# IDA-PBC for Polynomial and Mechanical Systems

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To Lia, Karla, and my Parents

*"It's not what you look at that matters; it's what you see."*

-Henry David Thoreau

*"Anyone who has never made a mistake has never tried anything new."*

-Albert Einstein

# Abstract


Since its introduction a little over 20 years ago, the Interconnection and Damping Assignment Passivity-based Control (IDA-PBC) has led to many theoretical extensions and practical applications. These controllers have been reported successful, among others, for *i)* their robust features, *ii)* exploiting the system's structural properties, and *iii)* their universal stabilization feature, reflected in the wide variety of applications ranging from electrical, mechanical, general nonlinear, and even infinite-dimensional systems. The major obstacle for implementing this control scheme is the satisfaction of the matching condition, which turns out to be a system of Partial Differential Equations (PDEs). Other significant problems include the dissipation condition in Underactuated Mechanical Systems (UMSs), splitting the controller scheme into two stages, and the incomplete theory for systems in implicit representation, i.e., systems described by Differential-algebraic Equations (DAEs).

In this context, the present work is divided into two parts. First, we introduce algebraic solutions for a class of nonlinear systems with polynomial structure. The proposed method avoids some of the standard problems of IDA-PBC previously mentioned and leads to conditions that can be recast as Sum of Squares (SOS) programs. Besides, it allows incorporating input saturation and minimization objectives to address the controller parameter selection. In the second part, we generalized the total energy shaping IDA-PBC to mechanical systems in implicit representation, i.e., with kinematic constraints. Furthermore, we introduce a heuristic formulation and some constructive methods to solve the PDEs of the matching conditions. Finally, we provide a method for eliminating kinematic constraints and constraint forces, i.e., reducing the implicit to explicit representation.

The contributions are validated in two second-order polynomial systems, a third-order rational system, the simple pendulum, the rolling disk, the Planar Vertical Takeoff-and-landing (PVTOL) aircraft, the cart-pole and the portal crane. Real experiments are carried out in the last two, both located at the Control Engineering Group's laboratory in the TU-Ilmenau.

# Kurzfassung

Seit seiner Einführung vor etwas mehr als 20 Jahren hat das Verfahren *Interconnection and Damping Assignment Passivity-based Control* (IDA-PBC) zu vielen theoretischen Erweiterungen und praktischen Anwendungen geführt. Diese Regler zeichen sich unter anderem durch *i)* robuste Eigenschaften, *ii)* die Ausnutzung der strukturellen Eigenschaften des Systems und *iii)* ihre universelle Stabilisierungseigenschaft aus. Dies spiegelt sich in der großen Vielfalt der Anwendungen von elektrischen, mechanischen, allgemein nichtlinearen und sogar unendlich-dimensionalen Systemen wider. Das größte Problem für die Implementierung dieser Regelverfahren ist die Erfüllung der sogenante *matching condition*, die sich als ein System von partiellen Differentialgleichungen herausstellt. Weitere wichtige Probleme sind die Dissipationsbedingung in unteraktuierten mechanischen Systemen, die Aufteilung der Regelverfahren in zwei Stufen und die unvollständige Theorie für Systeme in impliziter Darstellung, d.h. für Systeme, die durch differential-algebraische Gleichungen beschrieben werden.

In diesem Zusammenhang gliedert sich die vorliegende Arbeit in zwei Teile. Zunächst werden algebraische Lösungen für eine Klasse von nichtlinearen Systemen mit Polynomstruktur vorgestellt. Die vorgeschlagene Methode vermeidet einige der zuvor erwähnten Standardprobleme von IDA-PBC und führt zu Bedingungen, die als Summen-Quadrate-Programme umformuliert werden können. Außerdem erlaubt sie die Einbeziehung von Eingangssättigungs- und Minimierungszielen, um den Regler zu parametrieren. Im zweiten Teil verallgemeinern wir die Gesamtenergieumformung bei IDA-PBC auf mechanische Systeme in impliziter Darstellung, d.h. mit kinematischen Nebenbedingungen. Weiterhin stellen wir eine heuristische Formulierung und einige konstruktive Methoden zur Lösung der partiellen Differentialgleichungen der *matching conditions* vor. Schließlich stellen wir eine Methode zur Eliminierung von kinematischen Zwangsbedingungen und Zwangskräften vor, d. h. zur Reduzierung der impliziten auf die explizite Darstellung.

Die Beiträge werden in zwei Polynomsystemen zweiter Ordnung, einem rationalen System dritter Ordnung, dem einfachen Pendel, der rollenden Scheibe, dem PVTOL-Flugzeug, dem *cart-pole* System und dem Portalkran validiert. Für die beiden letztgenannten werden reale Experimente durchgeführt, die beide im Labor des Fachgebiets Regelungstechnik an der TU-Ilmenau angesiedelt sind.

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# Notation and Acronyms

Throughout this dissertation, lower-case Roman letters refer to vectors, upper-case Roman letters stand for matrices, upper-case script or blackboard letters denote sets or manifolds, and upright bold letters are used to represent variables or functions from the implicit model representation. For instance:

| | |
|---|---|
| $x$, $q$, $r$, $f$, $h$ | Vectors and vector-valued functions |
| $A$, $M$, $F$, $\Gamma$ | Matrices and matrix-valued functions |
| $\mathbb{R}$, $\mathbb{S}^n$, $\mathcal{X}$, $\mathcal{E}$ | Sets and manifolds |
| $\mathbf{M}$, $\mathbf{G}$ | Matrices and matrix-valued functions from an implicit model |
| $H$, $\mathbf{H}$, $g$ | Some exceptions caused by common conventions |

If the same symbol denotes a variable or a function simultaneously, we will use the dot notation to highlight this difference, e.g., $x$ are the states of a dynamical system while $x(\cdot)$ denotes its solution or integral curve. Besides, for multivariable calculus, we employ the matrix calculus notation and the numerator layout (also known as Jacobian formulation), meaning that the partial derivative $\frac{\partial y}{\partial x}$ is arranged as $\frac{\partial y}{\partial x} = \begin{bmatrix} \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \dots \end{bmatrix}$, where $y$ is a column vector (or scalar) function. The reader is additionally advised to look briefly at the below-stated notation and acronyms.

## Frequent Symbols

| | |
|---|---|
| $\mathbb{R}$ | set of real numbers |
| $\mathbb{R}^n$ | set of real column vectors of length $n$ |
| $\mathbb{R}^{n\times m}$ | set of $n \times m$ real matrices ($n$ rows and $m$ columns) |
| $\mathbb{R}_d^{n\times m}$ | set of $n \times m$ real matrices with rank $d$ |
| $\mathbb{R}_n^{n\times n}$ | set of square and invertible matrices of size $n$, or equivalently general linear group of degree $n$ over $\mathbb{R}$ |
| $\mathbb{S}^n$ | $n$-dimensional sphere |
| $\mathcal{C}^0$ | set of continuous functions |
| $\mathcal{C}^k$ | set of functions with $k$-continuous derivatives |
| $\mathcal{I}$ | real interval |

| | |
|---|---|
| $I_n$ | identity matrix of size $n$ |
| $0_n$, $0_{n\times m}$ | column vector of size $n$ with elements equal to 0, matrix of size $n \times m$ with elements equal to 0 |
| $:=$ | equal by definition |
| $\equiv$ | identical equality, i.e., $f(x) \equiv g(x)$ imply $f(x) = g(x)$ for all $x$ |
| $e_i$, $\bar{e}_i$, $\hat{e}_i$ | column vectors with the $i$-th element equal to one and zeros elsewhere, or equivalently, standard basis vectors for a Euclidean space |
| $q$, $p$, $H$, $V$ | generalized coordinates, generalized momenta, Hamiltonian, and potential energy of mechanical systems in explicit representation |
| $r$, $\rho$, $\mathbf{H}$, $\mathbf{V}$ | coordinates, momenta, Hamiltonian, and potential energy of mechanical systems in implicit representation |
| $\square$ | end of a proof |
| $\triangle$ | end of an example |

# Superscripts, Subscripts and Accents

| | |
|---|---|
| $(\cdot)^\top$ | transpose of a matrix or vector |
| $(\cdot)^*$ | conjugate transpose |
| $(\cdot)^{-1}$ | inverse of a non-singular square matrix |
| $(\cdot)^{-\top}$ | $((\cdot)^{-1})^\top$ or $\left((\cdot)^\top\right)^{-1}$ |
| $(\cdot)^{\mathrm{g}}$ | generalized inverse |
| $(\cdot)^+$ | Moore–Penrose inverse |
| $(\cdot)^{\mathrm{s}}$ | $(\cdot) + (\cdot)^\top$ |
| $(\cdot)^\perp$ | orthogonal complement of a subspace |
| $(\cdot)_\perp$, $(\cdot)_{\top\!\!\!\perp}$ | left and right annihilators, respectively, which are full rank for convenience |
| $(\cdot)_\perp^\top$, $(\cdot)_{\top\!\!\!\perp}^\top$ | $((\cdot)_\perp)^\top$, $((\cdot)_{\top\!\!\!\perp})^\top$ |
| $(\cdot)_{\mathrm{d}}$ | desired constant or function |
| $(\cdot)^\star$ | value on an equilibrum point |
| $\dot{(\cdot)}$ | first derivative of a (vector-valued) function with respect to time |
| $\ddot{(\cdot)}$ | second derivative of a (vector-valued) function with respect to time |

# Operators and Relations

| | |
|---|---|
| $\mathrm{diag}(A_1, \ldots, A_k)$ | (block) diagonal matrix with entries given by $A_1, \ldots, A_k$, or equivalently, the direct sum $A_1 \oplus \ldots \oplus A_k$ |
| $\mathrm{vec}(f_1, \ldots, f_k)$ | column vector build with the scalars or vectors $f_1, \ldots, f_k$ |
| $\mathrm{col}_i(A)$, $\mathrm{row}_i(A)$ | $i$-th column of A, $i$-th row of A |

| | |
|---|---|
| $\text{elem}_{i,j}(A)$ | element of the $i$-th column and $j$-th row of A |
| $\text{rank}(A)$ | rank of a real matrix $A$ |
| $\det(A)$ | determinant of a real square matrix $A$ |
| $\text{trace}(A)$ | trace of a real square matrix $A$ |
| $\text{Colsp}(A)$ | column space of a matrix $A$, i.e., span of the columns of $A$ |
| $\text{Rowsp}(A)$ | row space of a matrix $A$, i.e., span of the rows of $A$ |
| $\text{Null}(A)$ | null space of a matrix $A$ |
| $\text{int}(\mathcal{A})$ | interior of a set $\mathcal{A}$ |
| $\partial\mathcal{A}$ | boundary of a set $\mathcal{A}$ |
| $\text{span}\{f_1,\ldots,f_k\}$ | span of the set of vectors $\{f_1,\ldots,f_k\}$ |
| $\dim(x)$ | dimension of a real vector $x$ |
| $\text{atan2}(y,x)$ | four-quadrant inverse tangent of $y$ and $x$ |
| $\binom{n}{k}$ | binomial coefficient, i.e., $\frac{n!}{k!(n-k)!}$ |
| $M/A$ | (generalized) Schur complement of (the block) $A$ in the matrix $M$ |
| $A^{\frac{1}{2}}$ | $A^{\frac{1}{2}}A^{\frac{1}{2}} = A$ |
| $\lambda_i(A)$ | $i$-th eigenvalue of $A$ |
| $\sigma_{\max}(A)$, $\sigma_{\min}(A)$ | maximum, minimum singular value of $A$ |
| $\log_e(a)$, $\ln(a)$ | natural logarithm of $a$ |
| $\arg\min\limits_{x\in\mathcal{X}} f(x)$ | argument $x \in \mathcal{X}$ that minimizes the function $f$ |
| $\lVert x\rVert_2$ , $\lVert A\rVert_2$ | Euclidean norm of a real vector $x$, spectral norm of a real matrix $A$ |
| $\frac{\partial f}{\partial x}$ | partial derivative of $f$ with respect to $x$ |
| $\frac{\partial^\top f}{\partial x}$ | $\left(\frac{\partial f}{\partial x}\right)^\top$ |
| $\dfrac{\partial^2\gamma}{\partial x^2}$ | Hessian (second order partial derivative) of a function $\gamma$ w.r.t. $x$ |
| $\mathcal{L}_f^n h(x)$ | $n$-th order Lie (or directional) derivative of $h$ along $f$ evaluated at $x$, i.e., $\mathcal{L}_f^0 h(x) = h(x)$, $\mathcal{L}_f^1 h(x) = \mathcal{L}_f h(x) = \frac{\partial h}{\partial x}(x)f(x)$, $\mathcal{L}_f^2 h(x) = \mathcal{L}_f\mathcal{L}_f h(x) = \frac{\partial \mathcal{L}_f h}{\partial x}(x)f(x)$, ..., $\mathcal{L}_f^n h(x) = \frac{\partial \mathcal{L}_f^{n-1} h}{\partial x}(x)f(x)$ |
| $[f,h]\,(x)$ | Lie bracket of $f$ and $h$ evaluated at $x$, i.e., $\frac{\partial h}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)h(x)$ |
| $f\|_{x=x^\star}$ | evaluation of $f$ in $x^\star$, i.e., $f(x^\star)$ |
| $f\|_{\mathcal{X}}$ | restriction of $f$ in the set $\mathcal{X}$ |
| $\mathcal{X}\times\mathcal{Y}$ | Cartesian product of sets $\mathcal{X}$ and $\mathcal{Y}$ |
| $A\otimes B$ | Kronecker product of $A$ and $B$ |
| $a \geq b$, $a > b$ | $a$ is greater than or equal $b$, $a$ is greater than $b$ |
| $a \leq b$, $a < b$ | $a$ is less than or equal $b$, $a$ is less than $b$ |
| $A \succeq B$, $A \succ B$ | $A - B$ is positive semidefinite, $A - B$ is positive definite |
| $A \preceq B$, $A \prec B$ | $B - A$ is positive semidefinite, $B - A$ is positive definite |

# Acronyms

| | |
|---|---|
| **LHS** | Left-hand Side |
| **RHS** | Right-hand Side |
| **ODE** | Ordinary Differential Equation |
| **DAE** | Differential-algebraic Equation |
| **PDE** | Partial Differential Equation |
| **UMS** | Underactuated Mechanical System |
| **FMS** | Fully-actuated Mechanical System |
| **DoF** | Degrees of Freedom |
| **PVTOL** | Planar Vertical Takeoff-and-landing |
| **PBC** | Passivity-based Control |
| **IDA-PBC** | Interconnection and Damping Assignment Passivity-based Control |
| **PD** | Proportional-derivative |
| **PID** | Proportional-integral-derivative |
| **PID-PBC** | Proportional-integral-derivative Passivity-based Control |
| **CL** | Controller Lagrangians |
| **ZSD** | Zero State Detectable |
| **ZSO** | Zero State Observable |
| **SOS** | Sum of Squares |
| **SDP** | Semidefinite Programming |
| **LMI** | Linear Matrix Inequality |
| **LQR** | Linear Quadratic Regulator |
| **LTI** | Linear Time-invariant |
| **PFL** | Partial Feedback Linearization |
| **KYP** | Kalman-Yacubovitch-Popov |

# I

# Preliminaries

# Chapter 1

# Introduction

Energy is a fundamental property of physical systems that describes their dynamic behavior to such an extent that these systems can be considered energy transformation devices [1]. For example, the total mechanical energy (kinetic plus potential) of a falling object is dissipated (transformed and transferred) in the presence of air friction, and it determines the object's velocity before hitting the ground. This is the starting point of **passivity** and **port-Hamiltonian** systems, which are key components in multi-physics modeling and control. In a nutshell, a passive system is a dynamic system that cannot possess more energy than the one is supplied to it from its inputs, whereas a port-Hamiltonian system is a dynamic system defined by a power conserving interconnection structure, a dissipation object, and an energy-storing function called Hamiltonian [2]. Besides, port-Hamiltonian systems are passive whenever their Hamiltonian is lower bounded but the converse is not always true [3]. In both cases, the "energy" term is not necessarily a physical property but an abstract generalization.

## 1.1 Motivation and Literature Review

Ortega and Spong coined the term **Passivity-based Control (PBC)** in [4] to describe controller design methodologies that achieve stabilization by rendering the closed-loop passive. The main motivation for PBC stems from three facts.

1. There is a strong link between passivity and stability [5–7].

2. The scheme may take advantage of the system's structural properties [8, 9].

3. Passivity is an input-output feature, i.e., it is independent of the notion of states [3].

Several strategies within the framework PBC have been proposed in the last 40 years. For example the Proportional-derivative (PD) controller [8, 10] and the potential energy shaping [11] for fully actuated mechanical systems, the energy-balancing PBC [12, 13], the energy-Casimir methods [3, 14], backstepping and forwarding [5], the controlled Lagrangians [15–17], the **Proportional-integral-derivative Passivity-based Control (PID-PBC)** [18–21], and the **Interconnection and Damping Assignment Passivity-based Control (IDA-PBC)** [22–24], to name but a few. Although not all of them were initially proposed within the framework of passivity, they all belong to the PBC group.

### 1.1.1 Why IDA-PBC?

Since its introduction, the IDA-PBC has led to many theoretical extensions and practical applications [13, 23–28]. In its most general description, this strategy achieves closed-loop stabilization by transforming a nonlinear system into a port-Hamiltonian system with some desired dissipation, interconnection structure, and lower bounded Hamiltonian. These controllers solve some of the intrinsic problems of other PBC formulations, such as

- the dissipation obstacle in energy-balancing PBC [12],
- the restrictive total energy shaping in energy-Casimir methods [12], and
- the reduced system class of backstepping (pure-feedback form), forwarding (pure-feedforward form) and controlled Lagrangians (Euler-Lagrange systems) [23, 24, 29].

Nevertheless, the above points are merely a product of the universal stabilization feature of IDA-PBC [13], reflected in the wide variety of applications ranging from electrical [30–32], mechanical [22, 33–35], electromechanical [25, 35–37], general nonlinear [23, 26], and even infinite dimensional systems [38]. Another strength that cannot be overlooked is the inherent robustness of IDA-PBC [39] that can be enhanced with a dynamic extension, designed under adaptive control [40–45] or integral action control [46–50].

### 1.1.2 Obstacles in the Application of IDA-PBC

The standard IDA-PBC method requires a two-stage procedure: **energy shaping** and **damping injection**. The energy shaping stage renders the nominal system into a lossless port-Hamiltonian system under the satisfaction of the so-called **matching condition**, which is a system of nonhomogeneous and quasilinear Partial Differential Equations (PDEs) with unknowns in the target Hamiltonian and the desired interconnection structure and whose solutions are not a simple task, see [51] for example. In the next stage, we inject damping to render the closed-loop passive and guarantee stabilization whenever the Hamiltonian has a strict minimum in the desired state.

### The Matching Condition

Solving the matching condition persists in being a stumbling block in IDA-PBC. Great effort has been devoted to solve, simplify or completely avoid the PDEs. The simplest idea goes back to [52], and it consists in *fixing the desired energy function (Hamiltonian) so that the PDEs are reduced to algebraic equations.* This approach is then adapted to IDA-PBC in [23, 25] for the control of a micro-electromechanical system and an induction motor. By using algebraic solutions, Nunna et al. [26, 53, 54] introduce a dynamic extension to further simplify the matching conditions, testing their results in three third-order systems: a magnetic levitated ball, an electrostatic microactuator and a prey-predator system. However, the resulting closed-loop may lose the port-Hamiltonian structure.

Changing the coordinates and modifying the target dynamics has been proposed in [52, 55] to reduce or eliminate PDEs in port-Hamiltonian systems. In [56], Acosta and Astolfi construct a dynamic extension and an approximate integral of the target Hamiltonian as an alternative to handle the PDEs with major applications in strict-feedback and -feedforward systems. In another perspective, for some systems, we can impose sufficient conditions to ensure that the PDEs are reduced to simple integrals. This formulation has been developed in [12, 57, 58] for the control of a class of Underactuated Mechanical Systems (UMSs) with underactuation degree one and a magnetic levitated ball. Recently, in [59], the authors transform the nonhomogeneous PDEs to the corresponding Pfaffian differential equations, simplifying the solution task for third-order systems. Their results are verified in a magnetic levitation system, a Pendubot and an underactuated cable-driven robot.

**Underactuation** is a technical term used in robotics and control theory to describe mechanical systems that possess a lower number of independent actuators (or control inputs) than Degrees of Freedom (DoF) [60]. The distinction is fundamental since it poses a restriction to manipulate (or control) the instantaneous accelerations, meaning that **Fully-actuated Mechanical Systems (FMSs)** can follow arbitrary trajectories while **Underactuated Mechanical Systems (UMSs)** not [61–65]. For FMSs, the matching condition of IDA-PBC is trivial, but for UMSs it is usually split twofold: a kinetic energy shaping, which includes quasilinear PDEs, and a potential shaping part, which are linear PDEs whenever the kinetic shaping has a solution first. In [66], it is used the so-called $\lambda$-method to reduce the matching conditions to linear PDEs. The contribution is actually developed for controlled Lagrangians, but the results can be easily extrapolated since this formulation is strictly contained in IDA-PBC [17, 67]. In [68], Mahindrakar et al. proposed a constructive approach for the potential energy shaping of a class of UMSs with two DoF, obtaining for the first time swing-up and stabilization of the Acrobot without using two distinct controllers. By using new passive outputs, in [20, 69], an alternative constructive method that completely avoids PDEs for a class of UMSs is introduced. The approach has been further developed under the name PID-PBC [70].

### Simultaneous IDA-PBC

Splitting the IDA-PBC in energy shaping and damping injection is not always the best option. Batlle et al. [25] prove that such a partition prevents the application of IDA-PBC in a simple induction motor regulation, and to overcome the problem, they propose a one-stage approach called simultaneous IDA-PBC (SIDA-PBC). Inspired in [25], Donaire et al. [24] introduce the use of generalized forces, extending the SIDA-PBC application in UMSs at the cost of adding flexibility and complexity in the PDEs.

### The Dissipation Condition

Another adversity in the IDA-PBC design is the dissipation condition. This problem appears in mechanical systems when using kinetic energy shaping [27], and therefore also in general port-Hamiltonian systems with total Hamiltonian shaping. As a remedy, [71] propose to overcome the dissipation condition for a class of mechanical systems by adding a cross term between coordinates and momenta in the desired Hamiltonian. However, the stability proof relies on linearity, and it is not analyzed whether nonlinearities outside of the local point contribute to or obstruct stability.

### Implicit Mechanical Systems

At least two different representations can be used when modeling port-Hamiltonian systems [72]:

1. The **implicit representation**, where system models are obtained by aggregating simpler subsystems and the dynamics are described by Differential-algebraic Equations (DAEs) with the interconnections expressed as algebraic constraints.

2. The **explicit representation**, where the interconnections are simplified and the system is handled as a whole with a model described by Ordinary Differential Equations (ODEs).

For example, in the framework of implicit mechanical systems, interconnections can be joint constraints, distances between points of a rigid body, a non-slip condition in wheeled robots, etc. As discussed in [73], most ODEs that we encounter in applications are actually simplified DAEs, and this reduction can

- produce less meaningful variables (physical quantities are usually found DAE models)
- require different explicit models,
- be numerically inefficient (most leading software generates a differential-algebraic model whenever constraints are present), and

- be time consuming or impossible to obtain.

In this regard, the IDA-PBC has been intensively studied on explicit systems, capturing a wide range of applications. However, only few research has been devoted to IDA-PBC in implicit systems. The first contribution goes back as far as [74] for the control of nonholonomic systems with controlled Lagrangians. The approach is then extended and adapted to IDA-PBC in [75]. Motivated by discretized infinite-dimensional systems, Macchelli [76] introduces a general IDA-PBC approach, conditions of which are stated in image and kernel Dirac representations with linear maps. This scheme appears very generic such that for UMSs almost no advantages may be explored. Therefore, Castaños and Gromov [77] take a closer look and focus on UMSs with holonomic constraints and a representation equivalent to a Dirac structure given by a combination of hybrid and constrained input-output representations, see [78]. Their algorithm does not modify the interconnection structure and dissipation object, but, for a class of holonomic systems, is able to reduce the matching condition to a simple quadratic programming problem. In other words, by using implicit formulations, we may also avoid the PDEs of the matching conditions.

#### Input Saturation

It is well-known that input saturation can cause performance losses or even lead to closed-loop instability. Perhaps the earliest inputs saturation work on PBC are due to Loria et al. [79], where the authors present a dynamic output feedback controller with input saturation for a class of FMSs. Escobar et al. [80] extend and implement this algorithm in the TORA robot, which is an UMS. From the variable structure point of view, Machelli [81, 82] develops an approach to energy shaping that includes input saturation for a class of port-Hamiltonian systems. Later, Åström et al. [83] study the energy shaping with input saturation for swinging up a pendulum. The works [84, 85] focus only on saturating the damping injection term. Not long ago, Sprangers et al. [86] studied a reinforcement learning method for energy shaping showing robust properties under input saturation.

### 1.1.3 Polynomial Systems and Sum of Squares

The celebrated book of Boyd et al. [87] has laid open the wide range of control problems that can be stated as Linear Matrix Inequalities (LMIs). Most of the theory was focused first on linear systems with nonlinearities modeled as uncertainty [88]. The contribution of LMIs in nonlinear control roughly started with the Sum of Squares (SOS) decomposition, which is a computationally tractable method to certify the nonnegativity of polynomial functions and matrices. Since most nonlinear systems can be locally approximated [89] or parameterized [90] with polynomial or rational functions, the SOS approach give new perspectives for solving problems such as

- construction of Lyapunov Functions [91],
- optimal control [92],
- $\mathcal{H}_\infty$ control with policy iteration [93],
- stabilization of uncertain systems [94],
- controller synthesis with actuator saturation [95],
- control of UMSs [96, 97], and many more.

## 1.2 Contributions of this Thesis

This work has two main contributions. First, for a class of nonlinear systems, we introduce algebraic solutions to avoid some of the standard problems of IDA-PBC: the necessity of solving PDEs, meeting the dissipation condition, and splitting the controller scheme into two stages. The proposed method leads to conditions that can be recast as SOS programs. Besides, we consider input saturation and four minimization objectives, including local optimal performance assignment, to address the great flexibility in the controller parameter selection. The results are verified in two second-order polynomial systems, a third-order rational system with two inputs, and the well-known cart-pole. This contribution leads to the publications [98, 99]

The second contribution is the extension of the total energy shaping IDA-PBC to mechanical systems with kinematic constraints (holonomic and nonholonomic). We combine the formulations of [75] and [77] into a unique scheme and remove the conservative assumptions that restricts the energy shaping and the scope of application. To solve the PDEs of the matching conditions, we introduce *i)* a heuristic formulation based on Sum of Squares (SOS) programs, and *ii)* some constructive methods that can also be employed in the explicit representation. Furthermore, it is shown that, based on the full state information control and two additional conditions, an output feedback controller might be obtained, reducing measurement requirements. Finally, we provide a method for eliminating kinematic constraints and constraint forces, which allows for comparing the approaches from explicit and implicit representations. The controller strategies are validated on two fully-actuated and three underactuated benchmark examples: the rolling disk, the simple pendulum, the Planar Vertical Takeoff-and-landing (PVTOL) aircraft, the portal crane and the cart-pole. Real experiments are carried out in the last two, both located at the Control Engineering Group's laboratory in the TU Ilmenau. This contribution leads to the publications [100–104]

## 1.3 Outline of the Thesis

The thesis is divided into four parts, where the second and third correspond to the contributions we discuss in the previous section, respectively. Outline of the parts can be consulted below.

### Part I

***Chapter 2:*** This chapter provides an overview of the fundamental concepts that this dissertation entails. It discusses stability of nonlinear autonomous systems, the SOS decomposition, manifolds and vector fields, DAEs from a geometric perspective, Lagrangian and Hamiltonian mechanics with kinematic constraints, and a closer look at passivity and port-Hamiltonian systems.

### Part II

***Chapter 3:*** This chapter recalls the standard formulation of IDA-PBC for nonlinear affine systems while introducing the use of generalized inverses to provide greater flexibility in the controller expression. Furthermore, for a class of systems with polynomial structure, we present a new algebraic solution, conditions of which can be restated as SOS programs. An analysis of the region of convergence and a comparison with Linear Time-invariant (LTI) systems is also addressed in this chapter. The results are verified in two second-order systems: one polynomial and one rational.

***Chapter 4:*** In this chapter, the new algebraic solution of Chapter 3 is extended by incorporating input saturation and four main optimization objectives: volume maximization for the region of convergence, minimization of the control action, and standard and generalized $\mathcal{H}_2$ local optimal performance assignment. We test the results in a second-order polynomial system, a third-order rational system with two inputs and the cart-pole.

### Part III

***Chapter 5:*** The chapter addresses the total energy shaping IDA-PBC for mechanical systems with and without kinematic constraints. The innovations are *i)* the introduction of generalized inverses (to offer greater flexibility in the controller expression), *ii)* the comparison between the standard and simultaneous IDA-PBC, *iii)* the unification of [74, 75, 105] into a unique scheme while removing three conservative assumptions that restrict the kinetic energy shaping and the scope of application for constrained mechanical systems, and *iv)* the inclusion of non-classical constrained systems, i.e., systems where the constraint forces can perform work.

***Chapter 6:*** Under the developments of Chapter 5 for constrained mechanical systems, in this chapter, we analyze the controller synthesis for Fully-actuated Mechanical Systems (FMSs), testing the results in a simple pendulum and a vertical rolling disk. For Underactuated Mechanical Systems (UMSs), we introduce a heuristic solution and some constructive methods to solve the PDEs arising in the matching condition. Furthermore, we generalized (by including constrained systems) the well-known result that the PBC design can obviate velocity measurement for some unconstrained systems, i.e., we present an output feedback controller for implicit systems. Finally, to provide equivalence between explicit and implicit representations, we eliminate the kinematic constraints and constraint forces of systems *i)* whose constraint forces are allowed to perform work, *ii)* that possess holonomic and nonholonomic constraints simultaneously, and *iii)* that have a closed-loop representation.

***Chapter 7:*** This chapter deals with the controller design of Chapters 5 and 6 on three underactuated benchmark examples with holonomic constraints: the PVTOL aircraft, the portal crane, the cart-pole. Real experiments with a dSPACE are carried out in the last two.

## Part IV

***Chapter 8 and Appendices A and B:*** These chapters provide the conclusions, future works, standard lemmas employed throughout this thesis, and the proof of propositions and lemmas of lesser relevance.

# Chapter 2

# Theoretical Preliminaries

This chapter provides an overview of the fundamental concepts that this dissertation entails. Section 2.1 is devoted to studying the stability of nonlinear autonomous systems in the sense of Lyapunov as well as the extensions under invariant sets. In Section 2.2, we review the SOS decomposition, which is as a sufficient and computationally tractable condition to certify the nonnegativity of polynomial functions. Section 2.3 is mainly concerned with implicitly defined manifolds, and vector fields. It lays the foundations to analyze DAEs from a geometric perspective in Section 2.4 and to understand the Lagrangian and Hamiltonian mechanics with kinematic constraints (holonomic and nonholonomic) discussed in Section 2.5. In addition, for each set of (non-independent or constraint-free) coordinates that completely specify a mechanical system, it allows us to obtain equations of motion that are equivalent for each selected set of coordinates. We close this preliminary chapter by presenting the concepts of passivity and port-Hamiltonian systems in Section 2.6.

## 2.1 Stability of Nonlinear Affine Systems

System stability is one of the pillars for the design of nearly any control algorithm. Roughly speaking, an equilibrium point of a dynamical system is stable if every solution starting in a vicinity of the equilibrium remains nearby for all future time. In the following section, we formalize this theory for autonomous systems and state the basic Lyapunov theorems as well as their extensions under the invariance principle. The concepts presented here are mainly borrowed from [106–110].

We begin by considering a nonlinear dynamical system of the form

$$\dot{x} = f(x), \tag{2.1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^{n_x}$ are the states, $\mathcal{X}$ is an open and connected set, and $f : \mathcal{X} \to \mathbb{R}^{n_x}$ is a **locally Lipschitz continuous** function. A **solution** (or **integral curve**) of (2.1) over an interval $\mathcal{I} \subset \mathbb{R}$ is a differentiable map, represented by $x : \mathcal{I} \to \mathcal{X}$ and $t \mapsto x(t)$, with the property that $\dot{x}(t) = f(x(t))$ holds for all $t \in \mathcal{I}$. Besides, due to the Lipschitz condition, the solution of the initial-value problem exists and is unique, see [110]. We call $x^\star$ an **equilibrium point** of the dynamical system (2.1) if $f(x^\star) = 0$, or equivalently if $x(t_p) = x^\star \implies x(t) = x^\star$ for all $t \neq t_p$.

**Definition 2.1** (Lyapunov Stability)**.** *The equilibrium $x^\star$ of (2.1) is*

- ***(Lyapunov) stable** if, for any neighborhood[1] $\mathcal{X}_\epsilon$ of $x^\star$, there exists a neighborhood $\mathcal{X}_\delta$ of $x^\star$ satisfying*

$$x(t_0) \in \mathcal{X}_\delta \quad \implies \quad x(t) \in \mathcal{X}_\epsilon \qquad \forall t \geq t_0;$$

- ***(locally) asymptotically stable** if it is stable and $\mathcal{X}_\delta$ can be chosen such that*

$$x(t_0) \in \mathcal{X}_\delta \quad \implies \quad \lim_{t \to \infty} x(t) = x^\star;$$

- ***globally asymptotically stable** if it is asymptotically stable and $\mathcal{X}_\delta = \mathbb{R}^{n_x}$; and*
- ***unstable** if it not stable.*

### 2.1.1 Lyapunov's Theorems

The following results, called the direct and indirect methods of Lyapunov, provide a simple framework to determine the stability of an equilibrium point in the system (2.1). The direct method relies on finding a so-called Lyapunov function, which can be interpreted as the generalization of energy functions, while the indirect one analyzes the linearized system.

**Theorem 2.1** (Lyapunov's direct method)**.** *Consider the autonomous system (2.1) and the equilibrium point $x^\star$. Suppose there exists a $\mathcal{C}^1$ (continuously differentiable) function $V : \mathcal{X} \to \mathbb{R}$ such that*

$$V(x^\star) = 0, \tag{2.2a}$$

$$V(x) > 0 \qquad \forall x \in \mathcal{X} - \{x^\star\}, \tag{2.2b}$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \qquad \forall x \in \mathcal{X}. \tag{2.2c}$$

*Then, the equilibrium is (Lyapunov) stable. If additionally,*

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \qquad \forall x \in \mathcal{X} - \{x^\star\}, \tag{2.2d}$$

[1] Let $p$ be a point in a topological space $\mathcal{M}$. A neighborhood of $p$ is a open subset of $\mathcal{M}$ for which $p$ belongs.

*then it is asymptotically stable.*

**Theorem 2.2** (Lyapunov's indirect method)**.** *Let $x^\star$ be an equilibrium point of* (2.1) *with $f \in \mathcal{C}^1$. Then, the equilibrium is*

*i) asymptotically stable if the real part of every eigenvalue in $\frac{\partial f}{\partial x}\big|_{x=x^\star}$ is negative, and*

*ii) unstable if the real part of some eigenvalue of $\frac{\partial f}{\partial x}\big|_{x=x^\star}$ is positive.*

A $\mathcal{C}^1$ function $V$ verifying (2.2a)–(2.2b) is called a **candidate Lyapunov function** for (2.1) in the equilibrium $x^\star$. If $V$ also fulfills (2.2c), then it is a **Lyapunov function**. Observe that the conditions of Theorem 2.1 are only sufficient, meaning that if a candidate function does not satisfy (2.2c) or (2.2d), it does not imply that the equilibrium is unstable or stable but not asymptotically, it just means that the candidate function is inadequate. In general, Lyapunov functions are non-unique and there is not a universal approach to find them, but there are some cases in which the selection is relatively natural or intuitive. For instance, in mechanical systems, the total energy (kinetic plus potential) is a standard choice. If such a selection is not straightforward, there are various methods for constructing $V$, e.g., the variable gradient, Krasovskii's, Zubov's, energy-Casimir, composite functions, among others. An eager reader can consult [5, 107] for a comprehensive discussion on some of the previously mentioned. Unlike Lyapunov's direct method, Theorem 2.2 avoids Lyapunov functions by calculating the Jacobian of $f$ and evaluating its eigenvalues, i.e., the qualitative behavior of the nonlinear system near the equilibrium point. If the Jacobian matrix has at least one eigenvalue on the imaginary axis while the rest belong to the left half-plane, stability cannot be establish with this theorem because the linearization could behave differently from the original system.

It is important to remark that the conditions of Theorem 2.1 are unable to ensure that every solution with initial conditions in $\mathcal{X}$ will approach $x^\star$ or even stay at $\mathcal{X}$ for all $t > t_0$. However, if we find a constant $d$ such that the set

$$\Omega_d := \{x \in \mathcal{X} \mid V(x) \leq d\}$$

is bounded, then every trajectory starting in $\Omega_d \subset \mathcal{X}$ will remain in $\Omega_d$ because $\dot{V}(x) \leq 0$, or equivalently $V(x(t)) \leq V(x(t_0)) \leq d$ for all $t \geq t_0$. In view of the above, we say that a set $\mathcal{X}_\mathrm{R} \subset \mathcal{X}$ is a **region of convergence** (or **attraction**) of $x^\star$ if $x(t) \to x^\star$ as $t \to \infty$ for every solution starting in $\mathcal{X}_\mathrm{R}$ (not necessarily of the form of $\Omega_d$). From the proof of Theorem 2.1, see [107], we can conclude that $\Omega_d$ is actually a region of attraction of $x^\star$ if (2.2d) holds and $x^\star$ is the only equilibrium in $\Omega_d$. Besides, the region of convergence covers to the whole space $\mathbb{R}^{n_x}$ if $\mathcal{X} = \mathbb{R}^{n_x}$ and $\Omega_d$ is bounded for every value of $d > 0$. The latter can be guarantee if $V$ is **radially unbounded**, i.e.,

$$V(x) \to \infty \qquad \text{as} \qquad \|x\| \to \infty. \tag{2.3}$$

The next theorem provided by Barbashin and Krasovskii [109, p. 5226] summarizes this insight.

**Theorem 2.3.** *Let the conditions of Theorem 2.1 for asymptotic stability be fulfilled. Assume besides that $\mathcal{X} = \mathbb{R}^{n_x}$ and $V$ is radially unbounded, then the origin of (2.1) is globally asymptotically stable.*

### 2.1.2 Invariant Set Stability Theorems

To establish asymptotic stability using Theorem 2.1, we require to find a Lyapunov function whose time derivative $\dot{V}$ is **negative definite**[2] about $x^\star$. As demonstrated in the theorem below this condition can be relaxed while ensuring asymptotic stability if no solution other than $x(t) \equiv x^\star$ can stay indefinitely at the points where $\dot{V}(x) = 0$. Before we state the theorem, let us introduce some necessary terminology. A set $\mathcal{X}_\mathrm{I} \subset \mathcal{X}$ is called **invariant** with respect to (2.1) if, for any initial state $x(t_0)$ inside of $\mathcal{X}_\mathrm{I}$, the solutions remain in $\mathcal{X}_\mathrm{I}$ for all past and future time, i.e., $x(t_0) \in \mathcal{X}_\mathrm{I} \implies x(t) \in \mathcal{X}_\mathrm{I}\ \forall t \in \mathbb{R}$. It is called **positively invariant** if the solutions remain in $\mathcal{X}_\mathrm{I}$ for all future time, i.e., $x(t_0) \in \mathcal{X}_\mathrm{I} \implies x(t) \in \mathcal{X}_\mathrm{I}\ \forall t \geq t_0$. The region of convergence $\mathcal{X}_\mathrm{R}$ and the bounded set $\Omega_d$ (of the previous section) are examples of positively invariant sets.

**Theorem 2.4** (LaSalle's invariance principle)**.** *Let $\mathcal{X}_\mathrm{I} \subset \mathcal{X}$ be a compact[3] and positively invariant set of (2.1). Assume there exists a $\mathcal{C}^1$ function $V : \mathcal{X}_\mathrm{I} \to \mathbb{R}$ with $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0$. Let $\mathcal{X}_\mathrm{L}$ be the largest invariant set of (2.1) contained in*

$$\left\{ x \in \mathcal{X}_\mathrm{I} \mid \tfrac{\partial V}{\partial x} f(x) = 0 \right\}.$$

*Then, every solution of (2.1) starting in $\mathcal{X}_\mathrm{I}$ approaches $\mathcal{X}_\mathrm{L}$ as $t \to \infty$.*

The previous theorem not only avoids the negative definite condition on $\dot{V}$, but it also extends the Lyapunov direct method (Theorem 2.1) in four additional ways: First, it allows **sign-indefinite** functions $V$. Second, we can guarantee convergence to an invariant set rather than just an equilibrium point. Third, it gives an estimate of the region of convergence which is more comprehensive than $\Omega_d$. And fourth, since $\mathcal{X}_\mathrm{I}$ is a compact set, it may take the form of a manifold, which is not always an open subset of $\mathbb{R}^{n_x}$ (with its boundary) as it is in the classical Lyapunov theory. In other words, we can work with **ODEs on manifolds** (also known as **ODEs with invariants**, see [112, 113]).

When $V$ is selected as a Lyapunov function, we have the following corollary that historically predates LaSalle's invariance principle.

[2] We recall that a function $H : \mathcal{X} \to \mathbb{R}$ is said to be **positive definite** about $x^\star$ if $H(x^\star) = 0$ and $H(x) > 0$ for all $x \in \mathcal{X} - \{x^\star\}$, and it is said to be **positive semidefinite** if $H(x) \geq 0$ for all $x \in \mathcal{X}$. **Negative definite** and **negative semidefinite** functions are defined analogously. Finally, a function $H$ is called **sign-indefinite** if it is neither positive nor negative semidefinite.

[3] A set is called compact if it is closed and bounded [111].

**Corollary 2.1** (Barbashin-Krasovskii-LaSalle)**.** *Consider the autonomous system* (2.1) *and assume there exists a* $\mathcal{C}^1$ *function* $V : \mathcal{X} \to \mathbb{R}$ *such that*

$$\begin{aligned} V(x^\star) &= 0, & \\ V(x) &> 0 & \forall x \in \mathcal{X} - \{x^\star\}, \\ \dot{V}(x) = \frac{\partial V}{\partial x} f(x) &\leq 0 & \forall x \in \mathcal{X}. \end{aligned}$$

*Then, every bounded solution* $x(\cdot)$ *approaches the largest invariant set of* (2.1) *contained in* $\left\{x \in \mathcal{X} \mid \frac{\partial V}{\partial x} f(x) = 0\right\}$*. If this invariant set posses no other than the trivial solution* $x(t) \equiv x^\star$*, then* $x^\star$ *is asymptotically stable. The stability properties are global if* $V$ *is also radially unbounded.*

## 2.2 Sum of Squares

Certifying the nonnegativity of a polynomial function is, in general, an NP-hard problem [114]. On the other hand, a polynomial function is nonnegative if it can be written as a Sum of Squares (SOS) of polynomials, and this task is now equivalent to a semidefinite program that can be solved numerically in polynomial-time. A wide variety of control problems can be formulated under the framework of SOS, e.g. Lyapunov stability analysis. In this section, we briefly recall the SOS method and its connection with nonlinear stability. The ideas discussed here are essentially extracted from [115–119]. Due to space limitations, it is not our purpose to review Linear Matrix Inequalities (LMIs) or Semidefinite Programming (SDP). For that end, the reader can consult [87, 88, 120, 121].

### 2.2.1 SOS Decomposition

We proceed by introducing the required terminology. A **monomial** in $x \in \mathbb{R}^n$ is a product given by

$$x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are nonnegative integers and $x_i$ is the $i$-th element of $x$. A (multivariate) **polynomial** $p$ in $x$ is a linear combination of monomials in $x$, i.e.,

$$p(x) = \sum_{\alpha} a_\alpha x^\alpha,$$

where the constants $a_\alpha \in \mathbb{R}$ are called **coefficients**. The (total) **degree** of a polynomial $p$, denoted by $\deg(f)$, is the maximum number $|\alpha| := \sum_{i=1}^{n} \alpha_i$ with nonzero coefficient $a_\alpha$.

**Definition 2.2** (SOS polynomial)**.** *A multivariate polynomial $p : \mathbb{R}^n \to \mathbb{R}$ in $x$ is said to be* ***SOS*** *(or to accept a SOS decomposition) if there are polynomials $g_1, ..., g_k$ (in $x$) such that*

$$p(x) = \sum_{i=1}^{k} g_i^2(x).$$

The subsequent theorem states an equivalent characterization of SOS polynomials.

**Theorem 2.5.** *A polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ of degree $2d$ is SOS if and only if there exists a positive semidefinite matrix $Q$ such that*

$$p(x) = Z^\top(x)QZ(x) \qquad \forall x \in \mathbb{R}^n \tag{2.4}$$

*where $Z$ is a vector of monomials in $x$ of degree less than or equal to $d$.*

Theorem 2.5 implies that finding a SOS decomposition for any polynomial $p$ can be recast as a semidefinite program, which is solved in polynomial time using interior-point methods [91]. This recasting process involves two steps. First, a suitable selection of $Z$ such that the monomials of $Z^\top QZ$ are enough to represent $p$. Note that the size of $Z$ is bounded by $\binom{n+d}{n}$.[4] In the second step, to fulfill (2.4), we compare all coefficients of the corresponding polynomials to extract constraints of the form

$$F(Q) = 0, \tag{2.5}$$

where $F$ is affine on the elements of $Q$. Now, the semidefinite program can be expressed as

$$\begin{array}{ll} \text{find} & Q \\ \text{subject to} & (2.5), \quad Q = Q^\top \quad \text{and} \quad Q \succeq 0. \end{array}$$

*Example* 2.1. Suppose we want to determine if the polynomial

$$p(x) = x_1^2x_2^4 - 2x_1^3x_2^3 + 2x_1^4x_2^2 - 2x_1x_2^5 + 2x_2^6$$

is SOS. Selecting $Z(x) = \text{vec}(x_1^2x_2, x_1x_2^2, x_2^3)$ gives

$$\begin{aligned} p(x) &= Z^\top(x) \underbrace{\begin{bmatrix} q_1 & q_2 & q_4 \\ q_2 & q_3 & q_5 \\ q_4 & q_5 & q_6 \end{bmatrix}}_{Q} Z(x) \\ &= q_1x_1^4x_2^2 + 2q_2x_1^3x_2^3 + q_3x_1^2x_2^4 + 2q_4x_1^2x_2^4 + 2q_5x_1x_2^5 + q_6x_2^6. \end{aligned}$$

[4]The maximum number of monomials required to identify a polynomial of degree $d$ in $n$ variables is $\binom{n+d}{n}$.

The previous equality holds for all $x \in \mathbb{R}^2$ if and only if

$$q_1 = 2, \qquad q_2 = -1, \qquad q_3 = 1 - 2q_4, \qquad q_5 = -1, \qquad q_6 = 2. \tag{2.6}$$

Consequently, $p$ is SOS if and only if we can find $Q \succeq 0$ verifying (2.6). Let $q_4 = 0$, then $Q$ is positive semidefinite and it admits the factorization $Q = L^\top L$ with

$$L = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Hence, $p(x) = Z^\top(x)L^\top LZ(x) = \|LZ(x)\|_2^2 = (x_1^2x_2 - x_1x_2^2 + x_2^3)^2 + (x_2^3 - x_1^2x_2)^2$. △

The Definition 2.2 and Theorem 2.5 can be naturally extended to polynomial matrices as follows.

**Definition 2.3** (Matrix SOS)**.** *A polynomial matrix $S : \mathbb{R}^n \to \mathbb{R}^{p\times p}$ is SOS if there exists a polynomial matrix $T$, not necessary square, such that*

$$S(x) = T^\top(x)T(x).$$

**Theorem 2.6.** *A polynomial matrix $S : \mathbb{R}^n \to \mathbb{R}^{p\times p}$ is SOS if and only if there exists a positive semidefinite matrix $Q$ such that*

$$S(x) = (I_p \otimes Z(x))^\top Q\,(I_p \otimes Z(x))\,, \tag{2.7}$$

*where $Z$ is a column of monomials and $\otimes$ represents the Kronecker product.*

We are ready to define **SOS programs** as convex optimization problems in the form:

$$\begin{array}{lll} \text{minimize} & b^\top y & \\ \text{subject to} & F_i(x,y) = 0 & i = \{1,2,\ldots,k\}\,, \\ & S_j(x,y) \text{ is SOS} & j = \{1,2,\ldots,l\}\,, \end{array}$$

where $y \in \mathbb{R}^m$ are the decision variables, $b \in \mathbb{R}^m$ is a constant, $F_i$ and $S_j$ are polynomial in $x$ but affine in $y$, and $S_j(x,y) = S_j^\top(x,y)$. Note, from Theorem 2.6, that linear matrix inequalities $P(y) \succeq 0$ can be included in a SOS program, because they belong to the class of matrix SOS.

While the recasting process (SOS to SDP) in Example 2.1 is feasible by hand, the overall situation is quite cumbersome. As a result, several software packages such as SOSTOOLS [122], SparsePOP [123], GpoSolver [124], and SOSOPT [125], to name but a few, have been developed. These implementations *i)* automate the process of conversion from a SOS program to a semidefinite program, *ii)* call a SDP solver, and *iii)* present the solution in the original SOS program. Besides, they take advantage of sparsity, symmetry and different representations to reduce computational cost and remove numerical ill-conditioning in the semidefinite program [126–129]. Although newer alternatives claim to be more efficient by solving the SOS decomposition without SDP [130, 131], these methods currently have no software package to ease the general SOS decomposition task.

### 2.2.2 Stability with SOS

With these tools in mind and given that every SOS polynomial (or polynomial matrix) is undoubtedly positive semidefinite, we can now replace any polynomial inequality with an SOS decomposition. However, this is just a sufficient condition, meaning that there might be positive semidefinite polynomials that are not SOS. In fact, Hilbert showed in [132] that aside from *i)* univariate polynomials, *ii)* quadratic polynomials or *iii)* bivariate quartics, there exist polynomials that fit into this group. The first historical example is the Motzkin polynomial

$$p(x) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6,$$

which is not SOS but its nonnegativity follows from the arithmetic-geometric inequality. A similar situation is observed for polynomial matrices [117]. We will not address the question of how conservative is decomposing a polynomial with SOS, but it is worth mentioning that some results suggest it is not too conservative [133, 134]. The proposition below restates the Barbashin-Krasovskii-LaSalle Corollary 2.1 under the SOS framework.

**Proposition 2.1.** *Consider the autonomous system* (2.1) *and the equilibrium point* $x^\star$*. Suppose there exists a polynomial function* $V : \mathbb{R}^{n_x} \to \mathbb{R}$ *with* $V(x^\star) = 0$ *such that*

$$V(x) - \Psi(x) \text{ is SOS} \tag{2.8a}$$

$$-\frac{\partial V}{\partial x} f(x) \text{ is SOS,} \tag{2.8b}$$

*where* $\Psi : \mathbb{R}^{n_x} \to \mathbb{R}$ *is a given polynomial function that is positive definite about* $x^\star$*. Then, the equilibrium is (Lyapunov) stable and every bounded solution* $x(\cdot)$ *approaches the largest invariant set of* (2.1) *contained in* $\left\{x \in \mathbb{R}^{n_x} \mid \frac{\partial V}{\partial x} f(x) = 0\right\}$*. If this invariant set posses no other than the trivial solution* $x(t) \equiv x^\star$*, then* $x^\star$ *is asymptotically stable. The stability properties are global if* $V$ *is also radially unbounded.*

**Proof.** From (2.8) and $\Psi(x)$ being positive definite about $x^\star$, we have

$$V(x) \geq \Psi(x) > 0 \qquad \forall x \in \mathbb{R}^{n_x} - \{x^\star\}\,.$$
$$\dot{V}(x) \leq 0 \qquad \forall x \in \mathbb{R}^{n_x}.$$

The proof follows from a straightforward application of Theorem 2.1 and Corollary 2.1. □

## 2.3 Manifolds and Vector Fields

Throughout Part III, we will work with mechanical systems described by DAEs. For those systems, we use implicitly defined manifolds as an intrinsic tool to handle kinematic constraints. In essence, manifolds are an abstract generalization of curves and surfaces to arbitrary dimensions with the property that locally, they are open subsets of Euclidean space. Simple examples are $n$-spheres, paraboloids, ellipsoids and also Euclidean Spaces. The purpose of this section is to provide a basic understanding of smooth manifolds, vector fields, distributions, integral manifolds and conservative vector fields. The content is mainly extracted from [135–139].

**Definition 2.4** (Manifolds)**.** *Let $\mathcal{M}$ be a Hausdorff* [5] ***topological space*** *with a countable basis of open sets (second-countable). A pair $(\mathcal{N}, \Psi)$ is called a **coordinate chart** on $\mathcal{M}$ if $\mathcal{N}$ is an open subset of $\mathcal{M}$ and $\Psi$ is a homeomorphism*[6] *from $\mathcal{N}$ to an open subset of a Banach space. The space $\mathcal{M}$ is said to be a **manifold** of dimension $m$ (or an $m$-dimensional manifold or just an $m$-manifold) if it is locally Euclidean of dimension $m$, i.e., for every point $p$ in $\mathcal{M}$ there exists a **coordinate chart** $(\mathcal{N}, \Psi)$ on $\mathcal{M}$ such that $p \in \mathcal{N}$ and $\Psi(\mathcal{N})$ is an open subset of $\mathbb{R}^m$.*

For simplicity, let us think of $\mathcal{M}$ as a **topological subspace**[7] of an Euclidean space, which means that $\mathcal{M}$ is Hausdorff and second-countable, see [135, Proposition A.17]. Then, $\mathcal{M}$ is a manifold if we can find a collection of charts whose domain covers $\mathcal{M}$.

*Example* 2.2 (Open subsets). Any open subset of $\mathbb{R}^n$ is a manifold of dimension $n$ because it is a topological subspace of $\mathbb{R}^n$ that is homeomorphic to $\mathbb{R}^n$ with a single coordinate chart. △

*Example* 2.3 (Unit circle). A more interesting case is the unit circle

$$\mathbb{S}^1 := \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = 1\right\},$$

which is also a topological subspace of $\mathbb{R}^2$. Here we can define the charts $(\mathcal{N}_1, \Psi_1)$ and $(\mathcal{N}_2, \Psi_2)$, where

$$\mathcal{N}_1 := \mathcal{M} - \{(-1, 0)\}, \qquad t_1 = \Psi_1(x, y) := \frac{y}{1 + x},$$

[5] A topological space $\mathcal{M}$ is Hausdorff if for each pair of different points $p$, $q \in \mathcal{M}$, there are open subsets $\mathcal{P}$, $\mathcal{Q} \in \mathcal{M}$ provided $p \in \mathcal{P}$, $q \in \mathcal{Q}$ and $\mathcal{P} \cap \mathcal{Q} = \emptyset$.

[6] A continuous map between two topological spaces is said to be a homeomorphism if it is bijective and its inverse is also continuous.

[7] Let $\mathcal{X}$ be a topological space with topology $\tau$. We say that $\mathcal{S}$ is a (topological) subspace of $\mathcal{X}$ if $\mathcal{S}$ is an arbitrary subset of $\mathcal{X}$ endowed with the subspace topology $\tau|_{\mathcal{S}} = \{\mathcal{U} \in \tau \mid \mathcal{U} \cap \mathcal{S}\}$. In other words, the topology of the subset $\mathcal{S}$ is inherited (or induced) from $\mathcal{X}$.

$$\mathcal{N}_2 := \mathcal{M} - \{(1,0)\}\,, \qquad t_2 = \Psi_2(x,y) := \frac{y}{1-x},$$

and $t_1$ stems from the intersection of the vertical line $x = 0$ and the line that passes through the points $(-1,0)$ and $(x,y)$, see Figure 2.1. The parametrization $t_2$ is obtained analogously with the point $(1,0)$. Since $\Psi_i$ are homeomorphisms and $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{M}$, then $\mathbb{S}^1$ is a 1-dimensional manifold. △

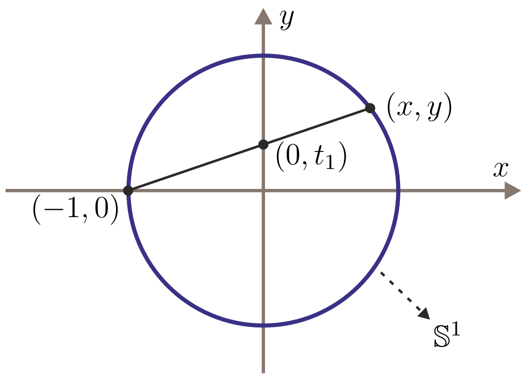



**Figure 2.1.** – Circle parametrization with $t_1$.

### 2.3.1 Smooth Manifolds and Implicitly Defined Submanifolds

When working with manifolds, we usually require to do calculus on them, for instance, integration or differentiation of a curve on a given manifold. Besides, manifolds are mostly defined as subsets of other manifolds, especially as level sets.[8] In this regard, we focus on the class of smooth manifolds, and in particular, on the regular (or embedded) submanifolds. Before presenting the main result, in Lemma 2.1, whereby we construct regular submanifolds from level sets, we introduce the required terminology.

**Definition 2.5** (Smooth manifolds)**.** *A manifold $\mathcal{M}$ is endowed with a smooth structure (or simply $\mathcal{M}$ is a smooth manifold) if for any two charts $(\mathcal{N}_i, \Psi_i)$ and $(\mathcal{N}_j, \Psi_j)$ on $\mathcal{M}$ with $\mathcal{N}_i \cap \mathcal{N}_j \neq \emptyset$, the* ***overlap map***

$$\Psi_i \circ \Psi_j^{-1} : \Psi_j(\mathcal{N}_i \cap \mathcal{N}_j) \to \Psi_i(\mathcal{N}_i \cap \mathcal{N}_j)$$

*is a* ***diffeomorphism****,[9] see Figure 2.2.*

**Definition 2.6** (Submanifolds)**.** *Let $\mathcal{M}$ be a smooth manifold and $\mathcal{K}$ be a subset of $\mathcal{M}$. A subset $\mathcal{K}$ is a $k$-dimensional* ***immersed submanifold*** *of $\mathcal{M}$ if $\mathcal{K}$ is also a smooth manifold*

[8] By a level set, we mean a set of the form $\mathcal{S}_c := \{x \in \mathcal{M} \mid \gamma(x) = c\}$ for some constant $c$ and mapping $\gamma : \mathcal{M} \to \mathcal{N}$

[9] A map $f$ between two smooth manifolds is called a diffeomorphism if it is a homeomorphism and both $f$ and its inverse are of class $\mathcal{C}^\infty$ (smooth).

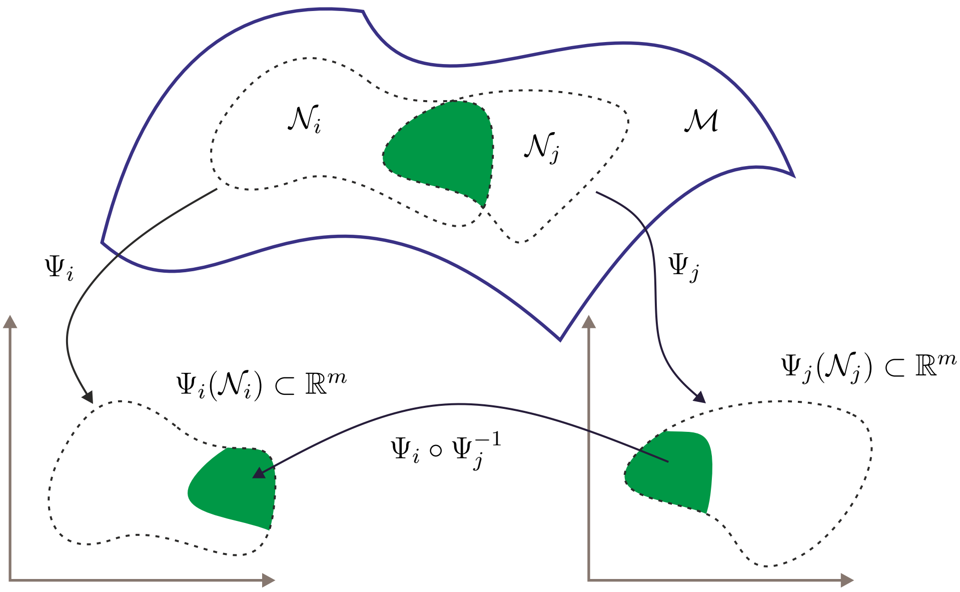



**Figure 2.2.** – Manifold $\mathcal{M}$ with a nonempty intersection of two coordinate charts.

*of dimension $k$. It is a $k$-dimensional* ***regular*** *(or* ***embedded****)* ***submanifold*** *of $\mathcal{M}$ if for every point $p \in \mathcal{K}$, there exists a chart $(\mathcal{N}, \Psi)$ of $\mathcal{M}$ with $p \in \mathcal{N}$ verifying*

$$\Psi(\mathcal{N} \cap \mathcal{K}) = \Psi(\mathcal{N}) \cap \left(\mathbb{R}^k \times \{0\}\right).$$

From the definitions, a smooth manifold can be understood as a manifold for which an arbitrary change of coordinates is always given by smooth functions. On the other hand, a regular submanifold is an immersed submanifold[10] that is also a topological subspace of the smooth manifold where it is contained, i.e., it inherits the subspace topology. For instance, the unit circle of Example 2.3 is a regular submanifold of $\mathbb{R}^2$ (see Example 2.4), whereas the **figure-eight** defined by $\{(\sin 2t, \sin t) \in \mathbb{R}^2 \mid t \in (-\pi, \pi)\}$ is an immersed submanifold of $\mathbb{R}^2$ but not a regular one because it is a smooth manifold subset of $\mathbb{R}^2$ without the subspace topology, see [135].

If $\mathcal{M}$ is a smooth $m$-manifold. The **tangent space** to $\mathcal{M}$ at the point $p \in \mathcal{M}$, denoted by $T_p\mathcal{M}$, is the linear subspace of dimension $m$ spanned by the **tangent vectors**[11] to all possible curves in $\mathcal{M}$ that passes through $p$. In other words, the tangent space is the linear approximation of a manifold at some point. Given a smooth mapping $\Phi : \mathcal{M} \to \mathcal{N}$, the **rank** of $\Phi$ at $p \in \mathcal{M}$ is the rank of the linear map $\mathrm{d}\Phi_p : T_p\mathcal{M} \to T_{\Phi(p)}\mathcal{N}$, i.e., the rank of the Jacobian matrix of $\Phi$ at $p$ bounded from above by the minimum dimension of $\mathcal{M}$ and $\mathcal{N}$. A **level set** $\mathcal{S}_c = \{p \in \mathcal{M} \mid \Phi(p) = c\}$ is called **regular** if $\mathrm{d}\Phi_p$ is surjective for each

[10] Immersed submanifolds are also known as smooth submanifolds.

[11] Let $\gamma : \mathcal{I} \subset \mathbb{R} \to \mathcal{M}$ be a $\mathcal{C}^1$ curve in a manifold $\mathcal{M}$ with $p = \gamma(t_p)$. The tangent vector to $\gamma$ at the point $p$ is the vector $\frac{\mathrm{d}\gamma}{\mathrm{d}t}\big|_{t=t_p}$.

$p \in \mathcal{S}_c$, meaning that the rank of $\Phi$ is equal to the dimension of $\mathcal{N}$ for all $p \in \mathcal{S}_c$. We are now ready to construct smooth manifolds from level sets.

**Lemma 2.1** (Implicit definition of submanifolds)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds of dimension $m$ and $n$, respectively, and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a smooth mapping. Every regular level set of $\Phi$ is a properly embedded submanifold of dimension $m - n$.*

*Example* 2.4 (Unit sphere of dimension $n$). Let us consider the unit $n$-sphere $\mathbb{S}^n := \left\{x \in \mathbb{R}^{n+1} \mid \|x\|_2^2 = 1\right\}$. Clearly, $\mathbb{S}^n$ is a level set where $\Phi$ can be selected as $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\Phi(x) := \frac{1}{2}\|x\|_2^2$. Given that Euclidean spaces are smooth manifolds and the Jacobian of $\Phi$ has rank 1 whenever $x \neq 0$, the level set $\mathbb{S}^n$ is a regular submanifold of $\mathbb{R}^{n+1}$ with dimension $n$. △

By using Lemma 2.1 in the $n$-sphere, not only did we give a much simpler proof that $\mathbb{S}^1$ is a manifold, but we also show that it is a smooth one embedded in $\mathbb{R}^2$.

