mathcal{F}$ this leads to

$$\||D_P(k)|^{1/2} R_{P,\Lambda}(\lambda)^{1/2}\psi\| \le \||D_P(k)|^{1/2} B_P^{-1/2}\| \|B_P^{1/2} R_{P,\Lambda}(\lambda)^{1/2}\|$$
$$\le \frac{1}{4a}\|A_{\frac{2}{3}} R_{P,\Lambda}(\lambda)^{1/2}\psi\| + \frac{1}{4}\|\psi\| \le \frac{1}{2}\|\psi\|.$$

The operator corresponding to the form $\mathfrak{q}_{D_P(k)}(R_{P,\Lambda}^{1/2}\cdot)$ is

$$\mathcal{Z}_{\Lambda,\lambda}(k) = (|D_P(k)|^{1/2} R_{P,\Lambda}(\lambda)^{1/2})^* \operatorname{sign}(D_P(k))|D_P(k)|^{1/2} R_{P,\Lambda}(\lambda)^{1/2},$$

which satisfies $\|\mathcal{Z}_{\Lambda,\lambda}(k)\| < \frac{1}{4}$ for all $\Lambda \geq \Lambda_0$ by above considerations.

Using (2.10), we now have

$$R_{P+k,\Lambda}(\lambda) = \sum_{n=0}^{\infty} R_{P,\Lambda}(\lambda)^{1/2} \mathcal{Z}_{\Lambda,\lambda}(k)^n R_{P,\Lambda}(\lambda)^{1/2} \qquad \text{for } \Lambda \in [\Lambda_0, \infty],$$

while by Lemma A.91

$$(H_k - \lambda)^{-1} = \sum_{n=0}^{\infty} R_{P,\infty}(\lambda)^{1/2} \mathcal{Z}_{\infty,\lambda}(k)^n R_{P,\infty}(\lambda)^{1/2}.$$

As $R_{P+k,\Lambda}(\lambda)$ converges strongly to $R_{P+k,\infty}(\lambda)$ and $\|\mathcal{Z}_{\Lambda,\lambda}(k)\| < \frac{1}{4}$ uniformly in Λ, it only remains to prove

$$\operatorname*{s-lim}_{\Lambda\to\infty} \mathcal{Z}_{\Lambda,\lambda}(k) = \mathcal{Z}_{\infty,\lambda}(k)$$

However, this directly follows from Theorem 2.1 (ii) and Lemma 2.22 and the proof is complete. □

Proof of Theorem 1.8 (ii). We set $\mathscr{D}_s = \mathcal{D}(A_s) = \mathcal{D}(\mathsf{d}\Gamma(\omega)^{1/2}) \cap \mathcal{D}(|\mathsf{d}\Gamma(m)|^s)$ and note that $\mathfrak{Q}_{\mathsf{N}} = \mathscr{D}_1 \subset \mathcal{D}(D_{P_1}(P_2 - P_1)) = \mathcal{D}(\mathsf{d}\Gamma(2(P_2 - P_1) \cdot m)$, by (2.9) and Lemmas 2.8 and B.17. Further, recall $\mathcal{Q}(H_\infty(P_1)) = \mathcal{Q}(H_\infty(P_2)) = W(B_{K,\infty})^* \mathscr{D}_1$ from Theorem 2.1.

First, assume $(P_2 - P_1) \cdot \mathbf{m} B_{K,\infty} \in L^2(\mathbb{R}^d)$. Then, by Theorem 2.14,

$$W(B_{K,\infty})^* \mathscr{D}_1 \subset \mathcal{D}(D_{P_1}(P_2 - P_1)).$$

Hence, Theorem 1.8 (i) yields

$$\mathfrak{q}_{H_\infty(P_2)}(\psi, \phi) = \mathfrak{q}_{H_\infty(P_1)}(\psi, \phi) + \langle D_{P_1}(P_2 - P_1)\psi, \phi\rangle \quad \text{for all } \psi, \phi \in W(B_{K,\infty})^* \mathscr{D}_1.$$

If we fix $\psi \in W(B_{K,\infty})^* \mathscr{D}_1$, the map $\phi \mapsto \mathfrak{q}_{H_\infty(P_1)}(\psi, \phi)$ is continuous if and only if the map $\phi \mapsto \mathfrak{q}_{H_\infty(P_2)}(\psi, \phi)$ is continuous. Hence, Lemma A.86 yields $\mathcal{D}(H_\infty(P_1)) = \mathcal{D}(H_\infty(P_2))$ and $H_\infty(P_2) = H_\infty(P_1) + D_{P_1}(P_2 - P_1)$.

We move to the case $(P_2 - P_1) \cdot \mathbf{m} B_{K,\infty} \notin L^2(\mathbb{R}^d)$. Assume $\psi \in \mathcal{D}(H_\infty(P_1)) \cap \mathcal{D}(H_\infty(P_2))$. Then it follows from Theorem 1.8 (i) and Lemma A.86 that the map

$$W(B_{K,\infty})^* \mathscr{D}_1 \ni \phi \mapsto \mathfrak{q}_{D_{P_1}(P_2 - P_1)}(\psi, \phi)$$

is continuous. If we can prove $W(B_{K,\infty})^* \mathscr{D}_1$ is a form core for $\mathsf{d}\Gamma((P_2 - P_1)\mathbf{m})$, we can deduce $\psi \in \mathcal{D}(D_{P_1}(P_2 - P_1)$, by Lemma A.86. From Theorem 2.14, we then find

$$\psi \in \mathcal{D}(\mathsf{d}\Gamma((P_2 - P_1)\mathbf{m})) \cap W(B_{K,\infty})^* \mathscr{D}_1 = \{0\},$$

which proves the statement.

As $A_{\frac{2}{3}}$ dominates $|D_{P_1}(P_2 - P_1)|^{1/2}$ by Lemma 2.22 and as $A_{\frac{2}{3}}$ strongly commutes with $|D_{P_1}(P_2 - P_1)|^{1/2}$, we see that any core for $A_{\frac{2}{3}}$ is a core for $|D_{P_1}(P_2 - P_1)|^{1/2}$ by Lemma A.70. Now, we note $A_{\frac{2}{3}}$ commutes with A_1 and is A_1-bounded, since $\mathscr{D}_1 \subset \mathscr{D}_{\frac{2}{3}}$, so $\mathscr{D}_1$ is a core for $A_{\frac{2}{3}}$ by Lemma A.70. Further, by Theorem 2.14 and Lemma 2.5, we know $W(B_{K,\infty})^*$ maps $\mathscr{D}_{\frac{2}{3}}$ continuously onto $\mathscr{D}_{\frac{2}{3}}$, so $W(B^*_{K,\infty})\mathscr{D}_1$ is a core for $A_{\frac{2}{3}}$ and hence for $|D_{P_1}(P_2 - P_1)|^{1/2}$. This finishes the proof. □

3. Absence of Ground States in the Nelson Model

In this chapter, we prove Theorem 1.7. Throughout, we will assume that either $\Lambda \in [0, \infty)$ and Hypothesis N0 or $\Lambda = \infty$ and Hypothesis NR hold, without further mention. In the proofs, we will usually distinguish these two cases, when necessary. Further, we will assume Hypothesis NA holds. To simplify notation, we drop the lower index N of H_{N} in this chapter.

Let us give a walkthrough of this chapter. We will start out by proving an energy inequality for the ground state energies at different total momentum. To that end, as in (2.15), we denote

$$\Sigma_\Lambda(P) = \inf \sigma(H_\Lambda(P)).$$

The energy inequality is the following lemma.

Lemma 3.1. *For all $P \in \mathbb{R}^d$ and $k \in \mathbb{R}^d$ with $k \nparallel P$*

$$\Sigma_\Lambda(P - k) + \omega(k) > \Sigma_\Lambda(P).$$

This allows us to define the bounded operator

$$Q_\Lambda(k, P) = \omega(k)(H_\Lambda(P + k) - \Sigma_\Lambda(P) + \omega(k))^{-1} \qquad \text{for } k \in \mathbb{R}^d \setminus \mathbb{R}P. \tag{3.1}$$

The proof of Theorem 1.7 is obtained in two propositions.

The first one is a so-called *pull-through formula*. Similar statements can, e.g., be found in [Frö73, BFS98a, Gér00]. Therefore, we denote by a_k the so-called pointwise annihilation operator $a_k : \mathcal{F}^{(n+1)} \to \mathcal{F}^{(n)}$ acting as

$$a_k f(k_1, \ldots, k_n) = \sqrt{n+1} f(k, k_1, \ldots, k_n). \tag{3.2}$$

By the Fubini-Tonelli theorem, this prescription is well-defined for almost every $k \in \mathbb{R}^d$. For $\psi \in \mathcal{F}$, we will write $a_k\psi = (a_k\psi^{(n)})_{n\in\mathbb{N}} \in \bigtimes_{n\in\mathbb{N}_0} \mathcal{F}^{(n)}$. For more details, we refer to Appendix B.6 and note that $a_k\psi = A\psi(k)$ in the notation used therein.

Proposition 3.2. *Let $P \in \mathbb{R}^d$ and assume ψ is a ground state for $H_\Lambda(P)$. Then $a_k\psi \in \mathcal{F}$ for almost every $k \in \mathbb{R}^d$ and*

$$a_k\psi = -v_\Lambda(k) Q_\Lambda(k, P)\psi \qquad \textit{for almost every } k \in \mathbb{R}^d \setminus \mathbb{R}P.$$

The second proposition describes the behavior of $Q_\Lambda(k, P)$ for small k. To that end, we introduce some more notation. First of all, as in (2.11), let B_P be defined as

$$B_P = (H_0(P) + 1)^{1/2}.$$

Further, we define the ground state projection

$$\mathscr{P}_\Lambda(P) = \mathsf{P}_{H_\Lambda(P)}(\{\Sigma_\Lambda(P)\}), \tag{3.3}$$

where $\mathsf{P}_{H_\Lambda(P)}$ denotes the spectral measure associated to $H_\Lambda(P)$ as defined in Definition A.55, cf. also Lemma A.56 (iii).

By Theorem 2.1 and Lemmas 2.5 and 2.22, $B_P\mathscr{P}_\Lambda(P)$ is bounded. Further, for any $i = 1, \dots, d$, the operator $(P_i - \mathsf{d}\Gamma(m_i))B_P^{-1}$ is bounded by Lemma 2.8 and selfadjoint (as they act on $\mathcal{F}^{(n)}$ as a real multiplication operator), so we can define

$$\mathbf{V}_\Lambda(P) = 2C_\omega(B_P^{1/2}\mathscr{P}_\Lambda(P))^*(P - \mathsf{d}\Gamma(\mathbf{m}))B_P^{-1}(B_P^{1/2}\mathscr{P}_\Lambda(P)) \tag{3.4}$$

as a vector of bounded and selfadjoint operators. Here, C_ω is the constant defined in Hypothesis NA (v).

For $k \in \mathbb{R}^d \setminus \{0\}$, we introduce the notation $\hat{k} = k/|k|$. Further, for $P \in \mathbb{R}^d$ and $\varepsilon \in (0,1)$, we define

$$S_\varepsilon(P) = \left\{k \in \mathbb{R}^d \setminus \{0\} : 2|\hat{k} \cdot P| < (1-\varepsilon)|P|\right\}. \tag{3.5}$$

Then, we need the following characterization of $Q_\Lambda(k,P)$ for small k.

Proposition 3.3. *Let $P \in \mathbb{R}^d$ and $\varepsilon \in (0,1)$. For $k \in S_\varepsilon(P)$, the operator $1 - \hat{k} \cdot \mathbf{V}_\Lambda(P)$ is invertible. Further,*

$$\operatorname*{w-lim}_{\substack{k\to 0\\ k\in S_\varepsilon(P)\cap S_\varepsilon(-P)}} \Big(Q_\Lambda(k,P) - \big(1 - \hat{k}\cdot\mathbf{V}_\Lambda(P)\big)^{-1}\mathscr{P}_\Lambda(P)\Big) = 0.$$

A similar proof for the absence of ground states was developed in [HH08] for the Pauli-Fierz model. Therein, the construction of $\mathbf{V}_\Lambda(P)$ used the (non-vanishing) gradient of the ground state energy. As differentiability is not available for arbitrary momentum in the Nelson model, Dam [Dam20] provided a different construction for the case $\Lambda < \infty$, where he used the rotation invariance and the non-degeneracy of the ground state, i.e., that $\dim \operatorname{ran} \mathscr{P}_\Lambda(P) = 1$. Our proof omits the latter assumption from the method and extends the result to the case $\Lambda = \infty$, by using the regularity results from the previous chapter.

This chapter is structured as follows. We start out by proving well-known statements, which lead to Lemma 3.1. Explicitly, these statements are that the ground state energy of the operators $H_\Lambda(P)$ attains a minimum at total momentum $P = 0$, which we prove in Section 3.1, and an Hunziker-van Winter-Zhislin (HVZ) theorem in Section 3.2. Combined with the rotation invariance from Hypothesis NA, we can then prove Lemma 3.1 and Proposition 3.3 in Section 3.3. In Section 3.4, we prove the pull-through formula Proposition 3.2. Combining these results, we can prove the absence of ground states in Section 3.5.

3.1. Energy Minimum at $P = 0$

The next lemma and its proof are essentially due to Gross [Gro72].

Lemma 3.4. *For all $P \in \mathbb{R}^d$, we have $\Sigma_\Lambda(P) \geq \Sigma_\Lambda(0)$.*

Proof. First, we note that it suffices to prove the statement for $\Lambda < \infty$, since the case $\Lambda = \infty$ then follows due to the norm resolvent convergence and Lemma A.76 (iv). Hence, we will now work under the assumption $\Lambda < \infty$.

By Lemma A.67, it suffices to prove

$$\|e^{-tH_\Lambda(P)}\| \leq \|e^{-tH_\Lambda(0)}\| \qquad \text{for some } t \in \mathbb{R}. \tag{3.6}$$

Let $L^2_{\mathbb{R}}(\mathbb{R}^d)$ be the set of real-valued L^2-functions and note $L^2(\mathbb{R}^d) = L^2_{\mathbb{R}}(\mathbb{R}^d) \oplus \mathsf{i}L^2_{\mathbb{R}}(\mathbb{R}^d)$. Let $\mathcal{Q}$ and Θ denote the probability space and isomorphism between $\mathcal{F}$ and $L^2(\mathcal{Q})$ corresponding to this decomposition, by Lemma B.30. Then, by Lemmas A.110 and B.32, the operator $\Theta e^{-t(\mathsf{d}\Gamma(\omega)+\varphi(v_\Lambda))}\Theta^*$ is positivitiy preserving on $L^2(\mathcal{Q})$ for all $t \geq 0$. Now, for $t \in \mathbb{R}^d$, let $f_t : \mathbb{R}^d \to [0,\infty)$ be chosen such that

$$e^{-tx^2} = \int_{\mathbb{R}^d} f_t(y) e^{-\mathsf{i}x\cdot y} \mathsf{d}y.$$

Then the spectral theorem combined with Fubini's theorem yield

$$e^{-t(P-\mathsf{d}\Gamma(\mathbf{m}))^2} = \int_{\mathbb{R}^d} f_t(y) e^{-\mathsf{i}P\cdot y} e^{\mathsf{i}y\cdot\mathsf{d}\Gamma(\mathbf{m})} \mathsf{d}y, \tag{3.7}$$

where we have a $\mathcal{B}(\mathcal{F})$-valued Bochner integral on the right hand side. Combining above observations with Lemma A.109 and the fact that $\Theta e^{\mathsf{i}y\cdot\mathsf{d}\Gamma(\mathbf{m})}\Theta^*$ is positivity preserving (cf. Lemma B.32), we hence obtain that, for any $t > 0$ and $\psi \in L^2(\mathcal{Q})$,

$$\begin{aligned}\left|\Theta e^{-t((P-\mathsf{d}\Gamma(\mathbf{m}))^2)} e^{-t(\mathsf{d}\Gamma(\omega)+\varphi(v_\Lambda))}\Theta^*\psi\right| &= \left|\int_{\mathbb{R}^d} f_t(y) e^{-\mathsf{i}P\cdot y}\Theta e^{\mathsf{i}y\cdot\mathsf{d}\Gamma(\mathbf{m})} e^{-t(\mathsf{d}\Gamma(\omega)+\varphi(v_\Lambda))}\Theta^*\psi \mathsf{d}y\right| \\ &\leq \Theta \int_{\mathbb{R}^d} f_t(y) e^{\mathsf{i}y\mathsf{d}\Gamma(\mathbf{m})} e^{-t(\mathsf{d}\Gamma(\omega)+\varphi(v_\Lambda))}\Theta^*|\psi|\mathsf{d}y \\ &= \Theta e^{-t|\mathsf{d}\Gamma(\mathbf{m})|^2} e^{-t(\mathsf{d}\Gamma(\omega)+\varphi(v_\Lambda))}\Theta^*|\psi|.\end{aligned}$$

We note that (3.7) especially implies $e^{-t|\mathsf{d}\Gamma(\mathbf{m})|^2}$ is positivity preserving. Hence, we can use induction and, for any $k \in \mathbb{N}$ and $t > 0$, obtain

$$\left|\Theta\left(e^{-\frac{t}{k}((P-\mathsf{d}\Gamma(\mathbf{m}))^2)} e^{-\frac{t}{k}(\mathsf{d}\Gamma(\omega)+\varphi(v_\Lambda))}\right)^k \Theta^*\psi\right| \leq \Theta\left(e^{-\frac{t}{k}((\mathsf{d}\Gamma(\mathbf{m}))^2)} e^{-\frac{t}{k}(\mathsf{d}\Gamma(\omega)+\varphi(v_\Lambda))}\right)^k \Theta^*|\psi|.$$

By the Trotter product formula (Theorem A.66), this yields

$$\left|\Theta e^{-tH_\Lambda(P)}\Theta^*\psi\right| \leq \Theta e^{-tH_\Lambda(0)}\Theta * |\psi| \qquad \text{for } t > 0.$$

Taking the L^2-norm and using that Θ is unitary, we have proved (3.6), which completes the proof. □

3.2. An HVZ Theorem

In this section, we prove an HVZ theorem for the Nelson model. Similar statements are proven, e.g., in [DG99, Amm00, Møl05, Dam20]

Proposition 3.5. *For all $P, k \in \mathbb{R}^d$, we have $\Sigma_\Lambda(P-k) + \omega(k) \in \sigma_{\mathsf{ess}}(H_\Lambda(P))$.*

Our proof follows the lines of [Dam20, Appendix A] and is split into several lemmas.

For a measurable set $A \subset \mathbb{R}^d$ and $\Lambda < \infty$, we define the operators

$$T^{(A)}_{\Lambda,0}(P) = (P - \mathsf{d}\Gamma(\mathbf{m}\restriction_{L^2(A)}))^2 + \mathsf{d}\Gamma(\omega\restriction_{L^2(A)}) + \varphi(v_\Lambda\chi_A) \quad \text{on } \mathcal{F}(L^2(A)) \text{ for } P \in \mathbb{R}^d. \tag{3.8}$$

Lemma 3.6. *For all measurable sets A, $\Lambda < \infty$ and $P \in \mathbb{R}^d$, the operator $T^{(A)}_{\Lambda,0}(P)$ is selfadjoint and bounded from below uniformly in P. Further, $P \mapsto T^{(A)}_{\Lambda,0}(P)$ is continuous in the norm resolvent sense.*

Proof. The first statement follows similar to Lemma 1.5, while the second is similar to Lemma 2.9. □

Now, by Lemma A.116, we can define

$$T^{(A)}_{\Lambda,n}(P) = \int^{\oplus}_{(A^c)^n} \Big(T^{(A)}_{\Lambda,0}(P - k_1 - \cdots - k_n) + \omega(k_1) + \cdots + \omega(k_n)\Big)\, \mathsf{d}(k_1, \ldots, k_n)$$

as selfadjoint lower-semibounded operators on $L^2((A^c)^n; \mathcal{F}(L^2(A)))$ for $n \in \mathbb{N}$, $P \in \mathbb{R}^d$.

The next lemma shows that the operators $T^{(A)}_{\Lambda,n}$ for $n \in \mathbb{N}_0$ describe the behavior of the Nelson Hamiltonian away from the interaction.

Lemma 3.7. *Assume that $A \subset \mathbb{R}^d$ is measurable, $\Lambda < \infty$ and $v_\Lambda = 0$ almost everywhere on A^c. Then there exists a unitary $U : \mathcal{F} \to \mathcal{F}(L^2(A)) \oplus \bigoplus_{n=1}^{\infty} L^2((A^c)^{\times n}; \mathcal{F}(L^2(A)))$ such that*

$$U H_\Lambda(P) U^* = \bigoplus_{n=0}^{\infty} T^{(A)}_{\Lambda,n}(P). \tag{3.9}$$

Proof. We explicitly define the unitary on coherent states (cf. Definition B.8) as

$$U\epsilon(f) = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (\chi_{A^c} f)^{\otimes_{\mathsf{s}} n} \epsilon(\chi_A f) \qquad \text{for } f \in L^2(\mathbb{R}^d),$$

where we use the natural identification $L^2((A^c)^{\times n}; \mathcal{F}(L^2(A))) \cong (L^2((A^c)))^{\otimes_{\mathsf{s}} n} \otimes \mathcal{F}(L^2(A)))$ (cf. Lemma A.103 and Remark B.3). By Definition B.8 and Lemmas B.9 and B.10, we can easily verify that this definition extends to a unitary U.

Now, let $f, g \in L^2(\mathbb{R}^d)$ be compactly supported. One easily verifies that $U\epsilon(f)$ is in the domain of the right hand side in (3.9) and, by a direct calculation using Lemmas B.15 (iv) and B.20 (iii) and that $\langle \chi_A f, v_\Lambda \chi_A g\rangle = \langle f, v_\Lambda g\rangle$ due to the assumptions, we obtain

$$\langle \epsilon(f), H_\Lambda(P)\epsilon(g)\rangle = \left\langle U\epsilon(f), \bigoplus_{n=0}^{\infty} T^{(A)}_{\Lambda,n}(P) U\epsilon(g) \right\rangle.$$

Since the compactly supported functions are dense in $L^2(\mathbb{R}^d)$, the set of coherent states generated from compactly supported functions, from now denoted as $\mathscr{E}$, is total (cf. Lemma B.10). Hence, if we can prove $\mathscr{E}$ spans a core for $H_\Lambda(P)$, the proof is complete by Lemma A.33, since the right hand side in (3.9) is selfadjoint by Lemma A.116. By the Kato-Rellich theorem (Theorem A.45), it suffices to prove $\mathscr{E}$ spans a core for $H_0(P)$.

To that end, let $f \in L^2(\mathbb{R}^d)$ be compactly supported and choose $R > 0$ such that $\operatorname{supp} f \subset B_R(0)$. From Definitions B.11 and B.18, for all $\ell \in \mathbb{N}_0$, we find

$$\|H_0(P)^\ell f^{\otimes_s n}\| \leq \left((|P| + nR)^2 + nC\right)^\ell \|f^{\otimes_s n}\| \leq \left((|P| + nR) + \sqrt{n}C\right)^{2\ell} \|f^{\otimes_s n}\|,$$

where $C = \sup\{\omega(k) : k \in B_R(0)\}$ exists by Hypothesis NA (ii). By Definition B.8, this implies $\epsilon(f) \in \mathcal{D}(H_0(P)^\ell)$ and

$$\sum_{\ell=0}^\infty \frac{\|H_0(P)^\ell \epsilon(f)\|}{(2\ell)!} \leq \sum_{\ell=0}^\infty \frac{1}{(2\ell)!} \sum_{n=0}^\infty \frac{\|H_0(P)^\ell f^{\otimes_s n}\|}{\sqrt{n!}} \leq \sum_{n=0}^\infty \frac{e^{(|P|+nR)+\sqrt{n}C}\|f\|^n}{\sqrt{n!}} < \infty.$$

Hence, $\epsilon(f)$ is semianalytic for $H_0(P)$ and the statement now follows by Theorem A.42. □

In the next lemma, we denote by $\mathcal{C}(A) \subset L^2(\mathbb{R}^d)$ the set of compactly supported functions with support inside of the measurable set A.

Lemma 3.8. *Fix $\Lambda < \infty$ and $P, k \in \mathbb{R}^d$. Then there exists a sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{R}^+$ with $\varepsilon_\ell \xrightarrow{\ell\to\infty} 0$ and a sequence of normed vectors $(\psi_\ell)_{\ell\in\mathbb{N}} \subset \mathscr{D}_{\mathsf{N}}$ with $\psi_\ell \in \mathcal{F}_{\mathsf{fin}}(\mathcal{C}(B_{\varepsilon_\ell}(k))^c)$ such that*

$$\|(H_\Lambda(P-k) - \Sigma_\Lambda(P-k))\psi_\ell\| \leq \frac{1}{\ell}, \tag{3.10}$$

$$\sup_{p\in B_{\varepsilon_\ell}(k)^c} \|(H_\Lambda(P-p) - H_\Lambda(P-k))\psi_\ell\| \leq \frac{1}{\ell}. \tag{3.11}$$

Proof. First, we observe that the set $\mathscr{B} := \bigcup_{\varepsilon\in\mathbb{R}^+} \mathcal{C}(B_\varepsilon(k)^c)$ is dense in $L^2(\mathbb{R}^d)$. This implies that the set $\mathcal{F}_{\mathsf{fin}}(\mathscr{B})$ is a core for $H_\Lambda(P-k)$, by an argument similar to the one presented in the end of the previous proof. Hence, for all $\ell \in \mathbb{N}$, there exists $\psi_\ell \in \mathcal{F}_{\mathsf{fin}}(\mathscr{B})$ such that (3.10) holds. By construction, it is further easy to check that there exists $\delta_\ell > 0$ such that $\psi_\ell \in \mathcal{F}_{\mathsf{fin}}(\mathcal{C}(B_{\delta_\ell}(k)^c))$. Now, for any $\phi \in \mathscr{D}_{\mathsf{N}} \subset \mathfrak{Q}_{\mathsf{N}} = \mathcal{D}(B_P)$, (2.10) and Lemma 2.8 yield

$$\|H_\Lambda(P+h)\phi - H_\Lambda(P)\phi\| \leq \|D_P(h)B_P^{-1}\|\|B_P\phi\| \leq 4|h|\|B_P\phi\| \qquad \text{for } |h| \leq 1,$$

so $h \mapsto H_\Lambda(P-k+h)\phi$ is continuous in a neighborhood of $h = 0$. This implies that there exists an $\varepsilon_\ell \in (0, \delta_\ell)$ such that (3.11) is satisfied and the proof is complete. □

Proof of Proposition 3.5. The proof goes in three steps.

Step 1. We prove the statement for $\Lambda < \infty$ and under the additional assumption that there exists an $\varepsilon_0 > 0$ such that $v_\Lambda = 0$ almost everywhere on $B_{\varepsilon_0}(k)$.

Let $(\psi_\ell)_{\ell\in\mathbb{N}}$ and $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ be sequences as in Lemma 3.8 and w.l.o.g. assume $\varepsilon_\ell < \varepsilon_0$ for all $\ell \in \mathbb{N}$. Further, let U_ℓ be the unitary from Lemma 3.7 corresponding to $A = B_{\varepsilon_\ell}(k)^c$ and let $P_\ell = \Gamma(\chi_{B_{\varepsilon_\ell}(k)^c})$ as operator from $\mathcal{F}$ to $\mathcal{F}(L^2(B_{\varepsilon_\ell}(k)^c))$. We now define

$$\phi_\ell = U_\ell^*\left(g_\ell^{\otimes_s \ell} P_\ell \psi_\ell\right) \qquad \text{with } g_\ell = \frac{\chi_{B_{\varepsilon_\ell}(k)\setminus B_{\varepsilon_{\ell+1}}(k)}}{\sqrt{\operatorname{vol}(B_{\varepsilon_\ell}(k) \setminus B_{\varepsilon_{\ell+1}}(k))}}.$$

It is easy to check that $\|\phi_\ell\| = 1$ for all $\ell \in \mathbb{N}$ and $\langle \phi_\ell, \phi_p \rangle = 0$ for $\ell \neq p$, by construction. Further, we can easily verify $\phi_\ell \in \mathcal{F}_{\mathsf{fin}}(\mathcal{C}(\mathbb{R}^d)) \subset \mathscr{D}_{\mathsf{N}}$, since $\psi_\ell \in \mathcal{F}_{\mathsf{fin}}(\mathcal{C}(\mathbb{R}^d))$, $g_\ell \in \mathcal{C}(\mathbb{R}^d)$.

Now, we use Lemma 3.7, (3.8) and $\|P_\ell\psi_\ell\| = 1$ to obtain

$$\begin{aligned} &\| \left(H_\Lambda(P) - \Sigma_\Lambda(P-k) - \omega(k)\right)\phi_\ell\|^2 \\ &= \int_{B_{\varepsilon_\ell}(k)} \left\| \left(T_{\Lambda,0}^{(B_{\varepsilon_\ell}(k))^c}(P-p) + \omega(p) - \Sigma_\Lambda(P-p) - \omega(k)\right) \mathscr{P}_\ell \psi_\ell \right\|^2 |g_\ell(p)|^2 \mathsf{d}p \\ &\le \int_{B_{\varepsilon_\ell}(k)} \| \left(H_\Lambda(P-p) - H_\Lambda(P-k)\right)\psi_\ell\|^2 |g_\ell(p)|^2 \mathsf{d}p \qquad (3.12) \\ &\quad + \int_{B_{\varepsilon_\ell}(k)} \| \left(H_\Lambda(P-k) - \Sigma_\Lambda(P-k)\right)\psi_\ell\|^2 |g_\ell(p)|^2 \mathsf{d}p \qquad (3.13) \\ &\quad + \int_{B_{\varepsilon_\ell}(k)} |\omega(p) - \omega(k)|^2 \, |g_\ell(p)|^2 \mathsf{d}p. \qquad (3.14) \end{aligned}$$

By the definition of g_ℓ and Lemma 3.8, we find (3.12) $\le 1/\ell$ and (3.13) $\le 1/\ell$. Further, (3.14) converges to zero as $\ell \to \infty$ by the continuity of ω (Hypothesis NA (ii)). This finishes the first step. ◊

Step 2. We now prove the statement for arbitrary v at fixed $\Lambda < \infty$.

To that end, let $v_\ell = \chi_{B_{1/\ell}(k)^c} v_\Lambda$, denote by $\hat{H}_\ell(P)$ the Nelson operator as defined in Definition 1.3 with v_Λ replaced by v_ℓ and write $\hat{\Sigma}_\ell(P) = \inf \sigma(\hat{H}_\ell(P))$. Using the resolvent identity (Lemma A.29) and the standard bound Lemma A.63, we have

$$\|(H_\Lambda(P) + \mathsf{i})^{-1} - (\hat{H}_\ell(P) + \mathsf{i})^{-1}\| \le \|\varphi(v_\Lambda - v_\ell)(H_\Lambda(P) + \mathsf{i})^{-1}\|.$$

Hence, by the bounds in Lemma B.20 (vii) and noting $\|(v_\Lambda - v_\ell)\omega^{-1/2}\| \xrightarrow{\ell\to\infty} 0$, the operator $\hat{H}_\ell(P)$ converges to $H_\Lambda(P)$ in the norm resolvent sense as $\ell \to \infty$. Now, by Lemma A.76 (iv), $\hat{\Sigma}_\ell(P)$ converges to $\Sigma_\Lambda(P)$ and hence the statement follows, by Lemma A.76 (vi). ◊

Step 3. It remains to treat the case $\Lambda = \infty$. However, due to the norm resolvent convergence of $H_\Lambda(P)$ to $H_\infty(P)$ (Lemma 1.6) the statement follows similar to the argument in Step 2. ◊ □

3.3. Rotation Invariance and Resolvent Bounds

In this section, we now utilize the rotation invariance assumption. We will work with the vector valued form

$$\mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\phi,\psi) = \left(\mathfrak{q}_{\mathsf{d}\Gamma(m_1)}(\phi,\psi), \ldots, \mathfrak{q}_{\mathsf{d}\Gamma(m_n)}(\phi,\psi)\right) \qquad (3.15)$$

defined for $\phi, \psi \in \mathcal{Q}(\mathsf{d}\Gamma(\mathbf{m})) = \mathcal{Q}(|\mathsf{d}\Gamma(\mathbf{m})|) = \bigcap_{i=1}^{d} \mathcal{Q}(\mathsf{d}\Gamma(m_i))$, by Lemmas A.87 and B.17. The following holds.

Lemma 3.9. *Let $k \in \mathbb{R}^d$ and $\psi, \phi \in \mathcal{Q}(B_0) \subset \mathcal{Q}(d\Gamma(k \cdot \mathbf{m}))$. Then, we have*

$$k \cdot \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\phi,\psi) = \mathfrak{q}_{\mathsf{d}\Gamma(k\cdot\mathbf{m})}(\phi,\psi).$$

Proof. This follows directly from

$$\mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\phi,\psi) = \sum_{n=1}^{\infty} \int_{\mathbb{R}^{nd}} (k_1 + ... + k_n)\overline{\phi^{(n)}(k_1,...,k_n)}\psi^{(n)}(k_1,...,k_n)\mathsf{d}(k_1,\dots,k_n). \qquad \square$$

The first statement is a simple implication of the rotation invariance assumption in Hypothesis NA (iii).

Lemma 3.10. *Let $O \in \mathbb{R}^{d\times d}$ be orthogonal and U_O be the associated rotation operator acting on $f \in \mathcal{H}$ as $U_O f(k) = f(Ok)$.*

(i) $\Gamma(U_O)\mathcal{Q}(B_0) = \mathcal{Q}(B_0)$ *and*

$$P \cdot \mathfrak{q}_{d\Gamma(m)}(\Gamma(U_O)\psi, \Gamma(U_O)\phi) = OP \cdot \mathfrak{q}_{d\Gamma(m)}(\psi,\phi) \qquad \text{for } \phi,\psi \in \mathcal{Q}(B_0),\ P \in \mathbb{R}^d.$$

(ii) $\Gamma(U_O)^* H_\Lambda(P)\Gamma(U_O) = H_\Lambda(OP)$ *holds for all* $\Lambda \in [0,\infty]$ *and* $P \in \mathbb{R}^d$. *Further,* $\Gamma(U_O)\mathcal{Q}(H_\Lambda(P)) = \mathcal{Q}(H_\Lambda(P))$.

Proof. Let $\Lambda < \infty$. By Hypothesis NA (iii) and Lemmas B.15 (x) and B.20, we find

$$\begin{aligned}
&\Gamma(U_O)^*\mathsf{d}\Gamma(\omega)\Gamma(U_O) = \mathsf{d}\Gamma(U_O^*\omega U_O) = \mathsf{d}\Gamma(\omega),\\
&\Gamma(U_O)^*\mathsf{d}\Gamma(P\cdot\mathbf{m})\Gamma(U_O) = \mathsf{d}\Gamma(U_O^*(P\cdot\mathbf{m})U_O) = \mathsf{d}\Gamma(OP\cdot\mathbf{m}),\\
&\Gamma(U_O)^*\varphi(v_\Lambda)\Gamma(U_O) = \varphi(U_O^* v_\Lambda) = \varphi(v_\Lambda).
\end{aligned}$$

Furthermore, we see

$$(U_O^{\otimes n})^*|k_1 + ... + k_n|^2 U_O^{\otimes n} = |Ok_1 + ... + Ok_n|^2 = |k_1 + ... + k_n|^2 \quad \text{for all } n \in \mathbb{N},$$

which proves $\Gamma(U_O)^*|\mathsf{d}\Gamma(\mathbf{m})|^2\Gamma(U_O) = |\mathsf{d}\Gamma(\mathbf{m})|^2$. Hence, by Definition 1.3, we have

$$\Gamma(U_O)^* H_\Lambda(P)\Gamma(U_O) = H_\Lambda(OP).$$

Taking limits yields the case $\Lambda = \infty$. This now implies

$$\Gamma(U_O)\mathcal{Q}(H_\Lambda(P)) = \mathcal{Q}(H_\Lambda(OP)) = \mathcal{Q}(H_\Lambda(P))$$

by Lemma 1.5 and Theorem 1.8 (i). Further, we note that $B_0^{1/2} = f(H_0(0))$ for a real function f (cf. (2.11)), so in particular $\Gamma(U_O)\mathcal{Q}(B_0) = \mathcal{Q}(B_0)$. Using Lemma 3.9, we now have

$$P \cdot \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\Gamma(U_O)\psi, \Gamma(U_O)\phi) = \mathfrak{q}_{\mathsf{d}\Gamma(P\cdot\mathbf{m})}(\Gamma(U_O)\psi, \Gamma(U_O)\phi) = OP \cdot \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi,\phi). \qquad \square$$

Although it is obvious from above lemma, we denote the rotation invariance of the ground state energy as a separate Corollary, as we will heavily use it in the proof of Lemma 3.1.

Corollary 3.11. *The function $P \mapsto \Sigma_\Lambda(P)$ is rotation invariant.*

The next two proofs are from [Dam20].

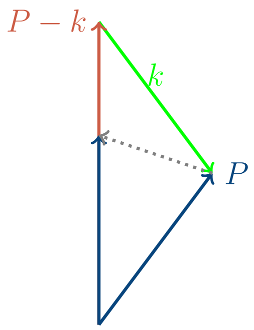

Figure 2: The role of rotation invariance in the proofs of Lemmas 3.1 and 3.12.

Proof of Lemma 3.1. We remark that this proof works for any choice of $\Lambda \in [0, \infty]$.

Observe that $\omega(k) > 0$ for all $k \neq 0$, by Hypotheses N0 (i) and NA (ii). Hence, using Lemma 3.4, we find

$$\Sigma_\Lambda(0-k) - \Sigma_\Lambda(0) \geq 0 > -\omega(k) \qquad \text{for all } k \neq 0,$$

which proves the case $P = 0$.

Now, we assume $P \geq 0$ and $k \in \mathbb{R}^d \setminus \mathbb{R}P$. The philosophy of our proof is sketched in Fig. 2. We rotate P onto $P-k$ and use the inverse triangle inequality as well as the strict monotonicity of ω.

By Corollary 3.11 and Proposition 3.5, we have

$$\begin{aligned} \Sigma_\Lambda(P-k) - \Sigma_\Lambda(P) &= \Sigma_\Lambda(P-k) - \Sigma_\Lambda\left(\frac{|P|}{|P-k|}(P-k)\right) \\ &\geq -\omega\left((|P-k| - |P|)\frac{P-k}{|P-k|}\right). \end{aligned} \tag{3.16}$$

Now, by the inverse triangle inequality $||P-k| - |P|| < |k|$ under our assumptions and hence using the strict monotonicity of ω (Hypothesis NA (ii)) and its rotation invariance (Hypothesis NA (iii)), we obtain

$$\Sigma_\Lambda(P-k) - \Sigma_\Lambda(P) > -\omega(k). \qquad \square$$

We further elaborate on the rotation-invariance arguments to obtain the following lemma.

Lemma 3.12. *For all $P \in \mathbb{R}^d$ and $\varepsilon \in (0,1)$ there exist constants $D = D(P,\varepsilon) < 1$ and $r = r(P,\varepsilon) > 0$ (independent of Λ) such that, for all $k \in B_r(0) \cap S_\varepsilon(P)$, we have*

$$\Sigma_\Lambda(P-k) - \Sigma_\Lambda(P) \geq -D\omega(k).$$

Proof. We observe

$$|P-k| - |P| = \frac{|P-k|^2 - |P|^2}{|P-k| + |P|} = \frac{|k| - 2\hat{k}\cdot P}{|P-k| + |P|}|k|.$$

Hence, for $\varepsilon \in (0,1)$ and $k \in S_\varepsilon(P)$, we find

$$\big||P-k| - |P|\big| \leq |k|\big(1 - \varepsilon + \tfrac{|k|}{|P|}\big).$$

Let $\delta \in (0, C_\omega^{-1})$ be arbitrary and choose r_δ such that (cf. Hypothesis NA (v))

$$\left| \frac{\omega(k)}{|k|} - C_\omega^{-1} \right| < \delta \qquad \text{for all } k \in B_{r_\delta}(0).$$

Let $e \in \mathbb{R}^d$ be an arbitrary vector of length 1. Using rotation invariance and monotonicity of ω again, we find for $|k| < \min\{r_\delta, |P|\varepsilon/2\}$

$$\omega((|P-k| - |P|)e) \leq \omega(|k|(1 - \varepsilon + \tfrac{|k|}{|P|})e) \leq (1 - \tfrac{\varepsilon}{2})(C_\omega^{-1} + \delta)|k| \leq \underbrace{(1 - \tfrac{\varepsilon}{2}) \frac{C_\omega^{-1} + \delta}{C_\omega^{-1} - \delta}}_{=: D_{\varepsilon,\delta}} \omega(k).$$

We observe that δ can be chosen sufficiently small for $D_{\varepsilon,\delta} < 1$. Combined with (3.16), this proves the statement. $\square$

We can now use these observations to further study $\mathbf{V}_\Lambda(P)$, as defined in (3.4).

Lemma 3.13. *Let $P \in \mathbb{R}^d$, $\varepsilon \in (0,1)$ and $k \in S_\varepsilon(P)$. Then, we have $\|\hat{k} \cdot \mathbf{V}_\Lambda(P)\| \leq \frac{1}{2}(1-\varepsilon)$ and hence the operator $1 - \hat{k} \cdot \mathbf{V}_\Lambda(P)$ is invertible.*

Proof. For $\psi \in \mathscr{P}_\Lambda(P)\mathcal{F}$, we define $v_\psi(P) = 2(P - \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi,\psi))$. As $\hat{k} \cdot \mathbf{V}_\Lambda(P)$ is selfadjoint, we have

$$\|\hat{k} \cdot \mathbf{V}_\Lambda(P)\| = \sup_{\substack{\psi \in \mathcal{F} \\ \|\psi\| = 1}} |\langle \psi, \hat{k} \cdot \mathbf{V}_\Lambda(P)\psi \rangle|.$$

Let $\psi \in \mathcal{F}$ and assume $\|\psi\| = 1$. Note that

$$\langle \psi, \hat{k} \cdot \mathbf{V}_\Lambda(P)\psi \rangle = \langle \mathscr{P}_\Lambda(P)\psi, \hat{k} \cdot \mathbf{V}_\Lambda(P)\mathscr{P}_\Lambda(P)\psi \rangle,$$

so we may assume $\mathscr{P}_\Lambda(P)\psi = \psi$. Now, setting $\mathbf{v}_\psi(P) = 2(P - \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi))$, we have

$$\langle \psi, \hat{k} \cdot \mathbf{V}_\Lambda(P)\psi \rangle = 2C_\omega \hat{k} \cdot (\|\psi\|^2 P - \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi,\psi)) = C_\omega \hat{k} \cdot \mathbf{v}_\psi(P).$$

Hence, it suffices to prove

$$|\hat{k} \cdot \mathbf{v}_\psi(P)| \leq \frac{1-\varepsilon}{2C_\omega}. \tag{3.17}$$

By (2.10) (or in the case $\Lambda = \infty$ Theorem 1.8 (i)) and Lemma 3.9, for all $h, \xi \in \mathbb{R}^d$ and $\phi \in \mathfrak{Q}_{\mathsf{N}}$, we have

$$\mathfrak{q}_{H_\Lambda(\xi+h)}(\phi) = \mathfrak{q}_{H_\Lambda(\xi)}(\phi) + 2h \cdot (\xi - \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\phi)) + |h|^2\|\phi\|^2. \tag{3.18}$$

In the case $\xi = P$, $\phi = \psi$ and using $\mathfrak{q}_{H_\Lambda(P+h)}(\psi) \geq \Sigma_\Lambda(P+h)$ as well as $\mathfrak{q}_{H_\Lambda(P)}(\psi) = \Sigma_\Lambda(P)$, this yields

$$\Sigma_\Lambda(P+h) - \Sigma_\Lambda(P) \leq h \cdot \mathbf{v}_\psi(P) + |h|^2 \qquad \text{for all } h \in \mathbb{R}^d. \tag{3.19}$$

If $P = 0$, the left hand side is non-negative by Lemma 3.4, so taking the limit $|h| \to 0$ we obtain $\hat{h} \cdot \mathbf{v}_\psi(0) \geq 0$ for all $h \in \mathbb{R}^d$. This directly implies $\mathbf{v}_\psi(0) = 0$ and hence (3.17).

From now, we can assume $P \neq 0$. By Proposition 3.5, we know that

$$\Sigma_\Lambda(P+h) - \Sigma_\Lambda(P) \geq -\omega(h).$$

Inserted into (3.19), this leads to

$$h \cdot \mathbf{v}_\psi(P) \geq -\omega(h) - |h|^2 \qquad \text{for all } h \in \mathbb{R}^d.$$

For $h \in \mathbb{R}^d \setminus \{0\}$, we divide by $|h|$ and take $|h| \to 0$ to obtain $\hat{h} \cdot \mathbf{v}_\psi(P) \geq -C_\omega^{-1}$ and hence $|\mathbf{v}_\psi(P)| \leq C_\omega^{-1}$. Let $O \in \mathbb{R}^{d\times d}$ be orthogonal. We insert $\xi = 0$, $h = P$ and $\phi = \Gamma(U_O)\psi$ as well as $\xi = P$, $h = -P$ and $\phi = \psi$ into (3.18). Using Lemma 3.10, this yields

$$\mathfrak{q}_{H_\Lambda(OP)}(\psi) + 2OP \cdot \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi) - |P|^2 = \mathfrak{q}_{H_\Lambda(0)} = \Sigma_\Lambda(P) + 2P \cdot \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi) - |P|^2.$$

Since $\mathfrak{q}_{H_\Lambda(OP)}(\psi) \geq \Sigma_\Lambda(OP) = \Sigma_\Lambda(P)$, by Corollary 3.11, we obtain

$$P \cdot \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi) \geq OP \cdot \mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi) \qquad \text{for all orthogonal } O \in \mathbb{R}^{d\times d}.$$

Hence, there is a constant $R_\psi \in \mathbb{R}$ such that $\mathfrak{q}_{\mathsf{d}\Gamma(\mathbf{m})}(\psi) = R_\psi P$. For all $k \in S_\varepsilon(P)$, this implies

$$|\hat{k} \cdot \mathbf{v}_\psi(P)| = |(1 - R_\psi)\hat{k} \cdot P| \leq \frac{1-\varepsilon}{2}|\hat{k}||(1-R_\psi)P| = \frac{1-\varepsilon}{2}|\mathbf{v}_\psi(P)| \leq \frac{1-\varepsilon}{2C_\omega}. \qquad \square$$

We will also need that $(1 - \hat{k} \cdot \mathbf{V}_\Lambda(P))^{-1}$ weakly converges to 0.

Lemma 3.14. *Let $P \in \mathbb{R}^d$, $R > 0$ and $\varepsilon \in (0,1)$.*
Further, let $\{o(k) \mid k \in B_R(0) \cap S_\varepsilon(P)\} \subset \mathcal{B}(\mathcal{F})$ satisfy $\operatorname*{w-lim}_{k\to 0} o(k) = 0$. Then

$$\operatorname*{w-lim}_{k\to 0} (1 - \hat{k} \cdot \mathbf{V}_\Lambda(P))^{-1} o(k) = 0.$$

Proof. Let $\phi, \psi \in \mathcal{F}$. By Lemma 3.13, we know $\|\hat{k} \cdot \mathbf{V}_\Lambda(P)\| < \frac{1}{2}(1-\varepsilon)$ so

$$\langle \psi, (1 - \hat{k} \cdot \mathbf{V}_\Lambda(P))^{-1} o(k)\phi\rangle = \sum_{n=0}^{\infty} \langle (\hat{k} \cdot \mathbf{V}_\Lambda(P))^n \psi, o(k)\phi\rangle.$$

By dominated convergence, it is enough to see each term in the sum converges to 0 as $k \to 0$. This follows from $|\hat{k}| = 1$ and

$$\langle (\hat{k} \cdot \mathbf{V}_\Lambda(P))^n \psi, o(k)\phi\rangle = \sum_{i_1=1}^{d} \dots \sum_{i_n=1}^{d} \hat{k}_{i_1} \dots \hat{k}_{i_n} \langle (\mathbf{V}_\Lambda(P))_{i_1} \dots (\mathbf{V}_\Lambda(P))_{i_n} \psi, o(k)\phi\rangle. \qquad \square$$

Apart from the operator $Q_\Lambda(k,P)$ as defined in (3.1), we also introduce the bounded operator

$$Q_\Lambda^{(0)}(k,P) = \omega(k)(H_\Lambda(P) - \Sigma_\Lambda(P) + \omega(k))^{-1} \qquad \text{for } k \in \mathbb{R}^d \setminus \{0\}. \tag{3.20}$$

The next lemmas collect some simple statements about these operators. Therein, we will use

$$\mathcal{D}(H_\Lambda(P_1)) \subset \mathcal{Q}(H_\Lambda(P_1)) \subset \mathcal{D}(B_{P_2}^{1/2}) \subset \mathcal{Q}(\mathsf{d}\Gamma(P_3 \cdot \mathbf{m})) \quad \text{for } P_1, P_2, P_3 \in \mathbb{R}^{\mathsf{d}}, \tag{3.21}$$

which (especially in the case $\Lambda = \infty$) follows from Theorem 1.8 (i) and Lemmas 2.8 and 2.22. We first consider $Q_\Lambda^{(0)}$.

Lemma 3.15. *Let $P \in \mathbb{R}^d$ and $R > 0$. Then the operator $B_P^{1/2} Q_\Lambda^{(0)}(k,P)$ is bounded uniformly in $k \in B_R(0) \setminus \{0\}$ and*

$$\operatorname*{s-lim}_{k\to 0} B_P^{1/2} Q_\Lambda^{(0)}(k,P)(1-\mathscr{P}_0(P)) = 0.$$

Proof. The statement follows from (3.21) and Lemma A.62. □

We move to investigating Q_Λ.

Lemma 3.16. *Let $\varepsilon \in (0,1)$ and $r > 0$ as in Lemma 3.12. Then the operators $Q_\Lambda(k,P)$ and $B_P^{1/2} Q_\Lambda(k,P)$ are bounded uniformly for all $k \in B_r(0) \cap S_\varepsilon(P)$.*

Proof. For all $k \in B_r(0) \cap S_\varepsilon(P)$, Lemma 3.12 yields $\|Q_\Lambda(k,P)\| \le (1-D)^{-1}$ for some $D \in (0,1)$, which proves the first uniform upper bound.

Now, note $B_P^{1/2} Q_\Lambda(k,P)$ is again bounded by (3.21). By Theorem 2.1 and Lemma 2.22, we can pick λ small enough such that $k \mapsto \|B_P(H_\Lambda(P+k)-\lambda)^{-1}\|$ is continuous and hence uniformly bounded by some constant C on $B_r(0)$. This leads to

$$\|B_P^{1/2} Q_\Lambda(k,P)\| \le C\|(H_\Lambda(P-k)-\lambda)Q_\Lambda(k,P)\|.$$

The uniform bound on $B_P^{1/2} Q_\Lambda(k,P)$ now follows from the one on $Q_\Lambda(k,P)$, since

$$(H_\Lambda(P-k)-\lambda)Q_\Lambda(k,P) = \omega(k) + (\lambda + \Sigma_\Lambda(P) - \omega(k))Q_\Lambda(k,P). \qquad \square$$

We now give an explicit connection between Q_Λ and $Q_\Lambda^{(0)}$. To that end, recall the definition (2.9) of $D_P(k)$.

Lemma 3.17. *For $P \in \mathbb{R}^d$ and $k \in \mathbb{R}^d \setminus \mathbb{R}P$, we have*

$$Q_\Lambda(k,P) = Q_\Lambda^{(0)}(k,P) + \frac{1}{\omega(k)}(B_P^{1/2} Q_\Lambda^{(0)}(k,P))^* D_P(k) B_P^{-1}(B_P^{1/2} Q_\Lambda(k,P)).$$

Proof. The statement for $\Lambda < \infty$ follows from the resolvent identity (Lemma A.29), (2.10) and the fact that B_P and $D_P(k)$ commute strongly. It remains to show we can take strong limits on both sides. To that end, for $a \in \{0,1\}$, it suffices to prove

$$\operatorname*{s-lim}_{\Lambda\to\infty} B_P^{a/2} Q_\Lambda^{(0)}(k,P) = B_P^{a/2} Q_\Lambda^{(0)}(k,P) \text{ and } \operatorname*{s-lim}_{\Lambda\to\infty} B_P^{a/2} Q_\Lambda(k,P) = B_P^{a/2} Q_\Lambda(k,P) \quad (3.22)$$

By Lemma A.76 and Theorem 2.1, we have $\lim_{\Lambda\to\infty} \Sigma_\Lambda(P) + E_\Lambda = \Sigma_\infty(P)$, so using

$$\begin{aligned}\exp(-t(H_\Lambda(P+h) - \Sigma_\Lambda(P))) &= \exp(-t(\Sigma_\Lambda(P)+E_\Lambda))\exp(-t(H_\Lambda(P+h)+E_\Lambda)) \\ &\xrightarrow{\Lambda\to\infty} \exp(-t(H_\infty(P+h) - \Sigma_\infty(P)))\end{aligned}$$

so the case $a = 0$ in (3.22) follows due to Lemma A.76.

Further, for $(Z_\Lambda, h) \in \{(Q_\Lambda^{(0)}(k,P), P), (Q_\Lambda(k,P), P+k)\}$ and $\lambda < \Sigma_\Lambda(P) + E_\Lambda$, the resolvent identity yields

$$B_P^{1/2} Z_\Lambda = B_P^{1/2}(H_\Lambda(h) + E_\Lambda - \lambda)^{-1}\left(1 - \frac{1}{\omega(k)}(\Sigma_\Lambda(P) + E_\Lambda - \lambda)\right) Z_\Lambda.$$

Hence, the case $a = 1$ of (3.22) follows by Theorem 2.1 and Lemma 2.22. □

In Proposition 3.3, we are interested in the weak limit of Q_Λ. An important ingredient in the proof is the following.

