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A note on uniquely 10-colorable graphs

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Abstract
Hadwiger conjectured that every graph of chromatic number k admits a clique minor of order k. Here we prove for $k \leq 10$, that every graph of chromatic number k with a unique k-coloring (up to the color names) admits a clique minor of order k. The proof does not rely on the Four Color Theorem.

KEYWORDS
coloring, clique minor, hadwiger conjecture, kempe-coloring

Mathematical Subject Classification
05c15, 05c40

A *clique minor* of a (simple, finite, undirected) graph G is a set of connected, nonempty, pairwise disjoint, pairwise adjacent subsets of $V(G)$, where a set $A \subseteq V(G)$ is *connected* if $G[A]$ is connected, and disjoint $A, B \subseteq V(G)$ are *adjacent* if there exists an edge $xy \in E(G)$ with $x \in A$ and $y \in B$. An *anticlique* of G is a set of pairwise nonadjacent vertices, and a *Kempe-coloring* of a graph G is a partition $\mathfrak{C}$ into anticliques such that any two of them induce a connected subgraph in G. In particular,

$$\text{for} \quad A \neq B \quad \text{from} \quad \mathfrak{C}, \text{every vertex from} \quad A \quad \text{has a neighbor in} \quad B. \tag{*}$$

The following facts are implicit in Section 4 from [2]. We add proofs for the sake of completeness. The *order* of a coloring as above is $|\mathfrak{C}|$

Lemma 1 (Kriesell [2]). *Every graph G with a Kempe-coloring of order k satisfies $|E(G)| \geq (k-1)|V(G)| - \binom{k}{2}$, with equality if and only if every pair of members of every Kempe-coloring of order k induces a tree.*

Proof. Let $\mathfrak{C}$ be a Kempe-coloring of order k of G and $A \neq B$ from $\mathfrak{C}$; then $|E(G[A \cup B])| \geq |A| + |B| - 1$ since $G[A \cup B]$ is a connected graph on $|A| + |B|$ vertices, with equality if and only if $G[A \cup B]$ is a tree. Since $G[A \cup B]$ and $G[A' \cup B']$ are edge-disjoint for $\{A, B\} \neq \{A', B'\}$ we get $|E(G)| = \sum |E(G[A \cup B])| \geq \sum (|A| + |B| - 1)$, where the sums are taken over all subsets $\{A, B\}$ of $\mathfrak{C}$ with $A \neq B$. Since every $X \in \mathfrak{C}$ occurs in exactly $k - 1$ of these sets, the latter sum equals $(k-1)|V(G)| - \binom{k}{2}$, with equality if and only if every two members of $\mathfrak{C}$ induce a tree, which proves the statement for $\mathfrak{C}$. As the latter bound is independent from the actual $\mathfrak{C}$, equality holds for $\mathfrak{C}$ if and only if it holds for *all* Kempe-colorings of order k, which proves the Lemma. □

Lemma 2 (Kriesell [2]). *Every graph with a Kempe-coloring of order k is $(k-1)$-connected.*

Proof. Let $\mathfrak{C}$ be a Kempe-coloring of order k of a graph G. Then $|V(G)| > k - 1$. Suppose, to the contrary, that there exists a separating vertex set T with $|T| < k - 1$. Then there exist $A \neq B$ in $\mathfrak{C}$ with $(A \cup B) \cap T = \emptyset$; since $G[A \cup B]$ is connected, $A \cup B \subseteq V(C)$ for some component C of $G - T$. Now take any $x \in V(G) \setminus (T \cup V(C))$. Then x is contained in some $Z \in \mathfrak{C}$ distinct from A (and B), but, obviously, x cannot have a neighbor in A, contradicting (*). □

An (H, k)-*cockade* is recursively defined as any graph isomorphic to H or any graph that can be obtained by taking the union of two (H, k)-cockades whose intersection is a complete graph on k vertices. The following is the main result from [3].

Theorem 1 [Song and Thomas [3]). *Every graph with $n > 8$ vertices and at least $7n - 27$ edges has a clique minor of order 9, unless it is isomorphic to $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}, 6)$-cockade.*

Now we are prepared to prove the main statement of this note.

Theorem 2. *Every graph with a Kempe-coloring of order 10 has a clique minor of order 10.*

Proof. Let $A \neq B$ be two color classes of a Kempe-coloring $\mathfrak{C}$ of order 10 of a graph G. Then $\mathfrak{C}' := \mathfrak{C} \setminus \{A, B\}$ is a Kempe-coloring of $G' := G - (A \cup B)$, of order 8. By Lemma 1, G' is a graph on $n' \geq 8$ vertices with at least $7n' - 28$ edges.