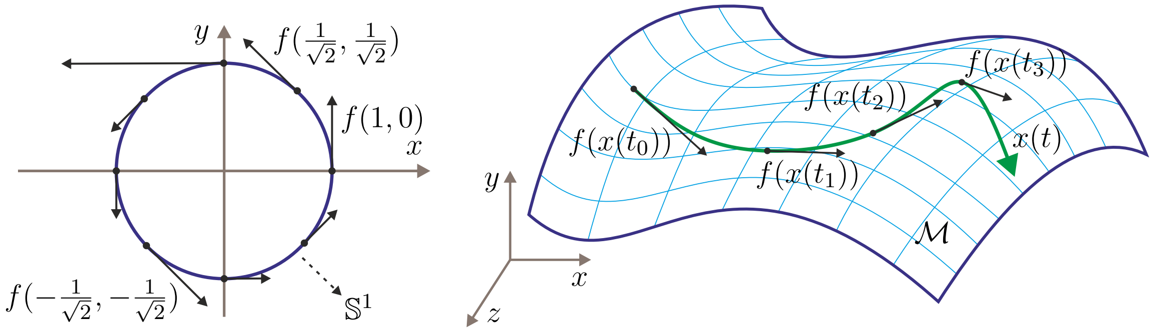
### 2.3.2 Vector Fields and Distributions

In its simplest form, a vector field in an open subset of Euclidean space is just a mapping that assigns a vector to each point in the subset. We will extend this idea to smooth manifolds as follows.

**Definition 2.7** (Vector fields and integral curves)**.** *Let $\mathcal{M}$ be a smooth manifold. A map $f$ is called a* ***vector field*** *on $\mathcal{M}$ if it assigns a tangent vector $f(x) \in T_x\mathcal{M}$ to every point $x \in \mathcal{M}$. A differentiable curve $x : \mathcal{I} \to \mathcal{M}$ with interval $\mathcal{I} \subset \mathbb{R}$ is said to be an* ***integral curve*** *of a vector field $f$ if it verifies $\dot{x}(t) = f(x(t))$ for all $t \in \mathcal{I}$.*

From Definition 2.7, we see that an integral curve $x(\cdot)$ of a vector field $f$ on a manifold $\mathcal{M}$ is just a solution of the dynamical system $\dot{x} = f(x)$ with the properties that $x(\cdot)$ remains in $\mathcal{M}$ and $f$ is the tangent vector to every solution. See Figure 2.3a for an example of a vector field on $\mathbb{S}^1$ and Figure 2.3b for an integral curve $x(\cdot)$ on a manifold $\mathcal{M}$. Given that $\mathcal{M}$ is locally diffeomorphic to Euclidean space and $f(x) \in T_x\mathcal{M}$, it can be demonstrated that the solution of the initial-value problem exists and is unique if $f$ is locally Lipschitz continuous for all $x \in \mathcal{M}$. This is essentially a restatement of the local existence and uniqueness theorem for ODEs. Observe now that an integral curve (generated by a vector field) is actually a 0- or 1-dimensional manifold. Similarly, we can extend this concept to higher dimensions in the definition below.

**Definition 2.8** (Distributions and integral manifolds)**.** *A* ***distribution*** *$\mathcal{D}$ on the manifold $\mathcal{M} \subset \mathbb{R}^n$ is a mapping that assigns to each $x \in \mathcal{M}$, a linear subspace $\mathcal{D}(x)$ of $T_x\mathcal{M}$. We say that $\mathcal{D}$ is* ***regular*** *if its dimension $\dim \mathcal{D}(x)$ remains constant for all $x \in \mathcal{M}$. An immersed submanifold $\mathcal{N}$ of $\mathcal{M}$ is called an* ***integral manifold*** *of the distribution $\mathcal{D}$ on*




**(a)** Vector field on $\mathbb{S}^1$ **(b)** Integral curve $x(\cdot)$ on $\mathcal{M}$ with $t_0 < t_1 < t_2 < t_3 < t$

**Figure 2.3.** – Subfigures showing a vector field and an integral curve.

*$\mathcal{M}$ if $T_x\mathcal{N} = \mathcal{D}(x)$ for every $x \in \mathcal{N}$. A distribution $\mathcal{D}$ on $\mathcal{M}$ is **integrable** if through each point $x \in \mathcal{M}$ there is an integral manifold of $\mathcal{D}$.*

In other words, a distribution is the generalization of a vector field while an integrable manifold is that of an integrable curve. Besides, we can **generate distributions** from a set of vector fields $f_1, f_2, \ldots, f_k$ on $\mathcal{M}$ as

$$\mathcal{D}_{f_1,\ldots,f_k}(x) := \operatorname{span}\{f_1(x), \ldots, f_k(x)\} \qquad \forall x \in \mathcal{M},$$

where $\dim \mathcal{D}_{f_1,\ldots,f_k}(x) = \operatorname{rank}\begin{bmatrix} f_1(x) & \ldots & f_k(x)\end{bmatrix}$ if $\mathcal{M}$ is an open subset of $\mathbb{R}^n$. We refer the reader to the textbooks [139–142] for more details on vector fields and distributions in ODEs and control. The above discussion raises an interesting question; namely, if every vector field is (locally) integrable, is every distribution also (locally) integrable? The answer to this question is given in Theorem 2.7 by requiring the involutive condition. A distribution $\mathcal{D}_{f_1,\ldots,f_k}$ on $\mathcal{M}$ is called **involutive** if $[f_i, f_j](x) \in \mathcal{D}_{f_1,\ldots,f_k}(x)$ for all $x \in \mathcal{M}$, where $[\cdot,\cdot]$ denotes the Lie bracket operator between two vector fields.

**Theorem 2.7** (Frobenius' Theorem)**.** *A distribution $\mathcal{D}$ is (completely) integrable if and only if it is involutive and regular.*

Perhaps the paramount application of Frobenius' Theorem 2.7 is in giving necessary and sufficient conditions for the solution existence of first-order homogeneous PDEs as presented in the next corollary.

**Corollary 2.2.** *Consider $B : \mathcal{X} \subset \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times m}$ with full rank $B(x)$. Let $\mathcal{D}_B(x)$ be a distribution generated by the columns of $B$, i.e., $\mathcal{D}_B(x) := \operatorname{Colsp} B(x)$. For every $x \in \mathcal{X}$, there exists a neighborhood $\mathcal{N}$ of $x$ such that the PDE*

$$\frac{\partial h}{\partial x}(x)B(x) = 0 \qquad\qquad \forall\, x \in \mathcal{N} \tag{2.9}$$

*has a solution in $h$ with rank $n_x - m$ if and only if the distribution $\mathcal{D}_B$ is involutive.*

### 2.3.3 Poincare's Lemma and Conservative Vector Fields

In few words, differential forms are alternating tensor fields on manifolds. They can be integrated over higher-dimensional submanifolds and generalize concepts such as the cross product, curl, divergence and Jacobian determinant. Given that there is a natural isomorphism between forms and vector fields, the **Poincare lemma**, which is technically formulated in the language of differential forms, provides an abstraction to all existing theorems of potential fields in multivariable calculus, namely the solutions $\phi$, $h$ and $g$ in the equations $\mathrm{grad}(\phi) = f$, $\mathrm{div}(h) = \alpha$ and $\mathrm{curl}(g) = k$, respectively [143]. Rather than presenting Poincare's Lemma together with a precise definition of differential forms, we concentrate on the particular case of solving the equation $\mathrm{grad}(\phi) = f$ (scalar potentials), thereby obtaining the well-known integrability condition on **conservative** (or **gradient**) **vector fields**, as stated in Lemma 2.2. An eager reader can consult [135, 144, 145] for a better understanding of differential forms.

**Lemma 2.2** (Gradient vector [23, 107])**.** *Let $f$ be a $\mathcal{C}^1$ (continuously differentiable) vector field on $\mathcal{X}$, an open subset of $\mathbb{R}^n$. There exists a function $\phi : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = \frac{\partial^\top \phi}{\partial x}$ if and only if the vector $f$ satisfies the integrability condition*

$$\frac{\partial f}{\partial x} = \frac{\partial^\top f}{\partial x} \qquad \forall x \in \mathcal{X}. \tag{2.10}$$

*If a function $\phi$ exists, then it can be calculated from the line integral*

$$\phi(x) = \int_0^x f^\top(\bar{x})\mathrm{d}\bar{x} + a = \int_0^1 f^\top(xs)x\mathrm{d}s + a,$$

*where $a \in \mathbb{R}$ is an arbitrary constant.*

## 2.4 Differential-algebraic Equations

When modeling, for example, physical systems with lumped-parameters (finite-dimensional), we usually get differential equations of the form[12]

$$F(\dot{y}, y) = 0, \tag{2.11}$$

where $y \in \mathcal{Y}$ are the states. If the Jacobian $\frac{\partial F}{\partial \dot{y}}$ is nonsingular, we can use the implicit function theorem to (locally) transform (2.11) into the **explicit ODE** $\dot{y} = f(y)$. However, if the Jacobian is singular, the transformation is not direct and (2.11) is said to be

[12]Non-autonomous systems fit into (2.11) by appending $t$ to $x$ and including $\dot{t} = 1$.

a **Differential-algebraic Equation (DAE)**. The term "algebraic" does not necessarily refer to polynomial equations but to non-differential ones.[13] The most fundamental types of DAEs in scientific or engineering problems are the **quasilinear DAEs**, given by

$$A(y)\dot{y} = B(y),$$

and the **semi-explicit DAEs**, given by

$$\dot{x} = g(x, z), \tag{2.12a}$$
$$0 = \Phi(x, z), \tag{2.12b}$$

where $A$, $B$, $g$ and $\Phi$ are functions of the states $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, and $A(y)$ is non-invertible. Relevant examples include electrical networks, finite dimensional multi-physic systems and mechanical systems with kinematic constraints [2, 147]. In the following, we briefly discuss the differential index of DAEs and the prevailing view of **DAEs as ODEs on manifolds**, which was pioneered by Rheinboldt in [148]. For a fuller treatment on DAEs from the "classical" (differential calculus and algebraic considerations) and geometric perspectives, the reader is referred to [73, 113, 149–152].[14]

### 2.4.1 Differentiation Index

The index is a classification tool for DAEs that measures a specific aspect of the system at hand. For example, the **perturbation index** measures the solution sensitivity of the perturbed system while the **differentiation index**, which we properly define below, measures the number of differentiations to render (2.11) explicit in $\dot{y}$. Although there are several notions of index (Kronecker, perturbation, tractability, to name but a few), we focus on the differentation index because it is the most popular classification tool for DAEs and is closely related with the geometric perspective.

**Definition 2.9.** *Consider the dynamical system* (2.11)*. Its* ***differentiation*** *(or* ***differential) index*** *is the minimal number $\mu$ such that, using only algebraic manipulations, the system of equations*

$$F(\dot{y}, y) = 0, \quad \frac{\mathrm{d}F}{\mathrm{d}t}(\dot{y}, y) = 0, \quad \ldots, \quad \frac{d^{\mu}F}{dt^{\mu}}(\dot{y}, y) = 0.$$

*can be transformed into an explicit ODE of the form $\dot{y} = f(y)$.*

[13] Some authors refer to (2.11) as an implicit ODE which is called regular or singular according to singularity of its Jacobian $\frac{\partial F}{\partial \dot{y}}$, see [146].

[14] Although the projector based analysis on DAEs [146] is an alternative approach with great relevance in the general theory, it does not take part of our discussions.

In other words, the differentiation index assumes that the nonlinear system (2.11) is eventually reducible to an ODE, called the **underlying ODE**, and that this transformation is accomplished in an iterative procedure with a finite number of steps (differentiations) denoted by $\mu$. Furthermore, after this reduction, the existence and uniqueness can be guaranteed with the usual Lipschitz continuity assumption.[15]

*Example* 2.5 (Index 1 semi-explicit DAEs). Let us consider the system (2.12) with nonsingular Jacobian $\frac{\partial \Phi}{\partial z}$. Note that (2.12a) is already explicit in $\dot{x}$ while (2.12b) is independent of $\dot{z}$. Thus, we differentiate (2.12b) w.r.t. the independent variable $t$ to obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\partial \Phi}{\partial x} g(x, z) + \frac{\partial \Phi}{\partial z} \dot{z} = 0. \tag{2.13}$$

Since $\frac{\partial \Phi}{\partial z}$ is invertible, the underlying ODE can be express as

$$\dot{x} = g(x, z), \tag{2.14a}$$

$$\dot{z} = -\left(\frac{\partial \Phi}{\partial z}\right)^{-1} \frac{\partial \Phi}{\partial x} g(x, z), \tag{2.14b}$$

meaning that our original system has differentiation index one. △

In Example 2.5, we assume that $\frac{\partial \Phi}{\partial z}$ is nonsingular. However, if it were singular, we would proceed by writing (2.12a), (2.13) in a semi-explicit form (e.g., by using coordinate change and algebraic manipulations) and differentiating the **hidden constraints**, which are the equations without $\dot{x}$ and $\dot{z}$.

*Example* 2.6 (Index 2 semi-explicit DAEs). Consider the system

$$\dot{x} = g(x, z), \tag{2.15a}$$

$$0 = \Phi(x), \tag{2.15b}$$

and assume $\frac{\partial \Phi}{\partial x} \frac{\partial g}{\partial z}$ is nonsingular. Differentiating (2.15b) gives the hidden constraint

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\partial \Phi}{\partial x} g(x, z) = 0. \tag{2.16}$$

In the next step, we differentiate this constraint and use the nonsingularity of $\frac{\partial \Phi}{\partial x} \frac{\partial g}{\partial z}$ to write the underlying ODE as

$$\dot{x} = g(x, z), \tag{2.17a}$$

$$\dot{z} = -\left(\frac{\partial \Phi}{\partial x} \frac{\partial g}{\partial z}\right)^{-1} \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial x} g(x, z)\right) g(x, z). \tag{2.17b}$$

[15] Geometrically, it can be shown that the transformation of DAEs into ODEs is indeed a reduction of manifolds. Consequently, the differentiation index is independent of the solution existence and uniqueness [113].

Hence, system (2.15) with nonsingular $\frac{\partial \Phi}{\partial x}\frac{\partial g}{\partial z}$ has index two. △

In Example 2.5, we observe that the ODE (2.14) ensures $\dot{\Phi}(x(t), z(t)) \equiv 0$ but not necessarily $\Phi(x(t), z(t)) \equiv 0$. Similarly, in Example 2.6, (2.17) ensures $\ddot{\Phi}(x(t), z(t)) \equiv 0$ but not $\dot{\Phi}(x(t), z(t)) \equiv 0$ or $\Phi(x(t)) \equiv 0$. This issue is solved with the mean value theorem by requiring that the initial conditions $(x(t_0), z(t_0))$ satisfy every constraint (including the hidden ones). Initial conditions with this property are said to be **consistent**.

### 2.4.2 DAEs on Manifolds

For sake of simplicity, let us consider the semi-explicit DAE of index 2 discussed in Example 2.6. By Lemma 2.1, the zero level set of $\Phi$ defines the regular submanifold

$$\mathcal{M} := \{x \in \mathcal{X} \mid \Phi(x) = 0\} \subset \mathbb{R}^{n_x}$$

of dimension $n_x - n_z$ whenever $\Phi$ is a smooth function and the Jacobian $\frac{\partial \Phi}{\partial x}$ has full rank on $\mathcal{M}$. Given that every solution $x(\cdot)$ must belong to $\mathcal{M}$, we deduce that $x(\cdot)$ is an integral curve of the vector field $g(x, z)$ on $\mathcal{M}$, i.e.,

$$g(x, z) \in T_x\mathcal{M} := \left\{ v \in \mathbb{R}^{n_x} \mid \frac{\partial \Phi}{\partial x} v = 0 \right\}.$$

Besides, from the implicit function theorem on (2.16), there exists (locally) a $\mathcal{C}^1$ function $\phi$ verifying $z = \phi(x)$, $\Phi(x, \phi(x)) = 0$ and that $\frac{\partial \phi}{\partial x} g(x, z)$ is equal to the right-hand side of (2.17b). In other words, $z$ can be interpreted as an **implicitly defined variable** that guarantees $g(x, z)$ being a vector field on $\mathcal{M}$. Figure 2.4 illustrates the situation with $g(x, z) := a(x) + b(x)z$. Now, if $\mathcal{M}$ is a smooth manifold, we have the following affirmations:

*i)* Every $\mathcal{C}^1$ solution $(x(\cdot), z(\cdot))$, which we assumed belongs to the open set $\mathcal{X} \times \mathcal{Z}$, is actually in the **reduced manifold** $\mathcal{X}_{\mathrm{R}} = \{(x, \phi(x)) \mid x \in \mathcal{M}\}$.

*ii)* The DAE (of Example 2.6) is in fact an ODE on $\mathcal{M}$ that is (locally) described by

$$\dot{x} = g(x, \phi(x)), \qquad z = \phi(x),$$

where $\mathcal{M}$ is called the **constrained state space**.

*iii)* For every point $x^\star$ in $\mathcal{M}$, there exists a coordinate chart $(\mathcal{N}^\star, \Psi^\star)$ on $\mathcal{M}$ with $x^\star \in \mathcal{N}^\star$ and **local coordinates** $\bar{x} := \Psi^\star(x)$ such that for all $x \in \mathcal{N}$ the ODE of Item *ii* is equivalent to

$$\dot{\bar{x}} = \frac{\partial \Psi^\star}{\partial x} g(x, \phi(x))\Big|_{x=\Psi^{\star-1}(\bar{x})}, \qquad z = \phi(\Psi^{\star-1}(\bar{x})).$$

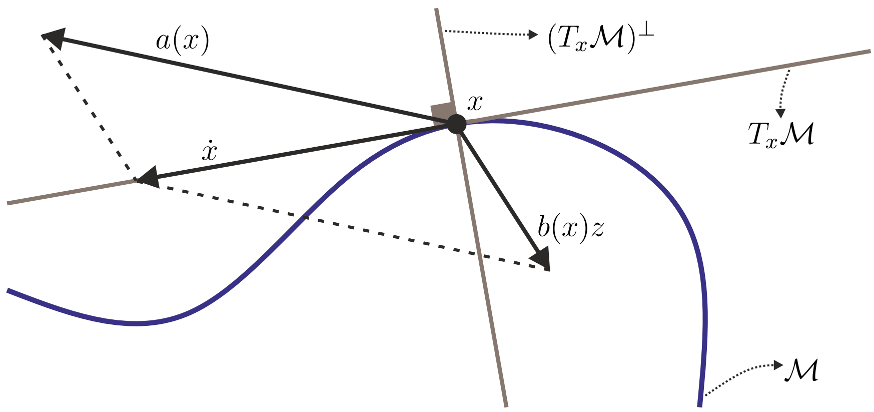



**Figure 2.4.** – Vector field $g(x, z) := a(x) + b(x)z \in T_x\mathcal{M}$.

## 2.5 Analytical Mechanics of Rigid Multibody Systems

A **rigid multibody system** is a collection of interconnected rigid bodies, where the interconnections are massless couplings and constraint elements. The approaches to model such systems can be classified into **Newton-Euler mechanics** and **analytical mechanics**. The former, as its name suggests, involves a recursive application of Newton's and Euler's laws. The latter has two dominant branches known as **Lagrangian** and **Hamiltonian mechanics**, which essentially use the kinetic and potential energy to derive motion equations. In this section, we focus on scleronomic (time independent [153]) mechanical systems and give an overview of the leading analytical methods under the presence of **kinematic constraints**.[16] For a more in-depth discussion on mechanics, the reader can consult the textbooks [154–158], and for a great emphasis on constraints, we refer to [10, 159–161].

### 2.5.1 Coordinates and Kinematic constraints

To describe the motion of a rigid multibody system, we start by specifying its geometry. More precisely, we select a set of **coordinates** (or **dynamical variables**) that fully defines the **system configuration**, which is the position and orientation of each body in the system. These coordinates, that throughout this document we denote with the vector $r \in \mathcal{R} \subset \mathbb{R}^{n_r}$, are non-unique and may also be subject to constrains of the form

$$\Phi(r) = 0 \qquad \forall r \in \mathcal{R}, \tag{2.18}$$

with $\Phi : \mathcal{R} \to \mathbb{R}^{n_\Phi}$, where $\mathcal{R}$ is an open and connected set. Any kinematic constraint that can be expressed as a functional relation between the coordinates alone, i.e., in the form of (2.18), is said to be **holonomic**.[17] Typical examples include joint constraints and

[16] A kinematic constraint is a functional relation between coordinates, velocities (or momenta) and time.
[17] Holonomic constraints are also called **geometric** because they impose a restriction on the system configuration.

distances between points of a rigid body. For simplicity, assume that the zero level set

$$\mathcal{R}_\Phi := \{r \in \mathcal{R} \mid 0 = \Phi(r)\}$$

is an immersed submanifold of $\mathcal{R}$ with dimension $n_q = n_r - n_\Phi$.[18] Then, $\mathcal{R}_\Phi$ is said to be a **configuration manifold** (or **space**) [162] for the given multibody system because each point in $\mathcal{R}_\Phi$ represents an admissible configuration. Although configuration manifolds are generally non-unique, they are all diffeomorphic, i.e., they share the same topology, which is inherent to the physical system and not to the coordinates we select to describe it. The dimension of $\mathcal{R}_\Phi$ represents the number of **DoF**, that is, the minimum number of independent (or constraint-free) coordinates that specify the system configuration. We call these minimal coordinates **generalized** and denote them by $q \in \mathcal{Q} \subset \mathbb{R}^{n_q}$, satisfying $r = \xi(q)$ for some surjective mapping $\xi : \mathcal{Q} \to \mathcal{R}_\Phi$.[19]

*Example* 2.7 (Simple Pendulum). Let us examine the simple one arm pendulum of Figure 2.5. It consists of a point mass $m$ attached at the end of a weightless bar (or rod) of length $l$. At the other end, there is a revolute joint constraining the mass position to a circle, which implies that the configuration manifold is $\mathbb{S}^1$ and that it can be parameterized by the angle $\theta \in \mathcal{Q} = \mathbb{R}$ (generalized coordinate). Alternatively, we can also select the Cartesian coordinates $r := \text{vec}(x_\text{p}, y_\text{p}) \in \mathcal{R} = \mathbb{R}^2$ on the position of $m$, obtaining the holonomic constraint $\Phi(x_\text{p}, y_\text{p}) := x_\text{p}^2 + y_\text{p}^2 - l^2 = 0$. Note that $\mathcal{R}_\Phi \cong \mathbb{S}^1$ while $\xi(\theta) := \text{vec}(l\cos\theta, l\sin\theta)$ is a surjection. △

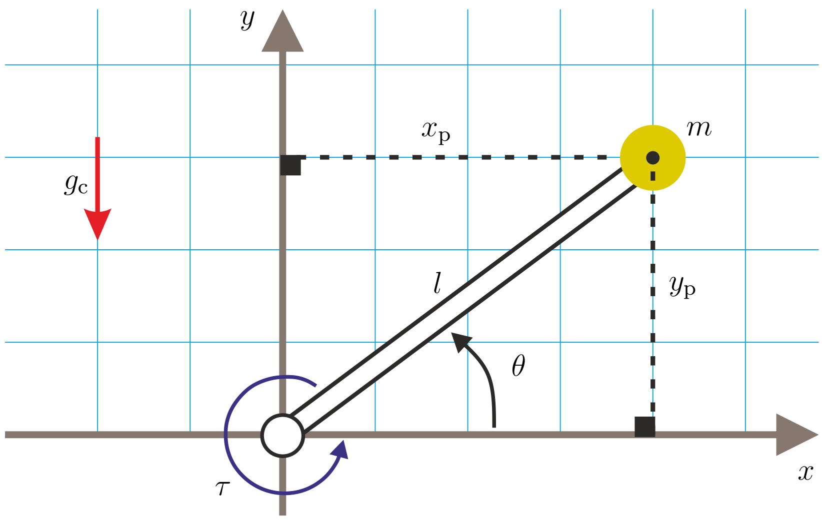



**Figure 2.5.** – Simple pendulum diagram.

[18] If the mapping $\Phi$ is smooth and has full rank in $\mathcal{R}_\Phi$, then $\mathcal{R}_\Phi$ is an embedded submanifold of $\mathcal{R}$ with dimension $n_r - n_\Phi$, see Lemma 2.1.

[19] We may require multiple sets of generalized coordinates to cover the whole configuration space.

Kinematic constraints that could not be represented as (2.18) are said to be **nonholonomic**. They have the property of restricting motion but not configuration. This is the case of inequality constraints $f(r, \dot{r}) \leq 0$ and constraints of the form $f(\dot{r}, r) = 0$ that cannot be integrated into (2.18). In this dissertation, we will consider nonholonomic constraints in Pfaffian form [163], namely

$$b^\top(r)\dot{r} = 0 \tag{2.19}$$

for some function $b$. Observe from Corollary 2.2 that the distribution $\mathcal{D}_b(r) := \operatorname{Null} b^\top(r)$ is not involutive. A common example of a nonholonomic constraint is the non-slip condition in wheeled robots.

**Definition 2.10.** *A **holonomic system** is a mechanical system whose constraints are all holonomic. Analogously, the system is **nonholonomic** if all its constraints are nonholonomic.*

### 2.5.2 Lagrangian Mechanics

The Lagrange equations of motion for rigid multibody systems have their fundamentals in **Hamilton's principle** for holonomic systems and the **Lagrange-d'Alembert formalism** for the general situation (holonomic and nonholonomic). These principles rely on the topology of the **configuration manifold** and are therefore independent of a specific set of coordinates.

To recall this concept, let us consider a rigid multibody system with coordinates $r \in \mathcal{R}$ (that uniquely defines the system configuration) subject to the holonomic constraints (2.18), defining an immersed submanifold $\mathcal{R}_\Phi$.

**Definition 2.11.** *The **Lagrangian** of a mechanical system is the difference between its kinetic and potential energy, namely*

$$\mathbf{L}(r, \dot{r}) := \frac{1}{2}\dot{r}^\top \mathbf{M}(r)\dot{r} - \mathbf{V}(r),$$

*where $\mathbf{V} : \mathcal{R} \to \mathbb{R}$ is the potential energy and $\mathbf{M} : \mathcal{R} \to \mathbb{R}^{n_r \times n_r}$ is the inertia matrix.*

The **Lagrange equations of the first kind** are given by

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial^\top \mathbf{L}}{\partial \dot{r}} - \frac{\partial^\top \mathbf{L}}{\partial r} + \frac{\partial^\top \mathbf{D}}{\partial \dot{r}} = \frac{\partial^\top \Phi}{\partial r}\lambda_\Phi + \mathbf{B}_{\mathrm{nh}}(r)\lambda_{\mathrm{nh}} + \mathbf{G}(r)u. \tag{2.20}$$

where $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the control input, $\mathbf{G} : \mathcal{R} \to \mathbb{R}^{n_r \times n_u}$ is the input matrix, $\mathbf{G}(r)u$ are the external control forces with the property that $\dot{r}^\top \mathbf{G}(r)u$ is power, $\mathbf{D} : \mathcal{R} \times \mathbb{R}^{n_r} \to \mathbb{R}$ is the **Rayleigh dissipation function**, which is usually defined by

$$\mathbf{D}(r, \dot{r}) := \dot{r}^\top \mathbf{R}(r)\dot{r} \tag{2.21}$$

with $\mathbf{R} : \mathcal{R} \to \mathbb{R}^{n_r \times n_r}$ being a positive semidefinite **dissipation matrix**, $\frac{\partial^\top \Phi}{\partial r}\lambda_\Phi$ are the **constraint forces** related to the holonomic constraints (2.18) while $\mathbf{B}_{\text{nh}}(r)\lambda_{\text{nh}}$ are the constraint forces related to the nonholonomic (or **non-integrable**) constraints

$$0 = \mathbf{B}_{\text{nh}}^\top(r)\dot{r} \qquad\qquad \forall r \in \mathcal{R},\, \dot{r} \in \mathbb{R}^{n_r}, \tag{2.22}$$

with $\mathbf{B}_{\text{nh}} : \mathcal{R} \to \mathbb{R}^{n_r \times n_{\text{nh}}}$, and $\lambda_\Phi \in \mathbb{R}^{n_\Phi}$ and $\lambda_{\text{nh}} \in \mathbb{R}^{n_{\text{nh}}}$ are **Lagrange multipliers** (or **implicit variables**). Observe that constraint forces are workless for any trajectory of the system, i.e.,

$$0 = \dot{r}^\top \frac{\partial^\top \Phi}{\partial r}\lambda_\Phi, \qquad\qquad 0 = \dot{r}^\top \mathbf{B}_{\text{nh}}(r)\lambda_{\text{nh}}$$

for every $\lambda_\Phi \in \mathbb{R}^{n_\Phi}$ and $\lambda_{\text{nh}} \in \mathbb{R}^{n_{\text{nh}}}$, which is consistent with the Lagrange-d'Alembert principle. By using the generalized coordinates $q \in \mathcal{Q}$ with $r = \xi(q)$, we can eliminate the redundant coordinates as well as the explicit appearance of constraint forces stemming from holonomic constraints. This results in the **Lagrange equations of the second kind** (also known as **Euler-Lagrange equations**)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^\top L}{\partial \dot{q}} - \frac{\partial^\top L}{\partial q} + \frac{\partial^\top D}{\partial \dot{q}} = B_{\text{nh}}\bar{\lambda}_{\text{nh}} + G(q)u, \tag{2.23}$$

where $L(q,\dot{q}) = \mathbf{L}(\xi(q), \frac{\partial \xi}{\partial q}\dot{q}) = \frac{1}{2}\dot{q}^\top \frac{\partial^\top \xi}{\partial q}\mathbf{M}(\xi(q))\frac{\partial \xi}{\partial q}\dot{q} - \mathbf{V}(\xi(q)) = \frac{1}{2}\dot{q}^\top M(q)\dot{q} - V(q)$, $G(q) = \frac{\partial^\top \xi}{\partial q}\mathbf{G}(\xi(q))$, $D(q,\dot{q}) = \mathbf{D}(\xi(q), \frac{\partial \xi}{\partial q}\dot{q}) = \dot{q}^\top \frac{\partial^\top \xi}{\partial q}\mathbf{R}(\xi(q))\frac{\partial \xi}{\partial q}\dot{q} = \dot{q}^\top R(q)\dot{q}$, $B_{\text{nh}}(q) = \frac{\partial^\top \xi}{\partial q}\mathbf{B}_{\text{nh}}(\xi(q))$, and (2.22) reduces to

$$0 = B_{\text{nh}}^\top(q)\dot{q} \qquad \forall q \in \mathcal{Q},\, \dot{q} \in \mathbb{R}^{n_q}. \tag{2.24}$$

We will not entail the debate of which model is better than the other, but we may point out that Euler-Lagrange equations are mainly used in control because they constitute a system ODEs in Euclidean space, and this situation also extends to systems with nonholonomic constraints, given that they can take the form of ODEs as well, see [3]. In contrast, the Lagrange equations of the first kind, which are a system of DAEs, are preferred in the simulation of complex systems because it might be impossible to obtain a set of generalized coordinates $q$ that is valid for the whole configuration space and $q$ do not necessarily have a physical significance [164]. We conclude this section, by providing a classification of mechanical systems in the following definition.

**Definition 2.12.** *A multibody system described by* (2.20) *is called* ***fully-actuated*** *if*

$$\operatorname{rank}\begin{bmatrix}\mathbf{G}(r) & \frac{\partial^\top \Phi}{\partial r} & \mathbf{B}_{\text{nh}}(r)\end{bmatrix} = n_r \qquad \forall r \in \mathcal{R}_\Phi,$$

*or in the representation* (2.23)*, if*

$$\operatorname{rank}\begin{bmatrix}G(q) & B_{\mathrm{nh}}(q)\end{bmatrix} = n_r - n_\Phi \qquad \forall q \in \mathcal{Q}.$$

*A system is said to be* ***underactuated*** *if it is not fully-actuated.*

Definition 2.12 is of great relevance in control because if a system is fully-actuated, we can arbitrarily modify the acceleration vector $\ddot{r}$ (or equivalently $\ddot{q}$) provided the resultant trajectory is consistent with the kinematic constraints. In the situation without nonholonomic constraints, this implies that $\ddot{q}$ can take any value, but $\ddot{r}$ must still agree with (2.18). On the other hand, if the system is underactuated, the assignment of $\ddot{r}$ (or $\ddot{q}$) is less flexible, meaning that the control law is far from trivial and must be carefully selected.

### 2.5.3 Hamiltonian Mechanics

The Lagrange equations presented above describe the behavior of a mechanical system with respect to its coordinates ($r$ or $q$) and velocities ($\dot{r}$ or $\dot{q}$). Similarly, in Hamiltonian mechanics, we derive the equations of motion but with the velocity variables replaced by their corresponding **conjugate momenta**

$$\rho := \frac{\partial^\top \mathbf{L}}{\partial \dot{r}} = \mathbf{M}(r)\dot{r}, \qquad p := \frac{\partial^\top L}{\partial \dot{q}} = M(q)\dot{q}.$$

Performing the above change of coordinates in the Lagrange equations (a process known as **Legendre transformation**) is always feasible if the Lagrangian is **regular** (or **nondegenerate**), i.e., if $\mathbf{M}$ is nonsingular. As a result, system (2.20) with Rayleigh dissipation (2.21) can be rewritten as

$$\begin{bmatrix}\dot{r}\\ \dot{\rho}\end{bmatrix} = \begin{bmatrix}0 & I_{n_r}\\ -I_{n_r} & -\mathbf{R}(r)\end{bmatrix}\begin{bmatrix}\frac{\partial^\top \mathbf{H}}{\partial r}\\ \frac{\partial^\top \mathbf{H}}{\partial \rho}\end{bmatrix} + \begin{bmatrix}0\\ \mathbf{B}(r)\end{bmatrix}\lambda + \begin{bmatrix}0\\ \mathbf{G}(r)\end{bmatrix}u, \tag{2.25a}$$

while the nonholonomic constraints (2.22) take the form

$$0 = \mathbf{B}_{\mathrm{nh}}^\top(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}, \tag{2.25b}$$

where $\mathbf{B}(r) := \begin{bmatrix}\frac{\partial^\top \Phi}{\partial r} & \mathbf{B}_{\mathrm{nh}}(r)\end{bmatrix}$, $\lambda = \operatorname{vec}(\lambda_\Phi, \lambda_{\mathrm{nh}})$ and $\mathbf{H} : \mathcal{R} \times \mathbb{R}^{n_r} \to \mathbb{R}$ is the **Hamiltonian** function defined as the total mechanical energy (kinetic plus potential), namely

$$\mathbf{H}(r, \rho) = \frac{1}{2}\rho^\top \mathbf{M}^{-1}(r)\rho + \mathbf{V}(r).$$

Similarly, system (2.23) and equations (2.24) can be expressed as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_{n_q} \\ -I_{n_q} & -R(q) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top H}{\partial q} \\ \frac{\partial^\top H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{\mathrm{nh}}(q) \end{bmatrix} \bar{\lambda}_{\mathrm{nh}} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u, \tag{2.26}$$
$$H(q,p) = \mathbf{H}(\xi(q), \mathbf{M}(\xi(q))\frac{\partial \xi}{\partial q} M^{-1}(q)p) = \frac{1}{2}p^\top M^{-1}(q)p + V(q),$$
$$0 = B_{\mathrm{nh}}^\top(q)\frac{\partial^\top H}{\partial p}.$$

Such equivalence between (2.25a) and (2.26) allows, e.g., to design a controller using one representation and to implement it using the other. For a thorough geometric analysis of the equivalence without nonholonomic constraints, the reader can consult [72].

### 2.5.4 Mechanical Systems as DAEs

We will apply the DAE concepts of Section 2.4 to mechanical systems with kinematic constraints (2.18) and (2.25b), whose motion is described by the Hamiltonian equations (2.25a). We exclude the analysis of (2.26) as it obeys the same procedure. Differentiating the holonomic constraints (2.18) w.r.t. time and rearranging with (2.25b), yields the **momentum level constraints**

$$0 = \mathbf{B}^\top(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}. \tag{2.27}$$

Differentiating them, results in the **hidden constraints**

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{B}^\top(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}\right) = \frac{\partial \mathbf{B}^\top \mathbf{M}^{-1}\rho}{\partial r}\dot{r} + \mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\dot{\rho} \\ &= \frac{\partial \mathbf{B}^\top \mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^\top \mathbf{H}}{\partial \rho} + \mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\left(-\frac{\partial^\top \mathbf{H}}{\partial r} - \mathbf{R}(r)\frac{\partial^\top \mathbf{H}}{\partial \rho} + \mathbf{G}(r)u + \mathbf{B}(r)\lambda\right). \end{aligned} \tag{2.28}$$

Hence, $\lambda$ has a unique solution from (2.28) if $\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ is nonsingular. Now, following Hairer and Wanner [149], the system has differentiation index three if we consider holonomic constraints and index two without them.[20] Note that if $\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ is singular, $\lambda$ could still be calculated but we will require additional derivatives in the constraints, i.e., the index is higher. The equivalent ODE on a manifold $\mathcal{X}_\mathrm{c}$ is thus given by (2.25a) with $\lambda$ calculated from the hidden constraints (2.28). The manifold $\mathcal{X}_\mathrm{c}$, which is known as the **constrained state space**, is now defined as

$$\mathcal{X}_\mathrm{c} := \left\{(r,\rho) \in \mathcal{R} \times \mathbb{R}^{n_r} \mid 0 = \Phi(r), 0 = \mathbf{B}^\top(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}\right\}.$$

[20] Our result agrees with [149] but differs from van der Schaft's [78] because of the chosen index.

Eventually, the solution existence and uniqueness of (2.25a) over an interval $\mathcal{I} \subset \mathbb{R}$ is reduced to the standard Lipschitz condition on the underlying ODE.

## 2.6 Dissipativity

Dissipative systems are dynamical systems for which its stored "energy" is less than or equal to the supplied one.[21] In other words, there is energy dissipation along the system trajectories and thus the name **dissipative**. Typical examples include mechanical and electrical systems, where the dissipation stems from friction and electrical resistance. This section gives a precise definition of dissipativity and passivity to then focus on stability and port-Hamiltonian systems. For a more in-depth discussion on the topic, the reader can consult [2, 3, 5, 165–167] from where most of the content is drawn.

### 2.6.1 Dissipative and Passive Systems

Let us consider a dynamical system of the form

$$\dot{x} = f(x) + g(x)u, \tag{2.29a}$$

$$y = h(x, u), \tag{2.29b}$$

where $x \in \mathcal{X} \subset \mathbb{R}^{n_x}$ are the states, $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the input and $y \in \mathcal{Y} \subset \mathbb{R}^{n_y}$ is the output. Besides, we assume that (2.29a) has a unique global solution to the initial-value problem.

**Definition 2.13** (Dissipativity)**.** *A system (2.29) is said to be* ***dissipative*** *with respect to the* ***supply rate*** *$\omega : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ if there exists a positive semidefinite function $S : \mathcal{X} \to \mathbb{R}$, called the* ***storage function****, such that*

$$S(x(T)) - S(x(t_0)) \leq \int_{t_0}^{T} \omega(u(t), y(t)) \mathrm{d}t < \infty \tag{2.30}$$

*for every $x(t_0) = x_0 \in \mathcal{X}$, $u \in \mathcal{U}$ and $T \geq t_0$. If (2.30) holds but $S$ is not necessarily positive definite, then (2.29) is called* ***cyclo-dissipative****.*

Inequality (2.30), which is called the **dissipation inequality**, states that the system's energy can only be increased with the supply rate $\omega$, namely a function of the input $u$ and output $y$. When $S$ is differentiable, (2.30) can be written as

$$\frac{\partial S}{\partial x} f(x) + \frac{\partial S}{\partial x} g(x) u \leq \omega(u, y) \qquad \forall x \in \mathcal{X},\, u \in \mathcal{U}, \tag{2.31}$$

[21] By energy, we do not necessarily refer to the physical property but the abstract generalization.

which is known as the **differential dissipation inequality**. Condition (2.31) is mostly used for stability analysis and control purposes because no knowledge of the system's trajectories is required. The storage function $S$ is not unique, and the set of all possible storage functions is convex. In fact, there is a function

$$S_{\mathrm{a}}(x_0) := \sup_{T\geq t_0,\, u\in\mathcal{U}} -\int_{t_0}^{T} \omega(u(t), y(t))\mathrm{d}t \geq 0 \qquad x(t_0) = x_0 \in \mathcal{X},$$

called the **available storage**, which can be understood as the maximum extractable energy of a dissipative system in the state $x_0$. The function $S_{\mathrm{a}}$ is itself a storage function and verifies

$$S(x) \geq S_{\mathrm{a}}(x) \qquad x \in \mathcal{X},$$

for any other storage function $S$ with the same supply rate, see [5, 167].

**Definition 2.14** (Passivity)**.** *System* (2.29) *with* $n_y = n_u$ *is said to be*

- ***passive*** *if it is dissipative w.r.t.* $\omega(u, y) := y^\top u$*,*
- ***output strictly passive*** *if it is dissipative w.r.t.* $\omega(u, y) := y^\top u - \delta(y)$ *for some positive definite function* $\delta : \mathcal{Y} \to \mathbb{R}$*, and*
- ***lossless*** *if it is passive and* (2.30) *holds with equality.*

***Cyclo-passive*** *systems are defined analogously.*

The differential dissipation inequality and Definition 2.14 lead to the following proposition.

**Proposition 2.2.** *Consider* (2.29) *without* ***throughput*** *(or a* ***feedthrough term****), i.e.,* $y = h(x)$*. The system is passive with a* $\mathcal{C}^1$ *storage function* $S$ *if and only if*

$$\frac{\partial S}{\partial x} f(x) \leq 0 \quad \text{and} \quad \frac{\partial S}{\partial x} g(x) = h(x) \qquad \forall x \in \mathcal{X}.$$

**Proof.** See [3, Proposition 4.1.2 and Corollary 4.1.5]. □

The conditions of Proposition 2.2 provide a characterization of input affine passive systems and represent the nonlinear version of the celebrated Kalman-Yakubovich-Popov lemma (see, e.g. [110]). As discussed by van der Schaft in [3], the definitions of dissipativity and passivity are not limited to ODEs, and they can be easily extended to dynamical systems with DAEs of the form

$$0 = F(\dot{x}, x, u), \tag{2.32a}$$

$$y = h(x, u), \tag{2.32b}$$

where state, input and output spaces remain as in (2.29). Here, the solutions $x(\cdot)$, which are assumed to belong to the open set $\mathcal{X}$, are actually in the reduced manifold $\mathcal{X}_\mathrm{R} \subset \mathcal{X} \subset \mathbb{R}^{n_x}$, see Section 2.4.2. In this situation the dissipation inequality (2.30) and $S(x) \geq 0$ are only required to hold in $\mathcal{X}_\mathrm{R}$.

### 2.6.2 Stability of Passive Systems and Damping Injection

A passive system is not inherently Lyapunov stable.[22] For this relation to hold we require additional conditions as demonstrated in Lemma 2.3. First, however, the following terminology is required. A system (2.32) with zero input ($u = 0$) is said to be **zero-state detectable** about the state $x^\star$ if, for every $x(t_0) \in \mathcal{X}$,

$$0 = h(x(t)) \qquad \forall t \geq t_0 \qquad \Longrightarrow \qquad \lim_{t\to\infty} x(t) = x^\star .$$

It is called **zero-state observable** (about $x^\star$) if, for every $x(t_0) \in \mathcal{X}$,

$$0 = h(x(t)) \qquad \forall t \geq t_0 \qquad \Longrightarrow \qquad x(t) \equiv x^\star .$$

The properties are said to be **local** if they are valid for any initial condition $x(t_0)$ in a neighborhood of $x^\star$.

**Lemma 2.3** (Stability of passive systems)**.** *Let the system* (2.29) *be passive with a* $\mathcal{C}^1$ *storage function* $S$*. Let* $x^\star$ *be an equilibrium point of* (2.29a) *with zero input (*$u = 0$*), i.e.* $f(x^\star) = 0$*. Let* $h$ *be* $\mathcal{C}^1$ *in* $u$ *for all* $x$*,* $h(x^\star, 0) = 0$ *and* $S(x^\star) = 0$*. The following statements hold for* (2.29a) *with zero input.*

*i) The equilibrium is stable if* $S$ *is positive definite about* $x^\star$*.*

*ii) The equilibrium is stable if* (2.29) *is (locally) zero-state detectable about* $x^\star$*.*

*iii) The equilibrium is asymptotically stable if* $S$ *is positive definite about* $x^\star$*, and* $x(t) \equiv x^\star$ *is the largest invariant set of* (2.29a) *with zero input contained in*

$$\left\{ x \in \mathcal{X} \mid \frac{\partial S}{\partial x} f(x) = 0 \right\} .$$

*iv) Suppose* (2.29) *is output strictly passive. The equilibrium is asymptotically stable if and only if* (2.29) *is (locally) zero-state detectable about* $x^\star$*.*

*Moreover, the properties are global if* $S$ *is radially unbounded and the detectability is not only local.*

[22] Consider the linear system $\dot{x}_1 = -x_1 + u$, $\dot{x}_2 = x_2 + u$ with storage function $S = \frac{1}{2}x_1^2$ and output $y = x_1$. Clearly the system with zero input is unstable, but $\dot{S} = -x_1^2 + x_1 u \leq yu$ which implies passivity.

**Proof.** The stability proof of *iii* follows from a direct application of the Barbashin-Krasovskii-LaSalle Corollary 2.1. The proof of *i*, *ii* and *iv* is along the same lines as [5, Proof of Theorem 2.28]. □

Lemma 2.3 is fundamental in Passivity-based Control (PBC), and in our case in IDA-PBC. The Theorem also supports the stronger condition of zero-state observability instead of detectability. The main idea for the controller design is to transform a nonlinear system into a passive system with a positive definite storage $S$ to have stability in $x^\star$, and then inject damping and use statements *iii* or *iv* to demonstrate asymptotic stability of $x^\star$. Clearly, Lemma 2.3 can be modified to obtain convergence to a set rather than an equilibrium point, but we will not prove this fact here. By **damping injection** we mean the **negative feedback interconnection** (see Figure 2.6) of a passive system without throughput and a locally Lipschitz function $\Psi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^{n_y}$ with $y^\top \Psi(x, y) \geq 0$, called **nonlinear damping**. From the dissipation inequality (2.30) it can be seen that any passive system with damping injection remains passive w.r.t the same storage function:

$$\begin{aligned} S(x(T)) - S(x(t_0)) &\leq \int_{t_0}^{T} y^\top(t)u(t)\mathrm{d}t = \int_{t_0}^{T} y^\top(t)\bar{u}(t)\mathrm{d}t - \int_{t_0}^{T} y^\top(t)\Psi(x(t), y(t))\mathrm{d}t \\ &\leq \int_{t_0}^{T} y^\top(t)\bar{u}(t)\mathrm{d}t. \end{aligned}$$

Furthermore, the closed-loop is rendered output strictly passive if there is a positive definite function $\delta : \mathcal{Y} \to \mathbb{R}$ such that $y^\top \Psi(x, y) \geq \delta(y)$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$:

$$\begin{aligned} S(x(t)) - S(x(t_0)) &\leq \int_{t_0}^{T} y^\top(t)u(t)\mathrm{d}t = \int_{t_0}^{T} y^\top(t)\bar{u}(t)\mathrm{d}t - \int_{t_0}^{T} y^\top(t)\Psi(x(t), y(t))\mathrm{d}t \\ &\leq \int_{t_0}^{T} y^\top(t)\bar{u}(t)\mathrm{d}t - \int_{t_0}^{T} \delta(y(t))\mathrm{d}t. \end{aligned}$$

In other words, any passive system can be converted into output strictly passive. Observe from *iii* in Lemma 2.3 that if (2.29) is output strictly passive, then

$$\left\{x \in \mathcal{X} \mid \frac{\partial S}{\partial x} f(x) = 0\right\} \subset \{x \in \mathcal{X} \mid h(x) = 0\}$$

with the zero-state detectability property implies that the largest invariant set of (2.29a) contained in $\left\{x \in \mathcal{X} \mid \frac{\partial S}{\partial x} f(x) = 0\right\}$ is $\{x^\star\}$, which is consistent with statement *iv*.

### 2.6.3 Port-Hamiltonian Systems

In a nutshell, port-Hamiltonian systems are cyclo-passive systems defined by a **Dirac structure** and two types of components: **energy-storing** and **energy-dissipating**. The energy-storing components are grouped into a single energy function $H : \mathcal{X} \to \mathbb{R}$, called

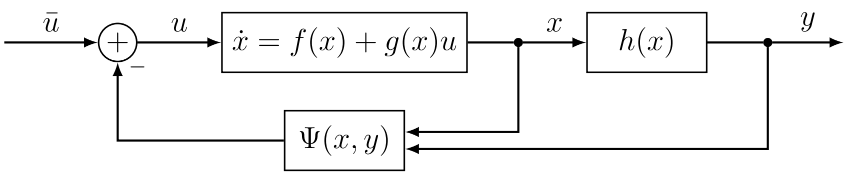



**Figure 2.6.** – Feedback interconnection of (2.29) with $-\Psi$.

the **Hamiltonian**. The energy-dissipating ones are constituted by static resistive elements. And, the Dirac structure is a geometric and power-conserving structure that **interconnects** the system components. Rather than taking the geometric viewpoint of Dirac structures—fundamental for a coordinate-free analysis of port-Hamiltonian systems, see [2, 78]—we employ the explicit and implicit input-state-output representations [3, 168].

**Definition 2.15.** *An **(explicit) input-state-output port-Hamiltonian system** is a system of the form*[23]

$$\dot{x} = W(x)\frac{\partial^\top H}{\partial x} + g(x)u, \tag{2.33a}$$

$$y = g^\top(x)\frac{\partial^\top H}{\partial x}, \tag{2.33b}$$

*where $x \in \mathcal{X} \subset \mathbb{R}^{n_x}$ are the states, $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the input (port), $y \in \mathcal{Y} = \mathcal{U}$ is the output (port), $g : \mathcal{X} \to \mathbb{R}^{n_x \times n_u}$ is the input matrix, $H : \mathcal{X} \to \mathbb{R}$ is the Hamiltonian or energy function, and $W : \mathcal{X} \to \mathbb{R}^{n_x \times n_x}$ fulfills the **energy-balance relation***

$$\frac{\partial H}{\partial x}W(x)\frac{\partial^\top H}{\partial x} \leq 0 \qquad \forall x \in \mathcal{X}. \tag{2.34}$$

Note that the time derivative of $H$ along the system's trajectories is

$$\dot{H}(x) = \frac{\partial H}{\partial x}\dot{x} = \frac{\partial H}{\partial x}\left(W(x)\frac{\partial^\top H}{\partial x} + g(x)u\right) \leq \frac{\partial H}{\partial x}g(x)u = y^\top u,$$

which implies cyclo-passivity, and passivity if $H(x) \geq 0$ for all $x \in \mathcal{X}$. The skew-symmetric portion of $W$ conserves power and is called **interconnection matrix**, whereas the symmetric portion (of $W$) dissipates energy and is thereby called **dissipation matrix**.

**Definition 2.16.** *An **implicit input-state-output port-Hamiltonian system** is a system of the form*

$$\dot{x} = W(x)\frac{\partial^\top H}{\partial x} + g(x)u + b(x)\lambda, \tag{2.35a}$$

$$0 = b^\top(x)\frac{\partial^\top H}{\partial x}, \tag{2.35b}$$

[23]When referring to this class of systems, the word "explicit" is usually omitted.

$$y = g^\top(x) \frac{\partial^\top H}{\partial x}, \tag{2.35c}$$

*where $x$, $u$, $y$ and $H$ are given as in Definition 2.15. Besides, $\lambda \in \mathbb{R}^{n_\lambda}$ is the implicit variable,* (2.35b) *represents the independent constraints with constant rank $b : \mathcal{X} \to \mathbb{R}^{n_x \times n_\lambda}$ and $W : \mathcal{X} \to \mathbb{R}^{n_x \times n_x}$ verifies the **energy-balance relation*** (2.34) *but for $x$ in the constrained state space $\mathcal{X}_\mathrm{c} \subset \mathcal{X}$.*

Similar to the explicit situation, the time derivative of $H$ along the trajectories is

$$\dot{H}(x) = \frac{\partial H}{\partial x}\left(W(x)\frac{\partial^\top H}{\partial x} + g(x)u + b(x)\lambda\right) = \frac{\partial H}{\partial x}W(x)\frac{\partial^\top H}{\partial x} + \frac{\partial H}{\partial x}g(x)u \leq y^\top u,$$

which implies cyclo-passivity, and passivity if $H(x) \geq 0$ for all $x \in \mathcal{X}_\mathrm{c}$. The implicit port-Hamiltonian system (2.35a)–(2.35b) is a system of DAEs and constitutes a generalization of the Hamiltonian equations for mechanical systems, see Section 2.5.3.

# II

# Polynomial Systems

# Chapter 3

# IDA-PBC for Systems with Polynomial Structure

Most nonlinear affine systems can be locally approximated or even parameterized with polynomial or rational functions, potentially simplifying their analysis and controller design. The purpose of this chapter is to introduce algebraic solutions for the well-known Interconnection and Damping Assignment Passivity-based Control (IDA-PBC) in a class of affine systems with polynomial structure. These classes do not involve solving Partial Differential Equations (PDEs), and the proposed method leads to conditions that can be recast as a Sum of Squares (SOS) program. The chapter is divided into two sections. In Section 3.1, we provide the general formulation of IDA-PBC for nonlinear affine systems. And, in Section 3.2, we introduce the algebraic solutions analyzing the region of convergence and some particular parameterizations for the desired Hamiltonian function.

## 3.1 Nonlinear Affine Systems

Generally speaking, the IDA-PBC aims to stabilize a nonlinear system by transforming it into a **desired** (or **target**) port-Hamiltonian system at the cost of satisfying the matching and stabilizing conditions.

### 3.1.1 Preliminaries

Before presenting the IDA-PBC, let us introduce the required concepts and terminology used extensively throughout this dissertation.

**Definition 3.1** (Annihilators). *Given $g : \mathcal{X} \to \mathbb{R}^{n_x \times n_u}$, a function $g_\perp$ with domain in $\mathcal{X}$ is called the **left annihilator** of $g$ if* [24]

$$\operatorname{Rowsp} g_\perp(x) = (\operatorname{Colsp} g(x))^\perp \qquad \forall x \in \mathcal{X},$$

*that is, $g_\perp(x)g(x) \equiv 0$ and $\operatorname{rank} g_\perp(x) + \operatorname{rank} g(x) \equiv n_x$. Similarly, a function $g_\top$ with domain in $\mathcal{X}$ is said to be the **right annihilator** of $g$ if*

$$\operatorname{Colsp} g_\top(x) = (\operatorname{Rowsp} g(x))^\perp \qquad \forall x \in \mathcal{X},$$

*that is, $g(x)g_\top(x) \equiv 0$ and $\operatorname{rank} g_\top(x) + \operatorname{rank} g(x) \equiv n_u$.*

For convenience, we will always consider $g_\perp(x)$ as a full rank matrix unless $\operatorname{rank} g(x) = n_x$, in which case $g_\perp$ is a zero matrix.[25] An analogous argument applies to $g_\top$.

**Definition 3.2** (Generalized inverse [169, 170]). *Let $g$ be a matrix-valued function with domain in $\mathcal{X}$. A function $g^{\mathrm{g}}$ is said to be a **generalized inverse** of $g$ if*

$$g(x)g^{\mathrm{g}}(x)g(x) = g(x) \qquad \forall x \in \mathcal{X}.$$

*A function $g^+$ is called a **Moore–Penrose inverse** (or **pseudoinverse**) of $g$ if it is a generalized inverse and satisfies*

$$g^+(x)g(x)g^+(x) = g^+(x), \quad \big(g(x)g^+(x)\big)^* = g(x)g^+(x), \quad \big(g^+(x)g(x)\big)^* = g^+(x)g(x)$$

*for every $x \in \mathcal{X}$.*

**Lemma 3.1** (Inverse uniqueness). *Let $g$ be a matrix-valued function. If its Moore–Penrose inverse $g^+$ exists, then it is unique.*

**Proof.** Point-wise extension of [169, Theorem 4.14] □

Suppose $g(x)$ can be decomposed as

$$g(x) = U_{\mathrm{r}}(x) \begin{bmatrix} \Sigma(x) & 0 \\ 0 & 0 \end{bmatrix} U_{\mathrm{l}}(x),$$

where $U_{\mathrm{r}}$, $U_{\mathrm{l}}$ and $\Sigma$ are square and nonsingular matrix-valued functions of appropriate size. Hence, according to [171], the set of all generalized inverses of $g$ reads

$$\left\{ U_{\mathrm{l}}^{-1}(x) \begin{bmatrix} \Sigma^{-1}(x) & R_1(x) \\ R_2(x) & R_3(x) \end{bmatrix} U_{\mathrm{r}}^{-1}(x) \mid R_1,\, R_2,\, R_3 \text{ arbitrary} \right\},$$

[24] The superscript $\perp$ denotes the orthogonal complement of a subspace.
[25] By definition, a zero matrix is not full rank.

and the uniquely defined Moore–Penrose inverse is

$$g^{+}(x) = U_{\mathrm{l}}^{-1}(x) \begin{bmatrix} \Sigma^{-1}(x) & 0 \\ 0 & 0 \end{bmatrix} U_{\mathrm{r}}^{-1}(x).$$

Clearly, if $g$ is constant, we can always compute its inverses by using the singular value decomposition with $U_{\mathrm{r}}$ and $U_{\mathrm{l}}$ being unitary matrices[26] and $\Sigma$ being diagonal with positive real numbers as elements. For other methods on calculating generalized inverses for constant and polynomial matrices, the reader can consult [172–176].

**Lemma 3.2.** *Consider $A : \mathcal{X} \to \mathbb{R}^{n\times s}$ and $G : \mathcal{X} \to \mathbb{R}^{n\times m}$. Then, the equation*

$$0 = A(x) + G(x)K(x) \qquad\qquad \forall x \in \mathcal{X}$$

*has a solution in $K : \mathcal{X} \to \mathbb{R}^{m\times s}$ if and only if*

$$0 = G_{\perp}(x)A(x) \qquad\qquad \forall x \in \mathcal{X}.$$

*Furthermore, all the solutions are of the form*

$$K(x) = -G^{\mathrm{g}}(x)A(x) + G_{\mathrm{I}}(x)\nu,$$

*where $\nu$ is arbitrary and of adequate size.*

**Proof.** The proof is a point-wise extension of [169, Corollary 5.2] and the properties of Lemma A.7, see also [177]. □

In particular, if $G(x)$ is full column rank for every $x \in \mathcal{X}$ and we choose $G^{\mathrm{g}}$ as the Moore–Penrose inverse, then $K(x) = -\left(G^{\top}(x)G(x)\right)^{-1} G^{\top}(x)A(x)$, which is a standard result in Passivity-based Control (PBC), see [178, Lemma 2.1] or [179, Lemma 2]. The advantage of using generalized inverses in Lemma 3.2 can be seen as follows. Suppose $G(x) = \begin{bmatrix} E(x) & I_{n_u} \end{bmatrix}^{\top}$ for some appropriate matrix-valued function $E$, then its Moore-Penrose inverse reads

$$G^{+}(x) = \left(E(x)E^{\top}(x) + I_{n_u}\right)^{-1} \begin{bmatrix} E(x) & I_{n_u} \end{bmatrix}.$$

Clearly, using $G^{+}$ to build $K$ can result in large expressions that may be difficult to compute because of the term $\left(E(x)E^{\top}(x) + I_{n_u}\right)^{-1}$. However, if we employ

$$G^{\mathrm{g}}(x) = \begin{bmatrix} 0 & I_{n_u} \end{bmatrix},$$

[26]A real matrix $U$ is said to be unitary if its inverse is equal to its transpose, i.e., $U^{\top}U = UU^{\top} = I$.

the aforementioned problem is avoided. In fact, we may use the high flexibility in the selection of $G^{\mathrm{g}}$ s.t. $K$ takes its most simple form; however, this is still an open problem in general, and the choice of $G^{\mathrm{g}}$ depends on the designer.

The lemma below extends the standard Finsler Lemma A.9 by providing equivalent statements for non-strict inequalities.

**Lemma 3.3** (Extended Finsler lemma). *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be given. The subsequent statements are equivalent:*

*i)* $y^\top A y \succeq 0$ *for all* $y \in \left\{y \in \mathbb{R}^n \mid B^\top y = 0\right\}$.

*ii)* $B_\perp A^{\mathrm{s}} B_\perp^\top \succeq 0$.

*iii)* *There exists* $K \in \mathbb{R}^{m\times n}$ *such that* $A^{\mathrm{s}} + BK + K^\top B^\top \succeq 0$.

*If statement iii holds, the solutions of $K$ are all of the form*

$$K = K_1 B^\top + \left(K_2 B_\perp - B^+\right) A^{\mathrm{s}} B_\perp^+ B_\perp + B_{\mathrm{I}} \nu, \tag{3.1a}$$

*where $K_1$ and $K_2$ are arbitrary matrices of adequate size satisfying*

$$B_{\mathrm{r}} \left(K_1^{\mathrm{s}} + B^+ A^{\mathrm{s}} B^{+\top} - K_2 B_\perp A^{\mathrm{s}} B_\perp^\top K_2^\top\right) B_{\mathrm{r}}^\top \succeq 0, \tag{3.1b}$$

*$\nu$ is also arbitrary and $(B_{\mathrm{l}}, B_{\mathrm{r}})$ are the full rank factors of $B$, i.e., $B = B_{\mathrm{l}} B_{\mathrm{r}}$. Suppose* $\operatorname{rank} B < n$ *and $y \neq 0$, then statements i–iii are also equivalent with strict inequalities and the solutions of $K$ are given by (3.1) with (3.1b) being also strict.*

**Proof.** See Appendix B.1. □

*Remark* 3.1. Matrix $K$ can be written as

$$K = K_1 B^\top + B_{\mathrm{I}} \bar{\nu}$$

if and only if there exist $K_1$ and $K_2$ satisfying (3.1b) and $0 = B_{\mathrm{r}}\left(K_2 B_\perp - B^+\right) A^{\mathrm{s}} B_\perp^\top$.

**Corollary 3.1.** *Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with symmetric $A$ and* $\operatorname{rank} B < n$. *Then,*

$$A + BKB^\top \succ 0$$

*has a solution in $K \in \mathbb{R}^{m\times m}$, if and only if*

$$B_\perp A B_\perp^\top \succ 0.$$

*If a solution exist, they are all of the form*

$$B_{\mathrm{r}} K B_{\mathrm{r}}^\top \succ B_{\mathrm{l}}^+ \left(A B_\perp^\top \left(B_\perp A B_\perp^\top\right)^{-1} B_\perp A - A\right) B_{\mathrm{l}}^{+\top},$$

*where* $(B_\mathrm{l}, B_\mathrm{r})$ *are the full-rank factors of* $B$*, i.e.,* $B = B_\mathrm{l} B_\mathrm{r}$*.*

**Lemma 3.4.** *Assume* $A : \mathcal{X} \to \mathbb{R}^{n\times n}$ *is partitioned as*

$$A(x) = \begin{bmatrix} B(x) & C(x) \end{bmatrix}.$$

*The following are equivalent for every* $x \in \mathcal{X}$*:*

*i) The inverse of* $A$ *exists and is given by*

$$A^{-1}(x) = \begin{bmatrix} (C_\perp(x)B(x))^{-1}\, C_\perp(x) \\ (B_\perp(x)C(x))^{-1}\, B_\perp(x) \end{bmatrix}. \tag{3.2}$$

*ii)* $B_\perp(x)C(x)$ *and* $C_\perp(x)B(x)$ *are nonsingular.*

*iii)* $A(x)$ *is nonsingular.*

**Proof.** See Appendix B.2. □

**Lemma 3.5** (Subsets)**.** *Consider the nonempty sets* $\mathcal{X}_\alpha := \{x \in \mathcal{X} \mid \alpha(x) \geq 0\}$ *and* $\mathcal{X}_\beta := \{x \in \mathcal{X} \mid \beta(x) \geq 0\}$ *for some functions* $\alpha$ *and* $\beta$*. If*

$$\alpha(x) \leq \beta(x) \qquad \forall x \in \mathcal{X}_\alpha,$$

*then* $\mathcal{X}_\alpha \subset \mathcal{X}_\beta$*.*

**Proof.** Suppose there exists an element $x^\star \in \mathcal{X}_\alpha$ that is not in $\mathcal{X}_\beta$. Then, $\alpha(x^\star) \geq 0 > \beta(x^\star)$, which is a contradiction, meaning that every element of $\mathcal{X}_\alpha$ belongs to $\mathcal{X}_\beta$. □

Let us consider the nonlinear and affine system

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathcal{X},\ u \in \mathbb{R}^{n_u}, \tag{3.3}$$

where $\mathcal{X}$ is an open and connected subset of $\mathbb{R}^{n_x}$, $f : \mathcal{X} \to \mathbb{R}^{n_x}$, $g : \mathcal{X} \to \mathbb{R}^{n_x \times n_u}$ has constant rank, and $f, g \in \mathcal{C}^1$.

**Definition 3.3.** *A point* $x^\star \in \mathcal{X}$ *is said to be an* ***admissible equilibrium*** *of* (3.3) *if there exists a control action* $u = u^\star \in \mathbb{R}^{n_u}$ *such that* $x^\star$ *is an equilibrium point.*

From this definition it is evident that $x^\star$ being an admissible equilibrium is a necessary condition for the asymptotic stabilization of the closed-loop in that point.