Lemma 3.18. *Let $P \in \mathbb{R}^d$ and $\varepsilon \in (0,1)$. Then*

$$\operatorname*{w-lim}_{\substack{k\to 0\\ k\in S_\varepsilon(P)}} Q_\Lambda(k,P)(1-\mathscr{P}_\Lambda(P)) = \operatorname*{w-lim}_{\substack{k\to 0\\ k\in S_\varepsilon(P)}} (1-\mathscr{P}_\Lambda(P))Q_\Lambda(k,P) = 0.$$

Proof. By taking adjoints, it suffices to prove one of the statements. Taking the adjoint in Lemma 3.17, we notice

$$\begin{aligned} &Q_\Lambda(k,P)(1-\mathscr{P}_\Lambda(P)) = (Q_\Lambda(k,P))^*(1-\mathscr{P}_\Lambda(P)) \\ &= Q_\Lambda^{(0)}(k,P)(1-\mathscr{P}_\Lambda(P)) + \frac{|k|}{\omega(k)}(B_P^{1/2}Q_\Lambda(k,P))^*\frac{D_P(k)B_P^{-1}}{|k|}B_P^{1/2}Q_\Lambda^{(0)}(k,P)(1-\mathscr{P}_\Lambda(P)). \end{aligned}$$

This goes to 0 strongly as $k\to 0$, by Lemmas 2.8, 3.15 and 3.16 and Hypothesis NA (v). □

We can now give the

Proof of Proposition 3.3. Throughout this proof, we assume $k \in S_\varepsilon(P)\cap S_\varepsilon(-P)$. First, we note

$$\operatorname*{w-lim}_{k\to 0}\big(Q_\Lambda(k,P) - \mathscr{P}_\Lambda(P)Q_\Lambda(k,P)\mathscr{P}_\Lambda(P)\big) = 0,$$

by Lemma 3.18. Using that

$$Q_\Lambda^{(0)}(k,P)\mathscr{P}_\Lambda(P) = \mathscr{P}_\Lambda(P) \quad \text{and} \quad \mathscr{P}_\Lambda(P)(B_P^{1/2}Q_\Lambda^{(0)}(k,P))^* = (B_P^{1/2}\mathscr{P}_\Lambda(P))^*$$

as well as Lemma 3.17, we get

$$\begin{aligned} \mathscr{P}_\Lambda(P)Q_\Lambda(k,P)\mathscr{P}_\Lambda(P) &= \mathscr{P}_\Lambda(P) + \frac{1}{\omega(k)}(B_P^{\frac12}\mathscr{P}_\Lambda(P))^*D_P(k)B_P^{-\frac12}Q_\Lambda(k,P)\mathscr{P}_\Lambda(P) \\ &= \mathscr{P}_\Lambda(P) + \frac{1}{\omega(k)}(B_P^{\frac12}\mathscr{P}_\Lambda(P))^*D_P(k)B_P^{-\frac12}\mathscr{P}_\Lambda(P)Q_\Lambda(k,P)\mathscr{P}_\Lambda(P) + o_1(k), \\ &= \mathscr{P}_\Lambda(P) + \hat{k}\cdot\mathbf{V}_\Lambda(P)\mathscr{P}_\Lambda(P)Q_\Lambda(k,P)\mathscr{P}_\Lambda(P) + o_1(k) + o_2(k), \quad \text{where} \\ o_1(k) &:= \frac{1}{\omega(k)}(B_P^{\frac12}\mathscr{P}_\Lambda(P))^*D_P(k)B_P^{-\frac12}(1-\mathscr{P}_\Lambda(P))Q_\Lambda(k,P)\mathscr{P}_\Lambda(P), \\ o_2(k) &:= \Big(\frac{1}{\omega(k)}(B_P^{\frac12}\mathscr{P}_\Lambda(P))^*D_P(k)B_P^{-\frac12}\mathscr{P}_\Lambda(P) - \hat{k}\cdot\mathbf{V}_\Lambda(P)\Big)\mathscr{P}_\Lambda(P)Q_\Lambda(k,P)\mathscr{P}_\Lambda(P). \end{aligned}$$

This leads to

$$\mathscr{P}_\Lambda(P)Q_\Lambda(k,P)\mathscr{P}_\Lambda(P) - (1-\hat{k}\cdot\mathbf{V}_\Lambda(P))^{-1}\mathscr{P}_\Lambda(P) = (1-\hat{k}\cdot\mathbf{V}_\Lambda(P))^{-1}(o_1(k)+o_2(k)).$$

By Lemma 3.14, it suffices to prove $\operatorname*{w-lim}_{k\to 0} o_1(k) = \operatorname*{w-lim}_{k\to 0} o_2(k) = 0$. Let $\phi,\psi\in\mathcal{F}$. By the definition (2.9), we have

$$(B_P^{\frac12}\mathscr{P}_\Lambda(P))^*D_P(k)B_P^{-\frac12}(1-\mathscr{P}_\Lambda(P))\psi = \sum_{i=1}^{\nu} k_i(B_P^{\frac12}\mathscr{P}_\Lambda(P))^*d\Gamma(m_i)B_P^{-\frac12}(1-\mathscr{P}_\Lambda(P))\psi$$

and hence

$$\langle \phi, o_1(k)\psi\rangle = \frac{|k|}{\omega(k)} \sum_{i=1}^{\nu} \hat{k}_i \langle B_P^{\frac{1}{2}} \mathscr{P}_\Lambda(P)\phi, d\Gamma(m_i) B_P^{-1} B_P^{\frac{1}{2}} (1 - \mathscr{P}_\Lambda(P)) Q_\Lambda(k,P) \mathscr{P}_\Lambda(P)\psi\rangle.$$

Using Lemmas 3.16, 3.18 and A.18, we see $\langle \phi, o_1(k)\psi\rangle \xrightarrow{k\to 0} 0$. We note that the uniform boundedness of $B_P^{1/2}(1 - \mathscr{P}_\Lambda(P))Q_\Lambda(k,P)$ necessary to apply Lemma A.18 follows from Lemmas 2.5, 2.22 and 3.16 and Theorem 2.1

Further, definition (3.4) yields

$$\hat{k} \cdot \mathbf{V}_\Lambda(P) = C_\omega |k|^{-1} (B_P^{\frac{1}{2}} \mathscr{P}_\Lambda(P))^* D_P(k) B_P^{-\frac{1}{2}} \mathscr{P}_\Lambda(P) - |k| \mathscr{P}_\Lambda(P),$$

so we have

$$o_2(k) = \left(\frac{|k|}{\omega(k)} - C_\omega\right) (B_P^{\frac{1}{2}} \mathscr{P}_\Lambda(P))^* \frac{D_P(k) B_P^{-1}}{|k|} B_P^{\frac{1}{2}} \mathscr{P}_\Lambda(P) Q_\Lambda(k,P) \mathscr{P}_\Lambda(P) - \frac{|k|^2}{\omega(k)} \mathscr{P}_\Lambda(P).$$

This converges to 0 in norm, due to Lemma 3.16 and Hypothesis NA (v). □

3.4. Pull-Through Formula

One ingredient described in the introduction of this chapter is still missing.

For $\Lambda < \infty$, we in fact prove a stronger pull-through formula than Proposition 3.2.

Lemma 3.19. *Let $\Lambda < \infty$, let $P \in \mathbb{R}^d$ and let $\psi \in \mathscr{D}_{\mathsf{N}}$. Then, for almost every $k \in \mathbb{R}^d$, both $a_k\psi$ and $a_k(H_\Lambda(P) - \Sigma_\Lambda(P))\psi$ are $\mathcal{F}$-valued and*

$$a_k\psi = Q_\Lambda(k,P) a_k (H_\Lambda(P) - \Sigma_\Lambda(P))\psi - v_\Lambda(k) Q_\Lambda(k,P)\psi \quad \text{for almost every } k \in \mathbb{R}^d \setminus \mathbb{R}P.$$

Proof. In this proof, which is from [Dam20], we use the notation from Appendix B.6. Further, we fix $\Lambda < \infty$, $P \in \mathbb{R}^d$ and $\psi \in \mathscr{D}_{\mathsf{N}}$.

We define the operators

$$\begin{aligned} H_+ &= |P - \mathbf{m} - \mathsf{d}\Gamma_+(\mathbf{m})|^2 + \mathsf{d}\Gamma_+(\omega) + \varphi_+(v_\Lambda) && \text{on } \mathcal{F}_+(L^2(\mathbb{R}^d)),\\ H_\oplus &= |P - \mathbf{m} - \mathsf{d}\Gamma_\oplus(\mathbf{m})|^2 + \mathsf{d}\Gamma_\oplus(\omega) + \omega + \varphi_\oplus(v_\Lambda) && \text{on } \mathscr{C}(\mathbb{R}^d). \end{aligned}$$

Using Lemmas B.37, B.39 and B.49, we see

$$(H_\oplus - \Sigma_\Lambda(P))A\psi = A(H_\Lambda(P) - \Sigma_\Lambda(P))\psi - M_v\psi.$$

Especially, there exists a zero-set $N \subset \mathbb{R}^d$ such that

$$(H_\oplus - \Sigma_\Lambda(P))A\psi(k) = (H_+ - \Sigma_\Lambda(P))A\psi(k) \in \mathcal{F} \qquad \text{for all } k \in N^c.$$

Now, let $M = \mathbb{R}P$ and let $\mathcal{C} \subset L^2(\mathbb{R}^d)$ denote the compactly supported functions. Since $\mathcal{F}_{\mathsf{fin}}(\mathcal{C})$ is a core for $H_\Lambda(P)$ (cf. the proof of Lemma 3.8), the set

$$D_k = \{\phi \in \mathcal{F}_{\mathsf{fin}}(L^2(\mathbb{R}^d)) : Q_\Lambda(k,P)\phi \in \mathcal{F}_{\mathsf{fin}}(L^2(\mathbb{R}^d))\}$$

is dense in $\mathcal{F}$ for any $k \in M^c$.

For $k \in M^c \cap N^c$ and $\phi \in D_k$, using Lemma B.40, we obtain

$$\begin{aligned}\langle \phi, A\psi(k)\rangle_+ &= \langle (H_\Lambda(P) - \Sigma_\Lambda(P) + \omega(k))Q_\Lambda(k,P)\phi, A\psi(k)\rangle_+ \\ &= \langle Q_\Lambda(k,P)\phi, (H_+ - \Sigma_\Lambda(P))A\psi(k)\rangle_+ \\ &= \langle Q_\Lambda(k,P)\phi, A(H_\Lambda(P) - \Sigma_\Lambda(P))\psi(k) - v(k)\psi\rangle_+ \\ &= \langle \phi, Q_\Lambda(k,P)A(H_\Lambda(P) - \Sigma_\Lambda(P))\psi(k) - v(k)\psi\rangle_+ .\end{aligned}$$

Combining this with Lemmas B.43 and B.48 finishes the proof. □

We can now give the

Proof of Proposition 3.2. Note that the case $\Lambda < \infty$ is a special case of Lemma 3.19. Hence, it remains to treat the case $\Lambda = \infty$. To that end, fix $P \in \mathbb{R}^d$ and assume ψ is a ground state of $H_\infty(P)$.

We set $E_\infty = 0$ and, for $\Lambda \in [0,\infty]$, we define the operator

$$\hat{H}_\Lambda(P,k) = H_\Lambda(P-k) - \Sigma_\Lambda(P) = H_\Lambda(P-k) + E_\Lambda - (\Sigma_\Lambda(P) + E_\Lambda).$$

Using Lemmas 1.6 and A.76, we see $\Sigma_\Lambda(P) + E_\Lambda$ converges to $\Sigma_\infty(P)$, so

$$\lim_{\Lambda\to\infty} e^{-t\hat{H}_\Lambda(P,k)} = \lim_{\Lambda\to\infty} e^{-t(H_\Lambda(P-k)+E_\Lambda)}e^{t(\Sigma_\Lambda(P)+E_\Lambda)} = e^{-t\hat{H}_\infty(P,k)} \quad \text{for all } t > 0,$$

which implies $\hat{H}_\Lambda(P,k)$ converges to $\hat{H}_\infty(P,k)$ in the norm resolvent sense (Lemma A.76). We pick η to be smooth and compactly supported such that $\eta(0) = 1$ and Λ_0 such that $\Sigma_\Lambda(P) + E_\Lambda - \Sigma_\infty(P) + 1 > 0$ for all $\Lambda > \Lambda_0$. Then, for $a \in \{0,1\}$ and $\Lambda \in (\Lambda_0,\infty)$, we define

$$\begin{aligned}\psi_\Lambda &= \eta(\hat{H}_\Lambda(P,0))\psi, \\ C_{a,\Lambda} &= \hat{H}_\Lambda(P,0)^a(\hat{H}_\Lambda(P,0) + E_\Lambda + \Sigma_\Lambda(P) - \Sigma_\infty(P) + 1)\eta(\hat{H}_\Lambda(P,0)), \\ B &= (\mathsf{d}\Gamma(\omega) + 1)^{1/2}, \\ D_\Lambda &= B(H_\Lambda(P) + E_\Lambda - \Sigma_\infty(P) + 1)^{-1}.\end{aligned}$$

By the functional calculus, Lemma 1.5 and Theorem 2.1, for $a \in \{0,1\}$ and $\Lambda \in (0,\infty)$, we see $\hat{H}_\Lambda(P,0)^a\psi_\Lambda \in \mathscr{D}_{\mathsf{N}} \subset \mathcal{D}(B)$ and that

$$B(\hat{H}_\Lambda(P,0))^a\psi_\Lambda = D_\Lambda C_{a,\Lambda}\psi.$$

From now, we abuse notation by setting $0^0 = 1$. Again using functional calculus, we see $C_{a,\Lambda}\psi$ converges to $0^a\psi$ as $\Lambda \to \infty$, so using Theorem 2.1 we find $D_\Lambda C_{a,\Lambda}\psi$ converges to $0^a\psi$. Hence, $(\hat{H}_\Lambda(P,0))^a\psi_\Lambda$ converges to $0^a\psi$ in B-norm.

By Lemma B.50, we see that $A\hat{H}_\Lambda(P,0)\psi_\Lambda$ is $\mathcal{F}$-valued. Therefore, we may apply Lemma 3.19 and find

$$\begin{aligned}(A\psi_\Lambda)(k) =& (H_\Lambda(P-k) - \Sigma_\Lambda(P) + \omega(k))^{-1}(A\hat{H}_\Lambda(P,0)\psi_\Lambda)(k) \\ &- v_\Lambda(k)(H_\Lambda(P-k) - \Sigma_\Lambda(P) + \omega(k))^{-1}\psi_\Lambda.\end{aligned}$$

Again using Lemma B.50, it follows that $\omega^{1/2}A\psi_\Lambda$ converges to $A\psi$ in $L^2(\mathbb{R}^d,\mathcal{F})$. Additionally, $A\hat{H}_\Lambda(P,0)\psi_\Lambda$ converges to 0 in $L^2(\mathbb{R}^d,\mathcal{F})$. As $\omega>0$ almost everywhere, we may pick elements $\Lambda_0<\Lambda_1<\Lambda_2<\cdots$ such that, for almost every k,

$$\lim_{n\to\infty}(A\psi_{\Lambda_n})(k)=(A\psi)(k),$$
$$\lim_{n\to\infty}(A\hat{H}_\Lambda(P,0)\psi_{\Lambda_n})(k)=0.$$

Now since, for all $k\in\mathbb{R}^d\setminus\mathbb{R}P$,

$$\lim_{\Lambda\to\infty}(H_\Lambda(P-k)-\Sigma_\Lambda(P)+\omega(k))^{-1}=(H_\infty(P-k)-\Sigma_\infty(P)+\omega(k))^{-1},$$

this finishes the proof. □

3.5. Proof of Absence of Ground States

We can now prove Theorem 1.7. The proof is an adaption of the one in [Dam20], which avoids any use of the non-degeneracy of the ground state energy.

Proof of Theorem 1.7.
The proof goes by contradiction. We fix $\Lambda\in(0,\infty]$, $P\in\mathbb{R}^d$ and assume there exists a $\psi_{\mathsf{gs}}\in\mathcal{F}$ such that $\|\psi_{\mathsf{gs}}\|=1$ and $H_\Lambda(P)\psi_{\mathsf{gs}}=\Sigma_\infty(P)\psi_{\mathsf{gs}}$.

Pick $\varepsilon=\frac{1}{2}$ and let $k\in S_\varepsilon(P)$. Then $\|\hat{k}\cdot\mathbf{V}_\Lambda(P)\|\leq\frac{1}{4}$ by Lemma 3.13, so a power expansion shows

$$\|1-(1-\hat{k}\cdot\mathbf{V}_\Lambda(P))^{-1}\|\leq\frac{\|\hat{k}\cdot\mathbf{V}_\Lambda(P)\|}{1-\|\hat{k}\cdot\mathbf{V}_\Lambda(P)\|}\leq\frac{1}{3}. \tag{3.23}$$

We denote the number operator $N=\mathsf{d}\Gamma(1)$ and choose a normalized element $\eta\in\mathcal{D}(N^{1/2})$ such that $|\langle\eta,\psi\rangle|>\frac{1}{2}$. Then the pull-through formula Proposition 3.2 shows

$$\langle\eta,a_k\psi_{\mathsf{gs}}\rangle=-\frac{v_\Lambda(k)}{\omega(k)}\langle\eta,Q_\Lambda(k,P)\psi_{\mathsf{gs}}\rangle\quad\text{for almost every }k\in\mathbb{R}^d.$$

Further, for $k\in S_\varepsilon(P)\cap S_\varepsilon(-P)$, Proposition 3.3 yields

$$\langle\eta,Q_\Lambda(k,P)\psi_{\mathsf{gs}}\rangle-\langle\eta,(1-\hat{k}\cdot\mathbf{V}_\Lambda(P))^{-1}\psi_{\mathsf{gs}}\rangle\xrightarrow{k\to0}0.$$

By (3.23), we now see $\big|\langle\eta,(1-\hat{k}\cdot\mathbf{V}_\Lambda(P))^{-1}\psi_{\mathsf{gs}}\rangle\big|>\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$. Hence, there is $R\in(0,\Lambda)$ such that

$$|\langle\eta,a_k\psi_{\mathsf{gs}}\rangle|\geq\frac{|v(k)|}{6\omega(k)}\qquad\text{for almost all }k\in S_\varepsilon(P)\cap S_\varepsilon(-P)\cap B_R(0)=:\tilde{B}_{\varepsilon,R}(\xi).$$

Further, since $S_\varepsilon(P)\cap S_\varepsilon(-P)$ is open, non-empty (due to $d\geq2$) and invariant under positive scalings, by rotation invariance of v and ω and the infrared-criticality assumption (Hypothesis NA), we obtain that

$$\int_{\tilde{B}_{\varepsilon,R}(P)}\frac{|v(k)|^2}{\omega(k)^2}\mathsf{d}k=\frac{\mathrm{vol}(\tilde{B}_{\varepsilon,1}(P))}{\mathrm{vol}(B_1(0))}\int_{B_R(0)}\frac{|v(k)|^2}{\omega(k)^2}\mathsf{d}k=\infty.$$

This proves that $\langle \eta, a_k \psi_{\mathsf{gs}} \rangle$ is not square-integrable.

On the other hand, using the Cauchy-Schwarz inequality, Definition B.11 and (3.2), we find

$$\begin{aligned} |\langle \eta, a_k \psi_{\mathsf{gs}} \rangle|^2 &\leq \|(N+1)^{1/2}\eta\|^2 \|(N+1)^{-1/2} a_k \psi_{\mathsf{gs}}\|^2 \\ &= \|(N+1)^{1/2}\eta\|^2 \sum_{n=1}^{\infty} \int_{\mathbb{R}^{(n-1)\nu}} |\psi_{\mathsf{gs}}^{(n)}(k, k_1, \ldots, k_{n-1})|^2 dk_1 \cdots dk_n, \end{aligned}$$

which is integrable by definition of the Fock space norm. Hence, we have arrived at a contradiction and the ground state ψ_{gs} cannot exist. □

4. Correlation Bounds in 1D Ising Models

In this intermediate chapter, we investigate correlation functions of one-dimensional Ising models. In the next chapter, we will then prove a Feyman-Kac-Nelson formula, which relates the continuous Ising model treated in Section 4.2 to the spin boson model. Especially, the correlation bound provided in Theorem 4.21 is an essential ingredient for our proof of Theorem 1.14. Nevertheless, the investigation of Ising models is justified by its own right, as we discuss as an introduction to this chapter.

The Ising model is a mathematical model of ferromagnetism and has been intensively investigated. The main concept is that the magnetic dipole moments are approximated by the values $\{+1, -1\}$, often referred to as Ising spins. It was originally proposed by Wilhelm Lenz [Len20] and treated by Ernst Ising in his PhD thesis [Isi25]. In his investigation, he treated a one-dimensional chain of Ising spins interacting with their nearest neighbors. The model has afterwards been generalized, for example positioning the Ising spins on different types of lattices or including more long-range interactions. We will restrict our attention to one-dimensional models, i.e., Ising spins positioned on the real axis.

In the first section, we will discuss Ising models defined on the lattice $\mathbb{Z}$ with long-range interactions. We will prove a correlation bound under the assumption that the interaction function is dominated by the nearest neighbor contribution. We will then introduce a continuous Ising model in Section 4.2. By proving that it can be understood as a scaling limit of the lattice model, where the nearest neighbor coupling becomes arbitrarily large, we obtain a correlation bound for the continuous setting, which is similar to the one proved for the discrete model. The bound holds for all interaction functions with L^1-norm smaller than a threshold.

4.1. The Ising Model on $\mathbb{Z}$

In this section, we introduce the discrete Ising model and prove an upper bound on correlation functions, which will be stated in Theorem 4.1. The novel aspect of this bound is that it can accomodate arbitrarily large nearest neighbor couplings.

Bounds on correlation functions of the Ising model have been studied throughout the literature, cf. [Gri67a, KS68, Gin70, Tho71, RT81] and references therein. They are, for example, used to prove the existence of the thermodynamic limit and of phase transitions in the Ising model, cf. [Gri67b, GMS67, Rue68, Dys69, KT69, FILS78, AN86].

We begin with a definition of the considered Ising model.

For $L \in \mathbb{N}$, let $\Lambda_L = \mathbb{Z} \cap [-L, +L]$ be the spin lattice and let $\mathcal{S}_L = \{-1, 1\}^{\Lambda_L}$ be the spin configuration space. For $\sigma = (\sigma_i)_{i \in \Lambda_L} \in \mathcal{S}_L$ and $A \subset \Lambda_L$, we write

$$\sigma_A = \prod_{i \in A} \sigma_i, \tag{4.1}$$

where we use the convention that $\sigma_\emptyset = 1$. For $J : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$, we define the corresponding Ising energy

$$E_{J,L}(\sigma) = -\sum_{A\subset\Lambda_L} J(A)\sigma_A \tag{4.2}$$

and the partition function

$$Z_{J,L} = \sum_{\sigma\in\mathcal{S}_L} \exp(-E_{J,L}(\sigma)). \tag{4.3}$$

In contrast to the standard definitions in statistical mechanics, we absorb the thermodynamic parameter β in the interaction function J. We remark that the case $J \geq 0$ is called ferromagnetic, while the case $J \leq 0$ is called anti-ferromagnetic. We will restrict our attention to the ferromagnetic case.

The expectation value of a function $f : \mathcal{S}_L \to \mathbb{R}$ is now defined as

$$\langle f\rangle_J^{(L)} = \frac{1}{Z_{J,L}} \sum_{\sigma\in\mathcal{S}_L} f(\sigma)\exp(-E_{J,L}(\sigma)). \tag{4.4}$$

For given $f : \mathcal{S}_L \to \mathbb{R}$ and $\widetilde{L} \geq L$, we denote the function $\widetilde{f} : \mathcal{S}_{\widetilde{L}} \to \mathbb{R}$ with $\widetilde{f}(\sigma) = f(\sigma|_{\Lambda_L})$ again by the same symbol f. Then, if the thermodynamic limit $L \to \infty$ exists, we will drop the superscript (L) and write

$$\langle f\rangle_J = \lim_{L\to\infty} \langle f\rangle_J^{(L)}. \tag{4.5}$$

Especially, we note that the existence of the thermodynamic limit of correlation functions $\langle\sigma_A\rangle_J$ for $J \geq 0$ and $A \subset \mathbb{Z}$ is well-known (cf. Corollary 4.7).

For a sequence $w = (w_k)_{k\in\mathbb{N}} \subset \mathbb{R}$, we define the associated pair interaction

$$J_w : \mathcal{P}(\mathbb{Z}) \to \mathbb{R} \quad \text{with} \quad \begin{cases} \{i,j\} \mapsto w_{|i-j|} & \text{for } i,j\in\mathbb{Z}, i\neq j, \\ A \mapsto 0 & \text{for any other } A\subset\mathbb{Z}. \end{cases} \tag{4.6}$$

In this section we prove the following theorem.

Theorem 4.1. *For every $\varepsilon \in (0, \frac{1}{10})$, there exists a constant $C_\varepsilon > 0$ such that for any $w = (w_k)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$ with $w \geq 0$ and*

$$\sum_{k=2}^{\infty} \tanh w_k \leq \varepsilon(1 - \tanh w_1), \tag{4.7}$$

we have

$$\sum_{i\in\mathbb{Z}} \langle\sigma_i\sigma_j\rangle_{J_w} \leq \frac{C_\varepsilon}{1 - \tanh w_1} \qquad \textit{for all } j \in \mathbb{Z}. \tag{4.8}$$

Remark 4.2. We note that for $v \in \ell^1(\mathbb{N})$ the sequence $w = \beta v$ satisfies the relation (4.7) for sufficiently small $\beta > 0$. Hence, our bound describes absence of long range order in the Ising model for any summable pair interaction provided the temperature is large enough.

Remark 4.3. We note that correlation estimates have already been shown a long time ago in [Dys69, RT81]. We generalize the result of [Dys69], in the sense that we can accomodate arbitrary large nearest neighbor couplings and obtain an analogous correlation bound. On the other hand the assumptions in [RT81] are weaker but their assertion is weaker as well. Explicitly, Rogers and Thompson prove the estimate

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \langle \sigma_i \sigma_j \rangle_{J_w} = 0 \qquad \text{under the assumption} \qquad \sum_{k=1}^{N} k w_k = o((\ln N)^{1/2}),$$

which shows the absence of long-range order. Note that under the stronger assumption (4.7), Theorem 4.1 implies the stronger correlation estimate

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{i,j=1}^{N} \langle \sigma_i \sigma_j \rangle_{J_w} < \infty.$$

Remark 4.4. Correlation bounds as above can be understood as bounds on the magnetic susceptibility of the Ising model. Explicitly, the above model in presence of an external magnetic field with strength $\mu \in \mathbb{R}$ is given by the interaction function

$$J_{w,\mu} = J_w + I_\mu, \qquad \text{with} \qquad I_\mu(\{i\}) = \mu \text{ and } I_\mu(A) = 0 \text{ in all other cases.}$$

Then, the magnetization of the model is given as

$$M_\mu(J_w, L) = \frac{1}{L} \partial_\mu \ln Z_{J_{w,\mu},L}$$

and its magnetic susceptibility is its derivative

$$X_\mu(J_w, L) = \partial_\mu M_\mu(J_w, L) = \frac{1}{L} \partial_\mu^2 \ln Z_{J_{w,\mu},L}.$$

From the definition (4.3), we directly obtain

$$X_\mu(J_w, L) = \frac{1}{L} \sum_{i,j=1}^{L} \langle \sigma_i \sigma_j \rangle_{J_w}^{(L)}.$$

Hence, the bound in Theorem 4.1 is a bound on the magnetic susceptibility, which is uniform in the length of the spin lattice.

The most of the remainder of this section is devoted to the proof of Theorem 4.1. In Section 4.1.1, we will recall the Griffiths' inequalities and prove some simple implications of these. We then use them to prove a correlation bound on finite lattices in Section 4.1.2. In Section 4.1.3, we give our proof of Theorem 4.1. In the last section, we recall the explicit calculations of the partition function and the correlation function for the Ising model only with nearest neighbor coupling, which we will utilize in Section 4.2.

4.1.1. Griffiths' Inequalities

Let us begin with recalling some well-known inequalities on correlation functions in the Ising model, which go back to Griffiths [Gri67a, Gri67c]. They were later generalized by Kelly and Sherman [KS68] and Ginibre [Gin70] and also referred to as the GKS (Griffiths-Kelly-Sherman) inequalities.

Lemma 4.5 (Griffiths' first inequality)**.** *Let $J : \mathcal{P}(\mathbb{Z}) \to [0,\infty)$, $L \in \mathbb{N}$ and $A \subset \Lambda_L$. Then*

$$\langle \sigma_A \rangle_J^{(L)} \geq 0.$$

Proof. This proof is essentially from [GJ87].

Let $n \in \mathbb{N}_0^{\Lambda_L}$ and denote

$$I(n) = \sum_{\sigma \in \mathcal{S}_L} \prod_{i \in \Lambda} \sigma_i^{n_i}.$$

Since for any $j \in \Lambda_L$, we have

$$I(n) = \sum_{\substack{\sigma \in \mathcal{S}_L \\ \sigma_j = 1}} \prod_{i \in \Lambda} \sigma_i^{n_i} + \sum_{\substack{\sigma \in \mathcal{S}_L \\ \sigma_j = -1}} \prod_{i \in \Lambda} \sigma_i^{n_i} = \frac{1 + (-1)^{n_j}}{2} I(n)$$

and hence $I(n) = 0$ if n_j is odd for any $j \in \Lambda_L$. Otherwise, the definition directly yields $I(n) = |\mathcal{S}_L| > 0$.

Now, we observe that the series expansion of the exponential in (4.3) yields

$$e^{-E_{J,L}(\sigma)} = \prod_{B \subset \Lambda_L} \sum_{k=0}^{\infty} \frac{J(B)^k \sigma_B^k}{k!} = \sum_{K \in \mathbb{N}_0^{\mathcal{P}(\Lambda_L)}} \prod_{B \subset \Lambda_L} \frac{J(B)^{k_B} \sigma_B^k}{k_B!}.$$

For $A \subset \Lambda_K$, we define the map $s_A : \Lambda_L \to \{0,1\}$ with $s_A(i) = 1$ if $i \in A$ and $s_A(i) = 0$ else. Thus, by inserting into (4.4), we obtain

$$\begin{aligned}
\langle \sigma_A \rangle_J^{(L)} &= Z_{J,L}^{-1} \sum_{\sigma \in \mathcal{S}_L} \sum_{K \in \mathbb{N}_0^{\mathcal{P}(\Lambda_L)}} \prod_{B \subset \Lambda_L} \frac{J(B)^{k_B} \sigma_A \sigma_B^k}{k_B!} \\
&= Z_{J,L}^{-1} \sum_{K \in \mathbb{N}_0^{\mathcal{P}(\Lambda_L)}} \prod_{B \subset \Lambda_L} \frac{J(B)^{k_B}}{k_B!} \sum_{\sigma \in \mathcal{S}_L} \prod_{i \in \Lambda} \sigma_i^{s_A(i) + k_B s_B(i)} \\
&= Z_{J,L}^{-1} \sum_{K \in \mathbb{N}_0^{\mathcal{P}(\Lambda_L)}} \prod_{B \subset \Lambda_L} \frac{J(B)^{k_B}}{k_B!} I(s_A(i) + k_B s_B(i)).
\end{aligned}$$

By our initial consideration and the assumption $J \geq 0$, every summand is nonnegative and the statement follows. □

The first Griffiths' inequality can be used to prove the second one.

Lemma 4.6 (Griffiths' second inequality)**.** *Let $J : \mathcal{P}(\mathbb{Z}) \to [0,\infty)$, $L \in \mathbb{N}$ and $A, B \subset \Lambda_L$. Then*

$$\langle \sigma_A \sigma_B \rangle_J^{(L)} \geq \langle \sigma_A \rangle_J^{(L)} \langle \sigma_B \rangle_J^{(L)}.$$

Proof. The argument presented here can be found in [FV17].

For a fixed spin configuration $\tau \in \mathcal{S}_L$, we define the interaction $J_\tau : \mathcal{P}(\mathbb{Z}) \to [0,\infty)$ as $J_\tau = J(C)(1 + \tau_C)$ for all $C \subset \Lambda_L$. By the definition (4.4), we have

$$\langle \sigma_A \sigma_B \rangle_J^{(L)} - \langle \sigma_A \rangle_J^{(L)} \langle \sigma_B \rangle_J^{(L)} = \frac{1}{Z_{J,L}^2} \sum_{\sigma, \sigma' \in \Lambda_L} \sigma_A (\sigma_B - \sigma'_B) e^{-E_{J,L}(\sigma) - E_{J,L}(\sigma')}.$$

Using the change of variables $\tau_i = \sigma_i \sigma_i'$ for $i \in \Lambda$ and observing

$$E_{J,L}(\sigma) + E_{J,L}(\sigma') = -\sum_{C \subset \Lambda_L} J(C)(\sigma_C + \sigma_C') = \sum_{C \subset \Lambda_L} J(C)\sigma_C(1 + \tau_C) = E_{J_\tau,L}(\sigma),$$

we obtain

$$\begin{aligned}\langle \sigma_A \sigma_B \rangle_J^{(L)} - \langle \sigma_A \rangle_J^{(L)} \langle \sigma_B \rangle_J^{(L)} &= \frac{1}{Z_{J,L}^2} \sum_{\tau \in \Lambda_L} (1 - \tau_B) \sum_{\sigma \in \Lambda_L} \sigma_A \sigma_B e^{-E_{J_\tau,L}(\sigma)} \\ &= \frac{1}{Z_{J,L}^2} \sum_{\tau \in \Lambda_L} (1 - \tau_B) \langle \sigma_A \sigma_B \rangle_{J_\tau}^{(L)} .\end{aligned}$$

Hence, the statement follows from Lemma 4.5. □

The second Griffiths' inequality directly yields that correlation functions are increasing in the coupling function and that the thermodynamic limit exists, see [Gri67a] for the first use of this argument.

Corollary 4.7. *Let $J : \mathcal{P}(\mathbb{Z}) \to [0, \infty)$ and $A \subset \mathbb{Z}$.*

(i) *If $\widetilde{J} : \mathcal{P}(\mathbb{Z}) \to [0, \infty)$ satisfies $\widetilde{J} \le J$, then $\langle \sigma_A \rangle_{\widetilde{J}}^{(L)} \le \langle \sigma_A \rangle_J^{(L)}$ for all $L \in \mathbb{N}$.*

(ii) *The thermodynamic limit $\langle \sigma_A \rangle_J$ exists.*

Proof. By the definition (4.4), we have

$$\frac{\partial}{\partial J(B)} \langle \sigma_A \rangle_J^{(L)} = \langle \sigma_A \sigma_B \rangle_J^{(L)} - \langle \sigma_A \rangle_J^{(L)} \langle \sigma_B \rangle_J^{(L)} \qquad \text{for any } B \subset \Lambda_L.$$

Hence, (i) follows from Lemma 4.6 since $\langle \sigma_A \rangle_J^{(L)}$ is increasing in J. This further implies that the expectation $\langle \sigma_A \rangle_J^{(L)}$ is nonnegative (Lemma 4.5), increasing in L (Part (i)), and bounded above by 1. Thus, (ii) follows by monotone convergence. □

Vice versa, it is possible to decrease the interaction and bound the difference by a correction term. This was first done in [Gri67c] to calculate the critical temperature of an Ising lattice. For the precise statement, we write the symmetric set difference as $AB = A \cup B \setminus (A \cap B)$ for $A, B \subset \mathbb{Z}$. We note that this implies $\sigma_A \sigma_B = \sigma_{AB}$. Further, if $\mathcal{A} \subset \mathcal{P}(\mathbb{Z})$, we define

$$\langle \cdot \rangle_{J;\mathcal{A}}^{(L)} := \langle \cdot \rangle_{I_\mathcal{A}}^{(L)} \quad \text{and} \quad \langle \cdot \rangle_{J;\mathcal{A}} := \langle \cdot \rangle_{I_\mathcal{A}}, \qquad \text{where } I_\mathcal{A}(A) = \begin{cases} J(A) & \text{for } A \notin \mathcal{A}, \\ 0 & \text{for } A \in \mathcal{A}. \end{cases} \tag{4.9}$$

By Corollary 4.7, it follows that

$$\langle \sigma_A \rangle_{J;\{B\}}^{(L)} \le \langle \sigma_A \rangle_J^{(L)} . \tag{4.10}$$

Lemma 4.8 (Griffiths' third inequality)**.** *Let $J : \mathcal{P}(\mathbb{Z}) \to [0, \infty)$ and assume $A, B \subset \mathbb{Z}$, $L \in \mathbb{N}$. Then*

$$\langle \sigma_A \rangle_J^{(L)} \le \langle \sigma_A \rangle_{J;\{B\}}^{(L)} + \tanh(J(B)) \langle \sigma_{AB} \rangle_{J;\{B\}}^{(L)} .$$

Before we prove above lemma, we state a related inequality due to Thompson [Tho71].

Lemma 4.9. *Let $J : \mathcal{P}(\mathbb{Z}) \to [0, \infty)$ and assume $A, B \subset \mathbb{Z}$, $L \in \mathbb{N}$. Then*

$$\langle \sigma_A \rangle_J^{(L)} \leq \tanh(J(B)) \langle \sigma_{AB} \rangle_J^{(L)} + (1 - \tanh^2(J(B))) \langle \sigma_A \rangle_{J;\{B\}}^{(L)} .$$

Proof of Lemmas 4.8 and 4.9. This proof is from [Tho71].

First observe that for $y \in \{\pm 1\}$ the series expansion of the exponential yields

$$e^{xy} = \cosh x + y \sinh x = \cosh x\,(1 + y \tanh x) \qquad \text{for any } x \in \mathbb{C}.$$

Then, the definition (4.2) implies

$$e^{-E_{J,L}(\sigma)} = e^{-J(A)\sigma_A} e^{-E_{J;\{A\},L}(\sigma)} = \cosh J(A)\,(1 + \tanh J(A)\sigma_A)\, e^{-E_{J;\{A\},L}(\sigma)}.$$

Inserting this into (4.3) and (4.4) now yields

$$\langle \sigma_A \rangle_J^{(L)} = \frac{\langle \sigma_A \rangle_{J;\{B\}}^{(L)} + \tanh J(B) \langle \sigma_A \sigma_B \rangle_{J;\{B\}}^{(L)}}{1 + \tanh J(B) \langle \sigma_B \rangle_{J;\{B\}}^{(L)}}. \tag{4.11}$$

Since $J \geq 0$, we can apply Lemma 4.5 and $\tanh J(B) \geq 0$, which proves Lemma 4.8. Further, using $\sigma_B^2 = 1$, (4.11) applied for AB yields

$$\langle \sigma_{AB} \rangle_J^{(L)} = \frac{\langle \sigma_{AB} \rangle_{J;\{B\}}^{(L)} + \tanh J(B) \langle \sigma_A \rangle_{J;\{B\}}^{(L)}}{1 + \tanh J(B) \langle \sigma_B \rangle_{J;\{B\}}^{(L)}}.$$

Rearranging and inserting $\langle \sigma_{AB} \rangle_{J;\{B\}}^{(L)}$ into (4.11), we obtain

$$\langle \sigma_A \rangle_J^{(L)} = \tanh(J(B)) \langle \sigma_{AB} \rangle_J^{(L)} + (1 - \tanh^2(J(B))) \frac{\langle \sigma_A \rangle_{J;\{B\}}^{(L)}}{1 + \tanh J(B) \langle \sigma_B \rangle_{J;\{B\}}^{(L)}}.$$

Again using Lemma 4.5 and $\tanh J(B) \geq 0$ finishes the proof. □

We will also utilize the simple fact that expectations involving uncoupled Ising spins always vanish. This is the content of the next lemma.

Lemma 4.10. *Let $L \in \mathbb{N}$, $i \in \Lambda_L$ and assume $J : \mathcal{P}(\mathbb{Z}) \to \mathbb{R}$ satisfies $J(A) = 0$ for all $A \subset \Lambda_L$ with $i \in A$. Then $\langle \sigma_B \rangle_J^{(L)} = 0$ for any $B \subset \Lambda_L$ with $i \in B$.*

Proof. We define $\phi_i : \mathcal{S}_L \to \mathcal{S}_L$ as $(\phi_i(\sigma))_k = -\sigma_k$, if $k = i$, and $(\phi_i(\sigma))_k = \sigma_k$, if $k \neq i$. By the assumptions, it follows that $E_{J,L}(\phi_i(\sigma)) = E_{J,L}(\sigma)$ for all $\sigma \in \mathcal{S}_L$. Further, if $i \in B$, we have $\sigma_B \circ \phi_i = -\sigma_B$. Together, we obtain

$$\langle \sigma_B \rangle_J^{(L)} = \langle \sigma_B \circ \phi_i \rangle_J^{(L)} = \langle -\sigma_B \rangle_J^{(L)} = -\langle \sigma_B \rangle_J^{(L)} .$$

This implies the claim. □

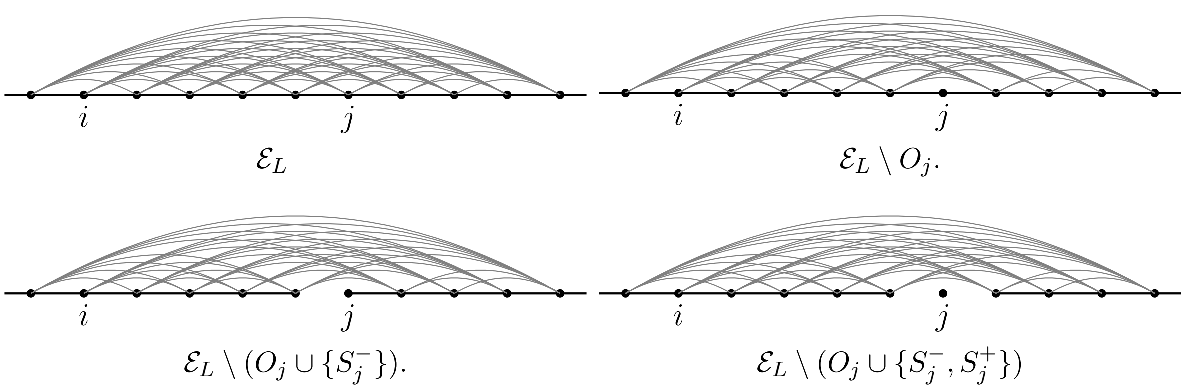

Figure 3: Illustration of the set $\mathcal{E}_L$, consisting of all edges with vertices in Λ_L, without the edges of the indicated sets.

4.1.2. Correlation Bound for Finite Chains

The major ingredient of the proof of Theorem 4.1 is the following correlation bound for finite Ising spin chains. Therein, we use the convention that $\langle \sigma_l \sigma_k \rangle_J^{(L)} = 0$ if l or k is not an element of Λ_L.

Lemma 4.11. *Let $L \in \mathbb{N}$ and $w = (w_k)_{k\in\mathbb{N}} \subset [0,\infty)$. We set $\tau_k = \tanh(w_k)$. If $i, j \in \Lambda_L$ with $i < j$, we have*

$$\langle \sigma_i \sigma_j \rangle_{J_w}^{(L)} \le \tau_1 \langle \sigma_i \sigma_{j-1} \rangle_{J_w}^{(L)} + \sum_{l=2}^{\infty} \sum_{s=\pm 1} \tau_l \langle \sigma_i \sigma_{j+sl} \rangle_{J_w}^{(L)} + (1-\tau_1^2) \sum_{b=1}^{\infty} \tau_1^b \sum_{l=2}^{\infty} \sum_{s=\pm 1} \tau_l \langle \sigma_i \sigma_{j+b+sl} \rangle_{J_w}^{(L)} .$$

If $i, j \in \Lambda_L$ with $i > j$, we have

$$\langle \sigma_i \sigma_j \rangle_{J_w}^{(L)} \le \tau_1 \langle \sigma_i \sigma_{j+1} \rangle_{J_w}^{(L)} + \sum_{l=2}^{\infty} \sum_{s=\pm 1} \tau_l \langle \sigma_i \sigma_{j+sl} \rangle_{J_w}^{(L)} + (1-\tau_1^2) \sum_{b=1}^{\infty} \tau_1^b \sum_{l=2}^{\infty} \sum_{s=\pm 1} \tau_l \langle \sigma_i \sigma_{j-b+sl} \rangle_{J_w}^{(L)} .$$

Proof. The philosophy of our proof is sketched in Fig. 3. We use the estimates from the previous section to reduce the number of interaction edges, in which j contributes. To that end, for $j \in \Lambda_L$, we define the sets

$$S_j^{\pm} = \{j, j \pm 1\} \quad \text{and} \quad O_j = \{\{j,k\} : k \in \mathbb{Z} \setminus \{j, j-1, j+1\}\}.$$

Note that $S_j^{\pm}$ contain the nearest neighbors of j, while O_j are all long-range pairs involving j. Throughout this proof, we drop the superscript (L) and the subscript J_w of expectation values. Moreover we assume $i < j$. The statement in the case $i > j$ can be obtained completely analogous.

By twice applying Lemma 4.8, we obtain

$$\begin{aligned} \langle \sigma_i \sigma_j \rangle &\le \langle \sigma_i \sigma_j \rangle_{;\{\{j,j-2\}\}} + \tau_2 \langle \sigma_i \sigma_{j-2} \rangle_{;\{\{j,j-2\}\}} \\ &\le \langle \sigma_i \sigma_j \rangle_{;\{\{j,j-2\},\{j,j+2\}\}} + \tau_2 \langle \sigma_i \sigma_{j+2} \rangle_{;\{\{j,j-2\},\{j,j+2\}\}} + \tau_2 \langle \sigma_i \sigma_{j-2} \rangle_{;\{\{j,j-2\}\}} . \end{aligned}$$

Combined with (4.10), this implies

$$\langle \sigma_i \sigma_j \rangle \le \tau_2 \left(\langle \sigma_i \sigma_{j-2} \rangle + \langle \sigma_i \sigma_{j+2} \rangle \right) + \langle \sigma_i \sigma_j \rangle_{;\{\{j,j-2\},\{j,j+2)\}\}} .$$

Iterating this argument, we arrive at

$$\langle\sigma_i\sigma_j\rangle \leq \sum_{l=2}^{\infty}\tau_l\sum_{s=\pm}\langle\sigma_i\sigma_{j+sl}\rangle + \langle\sigma_i\sigma_j\rangle_{;O_j}. \tag{4.12}$$

Then, Lemma 4.9 yields

$$\langle\sigma_i\sigma_j\rangle_{;O_j} \leq \tau_1\langle\sigma_i\sigma_{j-1}\rangle_{;O_j} + (1-\tau_1^2)\langle\sigma_i\sigma_j\rangle_{;O_j\cup\{S_j^-\}}. \tag{4.13}$$

The second term on the right hand side can be estimated by Lemmas 4.8 and 4.10

$$\langle\sigma_i\sigma_j\rangle_{;O_j\cup\{S_j^-\}} \leq \underbrace{\langle\sigma_i\sigma_j\rangle_{;O_j\cup\{S_j^-,S_j^+\}}}_{=0} + \tau_1\langle\sigma_i\sigma_{j+1}\rangle_{;O_j\cup\{S_j^-,S_j^+\}}. \tag{4.14}$$

Now applying (4.12) with j replaced by $j+1$ and again using (4.10), we obtain

$$\langle\sigma_i\sigma_{j+1}\rangle_{;O_j\cup\{S_j^-,S_j^+\}} \leq \sum_{l=2}^{\infty}\sum_{s=\pm1}\tau_l\langle\sigma_i\sigma_{j+1+sl}\rangle + \langle\sigma_i\sigma_{j+1}\rangle_{;O_j\cup O_{j+1}\cup\{S_j^-,S_j^+\}}. \tag{4.15}$$

As in (4.14), we use Lemmas 4.8 and 4.10, which yield

$$\begin{aligned}\langle\sigma_i\sigma_{j+1}\rangle_{;O_j\cup O_{j+1}\cup\{S_j^-,S_j^+\}} \leq{}& \overbrace{\langle\sigma_i\sigma_{j+1}\rangle_{;O_j\cup O_{j+1}\cup\{S_j^-,S_j^+,S_{j+1}^+\}}}^{=0}\\ &+\tau_1\langle\sigma_i\sigma_{j+1}\rangle_{;O_j\cup O_{j+1}\cup\{S_j^-,S_j^+,S_{j+1}^+\}}.\end{aligned} \tag{4.16}$$

Note that we hereby used $S_j^+ = S_{j+1}^-$. We now insert (4.16) into (4.15) and iterate the same arguments. As a result

$$\langle\sigma_i\sigma_{j+1}\rangle_{;O_j\cup\{S_j^-,S_j^+\}} \leq \sum_{b=1}^{\infty}\sum_{l=2}^{\infty}\sum_{s=\pm1}\tau_1^{b-1}\tau_l\langle\sigma_i\sigma_{j+b+sl}\rangle. \tag{4.17}$$

The statement now follows by combining (4.12), (4.13), (4.14) and (4.17). □

4.1.3. Proof of the Correlation Bound on $\mathbb{Z}$

We use the result from the previous section for the

Proof of Theorem 4.1. For the proof of the statement, we will use the estimate from Lemma 4.11. We need to take the limit $L\to\infty$ and sum over all $i\in\mathbb{Z}$. To show finiteness, we will make use of the translation-invariance of the model. Let us first assume that $w\in\ell^1(\mathbb{N})$ with $w\geq 0$ has compact support and let $K>0$ be such that

$$w_k = 0, \quad k\geq K. \tag{4.18}$$

As in Lemma 4.11, we shall use the notation $\tau_k=\tanh(w_k)$. We introduce a regularization parameter $\eta>0$ and define

$$\tau_{k,\eta} = e^{\eta k}\tau_k \quad\text{and}\quad \langle\sigma_i\sigma_j\rangle_{J_w}^{(L,\eta)} = e^{-\eta|i-j|}\langle\sigma_i\sigma_j\rangle_{J_w}^{(L)}. \tag{4.19}$$

Further, we define

$$M^-_{j,L}(\eta) = \sum_{i=-L}^{j-1} \langle \sigma_i \sigma_j \rangle^{(L,\eta)}_{J_w}, \quad M^+_{j,L}(\eta) = \sum_{i=j+1}^{L} \langle \sigma_i \sigma_j \rangle^{(L,\eta)}_{J_w},$$
$$M_{j,L}(\eta) = \sum_{i=-L}^{L} \langle \sigma_i \sigma_j \rangle^{(L,\eta)}_{J_w} = 1 + M^+_{j,L}(\eta) + M^-_{j,L}(\eta).$$

By the regularization (4.19) and Corollary 4.7 (ii), for any $\eta > 0$, the limits

$$M^\pm_j(\eta) = \lim_{L\to\infty} M^\pm_{j,L}(\eta) \quad \text{and} \quad M_j(\eta) = \lim_{L\to\infty} M_{j,L}(\eta)$$

exist. By translation-invariance of J_w, i.e., $\langle \sigma_i \sigma_j \rangle_{J_w} = \langle \sigma_{i+k} \sigma_{j+k} \rangle_{J_w}$ for any $k \in \mathbb{Z}$, it follows that $M_j(\eta)$ and $M^\pm_j(\eta)$ are independent of j and we shall write $M(\eta)$ for $M_j(\eta)$.

For $L \in \mathbb{N}$, we now multiply the inequalities in Lemma 4.11 with $e^{-\eta|i-j|}$ and use the triangle inequality, to obtain for $i \lessgtr j$

$$\begin{aligned}\langle \sigma_i \sigma_j \rangle^{(L,\eta)}_{J_w} \leq & \tau_1 \langle \sigma_i \sigma_{j\mp1} \rangle^{(L,\eta)}_{J_w} + \sum_{l=2}^{\infty} \sum_{s=\pm1} \tau_{l,\eta} \langle \sigma_i \sigma_{j+sl} \rangle^{(L,\eta)}_{J_w} \\ & + (1-\tau_1^2) \sum_{b=1}^{\infty} \tau^b_{1,\eta} \sum_{l=2}^{\infty} \sum_{s=\pm1} \tau_{l,\eta} \langle \sigma_i \sigma_{j\pm b+sl} \rangle^{(L,\eta)}_{J_w}.\end{aligned}$$

Adding the above expression for the cases $i > j$ and $j > i$, summing over all $i \in \Lambda_L$, using $\sigma_r^2 = 1$ for any $r \in \mathbb{Z}$ as well as Lemma 4.5, we find

$$\begin{aligned}M_{j,L}(\eta) \leq & 1 + \tau_1 \left(M^-_{j-1,L}(\eta) + 2 + M^+_{j+1,L}(\eta) \right) + \sum_{l=2}^{K} \tau_{l,\eta} \sum_{s=\pm1} M_{j+sl,L}(\eta) \\ & + \sum_{b=1}^{\infty} \tau^b_{1,\eta} (1-\tau_1^2) \sum_{l=2}^{K} \tau_{l,\eta} \sum_{s=\pm1} \left(M_{j+b+sl,L}(\eta) + M_{j-b+sl,L}(\eta) \right).\end{aligned} \tag{4.20}$$

Now we can take the limit $L \to \infty$. Since τ has compact support and $\eta > 0$, the expressions on the right hand side stay finite. Then, using the translation-invariance of J_w, we can drop the index j, and summing the geometric series $\sum_{b\in\mathbb{N}} \tau^b_{1,\eta}$, for sufficiently small $\eta > 0$, we obtain

$$M(\eta) \leq 1 + \tau_1 + M(\eta) \left(\tau_1 + 2 \sum_{l=2}^{K} \tau_{l,\eta} \left(1 + 2 \frac{1-\tau_1^2}{1-\tau_{1,\eta}} \right) \right). \tag{4.21}$$

We fix a parameter $D > 1$ such that $\varepsilon \in (0, (10D)^{-1})$. Since

$$1 < \frac{1-\tau_1^2}{1-\tau_1} = 1 + \tau_1 < 2, \tag{4.22}$$

we can choose $\eta_0 > 0$ such that $\frac{1-\tau_1^2}{1-\tau_{1,\eta_0}} < 2$ and $e^{K\eta_0} < D$. Then, for any $\eta \in (0, \eta_0)$, we obtain

$$\sum_{l=2}^{K} \tau_{l,\eta} \leq D \sum_{l=2}^{K} \tau_l \tag{4.23}$$

and

$$\tau_1 + 2\sum_{l=2}^{K} \tau_{l,\eta}\left(1 + 2\frac{1-\tau_1^2}{1-\tau_{1,\eta}}\right) < \tau_1 + 10D\sum_{l=2}^{\infty}\tau_l. \tag{4.24}$$

If

$$\sum_{l=2}^{\infty}\tau_l \leq \varepsilon(1-\tau_1), \tag{4.25}$$

the right hand side of (4.24) is smaller than 1, and we can take $M(\eta)$ in (4.21) to the left hand side. Thus, using (4.22) and (4.24), we find

$$M(\eta) \leq \frac{1+\tau_1}{1-\tau_1-2\sum_{l=2}^{K}\tau_{l,\eta}\left(1+2\frac{1-\tau_1^2}{1-\tau_{1,\eta}}\right)} \leq \frac{2}{1-\tau_1-10D\sum_{l=2}^{\infty}\tau_l} \leq \frac{2}{1-10D\varepsilon}\frac{1}{1-\tau_1}.$$

By monotone convergence, the limit $\eta \downarrow 0$ exists and

$$\sum_{i\in\mathbb{Z}}\langle\sigma_i\sigma_j\rangle_{J_w} = \lim_{\eta\downarrow 0} M(\eta) \leq \frac{2}{1-10D\varepsilon}\frac{1}{1-\tau_1}. \tag{4.26}$$

Thus, we have proven (4.26) for all nonnegative $w \in \ell^1(\mathbb{N})$ satisfying (4.18) and (4.25).