If $n' = 8$ then $V(G')$ is a clique of order 8, and, for every $x \in V(G')$, $G[\{x\} \cup A]$ and $G[\{x\} \cup B]$ are stars centered at x; therefore, if ab is any edge in $G[A \cup B]$, $V(G') \cup \{a, b\}$ is a clique of order 10. So we may assume that $n' \geq 9$.

Now let z be a leaf of any spanning tree of $G[A \cup B]$ or, equivalently, such that $G[(A \cup B) \setminus \{z\}]$ is connected. Without loss of generality, we may assume that $z \in A$, otherwise we swap the roles of A, B. Every $C \in \mathfrak{C}'$ contains a neighbor x_C of z in G by (*). If these eight vertices form a clique then one checks readily that $\{\{x_C\}\colon C \in \mathfrak{C}'\} \cup \{\{z\}, (A \cup B) \setminus \{z\}\}$ is a clique minor in G of order 10 (every vertex x_C has a neighbor in $B \subseteq (A \cup B) \setminus \{z\}$ by (*)). Therefore, we may assume that z has two distinct nonadjacent neighbors x, y in $V(G')$.

If $G' + xy$ has a clique minor $\mathfrak{K}$ of order 9 then we may assume without loss of generality that x is contained in some member Q of $\mathfrak{K}$, as $G' + xy$ is connected. Consequently, $(\mathfrak{K} \setminus \{Q\}) \cup \{Q \cup \{z\}, (A \cup B) \setminus \{z\}\}$ is a clique minor of G of order ten (no matter whether Q contains y or not).

Hence we may assume that $G' + xy$ has no clique minor of order 9. As $G' + xy$ has at least $n' \geq 9$ vertices and at least $7n' - 27$ edges, we know that $G' + xy$ is one of the exceptional graphs in Theorem 1. By Lemma 2, G' is 7-connected. Therefore, $G' + xy$ is 7-connected; consequently, it cannot be the union of two graphs on more than 6 vertices each, meeting in less than seven vertices. It follows that $G' + xy$ is isomorphic to either $K_{2,2,2,3,3}$ or $K_{1,2,2,2,2,2}$, and $n' = 11$ or $n' = 12$. Let $\mathfrak{B}$ be the set of single-vertex-sets in $\mathfrak{C}'$. From $n' \geq |\mathfrak{B}| + 2(8 - |\mathfrak{B}|)$ we infer $|\mathfrak{B}| \geq 16 - n'$, and, as $G[P \cup Q]$ is a star centered at the only vertex from P for all $P \in \mathfrak{B}$ and $Q \in \mathfrak{C}' \setminus \{P\}$, every vertex from $\bigcup \mathfrak{B}$ is adjacent to all others of G'. Consequently, G' — and hence $G' + xy$ — has at least $16 - n' \geq 4$ many vertices adjacent to all others. However, $K_{2,2,2,3,3}$ has no vertex adjacent to all others, and $K_{1,2,2,2,2,2}$ has only one, a contradiction, proving the Theorem. □

We may replace 10 in Theorem 2 by any nonnegative $k < 10$: Suppose that G has a Kempe-coloring $\mathfrak{C}$ of order k and consider the graph G^+ obtained from G by adding new vertices $a_{k+1}, \ldots, a_{10}$ and all edges from $a_i, i \in \{k + 1, \ldots, 10\}$ to any other vertex $x \in V(G) \cup \{a_{k+1}, \ldots, a_{10}\}$. Then $\mathfrak{C}^+ := \mathfrak{C} \cup \{\{a_{k+1}\}, \ldots, \{a_{10}\}\}$ is a Kempe-coloring of G^+ of order 10. By Theorem 1, G^+ has a clique minor $\mathfrak{K}$, and, as every a_i is contained in at most one member of $\mathfrak{K}$, the sets of $\mathfrak{K}$ not containing any of $a_{k+1}, \ldots, a_{10}$ form a clique minor of order at least k of G.

A *k-coloring* of G is a partition of $V(G)$ into at most k anticliques, and the *chromatic number* $\chi(G)$ is the minimum number k so that G admits a k-coloring. (Observe that if a graph G has a unique k-coloring then it has no $(k - 1)$-coloring unless it is a complete graph on less than k vertices, so that, up to these exceptions, $\chi(G) = k$).

Hadwiger conjectured that every graph of chromatic number k admits a clique minor of order k [1]. From Theorem 2 we infer the following.

Theorem 3. *For $k \leq 10$, every graph of chromatic number k with a unique k-coloring admits a clique minor of order k.*

Proof. Let $\mathfrak{C}$ be the unique k-coloring of G. Then $\mathfrak{C}$ is a Kempe-coloring of order k (cf. [2]), and the statement follows from Theorem 2. □

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