**Lemma 3.6.** *The state* $x^\star$ *is an admissible equilibrium of* (3.3) *if and only if*

$$x^\star \in \mathcal{X}_\mathrm{a} := \{x \in \mathcal{X} \mid g_\perp(x)f(x) = 0\}.$$

**Proof.** Direct application of Lemma 3.2 on $0 = f(x^\star) + g(x^\star)u^\star$. □

### 3.1.2 Controller Design

The following proposition extends the IDA-PBC methodology of [23, 29, 56] for affine nonlinear systems by replacing the Moore-Penrose inverse with generalized inverses, which are usually non-unique.

**Proposition 3.1.** *Let system (3.3) be given. The state feedback $u = u_{\text{ida}}(x)$, with*

$$u_{\text{ida}}(x) = g^{\text{g}}(x) \left( W_{\text{d}}(x) \frac{\partial^\top H_{\text{d}}}{\partial x} - f(x) \right) + g_{\mathbb{I}}(x)\nu \tag{3.4}$$

*and arbitrary $\nu$, transforms (3.3) into the* ***desired*** *or* ***target*** *system*

$$\dot{x} = W_{\text{d}}(x) \frac{\partial^\top H_{\text{d}}}{\partial x} \tag{3.5}$$

*if and only if the* ***matching condition***

$$0 = g_\perp(x) \left( W_{\text{d}}(x) \frac{\partial^\top H_{\text{d}}}{\partial x} - f(x) \right) \qquad \forall x \in \mathcal{X} \tag{3.6}$$

*holds. Here, $H_{\text{d}} : \mathcal{X} \to \mathbb{R}$, $H_{\text{d}} \in \mathcal{C}^1$ and $W_{\text{d}} : \mathcal{X} \to \mathbb{R}^{n_x \times n_x}$. The closed-loop system (3.5) is stable in the admissible equilibrium $x_{\text{d}}$ if*

$$H_{\text{d}}(x) - H_{\text{d}}(x_{\text{d}}) > 0 \qquad \forall x \in \mathcal{X} - \{x_{\text{d}}\}\,, \tag{3.7a}$$

$$\dot{H}_{\text{d}}(x) = \frac{\partial H_{\text{d}}}{\partial x} W_{\text{d}}(x) \frac{\partial^\top H_{\text{d}}}{\partial x} \leq 0 \qquad \forall x \in \mathcal{X}. \tag{3.7b}$$

*Let $\Omega_{\text{inv}}$ be the largest invariant set of (3.5) contained in*

$$\Omega := \left\{ x \in \mathcal{X} \mid \frac{\partial H_{\text{d}}}{\partial x} W_{\text{d}}(x) \frac{\partial^\top H_{\text{d}}}{\partial x} = 0 \right\}.$$

*The equilibrium $x_{\text{d}}$ is (locally) asymptotically stable if $\Omega_{\text{inv}}$ posses no other than the trivial solution $x(t) \equiv x_{\text{d}}$. Define $\mathcal{A}_c := \{x \in \mathbb{R}^{n_x} \,|\, H_{\text{d}}(x) \leq c\}$ with $c$ being the largest scalar such that $\mathcal{A}_c \subset \mathcal{X}$ is a compact set. Then, $\mathcal{A}_c$ is an estimate of the region of attraction.*

**Proof.** Equating the right-hand side of (3.3) and (3.5) yields

$$0 = W_{\text{d}}(x) \frac{\partial^\top H_{\text{d}}}{\partial x} - f(x) - g(x)u. \tag{3.8}$$

The control law (3.4) and matching condition (3.6) follow immediately from (3.8) with Lemma 3.2. Let $\tilde{H}_{\text{d}}(x) := H_{\text{d}}(x) - H_{\text{d}}(x_{\text{d}})$ and take its time derivative along the trajectories of (3.5):

$$\dot{\tilde{H}}_{\mathrm{d}}(x) = \frac{\partial H_{\mathrm{d}}}{\partial x}\dot{x} = \frac{\partial H_{\mathrm{d}}}{\partial x}W_{\mathrm{d}}(x)\frac{\partial^\top H_{\mathrm{d}}}{\partial x}.$$

In view of conditions (3.7), we deduce that $\tilde{H}_{\mathrm{d}}$ and $\dot{\tilde{H}}_{\mathrm{d}}$ are positive definite and negative semidefinite (about $x_{\mathrm{d}}$), respectively. Now, since $\tilde{H}_{\mathrm{d}} \in \mathcal{C}^1$, the equilibrium $x_{\mathrm{d}}$ is stable with Lyapunov function $\tilde{H}_{\mathrm{d}}$ (Theorem 2.1). Asymptotic stability and the region of attraction are obtained by invoking LaSalle's invariance principle, see Theorem 2.4. □

*Remark* 3.2. Suppose (3.7a) is not necessarily fulfilled, and let $y_{\mathrm{d}}$ be defined such that

$$y_{\mathrm{d}}^\top y_{\mathrm{d}} := -\frac{\partial H_{\mathrm{d}}}{\partial x}W_{\mathrm{d}}(x)\frac{\partial^\top H_{\mathrm{d}}}{\partial x}.$$

Then, asymptotic stability of $x_{\mathrm{d}}$ is achieved if and only if (3.5) with output $y_{\mathrm{d}}$ is zero-state detectable about $x_{\mathrm{d}}$, see Lemma 2.3.

*Remark* 3.3. Assume $H_{\mathrm{d}} \in \mathcal{C}^2$. The necessary and sufficient conditions (see [180, Proposition 1.3–1.4]) for $H_{\mathrm{d}}$ to have a strict local minimum in $x_{\mathrm{d}}$ are

$$\left.\frac{\partial H_{\mathrm{d}}}{\partial x}\right|_{x=x_{\mathrm{d}}} = 0, \qquad\qquad \left.\frac{\partial^2 H_{\mathrm{d}}}{\partial x^2}\right|_{x=x_{\mathrm{d}}} \succ 0.$$

*Remark* 3.4. Setting

$$W_{\mathrm{d}}(x) + W_{\mathrm{d}}^\top(x) \preceq 0 \qquad\qquad \forall x \in \mathcal{X}$$

is a sufficient but not necessary condition for (3.7b).

*Remark* 3.5. If $\mathcal{X} = \mathbb{R}^{n_x}$ and $\mathcal{A}_c$ is compact for every value of $c > 0$, i.e., $H_d$ is radially unbounded, then the asymptotic stability is global, see Section 2.1.

The crucial requirements for Proposition 3.1 are the satisfaction of the matching (3.6) and the stabilizing conditions (3.7) for some $g_\perp$, $H_{\mathrm{d}}$ and $W_{\mathrm{d}}$. Then, we select $g^{\mathrm{g}}$ and build the controller $u_{\mathrm{ida}}$. Note that a closed-loop verifying (3.7) is a port-Hamiltonian system that is passive with respect to the triplet[27] $\left\{\tilde{H}_{\mathrm{d}},\, u_{\mathrm{d}},\, y_{\mathrm{d}}\right\}$, where $\tilde{H}_{\mathrm{d}}(x) = H_{\mathrm{d}}(x) - H_{\mathrm{d}}(x_{\mathrm{d}})$, $y_d = g^\top(x)\frac{\partial^\top H_{\mathrm{d}}}{\partial x}$ and $u = u_{\mathrm{ida}}(x) + u_{\mathrm{d}}$, see Section 2.6. If the input matrix $g$ is full row rank, i.e., $\operatorname{rank} g(x) = n_x$, then $g_\perp$ is a zero matrix and the problem is trivial because the matching condition holds for an arbitrary closed-loop dynamics. Therefore, we can select any $H_{\mathrm{d}}$ and $W_{\mathrm{d}}$ satisfying (3.7). However, if $\operatorname{rank} g(x) < n_x$, the solvability of (3.6) is in general not a simple task [29], and we can classify its solution strategies in at least three groups [23].

[27]We follow the notation of [27] by calling a system (output strictly) passive with respect to the triplet $\{S,\, u,\, y\}$ if it is (output strictly) passive with storage function $S$, input $u$, and output $y$.

***Non-parameterized:*** Fixing $W_\text{d}$ and $g_\perp$ in the matching condition (3.6) yields a PDE whose general solution is the target Hamiltonian $H_\text{d}$, parameterized by arbitrary functions that are selected to satisfy the strict (global) minimum of $H_\text{d}$ in $x_\text{d}$.

***Algebraic:*** On the contrary, if $H_\text{d}$ is initially fixed, then (3.6) becomes an algebraic equation with unknowns in $W_\text{d}$ and $g_\perp$. We will use this perspective in the next section to address systems with polynomial structure in the matching and stabilizing conditions.

***Parameterized:*** This scheme is typically employed for UMSs, as showed in Chapter 5. It consists of exploiting the nominal system properties by restricting the target Hamiltonian $H_\text{d}$ to a particular class, which indirectly shapes the dissipation and interconnection matrix $W_\text{d}$. We employ this solution in Chapter 5 to extend the IDA-PBC to implicit mechanical systems.

In addition to the previous solutions strategies, we can assume that $W_\text{d}$ is partitioned as

$$W_\text{d}(x) = W_1(x) - g(x)K_\text{v}g^\top(x), \tag{3.9}$$

where $W_1 : \mathcal{X} \to \mathbb{R}^{n_x \times n_x}$ is skew-symmetric and $K_\text{v} \in \mathbb{R}^{n_u \times n_u}$ is constant, then the design procedure, known as **standard IDA-PBC**, is split in two distinct stages [13, 23, 25]: **energy shaping** and **damping injection**. In the energy shaping stage we solve the matching condition (3.6) for some functions $H_\text{d}$, $W_1$, and $g_\perp$ provided $H_\text{d}$ is positive definite about $x_\text{d}$. Clearly, the term $g(x)K_\text{v}g^\top(x)$ vanishes from the matching condition, which simplifies its solution. Later, in the damping injection (see Section 2.6.1), we choose $K_\text{v} + K_\text{v}^\top \succ 0$ to enforce (3.7b), obtaining a closed-loop that is output strictly passive w.r.t. $\left\{\tilde{H}_\text{d},\, u_\text{d},\, y_\text{d}\right\}$.

As pointed out in [25], partitioning $W_\text{d}$ as (3.9) is not without loss of generality because it reduces the system classes for which IDA-PBC is applicable. Therefore, whenever possible, we pursue the **one stage approach**, also called **Simultaneous IDA-PBC** [24, 25] because it involves searching directly for $W_\text{d}$, $g_\perp$ and a positive definite (about $x_\text{d}$) Hamiltonian $H_\text{d}$ such that conditions (3.6) and (3.7b) hold. Finally, Proposition 3.1 can be extended to *i)* non-autonomous systems (see [181, 182]), and *ii)* stabilization of an invariant set rather than an equilibrium point, see [183, 184] for instance, where a "Mexican hat" function is assigned to the target Hamiltonian.

## 3.2 Systems with Polynomial Structure

Polynomials may simplify analysis and controller design of nonlinear systems. In doing so, it is often preferred that all formulations remain polynomial and do not get rational. In this section, we take advantage of the algebraic solution and polynomial representation to recast the IDA-PBC problem as Sum of Squares (SOS) programs.

### 3.2.1 Controller Design

In view of Proposition 3.1, we consider affine systems of the form (3.3) and define the **region of interest** $\mathcal{X}_\beta := \{x \in \mathbb{R}^{n_x} \mid \beta(x) \geq 0\}$ with arbitrary function $\beta : \mathbb{R}^{n_x} \to \mathbb{R}$ such that $\mathcal{X}_\beta \subset \mathcal{X}$ and $\mathrm{int}(\mathcal{X}_\beta)$ is a connected set.[28] For instance, $\mathcal{X}_\beta$ is an ellipsoid if $\beta(x) = 1 - x^\top S_\beta x$ and $S_\beta \succ 0$, or $\mathcal{X}_\beta$ is the whole space $\mathbb{R}^{n_x}$ if $\beta(x) = 0$. We identify the system class by imposing

**Assumption 3.1.** *Given (3.3), there exist functions $\Lambda_0 : \mathcal{X} \to \mathbb{R}^{(n_x-m)\times n_z}$, $z : \mathcal{X} \to \mathbb{R}^{n_z}$ and $g_\perp : \mathcal{X} \to \mathbb{R}^{(n_x-m)\times n_x}$ verifying*

$$g_\perp(x) f(x) = \Lambda_0(x) z(x) \qquad \forall x \in \mathcal{X}_\beta \tag{3.10a}$$

$$z(x) = 0 \quad \Leftrightarrow \quad x = x_\mathrm{d}, \qquad \forall x \in \mathcal{X}_\beta \tag{3.10b}$$

*where $x_\mathrm{d}$ is an admissible equilibrium. Here, $n_z \geq n_x$ and $n_u \geq m := \mathrm{rank}\, g(x)$.*

By Lemma 3.6, the system (3.3) must satisfy $g_\perp(x_\mathrm{d}) f(x_\mathrm{d}) = 0$ in the admissible equilibrium $x_\mathrm{d}$, which means that (3.10a)–(3.10b) are not so conservative. Before stating our main result in Proposition 3.2, let us consider the target system (3.5) and select $W_\mathrm{d}$ as

$$W_\mathrm{d}(x) = \begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix}^{-1} \begin{bmatrix} F_0(x) \\ F_1(x) \end{bmatrix}, \tag{3.11a}$$

with nonsingular matrix $\left[\begin{smallmatrix} g_\perp(x) \\ \eta^\top(x) \end{smallmatrix}\right]$, for some functions $F_1, \eta : \mathcal{X} \to \mathbb{R}^{m\times n_x}$ and $F_0 : \mathcal{X} \to \mathbb{R}^{(n_x-m)\times n_x}$. We exploit the algebraic solution for the matching condition by fixing the Hamiltonian as

$$H_\mathrm{d}(x) = \frac{1}{2} z^\top(x) P^{-1} z(x) + \psi(\gamma(x)), \tag{3.11b}$$

where $P \in \mathbb{R}^{n_z \times n_z}$ is a positive definite constant matrix, $\psi$ is a $\mathcal{C}^2$ positive semidefinite function (about 0) and $\gamma$ is a function that verifies $F_0(x)\frac{\partial^\top \gamma}{\partial x} = 0$, see Corollary 2.2 and Lemma 2.2 for the solutions of $\gamma$.

**Proposition 3.2.** *Consider the system (3.3) verifying Assumption 3.1. Let (3.5) be the target system, where $W_\mathrm{d}$ and $H_\mathrm{d}$ are defined in (3.11) with $\gamma(x_\mathrm{d}) = 0$. Suppose $P$ and $F_0$ can be chosen such that*

$$\Lambda_0(x) P - F_0(x) \frac{\partial^\top z}{\partial x} = 0 \qquad \forall x \in \mathcal{X}, \tag{3.12}$$

[28] The interior of a set $\mathcal{A}$, denoted with $\mathrm{int}(\mathcal{A})$, is the union of all the open subset of $\mathcal{A}$ [111].

$$A_{\mathrm{F}}(x) := -g_{\perp}(x)F_0^{\top}(x) - F_0(x)g_{\perp}^{\top}(x) - \beta(x)S_{11}(x) \succeq 0 \qquad \forall x \in \mathcal{X} \tag{3.13}$$

*for some* $S_{11} : \mathcal{X} \to \mathbb{R}^{(n_x-m)\times(n_x-m)}$ *with* $S_{11}(x) \succeq 0$. *Then, for every* $\eta$ *provided* $\begin{bmatrix} g_{\perp}^{\top}(x) & \eta(x) \end{bmatrix}$ *is nonsingular, there exists* $F_1$ *verifying*

$$-\begin{bmatrix} g_{\perp}(x) \\ \eta^{\top}(x) \end{bmatrix} \begin{bmatrix} F_0(x) \\ F_1(x) \end{bmatrix}^{\top} - \begin{bmatrix} F_0(x) \\ F_1(x) \end{bmatrix} \begin{bmatrix} g_{\perp}(x) \\ \eta^{\top}(x) \end{bmatrix}^{\top} - \beta(x)S_1(x) \succeq 0 \qquad \forall x \in \mathcal{X} \tag{3.14}$$

*for some* $S_1 : \mathcal{X} \to \mathbb{R}^{n_x \times n_x}$ *with* $S_1(x) \succeq 0$. *The result also holds if* (3.13) *and* (3.14) *have strict inequalities. Under such a* $P$ *and* $F_1$, *feedback* $u = u_{\mathrm{ida}}(x)$ *with*

$$u_{\mathrm{ida}}(x) = \left(\eta^{\top}(x)g(x)\right)^{\mathrm{g}} \left(F_1(x)\frac{\partial^{\top} H_{\mathrm{d}}}{\partial x} - \eta^{\top}(x)f(x)\right) + g_{\mathcal{I}}(x)\nu \tag{3.15}$$

*and arbitrary* $\nu$ *renders system* (3.3) *into a stable (in* $x_{\mathrm{d}}$*) port-Hamiltonian system* (3.5). *Asymptotic stability in the equilibrium is demonstrated if the left-hand side of* (3.14) *is positive definite. Furthermore, a solution to* $F_1$ *verifying* (3.14), *or its strict version, is given by*

$$F_1(x) = -\begin{bmatrix} \eta^{\top}(x)F_0^{\top}(x) + K_2(x)A_{\mathrm{F}}(x), & K_1(x) \end{bmatrix} \begin{bmatrix} g_{\perp}^{\top}(x) & \eta(x) \end{bmatrix}^{-1} \tag{3.16a}$$

*for any* $K_1 : \mathcal{X} \to \mathbb{R}^{m \times m}$ *and* $K_2 : \mathcal{X} \to \mathbb{R}^{m \times (n_x - m)}$ *such that*

$$K_1(x) + K_1^{\top}(x) - K_2(x)A_{\mathrm{F}}(x)K_2^{\top}(x) \succ 0 \qquad \forall x \in \mathcal{X}. \tag{3.16b}$$

**Proof.** The proof is divided into three parts: *i)* transformation of (3.3) into (3.5), *ii)* stability analysis, and *iii)* existence of $F_1$. We will use Proposition 3.1 for the first two parts and Lemma 3.3 for the last one.

*i)* Pick $g^{\mathrm{g}}(x) := \left(\eta^{\top}(x)g(x)\right)^{\mathrm{g}} \eta^{\top}(x)$, which is a generalized inverse of $g$ because

$$\begin{bmatrix} g_{\perp}(x) \\ \eta^{\top}(x) \end{bmatrix} g(x)g^{\mathrm{g}}(x)g(x) = \begin{bmatrix} 0 \\ \eta^{\top}(x)g(x)\left(\eta^{\top}(x)g(x)\right)^{\mathrm{g}}\eta^{\top}(x)g(x) \end{bmatrix} = \begin{bmatrix} g_{\perp}(x) \\ \eta^{\top}(x) \end{bmatrix} g(x).$$

From (3.10a) and (3.11), the feedback (3.4) reads (3.15) and the matching condition (3.6) yields

$$\Lambda_0(x)z(x) = F_0(x)\frac{\partial^{\top} z}{\partial x}P^{-1}z(x) \qquad \forall x \in \mathcal{X}, \tag{3.17}$$

where (3.12) is a sufficient condition for (3.17), meaning that (3.15) transforms system (3.3) into (3.5).

*ii)* From (3.10b), $P \succ 0$, $\psi$ being positive semidefinite (about 0) and $\gamma(x_{\mathrm{d}}) = 0$, it follows that $H_{\mathrm{d}}$ is positive definite about $x_{\mathrm{d}}$ for all $x \in \mathcal{X}_{\beta}$. Next, multiply (3.14) on the left by

$\frac{\partial H_\mathrm{d}}{\partial x} G^{-1}(x)$ and on the right by $G^{-\top}(x)\frac{\partial^\top H_\mathrm{d}}{\partial x}$, with $G(x) = \left[\begin{smallmatrix} g_\perp(x) \\ \eta^\top(x) \end{smallmatrix}\right]$, obtaining

$$\dot{H}_\mathrm{d}(x) = \frac{\partial H_\mathrm{d}}{\partial x} W_\mathrm{d}(x) \frac{\partial^\top H_\mathrm{d}}{\partial x} \leq -\frac{1}{2}\beta(x)\frac{\partial H_\mathrm{d}}{\partial x} G^{-1}(x) S_1(x) G^{-\top}(x)\frac{\partial^\top H_\mathrm{d}}{\partial x} \qquad \forall x \in \mathcal{X}.$$

Since $S_1(x) \succeq 0$, we deduce that $\dot{H}_\mathrm{d}(x) \leq 0$ whenever $x \in \mathcal{X}_\beta$. Thus, the closed-loop with domain in $\mathrm{int}(\mathcal{X}_\beta)$, an open and connected subset of $\mathcal{X} \subset \mathbb{R}^{n_x}$, is stable in $x_\mathrm{d}$. Moreover, since $H_\mathrm{d} \in \mathcal{C}^2$ is positive definite, we have $\left.\frac{\partial H_\mathrm{d}}{\partial x}\right|_{x=x_\mathrm{d}} = 0$ and $\left.\frac{\partial^2 H_\mathrm{d}}{\partial x^2}\right|_{x=x_\mathrm{d}} \succ 0$, see [180, Section 1.3]. Consequently, there is a neighborhood of $x_\mathrm{d}$ subset of $\mathcal{X}_\beta$ s.t.

$$\left\{\tfrac{\partial H_\mathrm{d}}{\partial x} = 0\right\} = \{x_\mathrm{d}\}.$$

Now, If the left-hand side of (3.14) is positive definite, then asymptotic stability can be demonstrated with Lyapunov's direct method (Theorem 2.1) on such neighborhood.

*iii)* Let us decompose $S_1$ as

$$S_1(x) = \begin{bmatrix} S_{11}(x) & S_{12}(x) \\ S_{12}^\top(x) & S_{22}(x) \end{bmatrix}$$

with $S_{12}(x) \in \mathbb{R}^{(n_x-m)\times n_u}$ and $S_{22}(x) \in \mathbb{R}^{m\times m}$. Inequality (3.14) may be rewritten as

$$A_\mathrm{x}(x) - \begin{bmatrix} 0 \\ I_m \end{bmatrix} F_1(x) \begin{bmatrix} g_\perp^\top(x) & \eta(x) \end{bmatrix} - \begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix} F_1^\top(x) \begin{bmatrix} 0 & I_m \end{bmatrix} \succeq 0, \tag{3.18}$$

$$A_\mathrm{x}(x) = \begin{bmatrix} -F_0(x) g_\perp^\top(x) - g_\perp(x) F_0^\top(x) - \beta(x) S_{11}(x) & -F_0(x)\eta(x) - \beta(x) S_{12}(x) \\ -\eta^\top(x) F_0^\top(x) - \beta(x) S_{12}^\top(x) & -\beta(x) S_{22}(x) \end{bmatrix}.$$

Let $\eta$ satisfy the nonsingular condition on $\begin{bmatrix} g_\perp^\top(x) & \eta(x) \end{bmatrix}$. A direct application of Lemma 3.3 shows that (3.18) has a solution to $F_1$ if and only if so does (3.13). Besides, (3.16) gives a suitable solution provided $S_{12}(x) = 0$ and $S_{22}(x) = 0$. The same is true for strict inequalities in (3.13) and (3.18). □

For a class of affine systems, Proposition 3.2 exposes sufficient conditions to synthesize the stabilizing controller (3.15). The controller synthesis procedure requires to first select the region of interest $\mathcal{X}_\beta$ and functions $g_\perp$, $\Lambda_0$ and $z$ under Assumption 3.1. In the next step, we solve (3.12)–(3.13) for $P$, $F_0$ and $S_{11}$. If $g_\perp$, $\beta$, $F_0$, $S_{11}$, $\Lambda_0$ and $z$ are all polynomial, we can reformulate this task as the SOS program below.[29]

[29]Notice that the SOS decomposition is only a sufficient condition because there exist positive semidefinite polynomials that are not SOS, see Section 2.2.

**SOS Program 3.1.**

$$\begin{aligned}
\text{find} \quad & \text{the coefficients of } P,\ F_0,\ S_{11} \\
\text{subject to} \quad & \Lambda_0(x)P = F_0(x)\frac{\partial^\top z}{\partial x}, \\
& P - \epsilon_0 I_{n_z} && \text{is SOS,} \\
& S_{11} && \text{is SOS,} \\
& -g_\perp(x)F_0^\top(x) - F_0(x)g_\perp^\top(x) - \beta(x)S_{11}(x) - \epsilon_1 I_{n_x-m} && \text{is SOS,}
\end{aligned}$$

where $\epsilon_0 > 0$ and $\epsilon_1 \geq 0$ are user defined constants.

Later, we select $\eta$ and calculate $F_1$ from (3.16a) with some user-defined functions $K_1$ and $K_2$ (not necessarily polynomial) verifying (3.16b). However, instead of calculating $F_1$ afterwards, we may fix $\eta$ and search simultaneously for $P$, $F_0$, $F_1$ and $S_1$ satisfying the conditions (3.12) and (3.14). Similarly, if $g_\perp$, $\eta$, $\beta$, $F_0$, $F_1$, $S_1$, $\Lambda_0$ and $z$ are all polynomial, then the problem can be recast as

**SOS Program 3.2.**

$$\begin{aligned}
\text{find} \quad & \text{the coefficients of } P,\ F_0,\ F_1,\ S_1 \\
\text{subject to} \quad & \Lambda_0(x)P = F_0(x)\frac{\partial^\top z}{\partial x}, \\
& P - \epsilon_0 I_{n_z} && \text{is SOS,} \\
& S_1 && \text{is SOS,} \\
& -\begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix}\begin{bmatrix} F_0(x) \\ F_1(x) \end{bmatrix}^\top - \begin{bmatrix} F_0(x) \\ F_1(x) \end{bmatrix}\begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix}^\top - \beta(x)S_1(x) - \epsilon_1 I_{n_x} && \text{is SOS,}
\end{aligned}$$

where $\epsilon_0 > 0$ and $\epsilon_1 \geq 0$ are user defined constants.

In the SOS programs, the term $\epsilon_0 I_{n_z}$ establishes a minimum bound in $P$ and guarantees $P \succ 0$ while the terms $\epsilon_1 I_{n_x-m}$ and $\epsilon_1 I_{n_x}$ with $\epsilon_1 > 0$ are used to impose strict inequalities in (3.13) and (3.14), and thus guarantee asymptotic stability. Besides, if $\dim(z) = \dim(x)$ and $\frac{\partial z}{\partial x}$ is nonsingular in $\mathcal{X}_\beta$, then (3.12) has a unique solution in $F_0$ given by

$$F_0(x) = \Lambda_0(x)P\left(\frac{\partial z}{\partial x}\right)^{-\top}, \tag{3.19}$$

which can be introduced in the SOS Programs 3.1 and 3.2 whenever the right-hand side of (3.19) is polynomial for every $P$. Observe that $\Lambda_0$ and $z$ are not necessarily polynomial in this case. Lastly, with the obtained $P$, $F_0$ and $F_1$, we calculate $\gamma$, select $\psi$, $\left(\eta^\top(x)g(x)\right)^{\mathrm{g}}$, $g_\top$ and $\nu$, and build the controller (3.15).

In comparison, SOS Program 3.1 posses smaller LMIs, whereas SOS Program 3.2 defines $F_1$ as a polynomial function that is included in the optimization. Consequently, SOS Program 3.2 requires a higher computational cost (time) but allows imposing minimization objectives in $P$, $F_0$ and $F_1$ at the same time. Note that having a large number of unknown coefficients provides a great flexibility in the controller performance but may lead to an unpredictable behavior because the SOS Programs 3.1 and 3.2 have either none or infinite solutions. We can alleviate or even solve this issue by imposing constraints or minimization objectives on such new variables as we will discuss in Chapter 4. Given that the computational cost of SOS programs is upper bounded by a polynomial expression of the number of coefficients and linear constraints, we should aim (in practice) at a low polynomial order on the unknown functions $F_0$, $F_1$, $S_1$ and $S_{11}$. In this context, SOS Program 3.1 can be used as a fast indicator such that SOS Program 3.2 will work, but since $F_1$ is not necessarily polynomial,[30] this is experimentally still a good but not an unconditionally reliable reference.

Even tough, Proposition 3.2 impose only sufficient conditions, making it a bit conservative, it has the following advantages:

*i)* The matching condition is solved algebraically, avoiding the solution of PDEs.

*ii)* The controller design is not restricted to polynomial systems, i.e., $f$ and $g$ can posses non-polynomial elements (see Example 3.3 or Section 4.4).

*iii)* The terms $\beta(x)S_{11}(x)$ and $\beta(x)S_1(x)$ provide a relaxation for the local objectives.

*iv)* The polynomial structure allows to reformulate the matching and stabilizing conditions as the SOS Programs 3.1 or 3.2, simplifying algebraic analysis.

*v)* The design avoids the conservative partitioning of $W_{\mathrm{d}}$ as (3.9).

Throughout this dissertation, we will solve SOS programs with **SOSTOOLS** [122] because it is a free and third-party MATLAB toolbox that provides an intuitive environment to work with polynomial matrices, equality constraints, and minimization objectives. In addition it is compatible with many commercial and non-commercial Semidefinite Programming (SDP) solvers, namely SeDuMi, SDPT3, CSDP, SDPNAL, SDPNAL+ and SDPA.

*Example* 3.1. Consider the polynomial system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3.20}$$

[30] Proposition 3.2 does not guarantee the existence of a polynomial $F_1$, which is a stronger condition.

with $x := \mathrm{vec}(x_1, x_2) \in \mathcal{X} = \mathbb{R}^2$. We shall determine an asymptotically stabilizing controller with Proposition 3.2 and SOS Program 3.1. For this, we select the desired equilibrium

$$x_\mathrm{d} = \mathrm{vec}(x_1^\star, -(x_1^\star)^2) \in \mathcal{X}_a := \left\{x \in \mathbb{R}^2 \mid x_1^2 + x_2 = 0\right\}$$

and the region of interest $\mathcal{X}_\beta$ as the whole space $\mathbb{R}^2$ with $\beta(x) = 0$. Next, we pick

$$z(x) = \mathrm{vec}(x_1 - x_1^\star, x_1^2 + x_2), \qquad g_\perp = \begin{bmatrix}1 & 0\end{bmatrix}, \qquad \Lambda_0 = \begin{bmatrix}0 & 1\end{bmatrix}$$

to fulfill Assumption 3.1, i.e.,

$$g_\perp(x)f(x) = \Lambda_0 z(x) \qquad \text{and} \qquad z(x) = 0 \quad \Longleftrightarrow \quad x = x_\mathrm{d}.$$

At this point, we fix $\epsilon_0 = \epsilon_1 = 10^{-5}$, $S_{11}(x) = 0$ and employ the solution of $F_0$ given by (3.19), which is polynomial for every $P$ because $\frac{\partial z}{\partial x}$ is unimodular[31] and $\Lambda_0$ is constant. Solving the SOS Program 3.1 in SOSTOOLS and SDPT3 [185] yields

$$P = \begin{bmatrix} 5.079 & -1.935 \\ -1.935 & 5.079 \end{bmatrix} \succ 0,$$

i.e., (3.13) holds with a strict inequality. To calculate $\gamma$, we use Lemma 2.2 and search for a full-rank right annihilator of $F_0$ that is also a gradient vector:

$$F_0(x)\begin{bmatrix}2x_1 + 2.624 & 1\end{bmatrix}^\top = 0.$$

Then, integration of the right annihilator yields

$$\gamma(x) = \int_0^1 \begin{bmatrix}2x_1 s + 2.624 & 1\end{bmatrix} x\mathrm{d}s + a = x_1^2 + 2.624x_1 + x_2 + a,$$

where we pick $a$ such that $\gamma(x_\mathrm{d}) = 0$, i.e., $\gamma(x) = x_1^2 + 2.624(x_1 - x_1^\star) + x_2$. In the next step, we choose $\psi(\gamma) = 10^{-3}\gamma^4$, $\eta = g$ and $F_1$ from (3.16) with $K_2(x) = 0$. Hence, controller (3.15) reads

$$\begin{aligned} u_\mathrm{ida}(x) = {} & 1.17x_1^\star - 0.4458x_2 - 2.17x_1 - 0.0878K_1x_1 - 0.2303K_1x_2 \\ & + 0.0878K_1x_1^\star - 2.679x_1x_2 + 1.783x_1x_1^\star - 0.2303K_1x_1^2 \\ & - 1.783x_1^2x_2 + 0.6794x_1^2x_1^\star - 2.229x_1^2 - 3.359x_1^3 - 1.783x_1^4 \\ & - (\gamma(x))^3(0.004K_1 + 0.0813x_1 + 0.0533 + 0.031x_1^2) \end{aligned} \tag{3.21}$$

[31] A polynomial matrix is unimodular if it is square and its inverse is again a polynomial matrix. In other words, its determinant is always a non-zero constant.

and yields an asymptotically stable closed-loop in $x_\mathrm{d}$ for any $K_1(x) > 0$, where $(n^\top g)^\mathrm{g} = 1$ and $g_\bot(x)\nu = 0$.

Figure 3.1 shows the simulation results of system (3.20) with initial conditions $x(0) = \mathrm{vec}(2, -5)$ under the controller (3.21), where $K_1(x) = 5(x_1^2+1)^{-1}$ is chosen for illustration. We restrict the time span to 12 s, where the desired equilibrium is set to $x_\mathrm{d} = 0$ for $t \in [0, 6[$ s and $x_\mathrm{d} = \mathrm{vec}(2, -4)$ for $t \in [6, 12[$ s. Clearly, the states converge to $x_\mathrm{d}$ asymptotically and the desired Hamiltonian decreases monotonically as expected. △

### 3.2.2 Application to Linear Time-invariant Systems

Consider a Linear Time-invariant (LTI) system of the form

$$\dot{x} = Ax + Bu,$$

Assumption 3.1 is satisfied with

$$z(x) = x - x_\mathrm{d}, \qquad \beta(x) = 0, \qquad \Lambda_0 = B_\bot A,$$

for every $x_\mathrm{d}$ such that

$$0 = B_\bot A x_\mathrm{d},$$

see Lemma 3.6. From Proposition 3.2 with $\gamma(x) = 0$ and $\beta(x) = 0$, we conclude that $F_0 = B_\bot AP$ and there exists $F_1$ verifying (3.14) if $P$ can be chosen such that $P \succ 0$ and

$$A_\mathrm{F} := -B_\bot PA^\top B_\bot^\top - B_\bot APB_\bot^\top \succeq 0. \tag{3.22}$$

With such a $P$ and $F_1$, the controller (3.15) stabilizes the nominal system in the origin. Asymptotic stability is achieved if (3.22) holds with a strict inequality. Besides, a solution of $F_1$ is

$$F_1 = -\left[\eta^\top PA^\top B_\bot^\top + K_2 A_\mathrm{F}, \quad K_1\right]\left[B_\bot^\top, \quad \eta\right]^{-1}$$

for any $K_1$ and $K_2$ verifying

$$K_1 + K_1^\top - K_2 A_\mathrm{F} K_2^\top \succ 0.$$

Let $B$ be full column rank, and set $\eta = B$, $K_2 = 0$ and $K_1 = kB^\top B$ with $k > 0$, then the solution of $F_1$ reduces to

$$F_1 = -B^\top PA^\top B_\bot^+ B_\bot - kB^\top, \tag{3.23}$$

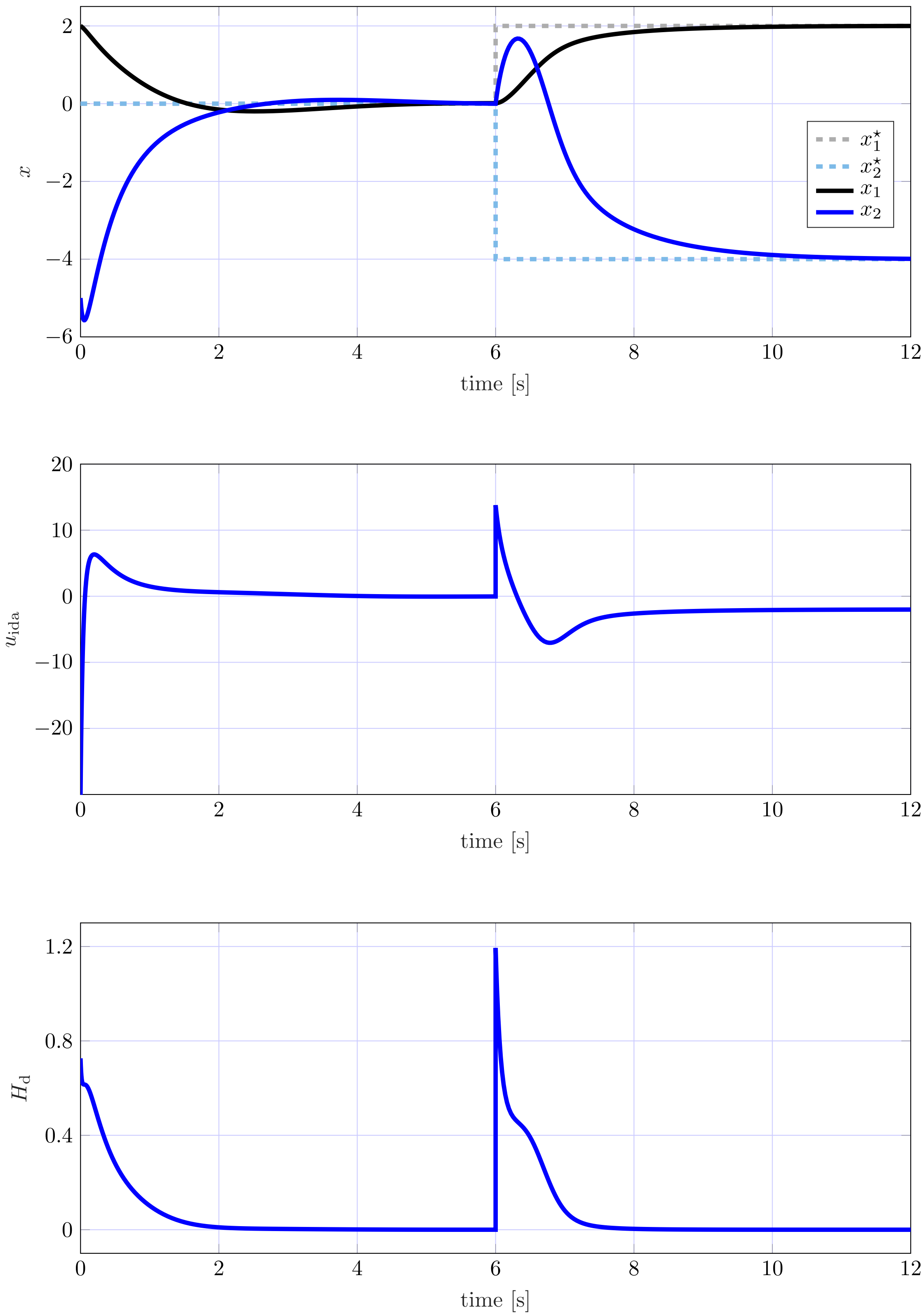



**Figure 3.1.** – Evolution of $x$, $u_{\text{ida}}$ and $H_d$ in Example 3.1.

where the inverse of $\begin{bmatrix} B_\perp^\top, & \eta \end{bmatrix}$ is computed from Lemma 3.4. Here, we can observe that equations (3.22) and (3.23) are equivalent to the results of [186, Proposition 7 and Remark 8]. In addition, according to [187, 188], solvability of $P$ for (3.22) is equivalent to stabilizability of the pair $(A, B)$.

### 3.2.3 Dimension of $z$ greater than $x$

Let us consider a function $z$ with a dimension greater than $n_x$ and take $z(x) = \text{vec}(x, x_\text{v})$ with $x_\text{v} : \mathcal{X} \to \mathbb{R}^{n_z - n_x}$. Then, $z$ contains the states $x$ and some functions of the states called $x_\text{v}$. It is interesting to see that this approach is equivalent to the adding of virtual states $x_\text{v}$ in the system (3.3) and then searching for a desired system with $z$ as the new states. Thus, the original and desired systems take the form

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{x}_\text{v} \end{bmatrix} = \begin{bmatrix} I_n \\ \frac{\partial x_\text{v}}{\partial x} \end{bmatrix} (f(x) + g(x) u_\text{ida}(x)) = \underbrace{\begin{bmatrix} I_n \\ \frac{\partial x_\text{v}}{\partial x} \end{bmatrix} \overbrace{\begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix}^{-1} \begin{bmatrix} F_0(x) \\ F_1(x) \end{bmatrix}}^{W_\text{d}(x)} \begin{bmatrix} I_n & \frac{\partial^\top x_\text{v}}{\partial x} \end{bmatrix}}_{=:\bar{W}_\text{d}(x)} \frac{\partial^\top H_\text{d}}{\partial z}.$$

The stability analysis given by $\bar{W}_\text{d}(x) + \bar{W}_\text{d}^\top(x) \preceq 0$ completes this equivalence. Note that the main advantage of choosing a vector $z$ with dimension greater than $n_x$ is in the selection of $\Lambda_0$, $g_\perp$ and $z$ from Assumption 3.1, which influence the algorithm solution. However, in this case we cannot achieve $\bar{W}_\text{d}(x) + \bar{W}_\text{d}^\top(x) \prec 0$ since $\operatorname{rank} W_\text{d} = n_x < n_z$.

### 3.2.4 Region of Convergence

The conditions of Proposition 3.2 cannot guarantee that every solution with initial conditions in $\mathcal{X}_\beta$ will approach $x_\text{d}$ or even stay at $\mathcal{X}_\beta$ for all $t > t_0$. The proposition below addresses this issue by providing invariant sets.

**Proposition 3.3.** *Suppose the conditions of Proposition 3.2 are satisfied with $\mathcal{X}_\beta = \mathbb{R}^{n_x}$, i.e., $\beta$ is a nonnegative constant. The stability is global if*

$$\|z(x)\| \to \infty \quad \textit{as} \quad \|x\| \to \infty.$$

*Suppose on the other hand that the conditions of Proposition 3.2 hold with $\psi(\gamma(x)) \equiv 0$ and $\beta(x) := 1 - z^\top(x) S_\beta z(x)$, where $S_\beta \in \mathbb{R}^{n_z \times n_z}$ is a constant matrix satisfying $S_\beta \succeq 0$. If*

$$I_{n_z} - 2c S_\beta^{\frac{1}{2}} P S_\beta^{\frac{1}{2}} \succeq 0, \tag{3.24}$$

*then* $\mathcal{A}_c := \{x \in \mathbb{R}^{n_x} \,|\, c \geq H_\mathrm{d}(x)\}$ *with positive constant* $c$ *is an invariant set of the closed-loop. If, in addition,* $x_\mathrm{d}$ *is asymptotically stable from Proposition 3.2 and*

$$\left\{x \in \mathcal{A}_c \mid \frac{\partial H_\mathrm{d}}{\partial x} = 0\right\} = \{x_\mathrm{d}\}\,, \tag{3.25}$$

*then* $\mathcal{A}_c$ *is also a region of attraction of* $x_\mathrm{d}$. *Lastly, a state* $x^\star$ *belongs to* $\mathcal{A}_c$ *if and only if*

$$2cP - z(x^\star)z^\top(x^\star) \succeq 0. \tag{3.26}$$

**Proof.** By definition of $\psi$, we can infer that $H_\mathrm{d}$ is radially unbounded if $\|z(x)\| \to \infty$ as $\|x\| \to \infty$, meaning that the global result stems from $\mathcal{X}_\beta = \mathbb{R}^{n_x}$, see Remark 3.5. For the local result, we use use the generalized Schur complements (see Lemma A.8) to express (3.24) as $(cP)^{-1} \succeq S_\beta$, which is a sufficient condition to guarantee $\mathcal{A}_c \subset \mathcal{X}_\beta$ whenever $\psi(\gamma(x)) \equiv 0$, see Lemma 3.5. Observe that $\mathcal{A}_c$ is closed by definition (see [189]) and bounded from $P \succ 0$ and $z$ being polynomial. Hence, from LaSalle's invariance principle (Theorem 2.4) and the properties of $H_\mathrm{d}$ and $\dot{H}_\mathrm{d}$ in $\mathcal{X}_\beta$, it follows that any trajectory starting in $\mathcal{A}_c$ remains in $\mathcal{A}_c$ whenever (3.24) holds. Besides, from (3.10b) and (3.25), we have

$$\dot{H}_\mathrm{d}(x) = 0 \qquad \Longleftrightarrow \qquad x = x_\mathrm{d},$$

meaning that $A_c$ is the region of attraction. Lastly, condition $x^\star \in \mathcal{A}_c$ reads $c \geq \frac{1}{2}z^\top(x^\star)P^{-1}z(x^\star)$, which by using the generalized Schur complements can be equivalently written as (3.26). □

*Remark* 3.6. A sufficient condition to guarantee (3.25) is $\frac{\partial z}{\partial x}$ being square and nonsingular in $\mathcal{X}_\beta$.

In the controller design, we usually select the region of interest $\mathcal{X}_\beta$ as the whole space $\mathbb{R}^{n_x}$, and if global stability cannot be ensured, we reduce $\mathcal{X}_\beta$, yielding at least local stability for adequate $\Lambda_0$, $z$ and $g_\perp$. In doing so, conditions (3.24) and (3.26) establish upper and lower bounds on $cP$, constraining the size of $\mathcal{A}_c$. Evidently, these conditions can be introduced in the SOS Programs 3.1 and 3.2 as

$$\text{LHSs of (3.24), (3.26) are SOS.}$$

*Example* 3.2 (Example 3.1, continued). Since $\beta(x) \equiv 0$ and $\|z(x)\| \to \infty$ as $\|x\| \to \infty$, we conclude that the asymptotic stability result is global. △

*Example* 3.3. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 x_2 \end{bmatrix} \frac{1}{(x_1^2+1)} + \begin{bmatrix} -x_1 \\ 1 \end{bmatrix} \frac{u}{x_1^2+1} \tag{3.27}$$

with $x := \mathrm{vec}(x_1, x_2) \in \mathcal{X} = \mathbb{R}^2$. Suppose we are now interested in finding an asymptotically stabilizing controller in the origin but with the SOS Program 3.2 in a region of interest $\mathcal{X}_\beta$ defined from

$$\beta(x) = 1 - z^\top(x) S_\beta z(x) \qquad \text{and} \qquad S_\beta = \mathrm{diag}(\frac{1}{2^2}, 1).$$

For this, we pick

$$x_\mathrm{d} = 0 \in \mathcal{X}_a := \left\{ x \in \mathbb{R}^2 \mid x_2 = 0 \right\}, \qquad z(x) = x, \qquad g_\perp(x) = \begin{bmatrix} 1 & x_1 \end{bmatrix}, \qquad \Lambda_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

to fulfill Assumption 3.1. Now, we fix $\epsilon_0 = \epsilon_1 = 10^{-5}$, set $F_1$ and $S_1$ as polynomials in $x$ with maximum degree one and two, respectively, and employ the solution of $F_0$ given by (3.19). Solving the SOS Program 3.2 in SOSTOOLS and SDPT3 including (3.24) and (3.26) with $c = 1$, $x^\star = \mathrm{vec}(1, -0.5)$, $\psi(\gamma(x)) \equiv 0$ and $\eta(x) = \mathrm{vec}(-x_1, 1)$, results in

$$P = \begin{bmatrix} 1.229 & -0.3921 \\ -0.3921 & 0.183 \end{bmatrix}, \qquad F_1(x) = \begin{bmatrix} 35.67x_1 - 0.2495 & 0.02693x_1 - 36.08 \end{bmatrix}.$$

Hence, the controller (3.15) reads

$$u_\mathrm{ida}(x) = -624.4x_2 - 199.3x_1 + 196.8x_1x_2 + 91.78x_1^2$$

and it yields from Propositions 3.2 and 3.3 a locally asymptotically stable closed-loop in the origin of (3.27) with region of attraction $\mathcal{A}_1$. Figure 3.2 shows the $x_1$–$x_2$ plane with $\mathcal{A}_1 \subset \mathcal{X}_\beta$ and the closed-loop trajectory with initial position $x(0) = x^\star$. Figure 3.3 illustrates the evolution of the corresponding states, which clearly converge to the origin asymptotically. △

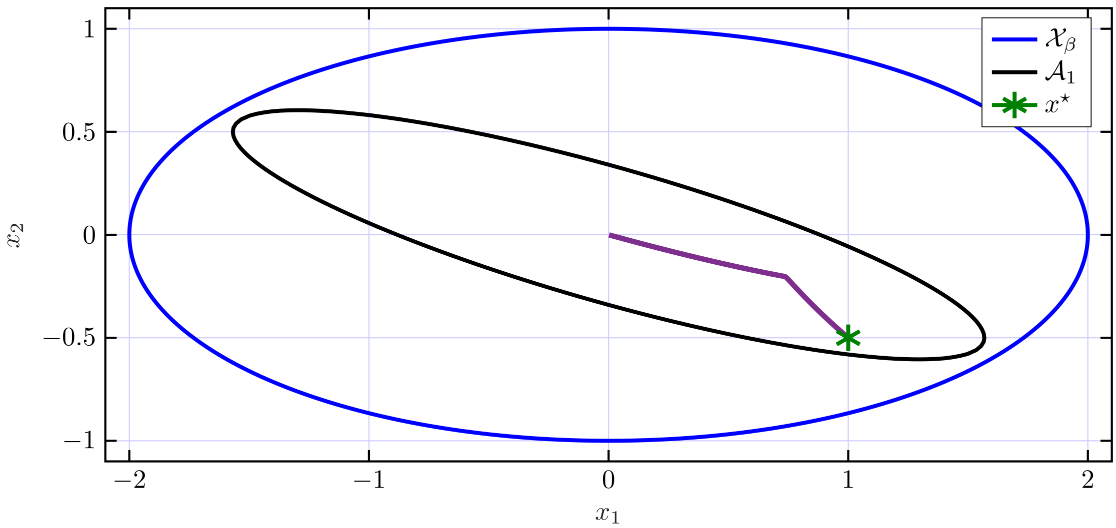



**Figure 3.2.** – Point $x^\star$ and sets $\mathcal{X}_\beta$ and $\mathcal{A}_1$ in Example 3.3.

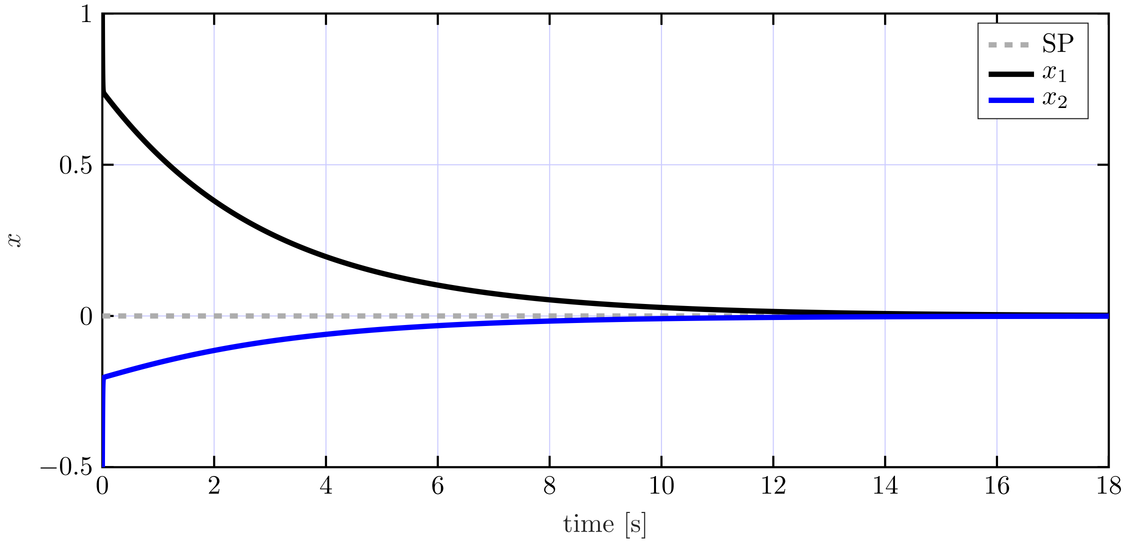



**Figure 3.3.** – Evolution of states in Example 3.3.

Chapter 4

# Algebraic IDA-PBC with Optimization and Input Saturation

In Chapter 3, we develop algebraic solutions for IDA-PBC in a class of affine systems with polynomial structure. The method solves the typical problems of IDA-PBC at the expense of an adequate parameterization and selection of the target Hamiltonian. In this chapter, the previous approach is extended in two directions. First, we incorporate input saturation using the polytope representation. And second, we include four minimization objectives in the SOS programs to address the great flexibility in the controller parameter selection. This chapter is divided in four sections. In Section 4.1 the input saturation problem is studied, whereas Section 4.2 discusses the minimization objectives. Finally, in Sections 4.3 and 4.4, we verify our results on a third-order rational system and the well-known cart-pole.

## 4.1 Input Saturation

In view of Propositions 3.2 and 3.3, we consider input saturation.

### 4.1.1 Restriction of the Control Action

Before presenting our main result in Proposition 4.1, let us define the lower and upper (state dependent) bounds in each input as the set

$$\mathcal{U}_{\mathrm{ida}}(x) := \Big\{ u \in \mathbb{R}^{n_u} \mid 1 - \big(E_i(x)u - d_i(x)\big)^\top U_i^{-1}(x)\big(E_i(x)u - d_i(x)\big) \geq 0,\ i = 1, \ldots, n_e \Big\}$$

for some functions $d_i : \mathcal{X} \to \mathbb{R}^{n_d}$, $U_i : \mathcal{X} \to \mathbb{R}^{n_d \times n_d}$ and $E_i : \mathcal{X} \to \mathbb{R}^{n_d \times n_u}$ with $U_i(x) \succ 0$. For instance, the set

$$\{u = \mathrm{vec}(u_1, u_2) \mid u_1 \in \mathbb{R},\ u_2 \in \mathbb{R},\ 0 \leq u_1 \leq 4,\ -3 \leq u_2 \leq 3\}$$

can be written in our notation with $E_1 = \begin{bmatrix}1 & 0\end{bmatrix}$, $U_1 = 2^2$, $d_1 = 2$, $E_2 = \begin{bmatrix}0 & 1\end{bmatrix}$, $U_2 = 3^2$, $d_2 = 0$ and $n_e = 2$. The following proposition ensures that the stabilizing feedback $u_{\text{ida}}$, as defined in (3.15), belongs to $\mathcal{U}_{\text{ida}}$.

**Proposition 4.1.** *Let all conditions of Propositions 3.2 and 3.3 for local (asymptotic) stability be satisfied. Assume there exist functions $\bar{S}_i : \mathcal{X} \to \mathbb{R}^{(n_d+n_z)\times(n_d+n_z)}$ and $\bar{\Lambda}_i : \mathcal{X} \to \mathbb{R}^{n_d \times n_z}$ with $\bar{S}_i(x) \succeq 0$ and $i = 1, \ldots, n_e$ such that*

$$\bar{E}_i(x)\eta^\top(x)f(x) + d_i(x) - \bar{\Lambda}_i(x)z(x) = 0 \quad \forall x \in \mathcal{X}, \tag{4.1}$$

$$\begin{bmatrix} U_i(x) & \bar{E}_i(x)F_1(x)\frac{\partial^\top z}{\partial x} - \bar{\Lambda}_i(x)P \\ \frac{\partial z}{\partial x}F_1^\top(x)\bar{E}_i^\top(x) - P\bar{\Lambda}_i^\top(x) & \frac{1}{2c}P \end{bmatrix} - \beta(x)\bar{S}_i(x) \succeq 0 \quad \forall x \in \mathcal{X}, \tag{4.2}$$

*where $\bar{E}_i(x) = E_i(x)\left(\eta^\top(x)g(x)\right)^{\mathrm{g}}$. Then, for any initial condition in $\mathcal{A}_c$ (as defined in Proposition 3.3), the stabilizing control law (3.15) is restricted to $\mathcal{U}_{\text{ida}}(x)$.*

**Proof.** From (4.2), $\bar{S}_i(x) \succeq 0$ and the definition of $\mathcal{X}_\beta$, we have

$$\begin{bmatrix} U_i(x) & \bar{E}_i(x)F_1(x)\frac{\partial^\top z}{\partial x} - \bar{\Lambda}_i(x)P \\ \frac{\partial z}{\partial x}F_1^\top(x)\bar{E}_i^\top(x) - P\bar{\Lambda}_i^\top(x) & \frac{1}{2c}P \end{bmatrix} \succeq 0 \qquad \forall x \in \mathcal{X}_\beta.$$

Using the generalized Schur complements (Lemma A.8) in the previous inequality yields

$$\frac{1}{2c}P - \left(\frac{\partial z}{\partial x}F_1^\top(x)\bar{E}_i^\top(x) - P\bar{\Lambda}_i^\top(x)\right)U_i^{-1}(x)\left(\bar{E}_i(x)F_1(x)\frac{\partial^\top z}{\partial x} - \bar{\Lambda}_i(x)P\right) \succeq 0 \qquad \forall x \in \mathcal{X}_\beta. \tag{4.3}$$

Hence, multiplying (4.3) on the right by $P^{-1}z(x)$ and on the left by its transpose and replacing (4.1) gives

$$\frac{1}{2}z^\top(x)(cP)^{-1}z(x) \geq \left(E_i(x)u_{\text{ida}}(x) - d_i(x)\right)^\top U_i^{-1}(x)\left(E_i(x)u_{\text{ida}}(x) - d_i(x)\right) \quad \forall x \in \mathcal{X}_\beta,$$

where $u_{\text{ida}}$ is as defined in (3.15). Now, from $\mathcal{A}_c \subset \mathcal{X}_\beta$ (see Proposition 3.3) and Lemma 3.5, it follows that

$$\mathcal{A}_c \subset \bar{\mathcal{U}}_i := \left\{x \in \mathbb{R}^{n_x} \mid 1 - \left(E_i(x)u_{\text{ida}}(x) - d_i(x)\right)^\top U_i^{-1}(x)\left(E_i(x)u_{\text{ida}}(x) - d_i(x)\right) \geq 0\right\}$$

for $i = 1, \ldots, n_e$, that is, $\mathcal{A}_c \subset \bigcap \bar{\mathcal{U}}_i$. Since $u = u_{\text{ida}}(x)$, we see that $x \in \bigcap \bar{\mathcal{U}}_i$ is equivalent to $u \in \mathcal{U}_{\text{ida}}(x)$. Consequently, from the results of Propositions 3.2 and 3.3, every trajectory starting in $\mathcal{A}_c$ remains in $\mathcal{A}_c$ and has a control action that belongs $\mathcal{U}_{\text{ida}}$, which completes the proof. □

To obtain the restriction of $u_{\text{ida}}$ in $\mathcal{U}_{\text{ida}}$, we first calculate $\bar{\Lambda}_i$ from (4.1), and then solve the conditions of Propositions 3.2 and 3.3 together with (4.2). Observe that if $\bar{\Lambda}_i$, $\bar{E}_i$, $U_i$,

$\frac{\partial z}{\partial x}$ and $\bar{S}_i$ are also polynomial, then (4.2) and $\bar{S}_i(x) \succeq 0$ can be easily introduced in the SOS Program 3.2. Similarly, if the condition (3.16b) and the solution of $F_1$ given by (3.16a) remains polynomial for some user-defined $K_2$, then we can replace such a solution in (4.2) and introduce (3.16b), (4.2) and $\bar{S}_i(x) \succeq 0$ in the SOS Program 3.1.

*Example* 4.1. Consider system (3.20). We shall test Propositions 3.2, 3.3 and 4.1 with the SOS Program 3.2 for synthesizing an IDA-PBC controller constrained to the set $\mathcal{U}_{\text{ida}} := \{u \in \mathbb{R} \mid -10 \le u \le 10\}$ in the region of interest $\mathcal{X}_\beta := \{x \in \mathbb{R}^2 \mid -3 \le x_1 \le 3\}$, i.e., $n_e = 1$, $E_1 = 1$, $d_1 = 0$, $U_1 = 10^2$ and $\beta(x) = 1 - 3^{-2}x_1^2$.

Similar to Example 3.1, we pick $x_{\text{d}} = 0$, $z(x) = \text{vec}(x_1, x_1^2 + x_2)$, $g_\perp = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\Lambda_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $\eta = g$ and $c = 1$, and employ the solution of $F_0$ given by (3.19). Since $\beta$ and $F_0$ are functions in $x_1$, we select for simplicity $F_1$, $S_1$ and $\bar{S}_1$ as polynomials in $x_1$ with maximum degree 1, 2 and 2, respectively. At this point, we impose a lower bound in $P$ by forcing $x^\star = \text{vec}(2, -5) \in \mathcal{A}_1$, i.e., condition (3.26). Let $\epsilon_0 = \epsilon_1 = 10^{-5}$, after solving the SOS Program 3.2 in SOSTOOLS and SDPT3 including (3.24), (3.26), (4.2) and $\bar{S}_1(x) \succeq 0$, where $\psi(\gamma(x)) \equiv 0$, $\bar{\Lambda}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\bar{E}_1 = 1$, we have

$$P = \begin{bmatrix} 3.695 & -0.2888 \\ -0.2888 & 1.105 \end{bmatrix}, \qquad F_1(x) = \begin{bmatrix} 0.01915x_1 + 0.3475 & -1.019x_1 - 6.412 \end{bmatrix}$$

and the asymptotically stabilizing (in $x_{\text{d}} = 0$) controller

$$u_{\text{ida}}(x) = -5.912x_2 - 1.368x_1 - 0.2946x_1x_2 + 0.0352x_1^2(x_2 + x_1^2) - 5.93x_1^2 - 0.2919x_1^3.$$

Figure 4.1 shows the $x_1$–$x_2$ plane with sets $\mathcal{A}_1 \subset \bar{\mathcal{U}}_1 = \left\{x \in \mathbb{R}^2 \mid 1 - u_{\text{ida}}^2(x)U_1^{-1} \ge 0\right\}$, $\mathcal{A}_1 \subset \mathcal{X}_\beta$ and the phase portrait of the closed-loop for 10 extreme initial positions $x(0)$ represented by symbol "$*$", where all trajectories converge to the origin as expected. Since $u = u_{\text{ida}}(x)$, note that $x \in \bar{\mathcal{U}}_1$ is equivalent to $u \in \mathcal{U}_{\text{ida}}(x)$. In addition, Figure 4.2 illustrates 5 seconds of respective control actions, which are all constrained to $\mathcal{U}_{\text{ida}}$. △

### 4.1.2 Saturation

Following the works of [95, 190, 191], among others, we are now ready to introduce the input saturation within the framework of algebraic IDA-PBC (Propositions 3.2 and 3.3) by using a modification of the **polytope** (or **polytopic**) saturation model.

**Proposition 4.2.** *Let the conditions of Propositions 3.2, 3.3 and 4.1 be satisfied for a system of the form (3.3) resulting in some matrices $P$, $F_0$, $F_1$ and a locally (asymptotically) stabilizing controller $u_{\text{ida}}$ that is constrained in $\mathcal{U}_{\text{ida}}$ for every initial condition in the region of attraction $\mathcal{A}_c$. Then, for all $i_k \in \{1, 2\}$ with $k = 1, \dots, m$, there exist functions*

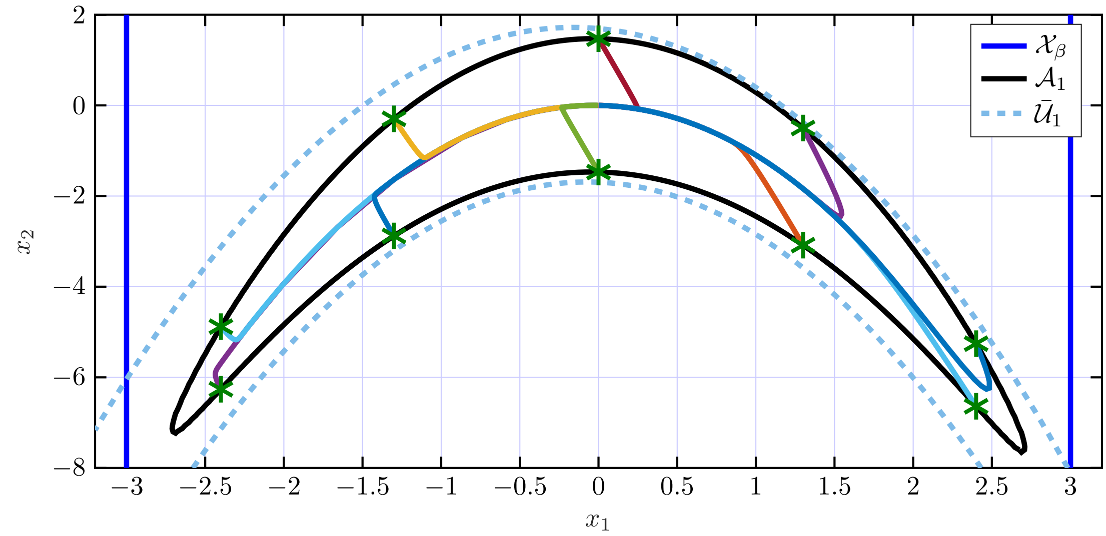



**Figure 4.1.** – Sets $\mathcal{A}_1 \subset \mathcal{X}_\beta$, $\mathcal{A}_1 \subset \bar{\mathcal{U}}_1$ and phase portrait for 10 extreme initial positions in Example 4.1.