Finally, let us consider general $w \in \ell^1(\mathbb{N})$ with $w \geq 0$ satisfying only (4.25). If $i, j \in \Lambda_L$, then as an immediate consequence of the definition (4.4)

$$\langle\sigma_i\sigma_j\rangle_{J_w}^{(L)} = \langle\sigma_i\sigma_j\rangle_{J_{w1_{[0,2L+1]}}}^{(L)}.$$

Since $w\chi_{[0,2L+1]}$ trivially satisfies (4.25) because w does, we find from (4.26) and monotonicity (Lemma 4.8) that, for all $N \in \mathbb{N}$, the estimate

$$\sum_{i=-N}^{N}\langle\sigma_i\sigma_j\rangle_{J_w}^{(L)} \leq \frac{2}{1-10D\varepsilon}\frac{1}{1-\tau_1}$$

holds. Thus, the bound (4.8) of the proposition follows from the above inequality, by first taking the limit $L \to \infty$ and then the limit $N \to \infty$. □

4.1.4. Ising Model with Nearest Neighbor Coupling

In this section, we consider the Ising model with nearest neighbor coupling and calculate the known partition function and correlation functions. These calculations are well-known, go back to Ising's PhD thesis [Isi25] and can be found in most textbooks covering the Ising model. We will make use of these results in our treatment of the continuous Ising model as scaling limit of discrete Ising models, in the next section.

For any constant $j \in \mathbb{R}$, we define the sequence $(w_k^{(j)})_{k\in\mathbb{N}} = (j, 0, \cdots)$ and consider the interaction function J_{w_j} as defined in (4.6).

Lemma 4.12. *Let $j \in \mathbb{R}$ and $L \in \mathbb{N}$.*

(i) *For $\sigma \in \mathcal{S}_L$ we write $n_\sigma = \left|\{i = -L, \ldots, L-1 : \sigma_i\sigma_{i+1} = -1\}\right|$. Then*

$$E_{J_{w^{(j)}},L}(\sigma) = 2j(n_\sigma - L).$$

(ii) $Z_{J_{w(j)},L} = 2\left(e^j + e^{-j}\right)^{2L}$.

(iii) *For* $n \in \mathbb{N}$ *and* $-L \leq i_1 \leq \cdots \leq i_n \leq L$*, we have*

$$\langle \sigma_{i_1} \cdots \sigma_{i_n} \rangle^{(L)}_{J_{w(j)}} = \begin{cases} \tanh(j)^{\sum_{k=1}^{N}(i_{2k}-i_{2k-1})} & \text{if } n = 2N, \\ 0 & \text{else.} \end{cases}$$

Proof. (i) follows directly from the definition. For the proof of (ii) and (iii), we use the change of variables

$$\sigma'_i = \sigma_i \sigma_{i+1} \quad \text{for } i = -L, \ldots, L-1, \quad \sigma'_L = \sigma_L.$$

Then, we have $E(\sigma)_{J_{w(j)},L} = -j \sum_{i=-L}^{L-1} \sigma'_i$ and hence

$$Z_{J_{w(j)},L} = \sum_{\sigma' \in \mathcal{S}} \prod_{i=-L}^{L-1} e^{j\sigma'_i} = 2 \prod_{i=-L}^{L-1} \sum_{\sigma'_i = \pm 1} e^{j\sigma'_i} = 2\left(e^j + e^{-j}\right)^{2L},$$

so (ii) is proved. Now, if $-L \leq i < j \leq L$, we observe

$$\sigma_i \sigma_j = (\sigma_i \sigma_{i+1})(\sigma_{i+1}\sigma_{i+2}) \cdots (\sigma_{j-1}\sigma_j) = \sigma'_i \sigma'_{i+1} \cdots \sigma'_{j-1}.$$

Assume $n = 2N$. Then, we have

$$\begin{aligned} Z_{J_{w(j)},L} \langle \sigma_{i_1} \cdots \sigma_{i_{2N}} \rangle^{(L)}_{J_{w(j)}} &= \sum_{\sigma \in \mathcal{S}} \sigma_{i_1} \cdots \sigma_{i_{2N}} e^{-E(\sigma)} \\ &= \sum_{\sigma \in \mathcal{S}} e^{-E(\sigma)} \prod_{a=1}^{N} (\sigma_{i_{2a-1}} \sigma_{i_{2a-1}+1}) \cdots (\sigma_{i_{2a}-1} \sigma_{i_{2a}}) \\ &= \sum_{\sigma' \in \mathcal{S}} \prod_{a=1}^{N} \sigma'_{i_{2a-1}} \cdots \sigma'_{i_{2a}-1} \prod_{i=-L}^{L-1} e^{j\sigma'_i}. \end{aligned}$$

Now, for $\ell \in \Lambda_L$, we set

$$s_\ell = \begin{cases} 1 & \text{if there is some } a \in \mathbb{N} \text{ with } i_{2a-1} \leq \ell < i_{2a}, \\ 0 & \text{else.} \end{cases}$$

and $S = \left|\{\ell \in \Lambda_L : s_\ell = 1\}\right| = |i_2 - i_1| + \cdots + |i_{2N} - i_{2N-1}|$. Inserting above, we obtain

$$\begin{aligned} Z_{J_{w(j)},L} \langle \sigma_{i_1} \cdots \sigma_{i_{2N}} \rangle^{(L)}_{J_{w(j)}} &= 2 \prod_{\ell=-L}^{L-1} \sum_{\sigma'_\ell = \pm 1} (\sigma'_\ell)^{s_\ell} e^{j\sigma'_\ell} \\ &= 2(e^j - e^{-j})^S (e^j + e^{-j})^{2L-S}. \end{aligned}$$

Combined with (ii), we obtain the identity (iii) for even n. It remains to consider the case that n is odd. The statement then, however, follows similar to the proof of Lemma 4.10 by the change of variables $\sigma \mapsto -\sigma$. □

4.2. The Ising Model on $\mathbb{R}$

In this section, we consider a one-dimensional continuous Ising model, which is described in terms of a jump process and a long range interaction given by a nonnegative integrable function. As in Section 4.1, we prove a correlation bound, which has the physical interpretation of a bound on the magnetic susceptibility.

The configuration space for the discrete Ising model is replaced by a jump process in the continuous case. For its definition, let us recall the notion of a continuous-time Markov process. Introductory literature on the topic includes [Res92, Ros07, Lig10].

Definition 4.13. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space, and let S be an arbitary finite set. A family $X = (X_t)_{t\geq 0}$ of S-valued random variables is called *continuous-time Markov process with state space* S if for all $0 \leq s \leq t$ and $x : [0, s) \to S$

$$\mathbb{P}[X_t = i | X_s = j, X_u = x(u), u \in [0, s)] = \mathbb{P}[X_t = i | X_s = j] \qquad \text{for all } i, j \in S.$$

Further, if $\mathbb{P}[X_{t+h} = i|X_t = j] = \mathbb{P}[X_h = i|X_0 = j]$ for all $i, j \in S$ and $t, h \geq 0$, we call X *homogeneous*. In this case, we call $p_t : S^2 \to [0, 1]$ with $p_t(i, j) = \mathbb{P}[X_t = j|X_0 = i]$ for $t \in [0, \infty)$ the *transition probability functions* of X. Further, $q_0 : S \to [0, 1]$ with $q_0(i) = \mathbb{P}[X_0 = i]$ is called *initial distribution*. The functions $t \mapsto X_t(\omega)$ for fixed $\omega \in \Omega$ are called *sample paths*.

Remark 4.14. We have intentionally restricted our attention to a finite state space here, to avoid any technical issues of more general settings.

Let us summarize some well-known statements.

Lemma 4.15. *Let S be a finite set.*

(i) *Let $p_t : S^2 \to [0, 1]$ for $t \in [0, \infty)$ be a collection of functions. Then p_t are the transition probability functions of a homogeneous continuous-time Markov process if and only if they satisfy*

$$\sum_{j\in S} p_t(i, j) = 1 \quad \textit{for all } t \in [0, \infty), \qquad \lim_{t\downarrow 0} p_t(i, i) = p_0(i, i) = 1 \tag{4.27}$$

and the Chapman-Kolmogorov equations

$$p_{t+s}(i, j) = \sum_{k\in S} p_t(i, k)p_s(k, j). \tag{4.28}$$

In this case, given a function $q_0 : S \to [0, 1]$ with

$$\sum_{i\in S} q_0(i) = 1, \tag{4.29}$$

then there exists a homogeneous continuous-time Markov process with initial distribution q_0 and transition probability functions p_t such that the sample paths are almost surely right-continuous.

(ii) *The sample paths of a continuous-time Markov process with finite state space have only finitely many jumps in any compact interval almost surely .*

Proof. An explicit construction for the existence statement in (i) can, for example, be found in [Lig10, Section 2.5]. Further, Lemma 4.15 (ii) is included in the statement of [Res92, Proposition 5.2.1] □

Our model treats the following jump process. In the definition $\delta_{x,y}$ denotes the usual Kronecker delta for $x, y \in \{\pm 1\}$.

Definition 4.16 (The jump process X on $[-T,T]$)**.** Fix some $T > 0$.
Let $(Y_t)_{t\geq 0}$ be a homogeneous continuous-time Markov process with state space $\{\pm 1\}$, initial distribution $q_0(1) = q_0(-1) = \frac{1}{2}$, transition probability

$$p_t(x,y) = \frac{1}{2}\left(1 + e^{-2t}\delta_{x,y} - e^{-2t}\delta_{x,-y}\right) \qquad \text{for } x, y \in \{\pm 1\}, t \geq 0$$

and right-continuous sample paths. Then, we define the jump process

$$X_t^{(T)} = Y_{t+T} \qquad \text{for all } t \in [-T, T].$$

We denote expectation values with respect to the probability distribution of $X^{(T)}$ as $\mathbb{E}_X^{(T)}$ and drop the upper index (T) of X_t inside of such expectation values.

Remark 4.17. It is easily checked that the transition probability satisfies the conditions (4.27), (4.28) and (4.29) and hence the definition makes sense by Lemma 4.15 (i).

Remark 4.18. In the literature, see for example [Abd11, HHL14], the process Y is usually explicitly constructed from a Poisson process as follows. Let $(N_t)_{t\geq 0}$ be a Poisson processes with intensity 1, i.e.,

$$\mathbb{P}[N_t = n] = \frac{t^n}{n!}e^{-t} \qquad \text{for } n \in \mathbb{N},\ t \geq 0. \tag{4.30}$$

Further, let B be a Bernoulli random variable with $\mathbb{P}[B = 1] = \mathbb{P}[B = -1] = \frac{1}{2}$. Then, we define

$$Y_t = (-1)^{N_t} B.$$

Now, $\mathbb{P}[Y_0 = x] = \mathbb{P}[B = x] = \frac{1}{2}$ for $x = \pm 1$ and

$$\mathbb{P}[Y_t = x | Y_s = x] = \mathbb{P}[N_{|t-s|} \text{ is even}] = \sum_{k=0}^{\infty} \frac{(|t-s|)^{2k}}{(2k)!} e^{-|t-s|} = \frac{1 + e^{-2|t-s|}}{2},$$

which reproduces the transition probability of Y. Since the Poisson process itself is a continuous-time Markov process, this also implies the Markov property. Similar to Lemma 4.15 (i), the Poisson process can be chosen right-continuous.

Remark 4.19. Vice versa to the previous remark, we can use Y to construct a Poisson process. Explicitly, if N_t denotes the number of jumps X_s has in the interval $s \in [0, t]$, then it follows easily from the assumptions that N has piecewise constant sample paths, jumps of size 1, and stationary as well as independent increments. This defines a Poisson process. See, for example, [Lig10] for more details.

To state the main result of this section, assume $I : \mathbb{R} \to \mathbb{R}$ is continuous. We then define the partition function

$$\mathscr{Z}_{I,T} = \mathbb{E}_X^{(T)}\left[\exp\left(\int_{-T}^{T}\int_{-T}^{T} I(t-s) X_t X_s \mathsf{d}t\mathsf{d}s\right)\right] \qquad \text{for } T > 0. \tag{4.31}$$

Remark 4.20. The integrals in (4.31) are Riemann integrals. By Lemma 4.15 (ii), the paths of X almost surely have only finitely many discontinuities on compact intervals. Hence, since I is continuous, the right hand side of (4.31) is well-defined.

Now assume O is a real-valued random variable defined on the same probability space as $X^{(T)}$. Then, we define the expectation values of the continuous Ising model as

$$[\![O]\!]_{I,T} = \frac{1}{\mathcal{Z}_{I,T}} \mathbb{E}_X^{(T)} \left[O \exp\left(\int_{-T}^{T} \int_{-T}^{T} I(t-s) X_t X_s \mathsf{d}t \mathsf{d}s \right) \right]. \tag{4.32}$$

Our correlation bound for the continuous Ising model is the following theorem.

Theorem 4.21. *For all $\varepsilon \in (0, \frac{1}{5})$, there exists $C_\varepsilon > 0$ such that for all continuous and even $I \in L^1(\mathbb{R})$ with $I \geq 0$ and $\|I\|_1 < \varepsilon$, we have*

$$\limsup_{T\to\infty} \frac{1}{T} \left[\!\!\left[\left(\int_{-T}^{T} X_t \mathsf{d}t \right)^2 \right]\!\!\right]_{I,T} \leq C_\varepsilon.$$

Remark 4.22. This result can easily be extended to an arbitrary intensity $\lambda > 0$ of the Poisson-distribution of the jump times, i.e., if (4.30) is replaced by

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \qquad \text{for } n \in \mathbb{N},\ t \geq 0.$$

Note that the constant C, however, is not independent of λ. This can be seen by a simple scaling argument.

Remark 4.23. Similar to Remark 4.4, the bound in Theorem 4.21 is a bound on the magnetic susceptibility of the Ising model. In presence of a constant magnetic field with strength $\mu \in \mathbb{R}$, the partition function is then given by

$$\mathcal{Z}_{I,T}^{(\mathsf{mag})}(\mu) = \mathbb{E}_X^{(T)} \left[\exp\left(\int_{-T}^{T} \int_{-T}^{T} I(t-s) X_t X_s \mathsf{d}t \mathsf{d}s + \mu \int_{-T}^{T} X_t \mathsf{d}t \right) \right].$$

The magnetization and magnetic susceptibility again are

$$\mathcal{M}_{I,T}(\mu) = \frac{1}{T} \partial_\mu \ln \mathcal{Z}_{I,T}^{(\mathsf{mag})}(\mu) \qquad \text{and} \qquad \mathcal{X}_{I,T}(\mu) = \partial_\mu \mathcal{M}_{I,T}(\mu).$$

Hence, we find

$$\mathcal{X}_{I,T}(0) = \frac{1}{T} \left[\!\!\left[\left(\int_{-T}^{T} X_t dt \right)^2 \right]\!\!\right]_{I,T},$$

which is the expression estimated uniformly in T in Theorem 4.21.

Remark 4.24. In the special case, where the integrable $I \geq 0$ satisfies the additional condition $I(t) \sim t^{-2}$ as $t \to \infty$, a bound as in Theorem 4.21 follows from [Spo89, Proposition 8.1] for the conditioned process with boundary conditions $X_T = X_{-T}$. The proof given in [Spo89] is based on results from percolation theory [AN86].

Remark 4.25. The bound in Theorem 4.21 is in general not expected to hold for arbitrary large $\varepsilon > 0$, as the following results indicate. The one-dimensional long-range Ising model with spins $\sigma_i = \pm 1$, $i \in \mathbb{Z}$ and interaction energy $\sum_{i,j} J(i-j)\sigma_i\sigma_j$ with $J(n) = n^{-\alpha}$ has a phase transition if $1 < \alpha \leq 2$. In that case the magnetic susceptibility diverges for sufficiently small temperatures. This was shown in [Dys69] for $1 < \alpha < 2$ and in [ACCN88] for $\alpha = 2$. It is reasonable to believe that such a divergence carries over to the continuous model, since the continuous model can be obtained by a scaling limit of the discrete model if an additional nearest neighbor coupling is imposed, cf. [SD85, Spo89] and Section 4.2.3.

In the remainder of this section, we will prove Theorem 4.21. In Section 4.2.1, we will recall the notion of weak convergence for measures and the Portmanteau theorem. Although these results are fairly standard in probability theory, we will give proofs for the aspects of interest for us, here. In Section 4.2.2, we will then define the path space of the stochastic process X and equip it with a suitable topology – the so-called *Skorokhod topology*. Then, we can give a characterization of weak convergence of a measure on this space. In Section 4.2.3, we will then use this formalism to prove that the continuous Ising model can in fact be obtained as the limit of a discrete one. This allows us to derive Theorem 4.21 from Theorem 4.1 in Section 4.2.4.

4.2.1. Weak Convergence of Probability Measures

In this section, we define the notion of weak convergence for measures and prove a part of the famous Portmanteau theorem. It is standard throughout the probability theory literature, see for example [Bil99, Kle20].

Definition 4.26. Let E be a metric space and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on E (with respect to the Borel sigma algebra on E). We say μ_n *weakly converges* to a probability measure μ on E and write $\mu = \operatorname*{w-lim}_{n\to\infty} \mu_n$ if

$$\int_E f \mathsf{d}\mu = \lim_{n\to\infty} \int_E f \mathsf{d}\mu_n \qquad \text{for all bounded and continuous functions } f : E \to \mathbb{R}.$$

The Portmanteau theorem contains a variety of equivalent statements for weak convergence, cf. [Kle20, Theorem 13.16]. We restrict ourselves to the equivalence, which we will employ. To that end, if $f : E \to \mathbb{R}$ is measurable, we denote by D_f the set of points at which f is discontinuous.

Theorem 4.27. *Let E be a metric space and let $(\mu_n)_{n\in\mathbb{N}}, \mu$ be probability measures on E. Then the following are equivalent:*

(i) $\mu = \operatorname*{w-lim}_{n\to\infty} \mu_n$.

(ii) $\int f \mathsf{d}\mu = \lim_{n\to\infty} \int f \mathsf{d}\mu_n$ *for all measurable and bounded functions f with $\mu(\mathsf{D}_f) = 0$.*

Proof. First, we note that the direction (ii)⇒(i) is trivial.

The proof of (i)⇒(ii) is divided into two steps.

Step 1. We show $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for any Borel set A with $\mu(\partial A) = 0$.

Let A be a Borel set satisfying $\mu(\partial A) = 0$. By the Urysohn lemma, for all $\varepsilon > 0$, we can pick a continuous function $\eta_\varepsilon : E \to [0,1]$ such that

$$\eta_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } \operatorname{dist}(A, \{x\}) > \varepsilon. \end{cases}$$

Since η_ε is continuous, the weak convergence of μ_n to μ and $\mu(\partial A) = 0$ imply

$$\limsup_{n\to\infty} \mu_n(A) \le \inf_{\varepsilon>0} \lim_{n\to\infty} \int \eta_\varepsilon \mathsf{d}\mu_n = \inf_{\varepsilon>0} \int \eta_\varepsilon \mathsf{d}\mu \le \inf_{\varepsilon>0} \mu\left(\bigcup_{x\in A} B_\varepsilon(x)\right) = \mu\left(\overline{A}\right) = \mu(A).$$

Replacing A by $E \setminus A$, we also find

$$\liminf_{n\to\infty} \mu_n(A) = 1 - \limsup_{n\to\infty} \mu_n(E \setminus A) \ge 1 - \mu(E \setminus A) = \mu(A).$$

This finishes the first step. ◊

Step 2. We now prove (ii).

Let $f : E \to \mathbb{R}$ be measurable and bounded and assume $\mu(\mathsf{D}_f) = 0$. W.l.o.g., we assume $f \ge 0$. Then, by the layer cake representation (cf. [LL01, Theorem 1.13], we have

$$\int_E f \mathsf{d}\nu = \int_0^\infty \nu\left(f^{-1}((t,\infty))\right) \mathsf{d}t \qquad \text{for any probability measure } \nu.$$

We note that $\partial f^{-1}((t,\infty)) \subset \{f^{-1}(\{t\}) \cup \mathsf{D}_f$. Since there are at most countably many $t \in [0,\infty)$ such that $\mu(f^{-1}(\{t\})) > 0$, this implies $\mu(\partial f^{-1}((t,\infty))) = 0$ for almost every $t \in \mathbb{R}$. Hence, applying Fatou's lemma and using Step 1, we find

$$\begin{aligned} \liminf_{n\to\infty} \int_E f \mathsf{d}\mu_n &= \liminf_{n\to\infty} \int_0^\infty \mu_n(f^{-1}((t,\infty)))\mathsf{d}t \\ &\ge \int_0^\infty \lim_{n\to\infty} \mu_n(f^{-1}((t,\infty)))\mathsf{d}t = \int_0^\infty \mu(f^{-1}((t,\infty))\mathsf{d}t = \int_E f \mathsf{d}\mu. \end{aligned}$$

Replacing f by $-f$ in above inequality finishes the proof. ◊ □

4.2.2. The Skorokhod Space

We now discuss the path space $\mathscr{D}_T$ of the stochastic process X and equip it with the Skorokhod metric. Then, we give a characterization of the weak convergence of probability measures on $\mathscr{D}_T$. This will be the key technical observation to use in the next section. Throughout this section, we fix a positive number $T > 0$.

Definition 4.28. We define $\mathscr{D}_T$ to be the set of all right-continuous $\omega : [-T,T] \to \{\pm 1\}$ with finitely many jumps. Further, let Φ_T denot the set of all continuous strictly increasing bijections $\varphi : [-T,T] \to [-T,T]$. Then, we define the *Skorokhod metric*

$$d(\omega,\nu) = \inf_{\varphi\in\Phi_T} \left(\|\varphi - \mathbb{1}\|_\infty + \|\omega - \nu \circ \varphi\|_\infty\right) \qquad \text{for } \omega,\nu \in \mathscr{D}_T.$$

The topology induced on $\mathscr{D}_T$ by d is called *Skorokhod topology*. Further, we equip $\mathscr{D}_T$ with the Borel σ-algebra.

We want to give a sufficient condition for a sequence of probability measures on $\mathscr{D}_T$ to converge weakly. To that end, for $k \in \mathbb{N}$ and $t \in [-T, T]^k$, we define the projections

$$\pi_t : \mathscr{D}_T \to \{\pm 1\}^k, \qquad \pi_t(\omega) = (\omega(t_1), \dots, \omega(t_k)) \text{ for all } \omega \in \mathscr{D}_T. \tag{4.33}$$

Further, let J_ω be the set of jumps of $\omega \in \mathscr{D}_T$. For $\varepsilon > 0$, we define the set

$$\Omega_\varepsilon = \{\omega \in \mathscr{D}_T : \exists t_1, t_2 \in (-T, T) : |t_2 - t_1| < \varepsilon, t_1, t_2 \in \mathsf{J}_\omega\}. \tag{4.34}$$

Also, if μ is a probability measure on $\mathscr{D}_T$, we define the set of all times for which the projection π_t is μ-almost everywhere continuous

$$\mathscr{T}_\mu = \{t \in [-T, T] : \mu\{\omega \in \mathscr{D}_T : t \in \mathsf{J}_\omega\} = 0\}. \tag{4.35}$$

We prove the following statement.

Theorem 4.29. *Let $(\mu_n)_{n\in\mathbb{N}}$, μ be probability measures on $\mathscr{D}_T$. Assume the following conditions hold:*

(i) $\mu \circ \pi_t^{-1} = \operatorname*{w-lim}_{n\to\infty} \mu_n \circ \pi_t^{-1}$ *for any* $t \in \mathscr{T}_\mu^k$, $k \in \mathbb{N}$.

(ii) $\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \mu_n(\Omega_\varepsilon) = 0$.

Then $\mu = \operatorname*{w-lim}_{n\to\infty} \mu_n$.

The criterion (i), which describes the weak convergence of probability measures on the discrete set $\{\pm 1\}^k$, is called the convergence of moments or finite dimensional distributions. For the proof of the above statement, we need to observe that (ii) is a translation of the notion of tightness for our space $\mathscr{D}_T$.

Definition 4.30. A family Π of probability measures on a topological space equipped with its Borel σ-algebra is called *tight* if for any $\varepsilon > 0$ there exists a compact set K such that

$$\mu(K) > 1 - \varepsilon \qquad \text{for all } \mu \in \Pi.$$

Further, it is called *relatively compact* if any sequence $(\mu_n)_{n\in\mathbb{N}} \subset \Pi$ has a weakly convergent subsequence.

The following statement is the famous Prohorov theorem. Proofs can be found in [Bil99, Theorem 5.1] or [Kle20, Theorem 13.29].

Proposition 4.31 (Prohorov's theorem)**.** *A tight family Π of probability measures on a metric space equipped with its Borel σ-algebra is relatively compact.*

Remark 4.32. Prohorov's theorem in the literature also contains a reverse statement. If the metric space is separable and complete, then any relatively compact family of probability measures is tight, cf. [Bil99, Theorem 5.2].

We want to characterize tightness on $\mathscr{D}_T$. To that end, we first give a characterization of compact sets.

Lemma 4.33. *If there exists $\delta > 0$ such that $\mathscr{K} \subset \Omega_\delta^c$, then $\mathscr{K}$ is relatively compact, i.e., its closure is compact. Especially, the sets Ω_ε^c for any $\varepsilon > 0$ are compact.*

Remark 4.34. In fact, the above characterization of relatively compact sets is necessary and sufficient, cf. [Bil99, Theorem 12.3].

Proof. It suffices to prove the especially part of the statement. To that end, fix $\varepsilon > 0$ and let $(\omega_n)_{n\in\mathbb{N}} \subset \Omega_\varepsilon^c$. Since any $\omega \in \Omega_\varepsilon^c$ can have at most $\lfloor T/\varepsilon \rfloor$ discontinuities, we can w.l.o.g assume that all ω_n have the same number $N \in \mathbb{N}$ of discontinuities and the same value at $-T$ (otherwise restrict to a subsequence). For $\omega = \omega_n$ for some $n \in \mathbb{N}$, we write $\mathsf{J}_\omega = \{t_\omega^{(1)}, \dots, t_\omega^{(N)}\}$ and assume $t_\omega^{(1)} \le \cdots \le t_\omega^{(N)}$. By the Bolzano-Weierstrass theorem, we can pick a subsequence $(\omega_{n_k})_{k\in\mathbb{N}}$ such that $((t_{\omega_{n_k}}^{(1)}, \dots, t_{\omega_{n_k}}^{(N)}))_k \subset [-T,T]^N$ converges to some vector $(t^{(1)}, \dots, t^{(N)}) \in [-T,T]^N$. Further, by construction $t^{(k)} - t^{(k-1)} \ge \varepsilon$ for all $k = 2, \dots, N$. Hence, it is straightforward to check that ω_{n_k} converges to the element $\omega_\infty \in \Omega_\varepsilon^c$ with the same starting value $\omega_\infty(-T)$ and discontinuities exactly at the positions $t^{(1)}, \dots, t^{(N)}$. This finishes the proof. □

This gives us the following characterization of tightness for probability measures on $\mathscr{D}_T$.

Lemma 4.35. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathscr{D}_T$ is tight if*

$$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \mu_n(\Omega_\varepsilon) = 0.$$

Remark 4.36. As above, this characterization is necessary and sufficient, cf. [Bil99, Theorem 13.2].

Proof. The statement directly follows by combining Definition 4.30 and Lemma 4.33. □

To conclude the proof of Theorem 4.29, we need to show that convergence of the finite dimensional distributions $\mu_n \circ \pi_t^{-1}$ to $\mu \circ \pi_t^{-1}$ for $t \in \mathscr{T}_\mu^k$ combined with tightness of the sequence $(\mu_n)_{n\in\mathbb{N}}$ yields weak convergence of μ_n to μ. In the proof, we will need the following characterization of $\mathscr{T}_\mu$.

Lemma 4.37. *Let μ be a probability measure on $\mathscr{D}_T$. Then the set $\mathscr{T}_\mu$ is cocountable and $\{0,1\} \subset \mathscr{T}_\mu$.*

Proof. First, we observe that $\mathscr{T}_\mu$ is the set of $t \in [0,1]$ such that π_t is μ-almost everywhere continuous. Since π_0 and π_1 are everywhere continuous, $0, 1 \in \mathscr{T}_\mu$ follows. Further, we note that $t \in (0,1) \setminus \mathscr{T}_\mu$ if and only if the set $\mathscr{J}_t = \{\omega \in \mathscr{D}_T : t \in \mathsf{J}_\omega\}$ has positive measure $\mu(\mathscr{J}_t) > 0$. If $\delta > 0$, there are only finitely many $t \in (0,1)$ with $\mu(\mathscr{J}_t) > 0$. Otherwise, $\mu\big(\bigcap_{t:\mu(\mathscr{J}_t)>0} \mathscr{J}_t\big) \ge \delta > 0$ would yield the existence of a path $\omega \in \mathscr{D}_T$ with infinitely many jumps, which is a contradiction to the definition of $\mathscr{D}_T$. Hence, there can be only countably many elements in $\mathscr{T}_\mu^c$. □

We can now deduce the following.

Lemma 4.38. *Let $(\mu_n)_{n\in\mathbb{N}}$ be a tight sequence of probability measures on $\mathscr{D}_T$ and let μ also be a probability measure on $\mathscr{D}_T$. Further, assume $\mu_n \circ \pi_t^{-1}$ weakly converges to $\mu \circ \pi_t^{-1}$ for any $t \in \mathscr{T}_\mu^k$, $k \in \mathbb{N}$. Then μ_n weakly converges to μ.*

Proof. This proof follows the lines of [Bil99, Theorem 13.1].

First, assume we have a weakly convergent subsequence $(\mu_{n_m})_{m\in\mathbb{N}}$. We denote its limit by $\widetilde{\mu}$. Since π_t is $\widetilde{\mu}$-almost everywhere continuous for $t \in \mathscr{T}_{\widetilde{\mu}}^k$, $k \in \mathbb{N}$, it follows from Theorem 4.27 that $\mu_{n_m} \circ \pi_t^{-1}$ weakly converges to $\widetilde{\mu} \circ \pi_t^{-1}$ for all $t \in \mathscr{T}_{\widetilde{\mu}}^k$, $k \in \mathbb{N}$. By the assumptions, this directly implies $\widetilde{\mu}\circ\pi_t^{-1}=\mu\circ\pi_t^{-1}$ for all $t\in(\mathscr{T}_\mu\cap\mathscr{T}_{\widetilde{\mu}})^k$, $k\in\mathbb{N}$. Now, observe that $\{\pi_t : t\in\mathcal{T}^k\}$ separates points in $\mathscr{D}_T$ whenever $\mathcal{T}$ is dense in $[-T,T]$, i.e., $\pi_t(\omega)=\pi_t(\widetilde{\omega})$ for all $t\in\mathcal{T}^k$ implies $\omega=\widetilde{\omega}$. Further, by Lemma 4.37, $\mathscr{T}_\mu\cap\mathscr{T}_{\widetilde{\mu}}$ is cocountable and hence dense in $[-T,T]$. This implies $\mu=\widetilde{\mu}$.

Now, by Prohorov's theorem (Proposition 4.31), any subsequence of (μ_n) has a weakly convergent subsequence. Hence, by the above considerations, it weakly converges to μ. This proves the statement. □

Proof of Theorem 4.29. The statement follows from Lemmas 4.35 and 4.38. □

4.2.3. The Continuum Limit of the Discrete Ising Model

In this section, we prove that the jump process X as defined in Definition 4.16 is the continuum limit of a discrete Ising model. The approach we use is based on the description in [SD85, Spo89]. To that end, we fix a $T>0$ throughout this section and use a parameter $\delta\in(0,\infty)$ as lattice spacing of the discrete Ising model. We define the map

$$\mathfrak{i}_\delta:\mathbb{R}\to\mathbb{N},\qquad t\mapsto\left\lfloor\frac{t}{\delta}+\frac{1}{2}\right\rfloor, \tag{4.36}$$

where $\lfloor\cdot\rfloor$ as usually denotes the integer part. We note that the interval $[-T,T]$ is mapped to the lattice $\Lambda_{L_\delta(T)}$ with length $L_\delta(T)=\mathfrak{i}_\delta(T)$. We set the nearest neighbor interaction on this lattice to be

$$j_\delta=-\frac{1}{2}\ln(\delta). \tag{4.37}$$

For a given even and continuous function $I:\mathbb{R}\to\mathbb{R}$, we define the corresponding pair interaction (cf. (4.6)) on the lattice as $w^{(\delta)}=(w_k^{(\delta)})_{k\in\mathbb{N}}$ with

$$w_k^{(\delta)}=\delta\int_{\delta(k-1)}^{\delta k}I(t)\mathsf{d}t. \tag{4.38}$$

We write the expectation values in the discrete Ising model given with these interactions as

$$\langle\!\langle\cdot\rangle\!\rangle_{\delta,T}^{(\mathrm{n})}:=\langle\cdot\rangle_{J_{(j_\delta,0,\dots)}}^{(L_\delta(T))}\qquad\text{and}\qquad\langle\!\langle\cdot\rangle\!\rangle_{\delta,T}:=\langle\cdot\rangle_{J_{(j_\delta,0,\dots)+w^{(\delta)}}}^{(L_\delta(T))}. \tag{4.39}$$

In this section we prove the following theorem.

Theorem 4.39. *Let $I:\mathbb{R}\to\mathbb{R}$ be even and continuous and let $T>0$. Then, for $-T\le t_1\le\cdots\le t_N\le T$, we have*

$$\lim_{\delta\downarrow0}\langle\!\langle\sigma_{\mathfrak{i}_\delta(t_1)}\cdots\sigma_{\mathfrak{i}_\delta(t_N)}\rangle\!\rangle_{\delta,T}=[\![X_{t_1}\cdots X_{t_N}]\!]_{I,T}\,.$$

As a first step of our proof, we need the following lemma.

Lemma 4.40. *Let $N\in\mathbb{N}$ and assume $-T\le t_1\le\cdots\le t_N\le T$. Then*

$$\mathbb{E}_X^{(T)}[X_{t_1}\cdots X_{t_N}]=e^{-2(|t_2-t_1|+\cdots+|t_N-t_{N-1}|)}\qquad\text{if } N \text{ is even}$$

and $\mathbb{E}_X^{(T)}[X_{t_1}\cdots X_{t_N}]=0$ if N is odd.

Proof. This proof is analogous to [Abd11, Lemma 1], where the construction from Remark 4.18 is used.

First, we observe that by construction the random variables X_{-T}, X_tX_s and X_uX_v are independent if $-T \leq t \leq s \leq u \leq v \leq T$ and that

$$\mathbb{E}_X^{(T)}[X_tX_s] = e^{-2(s-t)}.$$

This yields

$$\mathbb{E}_X^{(T)}\left[X_{t_1}\cdots X_{t_N}\right] = \mathbb{E}_X^{(T)}\left[(X_0)^N\right]\mathbb{E}_X^{(T)}\left[(X_{X_0X_{t_1}})^N\right]\prod_{k=1}^{N-1}\mathbb{E}_X^{(T)}\left[(X_{t_k}X_{t_{k+1}})^{N-k}\right].$$

By Definition 4.16, we have $\mathbb{E}_X^{(T)}[(X_{-T})^N] = 0$ if N is odd. Further, in the case that N is even, we obtain

$$\mathbb{E}_X^{(T)}\left[X_{t_1}\cdots X_{t_N}\right] = \prod_{k=1}^{N/2}\mathbb{E}_X^{(T)}\left[X_{t_{2k}}X_{t_{2k-1}}\right] = \prod_{k=1}^{N/2} e^{-2(t_{2k}-t_{2k-1})} = e^{-2\sum_{k=1}^{N/2}(t_{2k}-t_{2k-1})}. \qquad \square$$

We now use our explicit calculations on expectation values of discrete Ising models only with nearest neighbor interaction from Section 4.1.4 to prove the case $I = 0$ in Theorem 4.39.

Lemma 4.41. *Let $N \in \mathbb{N}$ and assume $-T \leq t_1 \leq \cdots \leq t_N \leq T$. Then*

$$\lim_{\delta\downarrow 0}\langle\!\langle \sigma_{\mathfrak{i}_\delta(t_1)}\cdots\sigma_{\mathfrak{i}_\delta(t_N)}\rangle\!\rangle_{\delta,T}^{(\mathrm{n})} = \mathbb{E}_X^{(T)}[X_{t_1}\cdots X_{t_N}].$$

Proof. If N is odd both sides vanish (Lemmas 4.12 and 4.41), so we assume N is even. Then, the definition (4.36) and Lemma 4.12 (iii) yield

$$\langle\!\langle \sigma_{\mathfrak{i}_\delta(t_1)}\cdots\sigma_{\mathfrak{i}_\delta(t_N)}\rangle\!\rangle_{\delta,T}^{(\mathrm{n})} = (\tanh j_\delta)^{|\mathfrak{i}_\delta(t_2)-\mathfrak{i}_\delta(t_1)|+\cdots+|\mathfrak{i}_\delta(t_N)-\mathfrak{i}_\delta(t_{N-1})|}.$$

Since (4.36) also yields $\frac{|u-v|}{\delta} - 1 \leq |\mathfrak{i}_\delta(u) - \mathfrak{i}_\delta(v)| \leq \frac{|u-v|}{\delta} + 1$ for all $u, v \in \mathbb{R}$, we obtain

$$\begin{aligned}\left[(\tanh j_\delta)^{\delta^{-1}}\right]^{|t_2-t_1|+\cdots+|t_N-t_{N-1}|-\delta N} &\leq \langle\!\langle \sigma_{\mathfrak{i}_\delta(t_1)}\cdots\sigma_{\mathfrak{i}_\delta(t_N)}\rangle\!\rangle_{\delta,T}^{(\mathrm{n})} \\ &\leq \left[(\tanh j_\delta)^{\delta^{-1}}\right]^{|t_2-t_1|+\cdots+|t_N-t_{N-1}|+\delta N}.\end{aligned}$$

Using $\lim_{\delta\downarrow 0}(\tanh j_\delta)^{\delta^{-1}} = e^{-2}$, the statement follows by Lemma 4.40. $\square$

To show Theorem 4.39 for nonzero I, we will use the notion of weak convergence of measures on the Skorokhod space as outlined in Sections 4.2.1 and 4.2.2. Explicitly, we will make use of the Portmanteau theorem (Theorem 4.27) and the sufficient condition for weak convergence from Theorem 4.29.

For the statement, we define $\mathfrak{s}_\delta : \mathcal{S}_{L_\delta(T)} \to \mathscr{D}_T$ by $\mathfrak{s}_\delta(\sigma) = \left[t \mapsto \sigma_{\mathfrak{i}_\delta(t)}\right]$.

Lemma 4.42. *Let $f : \mathscr{D}_T \to \mathbb{R}$ be bounded and continuous, let $N \in \mathbb{N}_0$ and assume $-T \leq t_1 \leq \cdots \leq t_N \leq T$. Then*

$$\lim_{\delta\downarrow 0}\left\langle\!\!\left\langle \sigma_{\mathfrak{i}_\delta(t_1)}\cdots\sigma_{\mathfrak{i}_\delta(t_N)} f(\mathfrak{s}_\delta(\sigma))\right\rangle\!\!\right\rangle_{\delta,T}^{(\mathrm{n})} = \mathbb{E}[X(t_1)\cdots X(t_N)f(X)].$$

Proof. First, we note that there exists a probability measure P_X on $\mathscr{D}_T$ such that for $\omega \in \mathscr{D}_T$ the jump process is given by $X_t(\omega) = \omega(t)$ for $t \in [-T, T]$ and for any measurable function $f : \mathscr{D}_T \to \mathbb{R}$

$$\mathbb{E}_X^{(T)}[f(X)] = \int_{\mathscr{D}_T} f(\omega) \mathsf{d} P_X(\omega).$$

Further, for $\delta > 0$, let P_δ be the pushforward measure on $\mathscr{D}_T$ obtained from the Ising probability measure on $\mathcal{S}_{L_\delta(T)}$ through the (obviously measurable) map $\mathfrak{s}_\delta$, i.e.,

$$P_\delta(A) = \sum_{\sigma \in \mathfrak{s}_\delta{}^{-1}(A)} \frac{e^{-E_{J_{j_\delta}, L_\delta(T)}(\sigma)}}{Z_{J_{j_\delta}, L_\delta(T)}} \qquad \text{for all measurable sets } A \subset \mathscr{D}_T. \tag{4.40}$$

This implies

$$\left\langle\!\left\langle \sigma_{\mathfrak{i}_\delta(t_1)} \cdots \sigma_{\mathfrak{i}_\delta(t_N)} f(\mathfrak{s}_\delta(\sigma)) \right\rangle\!\right\rangle_{\delta,T}^{(\mathrm{n})} = \int \omega(t_1) \cdots \omega(t_N) f(\omega) \mathsf{d} P_\delta(\omega).$$

We want to prove that P_δ weakly converges to P_X as $\delta \downarrow 0$ using Theorem 4.29.

First, we note that for any $k \in \mathbb{N}$ and $t \in [-T, T]^k$, the expectation values in Lemma 4.41 uniquely determine the probability measures $P_\delta \circ \pi_t^{-1}$ and $P_X \circ \pi_t^{-1}$ (cf. (4.33)), respectively, since every function on the set $\{\pm 1\}$ is given as a linear combination of the constant function one and the identity function. Hence, Lemma 4.41 implies the weak convergence of $P_\delta \circ \pi_t^{-1}$ to $P_X \circ \pi_t^{-1}$ as $\delta \downarrow 0$, so the condition (i) from Theorem 4.29 is satisfied.

Hence, it remains to prove

$$\lim_{\varepsilon \downarrow 0} \limsup_{\delta \downarrow 0} P_\delta(\Omega_\varepsilon) = 0., \tag{4.41}$$

To prove (4.41), we will assume

$$0 < \delta < \varepsilon < \min\{1, T\}. \tag{4.42}$$

For $\sigma \in \mathcal{S}_{L_\delta(T)}$, we denote by n_σ the number of sign changes (cf. Lemma 4.12 (i)). We observe that $\mathfrak{s}_\delta(\sigma) \in \Omega_\varepsilon$ if $n_\sigma > 2T/\varepsilon$. Otherwise, $\mathfrak{s}_\delta(\sigma) \notin \Omega_\varepsilon$ if and only if all sign changes have a distance of at least ε/δ. If $n_\sigma = k$ for some fixed $k \in \mathbb{N}$, then simple combinatorics yield that there are $\binom{2L_\delta(T) - (k-1)\lfloor \varepsilon/\delta \rfloor}{k}$ possibilities to position the sign changes such that all distances are larger than ε/δ.[1] Taking into account that an element $\sigma \in \mathcal{S}_{L_\delta(T)}$ is uniquely determined by the choice of the value $\sigma_{L_\delta(T)} \in \{\pm 1\}$ and the position of its sign changes, we obtain

$$\begin{aligned} &\left| \left\{ \sigma \in \mathfrak{s}_\delta{}^{-1}(\Omega_\varepsilon) : n_\sigma = k \right\} \right| \\ &\quad = \begin{cases} 2\binom{2L_\delta(T)}{k} & \text{for } k > \frac{2T}{\varepsilon}, \\ 2\binom{2L_\delta(T)}{k} - 2\binom{2L_\delta(T) - (k-1)\lfloor \varepsilon/\delta \rfloor}{k} & \text{for } 2 \le k \le \frac{2T}{\varepsilon}. \end{cases} \end{aligned} \tag{4.43}$$

[1] Explicitly, the combinatorial argument is as follows: In a chain of $N+1$ Ising spins, there are $\binom{N}{k}$ possibilities to position k sign changes. This is equal to the number of possibilities to choose $k+1$ positive integers $x_1, \ldots, x_{k+1}$ such that $x_1 + \cdots + x_{k+1} = N + 1$. Now, if the distance between any two sign changes shall be larger than m, this is equivalent to requiring $x_2, \ldots, x_k > m$. By the change of variables $y_1 = x_1$, $y_i = x_i - m$ for $i = 2, \ldots, k$, $y_{k+1} = x_{k+1}$, we find the number of possibilities to be equal to the number of possibilities to choose $y_1, \ldots, y_{k+1} \in \mathbb{N}$, such that $y_1 + \ldots + y_{k+1} = N + 1 - (k-1)m$. Recalling the initial argument, this is $\binom{N-(k-1)m}{k}$. In our case, we have $N = 2L_\delta(T)$ and $m = \lfloor \varepsilon/\delta \rfloor$.

From the definition of nearest neighbor coupling, it easily follows that (cf. Lemma 4.12 (i))

$$E_{J_{j_\delta},L_\delta(T)}(\sigma) = 2j_\delta(n_\sigma - L_\delta(T)) \qquad \text{for all } \sigma \in \mathcal{S}_{L_\delta(T)}.$$

Hence, combining (4.40) and (4.43) and summing over all possible numbers of spin changes, we obtain

$$\begin{aligned} P_\delta(\Omega_\varepsilon) = \sum_{k=2}^{\lfloor \frac{2T}{\varepsilon} \rfloor} & \left(\binom{2L_\delta(T)}{k} - \binom{2L_\delta(T) - (k-1)\lfloor \varepsilon/\delta \rfloor}{k} \right) \frac{2e^{2j_\delta(L_\delta(T)-k)}}{Z_{J_{j_\delta},L_\delta(T)}} \\ & + \sum_{k=\lfloor \frac{2T}{\varepsilon} \rfloor + 1}^{2L_\delta(T)} \binom{2L_\delta(T)}{k} \frac{2e^{2j_\delta(L_\delta(T)-k)}}{Z_{J_{j_\delta},L_\delta(T)}}. \end{aligned} \tag{4.44}$$

Since it is also possible to explicitly calculate the partition function for nearest neighbor coupling (cf. Lemma 4.12 (ii)), we have

$$\frac{2e^{2j_\delta L_\delta(T)}}{Z_{J_{j_\delta},L_\delta(T)}} = \left(\frac{e^{j_\delta}}{e^{j_\delta} + e^{-j_\delta}} \right)^{2L_\delta(T)} < 1. \tag{4.45}$$

Moreover, inserting the definition (4.37), we have $e^{-2j_\delta k} = \delta^k$ and hence

$$\binom{2L_\delta(T)}{k} e^{-2j_\delta k} \le \frac{(2L_\delta(T))^k}{k!} \delta^k \le \frac{(2T+\delta)^k}{k!} \qquad \text{for all } k \le 2L_\delta(T), \tag{4.46}$$

where we used $L_\delta(T) = \lfloor \frac{T}{\delta} + \frac{1}{2} \rfloor \le \frac{T}{\delta} + \frac{1}{2}$ in the last step. Now, from (4.42) it follows that

$$0 \le \frac{k}{2L_\delta(T)}(\varepsilon/\delta + 1) < 1 \qquad \text{if } k \in \mathbb{N} \text{ satisfies } k \le \frac{T}{2\varepsilon}.$$

Hence, along the same lines as the proof of (4.46), we can use Bernoulli's inequality for $k \le T/(2\varepsilon)$ and obtain

$$\begin{aligned} \left(\binom{2L_\delta(T)}{k} - \binom{2L_\delta(T) - (k-1)\lfloor \varepsilon/\delta \rfloor}{k} \right) & e^{-2j_\delta k} \\ & \le \frac{(2L_\delta(T))^k - (2L_\delta(T) - k(\varepsilon/\delta + 1))^k}{k!} \delta^k \\ & \le \frac{(2L_\delta(T))^k}{k!} \frac{k^2(\varepsilon/\delta + 1)}{2L_\delta(T)} \delta^k \\ & \le \frac{(2T+\delta)^{k-1}}{(k-1)!} k(\varepsilon + \delta). \end{aligned} \tag{4.47}$$

We can now insert (4.45), (4.46) and (4.47) into (4.44). Hence, for any $s_\varepsilon \in [0, \frac{1}{2\varepsilon}]$, we have

$$\begin{aligned} P_\delta(\Omega_\varepsilon) & \le \sum_{k=2}^{\lfloor T s_\varepsilon \rfloor} \frac{(2T+\delta)^{k-1}}{(k-1)!} k(\varepsilon+\delta) + \sum_{k=\lfloor T s_\varepsilon \rfloor + 1}^{2L_\delta(T)} \frac{(2T+\delta)^k}{k!} \\ & \le T s_\varepsilon (\varepsilon + \delta) e^{2T+\delta} + \sum_{k=\lfloor T s_\varepsilon \rfloor + 1}^{\infty} \frac{(2T+\delta)^k}{k!}, \end{aligned}$$

where we estimated the first half of the first sum in (4.44) by (4.47) and the second half using (4.46). Taking the limit $\delta \downarrow 0$, we observe

$$\limsup_{\delta\downarrow 0} P_\delta(\Omega_\varepsilon) \le T s_\varepsilon \varepsilon e^{2T} + \sum_{k=\lfloor T s_\varepsilon \rfloor +1}^{\infty} \frac{(2T)^k}{k!}.$$

We choose s_ε such that both $\lim_{\varepsilon\downarrow 0} s_\varepsilon = \infty$ and $\lim_{\varepsilon\downarrow 0} \varepsilon s_\varepsilon = 0$ hold, e.g., $s_\varepsilon = \frac{1}{2}\varepsilon^{-1/2}$. Then the summability of the second term proves (4.41) and hence Theorem 4.29 implies that P_δ weakly converges to P_X.

Since f is bounded and continuous, the statement for $N = 0$ directly follows from the definition of weak convergence. Further, observe that for any fixed $N \in \mathbb{N}$ and $t \in \mathbb{R}^N$ the function $\omega \to \pi_t(\omega) f(\omega)$ is only discontinuous at those ω having jumps exactly at the points given by the N-tuple t. Hence, the set of discontinuities has P_X-measure zero and the statement follows from Theorem 4.27. □

We apply above lemma to prove the expectation value in Theorem 4.39 is a limit of expectation values in the nearest neighbor Ising model.