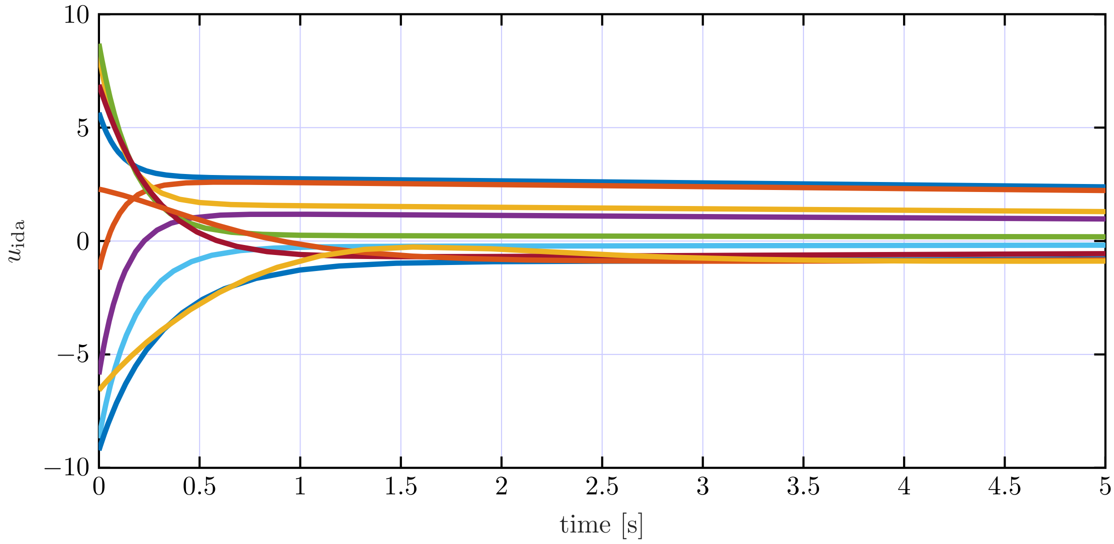



**Figure 4.2.** – Response of control signal $u_{\text{ida}} \in \mathcal{U}_{\text{ida}}$ in Example 4.1.

$F_2 : \mathcal{X} \to \mathbb{R}^{m\times n_x}$ *and* $\hat{S}_{i_1\ldots i_m} : \mathcal{X} \to \mathbb{R}^{n_x\times n_x}$ *with* $\hat{S}_{i_1\ldots i_m}(x) \succeq 0$ *verifying*

$$-\begin{bmatrix} F_0(x) \\ \text{row}_1(F_{i_1}(x)) \\ \vdots \\ \text{row}_m(F_{i_m}(x)) \end{bmatrix} \begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix}^\top - \begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix} \begin{bmatrix} F_0(x) \\ \text{row}_1(F_{i_1}(x)) \\ \vdots \\ \text{row}_m(F_{i_m}(x)) \end{bmatrix}^\top - \beta(x)\hat{S}_{i_1\ldots i_m}(x) \succeq 0 \quad \forall x \in \mathcal{X}. \tag{4.4}$$

*Under such an* $F_2$*, the feedback* $u = u_{\mathcal{F}\text{ida}}(x,\theta)$ *with*

$$u_{\mathcal{F}\text{ida}}(x,\Theta) = u_{\text{ida}}(x) + \left(\eta^\top(x)g(x)\right)^{\text{g}}\Theta\Big(F_2(x) - F_1(x)\Big)\frac{\partial^\top H_\text{d}}{\partial x}(x), \tag{4.5}$$

*and* $\Theta = \text{diag}(\theta_1,\dots,\theta_m)$ *renders the closed-loop stable in the desired equilibrium for any* $\theta_k \in [0,1]$, $k = 1,\dots,m$*. In addition, asymptotic stability is achieved if* (3.14) *and* (4.4) *hold with strict inequalities.*

**Proof.** Define

$$F_3(x,\Theta) := \Theta F_2(x) + (I_m - \Theta)F_1(x)$$

and replace $F_1$ with $F_3$ in (3.15) to obtain (4.5). From Proposition 3.2, the controller (4.5) stabilizes the system (3.3) in the desired equilibrium if (3.14) holds with $F_1$ substituted by $F_3$. For this, suppose $\Theta = \text{diag}(\theta_1,\dots,\theta_m)$ with $\theta_k \in [0,1]$, $k = 1,\dots,m$. Hence,

$$\mathcal{F}(x) := \{F_3(x,\Theta) \mid \theta_k \in \mathbb{R},\, 0 \le \theta_k \le 1,\, k = 1,\dots,m\}$$

is a convex polytope and we can define $\bar\theta_{k1} := 1 - \theta_k$ and $\bar\theta_{k2} := \theta_k$ to obtain

$$\sum_{i_k=1}^{2}\bar\theta_{ki_k} = 1, \qquad \bar\theta_{1i_1}\bar\theta_{2i_2}\dots\bar\theta_{mi_m} \ge 0, \qquad \sum_{i_1\dots i_m}\bar\theta_{1i_1}\bar\theta_{2i_2}\dots\bar\theta_{mi_m} = 1, \tag{4.6}$$

$$F_3(x,\Theta) = \begin{bmatrix}\sum_{i_1=1}^{2}\bar\theta_{1i_1}\,\text{row}_1(F_{i_1}(x))\\ \vdots \\ \sum_{i_m=1}^{2}\bar\theta_{mi_m}\,\text{row}_m(F_{i_m}(x))\end{bmatrix} = \sum_{i_1\dots i_m}\bar\theta_{1i_1}\bar\theta_{2i_2}\dots\bar\theta_{mi_m}\begin{bmatrix}\text{row}_1(F_{i_1}(x))\\ \vdots \\ \text{row}_m(F_{i_m}(x))\end{bmatrix},$$

where the second equality in $F_3$ results from multiplying its first row by

$$\left(\sum_{i_2=1}^{2}\bar\theta_{2i_2}\right)\left(\sum_{i_3=1}^{2}\bar\theta_{3i_3}\right)\dots\left(\sum_{i_m=1}^{2}\bar\theta_{mi_m}\right),$$

its second row by

$$\left(\sum_{i_1=1}^{2}\bar\theta_{1i_1}\right)\left(\sum_{i_3=1}^{2}\bar\theta_{3i_3}\right)\dots\left(\sum_{i_m=1}^{2}\bar\theta_{mi_m}\right)$$

and so on. Now, replacing $F_1$ by $F_3$ in (3.14), where we can define without loss of generality

$$S_1(x) = \sum_{i_1\dots i_m}\bar\theta_{1i_1}\bar\theta_{2i_2}\dots\bar\theta_{mi_m}\hat S_{i_1\dots i_m}(x) \succeq 0,$$

yields

$$\sum_{i_1\dots i_m}\bar\theta_{1i_1}\bar\theta_{2i_2}\dots\bar\theta_{mi_m}\left(-T^\top_{i_1\dots i_m}(x) - T_{i_1\dots i_m}(x) - \beta(x)\hat S_{i_1\dots i_m}(x)\right) \succeq 0 \qquad \forall x \in \mathcal{X}, \tag{4.7}$$

$$T_{i_1\dots i_m}(x) = \begin{bmatrix} F_0(x) \\ \mathrm{row}_1(F_{i_1}(x)) \\ \vdots \\ \mathrm{row}_m(F_{i_m}(x)) \end{bmatrix} \begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix}^\top .$$

Consequently, (4.4) is a sufficient condition for (4.7), meaning that (4.5) is a stabilizing controller for (3.3). The existence of $F_2$ can be demonstrated by setting $F_2(x) := F_1(x)$ and $\hat{S}_{i_1\dots i_m}(x) := S_1(x)$, in which case (4.4) reduces to (3.14) and $u_{\mathcal{F}\mathrm{ida}}(x,\Theta) \equiv u_{\mathrm{ida}}(x)$. The proof of asymptotic stability follows a similar procedure. □

*Remark* 4.1. Let $\theta_1 = \theta_2 = \ldots = \theta_m$, then $i_1 = i_2 = \ldots = i_m$ and (4.4) reduces to

$$-\begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix} \begin{bmatrix} F_0(x) \\ F_2(x) \end{bmatrix}^\top - \begin{bmatrix} F_0(x) \\ F_2(x) \end{bmatrix} \begin{bmatrix} g_\perp(x) \\ \eta^\top(x) \end{bmatrix}^\top - \beta(x)\hat{S}_2(x) \succeq 0 \qquad \forall x \in \mathcal{X}. \tag{4.8}$$

*Remark* 4.2. Let $F_1$ and $F_2$ be given by (3.16), then

$$F_i(x) = -\left[\eta^\top(x) F_0^\top(x) + K_{2i}(x) A_{\mathrm{F}}(x), \quad K_{1i}(x)\right] \left[g_\perp^\top(x) \quad \eta(x)\right]^{-1}, \qquad i = 1, 2$$

and (4.4) reduces to

$$\begin{bmatrix} \mathrm{row}_1(K_{1i_1}(x)) \\ \vdots \\ \mathrm{row}_m(K_{1i_m}(x)) \end{bmatrix} + \begin{bmatrix} \mathrm{row}_1(K_{1i_1}(x)) \\ \vdots \\ \mathrm{row}_m(K_{1i_m}(x)) \end{bmatrix}^\top - \begin{bmatrix} \mathrm{row}_1(K_{2i_1}(x)) \\ \vdots \\ \mathrm{row}_m(K_{2i_m}(x)) \end{bmatrix} A_{\mathrm{F}}(x) \begin{bmatrix} \mathrm{row}_1(K_{2i_1}(x)) \\ \vdots \\ \mathrm{row}_m(K_{2i_m}(x)) \end{bmatrix}^\top \succ 0$$

with $K_{11},\, K_{12} : \mathcal{X} \to \mathbb{R}^{m\times m}$ and $K_{21},\, K_{22} : \mathcal{X} \to \mathbb{R}^{m\times(n_x - m)}$.

The proposition implies that if there is a solution to the conditions of Propositions 3.2, 3.3 and 4.1, then there also exists an (asymptotically) stabilizing controller $u_{\mathcal{F}\mathrm{ida}}(x,\Theta)$, defined in (4.5), that verifies

$$u_{\mathcal{F}\mathrm{ida}}(x,0) = u_{\mathrm{ida}}(x) \in \mathcal{U}_{\mathrm{ida}}(x) \qquad \forall x \in \mathcal{A}_c.$$

If $F_2$ and $\hat{S}_{i_1\dots i_m}$ for all $i_k \in \{1,2\}$, $k = 1,\dots,m$, are polynomial, then the conditions of Proposition 4.2 can be presented as the following SOS program.

**SOS Program 4.1.**

| | |
|---|---|
| find | the coefficients of $F_2$, $\hat{S}_{i_1\dots i_m}$, $i_k \in \{1,2\}$, $k = 1,\dots,m$, |
| subject to | LHS of (4.4) minus $\epsilon_1 I_{n_x}$ is SOS for every $i_k \in \{1,2\}$, $k = 1,\dots,m$. |

Let us denote by $\overline{u} = \mathrm{vec}(\overline{u}_1, \ldots, \overline{u}_{n_u})$ and $\underline{u} = \mathrm{vec}(\underline{u}_1, \ldots, \underline{u}_{n_u})$ the maximum and minimum input values for the saturation, and define the set

$$\mathcal{U}_{\mathrm{sat}} := \{u = \mathrm{vec}(u_1, \ldots, u_{n_u}) \mid u_k \in \mathbb{R},\, \overline{u}_k \geq u_k \geq \underline{u}_k,\, k = 1, \ldots, n_u\}\,.$$

Hence, if $\mathcal{U}_{\mathrm{ida}}(x) \subset \mathcal{U}_{\mathrm{sat}}$ for all $x \in \mathcal{A}_c$, we can employ the feedback $u = u_{\mathcal{F}\mathrm{ida}}(x, \Theta)$ and define a saturation, for every $u_{\mathcal{F}\mathrm{ida}}(x, I_m) \notin \mathcal{U}_{\mathrm{sat}}$, by choosing $\Theta$ such that $u_{\mathcal{F}\mathrm{ida}}(x, \Theta) \in \partial\mathcal{U}_{\mathrm{sat}}$. Figure 4.3 illustrates the situation for a system with two inputs $u = \mathrm{vec}(u_1, u_2)$, where $\mathcal{U}_{\mathrm{ida}}$ is an ellipsoid contained in $\mathcal{U}_{\mathrm{sat}}$, $u_{\text{sat-i}} \in \partial\mathcal{U}_{\mathrm{sat}}$ and $u_{\text{sat-n}} \in \partial\mathcal{U}_{\mathrm{sat}}$ are examples for input saturation, and

$$\mathcal{U}_{\mathcal{F}}(x) := \{u_{\mathcal{F}\mathrm{ida}}(x, \Theta) \in \mathbb{R}^{n_u} \mid \theta_k \in \mathbb{R},\, 0 \leq \theta_k \leq 1,\, k = 1, \ldots, m\}\,.$$

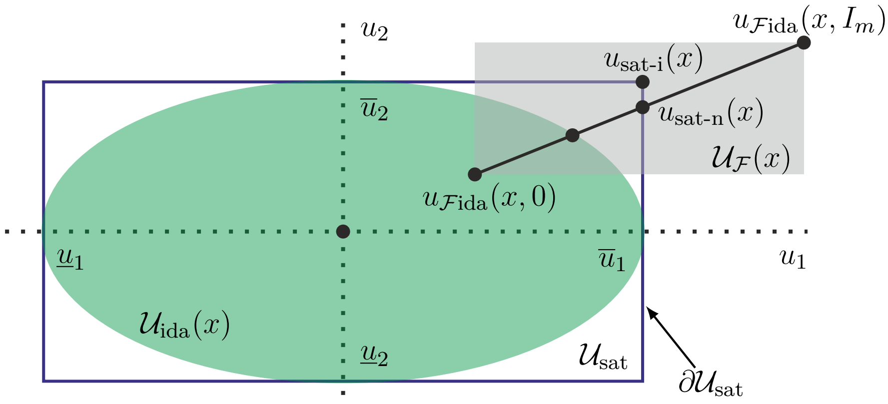



**Figure 4.3.** – Relations of $\mathcal{U}_{\mathrm{ida}}$, $\mathcal{U}_{\mathrm{sat}}$ and $\mathcal{U}_{\mathcal{F}}$ for a system with two inputs.

Suppose that

$$\left(\eta^\top(x)g(x)\right)^{\mathrm{g}} \Theta \equiv \Theta \left(\eta^\top(x)g(x)\right)^{\mathrm{g}}, \tag{4.9}$$

then $\mathcal{U}_{\mathcal{F}}(x)$ is an orthotope (or hyperrectangle[32]), and we can employ the independent input saturation of [95, 190–193], meaning that the controller is now

$$u_{\text{sat-i}}(x) = \mathrm{vec}(\bar{\rho}_1, \ldots, \bar{\rho}_{n_u}), \tag{4.10}$$
$$\bar{\rho}_k = \begin{cases} \overline{u}_k, & \text{if } \mathrm{row}_k(u_{\mathcal{F}\mathrm{ida}}(x, I_m)) > \overline{u}_k, \\ \underline{u}_k, & \text{if } \mathrm{row}_k(u_{\mathcal{F}\mathrm{ida}}(x, I_m)) < \underline{u}_k, \quad k = 1, \ldots n_u, \\ \mathrm{row}_k(u_{\mathcal{F}\mathrm{ida}}(x, I_m)), & \text{otherwise.} \end{cases}$$

for all $i_k \in \{1, 2\}$ with $k = 1, \ldots, m$

Implementation of $u_{\text{sat-i}}$ demands the solution of (4.4) for all the values of $i_k \in \{1, 2\}$ with $k = 1, \ldots, m$, i.e., solving $2^m$ inequalities while searching for $F_2$ and $2^m$ positive

[32]A hyperrectangle is the generalization of a rectangle for higher dimensions.

semidefinite polynomial matrices $\hat{S}_{i_1 \dots i_m}$. To guarantee condition (4.9) and reduce the number of matrices and inequalities, we can fix $\theta_1 = \theta_2 = \ldots = \theta_m$, see Remark 4.1. As a consequence the set $\mathcal{U}_{\mathcal{F}}(x)$ is a line between $u_{\mathcal{F}\text{ida}}(x, 0)$ and $u_{\mathcal{F}\text{ida}}(x, I_m)$ that intersect $\partial\mathcal{U}_{\text{sat}}$ in at least a point whenever $u_{\mathcal{F}\text{ida}}(x, I_m) \notin \mathcal{U}_{\text{sat}}$, see Figure 4.3. The resulting controller, which we denote by $u_{\text{sat-n}}$, can be written as

$$u_{\text{sat-n}}(x) := u_{\mathcal{F}\text{ida}}(x, \min(\rho_1, \ldots, \rho_{n_u}) I_m), \tag{4.11}$$

$$\rho_k = \begin{cases} \frac{\overline{u}_k - \text{row}_k(u_{\mathcal{F}\text{ida}}(x,0))}{|\text{row}_k(u_\delta(x))|}, & \text{if } \text{row}_k(u_{\mathcal{F}\text{ida}}(x, I_m)) > \overline{u}_k, \\ \frac{\text{row}_k(u_{\mathcal{F}\text{ida}}(x,0) - \underline{u}_k)}{|\text{row}_k(u_\delta(x))|}, & \text{if } \text{row}_k(u_{\mathcal{F}\text{ida}}(x, I_m)) < \underline{u}_k, \quad k = 1, \ldots n_u, \\ 1, & \text{otherwise}, \end{cases}$$

$$u_\delta(x) = u_{\mathcal{F}\text{ida}}(x, I_m) - u_{\mathcal{F}\text{ida}}(x, 0) = \left(\eta^\top(x) g(x)\right)^{\text{g}} \left(F_2(x) - F_1(x)\right) \frac{\partial^\top H_{\text{d}}}{\partial x}(x).$$

## 4.2 Minimization Objectives

If the conditions of Proposition 3.2 have a solution for $P$ and $F_1$, this solution is non-unique, meaning that some properties in the controller design can be further specified. In general, we can do so by imposing new constraints (e.g. with Propositions 3.3 and 4.1) and establishing a minimization objective. In this section, we take the second approach, providing four optimizations that can be included in the SOS Programs 3.1, 3.2 and 4.1. The lemma below will be used in the proof of the first two.

**Lemma 4.1.** *Let $A \in \mathbb{R}^{n \times n}$ be a positive definite matrix, then*

$$\text{trace}(I_n - A^{-1}) \leq \ln(\det(A)) \leq \text{trace}(A - I_n). \tag{4.12}$$

**Proof.** Since $A$ is positive definite, its eigenvalues are positive. Therefore, replacing the well-know identities $\text{trace}(A^k) = \sum_{i=1}^n \lambda_i^k(A)$ and $\det(A) = \prod_{i=1}^n \lambda_i(A)$ in (4.12) reads

$$\sum_{i=1}^n \frac{\lambda_i(A) - 1}{\lambda_i(A)} \leq \sum_{i=1}^n \ln(\lambda_i(A)) \leq \sum_{i=1}^n (\lambda_i(A) - 1)\,,$$

where $\lambda_i(A)$ is the $i$-th eigenvalue of $A$. Now, the proof is a direct consequence of the logarithm inequality [194, Section 4.5]

$$\frac{x-1}{x} \leq \ln(x) \leq x - 1 \qquad \forall x \in \mathbb{R},\ x \geq 0. \qquad \square$$

**Optimization 4.1** (Volume maximization of $\mathcal{A}_c$)**.**

$$\text{minimize} \quad \text{trace}(Y)$$

$$\text{subject to} \quad \begin{bmatrix} cP & I_{n_z} \\ I_{n_z} & Y \end{bmatrix} \succeq 0. \tag{4.13}$$

**Proof.** The set $\mathcal{A}_c$ with respect to the coordinates $\bar{z} = z(x)$ is an ellipsoid whose volume is proportional to $\sqrt{\det(cP)}$, see [87, pp. 48-49]. Therefore, maximizing the volume of $\mathcal{A}_c$ with $cP \succ 0$ and $\bar{z} = z(x)$ is equivalent to maximize $\ln(\det(cP))$. From Lemma 4.1, we achieve this objective by enlarging the minimum bound of $\ln(\det(cP))$, i.e., minimizing $\text{trace}(c^{-1}P^{-1})$, which is equivalent to Optimization 4.1 with Schur complement in (4.13). □

Optimization 4.1 maximizes the volume of $\mathcal{A}_c$ by maximizing the minimum bound of $cP$ given by $Y^{-1}$. Note that $\mathcal{A}_c$ is not necessarily an ellipsoid (see Figure 4.1) meaning that Optimization 4.1 gives only an approximation. This optimization is also used empirically in [191].

**Optimization 4.2** (Volume minimization of $\mathcal{U}_{\text{ida}}$)**.**

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^{n_e} \text{trace}(U_i) \\ &\text{subject to} && U_i \text{ is constant, } U_i - \epsilon_2 I_{n_d} \succeq 0 \text{ for all } i = 1, \ldots, n_e, \end{aligned} \tag{4.14}$$

where $\epsilon_2 > 0$ is a user defined constant.

**Proof.** The proof is along the same lines of Optimization 4.1, except that we consider the ellipsoids $\left\{\hat{u} \in \mathbb{R}^{n_d} \mid 1 \geq \frac{1}{2}\hat{u}^\top U_i^{-1}\hat{u}\right\}$ with $\hat{u} = E_i(x)u_{\text{ida}}(x) - d_i(x)$ and the upper bound of (4.12). □

Using the Schur complements in (4.2), for all $x \in \mathcal{X}_\beta$, reads

$$U_i - 2c\left(P^{-1}\bar{E}_i^\top(x) - \bar{\Lambda}_i^\top(x)\right)^\top P\left(P^{-1}\bar{E}_i^\top(x) - \bar{\Lambda}_i^\top(x)\right) \succeq 0.$$

Since $\left(\bar{E}_i(x)P^{-1} - \bar{\Lambda}_i(x)\right)z(x) = \bar{E}_i(x)u_{\text{ida}} - d_i(x)$, the previous inequality shows that minimization of $U_i$ (upper bound of $u$) implies minimizing $\bar{E}_i(x)u_{\text{ida}} - d_i(x)$ and the upper bound of $P$. As a consequence, it is require to have at least one minimum bound on $P$ like $P - \epsilon_0 I_{n_z} \succeq 0$, condition (3.26), or an optimization objective (see Optimization 4.1).

Before introducing Optimizations 4.3 and 4.4, let us consider the Jacobian linearization of system (3.3) about the admissible equilibrium $x_{\text{d}}$, including an exogenous disturbance $\bar{w} \in \mathbb{R}^{n_w}$ and a linear output $\bar{y} \in \mathbb{R}^{n_y}$ related to the $\mathcal{H}_2$ performance by means of $B_w \in \mathbb{R}^{n_x \times n_w}$, $C_x \in \mathbb{R}^{n_y \times n_x}$ and $D_u \in \mathbb{R}^{n_y \times n_u}$:

$$\dot{\bar{x}} = \left.\frac{\partial(f + gu)}{\partial x}\right|_{\substack{x=x_{\text{d}} \\ u=u_{\text{d}}}} \bar{x} + \left.\frac{\partial(f + gu)}{\partial u}\right|_{\substack{x=x_{\text{d}} \\ u=u_{\text{d}}}} \bar{u} + B_w \bar{w}$$

$$= \underbrace{\left( \left.\frac{\partial f}{\partial x}\right|_{x=x_\mathrm{d}} + \sum_{i=1}^{n_u} \mathrm{row}_i(u_\mathrm{d}) \left.\frac{\partial\, \mathrm{col}_i(g)}{\partial x}\right|_{x=x_\mathrm{d}} \right)}_{=:A_x} \bar{x} + \underbrace{g(x_\mathrm{d})}_{=:B_u} \bar{u} + B_w \bar{w},$$

$$\bar{y} = C_x \bar{x} + D_u \bar{u},$$

where $\bar{x} = x - x_\mathrm{d}$ and $\bar{u} = u - u_\mathrm{d}$.

**Optimization 4.3** (Standard $\mathcal{H}_2$ optimal local performance assignment)**.**

$$\begin{aligned} \text{minimize} \quad & \mathrm{trace}(D) \\ \text{subject to} \quad & -W_\mathrm{d}(x_\mathrm{d}) - W_\mathrm{d}^\top(x_\mathrm{d}) - B_w B_w^\top - \epsilon_3 I_{n_x} \succeq 0, & \text{(4.15a)} \\ & \left.\begin{bmatrix} D & C_1 \frac{\partial z}{\partial x}^+ P + C_2 \frac{\partial^\top z}{\partial x} \\ \left(C_1 \frac{\partial z}{\partial x}^+ P + C_2 \frac{\partial^\top z}{\partial x}\right)^\top & P \end{bmatrix}\right|_{x=x_\mathrm{d}} \succeq 0, & \text{(4.15b)} \end{aligned}$$

where $D \in \mathbb{R}^{n_y \times n_y}$, $\epsilon_3 > 0$ is a sufficiently small constant,

$$C_1 = C_x - D_u \left.\frac{\partial (\eta^\top g)^\mathrm{g} \eta^\top f}{\partial x}\right|_{x=x_\mathrm{d}}, \qquad C_2 = D_u \left(\eta^\top(x_\mathrm{d}) g(x_\mathrm{d})\right)^\mathrm{g} F_1(x_\mathrm{d}).$$

**Proof.** Define $u_\mathrm{d} = u_\mathrm{ida}(x_\mathrm{d})$, $K_x = \left.\frac{\partial u_\mathrm{ida}}{\partial x}\right|_{x=x_\mathrm{d}}$ and $\bar{u} = K_x \bar{x}$, with $u_\mathrm{ida}$ as defined in (3.15), then

$$\begin{aligned} \dot{\bar{x}} &= (A_x + B_u K_x)\, \bar{x} + B_w \bar{w} \\ &= \left( \left.\frac{\partial f}{\partial x}\right|_{x=x_\mathrm{d}} + \sum_{i=1}^{n_u} \mathrm{row}_i(u_\mathrm{ida}(x_\mathrm{d})) \left.\frac{\partial\, \mathrm{col}_i(g)}{\partial x}\right|_{x=x_\mathrm{d}} + g(x_\mathrm{d}) \left.\frac{\partial u_\mathrm{ida}}{\partial x}\right|_{x=x_\mathrm{d}} \right) \bar{x} + B_w \bar{w} \\ &= \left.\frac{\partial f + g u_\mathrm{ida}}{\partial x}\right|_{x=x_\mathrm{d}} \bar{x} + B_w \bar{w} = \left.\frac{\partial W_\mathrm{d} \frac{\partial^\top H_\mathrm{d}}{\partial x}}{\partial x}\right|_{x=x_\mathrm{d}} \bar{x} + B_w \bar{w} \\ &= W_\mathrm{d}(x_\mathrm{d}) \left.\left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)\right|_{x=x_\mathrm{d}} \bar{x} + B_w \bar{w}, \\ \bar{y} &= C_x \bar{x} + D_u \bar{u} = \left( C_x + D_u \left.\frac{\partial u_\mathrm{ida}}{\partial x}\right|_{x=x_\mathrm{d}} \right) \bar{x} = \left( C_1 + C_2 \left.\left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)\right|_{x=x_\mathrm{d}} \right) \bar{x}, \end{aligned}$$

where the last equality in $\dot{\bar{x}}$ and $\bar{y}$ are obtained from

$$\left.\frac{\partial^\top H_\mathrm{d}}{\partial x}\right|_{x=x_\mathrm{d}} = 0, \qquad \left.\frac{\partial^2 H_\mathrm{d}}{\partial x^2}\right|_{x=x_\mathrm{d}} = \left.\left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)\right|_{x=x_\mathrm{d}}$$

and the definitions of $C_1$ and $C_2$. From Lemma A.5, it follows that the linearized closed-loop is internally stable with a standard $\mathcal{H}_2$ performance $J_{\mathcal{H}_2} < \gamma$ if and only if there exists a matrix $Q \succ 0$ such that

$$-W_{\mathrm{d}}(x_{\mathrm{d}}) \left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)\bigg|_{x=x_{\mathrm{d}}} Q - Q \left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)\bigg|_{x=x_{\mathrm{d}}} W_{\mathrm{d}}^\top(x_{\mathrm{d}}) - B_w B_w^\top \succ 0, \quad (4.16)$$

$$\operatorname{trace}\left( \left( \left( C_1 + C_2 \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right) Q \left( C_1 + C_2 \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)^\top \right) \right)\bigg|_{x=x_{\mathrm{d}}} < \gamma. \quad (4.17)$$

Hence, the optimal performance is obtained by finding $C_2$ and $Q$ such that the upper bound $\gamma$ is minimized. Since $P$ and $\frac{\partial z}{\partial x}$ are full rank, let $Q = \left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)^{-1}\Big|_{x=x_{\mathrm{d}}}$ and let $D$ be a matrix such that

$$\left( C_1 + C_2 \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right) Q \left( C_1 + C_2 \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)^\top \bigg|_{x=x_{\mathrm{d}}} \preceq D. \quad (4.18)$$

Then, (4.15a) is a sufficient condition for (4.16), and minimizing $\gamma$ can be achieved by minimizing the trace of $D$ in (4.18). Using the Schur complements and the pseudoinverse of $\frac{\partial z}{\partial x}$, we can rewrite (4.18) as

$$\begin{aligned} 0 &\preceq \begin{bmatrix} D & C_1 + C_2 \left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)\Big|_{x=x_{\mathrm{d}}} \\ C_1^\top + \left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)\Big|_{x=x_{\mathrm{d}}} C_2^\top & \left( \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x} \right)\Big|_{x=x_{\mathrm{d}}} \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ 0 & \frac{\partial^\top z}{\partial x} P^{-1} \end{bmatrix} \begin{bmatrix} D & C_1 \frac{\partial z}{\partial x}^+ P + C_2 \frac{\partial^\top z}{\partial x} \\ \left( C_1 \frac{\partial z}{\partial x}^+ P + C_2 \frac{\partial^\top z}{\partial x} \right)^\top & P \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P^{-1} \frac{\partial z}{\partial x} \end{bmatrix}\Bigg|_{x=x_{\mathrm{d}}}. \end{aligned}$$

Consequently, (4.15b) is a sufficient condition for (4.18), and the proof is complete. □

*Remark* 4.3. The Linear Quadratic Regulator (LQR) problem, which consist of minimizing a performance index of the form

$$\min_{u \in \mathcal{L}_2[0,\infty)} \int_0^\infty \left( \bar{x}^\top(t) Q_{\mathrm{lqr}} \bar{x}(t) + \bar{u}^\top(t) R_{\mathrm{lqr}} \bar{u}(t) + \bar{x}^\top(t) N_{\mathrm{lqr}} \bar{u}(t) \right) \mathrm{d}t$$

with weighting matrices $Q_{\mathrm{lqr}} \in \mathbb{R}^{n_x \times n_x}$, $R_{\mathrm{lqr}} \in \mathbb{R}^{n_u \times n_u}$ and $N_{\mathrm{lqr}} \in \mathbb{R}^{n_x \times n_u}$, can be recast as the standard $\mathcal{H}_2$ optimal control (Optimization 4.3) by setting

$$B_w = I_{n_x}, \qquad \begin{bmatrix} C_x^\top \\ D_u^\top \end{bmatrix} \begin{bmatrix} C_x & D_u \end{bmatrix} = \begin{bmatrix} Q_{\mathrm{lqr}} & \frac{1}{2} N_{\mathrm{lqr}} \\ \frac{1}{2} N_{\mathrm{lqr}}^\top & R_{\mathrm{lqr}} \end{bmatrix},$$

see [195].

Optimization 4.3 assigns an standard $\mathcal{H}_2$ optimal local performance to the closed-loop system with controller (3.15). In other words, we can locally guarantee a maximum peak value of the output $\bar{y}$ in response to exogenous input $\bar{w}$ with unit energy. However, for

MIMO systems, the standard $\mathcal{H}_2$ problem is not consistent with the $\mathcal{H}_2$ norm interpretation, as opposed to the generalized $\mathcal{H}_2$, which is [196]. We obtain the generalized $\mathcal{H}_2$ performance as follows.

**Optimization 4.4** (Generalized $\mathcal{H}_2$ optimal local performance assignment)**.**

$$\begin{aligned} \text{minimize} \quad & \gamma \\ \text{subject to} \quad & -W_\text{d}(x_\text{d}) - W_\text{d}^\top(x_\text{d}) - B_w B_w^\top - \epsilon_3 I_{n_x} \succeq 0, && \text{(4.19a)} \\ & \left.\begin{bmatrix} I_{n_y}\gamma & C_1 \frac{\partial z}{\partial x}^+ P + C_2 \frac{\partial^\top z}{\partial x} \\ \left(C_1 \frac{\partial z}{\partial x}^+ P + C_2 \frac{\partial^\top z}{\partial x}\right)^\top & P \end{bmatrix}\right|_{x=x_\text{d}} \succeq 0, && \text{(4.19b)} \end{aligned}$$

where $C_1$ and $C_2$ are as defined in Optimization 4.3.

**Proof.** Along the same lines of the proof of Optimization 4.3 but with (4.17) replaced by

$$\sigma_\text{max}\left(\left.\left(C_1 + C_2 \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x}\right) Q \left(C_1 + C_2 \frac{\partial^\top z}{\partial x} P^{-1} \frac{\partial z}{\partial x}\right)^\top\right)\right|_{x=x_\text{d}} < \gamma,$$

and using Lemma A.2. □

Unlike Optimizations 4.1 and 4.2, Optimizations 4.3 and 4.4 can be used with global asymptotic stability. Lastly, we should point out that the optimizations presented here are only a fraction of the feasible optimizations objectives, see [197, 198] for instance, where objectives as $H_\infty$, passivity, asymptotic disturbance rejection, robust stability, among others, are employed. In addition, the optimizations can be combined in many cases, as shown below.

**Optimization 4.5** (Maximization of $\mathcal{A}_c$ with standard $\mathcal{H}_2$ local performance)**.**

$$\begin{aligned} \text{minimize} \quad & a_Y \operatorname{trace}(Y) + \operatorname{trace}(D) \\ \text{subject to} \quad & (4.13),\ (4.15) \end{aligned}$$

with $a_Y > 0$ being a user-defined constant.

**Optimization 4.6** (Maximization of $\mathcal{A}_c$, minimization of $\mathcal{U}_\text{ida}$ and standard $\mathcal{H}_2$ local performance)**.**

$$\begin{aligned} \text{minimize} \quad & a_Y \operatorname{trace}(Y) + \operatorname{trace}(D) + a_U \sum_{i=1}^{n_e} \operatorname{trace}(U_i) \\ \text{subject to} \quad & (4.13),\ (4.14),\ (4.15) \end{aligned}$$

with $a_Y, a_U > 0$ being user-defined constants.

The following table outlines the optimization objectives.

| Optimization | Description |
|---|---|
| Optimization 4.1 | Volume maximization for the region of convergence $\mathcal{A}_c$. |
| Optimization 4.2 | Minimization of the control action $u_{\text{ida}}$ for any initial condition in $\mathcal{A}_c$. |
| Optimization 4.3 | Standard $\mathcal{H}_2$ local optimal performance assignment in $x_{\text{d}}$. By using Remark 4.3, Optimization 4.3 reduces to the LQR local assignment. |
| Optimization 4.4 | Generalized $\mathcal{H}_2$ local optimal performance assignment in $x_{\text{d}}$. |
| Optimization 4.5 | Since maximizing the volume of $\mathcal{A}_c$ may yield poor or undesired performance, Optimization 4.5 combines Optimization 4.1 with the standard $\mathcal{H}_2$ local optimal performance assignment. |
| Optimization 4.6 | Since Optimizations 4.1 and 4.2 may yield poor or undesired performance, Optimization 4.6 combines them with the standard $\mathcal{H}_2$ local optimal performance assignment. |

**Table 4.1.** – Optimization objectives for IDA-PBC with SOS programs.

The subsequent algorithm summarizes the discussion of Sections 3.2.1, 3.2.4, 4.1 and 4.2 under SOS programs.

**Algorithm 4.1** IDA-PBC with SOS programs for polynomial systems.

**Require:** A nonlinear affine system of the form (3.3).

1: Select the region of interest $\mathcal{X}_\beta$ with a polynomial function $\beta$. Find polynomial $g_\perp$, $\Lambda_0$ and $z$ verifying Assumption 3.1 with $x_{\text{d}} \in \mathcal{X}_{\text{a}} := \{x \in \mathcal{X} \mid g_\perp(x) f(x) = 0\}$.

2: Pick $\epsilon_0$, $\epsilon_1$, the polynomial order for $F_0$ and $S_{11}$, and solve the SOS Program 3.1 e.g., with SOSTOOOLS and a SDP solver. In the case $\dim(z) = \dim(x)$, we can use (3.19) to compute the solution of $F_0$ and simplify the SOS program whenever the right-hand side of (3.19) is polynomial for every $P$. If solver converges proceed to next step, otherwise return to step 1 and reduce the region of interest $\mathcal{X}_\beta$ or select different functions $g_\perp$, $\Lambda_0$ and $z$.

3: Choose a polynomial $\eta$ and the polynomial order for $F_1$ and $S_1$ such that the order of $S_1$ is greater than or equal to the order of $S_{11}$, and solve the SOS Program 3.2.[33] Here, we can include additional constraints such as

*i)* determining the invariant set (or region of convergence) $\mathcal{A}_c$ with (3.24) and $\beta(x) := 1 - z^\top(x) S_\beta z(x)$ for some user defined $S_\beta \succeq 0$;

*ii)* introducing a desired state $x^\star$ in $\mathcal{A}_c$ with (3.26); and[34]

*iii)* restricting the stabilizing control law (3.15) to a set $\mathcal{U}_{\text{ida}}$ with (4.2), where $\bar{\Lambda}_i$ is obtained from (4.1).

[33] Since the computational cost of SOS programs depends on the number of unknown coefficients, we should aim at a low polynomial order for $F_0$, $F_1$ and $S_1$.

[34] The state $x^\star$ can represent a tentative initial condition.

Note that the last constraint requires setting the polynomial order of $\bar{S}_i$ with $i = 1, \ldots, n_e$. Similarly, we can consider the optimization objectives summarized in Table 4.1.

4: Calculate $\gamma$, and select $\psi$, $\left(\eta^\top(x)g(x)\right)^{\text{g}}$, $g_\perp$ and $\nu$ to build the feedback (3.15) whenever no saturation is required. Otherwise, choose the polynomial order of $F_2$, $\hat{S}_{i_1 \ldots i_m}$, $i_k \in \{1, 2\}$, $k = 1, \ldots, m$, and solve the SOS Program 4.1 to build the controllers (4.10) or (4.11). In SOS Program 4.1 we can also include additional constraints and optimization objectives.

5: For $\epsilon_1 > 0$, asymptotic stability in $\mathcal{A}_c$ can be verified if $\frac{\partial z}{\partial x}$ is square and nonsingular in $\mathcal{X}_\beta$, or more generally if (3.25) holds.

## 4.3 Application on a Third-order System

Consider the third-order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & 0 \\ -x_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_2 - x_3 - x_1x_2 - x_2^2 \\ x_1x_2 \\ x_1x_2 \end{bmatrix}}_{=:f(x)} + \underbrace{\begin{bmatrix} 1 & x_1 & 0 \\ -x_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=:g(x)} u, \tag{4.20}$$

where $x := \text{vec}(x_1, x_2, x_3) \in \mathcal{X} = \mathbb{R}^3$ and $u = \text{vec}(u_1, u_2)$. In the region of interest $\mathcal{X}_\beta$, defined from

$$\beta(x) = 1 - \frac{1}{2^2}x_1 - \frac{1}{2^2}x_2,$$

we wish to test Algorithm 4.1 to synthesize the IDA-PBC asymptotically stabilizing controllers $u_{\text{ida}}$ (without saturation) and $u_{\text{sat-n}}$ (including saturation), as defined in (3.15) and (4.11), respectively, which are both bounded to the set

$$\mathcal{U}_{\text{sat}} := \left\{u \in \mathbb{R}^2 \mid -3 \leq u_i \leq 3,\ i = 1, 2\right\}.$$

### 4.3.1 Algebraic IDA-PBC Design

**Step 1:** Assumption 3.1 is fulfilled by choosing

$$\Lambda_0 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, \quad g_\perp(x) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ -x_1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad z(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + x_1x_2 + x_2^2 \end{bmatrix}$$

and $x_{\text{d}} = \text{vec}(0, 0, 0) \in \mathcal{X}_a = \{x \in \mathcal{X} \mid x_2 - x_3 - x_1x_2 - x_2^2 = 0\}$. Note that $\frac{\partial z}{\partial x}$ is unimodular and $\Lambda_0$ is constant, meaning that $F_0$ has a unique and polynomial solution, see (3.19).

**Step 2:** We set $\epsilon_0 = \epsilon_1 = 10^{-5}$, $S_{11}$ with maximum polynomial order 2, and use SOS Program 3.1 as a fast indicator (that SOS Program 3.2 can work), which is met successfully because the SDP solver (from SOSTOOOLS) converges to a solution.

**Step 3:** To have a control action that remains in $\mathcal{U}_{\text{ida}} = \mathcal{U}_{\text{sat}}$, we consider the constraints (3.24), (4.1) and (4.2), where $S_\beta = \frac{1}{2^2}\operatorname{diag}(1,1,0)$, $\psi(\gamma(x)) \equiv 0$, $n_e = 2$, $d_1 = d_2 = 0$, $U_1 = U_2 = 3^2$, $E_1 = \begin{bmatrix}1 & 0\end{bmatrix}$ and $E_2 = \begin{bmatrix}0 & 1\end{bmatrix}$. Let

$$\eta^\top(x) = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & x_1 & 0\\ -x_1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},$$

then $\left(\eta^\top(x)g(x)\right)^{\text{g}} = I_2$, $\bar{E}_1 = E_1$, $\bar{E}_2 = E_2$ and a solution to (4.1) is

$$\bar{\Lambda}_1(x) = \begin{bmatrix}x_2 & 0 & 0\end{bmatrix}, \qquad \bar{\Lambda}_2(x) = \begin{bmatrix}0 & x_1 & 0\end{bmatrix}.$$

Since $\beta$, $F_0$, $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ are functions in $(x_1, x_2)$, we select for simplicity $F_1$, $S_1$, $\bar{S}_1$ and $\bar{S}_2$ as polynomials in $(x_1, x_2)$ with maximum degree 2. Furthermore, we include Optimization 4.5 to have a LQR optimal local performance while increasing the region of convergence with bounded input. Solving the SOS Program 3.2 with (3.24), (4.2) and Optimization 4.5 in SOSTOOOLS and SDPT3, where $c = 1$, $a = 20$, $Q_{\text{lqr}} = I_3$, $R_{\text{lqr}} = I_2$ and $\epsilon_3 = 10^{-5}$, yields

$$P = \begin{bmatrix}1.425 & 0.04322 & 0.5432\\ 0.04322 & 1.188 & 0.9377\\ 0.5432 & 0.9377 & 1.888\end{bmatrix}, \qquad F_1(x) = \begin{bmatrix}F_{1\text{a}}(x) & F_{1\text{b}}(x) & F_{1\text{c}}(x)\end{bmatrix},$$

$$F_{1\text{a}}(x) = \begin{bmatrix}0.3222x_1 + 0.3632x_2 + 0.05766x_1x_2 - 0.01062x_1^2 + 0.08363x_2^2 - 0.25\\ 0.1724x_2 - 0.1728x_1 - 0.6683x_1x_2 - 0.1107x_1^2 - 0.6614x_2^2 + 0.9502\end{bmatrix},$$

$$F_{1\text{b}}(x) = \begin{bmatrix}0.001546x_1^2 - 0.1326x_2 - 0.03595x_1x_2 - 0.156x_1 - 0.02119x_2^2 - 0.5\\ 0.6326x_1 + 0.435x_2 + 0.004657x_1x_2 - 0.1438x_1^2 + 0.2359x_2^2 - 0.04032\end{bmatrix},$$

$$F_{1\text{c}}(x) = \begin{bmatrix}0.6167x_1 + 0.6915x_2 + 0.145x_1x_2 + 0.3096x_1^2 - 0.2017x_2^2 + 0.04032\\ 1.059x_1 + 0.7465x_2 - 1.664x_1x_2 - 0.8575x_1^2 - 0.9628x_2^2 - 0.5\end{bmatrix}.$$

Figure 4.4 shows the sets $\mathcal{A}_1 \subset \mathcal{X}_\beta$, $\mathcal{A}_1 \subset \bar{\mathcal{U}}_1$ and $\mathcal{A}_1 \subset \bar{\mathcal{U}}_2$ in the planes $x_3 = -1$, $x_3 = 0$ and $x_3 = 1$. Here, $\bar{\mathcal{U}}_i = \left\{x \in \mathcal{X} \mid 1 - (E_i u_{\text{ida}}(x))^2\, U_i^{-1} \geq 0\right\}$ and $x \in \bigcap \bar{\mathcal{U}}_i$ is equivalent to $u \in \mathcal{U}_{\text{ida}}$.

**Steps 4 and 5:** Since $\left(\eta^\top(x)g(x)\right) = I_2$ and $g_\perp(x)\nu = 0$, we can build the controller $u_{\text{ida}}$ from (3.15) that renders the closed loop asymptotically stable in the origin from any initial condition in $\mathcal{A}_1$ because $\frac{\partial z}{\partial x}$ is nonsingular. On the other hand, we recall that $u_{\text{sat-n}}$ is obtained from $\theta_1 = \theta_2$. Hence, condition (4.4) reduces to (4.8), and solving the SOS

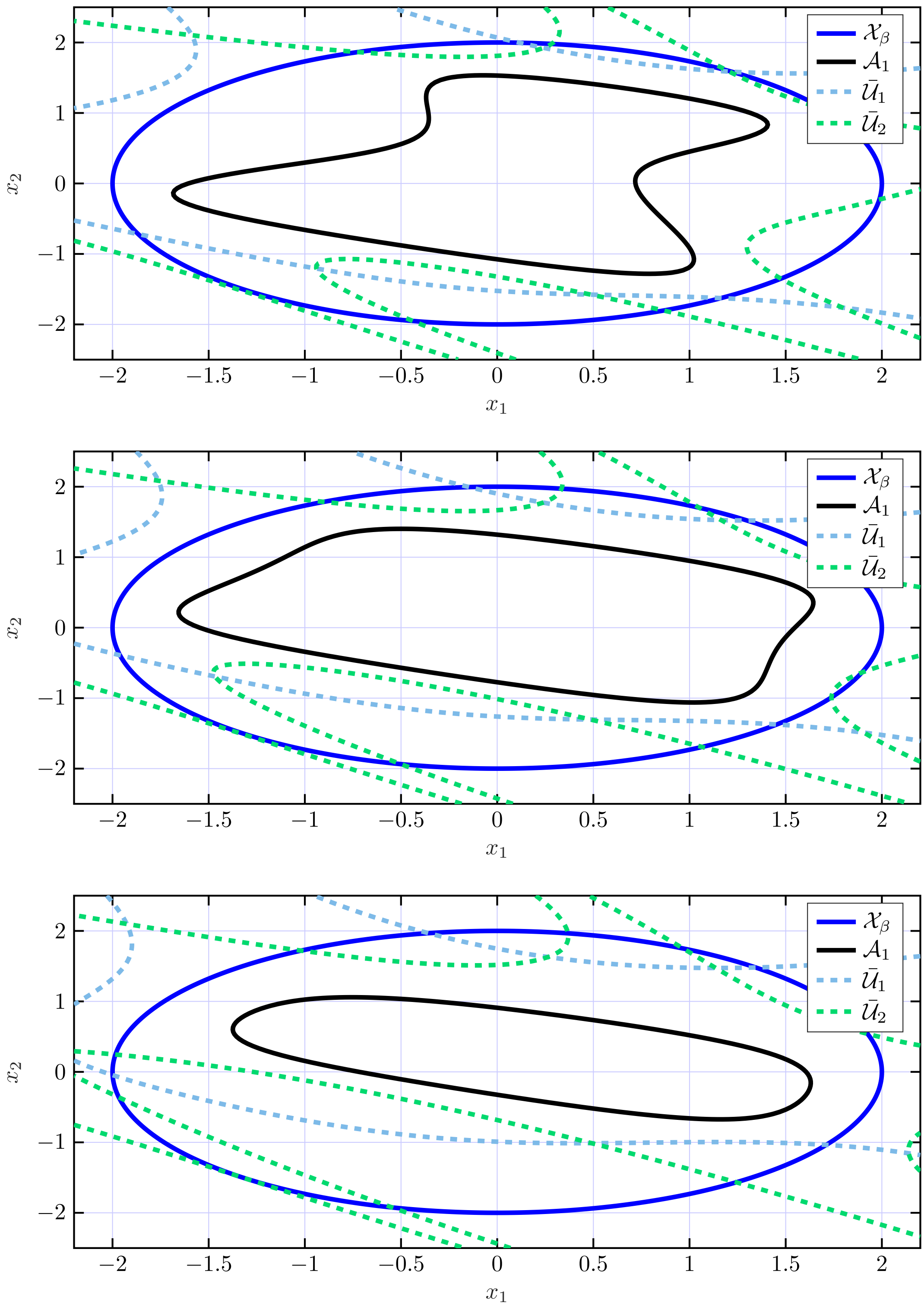



**Figure 4.4.** – Sets $\mathcal{A}_1 \subset \mathcal{X}_\beta$, $\mathcal{A}_1 \subset \bar{\mathcal{U}}_1$ and $\mathcal{A}_1 \subset \bar{\mathcal{U}}_2$ in the planes $x_3 = -1$ (upper plot), $x_3 = 0$ (middle plot) and $x_3 = 1$ (lower plot).

Program 4.1 with Optimization 4.3 in SOSTOOLS and SDPT3, where $F_2$ and $\hat{S}_2$ are polynomials in $(x_1, x_2)$ of maximum degree 2, results in $F_2(x) = \begin{bmatrix} F_{2a}(x) & F_{2b}(x) & F_{2c}(x) \end{bmatrix}$,

$$F_{2a}(x) = \begin{bmatrix} 2.29x_1 + 0.5178x_1^2 + 0.06412x_2^2 - 0.25 \\ 0.6146x_1 - 0.07858x_1^2 + 0.01402x_2^2 + 0.9502 \end{bmatrix},$$

$$F_{2b}(x) = \begin{bmatrix} -0.5276x_1 - 0.5677x_1^2 - 0.9799x_2^2 - 0.5 \\ 0.1337x_1 - 0.1029x_1^2 - 0.1132x_2^2 - 0.06251 \end{bmatrix},$$

$$F_{2c}(x) = \begin{bmatrix} 0.4467x_1 + 2.117x_1^2 + 0.6713x_2^2 + 0.06251 \\ -0.07331x_1 - 615.4x_1^2 - 615.9x_2^2 - 0.5 \end{bmatrix}.$$

At this point, we can also build the asymptotically stabilizing controller $u_{\text{sat-n}}$ from (4.11). Note that the region of convergence $\mathcal{A}_1$ remains the same since $P$ and $c$ are not modified with SOS Program 4.1.

### 4.3.2 Simulations

Figures 4.5 and 4.6 show the simulation results of the system (4.20) with initial condition $x(0) = \text{vec}(1.2, -1, 0) \in \mathcal{A}_1$ and controllers $u_{\text{ida}}$ and $u_{\text{sat-n}}$. It is clearly seen that all states will converge to the origin, and the controllers $u_{\text{ida}}$ and $u_{\text{sat-n}}$ remain in $\mathcal{U}_{\text{ida}} = \mathcal{U}_{\text{sat}}$. Furthermore, the saturation of $u_{\text{sat-n}}$ does not compromise the monotonically decreasing feature of the Hamiltonian, which is linked to the system's stability. Finally, we observe that the selected $F_2$ yields a faster settling time in the saturated controller $u_{\text{sat-n}}$. However, this feature does not hold for any $F_2$ obtained with SOS Program 4.1, and a suitable optimization objective has to be chosen.

## 4.4 Application on a Cart-pole System

In the previous examples, we considered a system that is naturally described with polynomial functions. We now address the cart-pole of Figure 4.7, which is an underactuated mechanical system that possess trigonometric functions in its model. The system is composed of a **cart** with mass $m_c$ sliding on a **runway** and a **simple one arm pendulum** attached to the cart. The cart moves along the $x$-axis and is actuated by the force $\tau$ while the pendulum is unactuated (free).

Our purpose is to synthesize an IDA-PBC controller without saturation that ensures the asymptotic stabilization of the pendulum's upright equilibrium as well as the cart at a desired position. With this objective, we device a model in explicit representation under Assumption 4.1, re-parametrize its trigonometric functions so as to obtain rational ones, and test Algorithm 4.1 with a local optimal performance assignment. The result is implemented in the test-bench of Figure 4.7, which is located at the laboratory of the Control

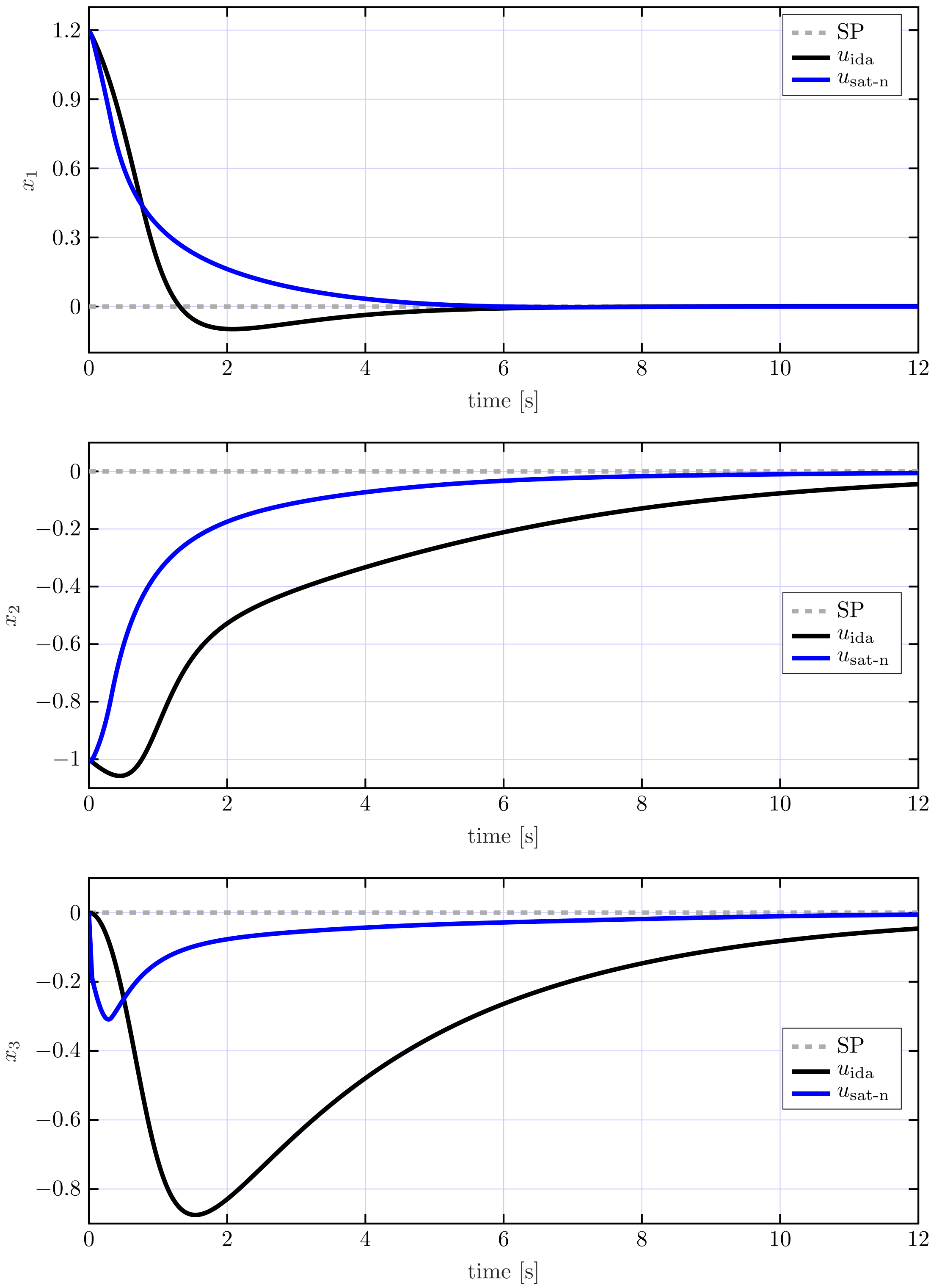



**Figure 4.5.** – States of the third-order system with initial conditions $x(0) = \text{vec}(1.2, -1, 0)$.

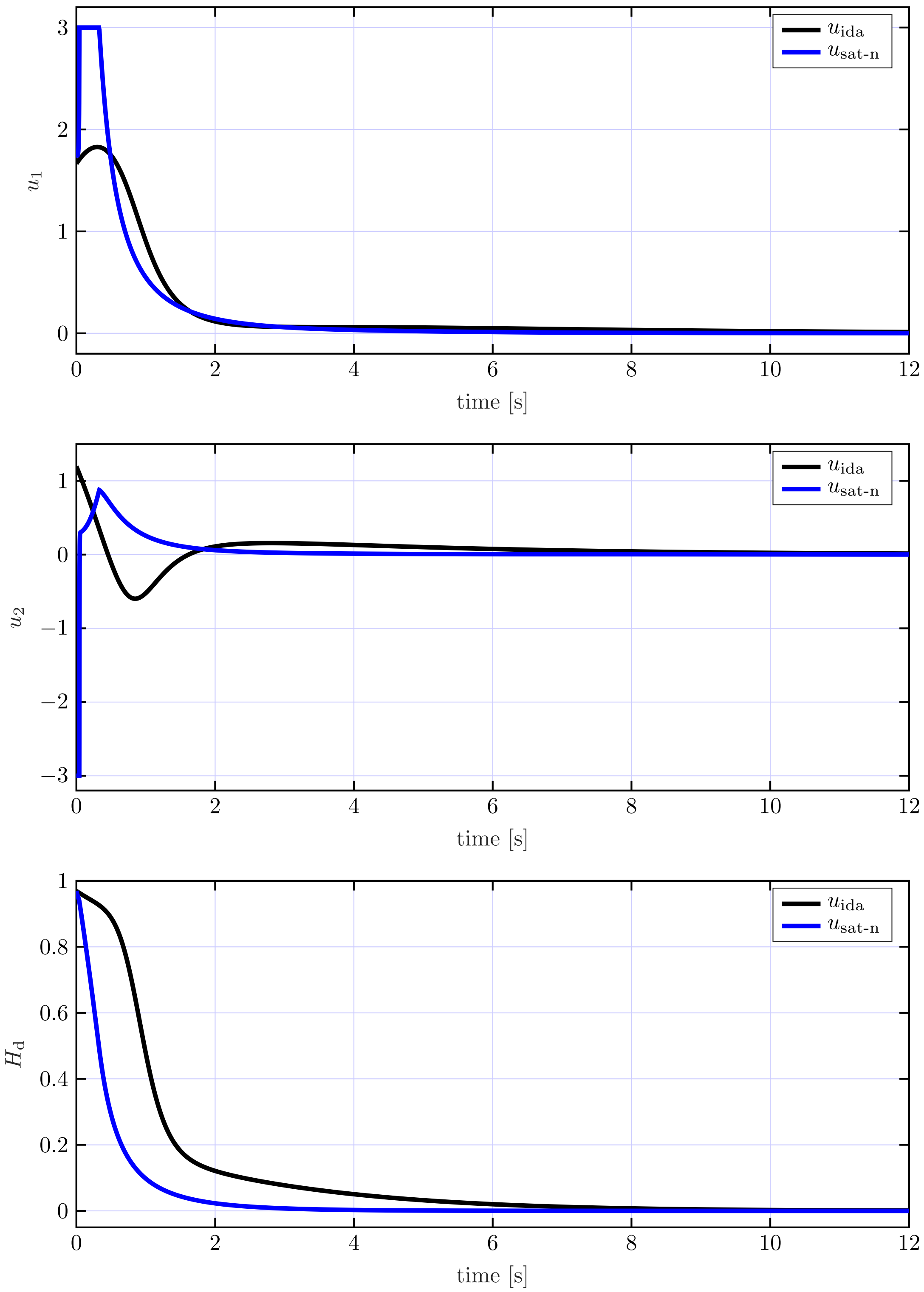



**Figure 4.6.** – Control action and Hamiltonian of the third-order system with initial conditions $x(0) = \text{vec}(1.2, -1, 0)$.

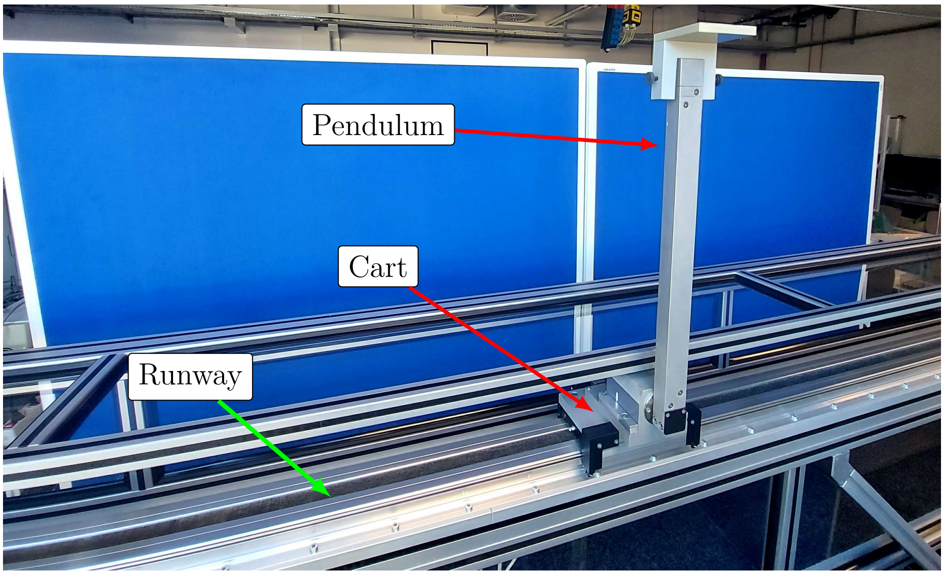



**Figure 4.7.** – Cart-pole of the Control Engineering Group at TU Ilmenau.

Engineering Group (Fachgebiet Regelungstechnik), Department of Computer Science and Automation, TU Ilmenau.

**Assumption 4.1.** *i) The pendulum's **rod** is massless and has a constant length $l$. ii) The pendulum's **bob** of mass $m_\mathrm{p}$ is a point mass. iii) The gravity of magnitude $g_\mathrm{c}$ points downwards (direction $-y$).* [35] *iv) The initial conditions are consistent (to be used in Chapters 5 to 7).*

Figure 4.8 illustrates the cart-pole schematic diagram, where $x_\mathrm{c}$ denotes the cart **pivot** position on the $x$-axis, $(x_\mathrm{p}, y_\mathrm{p})$ is the relative position of the bob with respect to the cart pivot and $\theta$ is the pendulum angle with respect to the $y$-axis. The real system is equipped with encoders in $x_\mathrm{c}$ and $\theta$, as well as three modes to move its servomotor: **current, velocity and position tracking**. The former is usually approximated to force input, meaning that controller design entails the identification of masses, moments of inertia, frictions, gear and belt features, and others, which are all non-error-free. On the other hand, we can use the velocity tracking with an integrator in its input to have an approximate representation of the cart-pole in Partial Feedback Linearization (PFL) with new input $u^* = \ddot{x}_\mathrm{c}^* \approx \ddot{x}_\mathrm{c}$, see Figure 4.9. This option avoids the cumbersome parameter identification because the PFL may yield a cart-pole model independent of masses and inertias (in the implicit and explicit representations), see [199]. How close our approximation is, clearly depends on the performance of the tracking controller, which consists of a Proportional-integral-derivative (PID) plus Feedforward. We discard the position tracking because it

[35] Changing the gravity direction is equivalent to tilt the whole system, that is, a cart-pole on an inclined plane, see [18].

is analogous to the velocity mode, but it may introduce some additional error due to its cascade nature.