Lemma 4.43. *Assume $I : [-T, T] \to \mathbb{R}$ is even and continuous and $w^{(\delta)}$ is as defined in (4.38). For $N \in \mathbb{N}_0$, let $-T \le t_1 \le \cdots \le t_N \le T$. Then*

$$\lim_{\delta\downarrow 0} \left\langle\!\!\left\langle \sigma_{\mathrm{i}_\delta(t_1)} \cdots \sigma_{\mathrm{i}_\delta(t_N)} \exp\left(\sum_{i,j\in\Lambda_{L_\delta(T)}} w^{(\delta)}_{|i-j|} \sigma_i \sigma_j \right) \right\rangle\!\!\right\rangle^{(\mathrm{n})}_{\delta,T}$$
$$= \mathbb{E}^{(T)}_X \left[X_{t_1} \cdots X_{t_N} \exp\left(\int_{-T}^{T} \int_{-T}^{T} I(t-s) X_s X_t \mathsf{d}s\mathsf{d}t \right) \right].$$

Proof. We define $f_0 : \mathscr{D}_T \to \mathbb{R}$ by

$$f_0(\omega) = \omega(t_1) \cdots \omega(t_N) e^{\mathcal{I}_0(\omega)}, \qquad \text{where} \quad \mathcal{I}_0(\omega) = \int_{-T}^{T} \int_{-T}^{T} I(t-s)\omega(t)\omega(s) \mathsf{d}s\mathsf{d}t.$$

It is straightforward to verify that $\mathcal{I}_0 : \mathscr{D}_T \to \mathbb{R}$ is bounded and continuous. Hence, we can apply Lemma 4.42 and obtain

$$\lim_{\delta\downarrow 0} \langle\!\langle f_0(\mathfrak{s}_\delta(\sigma)) \rangle\!\rangle^{(\mathrm{n})}_{\delta,T} = \mathbb{E}^{(T)}_X [f_0(X)]. \tag{4.48}$$

It remains to consider the left hand side and to analyze $f_0(\mathfrak{s}_\delta(\sigma))$. Further, for $\sigma \in \mathcal{S}_{L_\delta(T)}$, we define

$$g_\delta(\sigma) = \sigma_{\mathrm{i}_\delta(t_1)} \cdots \sigma_{\mathrm{i}_\delta(t_N)} e^{\mathcal{J}_\delta(\sigma)}, \qquad \text{where} \quad \mathcal{J}_\delta(\sigma) = \sum_{i,j\in\Lambda_{L_\delta(T)}} w^{(\delta)}_{|i-j|} \sigma_i \sigma_j.$$

Since continuous functions on compact intervals are uniformly continuous, there exists a $\delta_\varepsilon > 0$ for any $\varepsilon > 0$ such that

$$|I(t) - I(s)| < \varepsilon \qquad \text{for any } t, s \in [-T, T] \text{ with } |t-s| < \delta_\varepsilon. \tag{4.49}$$

Further, for any fixed $\delta > 0$, there exist $(\eta^{(\delta)}_{i,j})_{i,j\in\Lambda_{L_\delta(T)}} \subset [0,1]$ such that

$$w^{(\delta)}_{|i-j|} = \delta^2 I(\delta(|i-j| - \eta^{(\delta)}_{i,j})),$$

by the mean value theorem for integrals and the definition (4.38). Hence, for any $\delta > 0$ and $\sigma \in \mathcal{S}_{L_\delta(T)}$, we find

$$\begin{aligned} |\mathcal{J}_\delta(\sigma) - \mathcal{I}_0(\mathfrak{s}_\delta(\sigma))| &= \left| \sum_{i,j\in\Lambda_{L_\delta(T)}} \left[w^{(\delta)}_{|i-j|}\sigma_i\sigma_j - \int_{(i-\frac{1}{2})\delta}^{(i+\frac{1}{2})\delta} \int_{(j-\frac{1}{2})\delta}^{(j+\frac{1}{2})\delta} I(t-s)\sigma_i\sigma_j \mathsf{d}s\mathsf{d}t \right] \right| \\ &\leq \sum_{i,j\in\Lambda_{L_\delta(T)}} \delta^2 \sup\left\{ |I(\delta t) - I(\delta(|i-j| - \eta^{(\delta)}_{i,j}))| : t \in [|i-j|-1, |i-j|+1] \right\} \end{aligned} \tag{4.50}$$

Combining (4.49) and (4.50) as well as $L_\delta(T) \leq \frac{T}{\delta} + \frac{1}{2}$, for all $\delta \in (0, \delta_\varepsilon)$ and $\sigma \in \mathcal{S}_{L_\delta(T)}$, we have

$$|\mathcal{J}_\delta(\sigma) - \mathcal{I}_0(\mathfrak{s}_\delta(\sigma))| \leq (2L_\delta(T)+1)^2\delta^2\varepsilon \leq 4(T+\delta)^2\varepsilon. \tag{4.51}$$

Now, for all $\sigma \in \mathcal{S}_{L_\delta(T)}$, we have the algebraic identity

$$g_\delta(\sigma) - f_0(\mathfrak{s}_\delta(\sigma)) = f_0(\mathfrak{s}_\delta(\sigma)) \left(e^{\mathcal{J}_\delta(\sigma) - \mathcal{I}_0(\mathfrak{s}_\delta(\sigma))} - 1 \right).$$

Using this identity and (4.51), it follows that there exist constants C_1 and C_2 such that, for $\varepsilon > 0$ sufficiently small, $\delta \in (0, \delta_\varepsilon)$ and all $\sigma \in \mathcal{S}_{L_\delta(T)}$,

$$|g_\delta(\sigma) - f_0(\mathfrak{s}_\delta(\sigma))| \leq |f_0(\mathfrak{s}_\delta(\sigma))| C_1 \, |\mathcal{J}_\delta(\sigma) - \mathcal{I}_0(\mathfrak{s}_\delta(\sigma))| \leq C_2 e^{4T^2\|I\|_\infty}(T+\delta)^2\varepsilon.$$

Since $\sigma \in \mathcal{S}_{L_\delta(T)}$ was arbitrary, this estimate also holds for the expectation value, i.e.,

$$\left| \langle\!\langle g_\delta(\sigma) \rangle\!\rangle^{(\mathrm{n})}_{\delta,T} - \langle\!\langle f_0(\mathfrak{s}_\delta(\sigma)) \rangle\!\rangle^{(\mathrm{n})}_{\delta,T} \right| \leq C_2 e^{4T^2\|I\|_\infty}(T+\delta)^2\varepsilon. \tag{4.52}$$

Combining (4.48) and (4.52), the statement follows. □

It now remains to rewrite the Ising expectation value in above lemma as a correlation function.

Proof of Theorem 4.39. By the definition (4.4), we observe

$$\langle\!\langle \sigma_{\mathrm{i}_\delta(t_1)} \cdots \sigma_{\mathrm{i}_\delta(t_N)} \rangle\!\rangle_{\delta,T} = \frac{\left\langle\!\!\left\langle \sigma_{\mathrm{i}_\delta(t_1)} \cdots \sigma_{\mathrm{i}_\delta(t_N)} \exp\left(\sum_{i,j\in\Lambda_{L_\delta(T)}} w^{(\delta)}_{|i-j|}\sigma_i\sigma_j \right) \right\rangle\!\!\right\rangle^{(\mathrm{n})}_{\delta,T}}{\left\langle\!\!\left\langle \exp\left(\sum_{i,j\in\Lambda_{L_\delta(T)}} w^{(\delta)}_{|i-j|}\sigma_i\sigma_j \right) \right\rangle\!\!\right\rangle^{(\mathrm{n})}_{\delta,T}}.$$

Hence, the statement follows from Lemma 4.43 and the definition (4.31), (4.32). □

4.2.4. Proof of the Correlation Bound

We conclude this chapter, by combining Theorems 4.1 and 4.39 to the proof of Theorem 4.21. In the following, we use the definitions and notation from the previous section.

Proof of Theorem 4.21. Fix $T > 0$. Then using Fubini's theorem in the first equality and Theorem 4.39 in the second equality, we find

$$\begin{aligned}
&\frac{1}{\mathcal{Z}(I,T)}\mathbb{E}_X^{(T)}\left[\frac{1}{T}\left(\int_{-T}^{T} X_t dt\right)^2 \exp\left(\int_{-T}^{T}\int_{-T}^{T} X_t X_s I(t-s)\mathsf{d}t\mathsf{d}s\right)\right] \\
&\quad = \frac{1}{T\mathcal{Z}(I,T)}\int_{-T}^{T}\int_{-T}^{T}\mathbb{E}\left[X_u X_v \exp\left(\int_{-T}^{T}\int_{-T}^{T} X_t X_s I(t-s)\mathsf{d}t\mathsf{d}s\right)\right]\mathsf{d}u\mathsf{d}v \\
&\quad = \frac{1}{T}\lim_{\delta\downarrow 0}\int_{-T}^{T}\int_{-T}^{T}\langle\!\langle \sigma_{\mathrm{i}_\delta(u)}\sigma_{\mathrm{i}_\delta(v)}\rangle\!\rangle_{\delta,T}\mathsf{d}u\mathsf{d}v \\
&\quad = \lim_{\delta\downarrow 0}\frac{1}{T}\sum_{i,j\in\Lambda_{L_\delta(T)}}\delta^2\langle\!\langle\sigma_i\sigma_j\rangle\!\rangle_{\delta,T},
\end{aligned} \tag{4.53}$$

where we calculated the integral in the last step using that the integrand is a step function. To estimate (4.53), we want to use Theorem 4.1. First, we observe that by the definition (4.37)

$$\frac{1}{1-\tanh j_\delta} = \frac{e^{2j_\delta}+1}{2} < \frac{1}{\delta} \qquad \text{for any } \delta\in(0,1). \tag{4.54}$$

Further, using the definition (4.38), the fact that $I \in L^1(\mathbb{R})$ and that I is even, we have

$$w^{(\delta)} \in \ell^1 \qquad \text{and} \qquad \frac{2}{\delta}\|w^{(\delta)}\|_1 = \|I\|_1 \qquad \text{for all } \delta > 0.$$

Combining the above two relations, we find that for any constant $D > 1$ there exists $\delta_D > 0$ such that

$$\frac{\|w^{(\delta)}\|_1}{1-\tanh j_\delta} \le \frac{D}{2}\|I\|_1 \qquad \text{for all } \delta\in(0,\delta_D). \tag{4.55}$$

Now let $\varepsilon \in (0,\frac{1}{10})$ and C_ε be as in Theorem 4.1 and assume $\|I\|_1 < 2\varepsilon$. We can obviously fix $D > 1$ such that $\|I\|_1 \le 2\varepsilon/D$. For $\delta \in (0,\delta_D)$, it then follows from (4.55) that the assumption (4.7) holds, since $\tanh x \le x$ for $x \in [0,\infty)$. Hence, it follows from Theorem 4.1 that

$$\sum_{i\in\mathbb{Z}}\langle\!\langle\sigma_i\sigma_j\rangle\!\rangle_{\delta,T} \le \frac{C_\varepsilon}{1-\tanh(j_\delta)} < \frac{C_\varepsilon}{\delta} \qquad \text{for } \delta \text{ sufficiently small,}$$

where we used (4.54) in the second inequality. The last displayed inequality now implies

$$\sum_{i,j\in\Lambda_{L_\delta(T)}}\langle\!\langle\sigma_i\sigma_j\rangle\!\rangle_{\delta,T} < \frac{C_\varepsilon}{\delta}L_\delta(T).$$

Inserting this into (4.53) and using $L_\delta(T) \le \frac{T}{\delta}+\frac{1}{2}$ finishes the proof. □

5. FKN Formula for the Spin Boson Model with External Magnetic Field

In this chapter, we derive a Feynman-Kac-Nelson (FKN) formula for the spin boson model with external magnetic field. It expresses expectation values of the semigroup generated by the Hamiltonian as the expectation value of a Poisson-driven jump process and a Gaussian random process indexed by a real Hilbert space, obtained by an Euclidean extension of the dispersion relation of the bosons. Especially, when calculating expectation values with respect to the ground state of the free Hamiltonian, we can explicitly integrate out the boson field to prove Theorem 1.20 and Corollary 1.24. Adding a gap assumption, we can then express derivatives of the ground state energy in terms of correlation functions of a continuous Ising model in the proof of Theorem 1.25. In view of Theorem 4.21, this implies the second derivative of the ground state energy is bounded for sufficiently small coupling constants. This bound on the second derivative will be a key ingredient to our proof of Theorem 1.14.

The history of FKN-type theorems dates back to the work of Feynman and Kac [Fey05, Kac51]. Such functional integral respresentations were used to study the spectral properties of models in quantum field theory by Nelson [Nel73]. Since then, many authors have used this approach to study models of non-relativistic quantum field theory, see for example [GJ85, GJ87, Spo87, FFG97, Hir97, BHL$^+$02, BS05, HL08, BH09] and references therein. The spin boson model without an external magnetic field has been investigated using this approach in [SD85, FN88, Abd11] and recently in [HHL14]. In [Spo89] path measures for the spin boson model with magnetic field were studied by means of Gibbs measures.

This chapter is structured as follows. In Section 5.1, we derive the FKN formula Theorem 5.3 for $H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda,\mu)$ (cf. Definition 1.18). To our knowledge, the case $\mu \neq 0$ has not been treated in the literature yet. We then obtain Theorem 1.20, by integrating out the field degrees of freedom. This allows us to calculate expectation values with respect to the ground state $\Omega_\downarrow$ (cf. (1.8)) of the free operator $H_{\mathsf{SB}}^{(\mathsf{m})}(0,0)$ as the expectation value of a continuous Ising model. In Section 5.2, we then use the well-known connection between expectation values of the semigroup and the ground state energy to express the derivatives of the ground state energy with respect to the magnetic field strength as correlation functions of this continuous Ising model, under the assumption of a positive boson mass $m > 0$, i.e., $\omega(k) \geq m$ for almost all $k \in \mathbb{R}^d$.

Throughout this chapter, we assume Hypothesis SBF holds and drop the lower index SB as well as the upper index (m) and write $H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda,\mu) = H(\lambda,\mu)$.

5.1. The FKN Formula

In this section, we derive the FKN formula for the spin boson model interacting with an external magnetic field. The case without an external magnetic field has been treated in [HHL14] using results from [HL08]. The approach we use is closely related to the descriptions in the standard literature [Sim74, LHB11] and similar to [HHL14].

We recall that in Lemma 1.19, we proved $H(\lambda,\mu)$ is a selfadjoint lower-semibounded operator for any choice of $\lambda,\mu\in\mathbb{R}$. However, instead of considering $H(\lambda,\mu)$ directly, we apply the unitary

$$U = e^{\mathrm{i}\frac{\pi}{4}\sigma_y} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{5.1}$$

and define the transformed Hamilton operator

$$\widetilde{H}(\lambda,\mu) = \mathbb{1}+(U\otimes\mathbb{1})H(\lambda,\mu)(U\otimes\mathbb{1})^* = (\mathbb{1}-\sigma_x)\otimes\mathbb{1}+\mathbb{1}\otimes\mathsf{d}\Gamma(\omega)+\sigma_z\otimes(\lambda\varphi(v)+\mu\mathbb{1}), \tag{5.2}$$

where we used $U\sigma_zU^* = -\sigma_x$ and $U\sigma_xU^*=\sigma_z$.

Further, we define the Euclidean dispersion relation

$$\omega_{\mathsf{E}} : \mathbb{R}^{d+1}\to[0,\infty), \qquad \omega_{\mathsf{E}}(k,t) = \omega^2(k)+t^2 \tag{5.3}$$

and the Hilbert space of the Euclidean field as

$$\mathscr{E} = L^2(\mathbb{R}^{d+1},\omega_{\mathsf{E}}^{-1}(k,t)d(k,t)). \tag{5.4}$$

Let ϕ_{E} be the Gaussian random variable indexed by the real Hilbert space

$$\mathscr{R} = \{f\in\mathscr{E} : f(k,t) = \overline{f(-k,-t)}\} \tag{5.5}$$

on the probability space $(\mathcal{Q}_{\mathsf{E}},\Sigma_{\mathsf{E}},\mu_{\mathsf{E}})$ (cf. Lemma B.29) and denote expectation values w.r.t. μ_{E} as $\mathbb{E}_{\mathsf{E}}$. For details on the definitions, we refer the reader to Appendix B.5. We note $\mathscr{E}=\mathscr{R}\oplus\mathrm{i}\mathscr{R}$ in the sense of Remark B.31.

For $t\in\mathbb{R}$, we define

$$j_tf(k,s) = \frac{e^{-\mathrm{i}ts}}{\sqrt{\pi}}\omega^{1/2}(k)f(k). \tag{5.6}$$

Lemma 5.1.

(i) (5.6) *defines an isometry* $j_t : L^2(\mathbb{R}^d)\to\mathscr{E}$ *for any* $t\in\mathbb{R}$.

(ii) *If* $f\in L^2(\mathbb{R}^d)$ *satisfies* $f(k)=\overline{f(-k)}$, *then* $j_tf\in\mathscr{R}$.

(iii) $j_s^*j_t = e^{-|t-s|\omega}$ *for all* $s,t\in\mathbb{R}$.

Proof. The statements follow by the direct calculation

$$\langle j_sf,j_tg\rangle_{\mathscr{E}} = \int_{\mathbb{R}^d}\overline{f(k)}g(k)\int_{\mathbb{R}}e^{-\mathrm{i}(t-s)\tau}\frac{\omega(k)}{\omega^2(k)+\tau^2}\frac{\mathsf{d}\tau}{\pi}\mathsf{d}k = \int_{\mathbb{R}^d}\overline{f(k)}e^{-|t-s|\omega(k)}g(k)\mathsf{d}k. \quad \square$$

Remark 5.2. In the literature (5.6) is often defined via the Fourier transform $\breve{j_tf} = \delta_t\otimes\check{f}$.

We set

$$\widetilde{I}_t : \mathcal{F} \to L^2(\mathcal{Q}_{\mathsf{E}}), \qquad \psi \mapsto \Theta_{\mathscr{R}}\Gamma(j_t)\psi, \tag{5.7}$$

where $\Theta_{\mathscr{R}}$ denotes the Wiener-Itô-Segal isomorphism introduced in Lemma B.30 and $\Gamma(j_t)$ is the second quantization of the contraction operator j_t, as defined in Definition B.14. Further, we define the isometry $\iota : \mathbb{C}^2 \to L^2(\{\pm 1\}, \mu_{1/2})$, with $\mu_{1/2}(\{s\}) = \frac{1}{2}$ for $s \in \{\pm 1\}$, by

$$(\iota v)(+1) = \sqrt{2}v_1 \text{ and } (\iota v)(-1) = \sqrt{2}v_2.$$

We define the map $I_t := \iota \otimes \widetilde{I}_t$, where (cf. Lemma A.103)

$$I_t : \mathbb{C}^2 \otimes \mathcal{F} \to L^2(\{\pm 1\}, \mu_{1/2}) \otimes L^2(\mathcal{Q}_{\mathsf{E}}) \cong L^2(\{\pm 1\}, \mu_{1/2}; L^2(\mathcal{Q}_{\mathsf{E}})).$$

The FKN formula for the spin boson model with external magnetic field is now stated in the following theorem, where we use the jump process X as defined in Definition 4.16. Here, we drop the upper index (T) for expectation values, since for our purposes we only need X to be defined on the interval $[0, T]$.

Theorem 5.3. *For all $\Phi, \Psi \in \mathbb{C}^2 \otimes \mathcal{F}$ and $\lambda, \mu \in \mathbb{R}$, we have*

$$\left\langle \Phi, e^{-T\widetilde{H}(\lambda,\mu)}\Psi \right\rangle = \mathbb{E}_X \mathbb{E}_{\mathsf{E}} \left[\overline{I_0 \Phi(X_0)} \exp\left(-\lambda \int_0^T \phi_{\mathsf{E}}\,(j_t v)\, X_t \mathsf{d}t - \mu \int_0^T X_t \mathsf{d}t \right) I_T \Psi(X_T) \right].$$

We note that the integrability of the right hand side in above theorem follows from the identity

$$\mathbb{E}\left[\exp(Z)\right] = \exp\left(\frac{1}{2}\mathbb{E}[Z^2] \right), \tag{5.8}$$

which holds for any Gaussian random variable Z (see for example [Sim74, (I.17)]). We outline the argument in the remark below.

Remark 5.4. Let $\mathscr{D}_{\mathsf{f}}$ denote the set of right-continuous functions $x : [0, T] \to \{\pm 1\}$ with finitely many jumps and let μ_X be the measure induced on $\mathscr{D}_{\mathsf{f}}$ by X, cf. Lemma 4.15 (ii) and Definition 4.28. By (5.6), the map $[0, T] \to \mathscr{E}, t \mapsto j_t v$ is strongly continuous. Hence, by Definition B.27, the map $\mathbb{R} \to L^2(\mathcal{Q}_{\mathsf{E}}), t \mapsto \phi_{\mathsf{E}}(j_t v)$ is continuous. Thus, for $(x_t)_{t\in[0,T]} \in \mathscr{D}_{\mathsf{f}}$, the function $t \mapsto \phi_{\mathsf{E}}(j_t v)x_t$ is a piecewise continuous $L^2(\mathcal{Q}_{\mathsf{E}})$-valued function on compact intervals of $[0, \infty)$. This implies that the integral over t exists as an $L^2(\mathcal{Q}_{\mathsf{E}})$-valued Riemann integral μ_X-almost surely. Since Riemann integrals are given as limits of sums, the measurability with respect to the product measure $\mu_X \otimes \mu_{\mathsf{E}}$ follows. In fact, again fixing $x \in \mathscr{D}_{\mathsf{f}}$ and using Fubini's theorem as well as Hölder's inequality, one can prove that the integral $\int_0^T \phi_{\mathsf{E}}(j_t v)x_t dt$ can also be calculated as Lebesgue-integral evaluated μ_{E}-almost everywhere pointwise in $\mathcal{Q}_{\mathsf{E}}$ with the same result. This is outlined in Lemma 5.5 below. Furthermore, $\int_0^T \phi_{\mathsf{E}}(j_t v)x_t dt$ is a Gaussian random variable, since L^2-limits of linear combinations of Gaussians are Gaussian. We conclude that the right hand side of the FKN formula is finite, since exponentials of Gaussian random variables are integrable, cf. (5.8).

In the above remark, we use the following lemma with $f(t) = \phi_{\mathsf{E}}(j_t v)x_t$. Although the proof is simple, we give it here for completeness.

Lemma 5.5. *Let $(\mathcal{Q},\Sigma,\mu)$ be a probability space and assume $t\mapsto f_t\in L^2(\mathcal{Q})$ is piecewise continuous on the interval $[0,T]$. Then $[t\mapsto f_t(q)]\in L^1([0,T])$ for almost every $q\in\mathcal{Q}$ and*

$$\int_0^T f_t(q)\mathrm{d}t = \left(\int_0^T f_t\mathrm{d}t\right)(q) \qquad \text{for almost every } q\in\mathcal{Q}, \tag{5.9}$$

where the integral on the right hand side is the $L^2(\mathcal{Q})$-valued Riemann integral.

Proof. Using Fubini's theorem and Hölder's inequality, we find

$$\int_{\mathcal{Q}}\int_0^T |f_t(q)|\mathrm{d}t\mathrm{d}\mu(q) = \int_0^T\int_{\mathcal{Q}} |f_t(q)|\mathrm{d}\mu(q)\mathrm{d}t \le \int_0^T \|f_t\|_{L^2(\mathcal{Q})}\mathrm{d}t<\infty.$$

Hence, for μ-almost all $q\in\mathcal{Q}$, the map $t\mapsto f_t(q)$ is Lebesgue-integrable. Let $f_{\mathsf{s},t}$ be an $L^2(\mathcal{Q})$–valued step function. Then, using the triangle inequality, Fubini's theorem and Hölder's inequality, we find

$$\begin{aligned}
&\int_{\mathcal{Q}}\left|\int_0^T f_t(q)\mathrm{d}t - \left(\int_0^T f_t\mathrm{d}t\right)(q)\right|\mathrm{d}\mu(q)\\
&\quad\le \int_{\mathcal{Q}}\left|\int_0^T f_t(q)\mathrm{d}t - \int_0^T f_{\mathsf{s},t}(q)\mathrm{d}t\right|\mathrm{d}\mu(q) + \int_{\mathcal{Q}}\left|\left(\int_0^T f_{\mathsf{s},t}\right)(q)\mathrm{d}t - \left(\int_0^T f_t\mathrm{d}t\right)(q)\right|\mathrm{d}\mu(q)\\
&\quad\le \int_{\mathcal{Q}}\int_0^T |f_t(q)-f_{\mathsf{s},t}(q)|\,\mathrm{d}t\mathrm{d}\mu(q) + \left\|\int_0^T f_t\mathrm{d}t - \int_0^T f_{\mathsf{s},t}\mathrm{d}t\right\|_{L^1(\mathcal{Q})}\\
&\quad\le \int_0^T\int_{\mathcal{Q}} |f_t(q)-f_{\mathsf{s},t}(q)|\,\mathrm{d}\mu(q)\mathrm{d}t + \left\|\int_0^T f_t\mathrm{d}t - \int_0^T f_{\mathsf{s},t}\mathrm{d}t\right\|_{L^2(\mathcal{Q})}\\
&\quad\le 2\int_0^T \|f_t-f_{\mathsf{s},t}\|_{L^2(\mathcal{Q})}\mathrm{d}t.
\end{aligned}$$

By the piecewise $L^2(\mathcal{Q})$-continuity of $t\mapsto f_t$, the right hand side can be made arbitraritly small by making the mesh of the Riemann sum arbitrarily small. This implies (5.9). ☐

We now prove Theorem 5.3. To that end, we first derive a FKN formula for the spin part, which is described by the jump process.

Lemma 5.6. *Let $n\in\mathbb{N}$ and $t_1,\dots,t_n\ge 0$. We set $s_k=\sum_{i=1}^{k} t_k$ for $k=1,\dots,n$. Then, for all $v,w\in\mathbb{C}^2$ and $f_0,f_1,\dots,f_n:\{\pm1\}\to\mathbb{C}$, we have*

$$\begin{aligned}
&e^{-s_n}\left\langle w, f_0(\sigma_z)e^{t_1\sigma_x}f_1(\sigma_z)e^{t_2\sigma_x}\cdots e^{t_n\sigma_x}f_n(\sigma_z)v\right\rangle\\
&\qquad\qquad = \mathbb{E}_X\left[\overline{\iota w(X_0)}f_0(X_0)f_1(X_{s_1})\cdots f_n(X_{s_n})\iota v(X_{s_n})\right].
\end{aligned}$$

Proof. Since any function $f:\{\pm1\}\to\mathbb{C}$ is a linear combination of the identity and the constant function 1, it suffices to consider the case $f_0=f_1=\cdots=f_n=\mathrm{id}$. Further, due to bilinearity, it suffices to choose w and v to be arbitrary basis vectors. We here use the basis consisting of eigenvectors of σ_x, i.e., $e_1=\frac{1}{\sqrt2}(1,1)$ and $e_2=\frac{1}{\sqrt2}(1,-1)$. Then

$$\sigma_x e_1 = e_1,\quad \sigma_x e_2=-e_2,\quad \sigma_z e_1 = e_2,\quad\text{and}\quad \sigma_z e_2 = e_1$$

and hence

$$\langle e_1, \sigma_z e^{t_1\sigma_x}\cdots e^{t_n\sigma_x}\sigma_z e_1\rangle = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \exp\Big(\sum_{j=1}^{n}(-1)^j t_j\Big) & \text{if } n \text{ is odd,} \end{cases}$$

$$\langle e_2, \sigma_z e^{t_1\sigma_x}\cdots e^{t_n\sigma_x}\sigma_z e_1\rangle = \begin{cases} \exp\Big(-\sum_{j=1}^{n}(-1)^j t_j\Big) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

$$\langle e_2, \sigma_z e^{t_1\sigma_x}\cdots e^{t_n\sigma_x}\sigma_z e_2\rangle = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \exp\Big(-\sum_{j=1}^{n}(-1)^j t_j\Big) & \text{if } n \text{ is odd.} \end{cases}$$

From Lemma 4.41, we recall that, for $0 \le j \le k \le n$ and setting $s_0 = 0$,

$$\mathbb{E}_X\left[X_{s_j}\cdots X_{s_k}\right] = \begin{cases} 0 & \text{if } k-j \text{ is even,} \\ e^{-2(t_{j+1}+t_{j+3}+\cdots+t_{k-2}+t_k)} & \text{if } k-j \text{ is odd.} \end{cases}$$

Combined, this yields

$$\begin{aligned} e^{-s_n}\langle e_1, \sigma_z e^{t_1\sigma_x}\cdots e^{t_n\sigma_x}\sigma_z e_1\rangle &= \mathbb{E}_X\left[X_0 X_{s_1}\cdots X_{s_n}\right], \\ e^{-s_n}\langle e_2, \sigma_z e^{t_1\sigma_x}\cdots e^{t_n\sigma_x}\sigma_z e_1\rangle &= \mathbb{E}_X\left[X_{s_1}\cdots X_{s_n}\right], \\ e^{-s_n}\langle e_2, \sigma_z e^{t_1\sigma_x}\cdots e^{t_n\sigma_x}\sigma_z e_2\rangle &= \mathbb{E}_X\left[X_{s_1}\cdots X_{s_{n-1}}\right]. \end{aligned}$$

Observing that $\iota e_1(x) = 1$ and $\iota e_2(x) = x$ for $x = \pm 1$ finishes the proof. □

We now move to proving the FKN formula for the field part. For $I \subset \mathbb{R}$, let e_I denote the projection onto $\overline{\operatorname{span}\{f \in \mathscr{E} : f \in \operatorname{ran}(j_t) \text{ for some } t \in I\}}$. Further, set $e_t = e_{\{t\}}$.

Lemma 5.7. *Assume $a \le b \le t \le c \le d$. Then*

(i) $e_t = j_t j_t^*$,

(ii) $e_a e_b e_c = e_a e_c$,

(iii) $e_{[a,b]} e_t e_{[c,d]} = e_{[a,b]} e_{[c,d]}$.

Proof. Lemma 5.1 (i) and the definition of $e_{\{t\}}$ directly imply (i). Further, (ii) follows from Lemma 5.1 (iii), by

$$e_a e_b e_c = j_a j_a^* j_b j_b^* j_c j_c^* = j_a e^{-(b-a)\omega} e^{-(c-b)\omega} j_c^* = j_a e^{-(c-a)\omega} j_c^* = j_a j_a^* j_c j_c^* = e_a e_c.$$

To prove (iii), let $f, g \in \mathscr{E}$. By the definition, there exist sequences of times $(t_k)_{k\in\mathbb{N}} \subset [a,b]$ and $(s_m)_{m\in\mathbb{N}} \subset [c,d]$ and functions $f_k \in \operatorname{ran}(j_{t_k}) = \operatorname{ran}(e_{t_k})$, $g_m \in \operatorname{ran}(j_{s_m}) = \operatorname{ran}(e_{s_m})$ such that

$$e_{[a,b]} f = \sum_{k=1}^{\infty} f_k \qquad \text{and} \qquad e_{[c,d]} g = \sum_{m=1}^{\infty} g_m.$$

Hence, we can apply (ii) and obtain

$$\langle e_{[a,b]} e_t e_{[c,d]} g, f\rangle = \sum_{k,m=1}^{\infty} \langle e_t g_m, f_k\rangle = \sum_{k,m=1}^{\infty} \langle g_m, f_k\rangle = \langle e_{[a,b]} e_{[c,d]} g, f\rangle.$$

Since f and g were arbitrary, this proves the statement. □

Now, for $t \in \mathbb{R}$ and $I \subset \mathbb{R}$, let

$$J_t = \Gamma(j_t), \qquad E_t = \Gamma(e_t) \qquad \text{and} \qquad E_I = \Gamma(e_I). \tag{5.10}$$

Then the next statement in large parts follows directly from Lemmas 5.1 and 5.7 and standard Fock space properties.

Lemma 5.8. *Assume $a \le b \le t \le c \le d$ and $I \subset \mathbb{R}$.*

(i) *E_I is the orthogonal projection onto $\overline{\operatorname{span}\{f \in \mathcal{F}(\mathscr{E}) : f \in \operatorname{ran}(J_t) \text{ for some } t \in I\}}$.*

(ii) $E_t = J_t J_t^*$

(iii) $E_a E_b E_c = E_a E_c$

(iv) $E_{[a,b]} E_t E_{[c,d]} = E_{[a,b]} E_{[c,d]}$

(v) *For all $F \in \operatorname{ran}(E_{[a,b]})$ and $G \in \operatorname{ran}(E_{[c,d]})$, we have $\langle F, E_t G\rangle = \langle F, G\rangle$.*

(vi) *$J_s^* J_t = e^{-|t-s| \mathsf{d}\Gamma(\omega)}$ for all $s \in \mathbb{R}$.*

(vii) *$J_t \varphi(f) J_t^* = E_t \varphi(j_t f) E_t = \varphi(j_t f) E_t$ for all $f \in L^2(\mathbb{R}^d)$.*

(viii) *$J_t G(\varphi(f)) J_t^* = E_t G(\varphi(j_t f)) E_t = G(\varphi(j_t f)) E_t$ for all $f \in L^2(\mathbb{R}^d)$ and bounded measurable functions G on $\mathbb{R}$.*

Proof. All statements except for (v)–(viii) follow trivially from Lemmas 5.7, B.15 and the definitions. (v) follows from (iv), by the simple calculation

$$\begin{aligned}\langle F, E_t G\rangle &= \langle E_{[a,b]} F, E_t E_{[c,d]} G\rangle = \langle F, E_{[a,b]} E_t E_{[c,d]} G\rangle \\ &= \langle F, E_{[a,b]} E_{[c,d]} G\rangle = \langle E_{[a,b]} F, E_{[c,d]} G\rangle \\ &= \langle F, G\rangle .\end{aligned}$$

(vi) and (vii) follow by combining Lemmas 5.1 (iii) and B.20 (v) and (vi). Repeated application of (vii) shows that (viii) holds for G a polynomial. That it holds for arbitrary bounded measurable G follows from the measurable functional calculus [RS72]. □

We can now give the

Proof of Theorem 5.3. Throughout this proof, we drop tensor products with the identity in our notation. Further, for the convenience of the reader, we explicitly state in which Hilbert space the inner product is taken.

Let

$$\eta_K(x) = \begin{cases} \min\{x, K\} & \text{if } x \ge 0, \\ \max\{x, -K\} & \text{if } x < 0. \end{cases}$$

Further, let $\varphi_K(v) = \eta_K(\varphi(v))$, $\phi_{\mathsf{E},K}(j_t v) = \eta_K(\phi_{\mathsf{E}}(j_t v))$ and $\widetilde{H}_K(\lambda, \mu)$ as in (5.2) with φ replaced by φ_K. Since $\widetilde{H}_K(\lambda, \mu)$ is lower-semibounded and φ_K is bounded, we can use the Trotter product formula (cf. Theorem A.66) and Lemma 5.8 (vi) and (vii) (where the exponential is considered on the eigenspaces of σ_x) to obtain

$$\begin{aligned}
e^T \left\langle \Phi, e^{-T\widetilde{H}_K(\lambda,\mu)}\Psi \right\rangle_{\mathcal{H}} &= \lim_{N\to\infty} \left\langle \Phi, \left(e^{-\frac{T}{N}\mathsf{d}\Gamma(\omega)} e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\varphi_K(v)+\mu)}\right)^n \Psi \right\rangle_{\mathcal{H}} \\
&= \lim_{N\to\infty} \left\langle \Phi, \prod_{k=1}^{N} \left(J^*_{(k-1)\frac{T}{N}} J_{k\frac{T}{N}} e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\varphi_K(v)+\mu)}\right) \Psi \right\rangle_{\mathcal{H}} \\
&= \lim_{N\to\infty} \left\langle J_0\Phi, \prod_{k=1}^{N} \left(J_{k\frac{T}{N}} e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\varphi_K(v)+\mu)} J^*_{k\frac{T}{N}}\right) J_T\Psi \right\rangle_{\mathbb{C}^2\otimes\mathcal{F}(\mathscr{E})} \\
&= \lim_{N\to\infty} \left\langle J_0\Phi, \prod_{k=1}^{N} \left(E_{k\frac{T}{N}} e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\varphi_K(j_{k\frac{T}{N}}v)+\mu)} E_{k\frac{T}{N}}\right) J_T\Psi \right\rangle_{\mathbb{C}^2\otimes\mathcal{F}(\mathscr{E})}.
\end{aligned}$$

Now we make iterated use of Lemma 5.8 (v). Explicitly, by Lemma 5.8 (viii), the vector to the left of any $E_{k\frac{T}{N}}$, i.e.,

$$\prod_{j=0}^{k-1} \left(E_{j\frac{T}{N}} e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\varphi_K(j_{k\frac{T}{N}}v)+\mu)} E_{j\frac{T}{N}}\right) J_0\Phi \in \operatorname{ran}(E_{(k-1)\frac{T}{N}}),$$

$$e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\varphi_K(j_{k\frac{T}{N}}v)+\mu)} \prod_{j=0}^{k-1} \left(E_{j\frac{T}{N}} e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\varphi_K(j_{k\frac{T}{N}}v)+\mu)} E_{j\frac{T}{N}}\right) J_0\Phi \in \operatorname{ran}(E_{k\frac{T}{N}})$$

is an element of $\operatorname{ran}(E_{[0,k\frac{T}{N}]})$. Equivalently, the vector to the right is an element of $\operatorname{ran}(E_{[k\frac{T}{N},T]})$. Hence, we can drop all the factors $E_{k\frac{T}{N}}$. Then, using Lemma B.30 and (5.7), we derive

$$\begin{aligned}
e^T \left\langle \Phi, e^{-T\widetilde{H}_K(\lambda,\mu)}\Psi \right\rangle_{\mathcal{H}} &= \lim_{N\to\infty} \left\langle J_0\Phi, \prod_{k=1}^{N} \left(e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\varphi_K(j_{k\frac{T}{N}}v)+\mu)}\right) J_T\Psi \right\rangle_{\mathbb{C}^2\otimes\mathcal{F}(\mathscr{E})} \\
&= \lim_{N\to\infty} \left\langle I_0\Phi, \prod_{k=1}^{N} \left(e^{-\frac{T}{N}\sigma_z} e^{-\frac{T}{N}\sigma_x\otimes(\lambda\phi_{\mathsf{E},K}(j_{k\frac{T}{N}}v)+\mu)}\right) I_T\Psi \right\rangle_{\mathbb{C}^2\otimes L^2(\mathcal{Q}_{\mathsf{E}})}.
\end{aligned}$$

Hence, we can apply Lemma 5.6 to obtain

$$\left\langle \Phi, e^{-T\widetilde{H}_K(\lambda,\mu)}\Psi \right\rangle_{\mathcal{H}} = \lim_{N\to\infty} \mathbb{E}_X\mathbb{E}_{\mathsf{E}} \left[\overline{I_0\Phi(X_0)} e^{-\frac{T}{N}\sum_{k=1}^{N}\left(\lambda\phi_{\mathsf{E},K}(j_{k\frac{T}{N}}v)+\mu\right)X_{k\frac{T}{N}}} I_T\Psi(X_T)\right]. \quad (5.11)$$

Since η_K is Lipschitz continuous, it follows that $t\mapsto\phi_{\mathsf{E},K}(j_t v)$ is an $L^2(\mathcal{Q}_{\mathsf{E}})$-valued continuous function. Thus, the sum in the exponential in (5.11) converges to an $L^2(\mathcal{Q}_{\mathsf{E}})$-valued Riemann integral. By possibly going over to a subsequence the Riemann sum converges $\mu_X\otimes\mu_{\mathsf{E}}$-almost everywhere. Thus, it follows by dominated convergence that

$$\left\langle \Phi, e^{-T\widetilde{H}_K(\lambda,\mu)}\Psi \right\rangle_{\mathcal{H}} = \mathbb{E}_X\mathbb{E}_{\mathsf{E}} \left[\overline{I_0\Phi(X_0)} e^{-\lambda\int_0^T \phi_{\mathsf{E},K}(j_t v)X_t dt - \mu\int_0^T X_t dt} I_T\Psi(X_T)\right]. \quad (5.12)$$

(Alternatively, the convergence could also be deduced by estimating the expectation.) Since $\varphi(v)$ is bounded with respect to $\mathsf{d}\Gamma(\omega)$ (cf. Lemma B.22), the spectral theorem implies that $\widetilde{H}_K(\lambda,\mu)$ converges to $\widetilde{H}(\lambda,\mu)$ in the strong resolvent sense and hence the left hand side of above equation converges to $\left\langle \Phi, e^{-T\widetilde{H}(\lambda,\mu)}\Psi\right\rangle$ as $K \to \infty$. On the other hand, using that for $\mu_X \otimes \mu_{\mathsf{E}}$-almost every $(x,q) \in \mathscr{D}_{\mathsf{f}} \times \mathcal{Q}_{\mathsf{E}}$ the function $t \mapsto (\phi_{\mathsf{E},K}(j_t v))(q)x_t$ is Lebesgue integrable, see Remark 5.4, it follows that $\int_0^T \phi_{\mathsf{E},K}(j_t v)x_t dt$ converges to $\int_0^T \phi_{\mathsf{E}}(j_t v)x_t dt$ almost everywhere. Hence, the right hand side of (5.12) converges to

$$\mathbb{E}_X\mathbb{E}_{\mathsf{E}}\left[\overline{I_0\Phi(X_0)}e^{-\lambda\int_0^T \phi_{\mathsf{E}}(j_t v)X_t dt-\mu\int_0^T X_t dt}I_T\Psi(X_T)\right]$$

as $K \to \infty$, by the dominated convergence theorem. For the majorant, we use that by Jensen's inequality

$$\begin{aligned}\exp(-\lambda\int_0^T \phi_{\mathsf{E},K}(j_t v)X_t dt) &\leq \frac{1}{T}\int_0^T \exp(-\lambda T\phi_{\mathsf{E},K}(j_t v)X_t)dt\\ &\leq \frac{1}{T}\int_0^T [\exp(-\lambda T\phi_{\mathsf{E}}(j_t v))+\exp(\lambda T\phi_{\mathsf{E}}(j_t v))]dt,\end{aligned}$$

where in the second line we used $\max\{e^x,1\} \leq e^x + e^{-x}$. Now the right hand side is integrable over $\mathcal{Q}_{\mathsf{E}}$-space by (5.8). This proves the statement. □

We can now prove Theorem 1.20. Especially, recall how the interaction function W was defined from ω and v in (1.9).

Proof of Theorem 1.20. First, observe that with U as in (5.1) we have (cf. Definition B.14 and Lemma B.30)

$$(I_t(U^*\otimes\mathbb{1})\Omega_\downarrow)(x) = 1 \qquad \text{for } x=\pm 1 \text{ and } t\in\mathbb{R}.$$

Hence, Theorem 5.3 implies

$$e^{-T}\left\langle\Omega_\downarrow, e^{-TH(\lambda,\mu)}\Omega_\downarrow\right\rangle = \mathbb{E}_X\left[\mathbb{E}_{\mathsf{E}}\left[\exp\left(-\lambda\int_0^T \phi_{\mathsf{E}}(j_t v)X_t \mathsf{d}t\right)\right]\exp\left(-\mu\int_0^T X_t\mathsf{d}t\right)\right]. \tag{5.13}$$

Now, assume x is some path of X. By (5.8), we have

$$\mathbb{E}_{\mathsf{E}}\left[\exp\left(-\lambda\int_0^T \phi_{\mathsf{E}}(j_t v)x_t\mathsf{d}t\right)\right] = \frac{\lambda^2}{2}\mathbb{E}_{\mathsf{E}}\left[\left(\int_0^T \phi_{\mathsf{E}}(j_t v)x_t\mathsf{d}t\right)^2\right]. \tag{5.14}$$

Then Fubini's theorem and the definition of the $\mathscr{R}$-indexed Gaussian process (cf. Definition B.27 (v)) yield

$$\begin{aligned}\mathbb{E}_{\mathsf{E}}\left[\left(\int_0^T \phi_{\mathsf{E}}(j_t v)x_t\mathsf{d}t\right)^2\right] &= \int_0^T\int_0^T \mathbb{E}_{\mathsf{E}}\left[\phi_{\mathsf{E}}(j_t v)\phi_{\mathsf{E}}(j_s v)\right]x_t x_s\mathsf{d}t\mathsf{d}s\\ &= \int_0^T\int_0^T \langle j_t v, j_s v\rangle x_t x_s \mathsf{d}t\mathsf{d}s\\ &= 2\int_0^T\int_0^T W(t-s)x_t x_s\mathsf{d}t\mathsf{d}s,\end{aligned} \tag{5.15}$$

where we used $j_s^* j_t = e^{-|t-s|\omega}$ (cf. Lemma 5.1). Combining (5.13), (5.14) and (5.15) proves the statement. □

5.2. Ground State Energy of the Spin Boson Model

In this section, we study the ground state energy of the spin boson model

$$E(\lambda,\mu) = \inf \sigma(H(\lambda,\mu)). \tag{5.16}$$

In particular, we study the derivatives of the ground state energy with respect to the magnetic coupling strength μ. We remark that these derivatives can be used to obtain asymptotic expansions with respect to the magnetic coupling. Our arguments are inspired by [Dim74, Sim79].

For the investigation, we use the formula

$$E(\lambda,\mu) = -\lim_{T\to\infty} \frac{1}{T} \ln \left\langle \Omega_\downarrow, e^{-TH(\lambda,\mu)}\Omega_\downarrow \right\rangle, \tag{5.17}$$

which we justify below using positivity arguments. If the ground state energy is isolated from the rest of the spectrum, we can show that differentiation can be interchanged with the limit $T\to\infty$, by using spectral analysis. Then, using the formula from Theorem 1.20, we can express the derivatives of the ground state energy in terms of the correlation functions from Section 4.2. The corresponding formulas are collected in Theorem 1.25 and Corollary 5.12.

Lemma 5.9. *Equation* (5.17) *holds for all* $\lambda,\mu\in\mathbb{R}$.

Proof. By Definition 1.18 and (5.2), we have

$$e^{-T}\left\langle \Omega_\downarrow, e^{-TH(\lambda,\mu)}\Omega_\downarrow \right\rangle = \left\langle (U\otimes \mathbb{1})^*\Omega_\downarrow, e^{-T\widetilde{H}(\lambda,\mu)}(U\otimes\mathbb{1})^*\Omega_\downarrow \right\rangle.$$

Now let Θ be the natural isomorphism $\mathbb{C}^2\otimes\mathcal{F}\to L^2(\{1,2\}\times\mathcal{Q})$ corresponding to the decomposition $L^2(\mathbb{R}^d) = L^2_{\mathbb{R}}(\mathbb{R}^d)\oplus \mathrm{i}L^2_{\mathbb{R}}(\mathbb{R}^d)$, cf. Lemma B.30, i.e.,

$$\Theta(\alpha\otimes\psi) = \big((i,q)\mapsto \alpha_i\Theta_{L^2_{\mathbb{R}}(\mathbb{R}^d)}(\psi)(q)\big). \tag{5.18}$$

Then $\Theta(U\otimes\mathbb{1})^*\Omega_\downarrow = 1/\sqrt{2}$ is strictly positive. Hence, if $\Theta e^{-T\widetilde{H}(\lambda,\mu)}\Theta^*$ is positivity preserving, the statement follows by Lemma A.113. That statement is contained in Lemma 6.2 and we present a proof there. □

From here, we easily obtain the

Proof of Corollary 1.24. By the definition (1.10) and Theorem 1.20, we have

$$\mathfrak{Z}_T(\lambda,\mu) = e^{-T}\left\langle \Omega_\downarrow, e^{-TH(\lambda,\mu)}\Omega_\downarrow \right\rangle. \tag{5.19}$$

Hence, the statement follows from Lemma 5.9. □

The central statement of this section is that (5.17) carries over to the derivatives with respect to μ if $H(\lambda,\mu)$ has a spectral gap, i.e., $E(\lambda,\mu)$ is separated from the set $\sigma(H(\lambda,\mu))\setminus\{E(\lambda,\mu)\}$. It is known this especially holds for $H(\lambda,0)$ if $\operatorname{ess\,inf}_{k\in\mathbb{R}^d}\omega(k)>0$ [AH95], see also Section 6.2.

Theorem 5.10. *Let $\lambda, \mu_0 \in \mathbb{R}$ and suppose $H(\lambda,\mu_0)$ has a spectral gap. Then, for all $n \in \mathbb{N}$, the following derivatives exist and satisfy*

$$\partial_\mu^n E(\lambda,\mu)\big|_{\mu=\mu_0} = \lim_{T\to\infty} -\frac{1}{T}\partial_\mu^n \ln \left\langle \Omega_\downarrow, e^{-TH(\lambda,\mu)}\Omega_\downarrow \right\rangle\bigg|_{\mu=\mu_0}.$$

In the proof of Theorem 5.10, we will utilize the following lemma. The main argument of the proof further elaborates on above positivity argument and is deferred to Section 6.1.

Lemma 5.11. *Let $\lambda, \mu \in \mathbb{R}$. If $E(\lambda,\mu)$ is an eigenvalue of $H(\lambda,\mu)$, then the corresponding eigenspace is non-degenerate. In this case, if $\psi_{\lambda,\mu}$ is a ground state of $H(\lambda,\mu)$, then $\langle \psi_{\lambda,\mu}, \Omega_\downarrow\rangle \neq 0$.*

Proof. By the Perron-Frobenius-Faris theorem (Theorem A.112) as well as Lemma 6.2, if $E(\lambda,\mu)$ is an eigenvalue of $H(\lambda,\mu)$, then there exists a strictly positive $\phi_{\lambda,\mu} \in L^2(\{1,2\}\times \mathcal{Q}_{L^2_{\mathbb{R}}(\mathbb{R}^d)})$ such that the eigenspace corresponding to $E(\lambda,\mu)$ is spanned by $\Theta(U\otimes \mathbb{1})^*\phi_{\lambda,\mu}$, where Θ again is the defined as in (5.18). Since $\Theta(U\otimes\mathbb{1})^*\Omega_\downarrow$ is (strictly) positive, this proves the statement. □

We now give the

Proof of Theorem 5.10. As in Theorem 1.25, we denote by Π_n the set of all partitions of the set $\{1,\dots,n\}$ for the remainder of this section.

Throughout this proof, we fix λ, μ_0 as in the statement of the theorem. Further, for compact notation, we write

$$\mathbf{h}(\mu) = H(\lambda,\mu), \quad \mathbf{e}(\mu) = E(\lambda,\mu) \quad \text{and} \quad \mathbf{e}_T(\mu) = -\frac{1}{T}\ln\left\langle \Omega_\downarrow, e^{-Th(\mu)}\Omega_\downarrow\right\rangle.$$

Hence, we want to prove

$$\mathbf{e}^{(n)}(\mu_0) = \lim_{T\to\infty} \mathbf{e}_T^{(n)}(\mu_0) \qquad \text{for all } n\in\mathbb{N},$$

where $(\cdot)^{(n)}$ as usually denotes the n-th derivative.

We denote the spectral gap of $h(\mu)$ by $\delta > 0$, i.e., $\operatorname{dist}(\{e(\mu)\}, \sigma(h(\mu))\setminus\{e(\mu)\}) = \delta$. By standard perturbation theory (cf. Theorem A.46), the spectral gap implies that $\mu \mapsto \mathbf{e}(\mu)$ is analytic in a neighborhood of μ_0. Hence, we can choose an $\varepsilon > 0$ such that

$$|\mathbf{e}(\mu) - \mathbf{e}(\mu_0)| \le \frac{\delta}{4} \text{ and } \inf(\sigma(\mathbf{h}(\mu))\setminus \mathbf{e}(\mu)) \ge \mathbf{e}(\mu_0) + \frac{3}{4}\delta \text{ for } \mu \in (\mu_0-\varepsilon, \mu_0+\varepsilon), \quad (5.20)$$

where the second inequality can be obtained using a Neumann series, cf. (5.22), or alternatively it can be obtained from the lower boundedness of Lemma 1.13 and a compactness argument involving that the set of (μ, z), for which $\mathbf{h}(\mu) - z$ is invertible, is open, see [RS78, Theorem XII.7]. Henceforth, let $\mu \in (\mu_0 - \varepsilon, \mu_0+\varepsilon)$. Then, by (5.20) and Lemma A.58, we can write the ground state projection $P(\mu)$ of $\mathbf{h}(\mu)$ as

$$P(\mu) = \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_0} \frac{1}{z - \mathbf{h}(\mu)} dz,$$

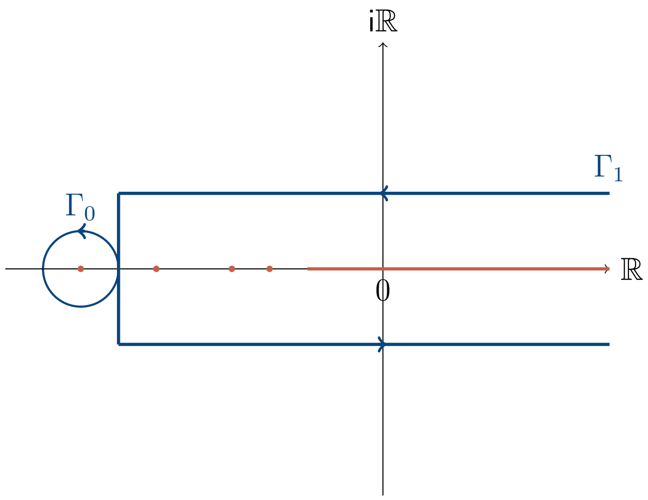

Figure 4: Illustration of the curves Γ_0 and Γ_1 (blue). The spectrum of $h(\mu)$ is displayed in red.

where Γ_0 is a curve encircling the point $\mathbf{e}(\mu_0)$ counterclockwise at a distance $\delta/2$. Further, let

$$\gamma_0 : [-1, +1] \to \mathbb{C}, \quad t \mapsto \mathbf{e}(\mu_0) + \frac{\delta}{2} - \mathsf{i}t,$$
$$\gamma_\pm : [0, \infty) \to \mathbb{C}, \quad t \mapsto \mathbf{e}(\mu_0) + \frac{\delta}{2} \pm \mathsf{i} + t$$

and define the curve $\Gamma_1 = -\gamma_+ + \gamma_0 + \gamma_-$ surrounding the set $\sigma(\mathbf{h}(\mu_0)) \setminus \{\mathbf{e}(\mu_0)\}$ (see Fig. 4). In view of (5.20), we can define

$$Q_T(\mu) := \frac{1}{2\pi\mathsf{i}} \int_{\Gamma_1} \frac{e^{T(\mathbf{e}(\mu)-z)}}{z - \mathbf{h}(\mu)} dz,$$

where the integral is understood as a Riemann integral with respect to the operator topology. The spectral theorem for the self-adjoint operator $\mathbf{h}(\mu)$ and Cauchy's integral formula yield

$$e^{-T(\mathbf{h}(\mu)-\mathbf{e}(\mu))} = P(\mu) + Q_T(\mu). \tag{5.21}$$

For $z \in \rho(\mathbf{h}(\mu_0))$ and μ in a neighborhood of μ_0 we have

$$\frac{1}{z - \mathbf{h}(\mu)} = \frac{1}{z - \mathbf{h}(\mu_0)} \sum_{k=0}^{\infty} \left((\mu - \mu_0)(\sigma_x \otimes \mathbb{1}) \frac{1}{z - \mathbf{h}(\mu_0)} \right)^k. \tag{5.22}$$

Using this expansion and the following bounds obtained from (5.20)

$$\begin{aligned} &\|(z - \mathbf{h}(\mu_0))^{-1}\| \leq \frac{2}{\delta} \quad \text{for } z \in \operatorname{ran}\Gamma_0 \cup \operatorname{ran}\Gamma_1, \\ &|e^{T(\mathbf{e}(\mu)-z)}| \leq \begin{cases} e^{-\frac{\delta}{4}T} & \text{for } z \in \operatorname{ran}\gamma_0, \\ e^{-\frac{\delta}{4}T} e^{-Tt} & \text{for } z = \gamma_\pm(t),\ t \in [0,\infty), \end{cases} \end{aligned} \tag{5.23}$$

we see that $P(\mu)$ and $Q_T(\mu)$ are real analytic for μ in a neighborhood of μ_0 and, moreover, that the integrals and derivatives with respect to μ can be interchanged due to the uniform convergence of the integrand on the curves Γ_0 and Γ_1. Hence, by virtue of (5.21), we see that the function $\mu \mapsto \langle \Omega_\downarrow, e^{-\mathbf{h}(\mu)}\Omega_\downarrow \rangle$ is real analytic on $(\mu_0 - \widetilde{\varepsilon}, \mu_0 + \widetilde{\varepsilon})$ for $\widetilde{\varepsilon} \in (0, \varepsilon)$ small enough.

Let ψ_μ be a normalized ground state of $\mathbf{h}(\mu)$. Then, by Lemma 5.11, we find

$$\langle \Omega_\downarrow, P(\mu)\Omega_\downarrow \rangle = |\langle \psi_\mu | \Omega_\downarrow \rangle|^2 > 0. \tag{5.24}$$

Further, by the spectral theorem and (5.20)

$$0 \le \langle \Omega_\downarrow, Q_T(\mu)\Omega_\downarrow \rangle = e^{-\frac{T\delta}{2}} \int_{\mathbf{e}(\mu)+\frac{1}{2}\delta}^{\infty} e^{T(\mathbf{e}(\mu)+\frac{1}{2}\delta - \lambda)} \mathsf{d}\langle \Omega_\downarrow, \mathsf{P}_{\mathbf{h}(\mu)}(\lambda)\Omega_\downarrow \rangle \le e^{-\frac{T\delta}{2}} \|\Omega_\downarrow\|^2. \tag{5.25}$$

By (5.21) and the definition of $\mathbf{e}_T(\mu)$, we have

$$\mathbf{e}(\mu) - \mathbf{e}_T(\mu) = \frac{1}{T} \ln \left(\langle \Omega_\downarrow, P(\mu)\Omega_\downarrow \rangle + \langle \Omega_\downarrow, Q_T(\mu)\Omega_\downarrow \rangle \right) \qquad \text{for } \mu \in (\mu_0 - \widetilde{\varepsilon}, \mu_0 + \widetilde{\varepsilon}).$$

Hence, we can calculate the n-th derivative of the expression on the left hand side at $\mu = \mu_0$, by taking the n-th derivative on the right hand side. Using the Faà di Bruno formula,[1] we find

$$\begin{aligned} &e^{(n)}(\mu) - e_T^{(n)}(\mu) \\ &\quad = \frac{-1}{T} \sum_{\mathfrak{P} \in \Pi_n} \frac{(-1)^{|\mathfrak{P}|}(|\mathfrak{P}| - 1)!}{\langle \Omega_\downarrow, (P(\mu) + Q_T(\mu))\Omega_\downarrow \rangle^{|\mathfrak{P}|}} \prod_{B \in \mathfrak{P}} \left\langle \Omega_\downarrow, (P^{(|B|)}(\mu) + Q_T^{(|B|)}(\mu))\Omega_\downarrow \right\rangle . \end{aligned}$$

By (5.24) and (5.25), the first factor is uniformly bounded in T. Hence, it remains to prove that $\left\langle \Omega_\downarrow, Q_T^{(k)}(\mu_0)\Omega_\downarrow \right\rangle$ is uniformly bounded in T for all $k = 1, \ldots, n$. Therefore, we explicitly calculate the derivative of $Q_T(\mu)$ at $\mu = \mu_0$. This is done by interchanging the integral with the derivative, which we justified above. Note that, by the series expansion (5.22), we have

$$\partial_\mu^k (z - \mathbf{h}(\mu))^{-1} = \frac{k!}{z - \mathbf{h}(\mu)} \left(\sigma_x \frac{1}{z - \mathbf{h}(\mu)} \right)^k \qquad \text{for } k \in \mathbb{N}_0.$$

Again using Faà di Bruno's formula (5.26) and the Leibniz rule, this yields

$$\begin{aligned} Q_T^{(k)}(\mu_0) &= \frac{1}{2\pi \mathsf{i}} \sum_{\ell=0}^{k} \binom{k}{\ell} \int_{\Gamma_1} \partial_\mu^\ell (e^{T(\mathbf{e}(\mu) - z)}) \partial_\mu^{k-\ell} (z - \mathbf{h}(\mu))^{-1} \mathsf{d}z \Big|_{\mu=\mu_0} \\ &= \frac{1}{2\pi \mathsf{i}} \sum_{\ell=0}^{k} \binom{k}{\ell} (k-\ell)! \\ &\qquad \times \underbrace{\left(\sum_{\mathfrak{P} \in \Pi_\ell} \prod_{B \in \mathfrak{P}} (T \mathbf{e}^{(|B|)}(\mu_0)) \right)}_{=: P_{k,\ell}(T)} \underbrace{\int_{\Gamma_1} e^{T(\mathbf{e}(\mu_0) - z)} \frac{1}{z - \mathbf{h}(\mu_0)} \left(\sigma_x \frac{1}{z - \mathbf{h}(\mu_0)} \right)^{k-\ell} \mathsf{d}z}_{=: I_{k,\ell}(T)} . \end{aligned}$$

[1] We use the following version of the Faà di Bruno formula, which can be found in [Har06].
Let $I \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^m$ be open and let $f : J \to \mathbb{R}$ and $g : \Omega \to J$ be n-times continuously differentiable functions. Then $f \circ g : \Omega \to \mathbb{R}$ is n-times continuously differentiable and for any choice of $k_1, \ldots, k_n \in \{1, \ldots, m\}$

$$\frac{\partial^n}{\partial x_{k_1} \cdots \partial x_{k_n}} (f \circ g) = \sum_{\mathfrak{P} \in \Pi_n} (f^{(|\mathfrak{P}|)} \circ g) \prod_{B \in \mathfrak{P}} \frac{\partial^{|B|} g}{\prod_{j \in B} \partial x_{k_j}}. \tag{5.26}$$

Applying the bounds (5.23), we find

$$\|I_{k,\ell}(T)\| \leq \left(\frac{2}{\delta}\right)^{k-\ell} e^{-\frac{\delta}{4}T}\left[\int_{-1}^{1} 1\mathsf{d}t + 2\int_{0}^{\infty} e^{-Tt}\mathsf{d}t\right].$$

Since $P_{k,\ell}(T)$ only grows polynomially in T, this implies $\|Q_T^{(k)}(\mu_0)\| \xrightarrow{T\to\infty} 0$ and especially proves $\left\langle \Omega_\downarrow, Q_T^{(k)}(\mu_0)\Omega_\downarrow\right\rangle$ is uniformly bounded in T. □

We can now give the

Proof of Theorem 1.25. First, we recall the definition of $\mathfrak{Z}_T(\lambda,\mu)$ in (1.10) and the notation $\langle\!\langle\cdot\rangle\!\rangle_{T,\lambda,\mu}$ from (1.11). By the dominated convergence theorem, one sees that $\mathfrak{Z}_T(\lambda,\mu)$ is infinitely often differentiable in μ and has the derivatives

$$\begin{aligned}\partial_\mu^n \mathfrak{Z}_T(\lambda,\mu) &= (-1)^n \mathbb{E}_X\left[\left(\int_0^T X_t\mathsf{d}t\right)^n \exp\left(\lambda^2\int_0^T\int_0^T W(t-s)X_tX_s\mathsf{d}s\mathsf{d}t - \mu\int_0^T X_t\mathsf{d}t\right)\right]\\ &= (-1)^n \mathfrak{Z}_T(\lambda,\mu)\left\langle\!\!\left\langle\left(\int_0^T X_t\mathsf{d}t\right)^n\right\rangle\!\!\right\rangle_{T,\lambda,\mu}. \end{aligned} \tag{5.27}$$

Further, first using Theorem 5.10 and the Faà di Bruno formula (5.26) to calculate the derivatives of the logarithm yields

$$\begin{aligned}\partial_\mu^n E(\lambda,\mu) &= -\lim_{T\to\infty}\frac{1}{T}\sum_{\mathfrak{P}\in\Pi_n}\frac{(-1)^{|\mathfrak{P}|-1}(|\mathfrak{P}|-1)!}{(\langle\Omega_\downarrow, e^{-TH(\lambda,\mu)}\Omega_\downarrow\rangle)^{|\mathfrak{P}|}}\prod_{B\in\mathfrak{P}}\partial_\mu^{|B|}\left\langle\Omega_\downarrow, e^{-TH(\lambda,\mu)}\Omega_\downarrow\right\rangle\\ &= -\lim_{T\to\infty}\frac{1}{T}\sum_{\mathfrak{P}\in\Pi_n}\frac{(-1)^{|\mathfrak{P}|-1}(|\mathfrak{P}|-1)!}{(\mathfrak{Z}_T(\lambda,\mu))^{|\mathfrak{P}|}}\prod_{B\in\mathfrak{P}}\partial_\mu^{|B|}\mathfrak{Z}_T(\lambda,\mu),\end{aligned} \tag{5.28}$$

where we inserted the identity (5.19) in the last line (which in turn follows from Theorem 1.20). Combining (5.27) and (5.28) proves the statement. □

To conclude this chapter, we express derivatives of the ground state energy in terms of the so-called Ursell functions [Per75] or cumulants. This allows us to use correlation inequalities to prove bounds on derivatives. In fact, we will use this in the next Chapter 6 to estimate the second derivative with respect to the magnetic field at zero. Given random variables $Y_1,\dots,Y_n$ on the measure space of the jump process X_t, we define the Ursell function

$$u_n(Y_1,\dots,Y_n) = \frac{\partial^n}{\partial h_1\cdots\partial h_n}\ln\left\langle\!\!\left\langle\exp\left(\sum_{j=1}^n h_iY_i\right)\right\rangle\!\!\right\rangle_{T,\lambda,\mu}\Bigg|_{h_i=0}, \tag{5.29}$$

where the expectation value $\langle\!\langle\cdot\rangle\!\rangle_{T,\lambda,\mu}$ is defined as in (1.11).

Corollary 5.12. *Let $\lambda,\mu\in\mathbb{R}$ and suppose $H(\lambda,\mu)$ has a spectral gap. Then, for all $n\in\mathbb{N}$, the following derivatives exist and satisfy*

$$\partial_\mu^n E(\lambda,\mu) = -\lim_{T\to\infty}\frac{1}{T}\int_{[0,T]^n} u_n(X_{s_1},\dots,X_{s_n})\mathsf{d}(s_1,\dots,s_n).$$

Proof. Using the multivariate Faà die Bruno formula (5.26) and the definition of the Ursell functions (5.29), we find

$$u_n\left(\int_0^T X_{s_1}\mathsf{d}s_1,\ldots,\int_0^T X_{s_n}\mathsf{d}s_n\right) = \sum_{\mathfrak{P}\in\Pi_n}(-1)^{|\mathfrak{P}|+n}(|\mathfrak{P}|-1)!\prod_{B\in\mathfrak{P}}\left\langle\left(\int_0^T X_t\mathsf{d}t\right)^{|B|}\right\rangle_{T,\lambda,\mu}.$$

Now, the Ursell functions are multilinear, cf. [Per75, Section 11], and by the dominated convergence theorem we can hence exchange the integrals with the expectation value, i.e.,

$$u_n\left(\int_0^T X_{s_1}\mathsf{d}s_1,\ldots,\int_0^T X_{s_n}\mathsf{d}s_n\right) = \int_0^T\cdots\int_0^T u_n\left(X_{s_1},\ldots,X_{s_n}\right)\mathsf{d}s_1\cdots\mathsf{d}s_n.$$

Inserting this into Theorem 1.25 finishes the proof. □

6. Existence of Ground States in the Spin Boson Model

In this chapter, we prove Theorem 1.14. The proof essentially consists of three ingredients. First, we prove the existence of a ground state under the assumption that the bosons are massive. Second, assuming a resolvent bound holds, we prove these ground states are embedded into a compact subset of $\mathbb{C}^2 \otimes \mathcal{F}$, which is independent of the boson mass. This allows us to take the mass to zero and obtain a ground state for the massless spin boson Hamiltonian in the limit. Third, we then use the results from Chapters 4 and 5 to prove the resolvent bound and hence conclude the proof of Theorem 1.14.

The chapter is structured as follows. In the preliminary Section 6.1, we will prove that a ground state of the spin boson Hamiltonian is unique if it exists, independent of a hypothetical boson mass. Then, in Section 6.2, we will prove that the ground state energy of the massive spin boson Hamiltonian is isolated from the essential spectrum and hence a unique ground state exists. Assuming the aforementioned resolvent bound holds, we derive infrared properties of these ground states in Section 6.3. We can then use these infrared bounds to construct a compact set containing ground states of the massive model for all boson masses. This allows us to prove existence of ground sates under this assumption in Section 6.4. We then combine the results from the previous chapters to prove that the resolvent bound holds under Hypothesis SBE and hence conclude the proof of Theorem 1.14.

Throughout this chapter, we assume Hypothesis SB0 holds and drop the lower index SB of the operators $H_{\mathsf{SB}}(\lambda)$ and $H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda, \mu)$.

6.1. Uniqueness of Ground States

In this section, we prove that ground states of the spin boson model with external magnetic field are unique.

Theorem 6.1. *Let $\lambda, \mu \in \mathbb{R}$. If $H^{(\mathsf{m})}(\lambda, \mu)$ has a ground state, then it is unique.*

Proofs of this statement for the case $\mu = 0$ can, for example, be found in [HH11b, DM20b]. We extend them to the case $\mu \neq 0$.

Our proof uses the Perron-Frobenius-Faris theorem (Theorem A.112) and the positivity of the operator $\widetilde{H}^{(\mathsf{m})}(\lambda, \mu)$ as defined in (5.2). Theorem 6.1 is a corollary of the following statement.

Lemma 6.2. *Let Θ be the natural unitary $\mathbb{C}^2 \otimes \mathcal{F} \to L^2(\{1,2\} \times \mathcal{Q})$ corresponding to the decomposition $L^2(\mathbb{R}^d) = L^2_{\mathbb{R}}(\mathbb{R}^d) \oplus \mathsf{i} L^2_{\mathbb{R}}(\mathbb{R}^d)$, cf. Lemma B.30 and* (5.18)*. Then, for all $t > 0$ and $\lambda, \mu \in \mathbb{R}$, the operator $\Theta e^{-t\widetilde{H}^{(\mathsf{m})}(\lambda,\mu)} \Theta^*$ is positivity improving.*

Proof. We want to apply the perturbative argument from Lemma A.111.

Let us first consider

$$e^{-T\Theta\widetilde{H}^{(\mathsf{m})}(0,\mu)\Theta^*} = e^{-T((1-\sigma_x)+\mu\sigma_z)}e^{-T(\Theta_{L^2_{\mathbb{R}}(\mathbb{R}^d)}\mathsf{d}\Gamma(\omega)\Theta^*_{L^2_{\mathbb{R}}(\mathbb{R}^d)})}. \tag{6.1}$$

Note that the first factor in (6.1) only acts on the variables $\{1,2\}$, and the second factor only acts on the variables in $\mathcal{Q}_{L^2_{\mathbb{R}}(\mathbb{R}^d)}$. The first factor on the right hand side of (6.1) can be calculated as

$$\exp\left(-T((1-\sigma_x)+\mu\sigma_z)\right) = \begin{pmatrix} e^{-(\mu+1)T} & e^T \\ e^{(\mu+1)T} & e^{-T} \end{pmatrix}$$

and hence is positivity improving on $L^2(\{1,2\})$, since all matrix elements are strictly positive. Further, the second factor on the right hand side of (6.1) is positivity improving by Lemmas B.30 and B.32 and Theorem A.112, since it has the unique strictly positive ground state eigenvector $\Theta_{L^2_{\mathbb{R}}(\mathbb{R}^d)}\Omega = 1$. Hence, the operator $e^{-T\Theta\widetilde{H}^{(\mathsf{m})}(0,\mu)\Theta^*}$ is positivity improving for all $T>0$.

Now, we consider the operator

$$V = \Theta(\sigma_x \otimes \varphi(v))\Theta^*.$$

As a multiplication operator, it is obvious that setting $V_\Lambda = \chi_{\{|\cdot|\le\Lambda\}}(V)V$ for $\Lambda>0$ the operator e^{-tV_Λ} is positivity preserving and satisfies $\langle f, e^{-tV_\Lambda}g\rangle = 0$ for all $f,g\in L^2_+(\mathbb{R}^d)$ with $\langle f,g\rangle = 0$.

Since V is also infinitesimally $\widetilde{H}^{(\mathsf{m})}(0,\mu)$-bounded, by Lemma B.22, the statement now follows from Lemma A.111. □

We now easily obtain the

Proof of Theorem 6.1. Let U be defined as in (5.1). Then, by the construction (5.2), $E(\lambda,\mu) = \inf\sigma(H^{(\mathsf{m})}(\lambda,\mu))$ is an eigenvalue of $H^{(\mathsf{m})}(\lambda,\mu)$ with eigenvector ψ if and only if $1+E(\lambda,\mu)$ is an eigenvalue of $\widetilde{H}^{(\mathsf{m})}(\lambda,\mu)$ with eigenvector $(U\otimes\mathbb{1})\psi$. Hence, the statement follows from Theorem A.112 and Lemma 6.2. □

6.2. Ground States in the Massive Spin Boson Model

In this section, we consider the massive spin boson model. We prove that there exists a ground state isolated from the essential spectrum for all values of the magnetic field. This is, in fact, stronger than the statement we need for our proof, since we therein only need the case $\mu = 0$.

We will from now write

$$E^{(\mathsf{m})}(\lambda,\mu) = \inf\sigma(H^{(\mathsf{m})}(\lambda,\mu)) \qquad \text{and} \qquad m_\omega = \operatorname*{ess\,inf}_{k\in\mathbb{R}^d}\omega(k). \tag{6.2}$$

Here, m_ω can be understood to be the boson mass.

The central statement of this section is the following theorem.

Theorem 6.3. *If $m_\omega > 0$, then $E^{(\mathsf{m})}(\lambda,\mu)$ is a simple eigenvalue of $H^{(\mathsf{m})}(\lambda,\mu)$ isolated from the essential spectrum for any values of $\lambda,\mu\in\mathbb{R}$.*

Remark 6.4. The statement for the case $\mu = 0$ can, for example, be obtained by combining the result in [AH95] with Theorem 6.1.

We obtain the above theorem as a corollary of the following.

Theorem 6.5. *For all $\lambda, \mu \in \mathbb{R}$, we have*

$$\inf \sigma_{\mathsf{ess}}(H^{(\mathsf{m})}(\lambda,\mu)) \geq E^{(\mathsf{m})}(\lambda,\mu) + m_\omega.$$

Remark 6.6. The statement can be seen as one half of the HVZ theorem for the spin boson model. Combined with a similar statement to the one in Proposition 3.5, it is possible to prove

$$\sigma_{\mathsf{ess}}(H^{(\mathsf{m})}(\lambda,\mu)) = [E^{(\mathsf{m})}(\lambda,\mu) + m_\omega, \infty).$$

Here, we restrict our attention to the proof of the lower bound.

A prominently found proof of HVZ theorems in the literature uses localization estimates. Heuristically, the argument therein can be seen as follows: One first confines the bosons to a ball of radius L in position space. As in the typical intuition of quantum mechanics, confined particles have discrete spectrum and to observe the essential spectrum one needs the presence of an unconfined particle. In the limit $L \to \infty$, the confined system behaves like the full Hamiltonian and hence the essential spectrum starts, when one free boson (which has at least the energy m_ω) is added to the system.

Applications of such localization techniques for the proof of HVZ theorems and hence for the existence of a ground state in the case of massive bosons can, for example, be found in [DG99, GLL01, Møl05, LMS07, HS20].

However, the use of localization estimates comes with a small downside. Explicitly, to bound the error terms obtained by confining the system to a ball of radius L, one needs to estimate the commutator of the multiplication operator ω and the Fourier multiplier $\eta(-\mathsf{i}\nabla/L)$, where η is a smooth and compactly supported function. Bounds on the commutator can be easily obtained, when ω is Lipschitz-continuous (cf. [HS20, Proof of Lemma 24]). However, for less regular choices of the dispersion relation, a generalization of the standard localization approach does not seem obvious.

Hence, we here use a related but slightly different approach, which we learned from [DM20b], allowing us to work directly in momentum space and without any regularity assumptions on ω going beyond Hypothesis SB0. The proof needs several approximation steps, so we start out with a convergence lemma.

Lemma 6.7. *Let $(\omega_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ be chosen such that Hypothesis SB0 with $\omega = \omega_k$ and $v = v_k$ is satisfied and define $H_k^{(\mathsf{m})}(\lambda,\mu)$ to be the operator defined in Definition 1.18 with $\omega = \omega_k$ and $v = v_k$. Further, assume*

$$\lim_{k\to\infty} \left\| \frac{\omega_k}{\omega} - 1 \right\|_\infty = \lim_{k\to\infty} \left\| \frac{\omega}{\omega_k} - 1 \right\|_\infty = \lim_{k\to\infty} \|v - v_k\|_2 = \lim_{k\to\infty} \left\|\omega^{-1/2} v - \omega_k^{-1/2} v_k\right\|_2 = 0.$$

Then, for all $\lambda, \mu \in \mathbb{R}$, the operators $H_k^{(\mathsf{m})}(\lambda,\mu)$ are uniformly bounded below and converge to $H^{(\mathsf{m})}(\lambda,\mu)$ in the norm resolvent sense.

Remark 6.8. If ω and ω_k are uniformly bounded above and below by some positive constants, then the uniform convergence assumptions are easily seen to be equivalent to $\|\omega_k - \omega\|_\infty \xrightarrow{k\to\infty} 0$.

Proof. Since the lower bound on $H^{(\mathsf{m})}$ is obtained from Lemma B.20 (vii) and the Kato-Rellich theorem (Theorem A.45), the uniform lower bound follows easily from the L^2-convergence assumptions.

Writing $\omega_\infty = \omega$ and assuming $f \in \mathcal{F}^{(n)} \cong L^2_{\mathsf{s}}(\mathbb{R}^{n\cdot d})$, we easily observe

$$\left\|\mathsf{d}\Gamma^{(n)}(\omega_k)f\right\| \le \left\|\frac{\omega_k}{\omega_{k'}}\right\|_\infty \left\|\mathsf{d}\Gamma^{(n)}(\omega_{k'})f\right\| \qquad \text{for } k,k' \in \mathbb{N}\cup\{\infty\}$$

from Definition B.11 and Remark B.13. Hence, we find $\mathcal{D}(\mathsf{d}\Gamma(\omega_k)) = \mathcal{D}(\mathsf{d}\Gamma(\omega))$ for all $k \in \mathbb{N}$, by Lemma B.15 (iii). Similarly, for $\psi \in \mathcal{D}(\omega)$, we have

$$\|(\mathsf{d}\Gamma(\omega_k) - \mathsf{d}\Gamma(\omega))\,\psi\| \le \left\|\frac{\omega_k}{\omega} - 1\right\|_\infty \|\mathsf{d}\Gamma(\omega)\psi\|.$$

Further, observe that the assumptions easily imply

$$\left\|\omega^{-1/2}(v - v_k)\right\|_2 \xrightarrow{k\to\infty} 0.$$

Now, by Lemmas 1.19 and A.44, $\left\|(B\otimes \mathsf{d}\Gamma(\omega))(H^{(\mathsf{m})}+\mathsf{i})^{-1}\right\|$ is bounded for any choice of the matrix B. Hence, using the resolvent identity (Lemma A.29) as well as the standard bounds Lemmas A.63 and B.20 (vii), we find

$$\begin{aligned}\left\|\left(H_k^{(\mathsf{m})}+\mathsf{i}\right)^{-1} - \left(H^{(\mathsf{m})}+\mathsf{i}\right)^{-1}\right\| \le{}& \left\|\frac{\omega_k}{\omega}-1\right\|_\infty \left\|(\mathbb{1}\otimes \mathsf{d}\Gamma(\omega))\left(H^{(\mathsf{m})}+\mathsf{i}\right)^{-1}\right\| \\ &+ |\lambda|\left\|\omega^{-1/2}(v_k - v)\right\|_2 \|(\sigma_x\otimes \mathsf{d}\Gamma(\omega))\left(H^{(\mathsf{m})}+\mathsf{i}\right)^{-1}\| \\ &+ |\lambda|\left\|v_k - v\right\|_2 \left\|(\sigma_x\otimes \mathbb{1})\left(H^{(\mathsf{m})}+\mathsf{i}\right)^{-1}\right\| \\ &\xrightarrow{k\to\infty} 0.\end{aligned}$$

□

Proof of Theorem 6.5. It suffices to treat the case $m_\omega > 0$, since the statement is trivial otherwise. The proof has three steps and we fix $\lambda, \mu \in \mathbb{R}$ throughout.

Step 1. We first prove the statement in a very simplified case: Assume $M \subset \mathbb{R}^d$ is a bounded and measurable set, $\omega\chi_M$ and $v\chi_M$ are simple functions on M and $v = 0$ almost everywhere on M^c.

Let M_k for $k = 1,\dots,N$ be a disjoint partition of M into measurable sets such that $\omega\restriction_{M_k}$ and $v\restriction_{M_k}$ are constant for each $k = 1,\dots,N$. We define

$$\mathcal{V} = \operatorname{span}\{\chi_{M_k} : k = 1,\dots,N\} \subset L^2(\mathbb{R}^d).$$

Since $\mathcal{V}$ is finite-dimensional, it is closed and we have the decomposition $L^2(\mathbb{R}^d) = \mathcal{V}\oplus\mathcal{V}^\perp$. Observing that by the assumptions $v \in \mathcal{V}$, we can define

$$T = \sigma_z \otimes \mathbb{1} + \mathbb{1}\otimes \mathsf{d}\Gamma(\omega) + \sigma_x \otimes (\lambda\varphi(v) + \mu\mathbb{1}) \quad \text{as operator on } \mathbb{C}^2\otimes\mathcal{F}(\mathcal{V}).$$

Combining the unitary map $\mathcal{F} = \mathcal{F}(\mathcal{V}\oplus\mathcal{V}^\perp) \to \mathcal{F}(\mathcal{V})\otimes\mathcal{F}(\mathcal{V}^\perp)$ from Lemma B.23 with the natural identification (see also Lemma 3.7)

$$\mathcal{F}(\mathcal{V})\otimes\mathcal{F}(\mathcal{V}^\perp) \cong \mathcal{F}(\mathcal{V}) \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}(\mathcal{V})\otimes\left(\mathcal{V}^\perp\right)^{\otimes_{\mathsf{s}} n},$$

we obtain a unitary $U : \mathbb{C}^2 \otimes \mathcal{F} \to (\mathbb{C}^2 \otimes \mathcal{F}(\mathcal{V})) \oplus \bigoplus_{n=1}^{\infty} (\mathbb{C}^2 \otimes \mathcal{F}(\mathcal{V})) \otimes \left(\mathcal{V}^\perp\right)^{\otimes_{\mathsf{s}} n}$. An explicit calculation then gives

$$UH^{(\mathsf{m})}(\lambda, \mu)U^* = T \oplus \bigoplus_{n=1}^{\infty} \left(T \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}\Gamma^{(n)}(\omega)\right). \tag{6.3}$$

Thus $\inf \sigma(T) \geq E^{(\mathsf{m})}(\lambda, \mu)$.

Now, assume $\gamma \in \sigma_{\mathsf{ess}}(H^{(\mathsf{m})}(\lambda, \mu))$. Then, by Weyl's criterion (Lemma A.37), there exists a normalized sequence $(\psi_n)_{n\in\mathbb{N}}$ weakly converging to zero such that

$$\lim_{n\to\infty} \left\| \left(H^{(\mathsf{m})}(\lambda, \mu) - \gamma\right) \psi_n \right\| = 0. \tag{6.4}$$

Inserting (6.4) into (6.3) and using $\omega \geq m_\omega$ almost everywhere, we find

$$\gamma \geq E^{(\mathsf{m})}(\lambda, \mu) + m_\omega + \left\langle S(U^*\psi_n)^{(0)}, S^{-1}T(U^*\psi_n)^{(0)} \right\rangle \quad \text{for all } n \in \mathbb{N}, \tag{6.5}$$

where $S = (\mathbb{1} \otimes \mathsf{d}\Gamma(\omega))$. By (6.4), $\|S(U^*\psi_n)^{(0)}\|$ is uniformly bounded in n.

We write

$$\mathcal{F}^{(\leq N)}(\mathcal{V}) = \bigoplus_{n=0}^{N} \mathcal{F}^{(n)}(\mathcal{V}).$$

The assumption $\omega_1 \geq m_\omega > 0$ implies that

$$\lim_{N\to\infty} S^{-1}T \restriction_{\mathcal{F}^{(\leq N)}(\mathcal{V})} = S^{-1}T.$$

Since $\mathcal{F}^{(\leq N)}(\mathcal{V})$ is finite-dimensional by construction $S^{-1}T \restriction_{\mathcal{F}^{(\leq N)}(\mathcal{V})}$ has finite rank for any $N \in \mathbb{N}$ and it follows that $S^{-1}T$ is compact, cf. Lemmas A.20 and A.21. Hence, applying Lemma A.22, the last term on the right hand side of (6.5) converges to zero as $n \to \infty$.

This finishes the first step. $\Diamond$

Step 2. We now relax the condition that ω and v must be simple: Assume $M \subset \mathbb{R}$ is a bounded measurable set, $\omega\chi_M$ is bounded and $v = 0$ almost everywhere on M^c.

By the simple function approximation lemma, we can pick a sequence $(\omega_k)_{k\in\mathbb{N}}$ of simple functions on M uniformly converging to ω. Outside of M, we set ω_k equal to ω. Further, w.l.o.g., we can assume that there exist constants $a, b > 0$ such that $a \leq \omega, \omega_k \leq b$ holds on M, by the assumptions that $m_\omega > 0$ and ω is bounded on M.

For given $k \in \mathbb{N}$, let $M_{k,i}, i = 1, \dots, N_k$ be a disjoint partition of M into measurable sets such that $\omega_k \restriction_{M_{k,i}}$ is constant for all $i = 1, \dots, N_k$. Further, w.l.o.g, we can assume that

$$\min_{i=1,\dots,N_k} \operatorname{diam}(M_{k,i}) \xrightarrow{k\to\infty} 0, \tag{6.6}$$

where diam denotes the usual diameter of a bounded set. Then, we define a projection P onto the simple functions with support in M by

$$P_k f = \sum_{i=1}^{N_k} \frac{\chi_{M_{k,i}}}{\operatorname{vol}(M_{k,i})} \int_{M_{k,i}} f(x) \mathsf{d}x,$$

which can be easily verified to be well-defined for any $f \in L^2(\mathbb{R}^d)$. If f is continuous and compactly supported on M, then it is straightforward to verify $P_k f \xrightarrow{k\to\infty} f$ in L^2-sense.

Since the continuous, compactly supported functions are dense in $L^2(M)$, this implies $\operatorname*{s-lim}_{k\to\infty} P_k = \mathbb{1}_{L^2(M)}$ as operators on $L^2(M)$.

We now define

$$v_k = \omega_k^{1/2} P_k(\omega^{-1/2} v)$$

and observe this directly implies $\omega_k^{-1/2} v$ goes to $\omega^{-1/2} v$ in L^2-sense. Further, a simple calculation yields

$$\begin{aligned}\|v_k - v\|_2^2 &\le \int_M |v_k - \omega_k^{1/2}\omega^{-1/2}v|^2 + \int_M |\omega_k^{1/2}\omega^{-1/2}v - v|^2 \\ &\le b\|P_k(\omega^{-1/2}v) - \omega^{-1/2}v\|_2^2 + \frac{1}{a}\|\omega_k^{1/2} - \omega^{1/2}\|_\infty \|v\|_2^2.\end{aligned}$$

By construction the right hand side goes to zero as $k \to \infty$.

Hence, all assumptions of Lemma 6.7 are satisfied and the operators $H_k^{(\mathsf{m})}(\lambda,\mu)$ obtained by inserting ω_k and v_k into Definition 1.18 are uniformly bounded below and converge to $H^{(\mathsf{m})}(\lambda,\mu)$ in the norm resolvent sense. Further, ω_k and v_k by construction satisfy the assumptions of Step 1. The statement now follows, since the uniform convergence of ω_k to ω implies m_{ω_k} converges to m_ω and the norm resolvent convergence and uniform lower boundedness imply convergence of the ground state energy and the infimum of the essential spectrum (cf. Lemma A.76). $\Diamond$

Step 3. We now move to the general case.

Let $R > 0$ and define

$$M_R = \big(\{k \in \mathbb{R}^d : v(k) \ne 0\} \cap \{k \in \mathbb{R}^d : \omega(k) < R\} \cap B_R(0)\big) \cup \{k \in \mathbb{R}^d : v(k) = 0\}.$$

Set $v_R = \chi_{M_R} v$. Then it is straightforward to verify that both v_R and $\omega^{-1/2}v_R$ converge to v and $\omega^{-1/2}v$ in L^2-sense, respectively. Hence, we can once more apply Lemma 6.7 to see that $H_R^{(\mathsf{m})}(\lambda,\mu)$ obtained by inserting ω and v_R in Definition 1.18 is uniformly bounded below and converges to $H^{(\mathsf{m})}(\lambda,\mu)$ in the norm resolvent sense. Since, ω and v_R also satisfy the assumptions of Step 2, the statement follows due to Lemma A.76. $\Diamond$ $\square$

We conclude this section with the

Proof of Theorem 6.3. By Theorem 6.5, we find $E^{(\mathsf{m})}(\lambda,\mu) \in \sigma_{\mathsf{d}}(H^{(\mathsf{m})}(\lambda,\mu))$. Hence, the statement follows from the definition of the discrete spectrum (Definition A.36) and Theorem 6.1. $\square$

6.3. Ground State Properties

We now want to consider massless bosons without an external magnetic field. In this case, we want to approximate ω by an infrared-regular version.

We will work under the following hypothesis, which is dependent of the coupling constant $\lambda \in \mathbb{R}$. To that end, if $(\omega_n)_{n\in\mathbb{N}}$ is a sequence of dispersion relations satisfying Hypothesis SB0 with $\omega = \omega_n$, we denote by $H_n(\lambda)$ the spin boson Hamiltonian as defined in Definition 1.11 with ω replaced by ω_n.

Hypothesis SBR(λ). We assume Hypothesis SB0 and the following:

(i) There exists $\alpha_1 > 0$ such that ω is locally α_1-Hölder continuous.

(ii) $\omega(k) \xrightarrow{|k|\to\infty} \infty$.

(iii) There exists $\epsilon > 0$ such that $\omega^{-1/2}v \in L^{2+\epsilon}(\mathbb{R}^d)$.

(iv) There exists $\alpha_2 > 0$ such that $\sup\limits_{|p|\le 1} \displaystyle\int_{\mathbb{R}^d} \frac{|v(k+p) - v(k)|}{\sqrt{\omega(k)}|p|^{\alpha_2}} \mathsf{d}k < \infty$.

(v) $\sup\limits_{|p|\le 1} \displaystyle\int_{\mathbb{R}^d} \frac{|v(k)|}{\sqrt{\omega(k)}\omega(k+p)} \mathsf{d}k < \infty$.

(vi) There exists a decreasing sequence $(\omega_n)_{n\in\mathbb{N}}$ of nonnegative measurable functions $\omega_n : \mathbb{R}^d \to \mathbb{R}$ converging uniformly to ω and satisfying
   - ω_n is locally α_1-Hölder continuous for all $n \in \mathbb{N}$, where α_1 is chosen as in (i),
   - $\inf_{k\in\mathbb{R}^d} \omega_n(k) > 0$.
   - We can choose normalized ground states $\psi_{\lambda,n}$ of $H_n(\lambda)$ such that there exists a constant $C_{\mathsf{R}} > 0$ with

$$(H_n(\lambda) - E_n(\lambda) + \omega_n(k))^{-1}\psi_{\lambda,n} \le C_{\mathsf{R}}\omega_n^{-1/2}(k) \qquad \text{for all } n \in \mathbb{N},\ k \in \mathbb{R}^d. \tag{6.7}$$

Remark 6.9. We note that the first two parts of (vi) are satisfied for the typical choice of a massive boson dispersion relation

$$\omega_n = \sqrt{m_n^2 + \omega^2}, \tag{6.8}$$

or also $\omega_n = \omega + m_n$, where $(m_n)_{n\in\mathbb{N}}$ is any sequence of positive numbers decreasing monotonically to zero. The constant m_n can be understood to be a boson mass. The result we prove is, however, independent of the specific choice of ω_n. Further, we emphasize that we have already proved the existence of the ground states $\psi_{\lambda,n}$ in Theorem 6.3. Hence, the restricting assumption in (vi) is the resolvent bound (6.7).

Remark 6.10. The assumptions from Hypothesis SBE are contained in those of Hypothesis SBR(λ), except for the parts Hypothesis SBE (ii) and (v), which are in turn easily recognized to be the assumptions in Hypothesis SBF. In fact, we will use the FKN formula, or more explicitly its implications for the derivative of the ground state energy given in Theorem 1.25 and Section 5.2, to prove the resolvent bound (6.7) for all values of λ smaller than the critical coupling constant in Theorem 1.14.

Throghout Section 6.3, we assume $\lambda \in \mathbb{R}$ is chosen such that Hypothesis SBR(λ) holds without further mention. Further, from now on, we write

$$E(\lambda) = \inf \sigma(H(\lambda)) \qquad \text{and} \qquad E_n(\lambda) = \inf \sigma(H_n(\lambda)) \qquad \text{for } n \in \mathbb{N}, \lambda \in \mathbb{R}. \tag{6.9}$$

In the next section, we prove the vectors $\psi_{\lambda,n}$ are a minimizing sequence for the operator $H_n(\lambda)$. Then, in Section 6.3.2, we prove infrared bounds on expectation values w.r.t. $\psi_{\lambda,n}$, which will be essential in the construction of a complex set containing $\psi_{\lambda,n}$ for all $n \in \mathbb{N}$.

6.3.1. Minimizing Sequence

In this section, we prove that $(\psi_{\lambda,n})_{n\in\mathbb{N}}$ is a minimizing sequence for $H(\lambda)$. To that end, we first note the following simple lemma.

Lemma 6.11. *We have*

(i) $H(\lambda) \le H_{n'}(\lambda) \le H_n(\lambda)$ *for* $n \le n'$,

(ii) $\lim\limits_{n\to\infty} E_n(\lambda) = E(\lambda)$.

Proof. (i) follows from the monotonicity of the sequence (ω_n) (cf. Hypothesis SBR(λ) (vi)) and Lemma B.15 (i). We set $N = \mathbb{1} \otimes \mathsf{d}\Gamma(1)$. Then, due to the uniform convergence of (ω_n), there is a sequence $(C_n) \subset \mathbb{R}^+$ satisfying $C_n \xrightarrow{n\to\infty} 0$ and $\omega_n \le \omega + C_n$. Hence,

$$\mathsf{d}\Gamma(\omega_n) \le \mathsf{d}\Gamma(\omega) + C_n \mathsf{d}\Gamma(1), \qquad \text{which implies} \qquad H_n(\lambda) \le H(\lambda) + C_n N.$$

On the other hand, let $\varepsilon > 0$ and fix $\varphi_\varepsilon \in \mathcal{D}(N) \cap \mathcal{D}(H_0)$ with $\|\varphi_\varepsilon\| = 1$ such that

$$\langle \varphi_\varepsilon, H(\lambda)\varphi_\varepsilon \rangle \le E + \varepsilon.$$

This is possible, since $\mathcal{D}(N) \cap \mathcal{D}(H_0)$ is a core for $\mathbb{1} \otimes \mathsf{d}\Gamma(\omega)$ and hence for H, by Lemmas B.15 (iii), B.22 and the Kato-Rellich theorem (Theorem A.45). Together with (i), we obtain

$$\begin{aligned} E(\lambda) \le E_n(\lambda) &\le \langle \varphi_\varepsilon, H_n(\lambda)\varphi_\varepsilon \rangle \\ &\le \langle \varphi_\varepsilon, H(\lambda)\varphi_\varepsilon \rangle + C_n \langle \varphi_\varepsilon, N\varphi_\varepsilon \rangle \\ &\le E(\lambda) + \varepsilon + C_n \langle \varphi_\varepsilon, N\varphi_\varepsilon \rangle \xrightarrow{n\to\infty} E(\lambda) + \varepsilon. \end{aligned}$$

Now, (ii) follows in the limit $\varepsilon \to 0$. □

A main ingredient of our proof for existence of ground states is the following lemma.

Lemma 6.12. *The sequence* $(\psi_{\lambda,n})_{n\in\mathbb{N}}$ *is minimizing for* $H(\lambda)$, *i.e.,*

$$0 \le \langle \psi_{\lambda,n}, (H(\lambda) - E(\lambda))\psi_{\lambda,n} \rangle \xrightarrow{n\to\infty} 0.$$

Proof. We use Lemma 6.11 and find

$$0 \le \langle \psi_{\lambda,n}, (H(\lambda) - E(\lambda))\psi_{\lambda,n} \rangle \le \langle \psi_{\lambda,n}, (H_n(\lambda) - E(\lambda))\psi_{\lambda,n} \rangle = E_n(\lambda) - E(\lambda) \xrightarrow{n\to\infty} 0. \quad \square$$

6.3.2. Infrared Bounds

In this section, we derive essential bounds on the ground states $\psi_{\lambda,n}$ which are uniform in $n \in \mathbb{N}$.

The first step is to connect the infrared behavior of $\psi_{\lambda,n}$ to the resolvent bound (6.7). This is done by a so-called pull-through formula, similar to the one stated in Proposition 3.2 and proved in Section 3.4. From (3.2), recall the definition of a_k. Further, for $n \in \mathbb{N}$, we define the operator

$$R_{\lambda,n}(k) = (H_n(\lambda) - E_n(\lambda) + \omega_n(k))^{-1} \quad \text{for } k \in \mathbb{R}^d, \tag{6.10}$$

which is bounded by Hypothesis SBR(λ) (vi), and the spectral theorem (or more precisely Lemma A.63) directly yields

$$\|R_{\lambda,n}(k)\| \leq \frac{1}{\omega_n(k)}. \tag{6.11}$$

In the statement of the pull-through formula, which is similar to [BFS98a, Gér00, DM20b], we write $\psi_{\lambda,n} = (\psi_{\lambda,n,1}, \psi_{\lambda,n,2})$ in the sense of the natural isomorphism $\mathbb{C}^2 \otimes \mathcal{F} \cong \mathcal{F} \oplus \mathcal{F}$ and denote

$$a_k\psi_{\lambda,n} = (a_k\psi_{\lambda,n,1}, a_k\psi_{\lambda,n,2}) \quad \text{and} \quad \sigma_x\psi_{\lambda,n} = (\sigma_x \otimes \mathbb{1})\,\psi_{\lambda,n} = (\psi_{\lambda,n,2}, \psi_{\lambda,n,1})\,. \tag{6.12}$$

The pull-through formula for the spin boson model is the next lemma.

Lemma 6.13. *Let $n \in \mathbb{N}$. Then, for almost every $k \in \mathbb{R}^d$, the vector $a_k\psi_n \in \mathbb{C}^2 \otimes \mathcal{F}$ and*

$$a_k\psi_{\lambda,n} = -v(k)R_{\lambda,n}(k)\sigma_x\psi_{\lambda,n}.$$

Proof. We again use the notation from Appendix B.6 and define the operators

$$\begin{aligned} H_+ &= \sigma_z + \mathsf{d}\Gamma_+(\omega) + \lambda\sigma_x\varphi_+(v) \qquad &&\text{on } \mathcal{F}_+ \times \mathcal{F}_+,\\ H_\oplus &= \sigma_z + \mathsf{d}\Gamma_\oplus(\omega) + \lambda\sigma_x\varphi_\oplus(v) &&\text{on } \mathscr{C}(\mathbb{R}^d) \times \mathscr{C}(\mathbb{R}^d), \end{aligned}$$

where in our notation the operators $\mathsf{d}\Gamma$ and φ act componentwise and the Pauli-matrices act as usually on vectors. Using Lemmas B.37, B.39 and B.49 and that $\psi_{\lambda,n}$ is a ground state of $H_n(\lambda)$, we find

$$(H_\oplus - E_n(\lambda))\,A\psi_{\lambda,n} = A(H_n(\lambda) - E_n(\lambda))\psi_{\lambda,n} - M_v\sigma_x\psi_{\lambda,n} = -M_v\sigma_x\psi_{\lambda,n}.$$

Especially, there exists a zero-set $N \subset \mathbb{R}^d$ such that

$$(H_\oplus - E_n(\lambda))\,A\psi_{\lambda,n}(k) = (H_+ - E_n(\lambda))\,A\psi_{\lambda,n}(k) \in \mathcal{F} \qquad \text{for all } k \in N^c.$$

Since $\mathbb{C}^2 \odot \mathcal{F}_{\mathsf{fin}}(L^2(\mathbb{R}^d))$ (where $\odot$ denotes the algebraic tensor product, Definition A.99) is a core for $\mathbb{1} \otimes \mathsf{d}\Gamma(\omega)$ (cf. Lemma B.15 (iii)), it is a core for $H(\lambda)$ by the Kato-Rellich theorem (Theorem A.45) and the set

$$D_k = \{\phi \in \mathbb{C}^2 \odot \mathcal{F}_{\mathsf{fin}}(L^2(\mathbb{R}^d)) : R_{\lambda,n}(k)\phi \in \mathcal{F}_{\mathsf{fin}}(L^2(\mathbb{R}^d))\}$$

is dense in $\mathbb{C}^2 \otimes \mathcal{F}$ for any $k \in N^c$. For $k \in N^c$ and $\phi \in D_k$, using Lemma B.40, we obtain

$$\begin{aligned} \langle \phi, A\psi_{\lambda,n}(k)\rangle_+ &= \langle (H_n(\lambda) - E_n(\lambda) + \omega_n(k))R_{\lambda,n}(k)\phi, A\psi_{\lambda,n}(k)\rangle_+ \\ &= \langle R_{\lambda,n}(k)\phi, (H_+ - E_n(\lambda) + \omega_n(k))A\psi_{\lambda,n}(k)\rangle_+ \\ &= \langle R_{\lambda,n}(k)\phi, -v(k)\sigma_x\psi_{\lambda,n}(k)\rangle_+ \\ &= \langle \phi, -v(k)R_{\lambda,n}(k)\sigma_x\psi_{\lambda,n}(k)\rangle_+\,. \end{aligned}$$

Combining this with Lemmas B.43 and B.48 finishes the proof. □

We combine the pull-through formula with the resolvent bound (6.7).

Lemma 6.14. *Let $B_1 = \{x \in \mathbb{R}^d : |x| \leq 1\}$.*

(i) *For all $n \in \mathbb{N}$ and almost all $k \in \mathbb{R}^d$, we have* $\|a_k\psi_{\lambda,n}\| \le C_{\mathsf{R}}\dfrac{|v(k)|}{\sqrt{\omega(k)}}$.

(ii) *There exist an $\alpha > 0$ and a measurable function $h : B_1 \times \mathbb{R}^d \to [0,\infty)$ with*

$$\sup_{p\in B_1} \|h(p,\cdot)\|_1 < \infty$$

such that for all $n \in \mathbb{N}$ and almost all $p \in B_1$ and $k \in \mathbb{R}^d$

$$\|a_{k+p}\psi_{\lambda,n} - a_k\psi_{\lambda,n}\| \le |p|^\alpha h(p,k).$$

Proof. (i) follows from Lemma 6.13 and (6.7) as well as the monotonicity of $(\omega_n)_{n\in\mathbb{N}}$.

Let α_1, α_2 as in Hypothesis SBR(λ). Then, we set $\alpha = \min\{\alpha_1,\alpha_2\}$ and

$$\tilde{h}(p,k) = \max\left\{ \frac{|v(k+p)-v(k)|}{|p|^\alpha\sqrt{\omega(k)}}, \frac{|v(k+p)|}{\omega(k)\sqrt{\omega(k+p)}} \right\}.$$

By Hypothesis SBR(λ) (iv) and (v), $\tilde{h}$ satisfies the above statements on h. Further, using the resolvent identity (Lemma A.29) and Lemma 6.13, we obtain

$$\begin{aligned}
a_{k+p}\psi_{\lambda,n} - a_k\psi_{\lambda,n} &= v(k)R_{\lambda,n}(k)\sigma_x\psi_{\lambda,n} - v(k+p)R_{\lambda,n}(k+p)\sigma_x\psi_{\lambda,n} \\
&= (v(k)-v(k+p))R_{\lambda,n}(k)\sigma_x\psi_{\lambda,n} \\
&\quad + v(k+p)(R_{\lambda,n}(k) - R_{\lambda,n}(k+p))\sigma_x\psi_{\lambda,n} \\
&= (v(k)-v(k+p))R_{\lambda,n}(k)\sigma_x\psi_{\lambda,n} && (6.13)\\
&\quad + v(k+p)R_{\lambda,n}(k)(\omega_n(k+p)-\omega_n(k))R_{\lambda,n}(k+p)\sigma_x\psi_{\lambda,n}. && (6.14)
\end{aligned}$$

By (6.7) and Hypothesis SBR(λ) (iv), we find

$$|(6.13)| \le C_{\mathsf{R}}\frac{|v(k+p)-v(k)|}{\sqrt{\omega(k)}} \le C_{\mathsf{R}}|p|^\alpha\tilde{h}(p,k).$$

Further, the local α_1-Hölder continuity of ω_n yields there is $C > 0$ such that

$$|(6.14)| \le C|p|^\alpha\tilde{h}(p,k).$$

This proves the statement for the function $h = (C_{\mathsf{R}} + C)\tilde{h}$. □

We use the above infrared bounds to derive an upper bound for the expectation values of the boson number operator and the free field energy

$$N = \mathbb{1}\otimes \mathsf{d}\Gamma(1) \quad \text{and} \quad H_{\mathrm{f}} = \mathbb{1}\otimes\mathsf{d}\Gamma(\omega) \tag{6.15}$$

with respect to the ground state $\psi_{\lambda,n}$.

Lemma 6.15. *For all $n \in \mathbb{N}$, we have $\psi_{\lambda,n} \in \mathcal{D}(N^{1/2}) \cap \mathcal{D}(H_{\mathrm{f}})$ and the inequalities $\langle N^{1/2}\psi_n, N^{1/2}\psi_n\rangle \le C_{\mathsf{R}}^2\|\omega^{-1/2}f\|^2$ and $\langle\psi_n, H_{\mathrm{f}}\psi_n\rangle \le C_{\mathsf{R}}^2\|f\|^2$.*

Proof. The property $\psi_{\lambda,n} \in \mathcal{D}(H_{\mathrm{f}})$ is contained in the domain statement in Lemma 1.13. The remaining statements follow from combining Lemmas 6.14 (i) and B.50. □

6.3.3. The Compactness Argument

In this section, we construct a compact set $K \subset \mathbb{C}^2 \otimes \mathcal{F}$ such that $(\psi_{\lambda,n})_{n\in\mathbb{N}} \subset K$.

Let us begin with the definition of K. To that end, assume y_i for $i = 1, \ldots, \ell$ is the position operator acting on $\psi^{(\ell)} \in \mathcal{F}^{(\ell)}$ as

$$\widehat{y_i\psi^{(\ell)}}(x_1, \ldots, x_\ell) = x_i\widehat{\psi^{(\ell)}}(x_1, \ldots, x_\ell), \tag{6.16}$$

where $\widehat{\cdot}$ denotes the Fourier transform. For $\delta > 0$, we now define a closed quadratic form q_δ acting on $\phi = (\phi_1, \phi_2) \in \mathcal{Q}(q_\delta) \subset \mathcal{H}$ with natural domain as

$$q_\delta(\phi) = \langle N^{1/2}\phi, N^{1/2}\phi\rangle + \sum_{\substack{\ell\in\mathbb{N}\\ s\in\{1,2\}}} \frac{1}{\ell^2}\sum_{i=1}^{\ell} \left\langle \phi_s^{(\ell)}, |y_i|^\delta \phi_s^{(\ell)}\right\rangle + \langle H_{\mathrm{f}}^{1/2}\phi, H_{\mathrm{f}}^{1/2}\phi\rangle, \tag{6.17}$$

where N and H_{f} are defined as in (6.15). Further, we define

$$K_{\delta,C} := \{\phi \in \mathcal{Q}(q_\delta) : \|\phi\| \le 1, q_\delta(\phi) \le C\} \qquad \text{for } C > 0. \tag{6.18}$$

We first verify that the set $K_{\delta,C}$ is compact.

Lemma 6.16. *For all $\delta, C > 0$ the set $K_{\delta,C}$ is compact.*

Proof. By definition, q_δ is nonnegative. Hence, there exists a self-adjoint positive operator T associated to q_δ, cf. Theorem A.85. By Lemma A.95, K_C is compact if and only if the i-th eigenvalues of T obtained by the min-max principle $\eta_i(T)$ (cf. Definition A.94) tend to infinity as $i \to \infty$.

To that end, we observe T preserves the ℓ-boson sectors $\mathbb{C}^2 \otimes \mathcal{F}^{(\ell)}$ and define the restriction $T_\ell = T \upharpoonright_{\mathbb{C}^2\otimes\mathcal{F}^{(\ell)}}$. Now, since $(\omega + 1)^{(\ell)}(K) \to \infty$ as $K \to \infty$ by Hypothesis SBR(λ) (ii), we can apply Rellich's criterion (Lemma A.96) and, combined with Lemma A.95, $\lim\limits_{i\to\infty} \eta_i(T_\ell) = \infty$ for all $\ell \in \mathbb{N}_0$. Further, since $T_\ell \ge \ell$, we have $\eta_i(T_\ell) \ge \ell$ by definition and therefore $\lim\limits_{i\to\infty} \eta_i(T) = \infty$. This finishes the proof. □

We need the following proposition to prove existence of ground states of $H(\lambda)$.