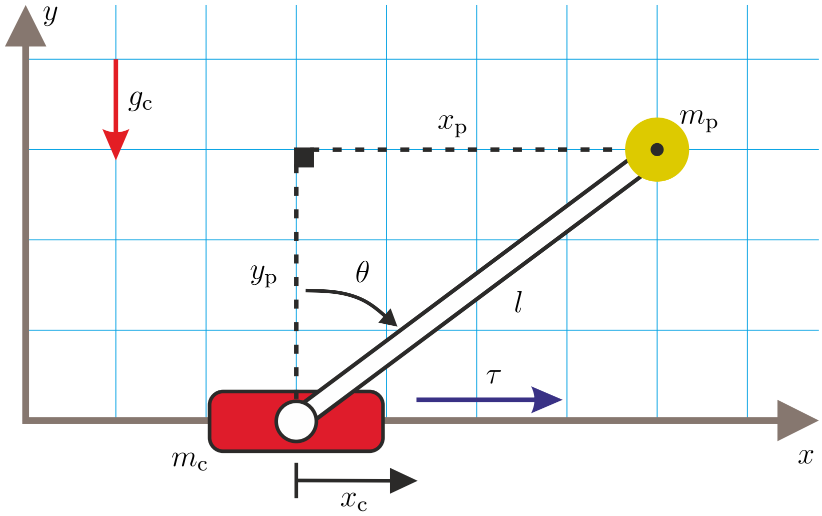



**Figure 4.8.** – Cart-pole diagram.

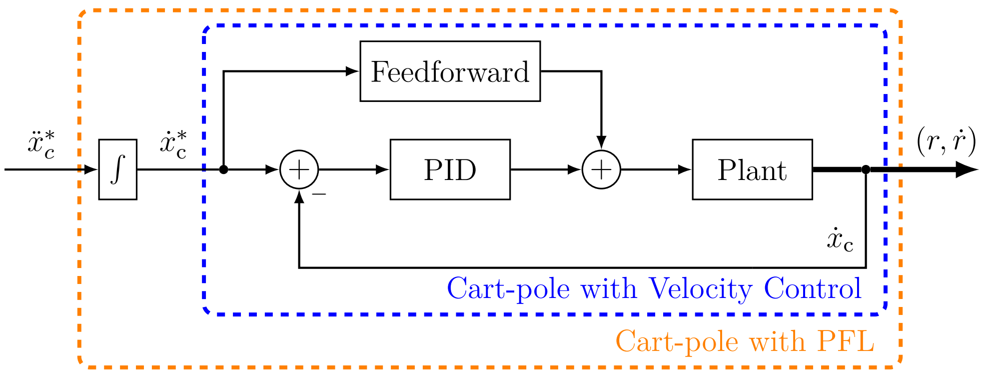



**Figure 4.9.** – Approximate cart-pole with PFL.

In the following, we will use the scheme of Figure 4.9, assuming that the PID plus Feedforward is well-tuned such that it corresponds to an acceptable approximation for practical purposes. It is important to remark that *the algebraic IDA-PBC introduced in this dissertation* do not necessarily require the cart-pole model in PFL (see e.g., [98]) but we employ such a representation to streamline design and implementation.

### 4.4.1 Explicit Model with PFL

The cart-pole has two DoF and its configuration space is given by $\mathbb{R} \times \mathbb{S}$. Therefore, taking $q := \mathrm{vec}(\theta, x_\mathrm{c}) \in \mathcal{Q} = \mathbb{R}^2$ as our generalized coordinates and using Assumption 4.1, we may calculate the kinetic and potential energy as[36]

$$E_k(q,\dot q) = \frac{m_c}{2}\dot x_c^2 + \frac{m_p}{2}\left(\dot x_c + \dot\theta l\cos\theta\right)^2 \quad \text{and} \quad \hat V(q) = g_c l m_p \cos\theta.$$

Hence, the Lagrange equations (of the second kind) with Rayleigh dissipation $\hat D(q,\dot q) = \frac{1}{2}c_\theta\dot\theta^2 + \frac{1}{2}c_c\dot x_c^2 \geq 0$ are

$$\hat M(q)\ddot q + \hat C(q,\dot q)\dot q + \hat R\dot q + \frac{\partial^\top \hat V}{\partial q} = \hat G\tau, \tag{4.21}$$

where $\hat R = \operatorname{diag}(c_\theta, c_c)$, $\hat G = \operatorname{vec}(0,1)$,

$$\hat M(q) = \begin{bmatrix} m_p l^2 & m_p l\cos\theta \\ m_p l\cos\theta & m_p + m_c \end{bmatrix}, \qquad \hat C(q,\dot q) = \begin{bmatrix} 0 & 0 \\ -m_p l\dot\theta\sin\theta & 0 \end{bmatrix}.$$

Following [200, 201], we obtain the partially linearized model

$$M\ddot q + R\dot q + \frac{\partial^\top V}{\partial q} = G(q)u, \tag{4.22}$$

where $M = I_2$, $V(q) = \frac{g_c}{l}\cos\theta$, $G(q) = \operatorname{vec}(-\frac{1}{l}\cos\theta, 1)$, and $R = \operatorname{diag}(\frac{c_\theta}{m_p l^2}, 0)$. Now, for $\theta \in \,]-\pi,\pi[\,$, we can parametrize the trigonometric functions so as to obtain the rational functions[37]

$$\cos\theta = \frac{1-\alpha^2}{1+\alpha^2}, \qquad \sin\theta = \frac{2\alpha}{1+\alpha^2}, \qquad \tan\frac{\theta}{2} = \alpha.$$

Hence, the cart-pole model (4.22) with states $x = \operatorname{vec}(\alpha, x_c, \dot\theta, \dot x_c)$ can be written as

$$\begin{bmatrix} \dot\alpha \\ \dot x_c \\ \ddot\theta \\ \ddot x_c \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+\alpha^2)\dot\theta \\ \dot x_c \\ \dfrac{2g_c\alpha}{l(1+\alpha^2)} - \dfrac{c_\theta\dot\theta}{m_p l^2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{\alpha^2-1}{l(1+\alpha^2)} \\ 1 \end{bmatrix} u. \tag{4.23}$$

Notice that (4.23) is independent of the mass $m_p$ whenever $\frac{c_\theta}{m_p l^2} \approx 0$.

### 4.4.2 Algebraic IDA-PBC Design

In this section, we test Algorithm 4.1 for the cart-pole with PFL described by (4.23).

[36] Since the cart is constrained to move horizontally, its center of mass can be described w.r.t. $x_c$.

[37] This parametrization is an exact change of coordinates, which is used in [129] to take advantage of SOS decomposition.

**Step 1:** With $\beta(x) = 0$, we select the region of interest $\mathcal{X}_\beta$ as the whole space $\mathcal{X} = \mathbb{R}^4$, and choose

$$\Lambda_0(x) = \begin{bmatrix} 0 & 0 & \frac{1}{2}(1+\alpha^2) & 0 \\ 0 & 0 & 0 & 1 \\ -2g_c & 0 & \dfrac{c_\theta(1+\alpha^2)}{m_p l} & 0 \end{bmatrix}, \qquad g_\perp(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -l(1+\alpha^2) & \alpha^2 - 1 \end{bmatrix},$$

$x_d = \mathrm{vec}(0, x_c^\star, 0, 0) \in \mathcal{X}_a = \left\{ x \in \mathcal{X} \mid 0 = \alpha = \dot{\theta} = \dot{x}_c \right\}$ and $z(x) = x - x_d$ verifying Assumption 3.1, where $x_c^\star$ is the desired cart position. Note that $\frac{\partial z}{\partial x}$ is unimodular and $\Lambda_0$ is polynomial, meaning that $F_0$ has a unique and polynomial solution, see (3.19).

**Step 2:** We pick $\epsilon_0 = \epsilon_1 = 10^{-5}$, $S_{11}$ with maximum polynomial order 2, and use SOS Program 3.1 as a fast indicator (that SOS Program 3.2 will work), observing that the SDP solver (from SOSTOOOLS) does not converge. Therefore, we reduce the region of interest to a subset of $-90° < \theta < 90°$, i.e., the pendulum is above the $x$-axis. For illustration we select $\beta(x) = 1 - 0.6^2\alpha^2$ or equivalently $-61.927° \le \theta \le 61.927°$, obtaining the desired solution.

**Step 3:** To determine the region of convergence, we consider the constraint (3.24), where $S_\beta = \frac{1}{0.6^2}\mathrm{diag}(1,0,0,0)$ and $\psi(\gamma(x)) \equiv 0$. Since $\beta$ and $F_0$ are functions in $\alpha$, we select for simplicity $F_1$ and $S_1$ as polynomials in $\alpha$ with maximum degree 2. Furthermore, we include Optimization 4.5 to have a LQR optimal local performance while increasing the region of convergence. Solving the SOS Program 3.2 with (3.24) and Optimization 4.5 in SOSTOOOLS and SDPT3, where $\eta(x) = \mathrm{vec}(0,0,0,1)$, $c = 0.5$, $a = 20$, $Q_{\mathrm{lqr}} = \mathrm{diag}(10I_2, I_2)$, $R_{\mathrm{lqr}} = 1$, $l = 0.4840\,\mathrm{m}$, $g_c = 9.81$, $c_\theta = 0$ and $\epsilon_3 = 10^{-5}$, yields

$$F_1(x) = \begin{bmatrix} 31.62\alpha^2 + 11.0 & -65.83\alpha^2 - 25.34 & 11.9\alpha^2 - 10.93 & -37.45\alpha^2 - 16.72 \end{bmatrix},$$
$$P = \begin{bmatrix} 0.36 & -1.037 & -1.317 & 0.1698 \\ -1.037 & 22.92 & 1.328 & -6.165 \\ -1.317 & 1.328 & 24.9 & -25.58 \\ 0.1698 & -6.165 & -25.58 & 44.34 \end{bmatrix}.$$

Although we have assumed negligible friction in the underactuated degree, i.e., $c_\theta = 0$, it is possible to solve the SOS Program 3.2 with $c_\theta \neq 0$, see [98]. This is an advantage with respect to the controller design with the standard IDA-PBC for underactuated mechanical systems (see Chapter 5), where the dissipation condition (5.7a) holds whenever $c_\theta = 0$. Figure 4.10 shows the sets $\mathcal{A}_{0.5} \subset \mathcal{X}_\beta$ in the planes $\dot{\theta} = -134.7°/\mathrm{s}$, $\dot{\theta} = 0°/\mathrm{s}$ and $\dot{\theta} = 134.7°/\mathrm{s}$ intersected with $\dot{x}_c = 0\,\mathrm{m/s}$.

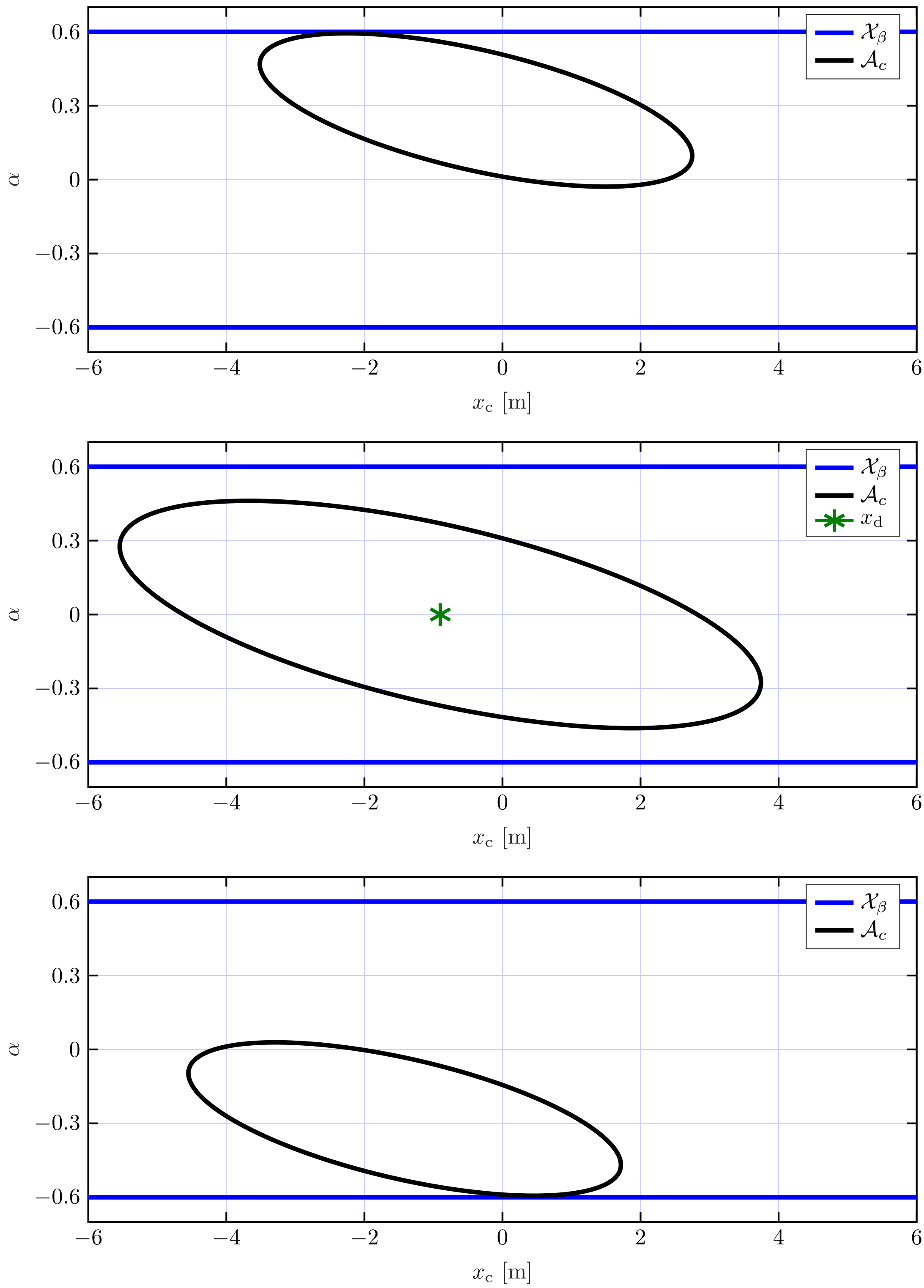



**Figure 4.10.** – Set point $x_d = \text{vec}(0, -0.9, 0, 0)$ and sets $\mathcal{A}_{0.5} \subset \mathcal{X}_\beta$ in the planes $\dot{\theta} = -134.7°/\text{s}$ (upper plot), $\dot{\theta} = 0°/\text{s}$ (middle plot) and $\dot{\theta} = 134.7°/\text{s}$ (lower plot) intersected with $\dot{x}_c = 0\,\text{m/s}$.

**Steps 4 and 5:** Since $\eta^\top(x)g(x) = 1$, $\frac{\partial z}{\partial x} = I_4$ and $g_\perp(x)\nu = 0$, the controller (3.15) reads

$$\begin{aligned} u_{\mathrm{ida}}(x) = 48.81\alpha + 4.146\dot{\theta} + 2.024\dot{x}_{\mathrm{c}} + 1.406(x_{\mathrm{c}} - x_{\mathrm{c}}^\star) + 23.86\alpha^2\dot{\theta} + 13.3\alpha^2\dot{x}_{\mathrm{c}} \\ + 8.004\alpha^2(x_{\mathrm{c}} - x_{\mathrm{c}}^\star) + 191.9\alpha^3 \end{aligned} \tag{4.24}$$

and it yields an asymptotically stable closed-loop in $x_{\mathrm{d}}$. The simulation and implementation results are displayed in Section 7.2.5, where we compare the controller (4.24), with the ones obtained from the developments of Chapters 5 and 6.

# III

# Mechanical Systems

# Chapter 5

# IDA-PBC for Mechanical Systems

The aim of this chapter is to introduce the total energy shaping IDA-PBC for mechanical systems with kinematic constraints (holonomic and nonholonomic). Since those systems are described by DAEs (see Section 2.5.4), we handle them from the geometric perspective, meaning that they are actually ODEs on a manifold. We will restrict our discussion to the general aspects of the IDA-PBC and leave the specific solutions and examples to Chapters 6 and 7. This chapter is divided in two sections. In Section 5.1, we review the total energy shaping IDA-PBC for unconstrained (or explicit) mechanical systems. Later, Section 5.2 encourages the controller design with constrained (or implicit) representations and introduces the corresponding IDA-PBC theory. Both situations (explicit and implicit designs) are discussed under the presence of dissipation and preliminary feedback.

## 5.1 Unconstrained Mechanical Systems

Let us recall the basic principles of the IDA-PBC for unconstrained mechanical systems, which was introduced in [22] and extended in [24, 27]. Consider the mechanical system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_{n_q} \\ -I_{n_q} & -R(q) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top H}{\partial q} \\ \frac{\partial^\top H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u, \tag{5.1}$$

where $q \in \mathcal{Q} \subset \mathbb{R}^{n_q}$ are the generalized coordinates, $\mathcal{Q}$ is an open and connected subset of $\mathbb{R}^{n_q}$, $p \in \mathbb{R}^{n_q}$ are the conjugate momenta defined as $p = M(q)\dot{q}$, $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the input, $G : \mathcal{Q} \to \mathbb{R}^{n_q \times n_u}$ is the input matrix with constant rank, $R : \mathcal{Q} \to \mathbb{R}^{n_q \times n_q}$ is the positive semidefinite dissipation matrix (obtained e.g. from the Rayleigh dissipation function) and $H$ is the Hamiltonian which represents the system's total energy (kinetic plus potential), meaning

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q),$$

with $M : \mathcal{Q} \to \mathbb{R}^{n_q \times n_q}_{n_q}$ being (the positive definite) inertia matrix and $V : \mathcal{Q} \to \mathbb{R}$ the potential energy.

### 5.1.1 Controller Design

Following Proposition 3.1, we will render (5.1) via state feedback $u = u_{\text{ida}}(q, p)$ into the port-Hamiltonian system (3.5) such that the closed-loop is (asymptotically) stable in the desired equilibrium $(q_\text{d}, 0)$, with $q_\text{d} \in \mathcal{Q}_\text{a} := \left\{ q \in \mathcal{Q} \mid G_\perp(q)\frac{\partial^\top V}{\partial q} = 0 \right\}$.[38] For this, we take advantage of the **parameterized IDA-PBC**, setting[39]

$$H_\text{d}(q, p) = \frac{1}{2} p^\top M_\text{d}^{-1}(q) p + V_\text{d}(q),$$

where $M_\text{d} : \mathcal{Q} \to \mathbb{R}^{n_q \times n_q}_{n_q}$ and $V_\text{d} : \mathcal{Q} \to \mathbb{R}$ are the desired inertia matrix and potential energy, respectively. Besides, we may impose a structure in the dissipation and interconnection matrix, writing the target system (3.5) as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & J(q) \\ -J^\top(q) & \Gamma_1(q, p) + \Gamma_2(q) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top H_\text{d}}{\partial q} \\ \frac{\partial^\top H_\text{d}}{\partial p} \end{bmatrix}, \tag{5.2}$$

for some $\Gamma_2 : \mathcal{Q} \to \mathbb{R}^{n_u \times n_q}$ and $\Gamma_1 : \mathcal{Q} \times \mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times n_q}$ with $\Gamma_1$ being linear in $p$. Now, the matching condition (3.6) can be reduced and decoupled w.r.t the dependency of p (quadratic in $p$, independent of $p$ and linear in $p$) as

$$0 = G_\perp(q) \left( \frac{1}{2} \frac{\partial^\top p^\top M^{-1} p}{\partial q} - M_\text{d}(q) M^{-1}(q) \frac{1}{2} \frac{\partial^\top p^\top M_\text{d}^{-1} p}{\partial q} + \Gamma_1(q, p) M_\text{d}^{-1}(q) p \right), \tag{5.3a}$$

$$0 = G_\perp(q) \left( \frac{\partial^\top V}{\partial q} - M_\text{d}(q) M^{-1}(q) \frac{\partial^\top V_\text{d}}{\partial q} \right), \tag{5.3b}$$

$$0 = G_\perp(q) \Big( R(q) M^{-1}(q) p + \Gamma_2(q) M_\text{d}^{-1}(q) p \Big), \tag{5.3c}$$

where $J(q) = M^{-1}(q) M_\text{d}(q)$. Equations (5.3), which are not trivial if (5.1) is underactuated ($\operatorname{rank} G < n_q$), are known as the matching conditions of the **kinetic energy, potential energy and dissipation**, respectively. Conditions (5.3a)–(5.3b) are a system of nonhomogeneous quasi-linear PDEs with unknowns in $\Gamma_1$, $M_\text{d}$ and $V_\text{d}$. On the other hand, (5.3c) is an algebraic equation, and its solution is given by

$$\Gamma_2(q) = -G(q) \bar{\Gamma}_2(q) J(q) - R(q) J(q)$$

[38] See Lemma 3.6 to determine the admissible equilibria set.

[39] The target Hamiltonian under the parameterized approach is not unique, see [71, 202], where the desired Energy is augmented by a mixed term of coordinates and momenta $p^\top \eta(q)$.

with arbitrary $\bar{\Gamma}_2 : \mathcal{Q} \to \mathbb{R}^{n_u \times n_q}$. It follows that the controller $u = u_{\text{ida}}(q, p)$ is

$$u_{\text{ida}}(q,p) = G^{\text{g}}(q)\left(\frac{\partial^\top H}{\partial q} - J^\top(q)\frac{\partial^\top H_{\text{d}}}{\partial q} + \Gamma_1(q,p)\frac{\partial^\top H_{\text{d}}}{\partial p}\right) - \bar{\Gamma}_2(q)\frac{\partial^\top H}{\partial p} + G_{\mathcal{I}}(q)\nu, \quad (5.4)$$

where $\nu$ is also arbitrary. Stability in the desired equilibrium $(q_{\text{d}}, 0)$ is now obtained if

$$M_{\text{d}}(q) \succ 0 \qquad \forall q \in \mathcal{Q}, \quad (5.5a)$$

$$V_{\text{d}}(q) - V_{\text{d}}(q_{\text{d}}) > 0 \qquad \forall q \in \mathcal{Q} - \{q_{\text{d}}\}, \quad (5.5b)$$

$$\frac{\partial H_{\text{d}}}{\partial p}\Big(G(q)\bar{\Gamma}_2(q)J(q) + R(q)J(q) - \Gamma_1(q,p)\Big)\frac{\partial^\top H_{\text{d}}}{\partial p} \geq 0 \qquad \forall\, (q,p) \in \mathcal{Q} \times \mathbb{R}^{n_q}. \quad (5.5c)$$

Here,

$$M_{\text{d}}(q_{\text{d}}) \succ 0, \qquad \left.\frac{\partial V_{\text{d}}}{\partial q}\right|_{q=q_{\text{d}}} = 0, \qquad \left.\frac{\partial^2 V_{\text{d}}}{\partial q^2}\right|_{q=q_{\text{d}}} \succ 0 \quad (5.6)$$

are the necessary and sufficient conditions (see [180, Proposition 1.3–1.4]) to locally guarantee the requirement for $H_{\text{d}} \in \mathcal{C}^2$, i.e., (5.5a)–(5.5b). Finally, asymptotic stability can be demonstrated if the largest invariant set of (5.2) contained in

$$\Omega = \left\{(q,p) \in \mathcal{Q} \times \mathbb{R}^{n_q} \mid \frac{\partial H_{\text{d}}}{\partial p}(q,p)\Big(G(q)\bar{\Gamma}_2(q)J(q) + R(q)J(q) - \Gamma_1(q,p)\Big)\frac{\partial^\top H_{\text{d}}}{\partial p} = 0\right\}$$

is no other than $\{(q_{\text{d}}, 0)\}$, or if (5.2) with output $y_{\text{d}}$, defined from

$$y_{\text{d}}^\top y_{\text{d}} := \frac{\partial H_{\text{d}}}{\partial p}(q,p)\Big(G(q)\bar{\Gamma}_2(q)J(q) + R(q)J(q) - \Gamma_1(q,p)\Big)\frac{\partial^\top H_{\text{d}}}{\partial p},$$

is zero-state detectable (or observable). Note that (5.4) uses $G^{\text{g}}$ (the generalized inverse of $G$) instead of $\left(G^\top(q)G(q)\right)^{-1} G^\top(q)$. The latter is a particular case of inverse (Moore–Penrose) that is used in the standard IDA-PBC literature, see [13, 48, 56, 58, 178, 203]. Observe that the freedom of $G^{\text{g}}$ and $\nu$ may simplify the final expression of (5.4) without impairing the system behavior. The procedure to apply the IDA-PBC method can be summarized in Algorithm 5.1.

**Algorithm 5.1** IDA-PBC for unconstrained mechanical systems.

**Require:** A mechanical system of the form (5.1).

1: Select $G_\perp$ and $q_{\text{d}} \in \mathcal{Q}_{\text{a}} := \left\{q \in \mathcal{Q} \mid G_\perp(q)\frac{\partial^\top V}{\partial q} = 0\right\}$.

2: Calculate a solution to $M_{\text{d}}$ (symmetric) and $\Gamma_1$ (linear in $p$) from the kinetic matching (5.3a). Here, $\Gamma_1$ is chosen to simplify the PDE solution.

3: With $M_{\text{d}}$ from step 2, calculate a general solution to $V_{\text{d}}$ from the potential matching (5.3b).

4: Select the arbitrary functions and parameters in $M_\mathrm{d}$, $V_\mathrm{d}$, $\Gamma_1$ and $\bar{\Gamma}_2$ such that the stabilizing conditions (5.5c) and (5.6) hold. If no solution exist, return to steps 2 or 3.
5: Select $G^\mathrm{g}$, $G_\mathcal{I}$ and $\nu$ to build the feedback (5.4).
6: (Local) asymptotic stability can be verified from

- the largest invariant set of (5.2) contained in $\Omega$,
- the (local) zero-state detectability of (5.2) with output $y_\mathrm{d}$, or
- Lyapunov's indirect method (Theorem 2.2).

### 5.1.2 Standard IDA-PBC and the Dissipation Condition

Taking a closer look to steps 2 and 4 of Algorithm 5.1, we observe that (5.5c) relies in general on the solution of $M_\mathrm{d}$ and $\Gamma_1$ obtained from (5.3a) and (5.6). In fact, we are solving matching and stabilizing conditions simultaneously, and thus the name **simultaneous IDA-PBC**, see [24].[40] To avoid this dependency, we impose the following assumptions.

**Assumption 5.1.** *System (5.1) is lossless or has a negligible dissipation ($R(q) = 0$).*

**Assumption 5.2.** $\Gamma_1$ *is skew-symmetric.*

This allows us to split the design procedure in two stages: **energy shaping and damping injection**. The former implies solving (5.3a)–(5.3b) and (5.6) for $M_\mathrm{d}$, $V_\mathrm{d}$ and $\Gamma_1$. The latter injects damping with

$$\bar{\Gamma}_2(q) = K_\mathrm{d} G^\top(q) J^{-1}(q), \qquad K_\mathrm{d} = K_\mathrm{d}^\top \succ 0,$$

to satisfy (5.5c) and yields an output strictly passive closed-loop, see Section 2.6.1. This twofold stage approach, known as the **standard IDA-PBC** [13, 22, 23, 203, 204], may fail to satisfy (5.5c) if Assumption 5.1 does not hold. Therefore, we can modify the damping injection stage by using Lemma 3.3, obtaining that there exists a $\bar{\Gamma}_2$ (not necessarily equal to $K_\mathrm{d} G^\top J^{-1}$) satisfying (5.5c) if and only if

$$\bar{A}(q) := G_\perp(q)\Big(R(q)J(q) + J^\top(q)R^\top(q)\Big)G_\perp^\top(q) \succeq 0 \qquad \forall q \in \mathcal{Q}. \tag{5.7a}$$

If a solution exists, then $\bar{\Gamma}_2$ can be express as

$$\bar{\Gamma}_2(q)J(q) = K_1(q)G^\top(q) + \Big(K_2(q)G_\perp(q) - G^+(q)\Big)A(q)\Big(G_\perp(q)\Big)^+ G_\perp(q) + G_\mathcal{I}(q)K_3(q), \tag{5.7b}$$

[40] Donaire et al. argued in [24] that the simultaneous IDA-PBC for mechanical systems may extend the application scope of the standard IDA-PBC. Nevertheless, we will see in Section 5.2.4 that this is not always the case.

where $K_1$, $K_2$ and $K_3$ are arbitrary functions of adequate size, verifying

$$G_{\mathrm{r}}(q)\left(K_1(q)+K_1^\top(q)+G^+(q)A(q)\left(G^+(q)\right)^\top-K_2(q)\bar{A}(q)K_2^\top(q)\right)G_{\mathrm{r}}^\top(q)\succ 0 \tag{5.7c}$$

for all $q\in\mathcal{Q}$. Here, $A(q)=R(q)J(q)+J^\top(q)R(q)$, $(G_{\mathrm{r}},G_{\mathrm{l}})$ are the full rank factors of $G$, i.e., $G(x)=G_{\mathrm{l}}(x)G_{\mathrm{r}}(x)$, and the strict inequality in (5.7c) ensures that the closed-loop is output strictly passive, meaning

$$\Omega\subset\left\{(q,p)\in\mathcal{Q}\times\mathbb{R}^{n_q}\mid 0=G^\top(q)M_{\mathrm{d}}^{-1}(q)p\right\}.$$

Condition (5.7a) is called the **dissipation inequality**, and it was introduced by Gómez-Estern and van der Schaft in [27, Prop. 3.1]. However, unlike them, we give the general solution of $\bar{\Gamma}_2$ instead of a particular one. Besides, from (5.7b) and Lemma 3.2, we can show that $\bar{\Gamma}_2$ may take the usual form $\bar{\Gamma}_2(q)=K_1G^\top(q)J^{-1}(q)$ if and only if there exists $K_2$ satisfying

$$0=G_{\mathrm{r}}(q)\Big(K_2(q)G_\perp(q)-G^+(q)\Big)\Big(R(q)J(q)+J^\top(q)R(q)\Big)G_\perp^\top(q)\qquad\forall q\in\mathcal{Q}.$$

The most straightforward examples of this situation are $R(q)=0$ and (5.7a) being a strict inequality.

### 5.1.3 Enlarging the Scope of Application

The previous IDA-PBC scheme for unconstrained mechanical systems in port-Hamiltonian representation can be extended to systems of the form

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix}=\begin{bmatrix}M^{-1}(q)p\\ -f_1(q)-f_2(q,p)-R(q)M^{-1}(q)p\end{bmatrix}+\begin{bmatrix}0\\ G(q)\end{bmatrix}u, \tag{5.8}$$

where $f_1:\mathcal{Q}\to\mathbb{R}^{n_q}$, $f_2:\mathcal{Q}\times\mathbb{R}^{n_q}\to\mathbb{R}^{n_q}$ is quadratic in $p$ and $M$ can be sign **indefinite** but nonsingular and symmetric.[41] The representation (5.8) may include mechanical systems with change of coordinates, preliminary feedback (e.g. partial feedback linearization), or both that cannot be written in the form (5.1);[42] however, it may also cover non-mechanical systems. The application of IDA-PBC for (5.8) follows straightforwardly by replacing $\frac{\partial^\top V}{\partial q}$ and $\frac{1}{2}\frac{\partial^\top p^\top M^{-1}p}{\partial q}$ with $f_1$ and $f_2$, respectively, in (5.3)–(5.5c). In the literature, shaping the energy of (5.8) is also known as the **Lyapunov direct method for mechanical systems**, see [24, 205–207].

[41] A square and symmetric matrix is called indefinite if it is neither positive nor negative semidefinite.

[42] System (5.8) preserves the port-Hamiltonian structure of (5.1) if $f_1(q)+f_2(q,p)+R_x(q)M^{-1}(q)p=-\frac{\partial^\top H}{\partial q}-R(q)\frac{\partial^\top H}{\partial p}$.

## 5.2 Constrained Mechanical Systems

### 5.2.1 A Motivating Example

Let us consider the cart-pole system without PFL discussed in Section 4.4 but with negligible friction in the pendulum. By selecting the generalized coordinates $q := \operatorname{vec}(\theta, x_\mathrm{c}) \in \mathcal{Q} = \mathbb{R}^2$, the port-Hamiltonian (explicit) model with Rayleigh dissipation $D(q, \dot{q}) = \frac{1}{2} c_\mathrm{c} \dot{x}_\mathrm{c}^2 \geq 0$ reads

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_2 \\ -I_2 & \hat{R} \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \hat{H}}{\partial q} \\ \frac{\partial^\top \hat{H}}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{G} \end{bmatrix} \nu, \tag{5.9}$$

where $\hat{G} = \operatorname{vec}(0, 1)$, $\hat{R} = \operatorname{vec}(0, c_\mathrm{c})$ and $\hat{H}$ is the cart-pole's total energy, which is

$$\hat{H}(q, p) = \frac{1}{2} p^\top \hat{M}^{-1}(q) p + g_\mathrm{c} l m_\mathrm{p} \cos\theta, \quad \hat{M}(q) = \begin{bmatrix} m_\mathrm{p} l^2 & m_\mathrm{p} l \cos\theta \\ m_\mathrm{p} l \cos\theta & m_\mathrm{p} + m_\mathrm{c} \end{bmatrix}.$$

Suppose we wish to design an IDA-PBC law that ensures the upright (asymptotic) stabilization of the pendulum as well as the cart at a desired position, i.e., $q_\mathrm{d} = \operatorname{vec}(0, x_\mathrm{c}^\star)$. For this purpose, we may use the theory for unconstrained mechanical systems developed in Section 5.1. In the first step, we select $\hat{G}_\perp = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and verify that $q_\mathrm{d} \in \mathcal{Q}_\mathrm{a}$. Next, we calculate $\hat{M}_\mathrm{d}$ from the quasi-linear PDE (5.3a). This step can be a significant obstacle in the application of IDA-PBC, so for simplicity, let us choose $\hat{M}_\mathrm{d}(q) = \hat{M}(q) \succ 0$ and $\hat{\Gamma}_1(q, p) = 0$, i.e., we are not shaping the kinetic energy. It follows that (5.3a) holds and the linear PDE (5.3b) reduces to an ODE that can be solved with a simple integration. As a result, we have $\hat{V}_\mathrm{d}(q) = g_\mathrm{c} l m_\mathrm{p} \cos\theta + \beta(x_\mathrm{c})$, where $\beta : \mathbb{R} \to \mathbb{R}$ is an arbitrary function. In the fourth step, we observe that there is no such $\beta$ that renders $q_\mathrm{d}$ a strict local minimum of $\hat{V}_\mathrm{d}$, see conditions (5.6). Consequently, we have to modify the kinetic energy ($\hat{M}_\mathrm{d} \neq \hat{M}$), which can prevent the reduction of (5.3b) to an ODE, that is, we have to solve PDEs or find a method that avoids them.

Instead of going forward with this viewpoint, suppose we choose the Cartesian coordinates $r = \operatorname{vec}(x_\mathrm{p}, y_\mathrm{p}, x_\mathrm{c})$, where $(x_\mathrm{p}, y_\mathrm{p})$ is the relative position of the pendulum w.r.t. the cart pivot, see Figure 4.8. This leads to the holonomic constraint

$$\Phi(r) := \frac{1}{2} \left( x_\mathrm{p}^2 + y_\mathrm{p}^2 - l^2 \right) = 0, \tag{5.10a}$$

the kinetic energy

$$\mathbf{E}_\mathrm{k}(r, \dot{r}) = \frac{m_\mathrm{c}}{2} \dot{x}_\mathrm{c}^2 + \frac{m_\mathrm{p}}{2} \dot{y}_\mathrm{p}^2 + \frac{m_\mathrm{p}}{2} (\dot{x}_\mathrm{c} + \dot{x}_\mathrm{p})^2 = \frac{1}{2} \dot{r}^\top \hat{\mathbf{M}} \dot{r}, \quad \text{with} \quad \hat{\mathbf{M}} = \begin{bmatrix} m_\mathrm{p} & 0 & m_\mathrm{p} \\ 0 & m_\mathrm{p} & 0 \\ m_\mathrm{p} & 0 & m_\mathrm{c} + m_\mathrm{p} \end{bmatrix},$$

and the potential energy

$$\hat{\mathbf{V}}(r) = g_{\mathrm{c}} m_{\mathrm{p}} y_{\mathrm{p}}.$$

Now, following Section 2.5.3, the Hamiltonian equations with external forces and constraints (5.10a) are

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ -I_3 & \hat{\mathbf{R}} \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \hat{\mathbf{H}}}{\partial r} \\ \frac{\partial^\top \hat{\mathbf{H}}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial^\top \Phi}{\partial r} \end{bmatrix} \hat{\lambda} + \begin{bmatrix} 0 \\ \hat{\mathbf{G}} \end{bmatrix} \tau, \tag{5.10b}$$
$$\hat{\mathbf{H}}(r,\rho) = \tfrac{1}{2}\rho^\top \hat{\mathbf{M}}^{-1}\rho + \hat{\mathbf{V}}(r),$$

where $\lambda$ is the implicit variable (that is calculated from the hidden constraints), $\hat{\mathbf{G}} = \mathrm{vec}(0,0,1)$ and $\hat{\mathbf{R}} = \mathrm{vec}(0,0,c_{\mathrm{c}})$ is obtained from the Rayleigh dissipation $\mathbf{D}(r,\dot{r}) = D(q,\dot{q}) = \frac{1}{2}c_{\mathrm{c}}\dot{x}_{\mathrm{c}}^2$. Stemming from the above, we observe that

- (5.9) has 4 states while (5.10b) has 6,
- (5.9) is an ODE while (5.10) is a DAE,
- the inertia matrix $M$ is state-dependent while $\hat{\mathbf{M}}$ is constant, and
- the potential energy $V$ is nonlinear while $\hat{\mathbf{V}}$ is linear.

In other words, after an adequate selection of coordinates, we can describe the system with DAEs and get simpler expressions in the inertia matrix and potential energy. The previous example raises the following questions.

1. Is it possible to extend the IDA-PBC to systems in implicit representation?
2. Since the nonlinear expressions of the matching conditions (5.3a)–(5.3b) arise from $M$ and $V$ being nonlinear, could we take advantage of the implicit model (5.10) (that has constant $\hat{\mathbf{M}}$, $\hat{\mathbf{G}}$ and linear $\hat{\mathbf{V}}$) to alleviate the task of solving PDEs (shaping the total energy)?

The answer to the first question is yes, and it can be traced back to [74, 75] for non-holonomic systems and [105] for the holonomic case. However, those works impose strong conditions in the target system, hindering the total energy shaping:

**Assumption 5.3** (Establish by [74, 75, 105])**.** *The constraint forces of the target and nominal port-Hamiltonian systems have the same direction.*

**Assumption 5.4** (Establish by [74, 75, 105])**.** *The nominal and desired inertia matrices are positive definite.*

**Assumption 5.5** (Establish by [105])**.** *The nominal and desired interconnection matrices are equal.*

In the remainder of this chapter we will drop Assumptions 5.3 to 5.5 by presenting a more general framework of the total energy shaping IDA-PBC for constrained mechanical systems that do not necessarily have a port-Hamiltonian representation. The second question, which is also motivated by Castaños and Gromov in [105], has an affirmative answer for a class of systems that includes, for instance, the cart pole, portal crane and PVTOL aircraft. The methods to solve these implicit matching conditions are discussed in Chapter 6.

### 5.2.2 System Class

We consider a general mechanical system of the form

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & I_{n_r} \\ -I_{n_r} & -\mathbf{R}(r) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}}{\partial r} \\ \frac{\partial^\top \mathbf{H}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{\mathbf{B}}(r) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ \mathbf{G}(r) \end{bmatrix} u, \tag{5.11a}$$

$$0 = \Phi(r), \tag{5.11b}$$

$$0 = \mathbf{B}_{\mathrm{nh}}^\top(r) \frac{\partial^\top \mathbf{H}}{\partial \rho}, \tag{5.11c}$$

where $r \in \mathcal{R} \subset \mathbb{R}^{n_r}$ are coordinates (that uniquely specify the system configuration), $\rho \in \mathbb{R}^{n_r}$ are the conjugate momenta defined as $\rho = \mathbf{M}(r)\dot{r}$, $\mathcal{R}$ is an open subset of $\mathbb{R}^{n_r}$, $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the input or control signal, $\mathbf{G} : \mathcal{R} \to \mathbb{R}^{n_r \times n_u}$ is the input matrix, $\mathbf{R} : \mathcal{R} \to \mathbb{R}^{n_r \times n_r}$ is the dissipation matrix (obtained e.g. from the Rayleigh dissipation function), $\bar{\mathbf{B}}(r)\lambda$ are the constraint forces with $\bar{\mathbf{B}} : \mathcal{R} \to \mathbb{R}^{n_r \times n_\lambda}$ and implicit variables $\lambda \in \mathbb{R}^{n_\lambda}$, (5.11b) and (5.11c) are the holonomic and nonholonomic constraints (respectively) with $\Phi : \mathcal{R} \to \mathbb{R}^{n_\Phi}$ and $\mathbf{B}_{\mathrm{nh}} : \mathcal{R} \to \mathbb{R}^{n_r \times (n_\lambda - n_\Phi)}$, and $\mathbf{H}$ is the Hamiltonian that represents the system's total energy, meaning

$$\mathbf{H}(r, \rho) = \tfrac{1}{2}\rho^\top \mathbf{M}^{-1}(r)\rho + \mathbf{V}(r)$$

with $\mathbf{M} : \mathcal{R} \to \mathbb{R}^{n_r \times n_r}_{n_r}$ being the nonsingular and symmetric inertia matrix, and $\mathbf{V} : \mathcal{R} \to \mathbb{R}$ the potential energy. Furthermore, the kinematic constraints (holonomic plus nonholonomic) define the level-sets

$$\mathcal{R}_\Phi := \{ r \in \mathcal{R} \mid 0 = \Phi(r) \}, \qquad \mathcal{X}_{\mathrm{c}} := \left\{ (r, \rho) \in \mathcal{R} \times \mathbb{R}^{n_r} \;\middle|\; 0 = \mathbf{B}^\top(r) \tfrac{\partial^\top \mathbf{H}}{\partial \rho},\ 0 = \Phi(r) \right\},$$

where $\mathbf{B}(r) = \begin{bmatrix} \frac{\partial^\top \Phi}{\partial r} & \mathbf{B}_{\mathrm{nh}}(r) \end{bmatrix}$. We identify the class by imposing

**Assumption 5.6.** *Mappings* $\Phi$ *and* $\frac{\partial \mathbf{H}}{\partial \rho}\mathbf{B}_{\mathrm{nh}}$ *are smooth. Matrix*

$$\Delta(r) := \mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\bar{\mathbf{B}}(r)$$

*is nonsingular for all* $r \in \mathcal{R}_\Phi$.

**Assumption 5.7.** *The initial conditions* $(r(t_0), \rho(t_0))$ *are consistent, i.e., they satisfy* (5.11b) *and the* ***momentum level constraints*** [43]

$$0 = \mathbf{B}^\top(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}. \tag{5.12}$$

**Assumption 5.8.** *The set* $\mathcal{R}_\Phi$ *is connected. The matrix*

$$\mathbf{N}(r) := \begin{bmatrix}\mathbf{G}(r) & \bar{\mathbf{B}}(r)\end{bmatrix}$$

*has constant rank and satisfies* $n_\lambda < \operatorname{rank}\mathbf{N}(r)$ *for every* $r \in \mathcal{R}_\Phi$*.*

By considering a general $\bar{\mathbf{B}}$ rather than $\operatorname{Colsp}\bar{\mathbf{B}}(r) \equiv \operatorname{Colsp}\mathbf{B}(r)$, we can include systems whose constraint forces do not necessarily satisfy the Lagrange-d'Alembert principle, meaning that they are allowed to do work along the system trajectories: $\frac{\partial \mathbf{H}}{\partial \rho}\bar{\mathbf{B}}(r)\lambda \neq 0$. Such systems have not been previously discussed in the PBC literature and they may result from Lagrangian or Hamiltonian dynamical systems with preliminary feedback, change of coordinates, or both. See Sections 7.1.5 and 7.1.6 for an example on the 5-DoF portal crane. Note that (5.11) is cyclo-passive w.r.t. to the triplet $\{\mathbf{H}, u, y\}$ with $y := \mathbf{G}^\top(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}$ whenever

$$\operatorname{Colsp}\bar{\mathbf{B}}(r) = \operatorname{Colsp}\mathbf{B}(r) \qquad \text{and} \qquad \frac{\partial \mathbf{H}}{\partial \rho}\mathbf{R}(r)\frac{\partial^\top \mathbf{H}}{\partial \rho} \geq 0 \qquad \forall (r,\rho) \in \mathcal{X}_\mathrm{c}.$$

From Assumption 5.6, the Jacobian of $\begin{bmatrix}\Phi \\ \mathbf{B}^\top \frac{\partial^\top \mathbf{H}}{\partial \rho}\end{bmatrix}$ exists and is full rank on $\mathcal{X}_\mathrm{c}$, implying that we have $n_\lambda$ smooth and independent kinematic constraints. Besides, by Lemma 2.1, the sets $\mathcal{R}_\Phi$ and $\mathcal{X}_\mathrm{c}$ are regular submanifolds of dimensions $n_r - n_\Phi$ and $2n_r - n_\lambda - n_\Phi$ that are embedded in $\mathcal{R}$ and $\mathcal{R} \times \mathbb{R}^{n_r}$, respectively. Differentiating the momentum constraints (5.12) along the system trajectories, yield the **hidden or secondary constraints**

$$0 = \frac{\partial \mathbf{B}^\top \mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^\top \mathbf{H}}{\partial \rho} + \mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\left(-\frac{\partial^\top \mathbf{H}}{\partial r} - \mathbf{R}(r)\frac{\partial^\top \mathbf{H}}{\partial \rho} + \mathbf{G}(r)u + \bar{\mathbf{B}}(r)\lambda\right), \tag{5.13}$$

where $\lambda$ can be uniquely calculated from Assumption 5.6.[44] Replacing such a solution in (5.11a) results in

$$\begin{bmatrix}\dot{r} \\ \dot{\rho}\end{bmatrix} = \begin{bmatrix}\mathbf{M}^{-1}(r)\rho \\ \mathbf{Z}(r)\left(-\frac{\partial^\top \mathbf{H}}{\partial r} - \mathbf{R}(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}\right) - \bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\partial \mathbf{B}^\top \mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^\top \mathbf{H}}{\partial \rho} + \mathbf{Z}(r)\mathbf{G}(r)u\end{bmatrix}, \tag{5.14}$$

[43] The momentum level constraints are obtained by differentiating the holonomic constraints (5.11b) w.r.t. time and rearranging them with the nonholonomic constraints (5.11c).

[44] Assumption 5.6 is a sufficient but not necessary condition to calculate $\lambda$. If $\Delta$ is singular, we may require higher-order derivatives of (5.12) to calculate $\lambda$, see Section 2.4.

where $\mathbf{Z}(r) = I_{n_r} - \bar{\mathbf{B}}(r)\Delta^{-1}(r)\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)$. It follows from the geometric perspective of DAEs that system (5.11) with consistent initial conditions (Assumption 5.7) is equivalent to the ODE (5.14) on the manifold $\mathcal{X}_\mathrm{c}$, i.e., the solutions $(r(\cdot), \rho(\cdot))$ of (5.11) are constrained to $\mathcal{X}_\mathrm{c}$. Besides, from Definition 2.9, the system has differentiation index 3 if at least one constraint is holonomic; otherwise, it has index 2. To be coherent with the terminology of Sections 2.4 and 2.5, we refer to $\mathcal{R}_\Phi$ as the **configuration manifold** (or **space**) and $\mathcal{X}_\mathrm{c}$ as the **constrained state-space**. The above can be summarized as

**Lemma 5.1** (Well-posedness)**.** *Consider the mechanical system (5.11) verifying Assumptions 5.6 and 5.7. Then, i) the system has $n_\lambda$ independent kinematic constraints and differential index 2 or 3, ii) $\lambda$ has a unique solution, iii) the DAE (5.11) is equivalent to the ODE (5.14) on $\mathcal{X}_\mathrm{c}$, and iv) the configuration manifold $\mathcal{R}_\Phi$ and constrained state space $\mathcal{X}_\mathrm{c}$ are regular (or embedded) submanifold of $\mathcal{R}$ and $\mathcal{R} \times \mathbb{R}^{n_r}$ with dimension $n_r - n_\Phi$ and $2n_r - n_\lambda - n_\Phi$, respectively.*

Finally, from Assumption 5.8, we exclude trivial systems with non-constant underactuation degree: If $\mathcal{R}_\Phi$ is not connected, then it is not path-connected (see [135, Proposition 1.11]), and we may have solutions of $r$ for which $\dot{r}(t)$ does not exist. Suppose $n_\lambda = \operatorname{rank} \mathbf{N}(r)$, then the input and constraint forces has the same direction, meaning that $u$ has no influence on the system trajectories because $\bar{\mathbf{B}}(r)\lambda$ dominates $\mathbf{G}(r)u$. And, if $n_\lambda = n_r$, the trajectories of (5.11) are reduced to $(r(t), \rho(t)) \equiv (r_\mathrm{c}, 0)$ for some constant $r_\mathrm{c}$.

### 5.2.3 Controller Design

Having defined the class of mechanical systems in implicit representation, our task now is to design a control law that transforms (5.11) into a constrained (or implicit) port-Hamiltonian system, which is (asymptotically) stable at the desired equilibrium $(r_\mathrm{d}, 0)$. The following result is instrumental to our subsequent discussions.

**Lemma 5.2** (Nonsingular matrices)**.** *Consider $A \in \mathbb{R}^{n\times n}_n$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times m}$ with $n > m$. The statements below are equivalent:*

*i)* $B^\top AC \in \mathbb{R}^{m\times m}_m$

*ii)* $C_\perp A^{-1} B_\perp^\top \in \mathbb{R}^{(n-m)\times(n-m)}_{(n-m)}$

*iii)* $\begin{bmatrix} C_\perp^\top & A^\top B \end{bmatrix} \in \mathbb{R}^{n\times n}_n$

*Suppose any (and hence all) of the assertions is satisfied, then*

$$I_n \equiv A^{-1}B_\perp^\top \left(C_\perp A^{-1} B_\perp^\top\right)^{-1} C_\perp + C\left(B^\top AC\right)^{-1} B^\top A.$$

**Proof.** See Appendix B.3. □

**Energy Shaping**

In spirit of Section 5.1.1, we employ the parameterized IDA-PBC, setting the target Hamiltonian as

$$\mathbf{H}_{\mathrm{d}}(r,\rho)=\tfrac{1}{2}\rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho+\mathbf{V}_{\mathrm{d}}(r),$$

where $\mathbf{M}_{\mathrm{d}}:\mathcal{R}\to\mathbb{R}^{n_r\times n_r}_{n_r}$ with $\mathbf{M}_{\mathrm{d}}(r)=\mathbf{M}_{\mathrm{d}}^{\top}(r)$ and $\mathbf{V}_{\mathrm{d}}:\mathcal{R}\to\mathbb{R}$ are the desired inertia matrix and potential energy, respectively. To streamline the controller design, we select a structure in the target system, writing

$$\begin{bmatrix}\dot{r}\\ \dot{\rho}\end{bmatrix}=\begin{bmatrix}0 & \mathbf{J}(r)\\ -\mathbf{J}^{\top}(r) & \mathbf{\Gamma}_1(r,\rho)+\mathbf{\Gamma}_2(r)\end{bmatrix}\begin{bmatrix}\frac{\partial^{\top}\mathbf{H}_{\mathrm{d}}}{\partial r}\\ \frac{\partial^{\top}\mathbf{H}_{\mathrm{d}}}{\partial \rho}\end{bmatrix}+\begin{bmatrix}0\\ \mathbf{B}_{\mathrm{d}}(r)\end{bmatrix}\lambda_{\mathrm{d}}, \tag{5.15}$$

where the state space remains the same, $\mathbf{B}_{\mathrm{d}}(r)\lambda_{\mathrm{d}}$ are the target constraint forces with $\mathbf{B}_{\mathrm{d}}:\mathcal{R}\to\mathbb{R}^{n_r\times n_\lambda}$ and implicit variables $\lambda_{\mathrm{d}}\in\mathbb{R}^{n_\lambda}$, $\mathbf{\Gamma}_1:\mathcal{R}\times\mathbb{R}^{n_r}\to\mathbb{R}^{n_r\times n_r}$ is linear in $\rho$, $\mathbf{\Gamma}_2:\mathcal{R}\to\mathbb{R}^{n_r\times n_r}$ is the target dissipation matrix (not necessarily symmetric and negative semidefinite), and $\mathbf{J}:\mathcal{R}\to\mathbb{R}^{n_r\times n_r}$ is a matrix that enables kinetic energy shaping. Observe that the kinematic constraints are a physical property of the nominal system (5.11), and therefore, they cannot be modified by control. The subsequent proposition shapes the total energy (kinetic plus potential) of the mechanical system (5.11).

**Assumption 5.9.** *Matrices* $\mathbf{J}$ *and* $\mathbf{M}_{\mathrm{d}}$ *satisfy*

$$\left(I_{n_r}-\mathbf{J}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{M}(r)\right)\mathbf{B}_{\perp}^{\top}(r)=0 \qquad \forall r\in\mathcal{R}_{\Phi}.$$

**Assumption 5.10.** *Matrix*

$$\Delta_{\mathrm{d}}(r):=\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\mathbf{B}_{\mathrm{d}}(r)$$

*is nonsingular for all* $r\in\mathcal{R}_{\Phi}$.

Clearly, a sufficient condition for Assumption 5.9 is $\mathbf{J}(r)=\mathbf{M}^{-1}(r)\mathbf{M}_{\mathrm{d}}(r)$.

**Proposition 5.1** (Implicit matching)**.** *Consider the implicit system* (5.11) *verifying Assumptions 5.6 to 5.8. Let* (5.15) *be the target system verifying Assumptions 5.9 and 5.10. System* (5.11) *can be transformed into* (5.15) *if and only if*

$$\begin{aligned}0=\;&\mathbf{N}_{\perp}(r)\left(\mathbf{Z}(r)\frac{1}{2}\frac{\partial^{\top}\rho^{\top}\mathbf{M}^{-1}\rho}{\partial r}+\mathbf{Z}_{\mathrm{d}}(r)\bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\partial\mathbf{B}^{\top}\mathbf{M}^{-1}\rho}{\partial r}\mathbf{M}^{-1}(r)\rho\right)\\&+\mathbf{N}_{\perp}(r)\mathbf{Z}_{\mathrm{d}}(r)\left(-\mathbf{J}^{\top}(r)\frac{1}{2}\frac{\partial^{\top}\rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}\rho}{\partial r}+\mathbf{\Gamma}_1(r,\rho)\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho\right),\end{aligned} \tag{5.16a}$$

$$0=\mathbf{N}_{\perp}(r)\left(\mathbf{Z}(r)\frac{\partial^{\top}\mathbf{V}}{\partial r}-\mathbf{Z}_{\mathrm{d}}(r)\mathbf{J}^{\top}(r)\frac{\partial^{\top}\mathbf{V}_{\mathrm{d}}}{\partial r}\right) \tag{5.16b}$$

*for all* $(r,\rho)\in\mathcal{X}_\mathrm{c}$*. Consequently, a solution of* $\mathbf{\Gamma}_2$ *is given by*

$$\mathbf{\Gamma}_2(r)\mathbf{M}_\mathrm{d}^{-1}(r)\rho = -\left(\bar{\mathbf{G}}(r)\bar{\mathbf{\Gamma}}_2(r)+\mathbf{Z}(r)\mathbf{R}(r)\right)\mathbf{M}^{-1}(r)\rho \qquad \forall(r,\rho)\in\mathcal{X}_\mathrm{c}, \tag{5.17}$$

*and the control laws are all of the form* $u=\mathbf{u}_\mathrm{ida}(r,\rho)$*,*

$$\begin{aligned}\mathbf{u}_\mathrm{ida}(r,\rho) = \bar{\mathbf{G}}^\mathrm{g}(r)&\left(\mathbf{Z}(r)\frac{\partial^\top\mathbf{H}}{\partial r}-\mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\frac{\partial^\top\mathbf{H}_\mathrm{d}}{\partial r}+\mathbf{Z}_\mathrm{d}(r)\mathbf{\Gamma}_1(r,\rho)\frac{\partial^\top\mathbf{H}_\mathrm{d}}{\partial\rho}\right)\\ &+\bar{\mathbf{G}}^\mathrm{g}(r)\mathbf{Z}_\mathrm{d}(r)\bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\partial\mathbf{B}^\top\mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^\top\mathbf{H}}{\partial\rho}-\bar{\mathbf{\Gamma}}_2(r)\frac{\partial^\top\mathbf{H}}{\partial\rho}+\bar{\mathbf{G}}_\perp(r)\nu.\end{aligned} \tag{5.18}$$

*Here,* $\mathbf{Z}(r)=I_{n_r}-\bar{\mathbf{B}}(r)\Delta^{-1}(r)\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)$*,* $\mathbf{Z}_\mathrm{d}(r)=I_{n_r}-\mathbf{B}_\mathrm{d}(r)\Delta_\mathrm{d}^{-1}(r)\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)$*,* $\bar{\mathbf{G}}(r)=\mathbf{Z}(r)\mathbf{G}(r)$*, and* $\nu$ *and* $\bar{\mathbf{\Gamma}}_2:\mathcal{R}_\Phi\to\mathbb{R}^{n_u\times n_r}$ *are both arbitrary.*

**Proof.** From Assumptions 5.6 to 5.8, system (5.11) is equivalent to the ODE (5.14) on the regular manifold $\mathcal{X}_\mathrm{c}$. Geometrically, it means that every solution $(r(\cdot),\rho(\cdot))$ of (5.11) belongs to $\mathcal{X}_\mathrm{c}$ while its derivative $(\dot r(\cdot),\dot\rho(\cdot))$ belongs to the tangent space of $\mathcal{X}_\mathrm{c}$, i.e.,

$$(\dot r,\dot\rho)\in T_{(r,\rho)}\mathcal{X}_\mathrm{c} := \left\{(s_r,s_\rho)\mid \mathbf{B}^\top(r)s_r=0,\,\frac{\partial^\top\mathbf{B}^\top\mathbf{M}^{-1}\rho}{\partial r}s_r+\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)s_\rho\right\}.$$

For clarity, we will omit the expression $(r,\rho)\in\mathcal{X}_\mathrm{c}$ in the rest of this proof, as it is implicitly included in all our derivations. From the tangent space of $\mathcal{X}_\mathrm{c}$, it follows that the trajectories of the closed-loop are consistent with the constrained state space if

$$0=\mathbf{B}^\top(r)\mathbf{J}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\rho, \tag{5.19a}$$

$$0=\frac{\partial\mathbf{B}^\top\mathbf{M}^{-1}\rho}{\partial r}\mathbf{J}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\rho+\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\mathbf{X}_\mathrm{d}(r,\rho)+\Delta_\mathrm{d}(r)\lambda_\mathrm{d}, \tag{5.19b}$$

where $\mathbf{X}_\mathrm{d}(r,\rho)=-\mathbf{J}^\top(r)\frac{\partial^\top\mathbf{H}_\mathrm{d}}{\partial r}+\left(\mathbf{\Gamma}_1(r,\rho)+\mathbf{\Gamma}_2(r)\right)\frac{\partial^\top\mathbf{H}_\mathrm{d}}{\partial\rho}$. Since $\mathcal{X}_\mathrm{c}$ can always be written as

$$\mathcal{X}_\mathrm{c}=\left\{(r,\rho)\mid r\in\mathcal{R}_\Phi,\,\bar\rho\in\mathbb{R}^{n_r-n_\lambda},\,\rho=\mathbf{M}(r)\mathbf{B}_\perp^\top(r)\bar\rho\right\}, \tag{5.20}$$

we conclude that Assumption 5.9 verifies (5.19a) while Assumption 5.10 guarantees the existence of a unique $\lambda_\mathrm{d}$ that meets (5.19b).

In the next step, we replace the solution of $\lambda_\mathrm{d}$ in the target system (5.15) to get the equivalent (target) ODE

$$\begin{bmatrix}\dot r\\ \dot\rho\end{bmatrix}=\begin{bmatrix}\mathbf{J}(r)\frac{\partial^\top\mathbf{H}_\mathrm{d}}{\partial\rho}\\ \mathbf{Z}_\mathrm{d}(r)\mathbf{X}_\mathrm{d}(r,\rho)-\mathbf{B}_\mathrm{d}(r)\Delta_\mathrm{d}^{-1}(r)\frac{\partial\mathbf{B}^\top\mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^\top\mathbf{H}}{\partial\rho}\end{bmatrix}. \tag{5.21}$$

At this point, system (5.11) can be transformed into (5.15) if and only if, for some feedback $u = u(r,\rho)$, the ODEs (5.14) and (5.21) have identical trajectories, or equivalently their vector fields match (are equal) in $\mathcal{X}_\mathrm{c}$. Such a matching process verifies Assumption 5.9 and results in

$$\bar{\mathbf{G}}(r)u = \mathbf{Z}_\mathrm{d}(r)\mathbf{X}_\mathrm{d}(r,\rho) - \mathbf{Z}(r)\mathbf{X}(r,\rho) + \mathbf{Z}_\mathrm{d}(r)\bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\partial \mathbf{B}^\top \mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^\top \mathbf{H}}{\partial \rho},$$

where $\mathbf{X}(r,\rho) = -\frac{\partial^\top \mathbf{H}}{\partial r} - \mathbf{R}(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}$. Besides, the above equation can always be decomposed w.r.t. its dependency on $\rho$ (quadratic in $\rho$, linear in $\rho$ and independent of $\rho$, respectively), obtaining

$$\begin{aligned}\bar{\mathbf{G}}(r)u_1(r,\rho) = \mathbf{Z}(r)\frac{1}{2}\frac{\partial^\top \rho^\top \mathbf{M}^{-1}\rho}{\partial r} + \mathbf{Z}_\mathrm{d}(r)\bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\partial \mathbf{B}^\top \mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^\top \mathbf{H}}{\partial \rho}\\ - \mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\frac{1}{2}\frac{\partial^\top \rho^\top \mathbf{M}_\mathrm{d}^{-1}\rho}{\partial r} + \mathbf{Z}_\mathrm{d}(r)\mathbf{\Gamma}_1(r,\rho)\frac{\partial^\top \mathbf{H}_\mathrm{d}}{\partial \rho},\end{aligned} \tag{5.22a}$$

$$\mathbf{Z}_\mathrm{d}(r)\mathbf{\Gamma}_2(r)\frac{\partial^\top \mathbf{H}_\mathrm{d}}{\partial \rho} = \bar{\mathbf{G}}(r)u_2(r,\rho) - \mathbf{Z}(r)\mathbf{R}(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}, \tag{5.22b}$$

$$\bar{\mathbf{G}}(r)u_3(r) = \mathbf{Z}(r)\frac{\partial^\top \mathbf{V}}{\partial r} - \mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\frac{\partial^\top \mathbf{V}_\mathrm{d}}{\partial r}, \tag{5.22c}$$

$$u = u_1(r,\rho) + u_2(r,\rho) + u_3(r), \tag{5.22d}$$

where we set (without loss of generality) $u_2(r,\rho) := -\bar{\mathbf{\Gamma}}_2(r)\frac{\partial^\top \mathbf{H}}{\partial \rho}$.