Proposition 6.17. *There are $\delta, C > 0$, such that $\psi_{\lambda,n} \in K_{\delta,C}$ for all $n \in \mathbb{N}$.*

For the proof the following lemma is essential. To that end, for $n \in \mathbb{N}$, $s \in \{1, 2\}$ and $y, k \in \mathbb{R}^d$, we introduce the notation

$$\begin{aligned} &\widehat{\psi^{(\ell)}_{\lambda,n,s}}(y) : (y_1, \ldots, y_{\ell-1}) \mapsto \psi^{(\ell)}_{\lambda,n,s}(y, y_1, \ldots, y_{\ell-1}),\\ &\psi^{(\ell)}_{\lambda,n,s}(k) : (k_1, \ldots, k_{\ell-1}) \mapsto \psi^{(\ell)}_{\lambda,n,s}(k, k_1, \ldots, k_{\ell-1}). \end{aligned} \tag{6.19}$$

Due to the Fubini-Tonelli theorem, we have $\widehat{\psi^{(\ell)}_{\lambda,n,s}}(y), \psi^{(\ell)}_{\lambda,n,s}(k) \in L^2(\mathbb{R}^{(\ell-1)d})$ for almost every $k, y \in \mathbb{R}^d$. Further, comparing with the definition (3.2), we observe

$$\psi^{(\ell)}_{\lambda,n,s}(k) = \frac{1}{\sqrt{\ell+1}}(a_k\psi_{\lambda,n,s})^{(\ell)}. \tag{6.20}$$

Lemma 6.18. *There exist $\delta > 0$ and $C > 0$ such that for all $p \in \mathbb{R}^d$ and $n, \ell \in \mathbb{N}$, $s \in \{1,2\}$*

$$\int_{\mathbb{R}^d} |1 - e^{-ipy}|^2 \left\| \widehat{\psi^{(\ell)}_{\lambda,n,s}}(y) \right\|^2_{L^2(\mathbb{R}^{(\ell-1)d})} \mathsf{d}y \leq \frac{C}{\ell+1} \min\left\{1, |p|^\delta\right\}. \tag{6.21}$$

We note that δ can be chosen as $\delta = \dfrac{\epsilon\alpha}{1+\epsilon}$, where the values $\alpha > 0$ and $\epsilon > 0$ are those from Lemma 6.14 (ii) and Hypothesis SBR(λ) (iii), respectively.

Proof. That the left hand side of (6.21) is bounded by a constant C uniformly in p follows easily due to the Fock space definition, since the Fourier transform preserves the L^2-norm. Hence, we can restrict our attention to the case $|p| \leq 1$. We note that

$$\begin{aligned}\int_{\mathbb{R}^d} |1 - e^{-ipy}|^2 \left\| \widehat{\psi^{(\ell)}_{\lambda,n,s}}(y) \right\|^2_{L^2(\mathbb{R}^{(\ell-1)d})} \mathsf{d}y &= \int_{\mathbb{R}^d} \|\psi^{(\ell)}_{\lambda,n,s}(k+p) - \psi^{(\ell)}_{\lambda,n,s}(k)\|^2 \mathsf{d}k \\ &= \frac{1}{\ell+1} \int_{\mathbb{R}^d} \|(a_{k+p}\psi_{\lambda,n,s})^{(\ell)} - (a_k\psi_{\lambda,n,s})^{(\ell)}\|^2 \mathsf{d}k,\end{aligned}$$

where we used (6.20). Let $\theta \in (0,1)$ and write

$$w(p,k) = \max\left\{ \frac{|v(k)|}{\omega(k)^{1/2}}, \frac{|v(k+p)|}{\omega(k+p)^{1/2}} \right\}.$$

By Lemma 6.14, we have some $C > 0$ such that

$$\left\|(a_{k+p}\psi_{\lambda,n,s})^{(\ell)} - (a_k\psi_{\lambda,n,s})^{(\ell)}\right\| \leq C|p|^{\theta\alpha} h(p,k)^\theta w(p,k)^{1-\theta}.$$

For $r, r' > 1$ with $\frac{1}{r} + \frac{1}{r'} = 1$, we now use Young's inequality $bc \leq b^r/r + c^{r'}/r'$ to obtain a constant $C_r > 0$ with

$$\left\|(a_{k+p}\psi_{\lambda,n,s})^{(\ell)} - (a_k\psi_{\lambda,n,s})^{(\ell)}\right\|^2 \leq C_r |p|^{2\theta\alpha} \left(h(p,k)^{2\theta r} + w(p,k)^{2(1-\theta)r'} \right). \tag{6.22}$$

Set $r = \frac{1}{2\theta}$. Then the first summand in (6.22) is integrable in k due to Lemma 6.14. Further, the exponent of the second summand equals

$$2(1-\theta)r' = 2(1-\theta)\left(1 - \frac{1}{r}\right)^{-1} = \frac{2(1-\theta)}{1-2\theta}.$$

Hence, we can choose $\theta > 0$ such that $\dfrac{2(1-\theta)}{1-2\theta} = 2 + \epsilon$. By Hypothesis SBR($\lambda$) (iii), it follows that (6.22) is integrable in k and the proof is complete. □

From here, we can prove an upper bound for the Fourier term in (6.17).

Lemma 6.19. *Let $\delta > 0$ be as in Lemma 6.18. Then there exists $C > 0$ such that for all $n, \ell \in \mathbb{N}$ and $s \in \{1,2\}$*

$$\int_{\mathbb{R}^{d\cdot\ell}} \sum_{i=1}^{\ell} |x_i|^{\delta/2} \left| \widehat{\psi^{(\ell)}_{\lambda,n,s}}(x_1,\dots,x_\ell) \right|^2 \mathsf{d}(x_1,\dots,x_\ell) \leq C.$$

Proof. From Lemma 6.18, we know that there exists a finite constant C such that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|1-e^{-ipy}|^2 \|\widehat{\psi^{(\ell)}_{\lambda,n,s}}(y)\|^2}{|p|^{\delta/2}} \mathsf{d}y \frac{\mathsf{d}p}{|p|^d} \le \frac{C}{\ell+1} .$$

After interchanging the order of integration and a change of integration variables $q = |y|p$, we find

$$\begin{aligned}\frac{C}{\ell+1} &\ge \int_{\mathbb{R}^d} \|\widehat{\psi^{(\ell)}_{\lambda,n,s}}(y)\|^2 \int_{\mathbb{R}^d} \frac{|1-e^{-ipy}|^2}{|p|^{\delta/2}} \frac{\mathsf{d}p}{|p|^d} \mathsf{d}y \\ &= \int_{\mathbb{R}^d} \|\widehat{\psi^{(\ell)}_{\lambda,n,s}}(y)\|^2 |y|^{\delta/2} \underbrace{\int_{\mathbb{R}^d} \frac{|1-e^{-iqy/|y|}|^2}{|q|^{\delta/2}} \frac{\mathsf{d}q}{|q|^d}}_{=:\, c} \mathsf{d}y ,\end{aligned}$$

where c is nonzero and does not depend on y. □

We can now conclude.

Proof of Proposition 6.17. Combine Lemmas 6.15 and 6.19. □

6.4. Proof of Existence

We can now state and prove our existence result for the massless spin boson model.

Theorem 6.20. *Assume $\lambda \in \mathbb{R}$ is chosen such that Hypothesis SBR(λ) holds. Then $E(\lambda)$ is a simple eigenvalue of $H(\lambda)$.*

Proof. Choose $\delta, C > 0$ as in Proposition 6.17. Then, combining Lemma 6.16 and Proposition 6.17, we know there exists a subsequence $(\psi_{\lambda,n_k})_{k\in\mathbb{N}}$, which converges to a normalized vector $\psi_{\lambda,\infty} \in K_{\delta,C}$. By construction and Lemma 1.13, we have

$$K_{\delta,C} \subset \mathcal{D}(H_f^{1/2}) = \mathcal{D}((H(\lambda) - E(\lambda))^{1/2}).$$

Since any closed quadratic form is lower-semicontinuous, cf. Lemma A.81, Lemma 6.12 now yields

$$\begin{aligned}\|(H(\lambda)-E(\lambda))^{1/2}\psi_{\lambda,\infty}\|^2 &= \left\langle (H(\lambda)-E(\lambda))^{1/2}\psi_{\lambda,\infty}, (H(\lambda)-E(\lambda))^{1/2}\psi_{\lambda,\infty}\right\rangle \\ &\le \liminf_{k\to\infty} \langle \psi_{\lambda,n_k}, (H(\lambda)-E(\lambda))\psi_{\lambda,n_k}\rangle = 0.\end{aligned}$$

Hence, $(H(\lambda) - E(\lambda))^{1/2}\psi_{\lambda,\infty} = 0$. This especially implies

$$(H(\lambda)-E(\lambda))^{1/2}\psi_{\lambda,\infty} \in \mathcal{D}((H(\lambda)-E(\lambda))^{1/2}),$$

which in turn gives $\psi_{\lambda,\infty} \in \mathcal{D}(H(\lambda))$, by Lemma A.61 (iii), and yields

$$H(\lambda)\psi_{\lambda,\infty} = E(\lambda)\psi_{\lambda,\infty}.$$

The uniqueness now follows from Theorem 6.1 and the proof is complete. □

To prove Theorem 1.14, we now need to verify that the assumptions of Hypothesis SBR(λ) are fulfilled for all λ sufficiently small. To that end, we from now assume Hypothesis SBE holds. Especially, we note that this implies all assumptions from Hypothesis SBF and Hypothesis SBR(λ) are fulfilled, except for Hypothesis SBR(λ) (vi). As described in Remark 6.9, we can now choose a sequence $(\omega_n)_{n\in\mathbb{N}}$ such that Hypothesis SBF is satisfied with $\omega = \omega_n$ and all assumptions excluding (6.7) from Hypothesis SBR(λ) (vi) are satisfied. Hence, if we can prove (6.7) holds under these assumptions for λ as in Theorem 1.14, we are done.

The first step into this direction is the following implication of Theorem 1.25. We denote

$$E_n^{(\mathsf{m})}(\lambda,\mu) = \inf \sigma(H_n^{(\mathsf{m})}(\lambda,\mu)) \qquad \text{for } \lambda,\mu\in\mathbb{R},\ n\in\mathbb{N}. \tag{6.23}$$

Further, we define W_n similar to (1.9), with ω replaced by ω_n. In the same way as (1.10) and (1.11), we hence define

$$\langle\!\langle Y \rangle\!\rangle_{n,T,\lambda} = \frac{\mathbb{E}_X\left[Y \exp\left(\lambda^2 \int_0^T \int_0^T W_n(t-s) X_t X_s \mathsf{d}t\mathsf{d}s\right)\right]}{\mathbb{E}_X\left[\exp\left(\lambda^2 \int_0^T \int_0^T W_n(t-s) X_t X_s \mathsf{d}t\mathsf{d}s\right)\right]}, \tag{6.24}$$

where we only consider the case $\mu = 0$.

Lemma 6.21. *For all $\lambda\in\mathbb{R}$ and $n\in\mathbb{N}$, the function $\mu\mapsto E_n^{(\mathsf{m})}(\lambda,\mu)$ is twice differentiable in a neighborhood of zero and has derivatives*

$$\partial_\mu E_n^{(\mathsf{m})}(\lambda,0) = 0 \qquad \textit{and} \qquad \partial_\mu^2 E_n^{(\mathsf{m})}(\lambda,0) = -\lim_{T\to\infty}\frac{1}{T}\left\langle\!\!\left\langle \left(\int_0^T X_t \mathsf{d}t\right)^2 \right\rangle\!\!\right\rangle_{n,T,\lambda}.$$

Proof. Due to the definition, we have $\inf_{k\in\mathbb{R}^d}\omega_n(k) > 0$ for all $n\in\mathbb{N}$ and hence $H(\lambda,0)$ has a spectral gap by Theorem 6.3. Thus, Theorem 1.25 is applicable. Now, observe that due to the so-called spin-flip-symmetry of the model, i.e., X and $-X$ being equivalent stochastic processes, which follows directly from the Definition 4.16 and $\mu = 0$, the expectation value

$$\left\langle\!\!\left\langle \int_0^T X_t \mathsf{d}t \right\rangle\!\!\right\rangle_{n,T,\lambda} = 0 \qquad \text{for any value of } T>0,$$

which implies the first derivative $\partial_\mu E_n^{(\mathsf{m})}(\lambda,0)$ vanishes. Further, by Theorem 1.25, we conclude

$$\partial_\mu^2 E(\lambda,0) = -\lim_{T\to\infty}\frac{1}{T}\left\langle\!\!\left\langle \left(\int_0^T X_t dt\right)^2 \right\rangle\!\!\right\rangle_{n,T,\lambda}. \qquad \square$$

Remark 6.22. The vanishing first derivative can also directly be proven from the spin-flip symmetry of the spin boson Hamiltonian with zero external magnetic field. For example, in [HHS21, Lemma 4.1], we used $E_n^{(\mathsf{m})}(\lambda,\mu) = E_n^{(\mathsf{m})}(\lambda,-\mu)$ to obtain $\partial_\mu E_n^{(\mathsf{m})}(\lambda,\mu) = 0$.

We now want to show that the second derivative in Lemma 6.21 is directly connected to the resolvent in (6.7). This is done in the next two lemmas.

Lemma 6.23. *For all $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we have $\langle \sigma_x \psi_{\lambda,n}, \psi_{\lambda,n} \rangle = 0$ and*

$$0 \leq \langle \sigma_x \psi_{\lambda,n}, (H_n(\lambda) - E_n(\lambda))^{-1} \sigma_x \psi_{\lambda,n} \rangle = -\frac{1}{2} \partial_\mu^2 E_n^{(\mathsf{m})}(\lambda, 0).$$

Proof. The proof uses second order analytic perturbation theory, as described in Theorem A.46. To that end, we fix $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.

The assumptions of Theorem A.46 are satisfied by $H_0 = H_n(\lambda)$ and $V = \sigma_x \otimes \mathbb{1}$, since $E_n(\lambda) \in \sigma_{\mathsf{d}}(H_n(\lambda))$ by Theorem 6.3. Hence, in some ball around $\eta = 0$, there exists a unique analytic function $e(\cdot)$ and a $\mathcal{D}(H_n(\lambda))$-valued analytic function $\phi(\cdot)$ such that $\phi(0) = \psi_{\lambda,n}$ and

$$(H_0 + \eta V)\phi(\eta) = e(\eta)\phi(\eta). \tag{6.25}$$

Thus, $e(0) = E_n(\lambda)$. The first derivative of (6.25) and the previous considerations yield

$$\sigma_x \psi_{\lambda,n} + H_n(\lambda)\phi'(0) = e'(0)\psi_{\lambda,n} + E_n(\lambda)\phi'(0).$$

Multiplying $\psi_{\lambda,n}$ from the left and using $\|\psi_{\lambda,n}\| = 1$ yields

$$e'(0) = \langle \psi_n, \sigma_x \psi_n \rangle .$$

Since $e'(0) = \partial_\mu E_n^{(\mathsf{m})}(\lambda, 0)$, Lemma 6.21 implies $\langle \psi_{\lambda,n}, \sigma_x \psi_{\lambda,n} \rangle = 0$. Using that $e'(0) = 0$, we can solve for the first derivative of the eigenvector and obtain that there exists $\alpha \in \mathbb{C}$ such that

$$\phi'(0) = -(H_n(\lambda) - E_n(\lambda))^{-1} \sigma_x \psi_{\lambda,n} + \alpha \psi_{\lambda,n},$$

since $\ker(H_n(\lambda) - E_n(\lambda)) = \operatorname{span}\{\psi_{\lambda,n}\}$ by Theorem 6.3. Now, taking the second derivative of (6.25) and using that there is no perturbation of quadratic order in η, we similarly obtain

$$e''(0) = 2 \langle \psi_{\lambda,n}, \sigma_x \phi'(0) \rangle .$$

Inserting the first derivative of the eigenvector, we obtain the statement. □

We can now state the desired connection.

Lemma 6.24. *For all $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $k \in \mathbb{R}^d$,*

$$\|R_{\lambda,n}(k)\sigma_x \psi_{\lambda,n}\| \leq \sqrt{-\partial_\mu^2 E_n^{(\mathsf{m})}(\lambda, 0)}\ \omega_n^{-1/2}(k).$$

Proof. By the product inequality, we have

$$\|R_{\lambda,n}(k)\sigma_x \psi_{\lambda,n}\| \leq \|R_{\lambda,n}(k)(H_n(\lambda) - E_n(\lambda))^{1/2}\| \|(H_n(\lambda) - E_n(\lambda))^{-1/2} \sigma_x \psi_{\lambda,n}\|. \tag{6.26}$$

By Lemma 6.23, the second factor on the right hand side can be estimated using

$$\|(H_n(\lambda) - E_n(\lambda))^{-1/2} \sigma_x \psi_n\| \leq \sqrt{-\partial_\mu^2 E_n^{(\mathsf{m})}(\lambda, 0)}\ .$$

It remains to estimate the first factor in (6.26). Using $\|R_{\lambda,n}(k)^{1/2}(H_n(\lambda) - E_n(\lambda))^{1/2}\| \leq 1$ and the trivial bound (6.11), we obtain

$$\|R_{\lambda,n}(k)(H_n(\lambda) - E_n(\lambda))^{1/2}\| \leq \|R_{\lambda,n}(k)^{1/2}\| \leq \omega_n^{-1/2}(k).$$ □

Recalling the formula for the second derivative in Lemma 6.21, we can obtain the resolvent bound (6.7) from Lemma 6.24 if we can bound the correlation functions of the continuous Ising model. This is our final lemma, before we can give the proof of Theorem 1.14.

Lemma 6.25. *If $\lambda \in \mathbb{R}$ with $|\lambda| < \frac{1}{\sqrt{5}}\|\omega^{-1/2}v\|^{-1}$, then there exists a constant $C_\chi > 0$ such that*

$$0 \ge \partial_\mu^2 E_n^{(\mathsf{m})}(\lambda,0) \ge -C_\chi.$$

Proof. Comparing (1.10) and (4.31), we see

$$\left\langle\!\!\left\langle \left(\int_0^{2T} X_t \mathsf{d}t \right)^2 \right\rangle\!\!\right\rangle_{n,2T,\lambda} = \left[\!\!\left[\left(\int_{-T}^{T} X_t \mathsf{d}t \right)^2 \right]\!\!\right]_{\lambda^2 W_n, T}. \tag{6.27}$$

Further, by the definition (1.9), the interaction function W_n is an element of $L^1(\mathbb{R})$ and satisfies

$$\|W_n\|_1 = \|\omega_n^{-1/2}v\|_2^2 \le \|\omega^{-1/2}v\|_2^2.$$

Assume $|\lambda| < 5^{-1/2}\|\omega^{-1/2}v\|^{-1}$. Then there exists an $\varepsilon \in (0, \frac{1}{5})$ such that $\lambda^2\|W\|_1 < \varepsilon$. Hence, by Theorem 4.21, there exists a $C_\chi > 0$ such that

$$\limsup_{T\to\infty} \frac{1}{T} \left[\!\!\left[\left(\int_{-T}^{T} X_t \mathsf{d}t \right)^2 \right]\!\!\right]_{\lambda^2 W_n, T} \le C_\chi.$$

Combined with Lemma 6.21, this proves the statement. □

We can hence conclude with the

Proof of Theorem 1.14. This now easily follows by combining Theorems 6.1 and 6.20 and Lemmas 6.24 and 6.25. □

A. Operators on Hilbert Spaces

In this appendix, we recall definitions and theorems from the theory of Hilbert space operators. Most of them are contained in standard textbooks, e.g., [RS72, Wei80, Sch12, Tes14], so we state them here without proofs. Further, for well-known statements, we refrain from giving a special reference. Statements which are non-standard are either proven or explicitly referenced from the literature.

Throughout this appendix, we assume $\mathcal{H}$ and $\mathcal{V}$ to be (complex) Hilbert spaces.

Direct Sums of Hilbert Spaces

For technical reasons, we start out with the definition of the direct sum of a family of Hilbert spaces.

Definition A.1. Let $\mathscr{I}$ be an arbitrary index set. The *direct sum* of a family $(\mathcal{H}_i)_{i\in\mathscr{I}}$ of Hilbert spaces is the Hilbert space given by

$$\bigoplus_{i\in\mathscr{I}}\mathcal{H}_i = \left\{(x_i)_{i\in\mathscr{I}} \in \bigtimes_{i\in\mathscr{I}}\mathcal{H}_i \,\middle|\, \sum_{i\in\mathscr{I}}\|x_i\|^2_{\mathcal{H}_i} < \infty\right\},$$

$$\left\langle (x_i)_{i\in\mathscr{I}}, (y_i)_{i\in\mathscr{I}}\right\rangle = \sum_{i\in\mathscr{I}}\langle x_i, y_i\rangle_{\mathcal{H}_i}.$$

Hilbert Space Operators

We can now move to the basic notions for operators from $\mathcal{H}$ to $\mathcal{V}$.

Definition A.2. We say T is an *operator from $\mathcal{H}$ to $\mathcal{V}$* with *domain* $\mathcal{D}(T)$ if $\mathcal{D}(T)$ is a subspace of $\mathcal{H}$ and $T : \mathcal{D}(T) \to \mathcal{V}$ is linear. If $\mathcal{H} = \mathcal{V}$, we say T is an *operator on $\mathcal{H}$*.
It has *range* $\operatorname{ran} T = \{Tx \in \mathcal{V} : x \in \mathcal{D}(T)\}$ and *kernel* $\ker T = \{x \in \mathcal{H} : Tx = 0\}$.
If D is a subspace of $\mathcal{D}(T)$, we define the *restriction* of T to D as the operator $T\restriction_D$ satisfying $\mathcal{D}(T\restriction_D) = D$ and $T\restriction_D x = Tx$ for all $x \in D$.
The operator T is called *densely defined* if $\mathcal{D}(T)$ is dense in $\mathcal{H}$.
T is *closed* if its *graph* $\mathscr{G}_T = \{(x, Tx) : x \in \mathcal{D}(T)\}$ is a closed subspace of the Hilbert space $\mathcal{H} \oplus \mathcal{V}$.
The operator T is called *invertible* if $\ker T = \{0\}$. In this case the operator T^{-1} with $\mathcal{D}(T^{-1}) = \operatorname{ran} T$ and $T^{-1}Tx = x$ for all $x \in \mathcal{D}(T)$ is called *inverse* of T.
The operator T is called *bounded* if there exists a constant $C > 0$ such that $\|Tx\|_{\mathcal{V}} \leq C\|x\|_{\mathcal{H}}$ for all $x \in \mathcal{D}(T)$.
A set $D \subset \mathcal{D}(T)$ is called a *core* for T if it is dense in $\mathcal{D}(T)$ equipped with the *T-norm* $\|x\|_T = \|x\|_{\mathcal{H}} + \|Tx\|_{\mathcal{V}}$.

To perform calculations with operators, we make the following definition.

Definition A.3. If S and T are operators from $\mathcal{H}$ to $\mathcal{V}$ and $\alpha \in \mathbb{C} \setminus \{0\}$, we define the operator $S + \alpha T$ by

$$\mathcal{D}(S + \alpha T) = \mathcal{D}(S) \cap \mathcal{D}(T), \qquad (S + \alpha T)x = Sx + \alpha Tx.$$

If S is an operator from $\mathcal{H}$ to $\mathcal{V}$ and T is an operator from $\mathcal{V}$ to a Hilbert space $\mathcal{Y}$, then we define the operator TS from $\mathcal{H}$ to $\mathcal{Y}$ by

$$\mathcal{D}(ST) = \{x \in \mathcal{D}(T) \mid Tx \in \mathcal{D}(S)\}, \qquad (ST)x = S(Tx).$$

If S and T are operators on $\mathcal{H}$, then we define their *commutator* $[S, T]$ by

$$\mathcal{D}([S,T]) = \mathcal{D}(ST) \cap \mathcal{D}(TS), \qquad [S,T]x = STx - TSx.$$

Often it is necessary to compare operators.

Definition A.4. If S and T are operators from $\mathcal{H}$ to $\mathcal{V}$, then $S = T$ if $\mathcal{D}(S) = \mathcal{D}(T)$ and $Sx = Tx$ for all $x \in \mathcal{D}(T)$. We say S is a *restriction* of T and write $S \subset T$ if $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $S = T\restriction_{\mathcal{D}(S)}$. In this case, we call T an *extension* of S. An operator is called *closable* if it has a closed extension.
If $D \subset \mathcal{D}(S) \cap \mathcal{D}(T)$ and $Sx = Tx$ for all $x \in D$, then we say $S = T$ *holds on* D.

We now want to define the minimal closed extension of a closable operator.

Lemma A.5. *If T is a closable operator, then there exists a unique closed operator $\overline{T}$ such that $\mathscr{G}_{\overline{T}} = \overline{\mathscr{G}_T}$.*

Definition A.6. If T is a closable operator from $\mathcal{H}$ to $\mathcal{V}$, the operator $\overline{T}$ from Lemma A.5 is called *closure* of T.

This gives us a new characterization of the core of a closed operator.

Lemma A.7. *If T is a closed operator and $D \subset \mathcal{D}(T)$, then D is a core for T if and only if $T = \overline{T\restriction_D}$.*

We move to the definition of the adjoint operator.

Lemma A.8. *If T is a densely defined operator from $\mathcal{H}$ to $\mathcal{V}$, there exists a unique operator T^* from $\mathcal{V}$ to $\mathcal{H}$ with graph $\mathcal{G}_{T^*} = \{(Tx, -x) : x \in \mathcal{D}(T)\}^{\perp}$. Further, T^* is closed and $\langle T^*y, x\rangle_{\mathcal{H}} = \langle y, Tx\rangle_{\mathcal{V}}$ for all $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$.*

Definition A.9. If T is a densely defined operator from $\mathcal{H}$ to $\mathcal{V}$, we call the unique operator T^* from Lemma A.8 the *adjoint operator of* T.

The adjoint provides a characterization of closable operators.

Lemma A.10. *A densely defined operator T from $\mathcal{H}$ to $\mathcal{V}$ is closable if and only if T^* is densely defined. In this case $\overline{T} = (T^*)^*$.*

Bounded Operators

Before studying more properties of unbounded operators, we treat bounded operators. We will usually assume them to be defined on all of $\mathcal{H}$, which is justified by the following lemma.

Lemma A.11. *A bounded operator T from $\mathcal{H}$ to $\mathcal{V}$ is closed if and only if $\mathcal{D}(T)$ is closed. Especially, a densely defined closed and bounded operator satisfies $\mathcal{D}(T) = \mathcal{H}$. Vice versa, if $\mathcal{D}(T) = \mathcal{H}$ then T is bounded.*

We now define the space of bounded operators.

Definition A.12. The set of everywhere defined bounded operators from $\mathcal{H}$ to $\mathcal{V}$ is denoted as $\mathcal{B}(\mathcal{H}, \mathcal{V})$. Further, we write $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{V})$, we define

$$\|T\|_{\mathcal{B}(\mathcal{H},\mathcal{V})} = \sup\{\|Tx\|_{\mathcal{V}} : x \in \mathcal{H}, \|x\|_{\mathcal{H}} = 1\}.$$

Lemma A.13. *The pair $(\mathcal{B}(\mathcal{H}, \mathcal{V}), \|\cdot\|_{\mathcal{B}(\mathcal{H},\mathcal{V})})$ is a Banach space.*

The adjoints of bounded operators are also bounded.

Lemma A.14. *If $B \in \mathcal{B}(\mathcal{H}, \mathcal{V})$, then $B^* \in \mathcal{B}(\mathcal{V}, \mathcal{H})$ and $\|B\|_{\mathcal{B}(\mathcal{H},\mathcal{V})} = \|B^*\|_{\mathcal{B}(\mathcal{V},\mathcal{H})}$. Further, $\|B\|^2_{\mathcal{B}(\mathcal{H},\mathcal{V})} = \|B^*B\|_{\mathcal{B}(\mathcal{H})} = \|BB^*\|_{\mathcal{B}(\mathcal{V})}$.*

Apart from the canonical notion of convergence in $\mathcal{B}(\mathcal{H}, \mathcal{V})$, we will also use weaker forms of convergence.

Definition A.15. Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H}, \mathcal{V})$ and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{V})$.
If $\|T - T_n\|_{\mathcal{B}(\mathcal{H},\mathcal{V})} \xrightarrow{n\to\infty} 0$, we say T_n converges to T in norm and write $\lim\limits_{n\to\infty} T_n = T$.
If $T_n x \xrightarrow[\mathcal{V}]{n\to\infty} Tx$ for every $x \in \mathcal{H}$, we say T_n *strongly converges* to T and write $\operatorname*{s-lim}\limits_{n\to\infty} T_n = T$.
If $\langle y, T_n x\rangle_{\mathcal{V}} \xrightarrow{n\to\infty} \langle y, Tx\rangle_{\mathcal{V}}$ for all $x \in \mathcal{H}$, $y \in \mathcal{V}$, we say T_n *weakly converges* to T and write $\operatorname*{w-lim}\limits_{n\to\infty} T_n = T$.

Remark A.16. Note that norm convergence is the natural notion of convergence in the Banach space $\mathcal{B}(\mathcal{H}, \mathcal{V})$.

It is simple to verify a hierarchy of these notions of convergence.

Lemma A.17. *Let $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H}, \mathcal{V})$ and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{V})$.*

(i) *If $\lim\limits_{n\to\infty} T_n = T$, then $\operatorname*{s-lim}\limits_{n\to\infty} T_n = T$.*

(ii) *If $\operatorname*{s-lim}\limits_{n\to\infty} T_n = T$, then $\operatorname*{w-lim}\limits_{n\to\infty} T_n = T$.*

We will need some further properties of strong and weak convergence, which we summarize in the following.

Lemma A.18. *Let $(T_n) \subset \mathcal{B}(\mathcal{H})$ and let $T \in \mathcal{B}(\mathcal{H})$. Further assume A is a closed operator such that $\mathcal{D}(A) \supset \operatorname{ran} T_n$ for all $n \in \mathbb{N}$ and $\|AT_n\|_{\mathcal{B}(\mathcal{H})}$ is uniformly bounded.*

(i) *If $\operatorname{w-lim} T_n = T$, then $\operatorname{ran} T \subset \mathcal{D}(A)$ and $\operatorname{w-lim} AT_n = AT$.*

(ii) *If $\operatorname{s-lim} T_n^* = T^*$, then $\operatorname{ran} T \subset \mathcal{D}(A)$ and $\operatorname{s-lim}(AT_n)^* = (AT_n)^*$.*

We briefly discuss compact operators.

Definition A.19. An operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{V})$ is called *compact* if the set TM is relatively compact in Y for any bounded $M \subset X$. Further, it is called *finite-rank* if $\dim \operatorname{ran} T < \infty$.

Lemma A.20. *If $T \in \mathcal{B}(\mathcal{H}, \mathcal{V})$ has finite rank, then T is compact.*

Sequences of compact operators converging in norm have a compact operator as their limit.

Lemma A.21. *If $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\mathcal{H}, \mathcal{V})$ is a sequence of compact operators and $\lim_{n\to\infty} T_n = T \in \mathcal{B}(\mathcal{H}, \mathcal{V})$, then T is compact.*

We will use the following statement about weakly convergent sequences. To that end, we recall the definition of the weak limit of a sequence on Hilbert spaces:

$$\operatorname*{w-lim}_{n\to\infty} x_n = x \iff \forall y \in \mathcal{H} : \lim_{n\to\infty} \langle y, x_n \rangle = \langle y, x \rangle \,. \tag{A.1}$$

Lemma A.22. *If $T \in \mathcal{B}(\mathcal{H}, \mathcal{V})$ is compact and $(x_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ is weakly congergent to $x \in \mathcal{H}$, then $Tx = \lim_{n\to\infty} Tx_n$.*

For later reference, we collect some simple properties and notions for bounded operators. To that end, recall that a function $f : \mathcal{H} \to \mathcal{V}$ is called an *isometry* if $\|f(x)\|_{\mathcal{V}} = \|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$ and a *contraction* if $\|f(x)\|_{\mathcal{V}} \leq \|x\|_{\mathcal{H}}$ for all $x \in \mathcal{H}$.

Lemma A.23. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{V})$.Then T is an isometry if and only if $T^*T = \mathbb{1}_{\mathcal{H}}$. Further, T is a contraction if and only if $\|T\|_{\mathcal{B}(\mathcal{H},\mathcal{V})} \leq 1$.*

The isometric isomorphisms between Hilbert spaces are called unitaries.

Definition A.24. An operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{V})$ is *unitary* if $U^*U = \mathbb{1}_{\mathcal{H}}$ and $UU^* = \mathbb{1}_{\mathcal{V}}$.

The Spectrum of an Operator

We move to defining the spectrum and the resolvent set of a closed operator.

Definition A.25. Let T be a closed operator on $\mathcal{H}$. The set

$$\varrho(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) = \{0\}, \operatorname{ran}(T - \lambda) = \mathcal{H}\}$$

is called resolvent set of T. Its complement $\sigma(T) = \varrho(T)^c$ is called *spectrum* of T.
For $\lambda \in \varrho(T)$ the operator $(T - \lambda)^{-1} \in \mathcal{B}(\mathcal{H})$ is called *resolvent* of T.

By definition, the spectrum decomposes into three different parts.

Lemma A.26. *For any closed operator T on $\mathcal{H}$, we have $\sigma(T) = \sigma_{\mathsf{p}}(T) \cup \sigma_{\mathsf{c}}(T) \cup \sigma_{\mathsf{r}}(T)$ with*

$$\begin{aligned}
\sigma_{\mathsf{p}}(T) &= \{\lambda \in \mathbb{C} : \ker(T - \lambda) \supsetneq \{0\}\}, \\
\sigma_{\mathsf{c}}(T) &= \{\lambda \in \mathbb{C} : \operatorname{ran}(T - \lambda) \textit{ is not closed}\}, \\
\sigma_{\mathsf{r}}(T) &= \{\lambda \in \mathbb{C} : \operatorname{ran}(T - \lambda) \textit{ is not dense}\}.
\end{aligned}$$

Definition A.27. The sets from the previous lemma are called *point spectrum* σ_{p}, *continuous spectrum* σ_{c} and *residual spectrum* σ_{r}. Further, the values $\lambda \in \sigma_{\mathsf{p}}(T)$ are called *eigenvalues* of the operator T with *multiplicity* $\dim\ker(T-\lambda)$. The vectors $v \in \ker(T-\lambda)$ are called *eigenvectors* of T corresponding to λ. If the multiplicity of an eigenvalue is one, then we call it *nondegenerate* and the corresponding eigenvector *unique*.

Remark A.28. The notion of uniqueness of an eigenvalue agrees with Definition 1.1.

When explicitly calculating resolvents, the following identity is often useful.

Lemma A.29 (Resolvent Identity)**.** *Let T and S be closed operators on $\mathcal{H}$ satisfying $\mathcal{D}(S) \subset \mathcal{D}(T)$ and let $\lambda \in \varrho(T) \cap \varrho(S)$. Then*

$$(T-\lambda)^{-1} - (S-\lambda)^{-1} = (T-\lambda)^{-1}(S-T)(S-\lambda)^{-1}.$$

Selfadjoint Operators

Apart from bounded operators, we will mostly be concerned with symmetric and selfadjoint operators. From now on, we will drop the index $\mathcal{H}$ in scalar products and norms.

Definition A.30. An operator A on $\mathcal{H}$ is *symmetric* if $\langle Ax, x\rangle = \langle x, Ax\rangle$ for all $x \in \mathcal{D}(A)$.

Symmetric operators can always be closed.

Lemma A.31. *A densely defined symmetric operator A is closable.*

Definition A.32. A densely defined symmetric operator A is called *essentially selfadjoint* if $\overline{A} = A^*$ and *selfadjoint* if $A = A^*$.

Two selfadjoint operators are equal if their restrictions to a core are.

Lemma A.33. *Assume A and B are selfadjoint operators. If $D \subset \mathcal{D}(A) \cap \mathcal{D}(B)$ is a core for A and $Ax = Bx$ for all $x \in D$, then $A = B$.*

Let us collect some properties of the spectrum of selfadjoint operators.

Lemma A.34. *If A is a selfadjoint operator on $\mathcal{H}$, then $\sigma_{\mathsf{r}}(A) = \emptyset$.*

Lemma A.35. *A closed symmetric operator A on $\mathcal{H}$ is selfadjoint if and only if $\sigma(A) \subset \mathbb{R}$.*

Apart from Lemma A.26, the following decomposition of the spectrum of a selfadjoint operator is useful. Here, we denote the distance of two sets by

$$\operatorname{dist}(M_1, M_2) = \inf\{|x-y| : x \in M_1, y \in M_2\} \qquad \text{for } M_1, M_2 \subset \mathbb{C}. \tag{A.2}$$

Definition A.36. If A is a selfadjoint operator the set $\sigma_{\mathsf{d}}(A)$ of all eigenvalues λ of finite multiplicity which are isolated from the rest of the spectrum, i.e., $\operatorname{dist}(\{\lambda\}, \sigma(A)\backslash\{\lambda\}) > 0$, is called *discrete spectrum* of A. Its complement is called *essential spectrum* $\sigma_{\mathsf{ess}}(A)$.

We need an equivalent characterization of the essential spectrum, which is known as *Weyl's Criterion*.

Lemma A.37 (Weyl's Criterion)**.** *Let A be a selfadjoint operator on $\mathcal{H}$ and let $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma_{\mathsf{ess}}(A)$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that*

$$\operatorname*{w-lim}_{n\to\infty} x_n = 0, \quad \liminf_{n\to\infty} \|x_n\| > 0, \quad \lim_{n\to\infty} \|(A-\lambda)x_n\| = 0.$$

Further, this sequence can be chosen to be orthonormal, i.e., $\langle x_n, x_m\rangle = 0$ if $n \neq m$ and $\|x_n\| = 1$ for all $n \in \mathbb{N}$.

Remark A.38. A sequence as in the previous lemma is often called *Weyl sequence* in the literature. We also note that any orthonormal sequence weakly converges to zero, by Bessel's inequality.

When we consider ground states, the operators need to be bounded from below.

Definition A.39. We call a selfadjoint operator A on $\mathcal{H}$ *lower-semibounded* by $c \in \mathbb{R}$ and write $A \geq c$ if $\langle x, Ax\rangle \geq c$ for all $x \in \mathcal{D}(A)$. Especially, we call A *positive* if $A \geq 0$. If A and B are selfadjoint operators and $\mathcal{D}(A) \subset \mathcal{D}(B)$, we write $A \geq B$ if $A - B \geq 0$.

Lemma A.40. *If A is a selfadjoint operator on $\mathcal{H}$ with $A \geq c \in \mathbb{R}$, then $\sigma(A) \geq c$.*

One way to prove a dense set is a core for a selfadjoint operator is the following.

Definition A.41. Let A be an operator on $\mathcal{H}$. We call a vector $x \in \mathcal{H}$ *semianalytic* for A if $x \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$ and

$$\sum_{n=0}^{\infty} \frac{\|A^n x\| t^n}{(2n)!} < \infty \qquad \text{for some } t > 0.$$

Theorem A.42 (Nussbaum-Masson-McClary Criterion [RS75, Theorem X.40])**.** *Let A be a lower-semibounded selfadjoint operator and $D \subset \mathcal{D}(A)$ a total set of semianalytic vectors. Then* span D *is a core for A.*

Relative Operator Bounds

In many of our applications it is important to have a notion of smallness for one operator against the other.

Definition A.43. Let A and B be operators on $\mathcal{H}$. B is called *A-bounded with relative bound (or A-bound)* $a \geq 0$ if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and there exists $b \geq 0$ such that

$$\|B\psi\| \leq a\|A\psi\| + b\|\psi\| \qquad \text{for all } \psi \in \mathcal{D}(A). \tag{A.3}$$

Especially, B is called *infinitesimally A-bounded* if it is A-bounded with relative bound ε for any choice of $\varepsilon > 0$.

We collect a variety of equivalent statements for relative boundedness of operators.

Lemma A.44 ([Tes14, Lemma 6.2])**.** *Let A be a closed operator and B a closable operator on $\mathcal{H}$. Then the following are equivalent:*

(i) *B is A-bounded.*

(ii) $\mathcal{D}(A) \subset \mathcal{D}(B)$.

(iii) $B(A-\lambda)^{-1}$ *is bounded for one (and hence all)* $\lambda \in \varrho(A)$.

In this case, any value larger than $\inf_{\lambda \in \varrho(A)} \|B(A-\lambda)^{-1}\|$ *is an* A*-bound of* B.

For us, the most important application of relative boundedness is the following selfadjointness criterion.

Theorem A.45 (Kato-Rellich, [RS75, Theorem X.12])**.**
Let A *be a selfadjoint operator and* B *a symmetric operator on* $\mathcal{H}$. *Further, assume* B *is* A*-bounded with relative bound* $a < 1$. *Then* $A + B$ *is a selfadjoint operator with domain* $\mathcal{D}(A)$ *and any core for* A *is a core for* $A+B$. *Further, if* $A \geq c \in \mathbb{R}$ *and* (A.3) *holds with* $a \in [0,1)$ *and* $b \geq 0$, *then* $A + B \geq c - \max\{b(1-a)^{-1}, a|C| + b\}$.

Analytic Perturbation Theory

Although the essentials of our proofs are non-perturbative, we do utilize a few results from perturbation theory in our treatment of the spin boson model with external magnetic field. Hence, we recall some facts from analytic perturbation theory. More details can be found in [Kat80, RS78]. Here, we restrict ourselves to the cases relevant for our application.

Theorem A.46. *Assume* H_0 *is a closed operator and* V *is an* H_0*-bounded operator. Let* $H(\eta) = H_0 + \eta V$ *for* $\eta \in \mathbb{C}$. *If* λ_0 *is an isolated nondegenerate eigenvalue of* H_0 *with eigenvector* ψ_0, *then there exists* $R > 0$, *a unique analytic function* $\lambda(\eta)$, *and a* $\mathcal{D}(H_0)$*-valued analytic function* $\psi(\eta)$ *for* $\eta \in \mathbb{C}$ *with* $|\eta| < R$ *such that* $\psi(0) = \psi_0$ *and*

$$H(\eta)\psi(\eta) = \lambda(\eta)\psi(\eta).$$

Proof. By [Kat80, §VII Theorem 2.6] and the Kato-Rellich theorem (Theorem A.45), $H(\eta)$ defines an analytic family of type (A) for $|\eta|$ sufficiently small. Therefore, the statement follows directly from [RS78, Theorem XII.8]. □

The Spectral Theorem

We now state the important spectral theorem. To that end, we first need to define the notion of a projection-valued measure.

Definition A.47. A non-zero operator $P \in \mathcal{B}(\mathcal{H})$ is called *projection* if $P^2 = P$. It is called *orthogonal projection* if it is selfadjoint.

We relate our definition to the classical notion of a projection, where we as usual denote

$$M^\perp = \{x \in \mathcal{H} \mid \forall y \in M : \langle x, y\rangle = 0\} \qquad \text{for } M \subset \mathcal{H}. \tag{A.4}$$

Lemma A.48. *For any closed subspace* $V \subset \mathcal{H}$, *there exists a unique orthogonal projection* P *satisfying* $\operatorname{ran} P = V$ *and* $\ker P = V^\perp$. *In this case* $Px = x$ *for all* $x \in V$.

Now, we can define projection-valued measures.

Definition A.49. Let (Ω, Σ) be a measurable space. A mapping P from Σ to the orthogonal projections on $\mathcal{H}$ is called *projection-valued measure* if $\mathsf{P}(\Omega) = \mathbb{1}$ and P is countably additive, i.e., for any sequence $(M_n)_{n\in\mathbb{N}} \subset \Sigma$ of pairwise disjoint sets

$$\mathsf{P}\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \operatorname*{s-lim}_{k\to\infty} \sum_{n=1}^{k} \mathsf{P}(M_n).$$

We call $\operatorname{supp} \mathsf{P} = \bigcap_{\substack{M\in\Sigma \\ \mathsf{P}(M)=\mathbb{1}}} M$ the *support* of P.

Projection valued measures give rise to complex measures by taking inner products.

Lemma A.50. *Let (Ω, Σ) be a measurable space and P a mapping from Σ to the orthogonal projections on $\mathcal{H}$. Then P is a projection-valued measure if and only if $\mathsf{P}(\Omega) = \mathbb{1}$ and for all $x \in \mathcal{H}$ the map $\langle x, \mathsf{P}(\cdot)x\rangle : \Sigma \to \mathbb{R}$ is a measure. In this case, for all $x, y \in \mathcal{H}$, the map $\langle y, \mathsf{P}(\cdot)x\rangle : \Sigma \to \mathbb{C}$ defines a complex measure.*

The spectral theorem holds for so-called normal operators.

Definition A.51. A densely defined operator T on $\mathcal{H}$ is called *normal* if $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $\|Tx\| = \|T^*x\|$ for all $x \in \mathcal{D}(T)$.

For the statement, we need to define an integral with respect to projection-valued measures.

Lemma A.52. *Let P be a projection-valued measure on $\mathcal{H}$ defined on the measurable space (Ω, Σ) and let $f : \Omega \to \mathbb{C} \cup \{\infty\}$ be a measurable function which is P-a.e. finite, i.e., $\mathsf{P}(f^{-1}(\{\infty\})) = 0$. Then there exists a unique normal operator $I_{\mathsf{P}}(f)$ on $\mathcal{H}$ such that*

$$\begin{aligned}
\mathcal{D}(I_{\mathsf{P}}(f)) &= \{x \in \mathcal{H} : f \in L^2(\Omega, \mathsf{d}\langle x, \mathsf{P}(\cdot)x\rangle)\}, \\
\langle y, I_{\mathsf{P}}(f)x\rangle &= \int_\Omega f(t)\mathsf{d}\langle y, \mathsf{P}(t)x\rangle \qquad \textit{for all } x \in \mathcal{D}(I_{\mathsf{P}}(f)),\ y \in \mathcal{H}.
\end{aligned}$$

Definition A.53. In the situation of Lemma A.52, we write $\int_\Omega f(t)\mathsf{dP}(t)$ for $I_{\mathsf{P}}(f)$.

In Lemma A.52, we have already seen that projection-valued measures generate normal operators from measurable functions. Vice versa, any normal operator gives rise to a projection-valued measure.

Theorem A.54 (Spectral Theorem)**.** *Let A be a normal operator on $\mathcal{H}$. Then there exists a unique projection-valued measure P_A on the Borel σ-algebra $\mathfrak{B}(\mathbb{C})$ such that*

$$A = \int_{\mathbb{C}} t\mathsf{dP}_A(t).$$

Definition A.55. If A is a normal operator on $\mathcal{H}$, we call the projection-valued measure P_A from Theorem A.54 the *spectral measure* associated with A.

We collect some properties of spectral measures.

Lemma A.56. *Let A be a normal operator on $\mathcal{H}$.*

(i) $\operatorname{supp} \mathsf{P}_A = \sigma(A)$.

(ii) $\operatorname{ran} \mathsf{P}_A(M) \subset \mathcal{D}(A)$ *for any* $M \in \mathfrak{B}(\mathbb{C})$.

(iii) $\mathsf{P}_A(\{\lambda\})$ *is the unique orthogonal projection onto* $\ker(A-\lambda)$ *(cf. Lemma A.48).*

The spectral theorem provides a variety of possibilities to recover projections onto eigenspaces of an operator from its resolvents. We use two versions of these.

Lemma A.57. *Let A be a selfadjoint operator and $\lambda \in \sigma(A)$. Further let $(\lambda_n)_{n\in\mathbb{N}} \subset \varrho(A)$ be such that $\lim\limits_{n\to\infty} \lambda_n = \lambda$. Then*

$$\mathsf{P}_A(\{\lambda\}) = \operatorname*{s-lim}_{n\to\infty} \lambda_n (A-\lambda_n)^{-1}.$$

Lemma A.58. *Let A be a selfadjoint operator and $\Gamma \subset \varrho(A)$ be a positively oriented Jordan curve with interior I. Then*

$$\mathsf{P}_A(I \cap \sigma(A)) = \int_\Gamma (A-z)^{-1} \mathsf{d}z,$$

where the integral is understood to be a Riemann integral on $\mathcal{B}(\mathcal{H})$ in norm.

Functional Calculus

The spectral theorem allows us to define a functional calculus.

Definition A.59 (Functional Calculus)**.** Let A be a normal operator on $\mathcal{H}$ and let $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be measurable and finite P_A-a.e. Then, we define

$$f(A) = \int_{\mathbb{C}} f(t) \mathsf{d}\mathsf{P}_A(t).$$

Remark A.60. Clearly, it suffices to define f on $\operatorname{supp} \mathsf{P}_A$ in the above definition, since we can set our function to infinity everywhere else. Especially, for selfadjoint operators A, we will usually use functions defined on $\mathbb{R}$ or in the case $A \geq c \in \mathbb{R}$ on $[c,\infty)$, cf. Lemmas A.35, A.40 and A.56 (i).

We collect some properties of the functional calculus.

Lemma A.61. *Let A be a selfadjoint operator on $\mathcal{H}$ and let $f, g : \mathbb{R} \to \mathbb{C} \cup \{\infty\}$ be measurable and finite P_A-a.e.*

(i) *$f(A)$ is bounded if and only if $f \in L^\infty(\mathbb{R}, \mathsf{P}_A)$. In this case $\|f(A)\| = \|f\|_{L^\infty(\mathbb{R},\mathsf{P}_A)}$.*

(ii) *We have $\mathsf{P}_A(M) = \chi_M(A)$ for any Borel set $M \subset \mathbb{R}$.*

(iii) *$fg(A)$ is the closure of $f(A)g(A)$.*

(iv) *If f is real-valued, then $f(A)$ is selfadjoint. Further, if $f \geq 0$ P_A-a.e., then $f(A) \geq 0$.*

Lemma A.62. *Let A, B be selfadjoint operators on $\mathcal{H}$ and let $\omega : \mathbb{R}^d \to \mathbb{R}$. Assume A is bounded below, B is A-bounded, ω is continuous, $\omega(0) = 0$ and $\omega(k) > 0$ for $k \neq 0$. We define $\lambda = \inf(\sigma(A))$ and $f(k) = \omega(k)B(A - \lambda + \omega(k))^{-1} \in \mathcal{B}(\mathcal{H})$ for $k \neq 0$. Then $k \mapsto f(k)$ is locally bounded and $\operatorname{s-lim}_{k\to 0} f(k) = B\mathsf{P}_A(\{\lambda\})$.*

Proof. By the assumptions and Lemma A.44, the operator $C = B(A-\lambda+1)^{-1}$ is bounded. Then the resolvent identity (Lemma A.29) yields

$$f(k) = \omega(k)C + C\omega(k)(A - \lambda + \omega(k))^{-1} - \omega(k)C\omega(k)(A - \lambda + \omega(k))^{-1}.$$

The first term converges to 0 in norm, since C is bounded. Further, by Lemma A.57, the last term converges to 0 strongly. Again using Lemma A.57, the middle term strongly converges to $C\mathsf{P}_A(\{\lambda\}) = B\mathsf{P}_A(\{\lambda\})$ by the definition of C, which proves the statement. □

The following upper bound is also a corollary of the functional calculus.

Lemma A.63. *If T is a normal operator and $\lambda \in \varrho(T)$, then*

$$\|(T - \lambda)^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq (\operatorname{dist}(\{\lambda\}, \sigma(T)))^{-1}.$$

Strongly Continuous Groups and Semigroups

Selfadjoint operators give rise to strongly continuous (semi-)groups of bounded operators, which we only briefly discuss here.

We start out with the simple observation that the unitary group leaves the domain of its generator invariant.

Lemma A.64 ([Sch12, Proposition 6.1])**.** *Let A be a selfadjoint operator on $\mathcal{H}$. Then $e^{\mathrm{i}tA}\mathcal{D}(A) \subset \mathcal{D}(A)$ for all $t \in \mathbb{R}$.*

Vice versa, a dense subspace which is invariant under the unitary group is a core.

Lemma A.65 ([RS72, Theorem VIII.11])**.** *Let A be a selfadjoint operator on $\mathcal{H}$ and $D \subset \mathcal{D}(A)$ be a dense subspace of $\mathcal{H}$. If $e^{\mathrm{i}tA}$ leaves D invariant for all $t \in \mathbb{R}$, then D is a core for A.*

The next statement is useful to calculate the unitary group and the semigroup, respectively.

Theorem A.66 (Trotter Product Formula, [RS72, Thm. VIII.30], [Sch12, Thm. 6.4])**.** *Let A and B be selfadjoint operators on $\mathcal{H}$ and assume $A + B$ is also selfadjoint. Then*

$$e^{\mathrm{i}t(A+B)} = \operatorname{s-lim}_{n\to\infty} \left(e^{\mathrm{i}tA/n}e^{\mathrm{i}tB/n}\right)^n \qquad \text{for all } t \in \mathbb{R}.$$

Further, if A and B are lower-semibounded, then

$$e^{-t(A+B)} = \operatorname{s-lim}_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n \qquad \text{for all } t \geq 0.$$

Finally, we state an application which helps us retrieve information on the ground state energy of a lower-semibounded operator.

Lemma A.67 ([Ara18, Lemma 1.5])**.** *Let A be a selfadjoint lower-semibounded operator. Then, for all $t \in \mathbb{R}$, e^{-tH} is a bounded positive selfadjoint operator and*

$$\|e^{-tA}\|_{\mathcal{B}(\mathcal{H})} = e^{-t\inf\sigma(A)}.$$

Strongly Commuting Operators

The commutator as defined in Definition A.3 does not contain enough information in the case of unbounded operators. Hence, we need the notion of strongly commuting operators.

Definition A.68. Let A and B be normal operators. Then, we say that A and B *strongly commute* if their spectral measures commute, i.e., $[\mathsf{P}_A(M), \mathsf{P}_B(N)] = 0$ for all Borel sets $M, N \in \mathfrak{B}(\mathbb{C})$

The next two statements are simple applications for selfadjoint operators.

Lemma A.69 ([Ara18, Corollary 1.6])**.** *Let A, B be selfadjoint operators with $A \geq a \in \mathbb{R}$ and $B \geq b \in \mathbb{R}$. If A and B strongly commute, then $A+B$ is selfadjoint and $A+B \geq a+b$.*

Lemma A.70. *Let A and B be strongly commuting selfadjoint operators on $\mathcal{H}$. If A is B-bounded and $D \subset \mathcal{H}$ is a core for B, then D is a core for A.*

Proof. Clearly any element in $\mathcal{D}(B)$ can be approximated in A-norm by elements in D, so it is enough to see $\mathcal{D}(B)$ is a core for A. Now, for any $\psi \in \mathcal{D}(A)$, we can choose the approximating sequence $\chi_{\{|\cdot| \leq n\}}(B)\psi$, which converges in A-norm since

$$A\chi_{\{|\cdot| \leq n\}}(B)\psi = \chi_{\{|\cdot| \leq n\}}(B)A\psi$$

by the assumptions. □

We also want to write functions of a family of pairwise strongly commuting normal operators. To that end, we first need to define product measures of projection-valued measures.

Lemma A.71 ([Sch12, Theorem 4.10])**.** *Let $k \in \mathbb{N}$ and let $\mathsf{P}_1, \ldots, \mathsf{P}_k$ be pairwise commuting projection-valued measures on $\mathbb{C}$, i.e., $[\mathsf{P}_i(M), \mathsf{P}_j(N)] = 0$ for all Borel sets $M, N \subset \mathbb{C}$ and $i, j \in \{1, \ldots, k\}$. Then there exists a unique projection-valued measure P on $\mathbb{C}^n$ such that*

$$\mathsf{P}(M_1 \times \cdots \times M_k) = \mathsf{P}_1(M_1) \cdots \mathsf{P}_k(M_k) \qquad \text{for all Borel sets } M_1, \ldots, M_k \subset \mathbb{C}.$$

Definition A.72. In the situation of above theorem, we call P the *product* of the projection-valued measures $\mathsf{P}_1, \ldots, \mathsf{P}_k$ and denote it by $\mathsf{P}_1 \otimes \cdots \otimes \mathsf{P}_k$.

We can finally give the desired generalization of the functional calculus from Definition A.59.

Definition A.73. Let $k \in \mathbb{N}$ and let $\mathbf{A} = (A_1, \ldots, A_k)$ be a family of pairwise strongly commuting normal operators. We then write $\mathcal{D}(\mathbf{A}) = \bigcap_{i=1}^k \mathcal{D}(A_i)$.
Further, for measurable $f : \mathbb{C}^k \to \mathbb{C}$, we define

$$f(\mathbf{A}) = \int_{\mathbb{C}^k} f(\lambda) \mathsf{d}\mathsf{P}_{A_1} \otimes \cdots \otimes \mathsf{P}_{A_k}(\lambda).$$

Remark A.74. The properties of the functional calculus from Lemma A.61 carry over to the above definition. We refrain from restating them here.

Convergence of Selfadjoint Operators

We introduced different notions of convergence for bounded operators, in Definition A.15. We will also need to treat the convergence of selfadjoint operators.

Definition A.75. Let $(A_n)_{n\in\mathbb{N}}, A$ be selfadjoint operators.
We say A_n converges to A in the *norm resolvent sense* if $(A+\mathrm{i})^{-1} = \lim_{n\to\infty}(A_n+\mathrm{i})^{-1}$.
We say A_n converges to A in the *strong resolvent sense* if $(A+\mathrm{i})^{-1} = \operatorname{s-lim}_{n\to\infty}(A_n+\mathrm{i})^{-1}$.

We collect some implications of norm resolvent convergence.

Lemma A.76.
Let $(A_n)_{n\in\mathbb{N}}$ be selfadjoint operators and let A be a selfadjoint operator.

(i) [RS72, Theorem VIII.19] *If $(A-\lambda)^{-1} = \lim_{n\to\infty}(A_n-\lambda)^{-1}$ for some $\lambda\in\mathbb{C}\setminus\mathbb{R}$, then A_n converges to A in the norm resolvent sense. Further, if there is $c\in\mathbb{R}$ such that $A\geq c$ and $A_n\geq c$ for all $n\in\mathbb{N}$, then we can also choose $\lambda<c$.*

(ii) [RS72, Theorem VIII.23] *Assume A_n converges to A in the norm resolvent sense. If $\lambda\notin\sigma(A)$ then $\lambda\notin\sigma(A_n)$ for n large enough and $(A_n-\lambda)^{-1}$ converges to $(A-\lambda)^{-1}$ in norm.*

(iii) [RS72, Theorem VIII.20] *If A_n converges to A in the norm resolvent sense and $f:\mathbb{R}\to\mathbb{R}$ is continuous and bounded, then $f(A_n)$ converges strongly to $f(A)$. If f is vanishing at $\pm\infty$ then convergence is in norm.*

(iv) [DM20a, Lemma 5.5] *Assume $A_n\geq\lambda\in\mathbb{R}$ for all $n\in\mathbb{N}$. Then A_n converges to A in the norm resolvent sense if and only if e^{-tA_n} converges to e^{-tA} in norm for all $t\geq 0$. In this case, $\inf\sigma(A_n)$ converges to $\inf\sigma(A)$.*

(v) [Tes14, Theorem 6.38] *Assume A_n converges to A in the norm resolvent sense. Then $\sigma(A) = \lim_{n\to\infty}\sigma(A_n)$.*

(vi) [Oli09, Proposition 11.4.31] *Assume A_n converges to A in the norm resolvent sense. If $(a,b)\cap\sigma_{\mathsf{ess}}(A_n)=\emptyset$ for all $n\in\mathbb{N}$, then $(a,b)\cap\sigma_{\mathsf{ess}}(A)=\emptyset$. If $(\lambda_n)_{n\in\mathbb{N}}\subset\mathbb{R}$ is a convergent sequence and $\lambda_n\in\sigma_{\mathsf{ess}}(A_n)$ for all $n\in\mathbb{N}$, then $\lim_{n\to\infty}\lambda_n\in\sigma_{\mathsf{ess}}(A)$.*

Remark A.77. The second part of (i) is not contained in the reference. It, however, follows by a similar proof.

Lemma A.78. *Let $(A_n)_{n\in\mathbb{N}}$ be a family of selfadjoint operators on $\mathcal{H}$ and assume there is $\lambda\in\mathbb{R}$ such that $A_n\geq\lambda$ for all $n\in\mathbb{N}$. Let A and B be selfadjoint operators on $\mathcal{H}$ and assume that $\mathcal{D}(|A_n|^{1/2})\subset\mathcal{D}(|B|^{1/2})$ for all $n\in\mathbb{N}$, that A_n converges to A in the norm resolvent sense and that $|B|^{1/2}$ has a bounded inverse. For $z<\inf\sigma(A_n)$, we define the bounded operator $C_{n,z}=|B|^{1/2}(A_n-z)^{-1/2}$. If $C_{n,z}C_{n,z}^*$ converges strongly for some $z_0<\lambda$, then $\mathcal{D}(|A|^{1/2})\subset\mathcal{D}(|B|^{1/2})$ and $\operatorname{s-lim}_{n\to\infty}C_{n,z}=|B|^{1/2}(A-z)^{-1/2}=:C_{\infty,z}$ for all $z<\inf\sigma(A)$. Further, $\operatorname{s-lim}_{n\to\infty}C_{n,z}C_{n,z}^*=C_{\infty,z}C_{\infty,z}^*$*

Proof. Note that $A \geq \lambda$, by Lemma A.76 (iv). Pick $z_0 < \lambda$ such that $C_{n,z_0} C_{n,z_0}^*$ strongly converges to a selfadjoint operator $C \in \mathcal{B}(\mathcal{H})$. For $\phi \in \mathcal{D}(|B|^{1/2})$ and $\psi \in \mathcal{H}$, we see

$$\langle |B|^{1/2}\phi, (A-z_0)^{-1/2}\psi\rangle \leq \lim_{n\to\infty} \langle \phi, C_{n,z_0} C_{n,z_0}^* \phi\rangle^{1/2} \|\psi\| \leq \|C\|^{1/2} \|\psi\| \|\phi\|,$$

showing $(A-z_0)^{-1/2}\psi \in \mathcal{D}(|B|^{1/2})$ and hence $\mathcal{D}(|A|^{1/2}) \subset \mathcal{D}(|B|^{1/2})$.

For $\phi, \psi \in \mathcal{D}(|B|^{1/2})$, the norm resolvent convergence of (A_n) also yields

$$\begin{aligned}\langle \phi, C_{\infty,z_0} C_{\infty,z_0}^* \psi\rangle &= \lim_{n\to\infty} \langle (A_n - z_0)^{-1/2}|B|^{1/2}\phi, (A_n - z_0)^{-1/2}|B|^{1/2}\psi\rangle \\ &= \lim_{n\to\infty} \langle \phi, C_{n,z_0} C_{n,z_0}^* \psi\rangle = \langle \phi, C\psi\rangle\end{aligned}$$

so $C_{\infty,z_0} C_{\infty,z_0}^* = C$.

Note that $\|C_{n,z_0}\|^2 = \|C_{n,z_0} C_{n,z_0}^*\|$ (Lemma A.14) is bounded uniformly in n by the uniform boundedness principle. Since $\mathcal{D}(|A|^{1/2})$ is dense, it is now enough to show $\lim_{n\to\infty} C_{n,z_0}\psi = C_{\infty,z_0}\psi$ for all $\psi \in \mathcal{D}(|A|^{1/2})$. Hence, using

$$\begin{aligned}C_{n,z_0}\psi =& C_{n,z_0} C_{n,z_0}^* |B|^{-1/2}(A-z_0)^{1/2}\psi \\ &+ C_{n,z_0}((A-z_0)^{-1/2} - (A_n - z_0)^{-1/2})(A-z_0)^{1/2}\psi,\end{aligned}$$

we see that $C_{n,z_0}\psi$ converges to $C|B|^{-1/2}(A-z_0)^{1/2}\psi = C_{\infty,z_0}\psi$ for all $\psi \in \mathcal{D}(|A|^{1/2})$ by Lemma A.76 (iii). For any other $z < \inf \sigma(A)$, we conclude that $z < \inf \sigma(A_n)$ for n large enough by Lemma A.76 (iv). Then, by Lemma A.76 (iii)

$$C_{n,z} = C_{n,z_0} \left(\frac{A_n - z_0}{A_n - z}\right)^{1/2} \xrightarrow[\mathrm{s}]{n\to\infty} C_{\infty,z_0} \left(\frac{A - z_0}{A - z}\right)^{1/2} = C_{\infty,z}. \qquad \square$$

Quadratic Forms

We also need some notions from the theory of quadratic forms.

Definition A.79. We say $\mathfrak{q}$ is a *(sesquilinear) form on* $\mathcal{H}$ with *form domain* $\mathcal{Q}(\mathfrak{q})$ if $\mathcal{Q}(\mathfrak{q})$ is a subspace of $\mathcal{H}$ and $\mathfrak{q} : \mathcal{Q}(\mathfrak{q}) \times \mathcal{Q}(\mathfrak{q}) \to \mathbb{C}$ is linear in the second and anti-linear in the first argument. We also write $\mathfrak{q}(x) = \mathfrak{q}(x,x)$ for $x \in \mathcal{Q}(\mathfrak{q})$ and then call *q quadratic form*. The form $\mathfrak{q}$ is called *symmetric* if $\mathfrak{q}(x,y) = \overline{\mathfrak{q}(y,x)}$ for all $x, y \in \mathcal{Q}(\mathfrak{q})$.
It is called *lower-semibounded* if there exists $C \in \mathbb{R}$ such that $\mathfrak{q}(x,x) \geq C\|x\|_{\mathcal{H}}$ for all $x \in \mathcal{Q}(\mathfrak{q})$. Especially, if $C = 0$ it is called *positive*.

Definition A.80. Let $\mathfrak{q}$ be a lower-semibounded symmetric form with lower bound C. We say $\mathfrak{q}$ is *closed* if $\mathcal{Q}(\mathfrak{q})$ with inner product $\langle x, y\rangle_{\mathfrak{q}} = \mathfrak{q}(x,y) + (1-C)\langle x, y\rangle_{\mathcal{H}}$ is a Hilbert space. In this case a subspace $D \subset \mathcal{Q}(\mathfrak{q})$ is called a *form core* of $\mathfrak{q}$ if it is dense in $(\mathcal{Q}(\mathfrak{q}), \langle\cdot,\cdot\rangle_{\mathfrak{q}})$.
We say $\mathfrak{q}$ is *closable* if there exists a closed lower-semibounded symmetric form $\widetilde{\mathfrak{q}}$ such that $\mathcal{Q}(\mathfrak{q}) \subset \mathcal{Q}(\widetilde{\mathfrak{q}})$ and $\mathfrak{q}(x,y) = \widetilde{\mathfrak{q}}(x,y)$ for all $x, y \in \mathcal{Q}(\mathfrak{q})$.

The following characterization of closed forms will be important for us.

Lemma A.81 ([Sch12, Proposition 10.1])**.** *Let* $\mathfrak{q}$ *be a lower-semibounded symmetric form. Then the following are equivalent:*

(i) $\mathfrak{q}$ *is closed.*

(ii) $\widetilde{\mathfrak{q}} : \mathcal{H} \to \mathbb{R} \cup \{\infty\}$ *with* $\widetilde{\mathfrak{q}}(x) = \mathfrak{q}(x)$ *for* $x \in \mathcal{Q}(\mathfrak{q})$ *and* $\widetilde{\mathfrak{q}}(x) = \infty$ *if* $x \notin \mathcal{Q}(q)$ *is lower-semicontinuous, i.e.,*

$$\widetilde{\mathfrak{q}}\left(\lim_{n\to\infty} x_n\right) \leq \liminf_{n\to\infty} \widetilde{\mathfrak{q}}(x_n) \qquad \text{holds for any convergent sequence } (x_n)_{n\in\mathbb{N}} \subset \mathcal{H}.$$

Remark A.82. From now on, we will speak of *closed forms* and mean closed lower-semibounded symmetric forms.

We conclude with a simple additivity statement.

Lemma A.83 ([Sch12, Corollary 10.2])**.** *Any finite sum of closed forms is closed.*

Forms Associated to Operators

We are mainly interested in the forms, which are associated to a selfadjoint operator.

Definition A.84. If A is a selfadjoint operator on $\mathcal{H}$, we call the form

$$\mathfrak{q}_A(x,y) = \langle |A|^{1/2}x, \operatorname{sign}(A)|A|^{1/2}y\rangle \quad \text{with form domain } \mathcal{Q}(\mathfrak{q}_A) = \mathcal{D}(|A|^{1/2}) =: \mathcal{Q}(A)$$

the *form associated with A*. We call $\mathcal{Q}(A)$ the *form domain of A*.

Theorem A.85. *The map* $A \mapsto \mathfrak{q}_A$ *is a bijection from the set of selfadjoint lower-semibounded operators on* $\mathcal{H}$ *to the closed forms on* $\mathcal{H}$*.*

We will need some relations between operator and form domains.

Lemma A.86 ([Wei80, Theorem 5.37])**.** *Let A be a selfadjoint operator on* $\mathcal{H}$ *and let* $\psi \in \mathcal{Q}(A)$*. Then the following are equivalent:*

(i) $\psi \in \mathcal{D}(A)$,

(ii) *The map* $\mathcal{Q}(A) \ni \phi \mapsto \mathfrak{q}_A(\psi, \phi)$ *is continuous.*

(iii) *There is a form core D of A such that* $D \ni \phi \mapsto \mathfrak{q}_A(\psi, \phi)$ *is continuous.*

Lemma A.87. *Let A and B be selfadjoint operators on* $\mathcal{H}$ *and let* $U \in \mathcal{B}(\mathcal{H})$ *be a unitary. If* $U\mathcal{D}(A) \subset \mathcal{D}(B)$ *then* $U\mathcal{Q}(A) \subset \mathcal{Q}(B)$.

Proof. The statement in the case $U = \mathbb{1}$ and A and B are positive can be found in [Wei80, Theorem 9.4]. Now observe $\mathcal{D}(U|A|U^*) = \mathcal{D}(UAU^*) = U\mathcal{D}(A) \subset \mathcal{D}(B) = \mathcal{D}(|B|)$, so we have $\mathcal{D}(U|A|^{1/2}U^*) \subset \mathcal{D}(|B|^{1/2}) = \mathcal{Q}(B)$. The claim now follows by observing $\mathcal{D}(U|A|^{1/2}U^*) = U\mathcal{Q}(A)$. □

Relative Form Bounds

Similar to relative operator bounds, we discuss relative form bounds.

Definition A.88. Let $\mathfrak{q}$ and $\mathfrak{v}$ be symmetric forms and assume that $\mathfrak{q}$ is lower-semibounded. We say $\mathfrak{v}$ is *relatively form bounded* with respect to $\mathfrak{q}$ with $\mathfrak{q}$-bound $a > 0$ if $\mathcal{Q}(\mathfrak{v}) \subset \mathcal{Q}(\mathfrak{q})$ and there exists $b \geq 0$ such that

$$|\mathfrak{v}(x)| \leq a|\mathfrak{q}(x)| + b\|x\|^2 \qquad \text{for all } x \in \mathcal{H}.$$

Further, if A and B are selfadjoint operators and A is lower-semibounded, we say $\mathfrak{v}$ and B are *A-form bounded* if $\mathfrak{v}$ and $\mathfrak{q}_B$ are relatively form bounded with respect to $\mathfrak{q}_A$, respectively.

Similar to Lemma A.44, we have the following lemma

Lemma A.89 ([Tes14, Lemma 6.28])**.** *Let A and B be selfadjoint and assume A is lower-semibounded. Then the following are equivalent:*

(i) *B is A-form bounded.*

(ii) *$\mathcal{Q}(A) \subset \mathcal{Q}(B)$.*

(iii) *$|B|^{1/2}(A - \lambda)^{-1/2}$ is bounded for one (and hence all) $\lambda \in \varrho(A)$.*

Our main application is the following generalization of the Kato-Rellich theorem, named after Kato, Lax, Lions, Milgram and Nelson.

Theorem A.90 (KLMN Theorem)**.** *Let A be a selfadjoint lower-semibounded operator and let $\mathfrak{q}$ be an A-bounded symmetric form with $\mathfrak{q}_A$-bound smaller than one. Then $\mathfrak{q}_A + \mathfrak{q}$ is a closed form on $\mathcal{Q}(A)$ and hence corresponds to a selfadjoint lower-semibounded operator with the same form domain as A. Explicitly, if $A \geq c \in \mathbb{R}$ and $|\mathfrak{q}(\psi)| \leq a|\mathfrak{q}_A(\psi)| + b\|\psi\|^2$ for all $\psi \in \mathcal{Q}(A)$ with $a \in (0,1)$ and $b \in \mathbb{R}$, then $\mathfrak{q}_A + \mathfrak{q} \geq (1-a)c - b$.*

Lemma A.91 ([Tes14, Theorem 6.25])**.** *Let A be a selfadjoint operator with $A \geq \lambda$, $\mathfrak{q}$ a symmetric form with $\mathcal{Q}(A) \subset \mathcal{Q}(\mathfrak{q})$ and assume $a, b \in \mathbb{R}$. The symmetric sesquilinear form $\mathfrak{q}((A-z)^{-1/2}x, (A-z)^{-1/2}x)$ for $\phi, \psi \in \mathcal{H}$ corresponds to a bounded operator $C(z)$ with $\|C(z)\| \leq a$ for $z < -ba^{-1} - \lambda$ if and only of*

$$\mathfrak{q}(\psi) \leq a\mathfrak{q}_A(\psi) + b\|\psi\|^2 \qquad \textit{for all } \psi \in \mathcal{Q}(\mathfrak{q}).$$

Further, if $a < 1$, then

$$(B - z)^{-1} = (A - z)^{-1/2}(1 + C(z))^{-1}(A - z)^{-1/2},$$

where B denotes the selfadjoint and lower-semibounded operator corresponding to $\mathfrak{q}_A + \mathfrak{q}$ by the KLMN theorem.

Weak Commutators

We will also use the weak commutator of operators.

Definition A.92. Let A and B be operators on $\mathcal{H}$. Then, we define the *weak commutator* as the form

$$\mathfrak{c}_{A,B}(\psi,\phi) = \langle A\psi, B\phi\rangle - \langle B\psi, A\phi\rangle \qquad \text{on the form domain } \mathcal{Q}(\mathfrak{c}_{A,B}) = \mathcal{D}(A)\cap\mathcal{D}(B).$$

Lemma A.93. *Let A and B be selfadjoint operators and assume that there is a set $D \subset \mathcal{D}(B)$ such that $e^{itB}D \subset \mathcal{D}(A)$ for all $t\in\mathbb{R}$ and $t\mapsto Ae^{itB}\psi$ is continuous for all $\psi\in D$. For fixed $\psi,\phi\in D$, we define the map $f:\mathbb{R}\to\mathbb{C}$ as*

$$f(t) = \langle \psi, e^{-itB}Ae^{itB}\phi\rangle\,.$$

Then f is continuously differentiable with derivative

$$f'(t) = \mathrm{i}\mathfrak{c}_{A,B}(e^{itB}\psi, e^{itB}\phi).$$

Proof. We easily calculate

$$f(t+h)-f(t) = \langle (e^{ihB}-1)e^{itB}\psi, Ae^{i(t+h)B}\phi\rangle + \langle Ae^{itB}\psi, (e^{ihB}-1)e^{itB}\phi\rangle\,.$$

The statement then directly follows using the continuity assumption. □

The Min-Max Principle and Compactness

We construct compact sets from operators. To that end, we define the following sequence for any lower-semibounded selfadjoint operator.

Definition A.94 (Min-Max Principle)**.** Let A be a selfadjoint lower-semibounded operator. Then, for $n\in\mathbb{N}$, we define

$$\eta_n(A) = \sup_{x_1,\ldots,x_n\in\mathcal{H}} \inf\left\{\langle x, Ax\rangle\middle| x\in\mathcal{D}(A)\cap\{x_1,\ldots,x_n\}^\perp, \|x\|=1\right\}$$

and call $\eta_n(A)$ the *n-th eigenvalue of A obtained by the min-max principle.*

We will need the following statement characterizing compact sets.

Lemma A.95 ([RS78, Theorem XIII.64])**.** *Let A be a selfadjoint lower-semibounded operator. Then $\{\psi\in\mathcal{Q}(A) : \|\psi\|\le 1, \mathfrak{q}_A(\psi)\le b\}$ is compact for all $b>0$ if and only if $\eta_n(A)\xrightarrow{n\to\infty}\infty$.*

Also, we note the following compactness criterion in L^2-spaces.

Lemma A.96 (Rellich's Criterion, [RS78, Theorem XIII.65])**.** *Let $F,G:\mathbb{R}^n\to[0,\infty)$ be measurable satisfying $\lim\limits_{x\to\infty}F(x) = \lim\limits_{x\to\infty}G(x)=\infty$. Then the set*

$$\left\{f\in L^2(\mathbb{R}^n)\middle|\|f\|_2\le 1, \|F^{1/2}f\|_2\le 1, \|G^{1/2}\widehat{f}\|\le 1\right\}$$

is a compact subset of $L^2(\mathbb{R}^n)$.

Remark A.97. Here, we understand F,G as selfadjoint multiplication operators on $L^2(\mathbb{R}^n)$ and denote the usual unitary Fourier transform of f by $\widehat{f}$.

Direct Sums of Operators

We return to the topic of direct sums and define direct sums of operators.

Definition A.98. Let $\mathscr{I}$ be an arbitrary index set, let $(\mathcal{H}_i)_{i\in\mathscr{I}}$ and $(\mathcal{V}_i)_{i\in\mathscr{I}}$ be families of Hilbert spaces and assume that A_i is an operator from $\mathcal{H}_i$ to $\mathcal{V}_i$ for each $i \in \mathscr{I}$. The *direct sum operator* $\bigoplus_{i\in\mathscr{I}} A_i$ is defined as the operator from $\bigoplus_{i\in\mathscr{I}} \mathcal{H}_i$ to $\bigoplus_{i\in\mathscr{I}} \mathcal{V}_i$ with

$$\mathcal{D}\left(\bigoplus_{i\in\mathscr{I}} A_i\right) = \left\{(x_i) \in \bigoplus_{i\in\mathscr{I}} \mathcal{H}_i \,\middle|\, x_i \in \mathcal{D}(A_i), \sum_{i\in\mathscr{I}} \|A_i x_i\|^2 < \infty\right\},$$

$$\bigoplus_{i\in\mathscr{I}} A_i (x_i)_{i\in\mathscr{I}} = (A_i x_i)_{i\in\mathscr{I}}.$$

Tensor Products

We now turn to the definition of tensor products. Throughout, we assume that $N \in \mathbb{N}$ and $\mathcal{H}_1, \dots, \mathcal{H}_N$ are complex Hilbert spaces.

Definition A.99. For a family of vectors $x_i \in \mathcal{H}_i$ for $i = 1, \dots, N$, we define the *pure or elementary tensor* to be the multi-linear form

$$\bigotimes_{i=1}^N x_i : \bigtimes_{i=1}^N \mathcal{H}_i \to \mathbb{C} \qquad \text{with} \qquad \bigotimes_{i=1}^N x_i(y_1, \dots, y_N) = \prod_{i=1}^N \langle x_i, y_i \rangle_{\mathcal{H}_i}.$$

In the case $\mathcal{H}_1 = \dots = \mathcal{H}_N$ and $x_1 = \dots = x_N$, we write $x^{\otimes N} = \bigotimes_{i=1}^N x$.
We call the subspace of the space of all $\mathbb{C}$-valued functions on $\bigtimes_{i=1}^N \mathcal{H}_i$ spanned by the elementary tensors

$$\bigodot_{i=1}^N \mathcal{H}_i = \operatorname{span}\left\{\bigotimes_{i=1}^N x_i \,\middle|\, (x_1, \dots, x_N) \in \bigtimes_{i=1}^N \mathcal{H}_i\right\}$$

the *algebraic tensor product* of $\mathcal{H}_1, \dots, \mathcal{H}_N$.

We can equip the algebraic tensor product with an inner product,

Lemma A.100. *There exists a unique inner product, that is, positive definite symmetric sesquilinear form,* $\langle \cdot, \cdot \rangle_\otimes$ *on* $\bigodot_{i=1}^N \mathcal{H}_i$ *such that*

$$\left\langle \bigotimes_{i=1}^N x_i, \bigotimes_{i=1}^N y_i \right\rangle_\otimes = \prod_{i=1}^N \langle x_i, y_i \rangle_{\mathcal{H}_i}.$$

We can now define the full tensor product.

Definition A.101. We call the completion of $\left(\bigodot_{i=1}^N \mathcal{H}_i, \langle \cdot, \cdot \rangle_\otimes\right)$ the *tensor product* of $\mathcal{H}_1, \dots, \mathcal{H}_N$ and denote it by $\bigotimes_{i=1}^N \mathcal{H}_i$. In the case $\mathcal{H} = \mathcal{H}_1 = \dots = \mathcal{H}_N$, we write $\mathcal{H}^{\otimes N} = \bigotimes_{i=1}^N \mathcal{H}$.

We discuss the case of L^2-spaces.

Lemma A.102. *Let $(\mathcal{M}_i, \Sigma_i, \mu_i)$ for $i = 1, \dots, N$ be σ-finite measure spaces. Then the map defined on pure tensors as $f_1 \otimes \cdots \otimes f_N \mapsto ((x_1, \dots, x_N) \mapsto f_1(x_1) \cdots f_N(x_N))$ extends to a unitary*

$$\bigotimes_{i=1}^{N} L^2(\mathcal{M}_i, \mu_i) \cong L^2\left(\bigtimes_{i=1}^{N} \mathcal{M}_i, \bigotimes_{i=1}^{n} \mu_i\right),$$

where $\bigotimes_{i=1}^{N} \mu_i$ denotes the product measure.

Lemma A.103. *Let $(\mathcal{M}, \Sigma, \mu)$ be a σ-finite measure space and assum $\mathcal{H}$ is separable. Then the map defined on pure tensors as $f \otimes \psi \mapsto (x \mapsto f(x)\psi)$ for $f \in L^2(\mathcal{M}, \mu)$ and $\psi \in \mathcal{H}$ extends to a unitary*

$$L^2(\mathcal{M}, \mu) \otimes \mathcal{H} \cong L^2(\mathcal{M}, \mu; \mathcal{H}),$$

where the right hand side denotes the Hilbert space of $\mathcal{H}$-valued L^2-functions on $\mathcal{M}$.

We now need to define operators on tensor products.

Definition A.104. Let A_i be an operator on $\mathcal{H}_i$ for each $i = 1, \dots, N$. Then, we define the *algebraic tensor product operator* as the unique linear operator on $\bigotimes_{i=1}^{N} \mathcal{H}_i$ satisfying

$$\mathcal{D}\left(\bigodot_{i=1}^{N} A_i\right) = \operatorname{span}\left\{\bigotimes_{i=1}^{N} x_i \,\middle|\, x_i \in \mathcal{D}(A_i)\right\}, \qquad \bigodot_{i=1}^{N} A_i \bigotimes_{i=1}^{N} x_i = \bigotimes_{i=1}^{N} A_i x_i.$$

Lemma A.105. *Assume A_i is a densely defined closable operator on $\mathcal{H}_i$ for each $i = 1, \dots, N$. Then $\bigodot_{i=1}^{N} A_i$ is densely defined and closable.*

Definition A.106. In the situation of Lemma A.105, we define the *tensor product operator* as

$$\bigotimes_{i=1}^{N} A_i = \overline{\bigodot_{i=1}^{N}} A_i.$$

In the case $\mathcal{H}_1 = \cdots = \mathcal{H}_N$ and $A = A_1 = \cdots = A_N$, we write $A^{\otimes N} = \bigotimes_{i=1}^{N} A$.

Lemma A.107 ([Ara18, Theorem 3.9])**.** *Assume A_i are selfadjoint operators on $\mathcal{H}_i$ for each $i = 1, \dots, N$. Then $\bigotimes_{i=1}^{N} A_i$ is selfadjoint. Further, if D_i are cores for A_i for each $i = 1, \dots, N$, then $\bigodot_{i=1}^{N} D_i$ is a core for $\bigotimes_{i=1}^{N} A_i$.*

Positivity on L^2-Spaces

One main example, which is important in this thesis are operators on L^2-spaces. Especially, we will use multiplication operators in several places, which we do not separately define. Here, we introduce the concepts of positivity.

Throughout, we assume that $(\mathcal{M}, \Sigma, \mu)$ is a σ-finite measure space.

Definition A.108. We call $f \in L^2(\mathcal{M})$ *(strictly) positive* if $(f > 0)$ $f \geq 0$ μ-a.e. The space of all (strictly) positive functions is denoted as $(L^2_{++}(\mathcal{M}))$ $L^2_+(\mathcal{M})$. An operator $A \in \mathcal{B}(L^2(\mathcal{M}))$ is called *positivity preserving* if $AL^2_+(\mathcal{M}) \subset L^2_+(\mathcal{M})$. Further, it is called *positivity improving* if $AL^2_+(\mathcal{M}) \setminus \{0\} \subset L^2_{++}(\mathcal{M})$.

The following statement follows easily.

Lemma A.109 ([Sim74, Lemma I.14])**.** *If A is a positivity preserving operator on $L^2(\mathcal{M})$, then for any $f \in L^2(\mathcal{M})$ the inequality $|Af| \leq A|f|$ holds almost everywhere.*

The next two lemmas give perturbative criteria for selfadjoint operators to have a positivity preserving and positivity improving semigroup, respectively.

Lemma A.110 ([RS78, Thm. XIII.45])**.** *Let H_0 and H be selfadjoint lower-semibounded operators on $L^2(\mathcal{M})$. Further assume there is a sequence of bounded multiplication operators $(V_n)_{n\in\mathbb{N}}$ such that $H_0 + V_n$ converges to H in strong resolvent sense and $H - V_n$ converges to H_0 in strong resolvent sense and $H - V_n$ and $H_0 + V_n$ are uniformly bounded from below. Then e^{-tH} is positivity preserving if and only if e^{-tH_0} is positivity preserving.*

Lemma A.111 ([Far72, Theorem 3],[RS78, Theorem XIII.44])**.** *Let H_0 and V be selfadjoint operators on $L^2(\mathcal{M})$ such that H_0 is lower-semibounded and V is H_0-bounded with relative bound smaller than 1. For $\Lambda > 0$, we write $V_\Lambda = V\chi_{\{|\cdot|\leq\Lambda\}}(V)$. If e^{-tH_0} is positivity improving for all $t > 0$, e^{-tV_Λ} is positivity preserving for all $t, \Lambda > 0$, and $\langle f, e^{-tV_\Lambda} g\rangle = 0$ for all $t > 0$ and $f, g \in L^2_+(\mathcal{M})$ with $\langle f, g\rangle = 0$, then $e^{-t(H_0+V)}$ is positivity improving for all $t > 0$.*

The next famous theorem gives a connection between uniqueness of ground states and positivity properties of operators.

Theorem A.112 (Perron-Frobenius-Faris, [Far72, Theorem 1],[RS78, Theorem XIII.44])**.** *Let H be a selfadjoint lower-semibounded operator on $L^2(\mathcal{M})$ and assume that e^{-tH} is positivity preserving and $E = \inf \sigma(H)$ is an eigenvalue. Then e^{-tH} is positivity improving for all $t > 0$ if and only if there is $f \in L^2_{++}(\mathcal{M})$ such that* $\ker(H - E) = \operatorname{span}\{f\}$.

Finally, we state a lemma which allows us to obtain the ground state energy of an operator acting on an L^2-space with positivity preserving semigroup from any strictly positive function.

Lemma A.113 ([MM18, Theorem C.1])**.** *Let H be a self-adjoint operator on $L^2(\mathcal{M})$. If e^{-TH} is positivity preserving for all $T \geq 0$ and $f \in L^2_{++}(\mathcal{M})$, then*

$$\inf \sigma(H) = -\lim_{T\to\infty} \frac{1}{T} \log \left\langle f, e^{-TH} f \right\rangle .$$

Remark A.114. In fact, the referenced statement is stronger. It suffices to assume that $e^{-\tau H} f \in L^2_{++}(\mathcal{M})$ for any fixed choice of $\tau \geq 0$. We only need the case $\tau = 0$.

Direct Integrals

As above assume $(\mathcal{M}, \Sigma, \mu)$ is a σ-finite measure space. We want to define a class of operators on the space of $\mathcal{H}$-valued square integrable functions $L^2(\mathcal{M}; \mathcal{H})$.

Definition A.115. A family $\{A(x) \mid x \in \mathcal{M}\}$ of selfadjoint operators on $\mathcal{H}$ is called *measurable* if $x \mapsto (A(x)+\mathsf{i})^{-1}$ is measurable (in the sense of the Borel σ-algebra $\mathfrak{B}(\mathcal{B}(\mathcal{H}))$). In this case, we define the operator $A = \int_{\mathcal{M}}^{\oplus} A(x)\mathsf{d}x$ on $L^2(\mathcal{M};\mathcal{H})$ as

$$\mathcal{D}(A) = \{\psi \in L^2(\mathcal{M};\mathcal{H}) | \psi(x) \in \mathcal{D}(A(x)) \text{ a.e., } x \mapsto \|A(x)\psi(x)\| \in L^2(\mathcal{M})\},$$
$$(A\psi)(x) = A(x)\psi(x).$$

Lemma A.116 ([RS78, Theorem XIII.85]). *If $\{A(x)|x \in \mathcal{M}\}$ is a measurable family of selfadjoint operators on $\mathcal{H}$, then $\int_{\mathcal{M}}^{\oplus} A(x)\mathsf{d}x$ is selfadjoint.*
Further, if $A(x) \geq c \in \mathbb{R}$ for almost all $x \in \mathcal{M}$, then $\int_{\mathcal{M}}^{\oplus} A(x)\mathsf{d}x \geq c$.

B. Fock Space Analysis

In this appendix, we construct the bosonic Fock space and recall standard properties. Introductory literature on the topic includes [RS72, Par92, Ara18]. Hence, most proofs are deferred to those books.

Throughout this appendix, let $\mathfrak{h}$ be a complex Hilbert space, which we also refer to as *one-particle space*.

B.1. The Bosonic Fock Space

Bosons are indistinguishable particles. In this sense, our n-particle states need to be symmetric in the exchange of variables. To make this precise, we define the symmetrization operator $S_n \in \mathcal{B}(\mathfrak{h}^{\otimes n})$ as acting on pure tensors as

$$S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} f_{\pi(1)} \otimes \cdots f_{\pi(n)},$$

where $\mathfrak{S}_n$ denotes the symmetric group on $\{1, \ldots, n\}$. It is easy to check that S_n extends to an orthogonal projection. We denote its range as

$$\mathfrak{h}^{\otimes_{\mathsf{s}} n} = \operatorname{ran} S_n.$$

This now allows us to define the bosonic Fock space.

Definition B.1 (Fock Space)**.** For $n \in \mathbb{N}_0$, we define the *n-particle space (over $\mathfrak{h}$)* as

$$\mathcal{F}^{(n)}(\mathfrak{h}) = \mathfrak{h}^{\otimes_{\mathsf{s}} n} \quad \text{for } n \in \mathbb{N} \quad \text{and} \quad \mathcal{F}^{(0)} = \mathbb{C}.$$

Further, we define the *bosonic Fock space over $\mathfrak{h}$* as

$$\mathcal{F}(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}(\mathfrak{h}).$$

We write an element $\psi \in \mathcal{F}(\mathfrak{h})$ as vector $\psi = (\psi^{(n)})_{n \in \mathbb{N}_0}$ with $\psi^{(n)} \in \mathcal{F}^{(n)}(\mathfrak{h})$ for all $n \in \mathbb{N}_0$.

Remark B.2. We will, throughout this thesis, slightly abuse notation and consider $\mathcal{F}^{(n)}(\mathfrak{h})$ to be closed subspaces of $\mathcal{F}(\mathfrak{h})$.

Remark B.3. Let us discuss the case $\mathfrak{h} = L^2(\mathcal{M})$ for some σ-finite measure space $(\mathcal{M}, \Sigma, \mu)$. We denote by $L^2_{\mathsf{s}}(\mathcal{M}^{\times n})$ the space of all functions $f \in L^2(\mathcal{M}^{\times n})$ satisfying

$$\forall \pi \in \mathfrak{S}_n : f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \qquad \text{for a.e. } (x_1, \ldots, x_n) \in \mathcal{M}^{\times n}$$

with respect to the product measure on $\mathcal{M}^{\times n}$. In the sense of Lemma A.102, we then have the identification

$$\mathcal{F}^{(n)}(L^2(\mathcal{M})) \cong L^2_{\mathsf{s}}(\mathcal{M}^{\times n}).$$

An important subspace of the full Fock space is the following.

Definition B.4. Assume D is a subspace of $\mathfrak{h}$. Then, we define the *finite particle subspace generated by* D

$$\mathcal{F}_{\mathsf{fin}}(D) = \operatorname{span}\{S_n(f_1 \otimes \cdots \otimes f_n) | n \in \mathbb{N}, f_1, \ldots, f_n \in D\}.$$

Remark B.5. We note that some authors also use the term finite particle subspace for the set

$$\left\{(\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}(\mathfrak{h}) \middle| \exists N \in \mathbb{N}\, \forall k \geq N : \psi^{(k)} = 0\right\}.$$

This is related to our finite particle subspace by taking the closure inside of each subspace $\mathcal{F}^{(n)}$ and then taking the union over all $n \in \mathbb{N}_0$.

Lemma B.6 ([Ara18, Lemma 4.4])**.** *If D is a dense subspace of $\mathfrak{h}$, then $\mathcal{F}_{\mathsf{fin}}(D)$ is dense in $\mathcal{F}$.*

We define some special types of vectors, starting with the vacuum.

Definition B.7. The *Fock vacuum* is the vector $\Omega = (1, 0, 0, \ldots)$.

For us, an important class of vectors will also be the so-called *coherent states.*

Definition B.8. For $f \in \mathfrak{h}$, we define the *exponential vector* as

$$\epsilon(f) = 1 \oplus \bigoplus_{n=1}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n}.$$

For $D \subset \mathfrak{h}$, we write $\mathcal{E}(D) = \{\epsilon(f) : f \in D\}$.

The following is an easy calculation from the above definition.

Lemma B.9. *For all $f, g \in \mathfrak{h}$, we have $\langle \epsilon(f), \epsilon(g)\rangle = e^{\langle f,g\rangle}$.*

As we will heavily use the exponential vectors by acting on them with operators, we will need the following.

Lemma B.10 ([Par92, Corollary 19.5])**.** *If D is dense in $\mathfrak{h}$, then $\mathcal{E}(D)$ is total in $\mathcal{F}(\mathfrak{h})$.*

B.2. Second Quantization Operators

In this appendix, we define a method to lift operators on the one-particle space $\mathfrak{h}$ to operators on $\mathcal{F}(\mathfrak{h})$.

Definition B.11. Let T be a densely defined closable operator on $\mathfrak{h}$. Then, we define

$$\mathsf{d}\Gamma(T) = \bigoplus_{n=0}^{\infty} \mathsf{d}\Gamma^{(n)}(T) \qquad \text{with } \mathsf{d}\Gamma^{(n)}(T) = \begin{cases} \overline{\sum_{k=1}^{n} (\mathbb{1})^{\otimes(k-1)} \otimes T \otimes (\mathbb{1})^{\otimes(n-k)}} & \text{for } n \in \mathbb{N}, \\ 0 & \text{for } n = 0 \end{cases}$$

as operator on $\mathcal{F}(\mathfrak{h})$.

Remark B.12. The $\mathsf{d}\Gamma$-operators are also called *differential second quantization operators* in the literature.

Remark B.13. Let us again discuss the case of L^2-spaces in the setting of Remark B.3. If $T : \mathcal{M} \to \mathbb{C}$ is measurable, i.e., a normal multiplication operator on $L^2(\mathcal{M})$, then in the sense of our previous considerations the n-particle operators are the multiplication operators on $\mathcal{F}^{(n)}(L^2(\mathcal{M})) = L^2_{\mathsf{s}}(\mathcal{M}^{\times n})$ given by

$$\mathsf{d}\Gamma^{(n)}(T)(x_1, \dots, x_n) = \sum_{i=1}^{n} T(x_i).$$

Let us define a second type of second quantization operator.

Definition B.14. Let S be a densely defined closable operator from $\mathfrak{h}$ to $\mathfrak{v}$. Then, we define

$$\Gamma(S) = \mathbb{1} \oplus \bigoplus_{n=1}^{\infty} S^{\otimes n}$$

as operator from $\mathcal{F}(\mathfrak{h})$ to $\mathcal{F}(\mathfrak{v})$.

We now collect some properties of the operators defined above.

Lemma B.15. *Let $\mathfrak{h}$, $\mathfrak{v}$, $\mathfrak{w}$ be complex Hilbert spaces, and let A, B, and C be densely defined closed operators on $\mathfrak{h}$, from $\mathfrak{h}$ to $\mathfrak{v}$, and from $\mathfrak{v}$ to $\mathfrak{w}$, respectively.*

(i) *If A is selfadjoint, then $\mathsf{d}\Gamma(A)$ is selfadjoint. Further, if $A \geq 0$, then $\mathsf{d}\Gamma(A) \geq 0$.*

(ii) *If A is selfadjoint and T is a selfadjoint operator on $\mathfrak{h}$ strongly commuting with A, then $\mathsf{d}\Gamma(A)$ and $\mathsf{d}\Gamma(T)$ strongly commute.*

(iii) *If D is a core for A, then $\mathcal{F}_{\mathsf{fin}}(D)$ is a core for $\mathsf{d}\Gamma(A)$.*

(iv) *If $g \in \mathcal{D}(A)$, then $\epsilon(g) \in \mathcal{D}(\mathsf{d}\Gamma(A))$ and*

$$\langle \epsilon(f), \mathsf{d}\Gamma(A)\epsilon(g) \rangle = \langle f, Ag \rangle \, e^{\langle f,g \rangle} \qquad \text{for all } f \in \mathfrak{h}.$$

Further, if $h \in \mathcal{D}(B)$, then

$$\langle \mathsf{d}\Gamma(A)\epsilon(g), \mathsf{d}\Gamma(B)\epsilon(h) \rangle = (\langle Ag, h \rangle \langle g, Bh \rangle + \langle Ag, Bh \rangle) e^{\langle g,h \rangle}.$$

(v) *If B is a contraction, then $\Gamma(B)$ is a contraction and $\Gamma(B)^* = \Gamma(B^*)$.*

(vi) *If B and C are contractions, then $\Gamma(B)\Gamma(C) = \Gamma(BC)$.*

(vii) *If B is unitary, so is $\Gamma(B)$.*

(viii) *If A is selfadjoint, then $e^{\mathsf{i}t\mathsf{d}\Gamma(A)} = \Gamma(e^{\mathsf{i}tA})$ for all $t \in \mathbb{R}$.*

(ix) *If A is selfadjoint and positive, then $e^{-t\mathsf{d}\Gamma(A)} = \Gamma(e^{-tA})$ for all $t \geq 0$.*

(x) *If U is unitary and A is selfadjoint, then $\Gamma(U)\mathsf{d}\Gamma(A)\Gamma(U)^* = \mathsf{d}\Gamma(UAU^*)$.*

(xi) *If $g \in \mathcal{D}(B)$, then $\epsilon(g) \in \mathcal{D}(\Gamma(B))$ and $\Gamma(B)\epsilon(g) = \epsilon(Bg)$.*

References for proofs. (i) [Ara18, Theorem 5.2] (ii) [Ara18, Proposition 5.4] (iii) [Ara18, Theorem 5.1] (iv) [Par92, Proposition 20.13] (v) [Ara18, Theorem 5.5] (vi),(vii) [Ara18, Theorem 5.6] (viii),(ix) [Ara18, Theorem 5.7] (x) [Ara18, Theorem 5.8] (xi) [Par92, (20.2),(20.4),(20.11)] □

In some places, we will need vector notation for the $\mathsf{d}\Gamma$-operators.

Definition B.16. If $\mathbf{A} = (A_1, \dots, A_k)$ for $k \in \mathbb{N}$ is a family of pairwise strongly commuting selfadjoint operators, we denote by $\mathsf{d}\Gamma(\mathbf{A}) = (\mathsf{d}\Gamma(A_1), \dots, \mathsf{d}\Gamma(A_k))$ the corresponding family of pairwise strongly commuting selfadjoint operators, cf. Definition A.73.

We use the following lemma when considering the fiber operators of the Nelson model.

Lemma B.17. *Fix some $k \in \mathbb{N}$ and assume that $\mathbf{A} = (A_1, \dots, A_k)$ is a family of pairwise strongly commuting selfadjoint operators. Then, for all $P \in \mathbb{R}^k$ and $s > 0$, we have $\mathcal{D}(|P - \mathsf{d}\Gamma(\mathbf{A})|^s) = \bigcap_{i=1}^k \mathcal{D}(|\mathsf{d}\Gamma(A_i)|^s)$.*

Proof. For all $x, P_1, P_2 \in \mathbb{R}^k$ we have the inequalities

$$|x - P_1|^{2s} \le 2^{2s}(|x - P_2|^{2s} + |P_2 - P_1|^{2s}) \qquad \text{and} \qquad |x_i|^{2s} \le |x|^{2s} \le k^{2s} \sum_{i=1}^k |x_i|^{2s}.$$

Hence, the statement follows from Definition A.73 and Lemma A.52. □

B.3. Creation, Annihilation and Field Operators

We now define the operators describing the particle-field interaction.

Definition B.18. Let $f \in \mathfrak{h}$.
We define $a(f)$ as the unique closed operator on $\mathcal{F}(\mathfrak{h})$ acting as

$$a(f)(S_n g_1 \otimes \cdots \otimes g_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \langle f, g_k \rangle \, S_{n-1}\big(g_1 \otimes \cdots \otimes \not{g_k} \otimes \cdots \otimes g_n\big)$$

and call $a(f)$ *annihilation operator (corresponding to f).*
Further, let $a^\dagger(f)$ be the unique closed operator acting as

$$a^\dagger(f)(S_n g_1 \otimes \cdots \otimes g_n) = \sqrt{n+1} S_{n+1}\big(f \otimes g_1 \otimes \cdots \otimes g_n\big)$$

and call it *creation operator (corresponding to f).*

Remark B.19. The definitions above directly extend to the dense set $\mathcal{F}_{\mathsf{fin}}(\mathfrak{h})$ (Lemma B.6). Further, by direct calculation on $\mathcal{F}_{\mathsf{fin}}(\mathfrak{h})$, the operators $a(f)$ and $a^\dagger(f)$ have densely defined adjoints. Hence, they are closable by Lemma A.10. Taking the closure then finishes the construction.

We collect some properties of these operators.

Lemma B.20. *Let $f, g \in \mathfrak{h}$.*

(i) *$\mathcal{D}(a(f)) = \mathcal{D}(a^\dagger(f))$ and $a(f)^* = a^\dagger(f)$.*

(ii) *If D is a dense subspace of $\mathfrak{h}$, then $\mathcal{F}_{\mathsf{fin}}(D)$ is a core for $a(f)$ and $a^\dagger(f)$.*

(iii) $\mathcal{E}(\mathfrak{h}) \subset \mathcal{D}(a(f))$ *and* $a(f)\epsilon(g) = \langle f, g\rangle \epsilon(g)$.

(iv) *On $\mathcal{F}_{\mathsf{fin}}(\mathfrak{h}) \cup \operatorname{span} \mathcal{E}(\mathfrak{h})$, the canonical commutation relations*

$$[a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0, \qquad [a(f), a^\dagger(g)] = \langle f, g\rangle$$

hold. Further, they hold in the sense of weak commutators (cf. Definition A.92).

(v) *If B is a contraction, then $\Gamma(B)a^\dagger(f) = a^\dagger(Bf)\Gamma(B)$ and $a(f)\Gamma(B)^* = \Gamma(B)^* a(Bf)$.*

(vi) *If B is an isometry from $\mathfrak{h}$ to $\mathfrak{v}$, i.e., $B^*B = \mathbb{1}$, then $\Gamma(B)a(f) = a(Bf)\Gamma(B)$.*

(vii) *If A is a positive and injective selfadjoint operator on $\mathfrak{h}$ and $f \in \mathcal{D}(A^{-1/2})$, then $\mathcal{D}(\mathsf{d}\Gamma(A)^{1/2}) \subset \mathcal{D}(a(f)) = \mathcal{D}(a^\dagger(f))$ and for all $\psi \in \mathcal{D}(\mathsf{d}\Gamma(A)^{1/2})$*

$$\begin{aligned}
\|a(f)\psi\| &\leq \|A^{-1/2}f\|\|\mathsf{d}\Gamma(A)^{1/2}\psi\|,\\
\|a^\dagger(f)\psi\| &\leq \|A^{-1/2}f\|\|\mathsf{d}\Gamma(A)^{1/2}\psi\| + \|f\|\|\psi\|.
\end{aligned}$$

(viii) *If A is a positive and injective selfadjoint operator on $\mathfrak{h}$ and $f \in \mathcal{D}(A^{-1/2}) \cap \mathcal{D}(A)$, then*

$$a(f)\mathcal{D}(\mathsf{d}\Gamma(A)) \cup a^\dagger(f)\mathcal{D}(\mathsf{d}\Gamma(A)) \subset \mathcal{D}(\mathsf{d}\Gamma(A)^{1/2}).$$

Further, $a(f)\mathcal{D}(\mathsf{d}\Gamma(A)^{3/2}) \cup a^\dagger(f)\mathcal{D}(\mathsf{d}\Gamma(A)^{3/2}) \subset \mathcal{D}(\mathsf{d}\Gamma(A))$ and

$$[\mathsf{d}\Gamma(A), a^\dagger(f)] = a^\dagger(Af), \qquad [\mathsf{d}\Gamma(A), a(f)] = -a(Af) \qquad \text{hold on } \mathcal{D}(\mathsf{d}\Gamma(A)^{3/2}).$$

References for proofs. (i) [Ara18, Lemma 5.4, Corollary 5.6] (ii) This follows directly by construction. (iii) [Par92, Proposition 20.12] (iv) The operator statement is [Ara18, Theorem 5.13] and [Par92, Proposition 20.12]. The weak commutator can be directly calculated. (vii) [Ara18, Theorem 5.16] (viii) [Ara18, Thm. 5.17, Lemma 5.12] □

Proof of (v), (vi). Let $g_1, \dots, g_n \in \mathfrak{h}$. The first statement follows from the calculation

$$\begin{aligned}
\Gamma(B)a^\dagger(f)S_n g_1 \otimes \cdots \otimes g_n &= \sqrt{n+1}\Gamma(B)S_{n+1} f \otimes g_1 \otimes \cdots \otimes g_n\\
&= \sqrt{n+1} S_{n+1} Bf \otimes Bg_1 \otimes \cdots \otimes Bg_n\\
&= a^\dagger(Bf)S_n Bg_1 \otimes \cdots \otimes Bg_n\\
&= a^\dagger(Bf)\Gamma(B)S_n g_1 \otimes \cdots \otimes g_n
\end{aligned}$$

and using Lemmas B.15 (iii) and B.20 (ii).