Suppose for the moment that $\left[\begin{smallmatrix}\mathbf{B}^\top \mathbf{M}^{-1}\\ \mathbf{N}_\perp\end{smallmatrix}\right]$ is a full-rank left annihilator of $\bar{\mathbf{G}}$, and $\mathbf{B}^\top\mathbf{M}^{-1}$ is the one of $\mathbf{Z}$ and $\mathbf{Z}_\mathrm{d}$. Then, by using Lemma 3.2, we can show that (5.22b) has always a solution in $\mathbf{\Gamma}_2$ for every $\bar{\mathbf{\Gamma}}_2$ and that (5.22a) and (5.22c) have a solution in $u_1$ and $u_3$, respectively, if and only if the matching conditions (5.16a)–(5.16b) hold. Furthermore, the general solution of $u$ is given by (5.18) while the general solution of $\mathbf{\Gamma}_2$ is given by

$$\mathbf{\Gamma}_2(r)\frac{\partial^\top \mathbf{H}_\mathrm{d}}{\partial \rho} = -\mathbf{Z}_\mathrm{d}^\mathrm{g}(r)\bar{\mathbf{G}}(r)\bar{\mathbf{\Gamma}}_2(r)\frac{\partial^\top \mathbf{H}}{\partial \rho} - \mathbf{Z}_\mathrm{d}^\mathrm{g}(r)\mathbf{Z}(r)\mathbf{R}(r)\frac{\partial^\top \mathbf{H}}{\partial \rho} + \mathbf{Z}_{\mathrm{d}\mathcal{I}}(r)\bar{\nu}.$$

Here, we recover (5.17) by choosing $\bar{\nu} = 0$ and $\mathbf{Z}_\mathrm{d}^\mathrm{g}(r) = \mathbf{Z}(r)$.[45]

The demonstration of $\mathbf{B}^\top\mathbf{M}^{-1}$ being a full-rank left annihilator of $\mathbf{Z}$ and $\mathbf{Z}_\mathrm{d}$, is a direct consequence of identities

$$\begin{bmatrix}\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\\ \bar{\mathbf{B}}_\perp(r)\end{bmatrix}\mathbf{Z}(r) = \begin{bmatrix}0\\ \bar{\mathbf{B}}_\perp(r)\end{bmatrix}, \quad \begin{bmatrix}\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\\ \bar{\mathbf{B}}_{\mathrm{d}\perp}(r)\end{bmatrix}\mathbf{Z}_\mathrm{d}(r) = \begin{bmatrix}0\\ \bar{\mathbf{B}}_{\mathrm{d}\perp}(r)\end{bmatrix},$$

[45] Note that $\mathbf{Z}(r)\mathbf{Z}(r) \equiv \mathbf{Z}(r)$ and $\mathbf{Z}_\mathrm{d}(r)\mathbf{Z}(r)\mathbf{Z}_\mathrm{d}(r) \equiv \mathbf{Z}_\mathrm{d}(r)$.

where $\begin{bmatrix}\mathbf{M}^{-1}(r)\mathbf{B}(r) & \bar{\mathbf{B}}_{\perp}^{\top}(r)\end{bmatrix}$ and $\begin{bmatrix}\mathbf{M}^{-1}(r)\mathbf{B}(r) & \bar{\mathbf{B}}_{\mathrm{d}\perp}^{\top}(r)\end{bmatrix}$ are nonsingular as a result of the nonsingularity condition in $\Delta$ and $\Delta_d$, see Lemma 5.2. It remains to prove that $\begin{bmatrix}\mathbf{B}^{\top}\mathbf{M}^{-1}\\ \mathbf{N}_{\perp}\end{bmatrix}$ is a full-rank left annihilator of $\bar{\mathbf{G}}$. For this, we employ Sylvester's inequality (Lemma A.1) and identity

$$\begin{bmatrix}\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\\ \bar{\mathbf{B}}_{\perp}(r)\end{bmatrix}\bar{\mathbf{G}}(r)=\begin{bmatrix}0\\ \bar{\mathbf{B}}_{\perp}(r)\mathbf{G}(r)\end{bmatrix}$$

to get $\operatorname{rank}(\mathbf{N}(r))-n_{\lambda}\leq\operatorname{rank}(\bar{\mathbf{B}}_{\perp}(r)\mathbf{N}(r))=\operatorname{rank}(\bar{\mathbf{B}}_{\perp}(r)\mathbf{G}(r))=\operatorname{rank}\bar{\mathbf{G}}(r)$, which (by the rank–nullity theorem) is equivalent to $\operatorname{rank}\bar{\mathbf{G}}_{\perp}(r)\leq n_r+n_{\lambda}-\operatorname{rank}\mathbf{N}(r)$. The proof is now completed with identities

$$\begin{bmatrix}\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\\ \mathbf{N}_{\perp}(r)\end{bmatrix}\bar{\mathbf{G}}(r)=0,\quad \begin{bmatrix}\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\\ \mathbf{N}_{\perp}(r)\end{bmatrix}\begin{bmatrix}\bar{\mathbf{B}}(r) & \mathbf{N}_{\perp}^{\top}(r)\end{bmatrix}=\begin{bmatrix}\Delta(r) & \star\\ 0 & \mathbf{N}_{\perp}(r)\mathbf{N}_{\perp}^{\top}(r)\end{bmatrix}$$

under the observation that $\begin{bmatrix}\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\\ \mathbf{N}_{\perp}(r)\end{bmatrix}$ annihilates $\bar{\mathbf{G}}$ and that it posses the maximum admissible rank of $\bar{\mathbf{G}}_{\perp}$, i.e., $\operatorname{rank}\begin{bmatrix}\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\\ \mathbf{N}_{\perp}(r)\end{bmatrix}=\operatorname{rank}\Delta(r)+\operatorname{rank}\mathbf{N}_{\perp}(r)\mathbf{N}_{\perp}^{\top}(r)=n_r+n_{\lambda}-\operatorname{rank}\mathbf{N}(r)$. □

*Remark* 5.1. If $\bar{\mathbf{G}}^{\mathrm{g}}(r)\begin{bmatrix}\bar{\mathbf{B}}(r) & \mathbf{B}_{\mathrm{d}}(r)\end{bmatrix}\equiv 0$, then (5.18) reduces to

$$\mathbf{u}_{\mathrm{ida}}(r,\rho)=\bar{\mathbf{G}}^{\mathrm{g}}(r)\left(\frac{\partial^{\top}\mathbf{H}}{\partial r}-\mathbf{J}^{\top}(r)\frac{\partial^{\top}\mathbf{H}_{\mathrm{d}}}{\partial r}+\mathbf{\Gamma}_1(r,\rho)\frac{\partial^{\top}\mathbf{H}_{\mathrm{d}}}{\partial\rho}\right)-\bar{\mathbf{\Gamma}}_2(r)\frac{\partial^{\top}\mathbf{H}}{\partial\rho}+\bar{\mathbf{G}}_{\perp}(r)\nu. \quad (5.23)$$

Proposition 5.1 transforms the well-posed implicit system (5.11) into the target system (5.15) at the expense of satisfying the **(implicit) matching conditions** (5.16). These conditions, which are split into the **matching of the kinetic energy** (5.16a) and the **matching of potential energy** (5.16b), are the backbone of the proposed energy shaping and represent a system of nonhomogeneous first-order quasi-linear PDEs with unknowns in $\mathbf{M}_{\mathrm{d}}$, $\mathbf{V}_{\mathrm{d}}$ and $\mathbf{\Gamma}_1$ but constrained to $\mathcal{X}_{\mathrm{c}}$.

A striking feature of Proposition 5.1 is that it does not require the target system (5.15) to be port-Hamiltonian, since the only condition imposed on $\mathbf{B}_{\mathrm{d}}$ is the nonsingularity of $\Delta_{\mathrm{d}}(r):=\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\mathbf{B}_{\mathrm{d}}(r)$.[46] Besides, comparing the implicit and explicit energy shaping perspectives, we notice that the control law (5.4) and matching conditions (5.3) of *the unconstrained situation can be viewed as a specific case of the implicit perspective* introduced in Proposition 5.1:

$$\begin{aligned}&r=q, && \mathbf{M}(r)=M(q), && \mathbf{J}(r)=J(q), && \mathbf{N}(r)=G(q), && \mathbf{B}(r)=0,\\ &\rho=p, && \mathbf{V}(r)=V(q), && \mathbf{R}(r)=R(q), && \mathbf{Z}_{\mathrm{d}}(r)=I_{n_r}.\end{aligned}$$

[46] System (5.15) is port-Hamiltonian if $0=\mathbf{B}_{\mathrm{d}}^{\top}(r)\frac{\partial^{\top}\mathbf{H}_{\mathrm{d}}}{\partial\rho}$ for all $(r,\rho)$ in $\mathcal{X}_{\mathrm{c}}$, i.e., if the constrained forces are workless for every $\lambda_d$, see Section 2.6.3.

This considerable resemblance will allow us to extrapolate most of the results for constrained systems to unconstrained ones, see e.g., Section 6.2. Besides, using the generalized inverse of $\bar{\mathbf{G}}$ instead of the Moore–Penrose inverse, we can simplify the final expression of (5.18), see Remark 5.1 and Sections 7.1.3, 7.1.6, 7.2.2, 7.2.3 and 7.3.3.

*Example* 5.1. (Example 2.7, continued) Consider the simple pendulum of Figure 2.5 with coordinates $r = \mathrm{vec}(x_\mathrm{p}, y_\mathrm{p}) \in \mathcal{R} = \mathbb{R}^2$, input torque $\tau$, holonomic constraint

$$\Phi(r) := \tfrac{1}{2}\left(x_\mathrm{p}^2 + y_\mathrm{p}^2 - l^2\right) = 0 \tag{5.24a}$$

and consistent initial conditions (Assumption 5.7). We wish to reshape the pendulum's energy with Proposition 5.1. For this, we employ the standard assumptions that the bob is a point mass while the rod is massless. Hence, the pendulum's total energy is given by

$$\mathbf{H}(r,\rho) = \tfrac{1}{2}\rho^\top \mathbf{M}^{-1}\rho + \mathbf{V}(r) = \tfrac{1}{2}\dot{r}^\top \mathbf{M}\dot{r} + \mathbf{V}(r) = \tfrac{m}{2}\left(\dot{x}_\mathrm{p}^2 + \dot{y}_\mathrm{p}^2\right) + g_\mathrm{c} m y,$$

and its constrained Hamiltonian dynamics is

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & I_2 \\ -I_2 & -\mathbf{R}(r) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}}{\partial r} \\ \frac{\partial^\top \mathbf{H}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial^\top \Phi}{\partial r} \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ \mathbf{G}(r) \end{bmatrix} \tau, \tag{5.24b}$$

where $\mathbf{M} = mI_2$, $g_\mathrm{c}$ is the gravity constant, and matrices $\mathbf{G} = \frac{1}{l^2}\,\mathrm{vec}(-y_\mathrm{p}, x_\mathrm{p})$ and $\mathbf{R} = \frac{1}{l^2} c_\theta I_2$ result from $\dot{r}^\top \mathbf{G}(r)\tau = \dot{\theta}\tau$, the Rayleigh dissipation $\mathbf{D}(r,\dot{r}) = \frac{1}{2} c_\theta \frac{\dot{x}_\mathrm{p}^2 + \dot{y}_\mathrm{p}^2}{l^2} \geq 0$ and $\frac{\dot{x}_\mathrm{p}^2 + \dot{y}_\mathrm{p}^2}{l^2} = \dot{\theta}^2$. We have the following observations regarding the implicit system (5.24).

- The set $\mathcal{R}_\Phi := \{r \in \mathcal{R} \mid 0 = \Phi(r)\} \cong \mathbb{S}^1$ is a smooth manifold (see Example 2.4), and

  $$\Delta(r) := \mathbf{B}^\top(r)\mathbf{M}^{-1}\bar{\mathbf{B}}(r) = \frac{x_\mathrm{p}^2 + y_\mathrm{p}^2}{m} = \frac{l^2}{m} > 0 \qquad \forall r \in \mathcal{R}_\Phi,$$

  i.e., Assumption 5.6 is verified.

- $\mathbf{N}(r) := \begin{bmatrix} \mathbf{G}(r) & \bar{\mathbf{B}}(r) \end{bmatrix} = \begin{bmatrix} \mathbf{G}(r) & \frac{\partial^\top \Phi}{\partial r} \end{bmatrix}$ is nonsingular for every $r \in \mathcal{R}_\Phi$, meaning that Assumption 5.8 holds and the pendulum is fully actuated (see Definition 2.12).

To fulfill Assumptions 5.9 and 5.10, let for simplicity $\mathbf{M}_\mathrm{d} = a_1 I_2$, $\mathbf{J} = \mathbf{M}^{-1}\mathbf{M}_\mathrm{d}$ and $\mathbf{B}_d(r) = \mathbf{J}^\top \mathbf{B}(r)$ with a nonzero constant $a_1$:

$$\Delta_\mathrm{d}(r) = a_1 \frac{x_\mathrm{p}^2 + y_\mathrm{p}^2}{m^2} = \frac{a_1 l^2}{m^2} \neq 0 \qquad \forall r \in \mathcal{R}_\Phi.$$

Since $\mathbf{N}$ is nonsingular, the implicit matching conditions (5.16) are trivial ($\mathbf{N}_\perp(r) = 0$) for any $a_1$, $\mathbf{V}_\mathrm{d}$ and $\mathbf{\Gamma}_1$. Consequently, all the conditions of Proposition 5.1 are fulfilled and feedback[47]

$$\mathbf{u}_{\mathrm{ida}}(r,\rho) = x_{\mathrm{p}} g_{\mathrm{c}} m + \frac{a_1}{m} \begin{bmatrix} y_{\mathrm{p}} & -x_{\mathrm{p}} \end{bmatrix} \frac{\partial^\top \mathbf{V}_{\mathrm{d}}}{\partial r} + \frac{1}{a_1} \begin{bmatrix} -y_{\mathrm{p}} & x_{\mathrm{p}} \end{bmatrix} \mathbf{\Gamma}_1(r,\rho)\rho - \frac{1}{m}\bar{\mathbf{\Gamma}}_2(r)\rho \tag{5.25}$$

transforms (5.24) into the system

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & \frac{a_1}{m} I_2 \\ -\frac{a_1}{m} I_2 & \mathbf{\Gamma}_1(r,\rho) + \mathbf{\Gamma}_2(r) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}_{\mathrm{d}}}{\partial r} \\ \frac{\partial^\top \mathbf{H}_{\mathrm{d}}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{a_1}{m}\mathbf{B}(r) \end{bmatrix} \lambda_{\mathrm{d}}, \tag{5.26a}$$

$$0 = \Phi(r) := \tfrac{1}{2}\left(x_{\mathrm{p}}^2 + y_{\mathrm{p}}^2 - l^2\right) \tag{5.26b}$$

with new energy

$$\mathbf{H}_{\mathrm{d}}(r,\rho) = \tfrac{1}{2}\rho^\top \mathbf{M}_{\mathrm{d}}^{-1}\rho + \mathbf{V}_{\mathrm{d}}(r),$$

where $\bar{\mathbf{\Gamma}}_2$ is arbitrary, $\mathbf{\Gamma}_2$ is defined in (5.17) and $\bar{\mathbf{G}}^{\mathrm{g}}$ is selected as $\bar{\mathbf{G}}^{\mathrm{g}}(r) = \begin{bmatrix} -y_{\mathrm{p}} & x_{\mathrm{p}} \end{bmatrix}$. △

### Admissible Equilibria

A necessary condition to asymptotically stabilize a point $(r_{\mathrm{d}}, 0)$ is that this point must be an admissible equilibrium. In this regard, Lemma 3.6 gives the admissible equilibria set for ODEs of the form (3.3) that naturally includes mechanical systems without constraints. The Lemma below gives such a set for the constrained case.

**Lemma 5.3.** *The state $(r^\star, \rho^\star)$ is an admissible equilibrium of (5.11) verifying Assumptions 5.6 to 5.8 if and only if*

$$\rho^\star = 0 \quad \text{and} \quad r^\star \in \mathcal{R}_{\mathrm{a}} := \left\{ r \in \mathcal{R}_\Phi \mid 0 = \mathbf{N}_\perp(r)\frac{\partial^\top \mathbf{V}}{\partial r} \right\}. \tag{5.27}$$

**Proof.** Let $(r^\star, \rho^\star)$ be an admissible equilibrium of the well-posed system (5.11), or equivalently, of the underlying ODE (5.14). Then, $\rho^\star = 0$, $r^\star \in \mathcal{R}_\Phi$ and

$$\exists u^\star \in \mathbb{R}^{n_u} \text{ s.t.} \quad 0 = -\mathbf{Z}(r^\star)\left.\frac{\partial^\top \mathbf{V}}{\partial r}\right|_{r=r^\star} + \bar{\mathbf{G}}(r^\star)u^\star. \tag{5.28}$$

Hence, from Lemmas 3.2 and 5.2 with $\begin{bmatrix} \mathbf{B}^\top \mathbf{M}^{-1} \\ \mathbf{N}_\perp \end{bmatrix}$ being a full-rank left annihilator of $\bar{\mathbf{G}}$ (see the proof of Proposition 5.1), we have

$$(5.28) \iff 0 = \mathbf{N}_\perp(r^\star)\left.\frac{\partial^\top \mathbf{V}_{\mathrm{d}}}{\partial r}\right|_{r=r^\star}, \tag{5.29}$$

[47] Controller (5.25) is actually the simplified version given in Remark 5.1.

and the necessity is established. Now, suppose (5.27) holds. Evaluating (5.14) in $(r^\star, 0)$ gives $\dot{r} = 0$ and

$$\dot{\rho} = -\mathbf{Z}(r^\star) \left.\frac{\partial^\top \mathbf{V}}{\partial r}\right|_{r=r^\star} + \bar{\mathbf{G}}(r^\star)u, \tag{5.30}$$

but we know from (5.29) that there exists $u = u^\star$ s.t. the right-hand side of (5.30) is zero, which completes the proof. □

**Stabilization**

The following proposition applies and extends the results of [38, 75, 77] to mechanical systems with kinematic constraints (holonomic and nonholonomic) for the (asymptotic) stabilization of an admissible equilibrium with total energy shaping IDA-PBC.

**Assumption 5.11.** *Matrices* $\mathbf{B}_\mathrm{d}$ *and* $\mathbf{M}_\mathrm{d}$ *satisfy*

$$\operatorname{Colsp}(\mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)) = \operatorname{Colsp}\mathbf{B}_\mathrm{d}(r) \qquad \forall r \in \mathcal{R}_\Phi .$$

A sufficient condition for Assumption 5.11 is $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$.

**Proposition 5.2** (Implicit IDA-PBC)**.** *Consider the system* (5.11) *under Assumptions 5.6 to 5.8. Let* (5.15) *be the target dynamics verifying Assumptions 5.9 and 5.11, and*

$$\mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r) \succ 0 \qquad \forall r \in \mathcal{R}_\Phi, \tag{5.31a}$$

$$\frac{\partial \mathbf{H}_\mathrm{d}}{\partial \rho}\Big(\bar{\mathbf{G}}(r)\bar{\boldsymbol{\Gamma}}_2(r)\mathbf{J}(r) + \mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r) - \boldsymbol{\Gamma}_1(r,\rho)\Big)\frac{\partial^\top \mathbf{H}_\mathrm{d}}{\partial \rho} \geq 0 \qquad \forall (r,\rho) \in \mathcal{X}_\mathrm{c}. \tag{5.31b}$$

*Then, Assumption 5.10 holds, and the feedback* (5.18) *transforms* (5.11) *into* (5.15) *if and only if the matching conditions* (5.16) *are fulfilled. Here,* $\boldsymbol{\Gamma}_2$ *is as specified in* (5.17)*. Furthermore, all bounded trajectories of* (5.15) *converge to the set* $\Omega_{\mathrm{inv}}$*, defined as the largest invariant set of the closed-loop contained in*

$$\Omega = \left\{(r,\rho) \in \mathcal{X}_\mathrm{c} \mid \frac{\partial \mathbf{H}_\mathrm{d}}{\partial \rho}\Big(\bar{\mathbf{G}}(r)\bar{\boldsymbol{\Gamma}}_2(r)\mathbf{J}(r) + \mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r) - \boldsymbol{\Gamma}_1(r,\rho)\Big)\frac{\partial^\top \mathbf{H}_\mathrm{d}}{\partial \rho} = 0\right\}.$$

*Suppose, in addition, that*

$$\mathbf{V}_\mathrm{d}(r) > \mathbf{V}_\mathrm{d}(r_\mathrm{d}) \qquad \forall r \in \mathcal{R}_\Phi - \{r_\mathrm{d}\}, \tag{5.31c}$$

*then* (5.15) *is stable in the admissible equilibrium* $(r_\mathrm{d}, 0)$*. (Local) asymptotic stability can be demonstrated whenever* $(r_\mathrm{d}, 0)$ *is an isolated point in* $\Omega_{\mathrm{inv}}$*.*[48]

[48] Let $\mathcal{X}$ be a topological set. A point $p$ is said to be isolated in a set $\mathcal{P} \subset \mathcal{X}$ if there is a neighborhood $\mathcal{N} \subset \mathcal{X}$ of $p$ with the property that $\mathcal{N} \cap \mathcal{P} = \{p\}$.

**Proof.** The proof falls naturally into three parts: *i)* transformation of (5.11) into (5.15), *ii)* application of LaSalle's Theorem 2.4, and *iii)* stability analysis.

*i)* From Lemma 5.2, inequality (5.31a) is sufficient for the nonsingularity of

$$\bar{\Delta}_{\mathrm{d}}(r) := \mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r).$$

This means that $\mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ is full rank and Assumption 5.11 can be equivalently written as

$$\mathbf{B}_{\mathrm{d}}(r) = \mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)\mathbf{K}_{\mathbf{B}}(r)$$

with an arbitrary function $\mathbf{K}_{\mathbf{B}} : \mathcal{R}_{\Phi} \to \mathbb{R}^{n_\lambda \times n_\lambda}_{n_\lambda}$. Now, $\Delta_{\mathrm{d}}$ takes the form $\Delta_{\mathrm{d}}(r) = \bar{\Delta}_{\mathrm{d}}(r)\mathbf{K}_{\mathbf{B}}(r)$, which is clearly nonsingular. Hence, all conditions of Proposition 5.1 are fulfilled, meaning that (5.11) with controller (5.18) is transformed into system (5.15).

*ii)* Taking the time derivative of $\mathbf{H}_{\mathrm{d}}$ along the trajectories of (5.15) gives

$$\begin{aligned}
\dot{\mathbf{H}}_{\mathrm{d}}\Big|_{\mathcal{X}_{\mathrm{c}}}(r,\rho) &= \rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}(r)\Big(\mathbf{\Gamma}_1(r,\rho) + \mathbf{\Gamma}_2(r)\Big)\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho\Big|_{\mathcal{X}_{\mathrm{c}}} + \rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{B}_{\mathrm{d}}(r)\lambda_{\mathrm{d}}\Big|_{\mathcal{X}_{\mathrm{c}}} \\
&= \rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}(r)\Big(\mathbf{\Gamma}_1(r,\rho) + \mathbf{\Gamma}_2(r)\Big)\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho\Big|_{\mathcal{X}_{\mathrm{c}}} \\
&= \rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}(r)\Big(\mathbf{\Gamma}_1(r,\rho) - \bar{\mathbf{G}}(r)\bar{\mathbf{\Gamma}}_2(r)\mathbf{J}(r) - \mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r)\Big)\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho\Big|_{\mathcal{X}_{\mathrm{c}}},
\end{aligned}$$

where the second equality results from $\mathbf{B}_{\mathrm{d}}(r) = \mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)\mathbf{K}_{\mathbf{B}}(r)$ and the last one from the solution of $\mathbf{\Gamma}_2$ obtained from (5.17). Notice that replacing the most general solution of $\mathbf{\Gamma}_2$ given by (5.22b) also yields the same result because $\rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{Z}_{\mathrm{d}}(r)\Big|_{\mathcal{X}_{\mathrm{c}}} = \rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}(r)\Big|_{\mathcal{X}_{\mathrm{c}}}$. Since (5.15) is actually the ODE (5.21) on $\mathcal{X}_{\mathrm{c}}$, it follows by Lemma A.4 that the bounded trajectories of (5.21) approach a positive limit set $L^+ \subset \mathcal{X}_{\mathrm{c}}$ that is a nonempty, compact, connected and invariant. Now, application of LaSalle's Theorem 2.4 on this ODE with $L^+$ and (5.31b), i.e., $\dot{\mathbf{H}}_{\mathrm{d}}\Big|_{\mathcal{X}_{\mathrm{c}}}(r,\rho) \leq 0$, shows that the bounded trajectories of the closed-loop approach $\Omega_{\mathrm{inv}}$ as time goes to infinity.

*iii)* Let $(\bar{\mathcal{N}}, \bar{\Psi})$ be a coordinate chart on $\mathcal{X}_{\mathrm{c}}$ with local coordinates $\bar{x}$ s.t. $\bar{\mathcal{N}}$ is a connected set and $(r_{\mathrm{d}}, 0) \in \bar{\mathcal{N}}$. Then, (5.15) can be locally rewritten as an ODE with coordinates $\bar{x}$ and domain $\bar{\Psi}(\bar{\mathcal{N}})$,[49] which is an open and connected set. Define $\bar{V}(\bar{x}) := \mathbf{H}_{\mathrm{d}}(\bar{\Psi}^{-1}(\bar{x})) - \mathbf{H}_{\mathrm{d}}(r_{\mathrm{d}}, 0)$. From Lyapunov's Theorem 2.1, the point $\bar{x}_{\mathrm{d}} = \bar{\Psi}(r_{\mathrm{d}}, 0)$ is an stable equilibrium of the local ODE if $\bar{V}$ is positive definite about $\bar{x}_{\mathrm{d}}$ and $\dot{\bar{V}}$ is negative semidefinite (about $\bar{x}_{\mathrm{d}}$). For this, rewrite $\mathcal{X}_{\mathrm{c}}$ as in (5.20), observing that the constrained target Hamiltonian can be express as

$$\mathbf{H}_{\mathrm{d}}|_{\mathcal{X}_{\mathrm{c}}}(r,\rho) = \tfrac{1}{2}\,\rho^{\top}\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho\Big|_{\mathcal{X}_{\mathrm{c}}} + \mathbf{V}_{\mathrm{d}}(r)|_{\mathcal{R}_{\Phi}},$$

[49]The ODE (5.21) on $\mathcal{X}_{\mathrm{c}} \subset \mathcal{R} \times \mathbb{R}^{n_r}$ has coordinates $(r,\rho)$ while the local one has coordinates $\bar{x} \in \bar{\Psi}(\bar{\mathcal{N}}) \subset \mathbb{R}^{2n_r - n_\lambda - n_\Phi}$, see Section 2.4.2.

$$= \tfrac{1}{2}\,\bar{\rho}^\top \mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r)\bar{\rho}\Big|_{\mathcal{R}_\Phi} + \mathbf{V}_\mathrm{d}(r)|_{\mathcal{R}_\Phi}\,.$$

Thus, the conditions (5.31) are sufficient for $\mathbf{H}_\mathrm{d}|_{\bar{\mathcal{N}}}$ being positive definite about $(r_\mathrm{d}, 0)$ and $\dot{\mathbf{H}}_\mathrm{d}\big|_{\bar{\mathcal{N}}}$ being negative semidefinite, meaning that $\bar{x}_\mathrm{d}$ is a stable equilibrium of the local ODE and $(r_\mathrm{d}, 0)$ is a stable equilibrium of (5.15).[50] If $(r_\mathrm{d}, 0)$ is an isolated point in $\Omega_\mathrm{inv}$, there is a neighborhood $\mathcal{N}_\mathrm{nh}$ of $(r_\mathrm{d}, 0)$ satisfying $\mathcal{N}_\mathrm{nh} \cap \Omega_\mathrm{inv} = \{(r_\mathrm{d}, 0)\}$. The asymptotic stability of $(r_\mathrm{d}, 0)$ is obtained by setting $L^+ \subset \bar{\mathcal{N}} \subset \mathcal{N}_\mathrm{nh}$. □

Proposition 5.2 is our primary tool for the controller synthesis of constrained mechanical systems. It extends the results of Proposition 5.1 by ensuring convergence to the invariant set $\Omega_\mathrm{inv}$ and even (asymptotic) stability of the desired equilibrium $(r_\mathrm{d}, 0)$, where the closed-loop is a port-Hamiltonian system. Overall, the method presented above does not require

$$\operatorname{Colsp}\mathbf{B}_\mathrm{d}(r) = \operatorname{Colsp}\mathbf{B}(r) = \operatorname{Colsp}\bar{\mathbf{B}}(r) \qquad \text{or} \qquad \mathbf{J}(r) = I_{n_r}.$$

Besides, $\mathbf{M}_\mathrm{d}$ can be sign-indefinite yet without compromising stability. In other words, Assumptions 5.3 to 5.5 introduced in [74, 75, 77, 105] are unnecessary, and our method can be regarded as a generalization of those works. Needless to say, it is possible to omit the kinetic energy shaping, i.e., setting $\mathbf{M}_\mathrm{d}(r) = \mathbf{M}(r)$, see Section 6.2.2.

The actual synthesis with this proposition is similar to the explicit IDA-PBC case. It starts by defining the structure of $\mathbf{J}$ and $\mathbf{B}_\mathrm{d}$ from Assumptions 5.9 and 5.11. Next, we obtain an appropriate solution of $\mathbf{M}_\mathrm{d}$, $\mathbf{\Gamma}_1$ and $\bar{\mathbf{\Gamma}}_2$ from (5.16a), (5.31a) and (5.31b), to finally solve $\mathbf{V}_\mathrm{d}$ from (5.16b) and (5.31c). If $\mathbf{V}_\mathrm{d}$ cannot be obtained, we repeat the procedure by searching for a different $\mathbf{M}_\mathrm{d}$. Unlike the explicit IDA-PBC, imposing

$$\mathbf{M}_\mathrm{d}(r) \succ 0 \qquad \forall r \in \mathcal{R}_\Phi$$

to guarantee (5.31a) may significantly reduce the system class or hinder the solution of $\mathbf{M}_\mathrm{d}$. For example, see Sections 7.2.2 and 7.2.4, where the upright position of the cart-pole's pendulum cannot be stabilize if $\mathbf{M}_\mathrm{d}$ is constant and positive definite, but it can if $\mathbf{M}_\mathrm{d}$ is *i)* constant and sign-indefinite, or *ii)* positive definite and state-dependent.

### Local IDA-PBC

To achieve (asymptotic) stability with Proposition 5.2 it is required that $\mathbf{H}_\mathrm{d}|_{\mathcal{X}_\mathrm{c}}$ has a global strict minimum in $(r_\mathrm{d}, 0)$, see (5.31a) and (5.31c). Nonetheless, this condition can

[50] Lyapunov's direct method (see Theorem 2.1) is originally developed for ODEs in the Euclidean space. If we use this formulation on the equivalent ODE (5.21), we obtain stronger conditions than (5.31), namely $\mathbf{H}_\mathrm{d}$ being positive definite and $\dot{\mathbf{H}}_\mathrm{d}$ being negative semidefinite for all $(r, \rho) \in \mathcal{R} \times \mathbb{R}^{n_r}$ (instead of $\mathcal{X}_\mathrm{c}$). Therefore, either we use the standard Lyapunov theorem as shown above or a Lyapunov formulation on manifolds (see [137, 162, 208]). Note that LaSalle's Theorem 2.4 does not require an additional formulation on manifolds because it employs compact (closed and bounded) sets.

be relaxed by selecting a local minimum instead (see proof of Proposition 5.2). Towards this goal, let us define the **Lagrange function**[51]

$$\mathcal{L}_\mathrm{d}(r,\rho,\delta,\mu) := \mathbf{H}_\mathrm{d}(r,\rho) + \delta^\top \mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\rho + \mu^\top \Phi(r)$$

with **Lagrange multipliers** $\mu \in \mathbb{R}^{n_\Phi}$ and $\delta \in \mathbb{R}^{n_\lambda}$, and constraints

$$\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\rho = 0 \qquad \text{and} \qquad \Phi(r) = 0.$$

According to the constrained minimization theory, see [180, Section 1.4], the point $(r_\mathrm{d}, 0)$ is a strict local minimum of $\mathbf{H}_\mathrm{d}|_{\mathcal{X}_\mathrm{c}} \in \mathcal{C}^2$ if and only if there exist vectors $\mu_\mathrm{d} \in \mathbb{R}^{n_\Phi}$ and $\delta_\mathrm{d} \in \mathbb{R}^{n_\lambda}$ such that

$$0 = \left.\frac{\partial \mathcal{L}_\mathrm{d}}{\partial r}\right|_{\substack{r=r_\mathrm{d},\,\rho=0\\ \delta=\delta_\mathrm{d},\,\mu=\mu_\mathrm{d}}}, \qquad 0 = \left.\frac{\partial \mathcal{L}_\mathrm{d}}{\partial \rho}\right|_{\substack{r=r_\mathrm{d},\,\rho=0\\ \delta=\delta_\mathrm{d},\,\mu=\mu_\mathrm{d}}}, \tag{5.32a}$$

$$\begin{bmatrix} z_r^\top & z_\rho^\top \end{bmatrix} \left.\begin{bmatrix} \frac{\partial^2 \mathcal{L}_\mathrm{d}}{\partial r^2} & \frac{\partial}{\partial \rho}\left(\frac{\partial^\top \mathcal{L}_\mathrm{d}}{\partial r}\right) \\ \frac{\partial}{\partial r}\left(\frac{\partial^\top \mathcal{L}_\mathrm{d}}{\partial \rho}\right) & \frac{\partial^2 \mathcal{L}_\mathrm{d}}{\partial \rho^2} \end{bmatrix}\right|_{\substack{r=r_\mathrm{d}\\ \rho=0\\ \delta=\delta_\mathrm{d}\\ \mu=\mu_\mathrm{d}}} \begin{bmatrix} z_r \\ z_\rho \end{bmatrix} > 0 \qquad \forall (z_r, z_\rho) \in \mathcal{Z}, \tag{5.32b}$$

where

$$\mathcal{Z} = \left\{ (z_r, z_\rho) \in \mathbb{R}^{n_r} \times \mathbb{R}^{n_r} \,\middle|\, 0 = \left.\frac{\partial \Phi}{\partial r}\right|_{r=r_\mathrm{d}} z_r,\; 0 = \mathbf{B}^\top(r_\mathrm{d})\mathbf{M}^{-1}(r_\mathrm{d}) z_\rho \right\},$$

$$\begin{bmatrix} \frac{\partial^\top \mathcal{L}_\mathrm{d}}{\partial r} \\ \frac{\partial^\top \mathcal{L}_\mathrm{d}}{\partial \rho} \end{bmatrix} = \begin{bmatrix} \frac{\partial^\top \mathbf{V}_\mathrm{d}}{\partial r} + \frac{\partial^\top \rho^\top \mathbf{M}_d^{-1}\rho}{\partial r} + \frac{\partial^\top \Phi}{\partial r}\mu + \frac{\partial^\top \rho^\top \mathbf{M}^{-1} b\delta}{\partial r} \\ \mathbf{M}^{-1}(r)\mathbf{B}(r)\delta + \mathbf{M}_\mathrm{d}^{-1}(r)\rho \end{bmatrix}.$$

Hence, (5.32) can be equivalently rewritten as

$$\left.\frac{\partial \mathbf{V}_\mathrm{d}}{\partial r}\right|_{r=r_\mathrm{d}} + \mu_\mathrm{d}^\top \left.\frac{\partial \Phi}{\partial r}\right|_{r=r_\mathrm{d}} = 0, \tag{5.33a}$$

$$\left.\left(\frac{\partial^\top \Phi}{\partial r}\right)_\perp \left(\frac{\partial^2\, \mathbf{V}_\mathrm{d}}{\partial r^2} + \frac{\partial^2 \mu_\mathrm{d}^\top \Phi}{\partial r^2}\right) \left(\frac{\partial^\top \Phi}{\partial r}\right)_\perp^\top\right|_{r=r_\mathrm{d}} \succ 0, \tag{5.33b}$$

$$\mathbf{B}_\perp(r_\mathrm{d})\mathbf{M}(r_\mathrm{d})\mathbf{M}_\mathrm{d}^{-1}(r_\mathrm{d})\mathbf{M}(r_\mathrm{d})\mathbf{B}_\perp^\top(r_\mathrm{d}) \succ 0, \tag{5.33c}$$

where (5.33b)–(5.33c) is obtained from (5.32b) with Finsler's Lemma A.9. Observe that (5.33a)–(5.33b) reduce to

$$\left.\frac{\partial \mathbf{V}_\mathrm{d}}{\partial r}\right|_{r=r_\mathrm{d}} = 0 \qquad \text{and} \qquad \left.\frac{\partial^2 \mathbf{V}_\mathrm{d}}{\partial r^2}\right|_{r=r_\mathrm{d}} \succ 0$$

[51] By a **Lagrange function**, we mean a function associated with the method of Lagrange multipliers. This concept should not be confused with the **Lagrangian**, which we use as kinetic minus potential energy, see Section 2.5.2.

if the nominal system does not have holonomic constraints. Letting $\mathbf{M}_\mathrm{d}(r_\mathrm{d})$ be sign-indefinite involves calculating its inverse to verify (5.33c), but this can be an arduous task depending on the size and form of $\mathbf{M}_\mathrm{d}$. Lemma 5.4 restates this problem by requiring the inverse of a square matrix of size $n_\lambda$ (number of constraints) instead of $\mathbf{M}_\mathrm{d}$.

**Lemma 5.4.** *Consider systems (5.11) and (5.15) with Assumptions 5.6 to 5.8. Let* $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$. *Then, (5.31a) holds if and only if*

$$\bar{\mathbf{B}}_\perp(r)\left(\mathbf{M}_\mathrm{d}(r) - \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)\bar{\Delta}_\mathrm{d}^{-1}(r)\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)\right)\bar{\mathbf{B}}_\perp^\top(r) \succ 0 \qquad (5.34)$$

*and* $\bar{\Delta}_\mathrm{d}(r) = \mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ *is nonsingular for all* $r \in \mathcal{R}_\Phi$.

**Proof.** See Appendix B.4. □

Now, we can express inequality (5.33c) as

$$\bar{\mathbf{B}}_\perp(r)\left(\mathbf{M}_\mathrm{d}(r) - \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)\bar{\Delta}_\mathrm{d}^{-1}(r)\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)\right)\bar{\mathbf{B}}_\perp^\top(r)\Big|_{r=r_\mathrm{d}} \succ 0, \qquad (5.35)$$

with $\bar{\Delta}_\mathrm{d}(r_\mathrm{d})$ being nonsingular. Note that if

$$\bar{\mathbf{B}}_\perp(r_\mathrm{d})\mathbf{M}_\mathrm{d}(r_\mathrm{d})\mathbf{M}^{-1}(r_\mathrm{d})\mathbf{B}(r_\mathrm{d}) = 0,$$

then (5.35) reduces to

$$\bar{\mathbf{B}}_\perp(r_\mathrm{d})\mathbf{M}_\mathrm{d}(r_\mathrm{d})\bar{\mathbf{B}}_\perp^\top(r_\mathrm{d}) \succ 0.$$

Applying the IDA-PBC method to constrained systems can be summarized as follows.

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**Algorithm 5.2** IDA-PBC for mechanical systems with kinematic constraints.

---

**Require:** A mechanical system of the form (5.11) verifying Assumptions 5.6 to 5.8.

1: Select $\mathbf{N}_\perp$ and $r_\mathrm{d} \in \mathcal{R}_\mathrm{a} := \left\{r \in \mathcal{R}_\Phi \mid 0 = \mathbf{N}_\perp(r)\frac{\partial^\top \mathbf{V}}{\partial r}\right\}$. Pick the structure of $\mathbf{J}$ and $\mathbf{B}_\mathrm{d}$ from Assumptions 5.9 and 5.11, e.g.,

$$\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r) \qquad \text{and} \qquad \mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r).$$

2: Compute a solution to $\mathbf{M}_\mathrm{d}$ (symmetric) and $\mathbf{\Gamma}_1$ (linear in $\rho$) from (5.16a). Here, $\mathbf{\Gamma}_1$ is chosen to simplify the PDE of the kinetic matching (5.16a).

3: With $\mathbf{M}_\mathrm{d}$ from step 2, calculate a general solution to $\mathbf{V}_\mathrm{d}$ from the potential matching (5.16b).

4: Select the arbitrary functions and parameters in $\mathbf{M}_\mathrm{d}$, $\mathbf{V}_\mathrm{d}$, $\mathbf{\Gamma}_1$ and $\bar{\mathbf{\Gamma}}_2$ such that the **stabilizing conditions** (5.31b) and (5.33) hold. To avoid the inverse of $\mathbf{M}_\mathrm{d}$, we can replace (5.33c) by (5.35) with $\bar{\Delta}_\mathrm{d}(r_\mathrm{d})$ being nonsingular. If no solution exists, return to steps 3 or 4.

5: (Local) asymptotic stability can be verified from the invariant set $\Omega_{\text{inv}}$ (as defined in Proposition 5.2).

6: Select $\bar{\mathbf{G}}^{\text{g}}$ and $\bar{\mathbf{G}}_{\top}\nu$ to build the feedback (5.18). The freedom in the selection of $\bar{\mathbf{G}}^{\text{g}}$ and $\nu$ can simplify the controller expression. For example, if

$$\bar{\mathbf{G}}^{\text{g}}(r)\begin{bmatrix}\bar{\mathbf{B}}(r) & \mathbf{B}_{\text{d}}(r)\end{bmatrix} \equiv 0,$$

then (5.18) reduces to (5.23).

### 5.2.4 Standard IDA-PBC and the Dissipation Condition

Similar to the unconstrained situation, we can observe that the solutions of $\mathbf{M}_{\text{d}}$ and $\mathbf{\Gamma}_1$ depend simultaneously on the matching condition (5.16a) and the stabilizing condition (5.2). This dependency on (5.2) can be eliminated by assuming skew-symmetry on $\mathbf{\Gamma}_1$. However, Donaire et al. [24] argue for the unconstrained case that assigning skew-symmetry may introduce conservatism in the design. The following propositions solve this query for both constrained and unconstrained systems.

**Proposition 5.3.** *Consider the system (5.11) under Assumptions 5.6 to 5.8. Let (5.15) be the target dynamics verifying (5.31a) and Assumptions 5.9 and 5.11, . The term $\bar{\mathbf{\Gamma}}_2$ has a solution in (5.31b) if and only if*

$$\rho^\top \mathbf{M}_{\text{d}}^{-1}(r)\mathbf{\Gamma}_1(r,\rho)\mathbf{M}_{\text{d}}^{-1}(r)\rho = 0 \qquad \forall (r,\rho)\in\mathcal{X}_{\text{c}}, \tag{5.36a}$$

$$\mathbf{N}_{\perp}(r)\Big(\mathbf{R}(r)\mathbf{M}^{-1}(r)\mathbf{M}_{\text{d}}(r)\mathbf{Z}_{\text{d}}^\top(r)\Big)^{\text{s}}\mathbf{N}_{\perp}^\top(r) \succeq 0 \qquad \forall r\in\mathcal{R}_{\Phi}, \tag{5.36b}$$

*Let $\rho$ be bounded, then (5.36) is a sufficient but not necessary condition for (5.31b). Furthermore, for any $\mathbf{\Gamma}_1$ verifying (5.36a), there is an equivalent target system with $\mathbf{\Gamma}_1$ replaced $\hat{\mathbf{\Gamma}}_1 : \mathcal{R}\times\mathbb{R}^{n_r}\to\mathbb{R}^{n_r\times n_r}$ such that $\hat{\mathbf{\Gamma}}_1$ is skew-symmetric and linear in $\rho$.*

The lemma below, whose proof is given in Appendix B.5, is an extension of the well-known result

$$0 = z^\top Q z \qquad \forall z\in\mathbb{R}^{n_z} \qquad \Longleftrightarrow \qquad Q + Q^\top = 0$$

and it will be used to demonstrate Proposition 5.3.

**Lemma 5.5.** *Consider $A\in\mathbb{R}^{n\times m}$ and $Q(z)=\sum_{i=1}^{n} Q_i\left(z^\top e_i\right)$ with $Q_i\in\mathbb{R}^{n\times n}$. Then,*

$$0 = z^\top Q(z) z \qquad \forall\, z\in\{z\in\mathbb{R}^n \mid z = Ay, y\in\mathbb{R}^m\} \tag{5.37}$$

$$\Longleftrightarrow \quad 0=\sum_{i=1}^{n} A^\top\Big(Q_i^{\text{s}}\left(\bar{e}_k^\top A^\top e_i\right) + e_i\bar{e}_k^\top A^\top Q_i^{\text{s}} + Q_i^{\text{s}} A\bar{e}_k e_i^\top\Big)A \qquad \forall k\in\{1,2,\ldots,m\}, \tag{5.38}$$

*where $e_i=\operatorname{col}_i(I_n)$ and $\bar{e}_k=\operatorname{col}_k(I_m)$.*

**Proof of Proposition 5.3.** From Assumption 5.11 and (5.31a), we can assert that $\Delta_\mathrm{d}$ is nonsingular and $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)\mathbf{K}_\mathbf{B}(r)$ for some function $\mathbf{K}_\mathbf{B} : \mathcal{R}_\Phi \to \mathbb{R}^{n_\lambda \times n_\lambda}_{n_\lambda}$, see the proof of Proposition 5.2. By definition of $\mathcal{X}_c$, we can write (without loss of generality) $\rho = \mathbf{M}(r)\mathbf{B}_\perp^\top(r)\bar{\rho}$ with $\bar{\rho} = h\tau \in \mathbb{R}^{n_r - n_\lambda}$, $\tau \in \mathbb{R}$ and $h \in \mathbb{S}^{n_r - n_\lambda - 1} = \{h \in \mathbb{R}^{n_r - n_\lambda} \mid \|h\|_2 = 1\}$, meaning that $\bar{\rho}$ is parameterized with magnitude $\tau$ and direction $h$. Then, from Assumption 5.9 and linearity of $\mathbf{\Gamma}_1$ in $\rho$, the condition (5.31b) reads

$$
\begin{aligned}
& \tau^2 \left(-\tau\kappa_1(r,h) + h^\top \kappa_2(r) h\right) \geq 0 && \forall \tau \in \mathbb{R},\ h \in \mathbb{S}^{n_r - n_\lambda - 1},\ r \in \mathcal{R}_\Phi, && (5.39\mathrm{a}) \\
\iff\ & \kappa_1(r,h) = 0, \quad \kappa_2(r) + \kappa_2^\top(r) \succeq 0 && \forall h \in \mathbb{S}^{n_r - n_\lambda - 1},\ r \in \mathcal{R}_\Phi, && (5.39\mathrm{b}) \\
\iff\ & (5.36\mathrm{a}), \quad \kappa_2(r) + \kappa_2^\top(r) \succeq 0 && \forall (r,\rho) \in \mathcal{X}_\mathrm{c}, && (5.39\mathrm{c})
\end{aligned}
$$

where $\kappa_1(r,h) = h^\top \mathbf{B}_{\mathrm{d}\perp}(r)\mathbf{\Gamma}_1(r, \mathbf{M}(r)\mathbf{B}_\perp^\top(r)h)\mathbf{B}_{\mathrm{d}\perp}^\top(r)h$, $\mathbf{B}_{\mathrm{d}\perp}(r) = \mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)$ and $\kappa_2(r) = \mathbf{B}_{\mathrm{d}\perp}(r)\Big(\bar{\mathbf{G}}(r)\bar{\mathbf{\Gamma}}_2(r) + \mathbf{Z}(r)\mathbf{R}(r)\Big)\mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)\mathbf{B}_{\mathrm{d}\perp}^\top(r)$. Observe that if $\rho$ is bounded, then $\tau$ is bounded and (5.39b) is a sufficient but not necessary condition for (5.39a). For clarity, we left out the expression $(r,\rho) \in \mathcal{X}_\mathrm{c}$ in the rest of this proof. From (5.39c), we have the following chain of implications.

$$
\begin{aligned}
& \exists \bar{\mathbf{\Gamma}}_2 \ \text{s.t.} \ \ \mathbf{B}_{\mathrm{d}\perp}(r)\Big(\mathbf{F}_1(r) + \bar{\mathbf{G}}(r)\bar{\mathbf{\Gamma}}_2(r)\mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)\Big)^{\mathrm{s}}\mathbf{B}_{\mathrm{d}\perp}^\top(r) \succeq 0 \\
\iff\ & \exists \hat{\mathbf{\Gamma}}_3, \hat{\mathbf{\Gamma}}_2 \ \text{s.t.} \ \ \mathbf{F}_1^\mathrm{s}(r) - \begin{bmatrix}\bar{\mathbf{G}}(r) & \mathbf{B}_\mathrm{d}(r)\end{bmatrix}\begin{bmatrix}\hat{\mathbf{\Gamma}}_2(r) \\ \hat{\mathbf{\Gamma}}_3(r)\end{bmatrix} - \begin{bmatrix}\hat{\mathbf{\Gamma}}_2(r) \\ \hat{\mathbf{\Gamma}}_3(r)\end{bmatrix}^\top \begin{bmatrix}\bar{\mathbf{G}}(r) & \mathbf{B}_\mathrm{d}(r)\end{bmatrix}^\top \succeq 0 \\
\iff\ & \mathbf{N}_\perp(r)\mathbf{Z}_\mathrm{d}(r)\Big(\mathbf{F}_1(r) + \mathbf{F}_1^\top(r)\Big)\mathbf{Z}_\mathrm{d}^\top(r)\mathbf{N}_\perp^\top(r) \succeq 0 \\
\iff\ & (5.36\mathrm{b}).
\end{aligned}
$$

where $\mathbf{F}_1(r) = \mathbf{Z}(r)\mathbf{R}(r)\mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)$ and $\hat{\mathbf{\Gamma}}_2(r) = \bar{\mathbf{\Gamma}}_2(r)\mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)$. Here, the first and second equivalences are a consequence of the extended Finsler Lemma 3.3 while the third one results from $\mathbf{N}_\perp(r)\mathbf{Z}_\mathrm{d}(r)\mathbf{Z}(r) = \mathbf{N}_\perp(r)$ and $\mathbf{N}_\perp(r)\mathbf{Z}_\mathrm{d}(r)$ being a full rank left annihilator of $\begin{bmatrix}\bar{\mathbf{G}}(r) & \mathbf{B}_\mathrm{d}(r)\end{bmatrix}$. To demonstrate the annihilator, let us consider the identities

$$
\begin{aligned}
\begin{bmatrix}\mathbf{B}^\top(r)\mathbf{M}^{-1}(r) \\ \bar{\mathbf{B}}_\perp(r)\end{bmatrix}\begin{bmatrix}\bar{\mathbf{G}}(r) & \mathbf{B}_\mathrm{d}(r)\end{bmatrix} &= \begin{bmatrix}0 & \Delta_\mathrm{d}(r) \\ \bar{\mathbf{B}}_\perp(r)\mathbf{G}(r) & \star\end{bmatrix}, \\
\mathbf{N}_\perp(r)\mathbf{Z}_\mathrm{d}(r)\begin{bmatrix}\mathbf{B}_\mathrm{d}(r) & \mathbf{M}(r)\mathbf{B}_\perp^\top(r)\end{bmatrix} &= \mathbf{N}_\perp(r)\begin{bmatrix}\bar{\mathbf{B}}(r) & \mathbf{M}(r)\mathbf{B}_\perp^\top(r)\end{bmatrix},
\end{aligned}
\tag{5.40}
$$

where matrices $\begin{bmatrix}\mathbf{B}_\mathrm{d}(r) & \mathbf{M}(r)\mathbf{B}_\perp^\top(r)\end{bmatrix}$ and $\begin{bmatrix}\bar{\mathbf{B}}_\perp^\top(r) & \mathbf{M}^{-1}(r)\mathbf{B}(r)\end{bmatrix}$ are nonsingular from Lemma 5.2 with Assumption 5.6 and nonsingularity of $\Delta_\mathrm{d}$. Hence, by using $\operatorname{rank}(\bar{\mathbf{B}}_\perp(r)\mathbf{G}(r)) = \operatorname{rank}(\mathbf{N}(r)) - n_\lambda$, which is obtained from the proof of Proposition 5.1 under Assumptions 5.6 and 5.8, we can infer that

$$
\operatorname{rank}\begin{bmatrix}\bar{\mathbf{G}}(r) & \mathbf{B}_\mathrm{d}(r)\end{bmatrix} = \operatorname{rank}\mathbf{N}(r), \qquad \operatorname{rank}\mathbf{N}_\perp(r)\mathbf{Z}_\mathrm{d}(r) = \operatorname{rank}\mathbf{N}_\perp(r).
$$

At this stage, the result is evident from $\mathbf{N}_\perp(r)\mathbf{Z}_\mathrm{d}(r)\left[\bar{\mathbf{G}}(r) \;\; \mathbf{B}_\mathrm{d}(r)\right] = 0$.

It remains to proof that for any $\mathbf{\Gamma}_1$ verifying (5.36a), there is an equivalent closed-loop with a skew-symmetric $\hat{\mathbf{\Gamma}}_1$. For this, note from the ODE (5.21) on $\mathcal{X}_\mathrm{c}$ that two target systems, one with $\mathbf{\Gamma}_1$ and another with $\hat{\mathbf{\Gamma}}_1$ instead, have identical trajectories iff

$$\mathbf{Z}_\mathrm{d}(r)\mathbf{\Gamma}_1(r,\rho)\mathbf{M}_\mathrm{d}^{-1}(r)\rho = \mathbf{Z}_\mathrm{d}(r)\hat{\mathbf{\Gamma}}_1(r,\rho)\mathbf{M}_\mathrm{d}^{-1}(r)\rho. \tag{5.41}$$

Left multiplying the above equation by the matrix

$$\left[\mathbf{M}^{-1}(r)\mathbf{B}(r) \quad \mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r)\right]^\top,$$

which is nonsingular from Lemma 5.2 with (5.31a), results in the equivalent expression

$$0 = \mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\left(\mathbf{\Gamma}_1(r,\rho)\mathbf{M}_\mathrm{d}^{-1}(r)\rho - \hat{\mathbf{\Gamma}}_1(r,\rho)\mathbf{M}_\mathrm{d}^{-1}(r)\rho\right). \tag{5.42}$$

Now, let us write (without loss of generality)

$$\begin{aligned}
\mathbf{\Gamma}_1(r,\rho) &:= \sum_{i=1}^{n_r} e_i\rho^\top\mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{Q}_i(r), \qquad & a_{jm}^k &:= \bar{e}_j^\top\mathbf{A}^\top(r)\sum_{i=1}^{n_r}\mathbf{Q}_i^\mathrm{s}(r)\left(\bar{e}_k^\top\mathbf{A}^\top(r)e_i\right)\mathbf{A}(r)\bar{e}_m,\\
\hat{\mathbf{\Gamma}}_1(r,\rho) &:= \sum_{i=1}^{n_r}\hat{\mathbf{Q}}_i(r)\left(e_i^\top\mathbf{M}_\mathrm{d}^{-1}(r)\rho\right), & b_{km}^j &:= \bar{e}_k^\top\mathbf{A}^\top(r)\sum_{i=1}^{n_r}\hat{\mathbf{Q}}_i(r)\left(\bar{e}_j^\top\mathbf{A}^\top(r)e_i\right)\mathbf{A}(r)\bar{e}_m,
\end{aligned}$$

where $\mathbf{Q}_i, \hat{\mathbf{Q}}_i : \mathcal{R} \to \mathbb{R}^{n_r\times n_r}$, $\mathbf{A}(r) = \mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r)$, $\bar{e}_k = \mathrm{col}_k(I_{n_r-n_\lambda})$, $e_i = \mathrm{col}_i(I_{n_r})$ and $a_{jm}^k$ satisfies

$$a_{jm}^k = a_{mj}^k \qquad \forall k,j,m \in \{1,\ldots,n_r-n_\lambda\}\,. \tag{5.43a}$$

Hence, (5.42) reads

$$\begin{aligned}
&\mathbf{A}^\top(r)\sum_{i=1}^{n_r} e_i\bar{\rho}^\top\mathbf{A}^\top(r)\mathbf{Q}_i(r)\mathbf{A}(r)\bar{\rho} = \mathbf{A}^\top(r)\sum_{i=1}^{n_r}\left(e_i^\top\mathbf{A}(r)\bar{\rho}\right)\hat{\mathbf{Q}}_i(r)\mathbf{A}(r)\bar{\rho} \qquad \forall\bar{\rho}\in\mathbb{R}^{n_r-n_\lambda}\\
\iff\;& \sum_{i=1}^{n_r}\mathbf{A}^\top(r)\mathbf{Q}_i^\mathrm{s}(r)\mathbf{A}(r)\left(e_i^\top\mathbf{A}(r)\bar{e}_k\right) = \mathbf{A}^\top(r)\sum_{i=1}^{n_r} e_i\bar{e}_k^\top\mathbf{A}^\top(r)\hat{\mathbf{Q}}_i(r)\mathbf{A}(r)\\
&\qquad\qquad + \mathbf{A}^\top(r)\sum_{i=1}^{n_r}\hat{\mathbf{Q}}_i^\top(r)\mathbf{A}(r)\bar{e}_k e_i^\top\mathbf{A}(r)\\
\iff\;& a_{jm}^k = b_{km}^j + b_{kj}^m \qquad \forall k,j,m\in\{1,\ldots,n_r-n_\lambda\}\,,
\end{aligned} \tag{5.43b}$$

while (5.36a) with Lemma 5.5 is equivalent to

$$a_{jm}^k + a_{km}^j + a_{jk}^m = 0 \qquad \forall k,j,m\in\{1,\ldots,n_r-n_\lambda\}\,, \tag{5.43c}$$

It follows that (5.43a), (5.43c) and

$$b^j_{km} = c\left(a^k_{jm} - a^m_{jk}\right) \qquad \forall k,j,m \in \{1,\dots,n_r - n_\lambda\}\,, \tag{5.43d}$$

imply (5.43b), or equivalently (5.41), if $c = \frac{1}{3}$. Rewriting (5.43d) in matrix notation

$$\begin{aligned} 0 &= \sum_{k=1}^{n_r-n_\lambda}\sum_{m=1}^{n_r-n_\lambda} \bar{e}_k \left(b^j_{km} - \frac{1}{3}\left(a^k_{jm} - a^m_{jk}\right)\right)\bar{e}_m^\top \\ &= \mathbf{A}^\top(r)\left(\left(\sum_{i=1}^{n_r}\hat{\mathbf{Q}}_i(r)\left(\bar{e}_j^\top\mathbf{A}^\top(r)e_i\right)\right) - \frac{1}{3}\left(\sum_{i=1}^{n_r} e_i\bar{e}_j^\top\mathbf{A}^\top(r)\mathbf{Q}_i^{\mathrm{s}}(r) - \mathbf{Q}_i^{\mathrm{s}}(r)\mathbf{A}(r)\bar{e}_je_i^\top\right)\right)\mathbf{A}^\top(r), \end{aligned}$$

to then multiply by $\bar{\rho}\left(\bar{e}_j^\top\bar{\rho}\right)$ on the right and sum over $j$ gives

$$\begin{aligned} 0 = \mathbf{A}^\top(r)&\underbrace{\sum_{i=1}^{n_r}\hat{\mathbf{Q}}_i(r)\left(e_i^\top\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho\right)}_{\hat{\mathbf{\Gamma}}_1(r,\rho)}\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho \\ &- \frac{1}{3}\mathbf{A}^\top(r)\left(\sum_{i=1}^{n_r} e_i\rho^\top\mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{Q}_i^{\mathrm{s}}(r) - \mathbf{Q}_i^{\mathrm{s}}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho e_i^\top\right)\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho. \end{aligned}$$

It follows that $\hat{\mathbf{\Gamma}}_1$ can take the skew-symmetric form

$$\hat{\mathbf{\Gamma}}_1(r,\rho) = \frac{1}{3}\left(\sum_{i=1}^{n_r} e_i\rho^\top\mathbf{M}_{\mathrm{d}}^{-1}(r)\left(\mathbf{Q}_i(r) + \mathbf{Q}_i^\top(r)\right) - \left(\mathbf{Q}_i(r) + \mathbf{Q}_i^\top(r)\right)\mathbf{M}_{\mathrm{d}}^{-1}(r)\rho e_i^\top\right)$$

and our claim is proved. □

To preserve the usual terminology, we refer to (5.36b) as the **dissipation condition** for mechanical systems with kinematic constraints. In this regard, Proposition 5.3 asserts that there exists a $\bar{\mathbf{\Gamma}}_2$ satisfying the stabilizing condition (5.31b) if an only if the dissipation condition holds and $\mathbf{\Gamma}_1$ verifies (5.36a), which implies that $\mathbf{\Gamma}_1$ can be skew-symmetric without loss of generality. On the other hand, if we relax (5.31b) to hold only locally (bounded $\rho$), then the skew-symmetry feature indeed introduces conservatism. For example, we may have $\mathbf{\Gamma}_1$ not satisfying (5.36a) and still be able to find $\bar{\mathbf{\Gamma}}_2$ that certifies (5.31b) in a compact set $\mathcal{X}_k = \{(r,\rho) \in \mathcal{X}_{\mathrm{c}} \mid \mathbf{H}_{\mathrm{d}}(r,\rho) \le k\}$ for some appropriate constant $k \in \mathbb{R}$ and energy function $\mathbf{H}_{\mathrm{d}}$. Note that $\dot{\mathbf{H}}_{\mathrm{d}}\Big|_{\mathcal{X}_k} \le 0$, and therefore, stability can be guaranteed inside of $\mathcal{X}_k$, but (overall) this may complicate the analysis and render smaller regions of convergence.

The statements of Proposition 5.3 can be extended to unconstrained mechanical systems as follows.

**Proposition 5.4.** *Consider the system* (5.1)*, and let* (5.2) *be the target dynamics. The term* $\bar{\Gamma}_2$ *has a solution in* (5.5c)*, i.e.,*

$$p^\top M_\mathrm{d}^{-1}(q)\Big(G(q)\bar{\Gamma}_2(q)J(q) + R(q)J(q) - \Gamma_1(q,p)\Big)M_\mathrm{d}^{-1}(q)p \geq 0 \qquad \forall\,(q,p) \in \mathcal{Q} \times \mathbb{R}^{n_q}$$

*if and only if*

$$p^\top M_\mathrm{d}^{-1}(q)\Gamma_1(q,p)M_\mathrm{d}^{-1}(q)p = 0 \qquad\qquad \forall (r,\rho) \in \mathcal{X}_\mathrm{c} \tag{5.44}$$

*and the dissipation condition* (5.7a) *holds. Let* $p$ *be bounded, then* (5.7a) *and* (5.44) *are sufficient but not necessary conditions for* (5.5c)*. Furthermore, for any* $\Gamma_1$ *verifying* (5.44)*, there is an equivalent target system with* $\Gamma_1$ *replaced* $\hat{\Gamma}_1 : \mathcal{Q} \times \mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times n_q}$ *such that* $\hat{\Gamma}_1$ *is skew-symmetric and linear in* $p$*.*

**Proof.** Along the same lines of the proof of Proposition 5.3. □

Proposition 5.4 shows that the simultaneous IDA-PBC, as presented by Donaire et al. [24] where (5.5c) is satisfied for all $p \in \mathbb{R}^{n_q}$, can be reframed as the standard IDA-PBC with the dissipation condition (5.7a).

*Example* 5.2 (PID-PBC). Donaire et al. [24] consider a class of unconstrained mechanical systems in feedback with the Proportional-integral-derivative Passivity-based Control (PID-PBC). Their resulting closed-loop is of the form (5.2), where

$$\begin{aligned}
\Gamma_2(q) &= -G(q)K_\mathrm{v}G^\top(q) = -G(q)\bar{\Gamma}_2(q)J(q),\\
\Gamma_1(q,p) &= M_\mathrm{d}(q)\begin{bmatrix} 0 & 2k_\mathrm{a}k_\mathrm{u}\frac{\partial m_\mathrm{x}p_\mathrm{u}}{\partial q_\mathrm{u}}m_\mathrm{uu}^{-1}(q_\mathrm{u}) \\ -k_\mathrm{a}k_\mathrm{u}m_\mathrm{uu}^{-1}(q_\mathrm{u})\frac{\partial^\top m_\mathrm{x}p_\mathrm{u}}{\partial q_\mathrm{u}} & L_\mathrm{uu}(q_\mathrm{u},p) - k_\mathrm{a}k_\mathrm{u}m_\mathrm{uu}^{-1}(q_\mathrm{u})\frac{\partial^\top m_\mathrm{x}^\top p_\mathrm{a}}{\partial q_\mathrm{u}}\end{bmatrix}M_\mathrm{d}(q),\\
L_\mathrm{uu}(q_\mathrm{u},p) &= -2k_\mathrm{u}^2 m_\mathrm{x}^\top(q_\mathrm{u})K_\mathrm{k}^{-1}\frac{\partial m_\mathrm{x}p_\mathrm{u}}{\partial q_\mathrm{u}}m_\mathrm{uu}^{-1}(q_\mathrm{u}) + k_\mathrm{u}^2 m_\mathrm{uu}^{-1}(q_\mathrm{u})\frac{\partial^\top m_\mathrm{x}^\top K_\mathrm{k}^{-1}m_\mathrm{x}p_\mathrm{u}}{\partial q_\mathrm{u}},\\
\dot{H}_\mathrm{d}(q,p) &= -p^\top M_\mathrm{d}^{-1}(q)G(q)K_\mathrm{v}G^\top(q)M_\mathrm{d}^{-1}(q)p.
\end{aligned}$$

Besides, $K_\mathrm{v} = K_\mathrm{v}^\top \succ 0$, $k_\mathrm{a}$, $k_\mathrm{u} \in \mathbb{R}$ and $K_\mathrm{k} = K_\mathrm{k}^\top \in \mathbb{R}^{n_u \times n_u}$ are controller parameters, $q = \mathrm{vec}(q_\mathrm{a}, q_\mathrm{u})$, $q_\mathrm{a} \in \mathbb{R}^{n_u}$ and $q_\mathrm{u} \in \mathbb{R}^{n_q - n_u}$ are the actuated and unactuated coordinates, respectively, $p = \mathrm{vec}(p_\mathrm{a}, p_\mathrm{u})$ are the momenta with $p_\mathrm{a} \in \mathbb{R}^{n_u}$ and $p_\mathrm{u} \in \mathbb{R}^{n_q - n_u}$, $m_\mathrm{uu} : \mathbb{R}^{n_q - n_u} \to \mathbb{R}^{(n_q - n_u)\times(n_q - n_u)}$ and $m_\mathrm{au} : \mathbb{R}^{n_q - n_u} \to \mathbb{R}^{n_u \times (n_q - n_u)}$ are submatrices of $M$, and $m_\mathrm{x}(q_\mathrm{u}) = K_\mathrm{k}m_\mathrm{au}(q_\mathrm{u})m_\mathrm{uu}^{-1}(q_\mathrm{u})$. There is also a particular structure on $M_\mathrm{d}$ and $V_\mathrm{d}$, but we omit them and other details for presentation easiness.