Similarly, using the isometry property of B, (vi) follows from

$$\begin{aligned}
\Gamma(B)a(f)S_n g_1 \otimes \cdots \otimes g_n &= \frac{1}{\sqrt{n}} \sum_{k=1}^n \langle f, g_k\rangle \Gamma(B) S_{n-1} g_1 \otimes \cdots \otimes \cancel{g_k} \otimes \cdots \otimes g_n\\
&= \frac{1}{\sqrt{n}} \sum_{k=1}^n \langle Bf, Bg_k\rangle S_{n-1} Bg_1 \otimes \cdots \otimes \cancel{Bg_k} \otimes \cdots \otimes Bg_n\\
&= a(Bf)S_n Bg_1 \otimes \cdots \otimes Bg_n\\
&= a(Bf)\Gamma(B)S_n g_1 \otimes \cdots \otimes g_n.
\end{aligned}$$

□

Definition B.21. For $f \in \mathfrak{h}$, we define the *(Segal) field operator*

$$\varphi(f) = \overline{a(f) + a^\dagger(f)}.$$

Lemma B.22. *Let $f \in \mathfrak{h}$.*

(i) *The operator $\varphi(f)$ is selfadjoint. Further, if D is a dense subspace of $\mathfrak{h}$, then both $\mathcal{F}_{\mathsf{fin}}(D)$ and* $\operatorname{span}\mathcal{E}(D)$ *are cores for $\varphi(f)$.*

(ii) *If A is a positive and injective selfadjoint operator and $f \in \mathcal{D}(A^{-1/2})$, then $\varphi(f)$ is $\mathsf{d}\Gamma(A)^{1/2}$-bounded. Especially, $\varphi(f)$ is infinitesimally $\mathsf{d}\Gamma(A)$-bounded.*

References for proofs. (i) [Ara18, Theorem 5.22], [Par92, Corollary 20.5] (ii) Follows from Lemma B.20 (vii). □

We will also make use of the following lemma

Lemma B.23. *Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be separable Hilbert spaces. Then, there exists a unique unitary $U : \mathcal{F}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) \to \mathcal{F}(\mathfrak{h}_1) \otimes \mathcal{F}(\mathfrak{h}_2)$ such that*

$$U\Omega = \Omega \otimes \Omega,$$
$$U\mathcal{F}_{\mathsf{fin}}(\mathfrak{h}_1 \oplus \mathfrak{h}_2) = \mathcal{F}_{\mathsf{fin}}(\mathfrak{h}_1) \odot \mathcal{F}_{\mathsf{fin}}(\mathfrak{h}_2), \quad \textit{(cf. Definition A.99)}$$
$$U(a^\#(f,g))U^* = \overline{a^\#(f) \otimes \mathbb{1} + \mathbb{1} \otimes a^\#(g)} \qquad \textit{for all } a^\# \in \{a, a^\dagger\},\ (f,g) \in \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

Further, for selfadjoint operators A and B on $\mathfrak{h}_1$ and $\mathfrak{h}_2$, we have

$$U\mathsf{d}\Gamma(A \oplus B)U^* = \overline{\mathsf{d}\Gamma(A) \otimes \mathbb{1} + \mathbb{1} \otimes \mathsf{d}\Gamma(B)}.$$

Similar, for contraction operators T and S on $\mathfrak{h}_1$ and $\mathfrak{h}_2$, we have

$$U\Gamma(T \oplus S)U^* = \Gamma(T) \otimes \Gamma(S).$$

Proof. Combine the statements from [Ara18, Theorems 5.38, 5.40, 5.41]. □

B.4. Weyl Operators

In our treatment of the Nelson model the Weyl operators have an important role. They are defined through their action on exponential vectors.

Definition B.24. For $f \in \mathfrak{h}$, we define the *Weyl operator* $W(f)$ as the unique closed operator acting as

$$W(f)\epsilon(g) = e^{-\frac{1}{2}\|f\|^2 - \langle f,g\rangle}\epsilon(f+g) \qquad \text{for all } g \in \mathfrak{h}.$$

Remark B.25. The Weyl operators in above definition are well-defined, since it yields a bounded operator on the total set $\mathcal{E}(\mathfrak{h})$ (Lemma B.10), cf. Lemma A.11.

Lemma B.26 ([Par92, Proposition 20.1])**.**

(i) *For all $f \in \mathfrak{h}$, $W(f)$ is unitary, $W(f)^* = W(-f)$ and $W(-\mathrm{i}f) = e^{-\mathrm{i}\varphi(f)}$.*

(ii) *For all $f, g \in \mathfrak{h}$, $W(f)W(g) = e^{-\mathrm{i}\,\mathrm{Im}\langle f,g\rangle}W(f+g)$.*

(iii) *The map $f \mapsto W(f)$ is continuous in the strong operator topology.*

B.5. $\mathcal{Q}$-Space

In some places, we need a representation of $\mathcal{F}(\mathfrak{h})$ as L^2-space over a probability space $(\mathcal{Q}, \Sigma, \mu)$. To that end, we first define Hilbert space indexed Gaussian random processes.

Definition B.27. Let $\mathfrak{r}$ be a real Hilbert space. A map ϕ from $\mathfrak{r}$ to the random variables on a probability space $(\mathcal{Q}, \Sigma, \mu)$ is called *Gaussian random process over* $\mathfrak{r}$, if the following holds:

(i) ϕ is $\mathbb{R}$-linear,

(ii) Σ is the minimal σ-field generated by $\{\phi(f) : f \in \mathfrak{r}\}$,

(iii) $\phi(f)$ is a Gaussian random variable for any $f \in \mathfrak{r}$, i.e., $\mu \circ \phi(f)^{-1}$ is normally distributed,

(iv) $\phi(f)$ has mean zero for any $f \in \mathfrak{r}$, i.e., $\int_\Omega \phi(f) \mathsf{d}\mu = 0$,

(v) the Gaussians have covariance $\int_\Omega \phi(f)\phi(g)\mathsf{d}\mu = \langle f, g\rangle_\mathfrak{r}$ for all $f, g \in \mathfrak{r}$.

Remark B.28. We fixed the mean of the random variables to zero and the covariance to $\langle v, w\rangle_\mathfrak{r}$, to make the statement of Lemma B.30 as simple as possible.

Lemma B.29. *For any real Hilbert space $\mathfrak{r}$ there exist a unique (up to isomorphism) probability space $(\mathcal{Q}_\mathfrak{r}, \Sigma_\mathfrak{r}, \mu_r)$ and a unique (again up to isomorphism) Gaussian random process $\phi_\mathfrak{r}$ indexed by $\mathfrak{r}$ on $(\mathcal{Q}_\mathfrak{r}, \Sigma_\mathfrak{r}, \mu_\mathfrak{r})$.*

Proof. See [Sim74, Theorems I.6 and I.9] or [LHB11, Prop. 5.6, Section 5.4]. □

The following isometry statement is also called Wiener-Itô-Segal isomorphism and can be found in [Sim74, Theorem I.11] and [LHB11, Prop. 5.7].

Lemma B.30. *There exists a unitary operator $\Theta_\mathfrak{r} : \mathcal{F}(\mathfrak{r} \oplus \mathrm{i}\mathfrak{r}) \to L^2(\mathcal{Q}_\mathfrak{r})$ such that*

(i) $\Theta_\mathfrak{r}\Omega = 1$,

(ii) $\Theta_\mathfrak{r}^*\phi_\mathfrak{r}(v)\Theta_\mathfrak{r} = \varphi(v)$ *for all* $v \in \mathfrak{r}$.

Remark B.31. By $\mathfrak{r} \oplus \mathrm{i}\mathfrak{r}$, we mean the complexification of $\mathfrak{r}$, i.e., the Hilbert space given by $\{(x, y) : x, y \in \mathfrak{r}\}$ with the usual addition, scalar multiplication

$$\alpha(x, y) = (x \operatorname{Re}\alpha - y \operatorname{Im}\alpha, x \operatorname{Im}\alpha + y \operatorname{Re}\alpha)$$

and inner product

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle_\mathfrak{r} + \langle y_1, y_2\rangle_\mathfrak{r} + \mathrm{i}(\langle x_1, y_2\rangle_\mathfrak{r} - \langle y_1, x_2\rangle_\mathfrak{r}).$$

Further, in (ii), we understand $\phi_\mathfrak{r}(v)$ as selfadjoint multiplication operator acting on $L^2(\mathcal{Q}_\mathfrak{r})$.

We will need the following positivity statement.

Lemma B.32 ([Sim74, Theorem I.12])**.** *If T is a contraction operator on $\mathfrak{r} \oplus \mathrm{i}\mathfrak{r}$, then $\Theta_\mathfrak{r}\Gamma(T)\Theta_\mathfrak{r}^*$ is positivity preserving on $L^2(\mathcal{Q}_\mathfrak{r})$. Especially, if A is selfadjoint on $\mathfrak{r} \oplus \mathrm{i}\mathfrak{r}$, then $\Theta_\mathfrak{r} e^{\mathrm{i}t\mathsf{d}\Gamma(A)}\Theta_\mathfrak{r}^*$ is positivity preserving for all $t \in \mathbb{R}$. If A is also positive, then $\Theta_\mathfrak{r} e^{-t\mathsf{d}\Gamma(A)}\Theta_\mathfrak{r}^*$ is positivity preserving for all $t \geq 0$.*

B.6. Pointwise Annihilation Operators

In this thesis, pull-through formulas are used for both the Nelson and the spin boson model. To prove them, we need to define pointwise annihilation operators and appropriately calculate their commutators with the operators defined above. Here, we give an overview of the approach in [DM20b, Appendix D]. All statements made here are proven therein.

Throughout, let $(\mathcal{M}, \Sigma, \mu)$ be a σ-finite measure space and $\mathfrak{h} = L^2(\mathcal{M})$.

Definition B.33. We define the *extended Fock space*

$$\mathcal{F}_+(\mathfrak{h}) = \bigtimes_{n=0}^{\infty} \mathcal{F}^{(n)}(\mathfrak{h})$$

and equip it with the metric

$$d_+(\psi, \phi) = \sum_{n=0}^{\infty} \frac{\|\psi^{(n)} - \phi^{(n)}\|}{2^n(1 + \|\psi^{(n)} - \phi^{(n)}\|)}.$$

Further, we define the coordinate projections $\mathcal{P}_n \psi = \psi^{(n)} \in \mathcal{F}^{(n)}$.

Although we cannot equip $\mathcal{F}_+$ with an appropriate inner product, we can pair elements of $\mathcal{F}_+$ with elements of the finite particle subspace.

Definition B.34. For $\phi \in \mathcal{F}_{\mathsf{fin}}(\mathfrak{h})$ and $\psi \in \mathcal{F}_+(\mathfrak{h})$, we define

$$\langle \phi, \psi \rangle_+ = \sum_{n=0}^{\infty} \langle \phi^{(n)}, \psi^{(n)} \rangle .$$

Remark B.35. Note that the infinite sum in the above definition has only finitely many non-zero summands, due to the definition of the finite particle subspace, and is hence well-defined.

We will need extended versions of the operators defined above. To that end, recall that $a(f)$ and $a^\dagger(f)$ map $\mathcal{F}^{(n)}$ to $\mathcal{F}^{(n-1)}$ and $\mathcal{F}^{(n+1)}$, respectively, and on this domain are bounded (and hence continuous) by $\sqrt{n}\|f\|$ and $\sqrt{n+1}\|f\|$, respectively.

Definition B.36. For $f \in \mathfrak{h}$, we define the continuous operators

$$a_+(f)\psi = \bigtimes_{n=1}^{\infty} a(f)\psi^{(n)} \qquad \text{and} \qquad a_+^\dagger(f)\psi = 0 \times \bigtimes_{n=0}^{\infty} a^\dagger(f)\psi^{(n)}.$$

Further, we define $\varphi_+(f) = a_+(f) + a_+^\dagger(f)$.

Although it is a trivial consequence of the definitions, we note the following lemma.

Lemma B.37. *For $f \in \mathfrak{h}$ and $\psi \in \mathcal{D}(a(f)) = \mathcal{D}(a^\dagger(f)) \subset \mathcal{F}(\mathfrak{h}) \subset \mathcal{F}_+(\mathfrak{h})$, we have*

$$a_+(f)\psi = a(f)\psi \qquad a_+^\dagger(f)\psi = a^\dagger(f)\psi.$$

Further, for $\psi \in \mathcal{D}(\varphi(f))$, we have $\varphi_+(f)\psi = \varphi(f)\psi$.

We also need second quantization operators. Here, we restrict ourselves to the case of selfadjoint multiplication operators, cf. Remark B.13.

Definition B.38. Let $k \in \mathbb{N}$, $\mathbf{A} = (A_1, \dots, A_k) : \mathcal{M} \to \mathbb{R}^k$ be a strongly commuting family of selfadjoint multiplication operators on $\mathfrak{h}$ and $f : \mathbb{R}^k \to \mathbb{C}$ measurable. Then, we define

$$f(\mathsf{d}\Gamma(\mathbf{A}))\psi = \bigtimes_{n=0}^{\infty} \mathsf{d}\Gamma^{(n)}(A)\psi^{(n)} \qquad \text{on the domain } \mathcal{D}(f(\mathsf{d}\Gamma(\mathbf{A}))) = \bigtimes_{n=0}^{\infty} \mathcal{D}(f(\mathsf{d}\Gamma^{(n)}(\mathbf{A}))).$$

Similar to Lemma B.37, we have the following.

Lemma B.39. *In the situation of Definition B.38, we have*

$$f(\mathsf{d}\Gamma_+(\mathbf{A}))\psi = f(\mathsf{d}\Gamma(\mathbf{A}))\psi \qquad \text{for all } \psi \in \mathcal{D}(f(\mathsf{d}\Gamma(\mathbf{A}))) \subset \mathcal{D}(f(\mathsf{d}\Gamma_+(\mathbf{A}))) \cap \mathcal{F}(\mathfrak{h}).$$

For the operators defined above, we can take adjoints w.r.t $\langle \cdot, \cdot \rangle_+$ similar to the adjoints on $\mathcal{F}(\mathfrak{h})$.

Lemma B.40. *Let* $f \in \mathfrak{h}$, $\mathbf{A} : \mathcal{M} \to \mathbb{R}^k$ *and* $g : \mathbb{R}^k \to \mathbb{C}$ *measurable. Then, we have*

$$\begin{aligned} &\langle \varphi(f)\psi, \phi \rangle_+ = \langle \psi, \varphi_+(f)\phi \rangle_+ \qquad \text{for all } \psi \in \mathcal{F}_{\mathsf{fin}}(\mathfrak{h}),\ \phi \in \mathcal{F}_+, \\ &\langle g(\mathsf{d}\Gamma(\mathbf{A}))\psi, \phi \rangle_+ = \langle \psi, \overline{g}(\mathsf{d}\Gamma_+(\mathbf{A}))\phi \rangle_+ \\ &\qquad\qquad \text{for all } \psi \in \mathcal{F}_{\mathsf{fin}}(\mathfrak{h}) \cap \mathcal{D}(g(\mathsf{d}\Gamma(\mathbf{A}))),\ \phi \in \mathcal{D}(\overline{g}(\mathsf{d}\Gamma_+(\mathbf{A}))). \end{aligned}$$

Apart from Fock space, we equip further subspaces of $\mathcal{F}_+(\mathfrak{h})$ with a norm.

Definition B.41. For $a \in \mathbb{R}$, we define

$$\mathcal{F}_{+,a} = \{\psi \in \mathcal{F}_+ | \|\psi\|_{+,a} < \infty\} \qquad \text{with the norm} \quad \|\psi\|_{+,a}^2 = \sum_{n \in \mathbb{N}_0} (1+n)^{3a} \|\psi^{(n)}\|^2.$$

Remark B.42. Obviously, we have $\mathcal{F}_{+,0}(\mathfrak{h}) = \mathcal{F}(\mathfrak{h})$.

We will need the following lemma

Lemma B.43. *Let* $a \leq 0$, $\phi \in \mathcal{F}_{+,a}(\mathfrak{h})$ *and* $D \subset \mathcal{F}_{\mathsf{fin}}(\mathfrak{h})$ *be dense in* $\mathcal{F}(\mathfrak{h})$. *If* $\langle \psi, \phi \rangle_+ = 0$ *for all* $\psi \in D$, *then* $\phi = 0$.

To define pointwise annihilation operators, we need to consider the space of $\mathcal{F}_+(\mathfrak{h})$-valued square-integrable functions on $\mathcal{M}$.

Definition B.44. Let

$$\mathscr{C}(\mathcal{M}) = \{f : \mathcal{M} \to \mathcal{F}_+(\mathfrak{h}) \mid \forall n \in \mathbb{N}_0 : \mathcal{P}_n f(\cdot) \in L^2(\mathcal{M}; \mathcal{F}^{(n)}(\mathfrak{h}))\}/\sim,$$

where we write $f \sim g$ if and only if $f = g$ almost everywhere.

Similar to the idea of a direct integral, we can take the operators φ_+ and $\mathsf{d}\Gamma_+(\mathbf{A})$ to operators on $\mathscr{C}(\mathcal{M})$.

Definition B.45. Let $f \in \mathfrak{h}$, $\mathbf{A} : \mathcal{M} \to \mathbb{R}^k$, $g : \mathbb{R}^k \to \mathbb{C}$ and $h : \mathcal{M} \to \mathbb{R}^k$. Then, we define

$$\begin{aligned}
&(\varphi_\oplus(f)\psi)(x) = \varphi_+(f)\psi(x) \qquad \text{for } \psi \in \mathscr{C}(\mathcal{M}),\ x \in \mathcal{M},\\
&\mathcal{D}(g(\mathsf{d}\Gamma_\oplus(\mathbf{A}) + h)) = \Big\{\psi \in \mathscr{C}(\mathcal{M}) \Big| \psi(\cdot) \in \mathcal{D}(g(\mathsf{d}\Gamma_+(\mathbf{A}) + h(x))) \text{ a.e.},\\
&\qquad\qquad \forall n \in \mathbb{N} : \int_{\mathcal{M}} \|\mathcal{P}_n g(\mathsf{d}\Gamma_+(\mathbf{A}) + h(x))\psi(x)\|^2 \mathsf{d}\mu(x) < \infty\Big\}\\
&(g(\mathsf{d}\Gamma_\oplus(\mathbf{A}) + h)\psi)(x) = g(\mathsf{d}\Gamma_+(\mathbf{A}) + h(x))\psi(x) \qquad \text{for } \psi \in \mathcal{D}(g(\mathsf{d}\Gamma_\oplus(\mathbf{A}) + h)).
\end{aligned}$$

We now define the pointwise annihilation operator.

Definition B.46. The *pointwise annihilation operator* is the operator $A : \mathcal{F}_+(L^2(\mathcal{M})) \to \mathcal{C}(\mathcal{M})$ with

$$\mathcal{P}_n(A\psi)(k) = \sqrt{n+1}(\mathcal{P}_{n+1}\psi)(k, \cdot, \cdots, \cdot).$$

Remark B.47. For $f \in \mathcal{F}^{(n+1)}$, the map $k \mapsto f(k, \cdot, \cdots, \cdot)$ is an element of $L^2(\mathcal{M}; \mathcal{F}^{(n)})$, by the Fubini-Tonelli theorem. Hence, the above definition defines $\mathcal{P}_n(A\psi)(k)$ for almost every $k \in \mathcal{M}$. Since countable unions of zero sets are again zero sets, the above prescription is well-defined.

The next lemma is a simple implication from the definitions.

Lemma B.48. *The pointwise annihilation operator A is a continuous operator from $\mathcal{F}_+(\mathfrak{h})$ to $\mathscr{C}(\mathcal{M})$ and if $\psi \in \mathcal{F}(\mathfrak{h})$, then $A\psi(k) \in \mathcal{F}_{+,-1/2}(\mathfrak{h})$ holds for almost every $k \in \mathcal{M}$.*

We can now explicitly calculate commutators, which are the main ingredient for the proof of our pull-through formulas.

Lemma B.49.

(i) *For all $f \in \mathfrak{h}$*

$$\varphi_\oplus(f)A\psi = A\varphi_+(f) - M_f.$$

(ii) *Let $\omega : \mathbb{R}^d \to \mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathbb{R}$ be measurable. Then, for all $\psi \in \mathcal{D}(g(\mathsf{d}\Gamma(\omega)))$, we have $A\psi \in \mathcal{D}(g(\mathsf{d}\Gamma_\oplus(\omega) + \omega)$ and*

$$g(\mathsf{d}\Gamma_\oplus(\omega) + \omega)A\psi = Af(\mathsf{d}\Gamma_+(\omega))\psi.$$

We will also need the following statement about second quantization operators on Fock space.

Lemma B.50. *Let $B : \mathcal{M} \to \mathbb{R}$ be measurable with $B \geq 0$. Then*

$$\psi \in \mathcal{D}(d\Gamma(B)^{\frac{1}{2}}) \iff B^{\frac{1}{2}}A\psi \in L^2(\mathcal{M}, \mathcal{F}).$$

Furthermore, for $\phi, \psi \in \mathcal{D}(d\Gamma(B)^{\frac{1}{2}})$, we have

$$\langle d\Gamma(B)^{\frac{1}{2}}\phi, d\Gamma(B)^{\frac{1}{2}}\psi\rangle = \int_{\mathcal{M}} B(k)\langle A\phi(k), A\psi(k)\rangle \mathsf{d}\mu(k),$$

and $A\psi(k) \in \mathcal{F}$ almost everywhere on $\{k \in \mathcal{M} : B(k) > 0\}$.

Nomenclature

General Symbols

$\mathbb{N}$	Positive integers
$\mathbb{N}_0$	Non-negative integers
$\mathbb{Z}$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
$\lfloor\cdot\rfloor$	Integer part of a real number
ln / log	Natural logarithm.
$\mathbb{C}$	Complex numbers
i	Imaginary unit
$\overline{z}$	Complex conjugate of $t \in \mathbb{C}$
$\mathfrak{S}_n$	Symmetric group on $\{1, \ldots, n\}$
Π_n	Set of partitions of $\{1, \ldots, n\}$
$\mathcal{P}(M)$	Power set of a set M
χ_M	Characteristic function of a set M
$\lvert M\rvert$, $\lvert k\rvert$	Cardinality of a finite set M and Euclidean norm of a vector $k \in \mathbb{R}^d$
$B_R(k)$	Open ball of radius $R > 0$ with center $k \in \mathbb{R}^d$

Banach and Hilbert Spaces

$\lVert\cdot\rVert$,$\langle\cdot,\cdot\rangle$	Norm and inner product (in given Banach / Hilbert spaces)
$\mathcal{H} \oplus \mathcal{V}$,$\mathcal{H} \otimes \mathcal{V}$	Direct sum and tensor product of Hilbert spaces
$\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}, \mathcal{V})$	Bounded operators on the Hilbert space $\mathcal{H}$ and from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{V}$, respectively (Definition A.12)
$\mathbb{1}$	Identity operator
$L^2(\mathcal{M}, \mu)$	Hilbert space of square-integrable functions over a measure space $(\mathcal{M}, \Sigma, \mu)$, if clear from the context the measure μ is dropped
$L^2(\mathcal{M}, \mu; \mathcal{H})$	Hilbert space of $\mathcal{H}$-valued square-integrable functions
$L^2_+(\mathcal{M}, \mu)$	Positive square-integrable functions (Definition A.108)
$L^2_{++}(\mathcal{M}, \mu)$	Strictly positive square-integrable functions (Definition A.108)
$L^2_{\mathbb{R}}(\mathcal{M}, \mu)$	Real-valued square-integrable functions
$\widehat{f}$	Unitary Fourier transform of $f \in L^2(\mathbb{R}^d)$

Operators on Hilbert Spaces

$\mathcal{D}(T)$	Domain of the operator T (Definition A.2)
$\overline{T}$	Closure of the operator T (Definition A.6)
T^*	Adjoint of the operator T (Definition A.9)

$[A, B]$	Commutator of the operators A and B (Definition A.3)
$\sigma(T)$	Spectrum of the operator T (Definition A.25)
$\sigma_{\mathsf{d}}(A)$	Discrete spectrum of the selfadjoint operator A (Definition A.36)
$\sigma_{\mathsf{ess}}(A)$	Essential spectrum of the selfadjoint operator A (Definition A.36)
P_A	Projection-valued measure of the normal operator A (Definition A.55)
$\mathcal{Q}(T)$	Form domain of the operator T (Definition A.84)
$\mathfrak{q}_A$	Quadratic form of the selfadjoint operator A (Definition A.84)
$\mathfrak{c}_{A,B}$	Weak commutator of the operators A and B (Definition A.92)

Fock Spaces

$\mathcal{F}(\mathfrak{h})$	Fock space over the Hilbert space $\mathfrak{h}$ (Definition B.1)
$\mathcal{F}^{(n)}(\mathfrak{h})$	n-particle space over the Hilbert space $\mathfrak{h}$ (Definition B.1)
$\mathcal{F}$, $\mathcal{F}^{(n)}$	Fock space and n-particle space over $L^2(\mathbb{R}^d)$
$\mathcal{F}_{\mathsf{fin}}(D)$	Finite particle subspace spanned by the subspace D of $\mathfrak{h}$ (Definition B.4)
Ω	Fock space vacuum ((1.8))
$\epsilon(f)$	Coherent state generated by $f \in \mathfrak{h}$ (Definition B.8)
$\mathcal{E}(D)$	Set of coherent states spanned by $D \subset \mathfrak{h}$ (Definition B.8)
$\mathsf{d}\Gamma(A), \Gamma(A)$	Second quantization operators of A (Definitions B.11 and B.14)
$a(f), a^\dagger(f)$	Annihilation and creation operator for $f \in \mathfrak{h}$ (Definition B.18)
$\varphi(f)$	(Segal) field operator for $f \in \mathfrak{h}$ (Definition B.21)
$W(f)$	Weyl operator for $f \in \mathfrak{h}$ (Definition B.24)
$\mathcal{Q}_{\mathfrak{r}}, \Theta_{\mathfrak{r}}, \phi_{\mathfrak{r}}$	$\mathcal{Q}$-space, Wiener-Itô-Segal isomorphism and Gaussian random variable associated with the decomposition $\mathfrak{h} = \mathfrak{r} \oplus \mathrm{i}\mathfrak{r}$ (Lemma B.30)

Nelson Model

$\mathbf{m}$	Momentum operator ((1.3))
$H_{\mathsf{N},\Lambda}(P)$	Translation-invariant Nelson Hamiltonian with sharp ultraviolet cutoff $\Lambda \in [0, \infty)$ and total momentum $P \in \mathbb{R}^d$ (Definition 1.3)
E_Λ	Self-energy of the Nelson Hamiltonian ((1.5))
$H_{\mathsf{N},\infty}(P)$	Renormalized translation-invariant Nelson Hamiltonian at total momentum $P \in \mathbb{R}^d$ (Lemma 1.6)
$\mathscr{D}_{\mathsf{N}}, \mathfrak{Q}_{\mathsf{N}}$	Domain and Form Domain of $H_{\mathsf{N},\Lambda}(P)$, $\Lambda < \infty$ ((1.4))

Spin Boson Model

$\sigma_x, \sigma_y, \sigma_z$	2×2 Pauli matrix ((1.7))
$H_{\mathsf{SB}}(\lambda)$	Spin boson Hamiltonian with coupling constant $\lambda \in \mathbb{R}$ (Definition 1.11)
$H_{\mathsf{SB}}^{(\mathsf{m})}(\lambda, \mu)$	Spin boson Hamiltonian with coupling constant $\lambda \in \mathbb{R}$ and external magnetic field $\mu \in \mathbb{R}$ (Definition 1.18)
$X_t, \mathbb{E}_X$	Poisson-driven jump process (Definition 4.16)
W	Interaction function of the continuous Ising model corresponding to the spin boson model with external magnetic field ((1.9))
$\langle\cdot\rangle_{T,\lambda,\mu}$	Expectation values in the continuous Ising model corresponding to the spin boson model with external magnetic field ((1.11))

List of Figures

Figure 1 Heuristic concept of the considered models 1

Figure 2 The role of rotation invariance in the proofs of Lemmas 3.1 and 3.12. . 40

Figure 3 Illustration of the set $\mathcal{E}_L$ 55

Figure 4 Illustration of the curves Γ_0 and Γ_1 85

Bibliography

[Abd11] A. Abdesselam. The Ground State Energy of the Massless Spin-Boson Model. *Ann. Henri Poincaré*, 12(7):1321–1347, 2011. doi:10.1007/s00023-011-0103-6.

[ACCN88] M. Aizenman, J. T. Chayes, L. Chayes, C. M. Newman. Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models. *J. Stat. Phys.*, 50(1):1–40, 1988. doi:10.1007/BF01022985.

[AH95] A. Arai, M. Hirokawa. On the Existence and Uniqueness of Ground States of the Spin-Boson Hamiltonian. *Hokkaido Univ. Prepr. Ser. Math.*, 309:2–20, 1995. doi:10.14943/83456.

[AH97] A. Arai, M. Hirokawa. On the Existence and Uniqueness of Ground States of a Generalized Spin-Boson Model. *J. Funct. Anal.*, 151(2):455–503, 1997. doi:10.1006/jfan.1997.3140.

[AH12] A. Abdesselam, D. Hasler. Analyticity of the Ground State Energy for Massless Nelson Models. *Commun. Math. Phys.*, 310(2):511–536, 2012. doi:10.1007/s00220-011-1407-6.

[AHH99] A. Arai, M. Hirokawa, F. Hiroshima. On the Absence of Eigenvectors of Hamiltonians in a Class of Massless Quantum Field Models without Infrared Cutoff. *J. Funct. Anal.*, 168(2):470–497, 1999. doi:10.1006/jfan.1999.3472.

[Ama91] A. Amann. Ground states of a spin-boson model. *Ann. Phys.*, 208(2):414–448, 1991. doi:10.1016/0003-4916(91)90302-O.

[Amm00] Z. Ammari. Asymptotic Completeness for a Renormalized Nonrelativistic Hamiltonian in Quantum Field Theory: The Nelson Model. *Math. Phys. Anal. Geom.*, 3(3):217–285, 2000. doi:10.1023/A:1011408618527.

[AN86] M. Aizenman, C. M. Newman. Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models. *Commun. Math. Phys.*, 107(4):611–647, 1986. doi:10.1007/BF01205489.

[Ara18] A. Arai. *Analysis on Fock Spaces and Mathematical Theory of Quantum Fields.* World Scientific, New Jersey, 2018. doi:10.1142/10367.

[BBKM17] V. Bach, M. Ballesteros, M. Könenberg, L. Menrath. Existence of ground state eigenvalues for the spin–boson model with critical infrared divergence and multiscale analysis. *J. Math. Anal. Appl.*, 453(2):773–797, 2017. doi:10.1016/j.jmaa.2017.03.075.

[BCFS07] V. Bach, T. Chen, J. Fröhlich, I. M. Sigal. The renormalized electron mass in non-relativistic quantum electrodynamics. *J. Funct. Anal.*, 243(2):426–535, 2007. doi:10.1016/j.jfa.2006.09.017.

[BDH19] M. Ballesteros, D.-A. Deckert, F. Hänle. Analyticity of resonances and eigenvalues and spectral properties of the massless Spin-Boson model. *J. Funct. Anal.*, 276(8):2524–2581, 2019. doi:10.1016/j.jfa.2019.02.008.

[BDP12] S. Bachmann, D.-A. Deckert, A. Pizzo. The mass shell of the Nelson model without cut-offs. *J. Funct. Anal.*, 263(5):1224–1282, 2012. doi:10.1016/j.jfa.2012.04.021.

[BFS98a] V. Bach, J. Fröhlich, I. M. Sigal. Quantum Electrodynamics of Confined Nonrelativistic Particles. *Adv. Math.*, 137(2):299–395, 1998. doi:10.1006/aima.1998.1734.

[BFS98b] V. Bach, J. Fröhlich, I. M. Sigal. Renormalization Group Analysis of Spectral Problems in Quantum Field Theory. *Adv. Math.*, 137(2):205–298, 1998. doi:10.1006/aima.1998.1733.

[BFS99] V. Bach, J. Fröhlich, I. M. Sigal. Spectral Analysis for Systems of Atoms and Molecules Coupled to the Quantized Radiation Field. *Commun. Math. Phys.*, 207(2):249–290, 1999. doi:10.1007/s002200050726.

[BH09] V. Betz, F. Hiroshima. Gibbs measures with double stochastic integrals on a path space. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 12(1):135–152, 2009. doi:10.1142/S0219025709003574.

[BHL$^+$02] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos, H. Spohn. Ground state properties of the Nelson Hamiltonian - A Gibbs measure-based approach. *Rev. Math. Phys.*, 14(02):173–198, 2002. doi:10.1142/S0129055X02001119.

[Bil99] P. Billingsley. *Convergence of Probability Measures.* Wiley Series in Probability and Statistics. John Wiley & Sons, New York, 2nd edn., 1999. doi:10.1002/9780470316962.

[BN37] F. Bloch, A. Nordsieck. Note on the Radiation Field of the Electron. *Phys. Rev.*, 52(2):54–59, 1937. doi:10.1103/PhysRev.52.54.

[BR97] O. Bratteli, D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2: Equilibrium States. Models in Quantum Statistical Mechanics.* Texts and Monographs in Physics. Springer, Berlin, 2nd edn., 1997. doi:10.1007/978-3-662-09089-3.

[BS05] V. Betz, H. Spohn. A central limit theorem for Gibbs measures relative to Brownian motion. *Probab. Theory Relat. Fields*, 131(3):459–478, 2005. doi:10.1007/s00440-004-0381-8.

[Can71] J. Cannon. Quantum field theoretic properties of a model of Nelson: Domain and eigenvector stability for perturbed linear operators. *J. Funct. Anal.*, 8(1):101–152, 1971. doi:10.1016/0022-1236(71)90023-1.

[Dam20] T. N. Dam. Absence of Ground States in the Translation Invariant Massless Nelson Model. *Ann. Henri Poincaré*, 21(8):2655–2679, 2020. doi:10.1007/s00023-020-00928-y.

[DG99] J. Dereziński, C. Gérard. Asymptotic Completeness in Quantum Field Theory. Massive Pauli-Fierz Hamiltonians. *Rev. Math. Phys.*, 11(4):383–450, 1999. doi:10.1142/S0129055X99000155.

[DG04] J. Dereziński, C. Gérard. Scattering theory of infrared divergent Pauli-Fierz Hamiltonians. *Ann. Henri Poincaré*, 5(3):523–577, 2004. doi:10.1007/s00023-004-0177-5.

[DH21] T. N. Dam, B. Hinrichs. Absence of Ground States in the Renormalized Massless Translation-Invariant Nelson Model. Preprint, arXiv:1909.07661v2, 2021.

[Dim74] J. Dimock. Asymptotic perturbation expansion in the $P(\phi)_2$ quantum field theory. *Commun. Math. Phys.*, 35(4):347–356, 1974. doi:10.1007/BF01646354.

[DM20a] T. N. Dam, J. S. Møller. Asymptotics in Spin-Boson type models. *Commun. Math. Phys.*, 374(3):1389–1415, 2020. doi:10.1007/s00220-020-03685-5.

[DM20b] T. N. Dam, J. S. Møller. Spin-Boson type models analysed using symmetries. *Kyoto J. Math.*, 60(4):1261–1332, 2020. doi:10.1215/21562261-2019-0062.

[Dys69] F. J. Dyson. Existence of a Phase-Transition in a One-Dimensional Ising Ferromagnet. *Commun. Math. Phys.*, 12(2):91–107, 1969. doi:10.1007/BF01645907.

[EL74] V. J. Emery, A. Luther. Low-temperature properties of the Kondo Hamiltonian. *Phys. Rev. B*, 9(1):215–226, 1974. doi:10.1103/PhysRevB.9.215.

[Far72] W. G. Faris. Invariant Cones and Uniqueness of the Ground State for Fermion Systems. *J. Math. Phys.*, 13(8):1285–1290, 1972. doi:10.1063/1.1666133.

[Fey05] R. P. Feynman. The Principle of Least Action in Quantum Mechanics, 1954. L. M. Brown (ed.), *Feynman's Thesis - A New Approach to Quantum Theory.* World Scientific, 2005. doi:10.1142/9789812567635_0001.

[FFG97] C. Fefferman, J. Fröhlich, G. M. Graf. Stability of Ultraviolet-Vutoff Quantum Electrodynamics with Non-Relativistic Matter. *Commun. Math. Phys.*, 190(2):309–330, 1997. doi:10.1007/s002200050243.

[FILS78] J. Fröhlich, R. Israel, E. H. Lieb, B. Simon. Phase transitions and reflection positivity. I. General theory and long range lattice models. *Commun. Math. Phys.*, 62(1):1–34, 1978. doi:10.1007/BF01940327.

[FN88] M. Fannes, B. Nachtergaele. Translating the spin-boson model into a classical system. *J. Math. Phys.*, 29(10):2288–2293, 1988. doi:10.1063/1.528109.

[FNV88] M. Fannes, B. Nachtergaele, A. Verbeure. The equilibrium states of the spin-boson model. *Commun. Math. Phys.*, 114(4):537–548, 1988. doi:10.1007/BF01229453.

[Frö73] J. Fröhlich. On the infrared problem in a model of scalar electrons and massless scalar bosons. *Ann. de l'Inst. Henri Poincaré*, 19(1):1–103, 1973.

[Frö74] J. Fröhlich. Existence of dressed one-electron states in a class of persistent models. *Fortschr. Phys.*, 22(3):159–198, 1974. doi:10.1002/prop.19740220304.

[FV17] S. Friedli, Y. Velenik. *Statistical mechanics of lattice systems: A Concrete Mathematical Introduction.* Cambridge University Press, Cambridge, 2017. doi:10.1017/9781316882603.

[Gér00] C. Gérard. On the Existence of Ground States for Massless Pauli-Fierz Hamiltonians. *Ann. Henri Poincaré*, 1(3):443–459, 2000. doi:10.1007/s000230050002.

[GH09] M. Griesemer, D. G. Hasler. Analytic Perturbation Theory and Renormalization Analysis of Matter Coupled to Quantized Radiation. *Ann. Henri Poincaré*, 10(3):577–621, 2009. doi:10.1007/s00023-009-0417-9.

[Gin70] J. Ginibre. General formulation of Griffiths' inequalities. *Commun. Math. Phys.*, 16(4):310–318, 1970. doi:10.1007/BF01646537.

[GJ85] J. Glimm, A. Jaffe. *Collected papers. Constructive Quantum Field Theory Selected Papers.* Birkhäuser, Boston, 1985.

[GJ87] J. Glimm, A. Jaffe. *Quantum Physics. A Functional Integral Point of View.* Springer, New York, 2nd edn., 1987. doi:10.1007/978-1-4612-4728-9.

[GLL01] M. Griesemer, E. Lieb, M. Loss. Ground states in non-relativistic quantum electrodynamics. *Invent. math.*, 145(3):557–595, 2001. doi:10.1007/s002220100159.

[GMS67] G. Gallavotti, S. Miracle-Sole. Statistical mechanics of lattice systems. *Commun. Math. Phys.*, 5(5):317–323, 1967. doi:10.1007/BF01646445.

[Gri67a] R. B. Griffiths. Correlations in Ising Ferromagnets. I. *J. Math. Phys.*, 8(3):478–483, 1967. doi:10.1063/1.1705219.

[Gri67b] R. B. Griffiths. Correlations in Ising Ferromagnets. II. External Magnetic Fields. *J. Math. Phys.*, 8(3):484–489, 1967. doi:10.1063/1.1705220.

[Gri67c] R. B. Griffiths. Correlations in Ising Ferromagnets. III. *Commun. Math. Phys.*, 6(2):121–127, 1967. doi:10.1007/BF01654128.

[Gro62] E. P. Gross. Particle-Like Solutions in Field Theory. *Ann. Phys.*, 19(2):219–233, 1962. doi:10.1016/0003-4916(62)90216-6.

[Gro72] L. Gross. Existence and uniqueness of physical ground states. *J. Funct. Anal.*, 10(1):52–109, 1972. doi:10.1016/0022-1236(72)90057-2.

[GW18] M. Griesemer, A. Wünsch. On the domain of the Nelson Hamiltonian. *J. Math. Phys.*, 59(4):042111, 2018. doi:10.1063/1.5018579.

[Har06] M. Hardy. Combinatorics of Partial Derivatives. *Electron. J. Comb.*, 13:R1, 2006. doi:10.37236/1027.

[HH08] D. Hasler, I. Herbst. Absence of Ground States for a Class of Translation Invariant Models of Non-relativistic QED. *Commun. Math. Phys.*, 279(3):769–787, 2008. doi:10.1007/s00220-008-0444-2.

[HH11a] D. Hasler, I. Herbst. Convergent expansions in non-relativistic qed: Analyticity of the ground state. *J. Funct. Anal.*, 261(11):3119–3154, 2011. doi:10.1016/j.jfa.2011.07.023.

[HH11b] D. Hasler, I. Herbst. Ground States in the Spin Boson Model. *Ann. Henri Poincaré*, 12(4):621–677, 2011. doi:10.1007/s00023-011-0091-6.

[HH11c] D. Hasler, I. Herbst. Smoothness and analyticity of perturbation expansions in qed. *Adv. Math.*, 228(6):3249–3299, 2011. doi:10.1016/j.aim.2011.08.007.

[HHL14] M. Hirokawa, F. Hiroshima, J. Lőrinczi. Spin-boson model through a Poisson-driven stochastic process. *Math. Z.*, 277(3):1165–1198, 2014. doi:10.1007/s00209-014-1299-1.

[HHS05] M. Hirokawa, F. Hiroshima, H. Spohn. Ground state for point particles interacting through a massless scalar Bose field. *Adv. Math.*, 191(2):339–392, 2005. doi:10.1016/j.aim.2004.03.011.

[HHS21] D. Hasler, B. Hinrichs, O. Siebert. On Existence of Ground States in the Spin Boson Model. *Commun. Math. Phys.*, 388(1):419–433, 2021. doi:10.1007/s00220-021-04185-w.

[HHS22a] D. Hasler, B. Hinrichs, O. Siebert. Correlation bound for a one-dimensional continuous long-range Ising model. *Stoch. Proc. Appl.*, 146:60–79, 2022. doi:10.1016/j.spa.2021.12.010.

[HHS22b] D. Hasler, B. Hinrichs, O. Siebert. FKN Formula and Ground State Energy for the Spin Boson Model with External Magnetic Field. Preprint, arXiv:2106.0869, to appear in *Ann. Henri Poincaré*, 2022.

[Hir97] F. Hiroshima. Functional integral representation of a model in quantum electrodynamics. *Rev. Math. Phys.*, 9(4):489–530, 1997. doi:10.1142/S0129055X97000208.

[HL08] F. Hiroshima, J. Lőrinczi. Functional integral representations of the Pauli-Fierz model with spin 1/2. *J. Funct. Anal.*, 254(8):2127–2185, 2008. doi:10.1016/j.jfa.2008.01.002.

[HM21] F. Hiroshima, O. Matte. Ground states and their associated path measures in the renormalized Nelson model. *Rev. Math. Phys.*, 2021. doi:10.1142/S0129055X22500027.

[HS95a] M. Hübner, H. Spohn. Radiative decay: nonperturbative approaches. *Rev. Math. Phys.*, 7(3):363–387, 1995. doi:10.1142/S0129055X95000165.

[HS95b] M. Hübner, H. Spohn. Spectral properties of the spin-boson Hamiltonian. *Ann. de l'Inst. Henri Poincaré*, 62(3):289–323, 1995.

[HS20] D. Hasler, O. Siebert. Ground States for translationally invariant Pauli-Fierz Models at zero Momentum. Preprint, arXiv:2007.01250, 2020.

[IN88] J. Z. Imbrie, C. M. Newman. An intermediate phase with slow decay of correlations in one-dimensional $1/|x-y|^2$ percolation, Ising and Potts models. *Comm. Math. Phys.*, 118(2):303–336, 1988. doi:10.1007/BF01218582.

[Isi25] E. Ising. Beitrag zur Theorie des Ferromagnetismus. *Z. Phys.*, 31(1):253–258, 1925. doi:10.1007/BF02980577.

[Kac51] M. Kac. On Some Connections between Probability Theory and Differential and Integral Equations. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, 189–215, 1951.

[Kat80] T. Kato. *Perturbation Theory for Linear Operators.* Classics in Mathematics, vol. 132. Springer, Berlin, 2nd edn., 1980. doi:10.1007/978-3-642-66282-9.

[Kle20] A. Klenke. *Probability Theory.* Universitext. Springer, 3rd edn., 2020. doi:10.1007/978-3-030-56402-5.

[KS68] D. G. Kelly, S. Sherman. General Griffiths' Inequalities on Correlations in Ising Ferromagnets. *J. Math. Phys.*, 9(3):466–484, 1968. doi:10.1063/1.1664600.

[KT69] M. Kac, C. J. Thompson. Critical Behavior of Several Lattice Models with Long-Range Interaction. *J. Math. Phys.*, 10(8):1373–1386, 1969. doi:10.1063/1.1664976.

[Lam21] J. Lampart. The Resolvent of the Nelson Hamiltonian Improves Positivity. *Math. Phys. Anal. Geom.*, 24(1):2, 2021. doi:10.1007/s11040-021-09374-6.

[LCD+87] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg, W. Zwerger. Dynamics of the dissipative two-state system. *Rev. Mod. Phys.*, 59(1):1–85, 1987.

[Len20] W. Lenz. Beitrag zum Verständnis der magnetischen Erscheinungen in festen Körpern. *Phys. Z.*, 21:613–615, 1920.

[LHB11] J. Lőrinczi, F. Hiroshima, V. Betz. *Feynman-Kac-Type Theorems and Gibbs Measures on Path Space.* De Gruyter Studies in Mathematics, vol. 34. De Gruyter, Berlin, 2011. doi:10.1515/9783110330397.

[Lig10] T. M. Liggett. *Continuous Time Markov Processes: An Introduction.* Graduate Studies in Mathematics, vol. 113. AMS, Providence, RI, 2010. doi:10.1090/gsm/113.

[LL01] E. H. Lieb, M. Loss. *Analysis.* Graduate Studies in Mathematics, vol. 14. AMS, Providence, 2nd edn., 2001. doi:10.1090/gsm/014.

[LLP53] T.-D. Lee, F. E. Low, D. Pines. The motion of slow electrons in a polar crystal. *Phys. Rev.*, 90(2):297–302, 1953. doi:10.1103/PhysRev.90.297.

[LMS02] J. Lőrinczi, R. A. Minlos, H. Spohn. The Infrared Behaviour in Nelson's Model of a Quantum Particle Coupled to a Massless Scalar Field. *Ann. Henri Poincaré*, 3(2):269–295, 2002. doi:10.1007/s00023-002-8617-6.

[LMS07] M. Loss, T. Miyao, H. Spohn. Lowest energy states in nonrelativistic QED: Atoms and ions in motion. *J. Funct. Anal.*, 243(2):353–393, 2007. doi:10.1016/j.jfa.2006.10.012.

[LS19] J. Lampart, J. Schmidt. On Nelson-Type Hamiltonians and Abstract Boundary Conditions. *Commun. Math. Phys.*, 367(2):629–663, 2019. doi:10.1007/s00220-019-03294-x.

[Miy21] T. Miyao. On Renormalized Hamiltonian Nets. *Ann. Henri Poincaré*, 22(9):2935–2973, 2021. doi:10.1007/s00023-021-01029-0.

[MM18] O. Matte, J. S. Møller. Feynman-Kac Formulas for the Ultra-Violet Renormalized Nelson Model. *Astérisque*, 404, 2018. doi:10.24033/ast.1054.

[Møl05] J. S. Møller. The Translation Invariant Massive Nelson Model: I. The Bottom of the Spectrum. *Ann. Henri Poincaré*, 6(6):1091–1135, 2005. doi:10.1007/s00023-005-0234-8.

[Nel64] E. Nelson. Interaction of Nonrelativistic Particles with a Quantized Scalar Field. *J. Math. Phys.*, 5(9):1190–1197, 1964. doi:10.1063/1.1704225.

[Nel73] E. Nelson. Quantum fields and Markoff fields. *Partial Differential Equations*, Proc. Sympos. Pure Math., vol. 23, 413–420. AMS, Berkeley, 1973. doi:10.1090/pspum/023.

[Neu32] J. von Neumann. *Mathematische Grundlagen der Quantenmechanik.* Grundlehren der Mathematischen Wissenschaften, vol. 38. Springer, Berlin, 1932. doi:10.1007/978-3-642-61409-5.

[Oli09] C. R. de Oliveira. *Intermediate Spectral Theory and Quantum Dynamics.* Progress in Mathematical Physics, vol. 54. Birkhäuser, Basel, 2009. doi:10.1007/978-3-7643-8795-2.

[Par92] K. R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus.* Monographs in Mathematics, vol. 85. Birkhäuser, Basel, 1992. doi:10.1007/978-3-0348-0566-7.

[Per75] J. K. Percus. Correlation inequalities for Ising spin lattices. *Commun. Math. Phys.*, 40(3):283–308, 1975. doi:10.1007/BF01610004.

[Piz03] A. Pizzo. One-particle (improper) States in Nelson's Massless Model. *Ann. Henri Poincaré*, 4(3):439–486, 2003. doi:10.1007/s00023-003-0136-6.

[Rek20] J. Reker. Existence of resonances for the spin-boson model with critical coupling function. *J. Math. Anal. Appl.*, 483(2):123628, 2020. doi:10.1016/j.jmaa.2019.123628.

[Res92] S. Resnick. *Adventures in Stochastic Processes.* Birkhäuser, 1992. doi:10.1007/978-1-4612-0387-2.

[Ros07] S. M. Ross. *Introduction to Probability Models.* Academic Press, Amsterdam, 9th edn., 2007. doi:10.1016/C2017-0-01324-1.

[RS72] M. Reed, B. Simon. *Functional Analysis.* Methods of Modern Mathematical Physics, vol. 1. Academic Press, San Diego, 1972.

[RS75] M. Reed, B. Simon. *Fourier Analysis, Self-Adjointness.* Methods of Modern Mathematical Physics, vol. 2. Academic Press, San Diego, 1975.

[RS78] M. Reed, B. Simon. *Analysis of Operators.* Methods of Modern Mathematical Physics, vol. 4. Academic Press, San Diego, 1978.

[RT81] J. B. Rogers, C. J. Thompson. Absence of Long-Range Order in One-Dimensional Spin Systems. *J. Stat. Phys.*, 25(4):669–678, 1981. doi:10.1007/BF01022361.

[Rue68] D. Ruelle. Statistical mechanics of a one-dimensional lattice gas. *Commun. Math. Phys.*, 9(4):267–278, 1968. doi:10.1007/BF01654281.

[Sch12] K. Schmüdgen. *Unbounded Self-adjoint Operators on Hilbert Space*. Graduate Texts in Mathematics, vol. 265. Springer, Dordrecht, 2012. doi:10.1007/978-94-007-4753-1.

[Sch21] J. Schmidt. The Massless Nelson Hamiltonian and its Domain. A. Michelangeli (ed.), *Mathematical Challenges of Zero-Range Physics*, Springer INdAM Series, vol. 42. Springer, 2021. doi:10.1007/978-3-030-60453-0.

[SD85] H. Spohn, R. Dümcke. Quantum tunneling with dissipation and the Ising model over $\mathbb{R}$. *J. Stat. Phys.*, 41(3):389–423, 1985. doi:10.1007/BF01009015.

[Sim74] B. Simon. *The $P(\phi)_2$ Euclidean (Quantum) Field Theory.* Princeton Series in Physics. Princeton University Press, Princeton, 1974.

[Sim79] B. Simon. *Functional Integration and Quantum Physics.* Pure and Applied Mathematics, vol. 86. Academic Press, New York, 1979.

[Spo87] H. Spohn. Effective mass of the polaron: A functional integral approach. *Ann. Phys.*, 175(2):278–318, 1987. doi:10.1016/0003-4916(87)90211-9.

[Spo89] H. Spohn. Ground State(s) of the Spin-Boson Hamiltonian. *Commun. Math. Phys.*, 123(2):277–304, 1989. doi:10.1007/BF01238859.

[Spo98] H. Spohn. Ground State of a Quantum Particle Coupled to a Scalar Bose Field. *Lett. Math. Phys.*, 44(1):9–16, 1998. doi:10.1023/A:1007473300274.

[Tes14] G. Teschl. *Mathematical Methods in Quantum Mechanics.* Graduate Studies in Mathematics, vol. 157. AMS, Providence, 2nd edn., 2014. doi:10.1090/gsm/157.

[Tho71] C. J. Thompson. Upper bounds for Ising model correlation functions. *Commun. Math. Phys.*, 24(1):61–66, 1971. doi:10.1007/BF01907034.

[Wei80] J. Weidmann. *Linear Operators in Hilbert Spaces.* Graduate Texts in Mathematics, vol. 68. Springer, New York, 1980. doi:10.1007/978-1-4612-6027-1.

Ehrenwörtliche Erklärung

Hiermit erkläre ich,

- dass mir die Promotionsordnung der Fakultät bekannt ist,
- dass ich die Dissertation selbst angefertigt habe, keine Textabschnitte oder Ergebnisse eines Dritten oder eigener Prüfungsarbeiten ohne Kennzeichnung übernommen und alle von mir benutzten Hilfsmittel, persönliche Mitteilungen und Quellen in meiner Arbeit angegeben habe,
- dass ich die Hilfe eines Promotionsberaters nicht in Anspruch genommen habe und dass Dritte weder unmittelbar noch mittelbar geldwerte Leistungen von mir für Arbeiten erhalten haben, die in Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen,
- dass ich die Dissertation noch nicht als Prüfungsarbeit für eine staatliche oder andere wissenschaftliche Prüfung eingereicht habe,
- dass ich weder die gleiche noch eine in wesentlichen Teilen ähnliche bzw. eine andere Abhandlung bereits bei einer anderen Hochschule als Dissertation eingereicht habe.

Bei der Auswahl des Materials sowie der Herstellung des Manuskripts haben mich durch ihr Mitwirken an Arbeiten, die Teil dieser Dissertation sind, folgende Personen unterstützt:

- Thomas Norman Dam ([DH21] bzw. Kapitel 2, 3),
- David Hasler und Oliver Siebert ([HHS21, HHS22a, HHS22b] bzw. Kapitel 4 – 6).

Jena, den 7. Januar 2022 Benjamin Hinrichs