Given that $\Gamma_1$ is not skew-symmetric, *they conclude that this problem belongs to the simultaneous IDA-PBC but not to the standard one*, see Section 5.1.2. However, from the form $\dot{H}_\mathrm{d}$ and the structure of (5.2), it is not difficult to see that $\Gamma_1$ also verifies (5.44), meaning the closed-loop can always be reformulate as the standard IDA-PBC, i.e., there exists an equivalent closed loop with skew-symmetric $\hat{\Gamma}_1$ verifying

$$\Gamma_1(q,p)M_{\mathrm{d}}^{-1}(q)p = \hat{\Gamma}_1(q,p)M_{\mathrm{d}}^{-1}(q)p. \tag{5.45}$$

In fact, after some straightforward manipulations and omitting the arguments $(q,p)$ for clarity, it follows that

$$\begin{aligned}
\Gamma_1 M_{\mathrm{d}}^{-1}p &= M_{\mathrm{d}}\begin{bmatrix} 2k_{\mathrm{a}}k_{\mathrm{u}}\frac{\partial m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}m_{\mathrm{uu}}^{-1}p_{\mathrm{u}} \\ -2k_{\mathrm{a}}k_{\mathrm{u}}m_{\mathrm{uu}}^{-1}\frac{\partial^\top p_{\mathrm{a}}^\top m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}} - 2k_{\mathrm{u}}^2 m_{\mathrm{x}}^\top K_{\mathrm{k}}^{-1}\frac{\partial m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}m_{\mathrm{uu}}^{-1}p_{\mathrm{u}} + k_{\mathrm{u}}^2 m_{\mathrm{uu}}^{-1}\frac{\partial^\top p_{\mathrm{u}}^\top m_{\mathrm{x}}^\top K_{\mathrm{k}}^{-1}m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}} \end{bmatrix} \\
&= M_{\mathrm{d}}\begin{bmatrix} 2k_{\mathrm{a}}k_{\mathrm{u}}\frac{\partial m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}m_{\mathrm{uu}}^{-1}p_{\mathrm{u}} \\ -2k_{\mathrm{a}}k_{\mathrm{u}}m_{\mathrm{uu}}^{-1}\frac{\partial^\top m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}p_{\mathrm{a}} + 2k_{\mathrm{u}}^2\left(m_{\mathrm{uu}}^{-1}\frac{\partial^\top m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}K_{\mathrm{k}}^{-1}m_{\mathrm{x}} - m_{\mathrm{x}}^\top K_{\mathrm{k}}^{-1}\frac{\partial m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}m_{\mathrm{uu}}^{-1}\right)p_{\mathrm{u}} \end{bmatrix} \\
&= M_{\mathrm{d}}\underbrace{\begin{bmatrix} 0 & 2k_{\mathrm{a}}k_{\mathrm{u}}\frac{\partial m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}m_{\mathrm{uu}}^{-1} \\ -2k_{\mathrm{a}}k_{\mathrm{u}}m_{\mathrm{uu}}^{-1}\frac{\partial^\top m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}} & 2k_{\mathrm{u}}^2\left(m_{\mathrm{uu}}^{-1}\frac{\partial^\top m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}K_{\mathrm{k}}^{-1}m_{\mathrm{x}} - m_{\mathrm{x}}^\top K_{\mathrm{k}}^{-1}\frac{\partial m_{\mathrm{x}}p_{\mathrm{u}}}{\partial q_{\mathrm{u}}}m_{\mathrm{uu}}^{-1}\right) \end{bmatrix}}_{M_{\mathrm{d}}^{-1}\hat{\Gamma}_1 M_{\mathrm{d}}^{-1}}\begin{bmatrix} p_{\mathrm{a}} \\ p_{\mathrm{u}} \end{bmatrix},
\end{aligned}$$

which verifies (5.45). Although, there are further extensions that enlarge the realm of applicability of the PID-PBC for UMSs [18, 19, 70], those results yield a closed-loop of the form (5.2) with $\Gamma_1$ satisfying (5.44). Consequently, they all can be reformulated as the standard IDA-PBC framework. △

### 5.2.5 Enlarging the Scope of Application

In our IDA-PBC formulation for constrained mechanical systems, we consider systems with constrain forces that are not necessarily workless. The objective is to include systems with change of coordinates, preliminary feedback (e.g., partial feedback linearization), or both. However, such mechanical systems may not preserve the structure of (5.11a), but have the more general form

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{-1}(r)\rho \\ -\mathbf{f}_1(r) - \mathbf{f}_2(r,\rho) - \mathbf{R}(r)\mathbf{M}^{-1}(r)\rho \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{G}(r) \end{bmatrix} u + \begin{bmatrix} 0 \\ \bar{\mathbf{B}}(r) \end{bmatrix} \bar{\lambda}, \tag{5.46}$$

where $\mathbf{f}_1 : \mathcal{R} \to \mathbb{R}^{n_r}$ and $\mathbf{f}_2 : \mathcal{R} \times \mathbb{R}^{n_r} \to \mathbb{R}^{n_r}$ is quadratic in $\rho$.[52] In fact, the representation (5.46) may also include non-mechanical systems. In this regard, we can straightforwardly incorporate this class by replacing the terms $\frac{\partial^\top \mathbf{V}}{\partial r}$ and $\frac{1}{2}\frac{\partial^\top \rho^\top \mathbf{M}^{-1}\rho}{\partial r}$ with $\mathbf{f}_1$ and $\mathbf{f}_2$, respectively, in all our previous derivations.

[52] System (5.46) can be written as (5.11a) if $\mathbf{f}_1(r) + \mathbf{f}_2(r,\rho) = \frac{\partial^\top \mathbf{H}}{\partial r}$.

# Chapter 6

# On the Solutions of IDA-PBC for Constrained Mechanical Systems

Chapter 5 develops the fundamentals of the total energy shaping IDA-PBC for constrained (or implicit) mechanical systems. The approach's success relies on our ability to solve the matching conditions (quasilinear PDEs) such that we have a sufficiently general target Hamiltonian that can verify the stabilizing conditions. In this chapter, we analyze these conditions (matching and stabilizing) more closely and provide some solutions.

Roughly speaking, a mechanical system is fully-actuated if the number of coordinates is less than or equal to the number of independent constraints and inputs; otherwise, it is underactuated, see Definition 2.12. Fully-actuated systems, which are studied in Section 6.1, stand out for the matching conditions' trivial satisfaction. We discuss underactuated systems in Section 6.2, where we introduce some heuristic and constructive methods to solve the PDEs of the matching conditions. Furthermore, we show that, based on the full state information control and two additional conditions, an output feedback controller may be obtained. We close the chapter, in Section 6.3, with the elimination of kinematic constraints and constraint forces to provide an equivalence between explicit and implicit representations.

## 6.1 Fully Actuated Systems

Let us synthesize an IDA-PBC controller for fully actuated mechanical systems with kinematic constraints on the basis of Algorithm 5.2. As usual, we will consider systems of the form (5.11), verifying Assumptions 5.6 to 5.8. We recall from Definition 2.12 that (5.11) is fully actuated if $\operatorname{rank}\mathbf{N}(r) = n_r$ for all $r \in \mathcal{R}_\Phi$. This means that *i)* $\mathbf{N}_\perp$ is a zero matrix or equivalently $\operatorname{rank}\mathbf{N}_\perp(r) = 0$, *ii)* the admissible equilibria set $\mathcal{R}_\mathrm{a} = \mathcal{R}_\Phi$, and *iii)* the matching conditions (5.16) are trivially guaranteed for every $\mathbf{M}_\mathrm{d}$, $\mathbf{V}_\mathrm{d}$ and $\mathbf{\Gamma}_1$. Now, setting

$\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)$ and $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ to satisfy Assumptions 5.9 and 5.11 completes steps 1–3. In step 4, we choose $\mathbf{M}_\mathrm{d}(r) \succ 0$, $\mathbf{\Gamma}_1(r,\rho) = -\mathbf{\Gamma}_1^\top(r,\rho)$ and

$$\mathbf{B}_\perp(r)\Big(\mathbf{J}^{-\top}(r)\bar{\mathbf{G}}(r)\bar{\mathbf{\Gamma}}_2(r) + \mathbf{J}^{-\top}(r)\bar{\mathbf{Z}}(r)\mathbf{R}(r)\Big)^{\mathrm{s}}\mathbf{B}_\perp^\top(r) \succ 0 \quad \forall r \in \mathcal{R}_\Phi \tag{6.1}$$

to fulfill (5.31a) and (5.31b), obtaining $\Omega = \{(r,\rho) \in \mathcal{X}_\mathrm{c} \mid \rho = 0\}$. The key aspect is that there always exist a $\bar{\mathbf{\Gamma}}_2$ verifying (6.1). This can be demonstrated in a similar way as in the proof of Proposition 5.3 but with Lemma A.9 instead of Lemma 3.3.

In the next step, let $(r(t), \rho(t))$ be a solution of the target system (5.15) that belongs identically to $\Omega$. We have

$$\rho(t) \equiv 0 \;\;\Rightarrow\;\; \dot{\rho}(t) \equiv \dot{r}(t) \equiv 0 \;\;\Rightarrow\;\; 0 \equiv \mathbf{Z}_\mathrm{d}(r(t))\mathbf{J}^\top(r(t))\frac{\partial^\top \mathbf{V}_\mathrm{d}}{\partial r} \;\;\Leftrightarrow\;\; 0 \equiv \mathbf{B}_\perp(r(t))\frac{\partial^\top \mathbf{V}_\mathrm{d}}{\partial r},$$

where the last equivalence is a direct consequence of Lemma 5.2, Assumption 5.11 and $\mathbf{M}_\mathrm{d}(r) \succ 0$. At this point, we can claim from Proposition 5.2 that every bounded trajectory of (5.11) in closed loop with (5.18) approaches the invariant set

$$\Omega_\mathrm{inv} := \left\{(r,0) \mid r \in \mathcal{R}_\Phi,\; r \text{ is constant},\; 0 = \mathbf{B}_\perp(r)\frac{\partial^\top \mathbf{V}_\mathrm{d}}{\partial r}\right\}. \tag{6.2}$$

Furthermore, if system (5.11) is holonomic and $\mathbf{V}_\mathrm{d} \in \mathcal{C}^2$, condition (5.33a) is equivalent to $0 = \mathbf{B}_\perp(r)\frac{\partial^\top \mathbf{V}_\mathrm{d}}{\partial r}$, which means that $\Omega_\mathrm{inv}$ is the set of all equilibria $(r^\star, 0)$ with $r^\star$ being the local minima or maxima of $\mathbf{V}_\mathrm{d}|_{\mathcal{R}_\Phi}$. Consequently, if $\mathbf{V}_\mathrm{d}|_{\mathcal{R}_\Phi} \in \mathcal{C}^2$ has a strict minimum in $r_\mathrm{d}$, i.e., conditions (5.31c) or (5.33a)–(5.33b) hold, then $(r_\mathrm{d}, 0)$ is a (locally) asymptotically stable equilibrium of the closed-loop.

Suppose, on the other hand, that (5.11) has nonholonomic constraints. Thus, $\Omega_\mathrm{inv}$ can have points that are not local extrema of $\mathbf{V}_\mathrm{d}|_{\mathcal{R}_\Phi}$, and we cannot guarantee asymptotic stability in $(r_\mathrm{d}, 0)$ but only a partial convergence to it. In fact, Brockett's necessary condition [209] implies that (both fully actuated and underactuated) systems of the form (5.11) with nonholonomic constraints are not asymptotically stabilizable through continuous time-invariant and static-state feedback, see [3, Corollary 6.4.7].[53] It is important to highlight that if we only consider nonholonomic systems, then $\Omega_\mathrm{inv}$ is equal to the invariant set obtained in [75, 215]. We summarize the above material in the next proposition.

**Proposition 6.1.** *Let (5.11) be a fully-actuated system satisfying Assumptions 5.6 to 5.8. Consider the system (5.15) with* $\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)$ *and* $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$. *Suppose* $\mathbf{M}_\mathrm{d}$ *is positive definite,* $\mathbf{\Gamma}_1$ *is skew-symmetric and* $\mathbf{V}_\mathrm{d} \in \mathcal{C}^2$ *satisfies (5.31c) or (5.33a)–(5.33b) for some* $r_\mathrm{d} \in \mathcal{R}_\Phi$. *Then, there is a function* $\bar{\mathbf{\Gamma}}_2$ *verifying (6.1) that entails the following affirmations: i) System (5.11) in closed-loop with feedback (5.18) is*

[53] A broader discussion on Brockett's necessary condition and the asymptotic stabilization of nonholonomic systems can be found in [210–214].

*stable in* $(r_\mathrm{d}, 0)$*. ii) Every bounded trajectory of the closed-loop approaches the invariant set* $\Omega_\mathrm{inv}$ *defined in* (6.2)*. iii) (Local) asymptotic stability of* $(r_\mathrm{d}, 0)$ *can be achieved if* (5.11) *is a holonomic system.*

*Example* 6.1. (Example 5.1, continued) Consider the simple pendulum of Examples 2.7 and 5.1, and suppose we intend to stabilize the equilibrium $(r_\mathrm{d}, 0)$ with Proposition 6.1. To this end, select $\mathbf{M}_\mathrm{d} = a_1 I_2 \succ 0$, $\mathbf{\Gamma}_1(r,\rho) = 0$, $\bar{\mathbf{\Gamma}}_2 = a_2 \mathbf{G}^\top(r)$, $\mathbf{J} = \mathbf{M}^{-1}\mathbf{M}_\mathrm{d}$, $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}\mathbf{M}^{-1}\mathbf{B}(r)$ and

$$\mathbf{V}_\mathrm{d}(r) = \frac{1}{2}(r - r_\mathrm{d})^\top A (r - r_\mathrm{d}) - (r - r_\mathrm{d})^\top \mathbf{B}(r_\mathrm{d}) a_3,$$

where $a_i$ and $A$ are constants. Hence, condition (6.1) is verified with $a_2 > -c_\theta l^2$, and $\mathbf{V}_\mathrm{d}|_{\mathcal{R}_\phi}$ has a strict local minimum in $r_\mathrm{d}$ if and only if (5.33a)–(5.33b) holds, i.e., $a_3 = \mu_\mathrm{d}$ and

$$\mathbf{B}_\perp(r_\mathrm{d})\,(A + I_2 a_3)\,\mathbf{B}_\perp^\top(r_\mathrm{d}) \succ 0. \tag{6.3}$$

See e.g., Figure 6.1. Since the system is holonomic, all conditions of Proposition 6.1 are

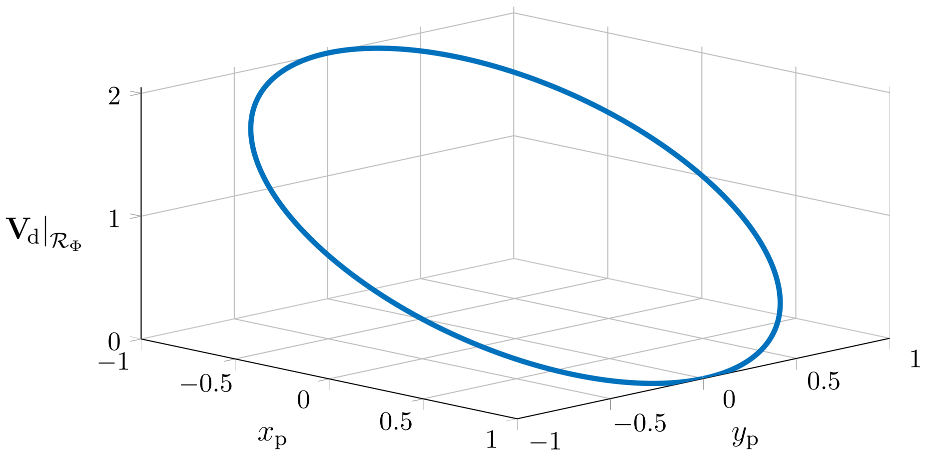


**Figure 6.1.** – Function $\mathbf{V}_\mathrm{d}|_{\mathcal{R}_\Phi}$ with $A = 0$, $l = 1$, $a_3 = 1$ and $r_\mathrm{d} = \mathrm{vec}(l, 0)$.

satisfied and $(r_\mathrm{d}, 0)$ is an asymptotically stable equilibrium of (5.24) in feedback with (5.25), which reads

$$\mathbf{u}_\mathrm{ida}(r,\dot r) = x_\mathrm{p} g_\mathrm{c} m + \frac{a_1}{m}\begin{bmatrix} y_\mathrm{p} & -x_\mathrm{p}\end{bmatrix}\Big(A(r - r_\mathrm{d}) - \mathbf{B}(r_\mathrm{d})a_3\Big) - \frac{a_2}{l^2}\left(x_\mathrm{p}\dot y_\mathrm{p} - y_\mathrm{p}\dot x_\mathrm{p}\right). \tag{6.4}$$

We have the following observations regarding (6.4). First, the design using global coordinates avoids undesirable behavior such as unwinding,[54] see [77] for an illustration on the simple

[54] By unwinding, we mean the unstable behavior of a closed-loop where a particular set of initial conditions, relatively close to the desired point in space, produces long trajectories before returning to the desired equilibrium [216, 217].

pendulum. And last, after setting $a_1 a_3 = \frac{m^2 g_c}{l}$, $a_2 = 0$ and $A = 0$, for $r_d = \mathrm{vec}(0, l)$, we recover the controller of [105]. △

*Example* 6.2. (Rolling disk) The vertical rolling disk or coin is a benchmark example in the analysis and control of mechanical systems with nonholonomic constraints [78, 159, 178, 212, 218]. It is also widely used as the basic representation of unicycle type robots [219–221]. This system, as shown in Figure 6.2, consists of a disk with symmetric mass distribution (constant density) that rolls on a horizontal plane without slipping and keeping its midplane vertical (not "falling"). Its configuration space is given by $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$, and therefore, we can choose the (generalized) coordinates $r = q = \mathrm{vec}(x_c, y_c, \theta_1, \theta_2) \in \mathcal{R} = \mathbb{R}^4$, where $(x_c, y_c)$ is the point of contact, $\theta_2$ is the orientation, and $\theta_1$ represents the rotation angle. The port-Hamiltonian model of the rolling disk subject to torques $\tau_1$ and $\tau_2$ is given by

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}}{\partial r} \\ \frac{\partial^\top \mathbf{H}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{B}(r) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ \mathbf{G}(r) \end{bmatrix} \tau, \tag{6.5a}$$

$$0 = \mathbf{B}^\top(r) \frac{\partial^\top \mathbf{H}}{\partial \rho}, \tag{6.5b}$$

where $\mathbf{H}(r, \rho) = \frac{1}{2}\rho^\top \mathbf{M}^{-1} \rho = \frac{1}{2}\dot{r}^\top \mathbf{M} \dot{r}$ is the Hamiltonian,

$$\mathbf{M} = \begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix}, \quad \bar{\mathbf{B}}(r) = \mathbf{B}(r) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -R_c \cos\theta_2 & -R_c \sin\theta_2 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{G}(r) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$m$ is the coin mass, $J_i$ is the moment of inertia passing through the coin center and relative to $\theta_i$, $R_c$ is the coin radius, and (6.5b) represents the no-slip constraints (of the coin w.r.t the plane), which are nonholonomic (nonintegrable). We leave the verification of Assumptions 5.6 to 5.8 to the reader.

To stabilize the equilibrium point $(r_d, 0)$ using Proposition 6.1, we set $\bar{\mathbf{\Gamma}}_2(r)\mathbf{J}(r) = \mathbf{K}_v(r)\bar{\mathbf{G}}^\top(r)$, $\mathbf{B}_d(r) = \mathbf{M}_d(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ and $\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_d(r)$ with arbitrary $\mathbf{M}_d(r) \succ 0$, $\mathbf{\Gamma}_1(r, \rho) = 0$ and $\mathbf{K}_v : \mathcal{R} \to \mathbb{R}^{n_u \times n_u}$ verifying $\mathbf{K}_v(r) + \mathbf{K}_v^\top(r) \succ 0$. Besides, given that the system is nonholonomic, we can pick

$$\mathbf{V}_d(r) = \frac{1}{2}(r - r_d)^\top A (r - r_d)$$

for some positive definite constant $A \in \mathbb{R}^{4\times 4}$. It follows that (6.5) in closed loop with (5.18) achieves stability in $(r_d, 0)$ and its trajectories converge to the set

$$\Omega_{\mathrm{inv}} := \left\{ (r, 0) \mid r \in \mathcal{R},\ r \text{ is constant},\ 0 = \begin{bmatrix} R_c \cos\theta_2 & R_c \sin\theta_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A (r - r_d) \right\}.$$

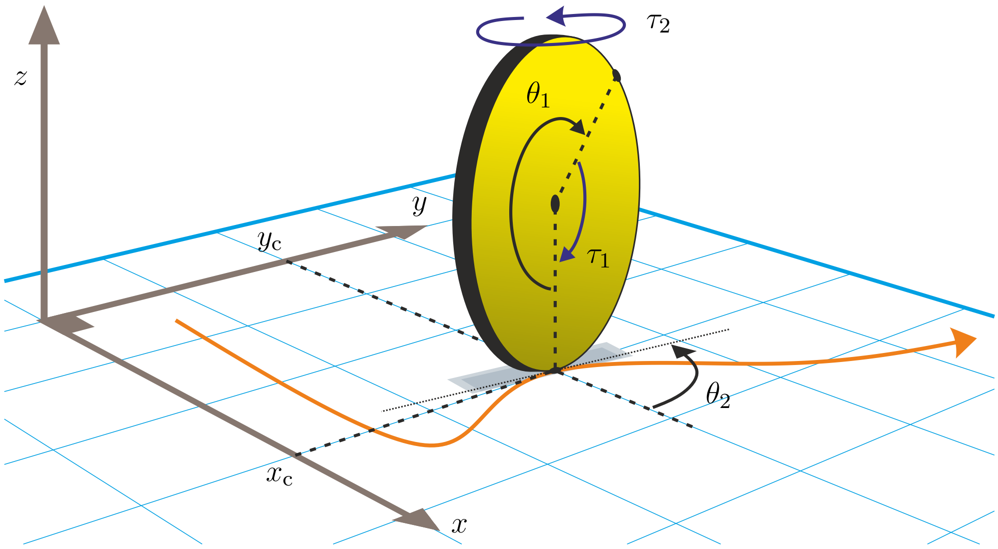


**Figure 6.2.** – Vertical rolling disk diagram.

For instance, let $r_d = 0$ and

$$A = \begin{bmatrix} a_1 + a_2 & 0 & -R_c a_1 & 0 \\ 0 & a_3 & 0 & 0 \\ -R_c a_1 & 0 & R_c^2 a_1 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$

with $a_i > 0$, then $\Omega_{\text{inv}}$ is reduced to $x_c = \theta_2 = 0$ with arbitrary but bounded $y_c$ and $\theta_1$. Similarly, if

$$A = \begin{bmatrix} a_1 & 0 & -R_c a_1 & 0 \\ 0 & a_3 & 0 & 0 \\ -R_c a_1 & 0 & a_2 + R_c^2 a_1 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$

with $a_i > 0$, then $\Omega_{\text{inv}}$ is reduced to $\theta_1 = \theta_2 = 0$ with arbitrary but bounded $x_c$ and $y_c$. △

## 6.2 Underactuated Systems

For underactuated mechanical systems, the matching conditions (5.16) are a system of nonhomogeneous quasilinear first-order PDEs with unknowns in $\mathbf{M}_d$, $\mathbf{V}_d$ and $\mathbf{\Gamma}_1$. Hypothetically, they can be solved with the **method of characteristics** [222, 223]; however, (in practice) it is not always feasible to find such a solution, see [17, 23, 58, 68, 69, 224, 225] for some solutions of the matching conditions in the explicit situation. In the following section, we will introduce *i)* a heuristic solution to the matching and stabilizing conditions by means of SOS programs,

*ii)* an equivalent representation of the kinetic matching (5.16a) that provides a characterization of the target inertia matrix, and

*iii)* a constructive method to solve the potential matching (5.16b) by using full-rank annihilators and the integrability condition of gradients.

### 6.2.1 A Heuristic Solution

Under an appropriate selection of coordinates, typically Euclidean, many mechanical systems in implicit representation possess a **polynomial characterization**.

**Definition 6.1.** *An implicit mechanical system of the form* (5.11) *is said to be polynomial (or possess a polynomial characterization) if the functions* $\mathbf{M}$*,* $\mathbf{V}$*,* $\mathbf{B}$ *and* $\mathbf{N}$ *are polynomial.*

For instance, the cart-pole system addressed in Section 5.2.1 has a linear potential energy, a quadratic (holonomic) constraint, a constant inertia matrix and a constant input matrix in the implicit representation, but its inertia matrix and potential energy are nonlinear when using generalized coordinates. In fact, *the polynomial characterization is rather unusual when a mechanical system is modeled in an explicit representation.* In this section, due to the inherent complexity of solving the matching and stabilizing conditions, we take advantage of the implicit systems with polynomial characterization to propose a heuristic solution that reformulates the IDA-PBC problem into an SOS program, see Section 2.2. The proposition below provides equivalent conditions to solve the IDA-PBC problem and is instrumental for our derivations.

**Proposition 6.2** (Equivalent IDA-PBC)**.** *Consider* (5.11) *verifying Assumptions 5.6 to 5.8. Let* (5.15) *be the target system with* $\mathbf{J}^{-\top}(r) = \mathbf{M}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)$ *and* $\mathbf{B}_{\mathrm{d}}(r) = \mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$*. Let* $r \mapsto \mathbf{K}_{\mathrm{e}}(r)$ *with domain in* $\mathcal{R}_\Phi$ *be a function s.t.* $\mathbf{N}_\perp(r)\mathbf{M}(r)\mathbf{K}_{\mathrm{e}}^\top(r)$ *is nonsingular and*

$$0 = \mathbf{K}_{\mathrm{e}}(r)\begin{bmatrix}\mathbf{J}^{-\top}(r)\bar{\mathbf{G}}(r) & \mathbf{B}(r)\end{bmatrix} \qquad \forall r \in \mathcal{R}_\Phi \tag{6.6}$$

*Suppose* (5.31a) *holds, then the matching conditions* (5.16) *are equivalent to*

$$\begin{aligned} 0 &= \mathbf{K}_{\mathrm{e}}(r)\mathbf{J}^{-\top}(r)\left(\mathbf{Z}(r)\frac{1}{2}\frac{\partial^\top \rho^\top \mathbf{M}^{-1}\rho}{\partial r} + \bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\partial \mathbf{B}^\top \mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^\top \mathbf{H}}{\partial \rho}\right) \\ &\quad + \mathbf{K}_{\mathrm{e}}(r)\left(-\frac{1}{2}\frac{\partial^\top \rho^\top \mathbf{M}_{\mathrm{d}}^{-1}\rho}{\partial r} + \bar{\mathbf{\Gamma}}_1(r,\rho)\mathbf{M}^{-1}(r)\rho\right) \qquad \forall (r,\rho) \in \mathcal{X}_{\mathrm{c}}, \end{aligned} \tag{6.7a}$$

$$0 = \mathbf{K}_{\mathrm{e}}(r)\left(\mathbf{J}^{-\top}(r)\mathbf{Z}(r)\frac{\partial^\top \mathbf{V}}{\partial r} - \frac{\partial^\top \mathbf{V}_{\mathrm{d}}}{\partial r}\right) \qquad \forall r \in \mathcal{R}_\Phi, \tag{6.7b}$$

*and the control law* (5.18) *can be expressed as*

$$\begin{aligned}\mathbf{u}_{\mathrm{ida}}(r,\rho) = \hat{\mathbf{G}}^{\mathrm{g}}(r)\mathbf{B}_{\perp}(r)\mathbf{J}^{-\top}(r)\left(\bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\partial \mathbf{B}^{\top}\mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^{\top}\mathbf{H}}{\partial \rho} + \mathbf{Z}(r)\frac{\partial^{\top}\mathbf{H}}{\partial r}\right) \\ + \hat{\mathbf{G}}^{\mathrm{g}}(r)\mathbf{B}_{\perp}(r)\left(-\frac{\partial^{\top}\mathbf{H}_{\mathrm{d}}}{\partial r} + \bar{\mathbf{\Gamma}}_1(r,\rho)\frac{\partial^{\top}\mathbf{H}}{\partial \rho}\right) - \bar{\mathbf{\Gamma}}_2(r)\frac{\partial^{\top}\mathbf{H}}{\partial \rho} + \bar{\mathbf{G}}_{\mathrm{I}}(r)\nu,\end{aligned} \tag{6.8}$$

*where* $\bar{\mathbf{\Gamma}}_1(r) = \mathbf{J}^{-\top}(r)\mathbf{\Gamma}_1(r)\mathbf{J}^{-1}(r)$ *and* $\hat{\mathbf{G}}(r) = \mathbf{B}_{\perp}(r)\mathbf{M}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)\bar{\mathbf{G}}(r)$. *Furthermore, inequalities* (5.31b) *and* (5.36b) *read*

$$\frac{\partial \mathbf{H}}{\partial \rho}\left(\mathbf{J}^{-\top}(r)\Big(\mathbf{Z}(r)\mathbf{R}(r) + \bar{\mathbf{G}}(r)\bar{\mathbf{\Gamma}}_2(r)\Big) - \bar{\mathbf{\Gamma}}_1(r,\rho)\right)\frac{\partial^{\top}\mathbf{H}}{\partial \rho} \geq 0 \qquad \forall (r,\rho)\in\mathcal{X}_{\mathrm{c}}, \tag{6.9}$$

$$\mathbf{K}_{\mathrm{e}}(r)\Big(\mathbf{J}^{-\top}(r)\mathbf{Z}(r)\mathbf{R}(r) + \mathbf{R}^{\top}(r)\mathbf{Z}^{\top}(r)\mathbf{J}^{-1}(r)\Big)\mathbf{K}_{\mathrm{e}}^{\top}(r) \succeq 0 \qquad \forall (r,\rho)\in\mathcal{X}_{\mathrm{c}}. \tag{6.10}$$

**Proof.** Recall from the proof of Proposition 5.3, Assumptions 5.6 and 5.8, and condition (5.31a) that $\begin{bmatrix}\bar{\mathbf{G}}(r) & \mathbf{B}_{\mathrm{d}}(r)\end{bmatrix}$ is the full-rank right annihilator of $\mathbf{N}_{\perp}(r)\mathbf{Z}_{\mathrm{d}}(r)$. It follows by Lemma 3.2 and condition (6.6) that the relation

$$\mathbf{K}_{\mathrm{e}}(r) = \bar{\mathbf{K}}(r)\mathbf{N}_{\perp}(r)\mathbf{Z}_{\mathrm{d}}(r)\mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r) \qquad \forall r\in\mathcal{R}_{\Phi} \tag{6.11}$$

holds with

$$\bar{\mathbf{K}}(r) = \mathbf{K}_{\mathrm{e}}(r)\mathbf{J}^{-\top}(r)\left(\mathbf{N}_{\perp}(r)\mathbf{Z}_{\mathrm{d}}(r)\right)^{\mathrm{g}}(r).$$

From (6.6), we can write without loss of generality $\mathbf{K}_{\mathrm{e}}(r) = \bar{\mathbf{K}}_{\mathrm{e}}(r)\mathbf{B}_{\perp}(r)$, meaning that

$$\mathbf{K}_{\mathrm{e}}(r)\mathbf{M}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{M}(r)\mathbf{K}_{\mathrm{e}}^{\top}(r) \succ 0 \qquad \forall r\in\mathcal{R}_{\Phi} \tag{6.12}$$

whenever (5.31a) is verified. Hence, from the nonsingularity of $\mathbf{N}_{\perp}(r)\mathbf{M}(r)\mathbf{K}_{\mathrm{e}}^{\top}(r)$ and (5.31a), we can always build a nonsingular $\bar{\mathbf{K}}$ by choosing

$$\left(\mathbf{N}_{\perp}(r)\mathbf{Z}_{\mathrm{d}}(r)\right)^{\mathrm{g}}(r) := \mathbf{M}(r)\mathbf{K}_{\mathrm{e}}^{\top}(r)\left(\mathbf{N}_{\perp}(r)\mathbf{M}(r)\mathbf{K}_{\mathrm{e}}^{\top}(r)\right)^{-1}.$$

Now, replacing $\mathbf{N}_{\perp}(r)\mathbf{Z}_{\mathrm{d}}(r)\mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)$ by $\bar{\mathbf{K}}^{-1}(r)\mathbf{K}_{\mathrm{e}}(r)$ in (5.16), (5.31b) and (5.36b) yields the equivalent expressions (6.7), (6.9) and (6.10), respectively. To obtain the control law (6.8), note that $\begin{bmatrix}\mathbf{M}^{-1}(r)\mathbf{B}(r) & \mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{M}(r)\mathbf{B}_{\perp}^{\top}(r)\end{bmatrix}$ is nonsingular from (5.31a), which implies that

$$\bar{\mathbf{G}}(r)\bar{\mathbf{G}}^{\mathrm{g}}(r)\bar{\mathbf{G}}(r) = \bar{\mathbf{G}}(r) \quad\Longleftrightarrow\quad \hat{\mathbf{G}}(r)\bar{\mathbf{G}}^{\mathrm{g}}(r)\bar{\mathbf{G}}(r) = \hat{\mathbf{G}}(r) \tag{6.13}$$

and that $\bar{\mathbf{G}}^{\mathrm{g}}(r)$ can be taken as $\bar{\mathbf{G}}^{\mathrm{g}}(r) = \hat{\mathbf{G}}^{\mathrm{g}}(r)\mathbf{B}_{\perp}(r)\mathbf{M}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)$. Replacing this selection in (5.18) gives (6.8), and the proof is complete. □

*Remark* 6.1. The condition

$$0 = \mathbf{K}_\mathrm{e}(r)\mathbf{J}^{-\top}(r)\mathbf{N}(r) \qquad \forall r \in \mathcal{R}_\Phi \tag{6.14}$$

guarantees $0 = \mathbf{K}_\mathrm{e}(r)\mathbf{J}^{-\top}(r)\bar{\mathbf{G}}(r)$ and renders simpler expressions in (6.7) and (6.10).

Proposition 6.2 asserts that (5.16), (5.18), (5.31b) and (5.36b) can be written as (6.7)–(6.10) where only $\mathbf{M}_\mathrm{d}^{-1}$ is present ($\mathbf{M}_\mathrm{d}$ has been removed) at the cost of introducing a new unknown matrix $\mathbf{K}_\mathrm{e}$ satisfying (6.6) with nonsingular $\mathbf{N}_\perp(r)\mathbf{M}(r)\mathbf{K}_\mathrm{e}^\top(r)$. Another contribution of Proposition 6.2 is that by fixing $\mathbf{K}_\mathrm{e}$, we reduce the problem of solving the quasilinear PDEs (5.16) to the one of solving the linear PDEs (6.7). Consequently, if we *i)* restrict our attention to polynomial systems, *ii)* assume that the solutions of $\bar{\mathbf{\Gamma}}_1$, $\mathbf{M}_\mathrm{d}^{-1}$ and $\mathbf{V}_\mathrm{d}$ are polynomial, and *iii)* choose $\mathbf{K}_\mathrm{e}$ verifying $0 = \mathbf{K}_\mathrm{e}(r)\mathbf{B}(r)$ with nonsingular $\mathbf{N}_\perp(r)\mathbf{M}(r)\mathbf{K}_\mathrm{e}^\top(r)$ such that the conditions (6.6)–(6.7) and (6.10) are all polynomial. Then, we can reframe the IDA-PBC controller synthesis as the following SOS program.

**SOS Program 6.1** (Heuristic solution)**.**

find: the coefficients of $\bar{\mathbf{\Gamma}}_1$, $\mathbf{M}_\mathrm{d}^{-1}$, $\mathbf{V}_\mathrm{d}$

subject to: (5.33a), (6.7), $0 = \mathbf{K}_\mathrm{e}(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{Z}(r)\mathbf{G}(r)$, $0 = \mathbf{V}_\mathrm{d}(r_\mathrm{d})$, LHSs of (5.33b), (5.33c), (6.10) are SOS,

where we add $-\epsilon I_{n_r - n_\Phi}$ and $-\epsilon I_{n_r - n_\lambda}$ with a sufficiently small $\epsilon > 0$ in the left-hand sides of (5.33b) and (5.33c), respectively, to guarantee the strict inequalities.

Observe that the SOS program cannot be formulated with the matching conditions (5.16) or by searching simultaneously for $\mathbf{K}_\mathrm{e}$, $\bar{\mathbf{\Gamma}}_1$, $\mathbf{M}_\mathrm{d}^{-1}$ and $\mathbf{V}_\mathrm{d}$ in (6.7) because their coefficients would not be subject to linear constraints.

Given that (5.33b)–(5.33c) are LMIs, there is no loss of generality by including them in the framework of SOS, see Section 2.2. Nevertheless, choosing $\mathbf{K}_\mathrm{e}$ in advance, searching only for polynomial solutions, and assuming that the left-hand side of (6.10) is SOS introduce some conservatism that, in the worst case, may lead to no solution whatsoever. To alleviate the first problem, we propose two options to pick $\mathbf{K}_\mathrm{e}$:

**Option 1.** Choose $\mathbf{K}_\mathrm{e}$ such that $\mathbf{N}_\perp(r)\mathbf{M}(r)\mathbf{K}_\mathrm{e}^\top(r)$ is nonsingular and

$$0 = \mathbf{K}_\mathrm{e}(r)\begin{bmatrix}\mathbf{G}(r) & \mathbf{B}(r)\end{bmatrix} \qquad \forall r \in \mathcal{R}_\Phi.$$

**Option 2.** Choose $\mathbf{K}_\mathrm{e}$ such that $0 = \mathbf{K}_\mathrm{e}(r)\mathbf{B}(r)$ and there exist functions $\mathbf{K}_\mathrm{a}$ and $\hat{\mathbf{V}}_\mathrm{d}$ of appropriate size verifying

$$\mathbf{K}_\mathrm{a}(r)\mathbf{N}_\perp(r)\frac{\partial^\top \mathbf{V}}{\partial r} - \mathbf{K}_\mathrm{e}(r)\frac{\partial^\top \hat{\mathbf{V}}_\mathrm{d}}{\partial r} = 0, \tag{6.15a}$$

$$\mathbf{K}_{\mathrm{a}}(r_{\mathrm{d}})\mathbf{N}_{\perp}(r_{\mathrm{d}})\mathbf{M}(r_{\mathrm{d}})\mathbf{K}_{\mathrm{e}}^{\top}(r_{\mathrm{d}}) \succ 0 \tag{6.15b}$$

for every $r \in \mathcal{R}_{\Phi}$, where $\hat{\mathbf{V}}_{\mathrm{d}}$ is polynomial and has a strict local minimum in $r_{\mathrm{d}}$.[55]

Option 1 is the most simple but also uncertain. It actually implies, from (6.11), that $\mathbf{M}_{\mathrm{d}}^{-1}$ is constrained by

$$0 = \mathbf{N}_{\perp}(r)\mathbf{Z}_{\mathrm{d}}(r)\mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{G}(r).$$

This solution can be implemented for relatively straightforward control objectives where the target potential field (gradient of $\mathbf{V}_{\mathrm{d}}$) does not change drastically, see e.g., the 4-DoF portal crane of Section 7.1.2. The typical steps where Option 1 might fail are two.

- The solution of $\mathbf{V}_{\mathrm{d}}$ resulting from the potential matching (6.7b) could be incompatible with $r_{\mathrm{d}}$ being the strict local minimum of $\mathbf{V}_{\mathrm{d}}|_{\mathcal{R}_{\Phi}}$, i.e., (5.33a)–(5.33b).
- The solution of $\mathbf{M}_{\mathrm{d}}^{-1}$ that guarantees (5.33a)–(5.33b) and (6.7b) could be incompatible with the necessary condition (6.12).

Option 2 overcomes the aforementioned problems, by selecting a $\mathbf{K}_{\mathrm{e}}$ that certifies the existence of $\mathbf{V}_{\mathrm{d}}$ and $\mathbf{M}_{\mathrm{d}}^{-1}$ verifying (5.33a)–(5.33b), (6.7b) and (6.12). Here, conditions (6.15) are an equivalent and simpler representation of (6.7b) and (6.12), see (6.11). We remove the nonsingularity condition of $\mathbf{N}_{\perp}(r)\mathbf{M}(r)\mathbf{K}_{\mathrm{e}}^{\top}(r)$ because (6.15b) already guarantees it locally. The underlying idea with Option 2 is to propose a tentative polynomial solution for $\hat{\mathbf{V}}_{\mathrm{d}}$ and then calculate $\mathbf{K}_{\mathrm{e}}$ and $\mathbf{K}_{\mathrm{a}}$ from (6.15a), but if the selection does not meet (6.15b), we change $\hat{\mathbf{V}}_{\mathrm{d}}$ and repeat the process. By using this option, we can achieve the upright stabilization of the cart-pole's pendulum (see Section 7.2.3), which otherwise with Option 1 is unfeasible.

In summary, the proposed method hinges on finding a suitable function $\mathbf{K}_{\mathrm{e}}$ and the computational cost of solving the SOS Program 6.1. Recall that an SOS program is solved in polynomial time, meaning that the computational cost (time) is upper bounded by a polynomial expression of the number of inputs, i.e., the number of coefficients and linear constraints in $\bar{\mathbf{\Gamma}}_1$, $\mathbf{M}_{\mathrm{d}}^{-1}$ and $\mathbf{V}_{\mathrm{d}}$. Similar to the algebraic approach of Chapter 3, having a large number of unknown coefficients (parameters) may lead to an unpredictable controller performance because the SOS Program 6.1 has either none or infinite solutions. But the result can be improved by imposing constraints or minimization objectives on such new variables. This implies, for practical purposes, that we should aim at a low polynomial order on $\mathbf{V}_{\mathrm{d}}$, $\mathbf{M}_{\mathrm{d}}^{-1}$ and $\bar{\mathbf{\Gamma}}_1$. Algorithm 6.1 merges the procedure to synthesize the IDA-PBC for constrained mechanical systems of Algorithm 5.2 with the proposed heuristic solution.

[55] $\mathbf{K}_{\mathrm{a}}$ does not have to be polynomial.

**Algorithm 6.1** Heuristic solution of IDA-PBC with SOS programs for mechanical systems with kinematic constraints.

**Require:** A UMS of the form (5.11) with a polynomial characterization and verifying Assumptions 5.6 to 5.8.

1: Select $\mathbf{N}_\perp$ and $r_\mathrm{d} \in \mathcal{R}_\mathrm{a} := \left\{ r \in \mathcal{R}_\Phi \mid 0 = \mathbf{N}_\perp(r)\frac{\partial^\top \mathbf{V}}{\partial r} \right\}$. Let $\mathbf{J}^{-\top}(r) = \mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)$ and $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$.

2: Pick the polynomial order for $\mathbf{M}_\mathrm{d}^{-1}$ (symmetric), $\mathbf{V}_\mathrm{d}$ and $\bar{\mathbf{\Gamma}}_1$ (skew-symmetric and linear in $\rho$).

3: Choose $\mathbf{K}_\mathrm{e}$ from either Option 1 or 2 such that the conditions (6.6)–(6.7) and (6.10) are all polynomial. From (6.6), we can replace $0 = \mathbf{K}_\mathrm{e}(r)\mathbf{J}^{-\top}(r)\bar{\mathbf{G}}(r)$ with the stronger condition (6.14) to simplify (6.7) and (6.10).

4: Select $\epsilon$, $\left(\frac{\partial^\top \Phi}{\partial r}\right)_\perp$ and $\mathbf{B}_\perp$, and solve the SOS Program 6.1. If desired, impose a minimization objective and additional constraints on the coefficients of $\mathbf{M}_\mathrm{d}^{-1}$, $\mathbf{V}_\mathrm{d}$ and $\bar{\mathbf{\Gamma}}_1$. This step can be computed e.g., with **SOSTOOLS**[56] and a SDP solver. If solver converges and $\mathbf{M}_\mathrm{d}^{-1}(r_\mathrm{d})$ is nonsingular proceed to next step, otherwise return to steps 2 or 3.

5: To build the feedback (6.8), select $\bar{\mathbf{\Gamma}}_2$ from (6.9),[57] $\hat{\mathbf{G}}^\mathrm{g}$ and $\bar{\mathbf{G}}_\top \nu$. If $\mathbf{R}(r) = 0$, then

$$\bar{\mathbf{\Gamma}}_2(r) := \mathbf{K}_\mathrm{v}(r)\bar{\mathbf{G}}^\top(r)\mathbf{J}^{-1}(r)$$

with $\mathbf{K}_\mathrm{v} : \mathcal{R}_\Phi \to \mathbb{R}^{n_u \times n_u}$, $\mathbf{K}_\mathrm{v}(r) + \mathbf{K}_\mathrm{v}^\top(r) \succeq 0$ verifies (6.9).

6: (Local) asymptotic stability can be verified from the invariant set $\Omega_\mathrm{inv}$ (as defined in Proposition 5.2).

### 6.2.2 Constructive Solutions

Instead of solving the matching and stabilization conditions simultaneously, as performed in the heuristic solution, here we take the standard procedure of Algorithm 5.2 whereby we first solve the kinetic matching, then the potential one, and finally take advantage of the arbitrary functions and free parameters to fulfill the stabilizing conditions. In this context, we provide three solutions for the kinetic matching conditions (5.16a) and one for the potential matching (5.16b).

#### Avoiding the Kinetic Energy Shaping

Suppose

$$\operatorname{Colsp} \mathbf{B}(r) \subset \operatorname{Colsp} \mathbf{N}(r) \qquad \forall r \in \mathcal{R}_\Phi. \tag{6.16}$$

[56]SOSTOOLS is a Matlab toolbox specialized on the SOS method, providing a simple environment to work with polynomial equalities and inequalities, see Section 2.2 and [122].

[57]The satisfaction of (6.10) from the SOS Program 6.1 certifies the existence of $\bar{\mathbf{\Gamma}}_2$ verifying (6.9).

Then, Assumption 5.9 and the kinetic matching (5.16a) can be satisfied with $\mathbf{M}_\mathrm{d}(r) = \mathbf{M}(r)$, $\mathbf{J}(r) = I_{n_r}$, $\mathbf{\Gamma}_1(r, \rho) = 0$ and $\mathbf{B}_\mathrm{d}(r) = \mathbf{B}(r)$. The proof is self-evident after noting that (6.16) with $\mathbf{B}_\mathrm{d}(r) = \mathbf{B}(r)$ is equivalent to $0 = \mathbf{N}_\perp(r)\mathbf{B}_\mathrm{d}(r)$. As one would expect, not shaping the kinetic energy in IDA-PBC reduces the scope of application. For instance, under this selection, we can stabilize the downward pendulum position of the cart-pole, but not the upward (see Section 7.2.2).

### Constant Target Inertia Matrix

For unconstrained mechanical systems verifying

$$0 = G_\perp(q)\frac{\partial^\top p^\top M^{-1} p}{\partial q} \qquad \forall (q, p) \in \mathcal{Q} \times \mathbb{R}^{n_q}, \tag{6.17}$$

a common approach to solve the kinetic matching (5.3a) with $J(q) = M^{-1}(q)M_\mathrm{d}(q)$ is to consider $\Gamma_1(q, p) = 0$ and a constant target inertia matrix $M_\mathrm{d}$, see e.g., [67, 68]. We can extrapolate this idea to the implicit situation if

$$\mathbf{N}_\perp(r)\frac{\partial^\top \rho^\top \mathbf{M}^{-1}\rho}{\partial r} = 0 \qquad \forall (r, \rho) \in \mathcal{R}_\Phi \times \mathbb{R}^{n_r}. \tag{6.18}$$

Hence, (5.16a) with $\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)$ and $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ can be satisfied by choosing $\mathbf{\Gamma}_1(r, \rho) = 0$ and a constant $\mathbf{M}_\mathrm{d}$ such that

$$0 = \mathbf{N}_\perp(r)\mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r) \qquad \forall r \in \mathcal{R}_\Phi. \tag{6.19}$$

Condition (6.18) includes systems with constant mass matrix $\mathbf{M}$ mostly modeled in a Euclidean space, which otherwise in an unconstrained representation may possess a state-dependent inertia matrix $M$, see Section 6.3. Similarly, even though $\mathbf{M}_\mathrm{d}$ is constant, the inertia matrix of the reduced closed-loop can be state-dependent. Looking back to the motivating example of Section 5.2.1, we observe that the cart-pole model in explicit representation does not fulfill (6.17). Nevertheless, in the implicit representation, the inertia matrix $\mathbf{M}$ is constant, and shaping its kinetic energy relies on satisfying the algebraic equation (6.19), see Section 7.2.2 for the upward stabilization of the cart-pole pendulum.

### A Characterization of the Target Inertia Matrix

Until now, we have provided two analytic solutions to the kinetic matching (5.16a). These approaches avoid the task of solving PDEs but have two significant disadvantages. First, they reduce the scope of application of the implicit IDA-PBC, because if step 4 in Algorithm 5.2 fails, we may be forced to seek out a state-dependent $\mathbf{M}_\mathrm{d}(r) \neq \mathbf{M}(r)$. And second, using kinetic shaping with non-constant $\mathbf{M}_\mathrm{d}$ increases the controller's versatility e.g., by enlarging the region of convergence. In this regard, the following result provides an equivalent expression of the matching condition (5.16a), replacing the task of solving PDEs with the one of solving DAEs. This equivalence reduces the complexity of obtaining a solution of $\mathbf{M}_\mathrm{d}$ in such a manner that we can introduce a state-dependent characterization of $\mathbf{M}_\mathrm{d}$ that solves (5.16a). The following lemma, whose proof is given in Appendix B.6, is used in the proof of Proposition 6.3.

**Lemma 6.1.** *Let $A : \mathcal{X} \to \mathbb{R}^{n\times n}$ and $G : \mathcal{X} \to \mathbb{R}^{n\times m}$ be given. Assume $A$ is symmetric and $G$ has constant rank. Then,*

$$0 = A(x) + G(x)K(x) + K^\top(x)G^\top(x) \qquad\qquad x \in \mathcal{X}$$

*has a solution in $K : \mathcal{X} \to \mathbb{R}^{m\times n}$ if and only if*

$$0 = G_\perp(x)A(x)G_\perp^\top(x) \qquad\qquad x \in \mathcal{X}.$$

*If a solution exists, then they are all of the form*

$$K(x) = G^\mathrm{g}(x)A(x)\Big(\tfrac{1}{2}G(x)G^\mathrm{g}(x) - I_n\Big)^\top + W(x)G^\top(x) + G_\perp(x)\hat{K},$$

*where $\hat{K}$ is arbitrary and of adequate size, while $W : \mathcal{X} \to \mathbb{R}^{m\times m}$ is arbitrary and skew-symmetric.*

**Proposition 6.3.** *Consider the system (5.11) verifying Assumptions 5.6 to 5.8. Let (5.15) be the target system with $\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)$, $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ and $\mathbf{\Gamma}_1$ being skew-symmetric. Assume $\mathbf{M}_d$ satisfies (5.31a). The implicit matching of the kinetic energy (5.16a) holds if and only if there exists a function $\mathbf{P} : \mathcal{R}_\Phi \times \mathbb{R}^{n_r} \to \mathbb{R}^{n_r\times n_r}$ such that*

$$0 = \mathbf{N}_\perp(r)\mathbf{Z}_\mathrm{d}(r)\left(\frac{\mathrm{d}\mathbf{M}_\mathrm{d}}{\mathrm{d}t} + \mathbf{P}(r,\dot{r})\mathbf{J}(r) + \mathbf{J}^\top(r)\mathbf{P}^\top(r,\dot{r})\right)\mathbf{Z}_\mathrm{d}^\top(r)\mathbf{N}_\perp^\top(r), \tag{6.20a}$$

$$0 = \mathbf{P}(r,\dot{r})\dot{r} + \mathbf{Z}(r)\frac{1}{2}\frac{\partial^\top \dot{r}^\top\mathbf{M}\dot{r}}{\partial r} - \bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\mathrm{d}\mathbf{B}^\top\mathbf{M}^{-1}}{\mathrm{d}t}\mathbf{M}(r)\dot{r} \tag{6.20b}$$

*for all $(r,\dot{r}) \in \bar{\mathcal{X}}_\mathrm{c} := \Big\{(r,\dot{r}) \in \mathcal{R}_\Phi \times \mathbb{R}^{n_r} \mid 0 = \mathbf{B}^\top(r)\dot{r}\Big\}$. Furthermore, the controller (5.18) can be express as*

$$\begin{aligned}\mathbf{u}_\mathrm{ida}(r,\rho) = \bar{\mathbf{G}}^\mathrm{g}(r)\left(\mathbf{Z}(r)\frac{\partial^\top\mathbf{V}}{\partial r} - \mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\frac{\partial^\top\mathbf{V}_\mathrm{d}}{\partial r}\right) - \bar{\mathbf{\Gamma}}_2(r)\frac{\partial^\top\mathbf{H}}{\partial\rho} + \bar{\mathbf{G}}_\perp(r)\nu \\ + \bar{\mathbf{G}}^\mathrm{g}(r)\mathbf{A}(r,\rho)\Big(I_{n_r} - \tfrac{1}{2}\bar{\mathbf{G}}(r)\bar{\mathbf{G}}^\mathrm{g}(r)\Big)^\top\frac{\partial^\top\mathbf{H}_\mathrm{d}}{\partial\rho} + \mathbf{K}_w(r,\rho)\bar{\mathbf{G}}^\top(r)\frac{\partial^\top\mathbf{H}_\mathrm{d}}{\partial\rho},\end{aligned} \tag{6.21}$$

*where* $\mathbf{K}_w : \mathcal{X}_{\rm c} \to \mathbb{R}^{n_u \times n_u}$ *is arbitrary and skew-symmetric, and*

$$\mathbf{A}(r,\rho) = \mathbf{Z}_{\rm d}(r)\left(\frac{\mathrm{d}\mathbf{M}_{\rm d}}{\mathrm{d}t} + \mathbf{P}(r,\dot r)\mathbf{J}(r) + \mathbf{J}^\top(r)\mathbf{P}^\top(r,\dot r)\right)\Big|_{\dot r = \mathbf{M}^{-1}(r)\rho} \mathbf{Z}_{\rm d}^\top(r).$$

**Proof.** The kinetic matching (5.16a) is the necessary and sufficient condition for the existence of $\mathbf{u}_1$ verifying (5.22a), see the proof of Proposition 5.1. Hence, by setting

$$\bar{\mathbf{P}}(r,\rho) := \mathbf{Z}(r)\frac{1}{2}\frac{\partial^\top \mathbf{M}^{-1}\rho}{\partial r}\mathbf{M}(r) + \bar{\mathbf{B}}(r)\Delta^{-1}(r)\frac{\partial \mathbf{B}^\top \mathbf{M}^{-1}\rho}{\partial r},$$

equation (5.22a) reads

$$\begin{aligned}
\bar{\mathbf{G}}(r)\mathbf{u}_1(r,\rho) &= \mathbf{Z}_{\rm d}(r)\left(\bar{\mathbf{P}}(r,\rho)\frac{\partial^\top \mathbf{H}}{\partial \rho} - \mathbf{J}^\top(r)\frac{1}{2}\frac{\partial^\top \rho^\top \mathbf{M}_{\rm d}^{-1}\rho}{\partial r} + \mathbf{\Gamma}_1(r,\rho)\frac{\partial^\top \mathbf{H}_{\rm d}}{\partial \rho}\right)\\
&= \mathbf{Z}_{\rm d}(r)\left(\bar{\mathbf{P}}(r,\rho)\frac{\partial^\top \mathbf{H}}{\partial \rho} + \mathbf{J}^\top(r)\frac{1}{2}\left.\frac{\partial^\top \hat\rho^\top \mathbf{M}_{\rm d}\hat\rho}{\partial r}\right|_{\hat\rho = \mathbf{M}_{\rm d}^{-1}\rho} + \mathbf{\Gamma}_1(r,\rho)\frac{\partial^\top \mathbf{H}_{\rm d}}{\partial \rho}\right)\\
&= \mathbf{Z}_{\rm d}(r)\left(\bar{\mathbf{P}}(r,\rho)\frac{\partial^\top \mathbf{H}}{\partial \rho} + \frac{1}{2}\left.\frac{\partial \mathbf{M}_{\rm d}\hat\rho}{\partial r}\right|_{\hat\rho = \mathbf{M}_{\rm d}^{-1}\rho}\frac{\partial^\top \mathbf{H}}{\partial \rho} + \frac{1}{2}\widehat{\mathbf{\Gamma}}_1(r,\rho)\frac{\partial^\top \mathbf{H}_{\rm d}}{\partial \rho}\right)\\
&= \mathbf{Z}_{\rm d}(r)\left(\bar{\mathbf{P}}(r,\rho)\mathbf{J}(r) + \frac{1}{2}\left.\frac{\mathrm{d}\mathbf{M}_{\rm d}}{\mathrm{d}t}\right|_{\dot r = \mathbf{M}^{-1}(r)\rho} + \frac{1}{2}\widehat{\mathbf{\Gamma}}_1(r,\rho)\right)\mathbf{Z}_{\rm d}^\top(r)\frac{\partial^\top \mathbf{H}_{\rm d}}{\partial \rho},
\end{aligned} \tag{6.22}$$

where the second equality results from

$$0 \equiv \frac{\partial \mathbf{M}_{\rm d}}{\partial r_i}\mathbf{M}_{\rm d}^{-1}(r) + \mathbf{M}_{\rm d}(r)\frac{\partial \mathbf{M}_{\rm d}^{-1}}{\partial r_i},$$

the third one from

$$\frac{1}{2}\widehat{\mathbf{\Gamma}}_1(r,\rho) := \mathbf{\Gamma}_1(r,\rho) + \frac{1}{2}\left(\mathbf{J}^\top(r)\left.\frac{\partial^\top \mathbf{M}_{\rm d}\hat\rho}{\partial r}\right|_{\hat\rho = \mathbf{M}_{\rm d}^{-1}\rho} - \left.\frac{\partial \mathbf{M}_{\rm d}\hat\rho}{\partial r}\right|_{\hat\rho = \mathbf{M}_{\rm d}^{-1}\rho}\mathbf{J}(r)\right),$$

and the last one from a simple factorization. We did left out the membership $(r,\rho) \in \mathcal{X}_{\rm c}$ for clarity. Let, without loss of generality, $\mathbf{u}_1(r,\rho) := \mathbf{K}(r,\rho)\frac{\partial^\top \mathbf{H}_{\rm d}}{\partial \rho}$ for some function $\mathbf{K} : \mathcal{R}_\Phi \to \mathbb{R}^{n_u \times n_r}$ that is linear in $\rho$ ($\mathbf{u}_1$ is quadratic in $\rho$). Consequently, (6.22) can be expressed as

$$\bar{\mathbf{G}}(r)\mathbf{K}(r,\rho) = \mathbf{Z}_{\rm d}(r)\left(\bar{\mathbf{P}}(r,\rho)\mathbf{J}(r) + \hat{\mathbf{P}}(r,\rho)\mathbf{J}(r) + \frac{1}{2}\left.\frac{\mathrm{d}\mathbf{M}_{\rm d}}{\mathrm{d}t}\right|_{\dot r = \mathbf{M}^{-1}(r)\rho} + \frac{1}{2}\widehat{\mathbf{\Gamma}}_1(r,\rho)\right)\mathbf{Z}_{\rm d}^\top(r),$$

for some function $\hat{\mathbf{P}} : \mathcal{X}_c \to \mathbb{R}^{n_r \times n_r}$ verifying $\hat{\mathbf{P}}(r,\rho)\mathbf{M}^{-1}(r)\rho = 0$, or equivalently as

$$\bar{\mathbf{G}}(r)\mathbf{K}(r,\rho) = \mathbf{Z}_{\rm d}(r)\left(\mathbf{P}(r,\rho)\mathbf{J}(r) + \frac{1}{2}\left.\frac{\mathrm{d}\mathbf{M}_{\rm d}}{\mathrm{d}t}\right|_{\dot r = \mathbf{M}^{-1}(r)\rho} + \frac{1}{2}\widehat{\mathbf{\Gamma}}_1(r,\rho)\right)\mathbf{Z}_{\rm d}^\top(r),$$

with $\mathbf{P}$ satisfying (6.20b). Splitting the above equation in its symmetric and skew-symmetric components gives

$$\bar{\mathbf{G}}(r)\mathbf{K}(r,\rho)+\mathbf{K}^\top(r,\rho)\bar{\mathbf{G}}^\top(r)=\mathbf{Z}_\mathrm{d}(r)\left(\frac{1}{2}\left.\frac{\mathrm{d}\mathbf{M}_\mathrm{d}}{\mathrm{d}t}\right|_{\dot{r}=\mathbf{M}^{-1}(r)\rho}+\mathbf{P}(r,\rho)\mathbf{J}(r)\right)^{\mathrm{s}}\mathbf{Z}_\mathrm{d}^\top(r), \quad (6.23a)$$

$$\bar{\mathbf{G}}(r)\mathbf{K}(r,\rho)-\mathbf{K}^\top(r,\rho)\bar{\mathbf{G}}^\top(r)=\mathbf{Z}_\mathrm{d}(r)\left(\frac{1}{2}\widehat{\mathbf{\Gamma}}_1(r,\rho)+\mathbf{P}(r,\rho)\mathbf{J}(r)\right)^{\mathrm{skew}}\mathbf{Z}_\mathrm{d}^\top(r). \quad (6.23b)$$

Hence, by Lemma 6.1 and $\bar{\mathbf{G}}_\perp(r):=\begin{bmatrix}\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\\ \mathbf{N}_\perp(r)\end{bmatrix}$, the equation (6.23a) has a solution for $\mathbf{K}$, i.e., $\mathbf{u}_1$, if and only if there exist $\mathbf{M}_\mathrm{d}$ and $\mathbf{P}$ satisfying (6.20a). If a solution exists, then $\mathbf{K}$ can be written as

$$\mathbf{K}(r,\rho)=\bar{\mathbf{G}}^{\mathrm{g}}(r)\mathbf{A}(r,\rho)\Big(I_{n_r}-\tfrac{1}{2}\bar{\mathbf{G}}(r)\bar{\mathbf{G}}^{\mathrm{g}}(r)\Big)^\top+\mathbf{K}_w(r,\rho)\bar{\mathbf{G}}^\top(r), \quad (6.24)$$

and (6.21) is obtained from (5.22d) with $\mathbf{u}_1(r,\rho)=\mathbf{K}(r,\rho)\frac{\partial^\top\mathbf{H}_\mathrm{d}}{\partial\rho}$. After multiplying (6.23b) on the left with the nonsingular matrix $\begin{bmatrix}\mathbf{M}^{-1}(r)\mathbf{B}(r) & \mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r)\end{bmatrix}^\top$ and on the right with its transpose,[58] we observe that there always exists a skew-symmetric $\mathbf{\Gamma}_1$ verifying (6.23b) with $\mathbf{K}$ from (6.24), which completes the proof. □

**Corollary 6.1.** *Consider the system* (5.11) *verifying* (6.18) *and Assumptions 5.6 to 5.8. Let* (5.15) *be the target system with* $\mathbf{J}(r)=\mathbf{M}^{-1}(r)\mathbf{M}_\mathrm{d}(r)$ *and* $\mathbf{B}_\mathrm{d}(r)=\mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$. *Let* $\mathbf{M}_\mathrm{d}$ *be defined as*

$$\mathbf{M}_\mathrm{d}(r):=\mathbf{M}_{\mathrm{d}1}+\mathbf{M}_{\mathrm{d}4}(r)\mathbf{M}_{\mathrm{d}2}\mathbf{M}_{\mathrm{d}4}^\top(r)+\mathbf{N}(r)\mathbf{M}_{\mathrm{d}3}(r)\mathbf{N}^\top(r), \quad (6.25)$$

*where* $\mathbf{M}_{\mathrm{d}1}$ *and* $\mathbf{M}_{\mathrm{d}2}$ *are arbitrary constants, and* $\mathbf{M}_{\mathrm{d}3}$ *and* $\mathbf{M}_{\mathrm{d}4}$ *are arbitrary functions. Suppose* $\mathbf{M}_\mathrm{d}$ *fulfills* (5.31a), (6.19) *and*

$$\mathbf{N}_\perp(r)\dot{\mathbf{M}}_{\mathrm{d}4}(r,\dot{r})=0 \qquad \forall(r,\dot{r})\in\bar{\mathcal{X}}_\mathrm{c}, \quad (6.26)$$

*where* $\bar{\mathcal{X}}_\mathrm{c}$ *is as defined in Proposition 6.3. Then, there exists a skew-symmetric* $\mathbf{\Gamma}_1$ *such that the kinetic matching* (5.16a) *holds. Furthermore, the control law is given by* (6.21) *for any* $\mathbf{P}$ *satisfying* (6.20b).

**Proof.** An immediate result of combining (6.25) and (6.20b) with (6.20a) for $\mathbf{M}_\mathrm{d}$ verifying (5.31a) and (6.19). □

Condition (6.20) is equivalent to the kinetic matching (5.16a) and it represents a system of DAEs with unknowns in $\mathbf{M}_\mathrm{d}$ and $\mathbf{P}$. Here, there are infinite solutions of $\mathbf{P}$ from (6.20b),

[58] The nonsingularity of $\begin{bmatrix}\mathbf{M}^{-1}(r)\mathbf{B}(r) & \mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r)\end{bmatrix}$ is a direct consequence of Lemma 5.2 and condition (5.31a).

and its freedom is used to solve $\mathbf{M}_\mathrm{d}$ from (6.20a). Furthermore, the characterization of $\mathbf{M}_\mathrm{d}$ given in Corollary 6.1 already includes the solution with constant $\mathbf{M}_\mathrm{d}$ introduced in the previous subsection, and it is therefore, a state-dependent generalization of that case. Note that such a characterization (of $\mathbf{M}_\mathrm{d}$) is only a sufficient condition for (5.16a), meaning that there may exist solutions of $\mathbf{M}_\mathrm{d}$ satisfying (6.20), but not (6.25). In Sections 7.2.4 and 7.3 we show examples using the characterization (6.25). We extend the results of Proposition 6.3 to the explicit situation as follows.

**Proposition 6.4.** *Consider systems* (5.1) *and* (5.2)*. The matching condition of the kinetic energy* (5.3a) *with* $\Gamma_1$ *being skew-symmetric holds if and only if there exists* $P : \mathcal{Q} \times \mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times n_q}$ *verifying*

$$0 = G_\perp(q)\left(\frac{\mathrm{d}M_\mathrm{d}}{\mathrm{d}t} + P(q,\dot q)J(q) + J^\top(q)P^\top(q,\dot q)\right)G_\perp^\top(q), \tag{6.27a}$$

$$0 = P(q,\dot q)\dot q + \frac{1}{2}\frac{\partial^\top \dot q^\top M \dot q}{\partial q} \tag{6.27b}$$

*for all* $(q,p) \in \mathcal{Q} \times \mathbb{R}^{n_q}$.

**Proof.** Along the same lines of the proof of Proposition 6.3. □

Similar to the implicit situation, there are infinite solutions of $P$ in (6.27b). For instance, we can take $P(q,\dot q) := -C^\top(q,\dot q)$, where $C$ is the centrifugal and Coriolis matrix obtained from the Christoffel symbols [10, 63, 226] corresponding to the inertia matrix $M$, that is,

$$C(q,\dot q) := \frac{1}{2}\left(\dot M + \frac{\partial M\dot q}{\partial q} - \frac{\partial^\top M \dot q}{\partial q}\right)$$

and it verifies

$$C(q,\dot q)\dot q = \dot M \dot q - \frac{1}{2}\frac{\partial^\top \dot q^\top M \dot q}{\partial q}, \quad \dot M = C(q,\dot q) + C^\top(q,\dot q) \quad \text{and}$$
$$\dot M - 2C(q,\dot q) \text{ is skew-symmetric.}$$

Then, (6.27a) is a sufficient conditions for (5.3a) and it is equivalent to the conditions of [227, 228] for the method of Controlled Lagrangians.[59]

*Example* 6.3. In [102], the authors apply the simplified IDA-PBC of Ryalat and Laila [57] on a 2-DoF portal crane (planar model) with partial feedback linearization described by (5.1) with $q = \mathrm{vec}(q_1, q_2)$, $M = I_2$, $R = 0$, $V(q) = -\bar g \cos(q_1)$ and $G(q) = \mathrm{vec}(-a\cos q_1, 1)$, where $a$ and $\bar g$ are system parameters. The approach of Ryalat and Laila consists of reducing the potential matching (5.3b) to a simple integral. This is achieved by proposing a candidate

[59] In [227, 228] the authors actually consider a Lagrangian system with gyroscopic forces; nonetheless, this situation can also be included by using (5.8) instead of (5.1) and replacing $\frac{1}{2}\frac{\partial^\top p^\top M^{-1} p}{\partial q}$ by $f_2$ in (6.27b).

$M_\text{d}$ that meets the previous goal and then imposing conditions on $M_\text{d}$ such that the kinetic matching (5.3a) holds with a skew-symmetric $\Gamma_1$. In other words, we assume that the candidate $M_\text{d}$ will satisfy (5.3a), and if it does not, we propose a new candidate.

In summary, [102] initially considers

$$M_\text{d}(q) = \begin{bmatrix} m_1(q_1) & m_2(q_1) \\ m_2(q_1) & m_3 \end{bmatrix}$$

for some function $m_1$ and constant $m_3$, where the potential matching (5.3b) is reduced to a simple integral whenever

$$m_2(q_1) = -am_3 \cos q_1, \qquad G_\perp(q) = \begin{bmatrix} 1 & a\cos q_1 \end{bmatrix}.$$

Under the above assumptions, it was feasible to obtain $m_1(q_1) = a^2 m_3 \cos(q_1)^2 + c_1$ satisfying (5.3a) with a skew-symmetric $\Gamma_1$ because the kinetic matching also reduces to a simple integral. However, this is not always the case and (5.3a) may involve solving PDEs. To avoid this as well as "guessing" a structure for $M_\text{d}$, we can alternatively start with a characterization of $M_\text{d}$ obtained from Proposition 6.4, and then impose conditions to reduce the potential matching. Therefore, in the crane example, we use Proposition 6.4 with $P = 0$ to prove the existence of a skew-symmetric $\Gamma_1$ and

$$M_\text{d}(q) = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} + G(q)m_3(q)G^\top(q) + \begin{bmatrix} -a\sin q_1 \\ q_1 \end{bmatrix} c_4 \begin{bmatrix} -a\sin q_1 & q_1 \end{bmatrix}$$

verifying the kinetic matching (5.3a). Here, $c_i \in \mathbb{R}$ are arbitrary constants and $m_3 : \mathcal{Q} \to \mathbb{R}$ is also arbitrary. Now, it is not difficult to see that the potential matching can be reduced to a simple integral if $c_2 = c_3 = c_4 = 0$. Besides, the work [102] assumes $m_3$ to be constant, but our characterization shows that such an assumption is unnecessary. Note that the second approach is also simpler and it can be applied to systems with more than two coordinates, which was a limitation in the Ryalat and Laila method. △

### Solving the Potential Matching

Once the solution of $\mathbf{M}_\text{d}$ is found, the next step is to solve the potential matching PDE (5.16b). For this purpose, let us define

$$\mathbf{S}(r) := \mathbf{N}_\perp(r)\mathbf{Z}_\text{d}(r)\mathbf{M}_\text{d}(r)\mathbf{M}^{-1}(r), \qquad \mathbf{Q}(r) := \begin{bmatrix} \mathbf{S}(r) & -\mathbf{N}_\perp(r)\frac{\partial \mathbf{V}}{\partial r} \end{bmatrix}. \tag{6.28}$$

Then, (5.16b) can be expressed as

$$0 = \mathbf{Q}(r) \begin{bmatrix} \frac{\partial \mathbf{V}_\text{d}}{\partial r} & 1 \end{bmatrix}^\top \qquad \forall r \in \mathcal{R}_\Phi,$$

or equivalently

$$\begin{bmatrix} \frac{\partial \mathbf{V}_\mathrm{d}}{\partial r} & 1 \end{bmatrix}^\top = \mathbf{Q}_\perp(r)\bar{\nu} \qquad\qquad \forall r \in \mathcal{R}_\Phi, \tag{6.29}$$

where $\bar{\nu}$ is arbitrary provided the last row of (6.29) holds. Using Assumption 5.6 and condition (5.31a), we know from the proof of Proposition 5.3 with $\mathbf{B}_\mathrm{d}(r) = \mathbf{M}_\mathrm{d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$ that $\mathbf{S}$ is full rank and $\mathbf{S}(r)\mathbf{B}(r) = 0$. Consequently, $\mathbf{Q}_\perp$ and $\mathbf{S}_\perp$ can be partitioned as

$$\mathbf{Q}_\perp(r) = \begin{bmatrix} \mathbf{S}_\perp(r) & \hat{\mathbf{S}}(r) \\ 0 & 1 \end{bmatrix}, \qquad\qquad \mathbf{S}_\perp(r) = \begin{bmatrix} \frac{\partial^\top \Phi}{\partial r} & \bar{\mathbf{S}}(r) \end{bmatrix} \tag{6.30}$$

for some $\bar{\mathbf{S}} : \mathcal{R} \to \mathbb{R}^{n_r \times n_\gamma}$ and $\hat{\mathbf{S}} : \mathcal{R} \to \mathbb{R}^{n_r}$ with $n_\gamma = \mathrm{rank}(\mathbf{N}(r)) - n_\Phi$. Given that $\begin{bmatrix} \frac{\partial \mathbf{V}_\mathrm{d}}{\partial r} & 1 \end{bmatrix}$ is a gradient and $\bar{\nu}$ is almost arbitrary, a solution of the PDE (6.29), which is the equivalent representation of (5.16b), can be calculated by searching for $\bar{\mathbf{S}}$ and $\hat{\mathbf{S}}$ such that their columns are gradient (or integrable) vector fields (see Lemma 2.2), i.e.,[60]

$$\frac{\partial\, \mathrm{col}_i(\bar{\mathbf{S}})}{\partial r} = \frac{\partial^\top \mathrm{col}_i(\bar{\mathbf{S}})}{\partial r}, \qquad\qquad \frac{\partial \hat{\mathbf{S}}}{\partial r} = \frac{\partial^\top \hat{\mathbf{S}}}{\partial r}, \qquad\qquad \forall r \in \mathcal{R}. \tag{6.31}$$

Now, from the chain rule and equations (6.29)–(6.31), we can build

$$\mathbf{V}_\mathrm{d}(r) = \int_0^1 \hat{\mathbf{S}}^\top(vr)\, r \mathrm{d}v + \beta(\gamma(r)), \qquad\qquad \gamma(r) = \int_0^1 \bar{\mathbf{S}}^\top(vr)\, r \mathrm{d}v, \tag{6.32}$$

where $\beta : \mathbb{R}^{n_\gamma} \to \mathbb{R}$ is user defined. It follows that the necessary and sufficient conditions for $r_\mathrm{d}$ being a strict local minimum of $\mathbf{V}_\mathrm{d}$ in $\mathcal{R}_\Phi$ (see Section 5.2.3) are

$$\hat{\mathbf{S}}(r_\mathrm{d}) + \bar{\mathbf{S}}(r_\mathrm{d}) \left.\frac{\partial^\top \beta}{\partial \gamma}\right|_{\gamma=\gamma(r_\mathrm{d})} + \left.\frac{\partial^\top \mu_\mathrm{d}^\top \Phi}{\partial r}\right|_{r=r_\mathrm{d}} = 0, \tag{6.33a}$$

$$\left(\frac{\partial^\top \Phi}{\partial r}\right)_\perp \left(\frac{\partial \hat{\mathbf{S}}}{\partial r} + \sum_{i=1}^{n_\gamma} \frac{\partial^\top \beta}{\partial \gamma_i} \frac{\partial\, \mathrm{col}_i(\bar{\mathbf{S}})}{\partial r} + \frac{\partial^2 \mu_\mathrm{d}^\top \Phi}{\partial r^2} + \bar{\mathbf{S}}(r) \frac{\partial^2 \beta}{\partial \gamma^2} \bar{\mathbf{S}}^\top(r)\right) \left(\frac{\partial^\top \Phi}{\partial r}\right)_\perp \Bigg|_{\substack{r=r_\mathrm{d} \\ \gamma=\gamma(r_\mathrm{d})}} \succ 0$$

for some suitable constant $\mu_\mathrm{d} \in \mathbb{R}^{n_\Phi}$. We can decouple the effects of $\frac{\partial^\top \beta}{\partial \gamma}$ and $\frac{\partial^2 \beta}{\partial \gamma^2}$ in the above inequality by using Finsler's Lemma A.9 and Corollary 3.1. This results in

$$\bar{A} := \mathbf{S}(r_\mathrm{d}) \left(\frac{\partial \hat{\mathbf{S}}}{\partial r} + \sum_{i=1}^{n_\gamma} \frac{\partial^\top \beta}{\partial \gamma_i} \frac{\partial\, \mathrm{col}_i(\bar{\mathbf{S}})}{\partial r} + \frac{\partial^2 \mu_\mathrm{d}^\top \Phi}{\partial r^2}\right)\Bigg|_{\substack{r=r_\mathrm{d} \\ \gamma=\gamma(r_\mathrm{d})}} \mathbf{S}^\top(r_\mathrm{d}) \succ 0, \tag{6.33b}$$

$$\left.\frac{\partial^2 \beta}{\partial \gamma^2}\right|_{\gamma=\gamma(r_\mathrm{d})} - \begin{bmatrix} 0 \\ I_{n_\gamma} \end{bmatrix}^\top \mathbf{S}_\perp^\dagger(r_\mathrm{d}) \left(A\mathbf{S}^\top(r_\mathrm{d})\bar{A}^{-1}\mathbf{S}(r_\mathrm{d})A - A\right) \left(\mathbf{S}_\perp^\dagger(r_\mathrm{d})\right)^\top \begin{bmatrix} 0 \\ I_{n_\gamma} \end{bmatrix} \succ 0, \tag{6.33c}$$

[60]We use $\mathrm{col}_i(A)$ to denote the $i$-th column of a matrix $A$.

where
$$A = \left( \frac{\partial \hat{\mathbf{S}}}{\partial r} + \sum_{i=1}^{n_\gamma} \frac{\partial^\top \beta}{\partial \gamma_i} \frac{\partial \operatorname{col}_i(\bar{\mathbf{S}})}{\partial r} + \frac{\partial^2 \mu_{\mathrm{d}}^\top \Phi}{\partial r^2} \right) \Bigg|_{\substack{r=r_{\mathrm{d}} \\ \gamma=\gamma(r_{\mathrm{d}})}} .$$

The proposition below encapsulates the previous reasoning together with the results of Sections 5.2.3 and 5.2.4.

**Proposition 6.5.** *Let (5.11) be an underactuated mechanical system verifying Assumptions 5.6 to 5.8. Consider the target system (5.15) with* $\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_{\mathrm{d}}(r)$, $\mathbf{B}_{\mathrm{d}}(r) = \mathbf{J}^\top(r)\mathbf{B}(r)$ *and* $\mathbf{\Gamma}_1$ *being skew-symmetric. Suppose the following conditions hold.*

- ***C6.1*** $\mathbf{M}_{\mathrm{d}}$ *and* $\mathbf{\Gamma}_1$ *satisfy (5.33c), the kinetic matching (5.16a) and the dissipation condition (5.36b).*
- ***C6.2*** $\mathbf{S}$ *and* $\mathbf{Q}$ *are as defined in (6.28).*
- ***C6.3*** *There are functions* $\bar{\mathbf{S}}$ *and* $\hat{\mathbf{S}}$*, obtained from (6.30), such that their columns are gradient vector fields, i.e., (6.31).*
- ***C6.4*** $\mathbf{V}_{\mathrm{d}}$ *is given by (6.32).*
- ***C6.5*** *The functions* $\beta$, $\bar{\mathbf{S}}$ *and* $\hat{\mathbf{S}}$ *verify (6.33a)–(6.33c).*

*Then, there exists a function* $\bar{\mathbf{\Gamma}}_2$ *verifying (5.31b) that leads to the following assertions: i) System (5.11) in closed-loop with feedback (5.18) is stable in* $(r_{\mathrm{d}}, 0)$*. ii) Every bounded trajectory of the closed-loop converge to* $\Omega_{\mathrm{inv}}$ *(set* $\Omega_{\mathrm{inv}}$ *as defined in Proposition 5.2). iii) (Local) asymptotic stability can be demonstrated whenever* $(r_{\mathrm{d}}, 0)$ *is an isolated point of* $\Omega_{\mathrm{inv}}$*.*

*Remark* 6.2. Given $\bar{\mathbf{S}}$ from (6.30), we can write

$$\bar{\mathbf{G}}(r)\bar{\mathbf{\Gamma}}_2(r) := \mathbf{Z}_{\mathrm{d}}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\hat{\mathbf{\Gamma}}_2(r)$$

or equivalently
$$\bar{\mathbf{\Gamma}}_2(r) := \bar{\mathbf{G}}^{\mathrm{g}}(r)\mathbf{Z}_{\mathrm{d}}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\hat{\mathbf{\Gamma}}_2(r)$$

for any function $\hat{\mathbf{\Gamma}}_2 : \mathcal{R} \to \mathbb{R}^{n_\gamma \times n_r}$. The proof is a direct consequence of Lemma 3.2 and $\begin{bmatrix} \mathbf{B}^\top(r)\mathbf{M}^{-1}(r) \\ \mathbf{N}_\perp(r) \end{bmatrix}$ being a full-rank left annihilator of $\bar{\mathbf{G}}$ (see the proof of Proposition 5.1).

Proposition 6.5 exploits of the freedom of $\mathbf{Q}_\perp$ to solve the potential matching PDEs and provides conditions to synthesize an IDA-PBC feedback. The key step of this proposition is finding $\bar{\mathbf{S}}$ and $\hat{\mathbf{S}}$ with the property that their columns are gradient vectors. The complexity of this task depends heavily on the solution of $\mathbf{M}_{\mathrm{d}}$. Therefore, we may attempt to find a characterization of $\mathbf{M}_{\mathrm{d}}$ from Proposition 6.3 or Corollary 6.1 and then select its free functions and parameters to facilitate the search of $\bar{\mathbf{S}}$ and $\hat{\mathbf{S}}$ (possibly constants). Besides, possessing the polynomial characterization (see Definition 6.1), can also help to simplify the search. This can be observed with more detail in the examples of Chapter 7.

To conclude this section, we remark that the proposed solution of the potential matching (5.16b) is not exclusive to constrained mechanical systems, and it can be employed to solve the potential matching (5.3b) for unconstrained systems. In such a situation,

$$S(q) := G_{\perp}(q)M_{\mathrm{d}}(q)M^{-1}(q), \quad Q(q) := \begin{bmatrix} S(q) & -G_{\perp}(q)\frac{\partial V}{\partial q} \end{bmatrix}, \quad Q_{\mathrm{I}}(q) = \begin{bmatrix} S_{\mathrm{I}}(q) & \hat{S}(q) \\ 0 & 1 \end{bmatrix}$$

and $S_{\mathrm{I}}$ has no partition because there are no constraints. Similarly, we search for $S_{\mathrm{I}}$ and $\hat{S}$ such that their columns are gradient vectors and build the target potential energy as

$$V_{\mathrm{d}}(q) = \int_0^1 \hat{S}^{\top}(vq)\, q\mathrm{d}v + \beta(\gamma(q)), \qquad \gamma(q) = \int_0^1 S_{\mathrm{I}}^{\top}(vq)\, q\mathrm{d}v,$$

where $\beta$ is an arbitrary function of adequate size. Under this result, $V_{\mathrm{d}}$ has a strict local minimum in $q_{\mathrm{d}}$ if an only if

$$\hat{S}(q_{\mathrm{d}}) + S_{\mathrm{I}}(q_{\mathrm{d}}) \left.\frac{\partial^{\top}\beta}{\partial\gamma}\right|_{\gamma=\gamma(q_{\mathrm{d}})} = 0,$$

$$\left.\left(\frac{\partial\hat{S}}{\partial q} + \sum_{i=1}^{n_\gamma} \frac{\partial^{\top}\beta}{\partial\gamma_i}\frac{\partial\,\mathrm{col}_i(S_{\mathrm{I}})}{\partial q} + S_{\mathrm{I}}(r)\frac{\partial^2\beta}{\partial\gamma^2}S_{\mathrm{I}}^{\top}(q)\right)\right|_{\substack{q=q_{\mathrm{d}}\\ \gamma=\gamma(q_{\mathrm{d}})}} \succ 0.$$

The following algorithm summarizes the discussion of Section 6.2.2.

---

**Algorithm 6.2** Constructive solution of IDA-PBC for constrained mechanical systems.

**Require:** A UMS of the form (5.11) verifying Assumptions 5.6 to 5.8.

1: Select $\mathbf{N}_{\perp}$ and $r_{\mathrm{d}} \in \mathcal{R}_{\mathrm{a}} := \left\{ r \in \mathcal{R}_{\Phi} \mid 0 = \mathbf{N}_{\perp}(r)\frac{\partial^{\top}\mathbf{V}}{\partial r} \right\}$. Let

$$\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_{\mathrm{d}}(r) \qquad \text{and} \qquad \mathbf{B}_{\mathrm{d}}(r) = \mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r).$$

2: Compute a solution to $\mathbf{M}_{\mathrm{d}}$ (symmetric) and $\mathbf{\Gamma}_1$ (skew-symmetric and linear in $\rho$) from the kinetic matching (5.16a) or its equivalent representation (6.20). If the mechanical system (5.11) satisfies (6.16) we may neglect to shape the kinetic energy by using $\mathbf{M}_{\mathrm{d}}(r) = \mathbf{M}(r)$ and $\mathbf{\Gamma}_1(r, \rho) = 0$. On the other hand, if (5.11) satisfies (6.18), we may use the characterization (6.25) and impose the condition (6.19) to satisfy (5.16a).

3: Calculate $\bar{\mathbf{S}}$ and $\hat{\mathbf{S}}$ from (6.30) provided their columns are gradient vector fields, i.e., (6.31).

4: Choose the arbitrary functions and parameters in $\beta$ and $\mathbf{M}_{\mathrm{d}}$ such that (5.33c), (5.36b) and (6.33a)–(6.33c) hold. Here, (5.33c) can be substituted by (5.35) with $\bar{\Delta}_{\mathrm{d}}(r_{\mathrm{d}})$ being nonsingular. The function $\beta$ can be defined, e.g., as

$$\beta(\gamma) := \frac{1}{2}(\gamma - \gamma^\star)^\top \mathbf{K}_\gamma(\gamma - \gamma^\star),$$

where $\gamma^\star$ and $\mathbf{K}_\gamma = \mathbf{K}_\gamma^\top \succeq 0$ are appropriate constants. If no solution exists, return to step 2.

5: To build the controller (5.18), with $\mathbf{V}_\mathrm{d}$ as defined in (6.32), pick $\bar{\mathbf{\Gamma}}_2$ from (5.31b),[61] $\bar{\mathbf{G}}^\mathrm{g}$ and $\bar{\mathbf{G}}_\mathcal{I}\nu$. Note that if $\bar{\mathbf{G}}^\mathrm{g}(r)\begin{bmatrix}\bar{\mathbf{B}}(r) & \mathbf{B}_\mathrm{d}(r)\end{bmatrix} \equiv 0$, then (5.18) reduces to (5.23). Besides, setting

$$\bar{\mathbf{\Gamma}}_2(r) := \bar{\mathbf{G}}^\mathrm{g}(r)\mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\mathbf{K}_\mathrm{v}(r)\bar{\mathbf{S}}^\top(r)$$

with $\mathbf{K}_\mathrm{v} : \mathcal{R}_\Phi \to \mathbb{R}^{n_\gamma \times n_\gamma}$, $\mathbf{K}_\mathrm{v}(r) + \mathbf{K}_\mathrm{v}^\top(r) \succeq 0$ fulfills (5.31b) if $\mathbf{R}(q) = 0$.

6: (Local) asymptotic stability can be verified from the invariant set $\Omega_\mathrm{inv}$ (as defined in Proposition 5.2).

### 6.2.3 Position Feedback

On the basis of the full-state feedback obtained from the analytic solution it is also possible (under two additional requirements) to implement a simple dynamic output-feedback (independent of $\rho$), extending the well-known result that in some system classes the PBC design can obviate velocity measurement [8, 229].

**Proposition 6.6** (Position-feedback)**.** *Let the conditions of Propositions 6.3 and 6.5 be satisfied with* $\bar{\mathbf{\Gamma}}_2(r) = 0$*,*

$$\frac{\partial \mathbf{H}_\mathrm{d}}{\partial \rho}\mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r)\frac{\partial^\top \mathbf{H}_\mathrm{d}}{\partial \rho} \geq 0, \tag{6.34a}$$

$$\bar{\mathbf{G}}^\mathrm{g}(r)\mathbf{A}(r,\rho)\Big(I_{n_r} - \tfrac{1}{2}\bar{\mathbf{G}}(r)\bar{\mathbf{G}}^\mathrm{g}(r)\Big)^\top \frac{\partial^\top \mathbf{H}_\mathrm{d}}{\partial \rho} + \mathbf{K}_w(r,\rho)\bar{\mathbf{G}}^\top(r)\frac{\partial^\top \mathbf{H}_\mathrm{d}}{\partial \rho} = 0 \tag{6.34b}$$

*for all* $(r,\rho) \in \mathcal{X}_\mathrm{c}$*. Then, the new control law* $u = \mathbf{u}_\mathrm{of}(r,\rho,\zeta)$*,*

$$\begin{aligned}\mathbf{u}_\mathrm{of}(r,\zeta) = \bar{\mathbf{G}}^\mathrm{g}(r)&\left(\mathbf{Z}(r)\frac{\partial^\top \mathbf{V}}{\partial r} - \mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\frac{\partial^\top \mathbf{V}_\mathrm{d}}{\partial r}\right) \\ &- \bar{\mathbf{G}}^\mathrm{g}(r)\mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\bar{\mathbf{K}}_\mathrm{u}\Big(\zeta + \gamma(r)\Big) + \bar{\mathbf{G}}_\mathcal{I}(r)\nu,\end{aligned} \tag{6.35a}$$

$$\dot{\zeta} = -\mathbf{\Lambda}_\zeta(r)\bar{\mathbf{K}}_\mathrm{u}\Big(\zeta + \gamma(r)\Big) \tag{6.35b}$$

*with* $\zeta \in \mathbb{R}^{n_\gamma}$ *stabilizes the system at* $(r_\mathrm{d}, 0)$*. Moreover, the closed loop is asymptotically stable if the equilibrium* $(r_\mathrm{d}, 0)$ *is an isolated point of the largest invariant set of* (5.15)

[61] The satisfaction of (5.36b) from step 4 certifies the existence of $\bar{\mathbf{\Gamma}}_2$ verifying (5.31b).

*contained in*

$$\Omega_{\rm of} := \left\{ (r,\rho) \in \mathcal{X}_{\rm c} \mid \frac{\partial \mathbf{H}_{\rm d}}{\partial \rho}\mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r)\frac{\partial^\top \mathbf{H}_{\rm d}}{\partial \rho} = 0,\ \gamma(r)\ \textit{is constant} \right\}$$

*with* $\mathbf{\Gamma}_2(r) = -\mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r)$. *Here,* $\bar{\mathbf{K}}_{\rm u} \in \mathbb{R}^{n_\gamma \times n_\gamma}$ *and* $\mathbf{\Lambda}_\zeta : \mathcal{R}_\Phi \to \mathbb{R}^{n_\gamma \times n_\gamma}$ *are user-defined provided* $\bar{\mathbf{K}}_{\rm u} = \bar{\mathbf{K}}_{\rm u}^\top \succ 0$ *and* $\mathbf{\Lambda}_\zeta(r) + \mathbf{\Lambda}_\zeta^\top(r) \succ 0$.

**Proof.** Let $\bar{\mathbf{\Gamma}}_2(r) = 0$. Suppose (6.34) and the conditions Propositions 6.3 and 6.5 are satisfied. Then, there is a solution of $\mathbf{M}_{\rm d}$, $\mathbf{\Gamma}_1$ and $\mathbf{V}_{\rm d}$ verifying (5.16) with $(r_{\rm d}, 0)$ being a strict local minimum of $\mathbf{H}_{\rm d}$. Besides, the control law $u = \bar{\mathbf{u}}_{\rm of}(r) + \bar{u}_{\rm x}$ with

$$\bar{\mathbf{u}}_{\rm of}(r) = \bar{\mathbf{G}}^{\rm g}(r)\left(\mathbf{Z}(r)\frac{\partial^\top \mathbf{V}}{\partial r} - \mathbf{Z}_{\rm d}(r)\mathbf{J}^\top(r)\frac{\partial^\top \mathbf{V}_{\rm d}}{\partial r}\right) + \bar{\mathbf{G}}_{\mathcal{I}}(r)\nu \tag{6.36}$$

transforms the nominal system (5.11) into

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{J}(r) \\ -\mathbf{J}^\top(r) & \mathbf{\Gamma}_1(r,\rho) - \mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}_{\rm d}}{\partial r} \\ \frac{\partial^\top \mathbf{H}_{\rm d}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{B}_{\rm d}(r) \end{bmatrix} \lambda_{\rm d} + \begin{bmatrix} 0 \\ \bar{\mathbf{G}} \end{bmatrix} \bar{u}_{\rm x}. \tag{6.37}$$

where

$$\dot{\mathbf{H}}_{\rm d}(r,\rho) = -\frac{\partial \mathbf{H}_{\rm d}}{\partial \rho}\mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r)\frac{\partial^\top \mathbf{H}_{\rm d}}{\partial \rho} + \frac{\partial \mathbf{H}_{\rm d}}{\partial \rho}\bar{\mathbf{G}}(r)\bar{u}_{\rm x} \leq \frac{\partial \mathbf{H}_{\rm d}}{\partial \rho}\bar{\mathbf{G}}(r)\bar{u}_{\rm x}.$$

Note that the constraint forces $\mathbf{B}_{\rm d}\lambda_{\rm d}$ are independent of the instantaneous value of $\bar{u}_{\rm x}$, see (5.19b). Let

$$\bar{\mathbf{H}}_{\rm d}(r,\rho,\tilde{\zeta}) := \frac{1}{2}\rho^\top \mathbf{M}_{\rm d}^{-1}(r)\rho + \mathbf{V}_{\rm d}(r) + \frac{1}{2}\tilde{\zeta}^\top \bar{\mathbf{K}}_{\rm u}\tilde{\zeta}$$

with $\tilde{\zeta} \in \mathbb{R}^{n_\gamma}$ and set

$$\bar{\mathbf{G}}(r)\bar{u}_{\rm x} = \mathbf{Z}_{\rm d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\bar{\mathbf{K}}_{\rm u}\tilde{\zeta}.$$

From Remark 6.2, the previous equality has always a solution in $\bar{u}_{\rm x}$, and it is given by

$$\bar{u}_{\rm x} = \bar{\mathbf{G}}^{\rm g}(r)\mathbf{Z}_{\rm d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\bar{\mathbf{K}}_{\rm u}\tilde{\zeta}.$$

Hence, by setting $\tilde{\zeta} := \zeta + \gamma(r)$, we can rewrite $\bar{\mathbf{u}}_{\rm of}(r) + \bar{u}_{\rm x}$ as (6.35a), and (6.37) with (6.35b) as

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \\ \dot{\tilde{\zeta}} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{J}(r) & 0 \\ -\mathbf{J}^\top(r) & \mathbf{\Gamma}_1(r,\rho) - \mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r) & -\mathbf{Z}_{\rm d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r) \\ 0 & \bar{\mathbf{S}}^\top(r)\mathbf{J}(r)\mathbf{Z}_{\rm d}^\top(r) & -\mathbf{\Lambda}_\zeta(r) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \bar{\mathbf{H}}_{\rm d}}{\partial r} \\ \frac{\partial^\top \bar{\mathbf{H}}_{\rm d}}{\partial \rho} \\ \frac{\partial^\top \bar{\mathbf{H}}_{\rm d}}{\partial \tilde{\zeta}} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{B}_{\rm d}(r) \\ 0 \end{bmatrix} \lambda_{\rm d} \tag{6.38}$$

where

$$\dot{\bar{\mathbf{H}}}_{\mathrm{d}}(r,\rho,\zeta) = -\frac{\partial \mathbf{H}_{\mathrm{d}}}{\partial \rho}\mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r)\frac{\partial^\top \mathbf{H}_{\mathrm{d}}}{\partial \rho} - \tilde{\zeta}^\top \bar{\mathbf{K}}_{\mathrm{u}}\boldsymbol{\Lambda}_\zeta(r)\bar{\mathbf{K}}_{\mathrm{u}}\tilde{\zeta} \leq 0.$$

Since $\bar{\mathbf{H}}_{\mathrm{d}}$ has strict local minimum in $(r_{\mathrm{d}}, 0, 0)$, (following the proof of Proposition 5.2) we conclude that the closed-loop (6.38) is stable in $(r_{\mathrm{d}}, 0, 0)$. Asymptotic stability can be demonstrated whenever the equilibrium is also an isolated point of the largest invariant set of (6.38) contained in

$$\bar{\Omega}_{\mathrm{of}} = \left\{ (r,\rho,\tilde{\zeta}) \in \mathcal{X}_{\mathrm{c}} \times \mathbb{R}^{n_\gamma} \mid \frac{\partial \mathbf{H}_{\mathrm{d}}}{\partial \rho}\mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r)\frac{\partial^\top \mathbf{H}_{\mathrm{d}}}{\partial \rho} + \tilde{\zeta}^\top \bar{\mathbf{K}}_{\mathrm{u}}\boldsymbol{\Lambda}_\zeta(r)\bar{\mathbf{K}}_{\mathrm{u}}\tilde{\zeta} = 0 \right\}.$$

Let $(r(t), \rho(t), \zeta(t))$ be a solution that belongs to such a set, then

$$\tilde{\zeta}(t) \equiv 0 \quad \Longrightarrow \quad \bar{u}_{\mathrm{x}}(t) \equiv 0,\ \dot{\tilde{\zeta}}(t) \equiv 0 \quad \Longrightarrow \quad \dot{\gamma}(t) \equiv 0,$$

that is, the largest invariant set of (6.38) contained in $\bar{\Omega}_{\mathrm{of}}$ is equal to the one of (5.15) contained in $\Omega_{\mathrm{of}}$ with $\boldsymbol{\Gamma}_2(r) = -\mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r)$, and the proof is complete. □

*Remark* 6.3. Let $\mathbf{M}$ and $\mathbf{M}_{\mathrm{d}}$ be constant. If

$$\operatorname{Colsp} \mathbf{B}_{\mathrm{d}}(r) = \operatorname{Colsp} \bar{\mathbf{B}}(r) \qquad \text{or equivalently} \qquad 0 = \bar{\mathbf{B}}_\perp(r)\mathbf{B}_{\mathrm{d}}(r), \tag{6.39}$$

then $\mathbf{Z}_{\mathrm{d}}(r)\bar{\mathbf{B}}(r) = 0$ and condition (6.34b) holds with $\mathbf{K}_w(r,\rho) = 0$.

## 6.3 From Implicit to Explicit Representation

Eliminating the constraint forces and kinematic constraints from the Hamiltonian equations of motion, i.e., constructing an equivalent mechanical system in explicit representation, has been studied in [3, 78, 178] for nonholonomic systems and in [72] for the holonomic situation. However, these approaches are inadequate for implicit systems that

- do not satisfy the Lagrange-d'Alembert principle (workless constraint forces),
- possess holonomic and nonholonomic constraints simultaneously, and
- have the closed-loop form (5.15).

In this section, we overcome the previous restrictions by introducing a more general framework to reduce implicit systems to their explicit representation. This reduction is used in Chapter 7 to compare the novel implicit controller with other authors' results in the explicit framework. Before introducing our main result in Proposition 6.7, let us consider the system

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{J}_\mathrm{e}(r) \\ -\mathbf{J}_\mathrm{e}^\top(r) & \mathbf{\Gamma}_\mathrm{e}(r,\rho) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}_\mathrm{e}}{\partial r} \\ \frac{\partial^\top \mathbf{H}_\mathrm{e}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{\mathbf{B}}_\mathrm{e}(r) \end{bmatrix} \lambda_\mathrm{e} + \begin{bmatrix} 0 \\ \mathbf{G}_\mathrm{e}(r) \end{bmatrix} u, \tag{6.40a}$$
$$\mathbf{H}_\mathrm{e}(r,\rho) = \tfrac{1}{2}\rho^\top \mathbf{M}_\mathrm{e}^{-1}(r)\rho + \mathbf{V}_\mathrm{e}(r)$$

subject to smooth holonomic constraints

$$0 = \Phi(r) \tag{6.40b}$$

and smooth momentum constraints (nonholonomic constraints and the time derivative of the holonomic constraints)

$$0 = \mathbf{B}_\mathrm{e}^\top(r)\frac{\partial^\top \mathbf{H}_\mathrm{e}}{\partial \rho}. \tag{6.40c}$$

Here, $r \in \mathcal{R} \subset \mathbb{R}^{n_r}$ are coordinates, $\rho \in \mathbb{R}^{n_r}$ are the momenta, $\mathcal{R}$ is an open subset of $\mathbb{R}^{n_r}$, $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$ is the input, $\mathbf{G}_\mathrm{e} : \mathcal{R} \to \mathbb{R}^{n_r \times n_u}$ is the input matrix, $\mathbf{M} : \mathcal{R} \to \mathbb{R}^{n_r \times n_r}_{n_r}$ is the nonsingular and symmetric inertia matrix, $\mathbf{V}_\mathrm{e} : \mathcal{R} \to \mathbb{R}$ is the potential energy, $\mathbf{J}_\mathrm{e} : \mathcal{R} \to \mathbb{R}^{n_r \times n_r}$, $\mathbf{\Gamma}_\mathrm{e} : \mathcal{R} \times \mathbb{R}^{n_r} \to \mathbb{R}^{n_r \times n_r}$, and $\bar{\mathbf{B}}_\mathrm{e}(r)\lambda_\mathrm{e}$ are the constraint forces with $\bar{\mathbf{B}}_\mathrm{e} : \mathcal{R} \to \mathbb{R}^{n_r \times n_\lambda}$ and implicit variables $\lambda_\mathrm{e} \in \mathbb{R}^{n_\lambda}$. For the constraints (6.40b)–(6.40c) with $\Phi : \mathcal{R} \to \mathbb{R}^{n_\Phi}$ and $\mathbf{B}_\mathrm{e} : \mathcal{R} \to \mathbb{R}^{n_r \times n_\lambda}$, we will assume that

$$\Delta_\mathrm{e}(r) := \mathbf{B}_\mathrm{e}^\top(r)\mathbf{M}_\mathrm{e}^{-1}(r)\bar{\mathbf{B}}_\mathrm{e}(r)$$

is nonsingular for all $r \in \mathcal{R}_\Phi := \{r \in \mathcal{R} \mid 0 = \Phi(r)\}$ and that the initial conditions are consistent, i.e.,

$$(r(t_0), \rho(t_0)) \in \mathcal{X}_\mathrm{c} := \left\{ (r,\rho) \in \mathcal{R} \times \mathbb{R}^{n_r} \;\middle|\; 0 = \mathbf{B}_\mathrm{e}^\top(r)\tfrac{\partial^\top \mathbf{H}_\mathrm{e}}{\partial \rho},\ 0 = \Phi(r) \right\}.$$

**Proposition 6.7** (Implicit reduction)**.** *Consider the implicit system* (6.40)*. Let* $(\mathcal{N}, \xi^{-1})$ *be a coordinate chart on* $\mathcal{R}_\Phi$ *with local coordinates* $q \in \xi^{-1}(\mathcal{N}) \subset \mathbb{R}^{n_r - n_\Phi}$*, i.e.,* $r = \xi(q)$*. Let* $T : \xi^{-1}(\mathcal{N}) \to \mathbb{R}^{(n_r - n_\lambda) \times n_r}$ *be a* $\mathcal{C}^1$ *full-rank left annihilator of* $\mathbf{B}_\mathrm{e} \circ \xi$*. Then, for all* $r \in \mathcal{N}$*, system* (6.40) *with* $s = T(q)\rho \in \mathbb{R}^{n_r - n_\lambda}$ *can be reduced to*

$$\begin{bmatrix} \dot{q} \\ \dot{s} \end{bmatrix} = \begin{bmatrix} 0 & J_\mathrm{e}(q) \\ -J_\mathrm{e}^\top(q) & \Gamma_\mathrm{e}(q,s) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top H_\mathrm{e}}{\partial q} \\ \frac{\partial^\top H_\mathrm{e}}{\partial s} \end{bmatrix} + \begin{bmatrix} 0 \\ T(q)f_\mathrm{e}(\xi(q), L(q)s) \end{bmatrix} + \begin{bmatrix} 0 \\ G_\mathrm{e}(q) \end{bmatrix} u \tag{6.41}$$
$$H_\mathrm{e}(q,s) = \frac{1}{2}s^\top \Big(T(q)\mathbf{M}_\mathrm{e}(\xi(q))T^\top(q)\Big)^{-1} s + \mathbf{V}_\mathrm{e}(\xi(q)) = \mathbf{H}_\mathrm{e}(\xi(q), L(q)s),$$
$$G_\mathrm{e}(q) = T(q)\mathbf{Z}_\mathrm{e}(\xi(q))\mathbf{G}_\mathrm{e}(\xi(q)), \qquad \mathbf{Z}_\mathrm{e}(r) = I_{n_r} - \bar{\mathbf{B}}_\mathrm{e}(r)\Delta_\mathrm{e}^{-1}(r)\mathbf{B}_\mathrm{e}^\top(r)\mathbf{M}_\mathrm{e}^{-1}(r),$$
$$J_\mathrm{e}(q) = \left(\frac{\partial \xi}{\partial q}\right)^{\mathrm{g}} \mathbf{J}_\mathrm{e}(\xi(q))T^\top(q), \qquad L(q) = \mathbf{M}_\mathrm{e}(\xi(q))T^\top(q)\Big(T(q)\mathbf{M}_\mathrm{e}(\xi(q))T^\top(q)\Big)^{-1},$$

$$\Gamma_{\mathrm{e}}(q,s) = \left(\frac{\partial T\rho}{\partial q} J_{\mathrm{e}}(q) - J_{\mathrm{e}}^{\top}(q)\frac{\partial^{\top} T\rho}{\partial q}\right)\bigg|_{\rho=L(q)s} + T(q)\mathbf{\Gamma}_{\mathrm{e}}(\xi(q), L(q)s)T^{\top}(q),$$
$$f_{\mathrm{e}}(r,\rho) = \bar{\mathbf{B}}_{\mathrm{e}}(r)\Delta_{\mathrm{e}}^{-1}(r)\mathbf{B}_{\mathrm{e}}^{\top}(r)\mathbf{M}_{\mathrm{e}}^{-1}(r)\left(\mathbf{J}_{\mathrm{e}}^{\top}(r)\frac{\partial^{\top}\mathbf{H}_{\mathrm{e}}}{\partial r} - \mathbf{\Gamma}_{\mathrm{e}}(r,\rho)\frac{\partial^{\top}\mathbf{H}_{\mathrm{e}}}{\partial \rho}\right)$$
$$- \bar{\mathbf{B}}_{\mathrm{e}}(r)\Delta_{\mathrm{e}}^{-1}(r)\frac{\partial \mathbf{B}_{\mathrm{e}}^{\top}\mathbf{M}_{\mathrm{e}}^{-1}\rho}{\partial r}\mathbf{J}_{\mathrm{e}}(r)\frac{\partial^{\top}\mathbf{H}_{\mathrm{e}}}{\partial \rho}.$$

**Proof.** Let $T_{\mathrm{x}} : \xi^{-1}(\mathcal{N}) \to \mathbb{R}^{n_\lambda \times n_r}$ and $D : \xi^{-1}(\mathcal{N}) \to \mathbb{R}^{n_r \times n_\Phi}$ be $\mathcal{C}^1$ arbitrary functions such that $\begin{bmatrix} T^{\top}(q) & T_{\mathrm{x}}^{\top}(q)\end{bmatrix}$ and $\begin{bmatrix} \frac{\partial \xi}{\partial q} & D(q)\end{bmatrix}$ are nonsingular for all $q \in \xi^{-1}(\mathcal{N})$. Consider the change of variables

$$r = \bar{\xi}(\bar{q}) := \xi(q) + D(q)q_{\mathrm{x}}, \qquad \bar{s} = \bar{T}(q)\rho := \begin{bmatrix} T(q) \\ T_{\mathrm{x}}(q)\end{bmatrix}\rho, \tag{6.42}$$

with $\bar{s} = \mathrm{vec}(s, s_{\mathrm{x}})$, $\bar{q} = \mathrm{vec}(q, q_{\mathrm{x}})$, $s_{\mathrm{x}} \in \mathbb{R}^{n_\lambda}$ and $q_{\mathrm{x}} \in \mathbb{R}^{n_\Phi}$. By definition, the map $(\bar{q}, \bar{s}) \mapsto (\bar{\xi}(\bar{q}), \bar{T}^{-1}(q)\bar{s})$ is a $\mathcal{C}^1$ diffeomorphism in a neighborhood of $q_{\mathrm{x}} = 0$ and for all $q \in \xi^{-1}(\mathcal{N})$ and $\bar{s} \in \mathbb{R}^{n_r}$, see [137, Thm. 2.5.2]. Write

$$\widetilde{H}_{\mathrm{e}}(\bar{q}, \bar{s}) := \mathbf{H}_{\mathrm{e}}(\bar{\xi}(\bar{q}), \bar{T}^{-1}(q)\bar{s}).$$

Application of the chain rule on $\widetilde{H}_{\mathrm{e}}$ gives

$$\frac{\partial \widetilde{H}_{\mathrm{e}}}{\partial \bar{q}} = \left(\frac{\partial \mathbf{H}_{\mathrm{e}}}{\partial r}\frac{\partial \bar{\xi}}{\partial \bar{q}} + \frac{\partial \mathbf{H}_{\mathrm{e}}}{\partial \rho}\frac{\partial \bar{T}^{-1}\bar{s}}{\partial \bar{q}}\right)\bigg|_{\substack{r=\bar{\xi}(\bar{q})\\ \rho=\bar{T}^{-1}(q)\bar{s}}}, \qquad \frac{\partial \widetilde{H}_{\mathrm{e}}}{\partial \bar{s}} = \frac{\partial \mathbf{H}_{\mathrm{e}}}{\partial \rho}\bigg|_{\substack{r=\bar{\xi}(\bar{q})\\ \rho=\bar{T}^{-1}(q)\bar{s}}}\bar{T}^{-1}(q),$$

or equivalently

$$\frac{\partial^{\top}\mathbf{H}_{\mathrm{e}}}{\partial r}\bigg|_{\substack{r=\bar{\xi}(\bar{q})\\ \rho=\bar{T}^{-1}(q)\bar{s}}} = \left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-\top}\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{q}} + \left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-\top}\frac{\partial^{\top}\bar{T}\rho}{\partial \bar{q}}\bigg|_{\rho=\bar{T}^{-1}(q)\bar{s}}\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{s}},$$
$$\frac{\partial^{\top}\mathbf{H}_{\mathrm{e}}}{\partial \rho}\bigg|_{\substack{r=\bar{\xi}(\bar{q})\\ \rho=\bar{T}^{-1}(q)\bar{s}}} = \bar{T}^{\top}(q)\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{s}},$$

where it was used $\frac{\partial \bar{T}}{\partial q_i}\bar{T}^{-1}(q) + \bar{T}(q)\frac{\partial \bar{T}^{-1}}{\partial q_i} = 0$ on $\frac{\partial \bar{T}^{-1}\bar{s}}{\partial \bar{q}}$. Since $\Delta_{\mathrm{e}}$ is nonsingular, $\lambda_{\mathrm{e}}$ is uniquely defined and every solution $(r(\cdot), \rho(\cdot))$ with consistent initial conditions will remain in $\mathcal{X}_{\mathrm{c}}$. Now, system (6.40) with $\lambda_{\mathrm{e}}$ calculated from the hidden constraints is an ODE on the manifold $\mathcal{X}_{\mathrm{c}}$, and employing the change of coordinates (6.42) on this ODE results in

$$\dot{\bar{q}} = \left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-1}\dot{r} = \left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-1}\mathbf{J}_{\mathrm{e}}(\bar{\xi}(\bar{q}))\frac{\partial^{\top}\mathbf{H}_{\mathrm{e}}}{\partial \rho}\bigg|_{\substack{r=\bar{\xi}(\bar{q})\\ \rho=\bar{T}^{-1}(q)\bar{s}}} = \left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-1}\mathbf{J}_{\mathrm{e}}(\bar{\xi}(\bar{q}))\bar{T}^{\top}(q)\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{s}}, \tag{6.43a}$$

$$
\begin{aligned}
\dot{\bar{s}} &= \frac{\partial \bar{T}\rho}{\partial \bar{q}}\dot{\bar{q}} + \bar{T}(q)\dot{\rho} \\
&= \left.\frac{\partial \bar{T}\rho}{\partial \bar{q}}\right|_{\rho=\bar{T}^{-1}(q)\bar{s}} \left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-1} \mathbf{J}_{\mathrm{e}}(\bar{\xi}(\bar{q}))\bar{T}^{\top}(q)\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{s}} - \bar{T}(q)\left.\left(\mathbf{J}_{\mathrm{e}}^{\top}(r)\frac{\partial^{\top}\mathbf{H}_{\mathrm{e}}}{\partial r}\right)\right|_{\substack{r=\bar{\xi}(\bar{q})\\ \rho=\bar{T}^{-1}(q)\bar{s}}} \\
&\quad + \bar{T}(q)\left.\left(\mathbf{\Gamma}_{\mathrm{e}}(r,\rho)\frac{\partial^{\top}\mathbf{H}_{\mathrm{e}}}{\partial \rho}\right)\right|_{\substack{r=\bar{\xi}(\bar{q})\\ \rho=\bar{T}^{-1}(q)\bar{s}}} + \bar{T}(q)f_{\mathrm{e}}(\bar{\xi}(\bar{q}),\bar{T}^{-1}(q)\bar{s}) \\
&\quad + \bar{T}(q)\mathbf{Z}_{\mathrm{e}}(\bar{\xi}(\bar{q}))\mathbf{G}_{\mathrm{e}}(\bar{\xi}(\bar{q}))u \\
&= \left.\frac{\partial \bar{T}\rho}{\partial \bar{q}}\right|_{\rho=\bar{T}^{-1}(q)\bar{s}} \left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-1} \mathbf{J}_{\mathrm{e}}(\bar{\xi}(\bar{q}))\bar{T}^{\top}(q)\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{s}} - \bar{T}(q)\mathbf{J}_{\mathrm{e}}^{\top}(\bar{\xi}(\bar{q}))\left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-\top}\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{q}} \\
&\quad - \bar{T}(q)\mathbf{J}_{\mathrm{e}}^{\top}(\bar{\xi}(\bar{q}))\left(\frac{\partial \bar{\xi}}{\partial \bar{q}}\right)^{-\top}\left.\frac{\partial^{\top}\bar{T}\rho}{\partial \bar{q}}\right|_{\rho=\bar{T}^{-1}(q)\bar{s}}\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{s}} + \bar{T}(q)f_{\mathrm{e}}(\bar{\xi}(\bar{q}),\bar{T}^{-1}(q)\bar{s}) \qquad (6.43b) \\
&\quad + \bar{T}(q)\mathbf{\Gamma}_{\mathrm{e}}(\bar{\xi}(\bar{q}),\bar{T}^{-1}(q)\bar{s})\bar{T}^{\top}(q)\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial \bar{s}} + \bar{T}(q)\mathbf{Z}_{\mathrm{e}}(\bar{\xi}(\bar{q}))\mathbf{G}_{\mathrm{e}}(\bar{\xi}(\bar{q}))u.
\end{aligned}
$$

Given that $(\mathcal{N},\xi^{-1})$ is a coordinate chart on $\mathcal{R}_\Phi$ and $\bar{\xi}$ is a diffeomorphism between open subsets of $\mathbb{R}^{n_r}$, then $q_{\mathrm{x}}(\cdot)$ must be zero for every solution $r(\cdot)\in\mathcal{R}_\Phi$. Then, by using $T(q)\mathbf{B}_{\mathrm{e}}(\xi(q))=0$ and the previous change of coordinates we have (6.40c) $\iff 0=\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial s_{\mathrm{x}}}$. Write $T_{\mathrm{x}}^{\top}(q) := \mathbf{M}_{\mathrm{e}}^{-1}(\xi(q))\mathbf{B}_{\mathrm{e}}(\xi(q))$, which is consistent with the nonsingularity of $\bar{T}$ (see Lemma 5.2 with nonsingular $\Delta_{\mathrm{e}}$ and the annihilating feature of $T$). Then, every solution $\rho(\cdot)\in\mathcal{X}_{\mathrm{c}}$ implies that $s_{\mathrm{x}}(\cdot)$ is zero. Let, without loss of generality,

$$
\mathbf{J}_{\mathrm{e}}(\xi(q))T^{\top}(q) = \frac{\partial \xi}{\partial q}J_{\mathrm{e}}(q) + D(q)J_{\mathrm{x}}(q) \qquad (6.44)
$$

for some functions $J_{\mathrm{e}}$ and $J_x$. From the constraints (6.40b)–(6.40c), $\left(\frac{\partial\xi}{\partial q}\right)_{\perp}$ can be selected as $\left.\frac{\partial\Phi}{\partial r}\right|_{r=\xi(q)}$ while $\operatorname{Rowsp}\left(\frac{\partial\Phi}{\partial r}\mathbf{J}_{\mathrm{e}}(r)\right)\subset\operatorname{Rowsp}\mathbf{B}_{\mathrm{e}}^{\top}(r)$. Hence, by Lemmas 3.2 and 3.4, $T(q)\mathbf{B}_{\mathrm{e}}(\xi(q))=0$ and the nonsingularity of $\begin{bmatrix}\frac{\partial\xi}{\partial q} & D(q)\end{bmatrix}$ we deduce that $J_{\mathrm{x}}(q)=0$, $J_{\mathrm{e}}(q)=\left(\frac{\partial\xi}{\partial q}\right)^{\mathrm{g}}\mathbf{J}_{\mathrm{e}}(\xi(q))T^{\top}(q)$ and that (6.44) can be expressed as

$$
\left.\left(\frac{\partial\bar{\xi}}{\partial\bar{q}}\right)^{-1}\right|_{q_{\mathrm{x}}=0}\mathbf{J}_{\mathrm{e}}(\xi(q))\bar{T}^{\top}(q) = \begin{bmatrix} J_{\mathrm{e}}(q) & \star \\ 0 & \star \end{bmatrix}, \qquad (6.45)
$$

where $\star$ denotes unspecified elements. At this point we can use $q_{\mathrm{x}}=0$, $s_{\mathrm{x}}=0$, $0=\frac{\partial^{\top}\widetilde{H}_{\mathrm{e}}}{\partial s_{\mathrm{x}}}$ and (6.45) to rewrite (6.43) as

$$
\dot{q} = J_{\mathrm{e}}(q)\frac{\partial\widetilde{H}_{\mathrm{e}}}{\partial s}, \qquad (6.46a)
$$

$$\dot{s} = \left.\frac{\partial T\rho}{\partial q}\right|_{\rho=L(q)s} J_{\mathrm{e}}(q)\frac{\partial \widetilde{H}_{\mathrm{e}}}{\partial s} - J_{\mathrm{e}}^{\top}(q)\frac{\partial^{\top} \widetilde{H}_{\mathrm{e}}}{\partial q} - J_{\mathrm{e}}^{\top}(q)\left.\frac{\partial^{\top} T\rho}{\partial q}\right|_{\rho=L(q)s}\frac{\partial \widetilde{H}_{\mathrm{e}}}{\partial s} \\ + T(q)\boldsymbol{\Gamma}_{\mathrm{e}}(\xi(q), L(q)s)T^{\top}(q)\frac{\partial \widetilde{H}_{\mathrm{e}}}{\partial s} + T(q)f_{\mathrm{e}}(\xi(q), L(q)s) + G_{\mathrm{e}}(q)u, \tag{6.46b}$$

where $L(q)s$ is calculated from Lemma 3.4 with $T$ being the full-rank left annihilator of $\mathbf{B}_{\mathrm{e}} \circ \xi$. The proof is complete, i.e., we show that (6.41) is equivalent to (6.46) by observing that

$$\left.\frac{\partial \widetilde{H}_{\mathrm{e}}}{\partial q}\right|_{\substack{q_{\mathrm{x}}=0\\ s_{\mathrm{x}}=0}} = \left.\left(\frac{\partial \mathbf{H}_{\mathrm{e}}}{\partial r}\frac{\partial \xi}{\partial q} + \frac{\partial \mathbf{H}_{\mathrm{e}}}{\partial \rho}\frac{\partial \bar{T}^{-1}\bar{s}}{\partial q}\right)\right|_{\substack{r=\xi(q)\\ \rho=\bar{T}^{-1}(q)\bar{s}\\ s_{\mathrm{x}}=0}} = \frac{\partial H_{\mathrm{e}}}{\partial q}, \qquad \left.\frac{\partial \widetilde{H}_{\mathrm{e}}}{\partial s}\right|_{\substack{q_{\mathrm{x}}=0\\ s_{\mathrm{x}}=0}} = \frac{\partial H_{\mathrm{e}}}{\partial s}. \qquad \square$$

Proposition 6.7 removes the kinematic constraints of the system (6.40), reducing it from an implicit model of $2n_r$ states to an explicit one whose number of states is given by the dimension of the constrained manifold $\mathcal{X}_{\mathrm{c}}$, namely $2n_r - n_\lambda - n_\Phi$. The new states $q$ represent generalized coordinates while the states $s$ are a projection of the conjugate momenta. We remark that the reduced system is non-unique ($\xi$, $T$ and $\left(\frac{\partial \xi}{\partial q}\right)^{\mathrm{g}}$ are user-defined), but all the reductions are locally equivalent because the system dynamics is restricted to a manifold, and the coordinates we choose to represent it are immaterial (geometric perspective). Specializing Proposition 6.7 to implicit holonomic systems and their corresponding closed-loops with IDA-PBC leads to the following two corollaries.

**Corollary 6.2.** *Let* (5.11) *be an holonomic system verifying Assumptions* 5.6 *and* 5.7*. Then, there exists a diffeomorphism $\xi : \mathcal{Q} \to \xi(\mathcal{Q})$, with $r = \xi(q)$ and $\mathcal{Q}$ being an open subset of $\mathbb{R}^{n_r - n_\Phi}$ such that* (5.11) *can be transformed into*

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I_{n_r-n_\Phi}\\ -I_{n_r-n_\Phi} & -\frac{\partial^{\top}\xi}{\partial q}\mathbf{R}(\xi(q))\frac{\partial \xi}{\partial q}\end{bmatrix}\begin{bmatrix}\frac{\partial^{\top} H}{\partial q}\\ \frac{\partial^{\top} H}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ \frac{\partial^{\top}\xi}{\partial q}f_{\mathrm{e}}(\xi(q), L(q)p)\end{bmatrix} + \begin{bmatrix}0\\ G(q)\end{bmatrix}u, \tag{6.47}$$

*where $H(q,p) = \frac{1}{2}p^{\top}M^{-1}(q)p + \mathbf{V}(\xi(q))$, $M(q) = \frac{\partial^{\top}\xi}{\partial q}\mathbf{M}(\xi(q))\frac{\partial \xi}{\partial q}$, $p = \frac{\partial^{\top}\xi}{\partial q}\rho$, $G(q) = \frac{\partial^{\top}\xi}{\partial q}\mathbf{Z}(\xi(q))\mathbf{G}(\xi(q))$, $L(q) = \mathbf{M}(\xi(q))\frac{\partial \xi}{\partial q}M^{-1}(q)$ and*

$$f_{\mathrm{e}}(r,\rho) = \bar{\mathbf{B}}(r)\Delta^{-1}(r)\left(\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\left(\frac{\partial^{\top}\mathbf{H}}{\partial r} - \mathbf{R}(r)\frac{\partial^{\top}\mathbf{H}}{\partial \rho}\right) - \frac{\partial \mathbf{B}^{\top}\mathbf{M}^{-1}\rho}{\partial r}\frac{\partial^{\top}\mathbf{H}}{\partial \rho}\right).$$

**Proof.** From Assumption 5.6, $\mathcal{R}_\Phi$ is a regular and smooth manifold of dimension $n_q$. Therefore, we may choose $(\mathcal{N}, \xi^{-1})$ as a coordinate chart on $\mathcal{R}_\Phi$ with local coordinates $q$, such that $\mathcal{Q} = \xi^{-1}(\mathcal{N})$ and $r = \xi(q)$. Since $\frac{\partial \xi}{\partial q}$ is a full-rank left annihilator of $\mathbf{B} \circ \xi$, we use Proposition 6.7 with $T^{\top}(q) = \frac{\partial \xi}{\partial q}$, to obtain a reduced system of the form of (6.41), where $J_{\mathrm{e}} = I_{n_r - n_\Phi}$ for any selection of $\left(\frac{\partial \xi}{\partial q}\right)^{\mathrm{g}}$. The equivalence with (6.47) is obtained from

$$\begin{aligned}
\Gamma_{\mathrm{e}}(q) &= -T(q)\mathbf{R}(\xi(q))T^\top(q) + \left(\frac{\partial T\rho}{\partial q} - \frac{\partial^\top T\rho}{\partial q}\right)\Bigg|_{\rho = L(q)s} \\
&= -\frac{\partial^\top \xi}{\partial q}\mathbf{R}(\xi(q))\frac{\partial \xi}{\partial q} + \sum_{i=1}^{n_r}\left(\frac{\partial^2 \xi_i}{\partial q^2} - \left(\frac{\partial^2 \xi_i}{\partial q^2}\right)^\top\right)\mathrm{row}_i(L(q)p) \\
&= -\frac{\partial^\top \xi}{\partial q}\mathbf{R}(\xi(q))\frac{\partial \xi}{\partial q},
\end{aligned}$$

where $\xi_i$ denotes the $i$-th element of $\xi$. □

**Corollary 6.3.** *Consider a holonomic system of the form* (5.11)*. Suppose the conditions of Proposition 5.2 are satisfied with* $\mathbf{J}(r) = \mathbf{M}^{-1}(r)\mathbf{M}_{\mathrm{d}}(r)$*, meaning that* (5.15) *is the closed-loop of* (5.11) *with feedback* $u = \mathbf{u}_{\mathrm{ida}}(r, \rho)$*. Then, there exists a diffeomorphism* $\xi : \mathcal{Q} \to \xi(\mathcal{Q})$*, with* $r = \xi(q)$ *and* $\mathcal{Q}$ *being an open subset of* $\mathbb{R}^{n_r - n_\Phi}$*, that locally transforms* (5.15) *into* (5.2)*, where*

$$\begin{gathered}
J(q) = M^{-1}(q)M_{\mathrm{d}}(q), \quad T^\top(q) = \mathbf{J}^{-1}(\xi(q))\frac{\partial \xi}{\partial q}J(q), \quad V_{\mathrm{d}}(q) = \mathbf{V}_{\mathrm{d}}(\xi(q)), \\
M(q)M_{\mathrm{d}}^{-1}(q)M(q) = \frac{\partial^\top \xi}{\partial q}\mathbf{M}(\xi(q))\mathbf{M}_{\mathrm{d}}^{-1}(\xi(q))\mathbf{M}(\xi(q))\frac{\partial \xi}{\partial q}, \\
\Gamma_1(q,p) = \left(\frac{\partial T\rho}{\partial q}J(q) - J^\top(q)\frac{\partial^\top T\rho}{\partial q}\right)\Bigg|_{\rho = L(q)p} + T(q)\mathbf{\Gamma}_1(\xi(q), L(q)p)T^\top(q), \\
\Gamma_2(q) = T(q)\mathbf{\Gamma}_2(\xi(q))T^\top(q),
\end{gathered}$$

*and* $M$*,* $L$ *are as defined in Corollary 6.2.*

Corollary 6.2 states that holonomic systems in the representation (5.11) can be reduced into a system of the form (6.47), which is port-Hamiltonian if

$$\frac{\partial^\top \xi}{\partial q}\bar{\mathbf{B}}(\xi(q)) = 0 \quad \text{and} \quad \frac{\partial^\top \xi}{\partial q}\left(\mathbf{R}(\xi(q)) + \mathbf{R}^\top(\xi(q))\right)\frac{\partial \xi}{\partial q} \succeq 0 \qquad q \in \mathcal{Q}.$$

Observe that the result is consistent with the equivalence of Section 2.5 whenever $\bar{\mathbf{B}}(r) = \frac{\partial^\top \Phi}{\partial r}$. Corollary 6.3 establishes that any holonomic system in closed-loop with IDA-PBC can be reduced into (5.2), which is the closed-loop of IDA-PBC for explicit systems. Consequently, for this class, there is a locally equivalent IDA-PBC feedback $u = u_{\mathrm{ida}}(q,p)$ with $u_{\mathrm{ida}}$ as defined in (5.4). Furthermore, instead of analyzing the invariant set $\Omega_{\mathrm{inv}}$ in Algorithms 6.1 and 6.2 to demonstrate asymptotic stability, we can now employ the Lyapunov indirect method (Theorem 2.2) on (6.47) with $u = \mathbf{u}_{\mathrm{ida}}(\xi(q), L(q)p)$ or equivalently in the reduction of Corollary 6.3. Chapter 7 will use these concepts to assign an optimal local performance in the controllers designed from the implicit representation.

# Chapter 7

# Applications on Mechanical Systems with Holonomic Constrains

In Chapters 5 and 6, we develop the total energy shaping IDA-PBC for mechanical systems with kinematic constraints (holonomic and nonholonomic) and provide some solutions to the matching conditions of UMSs. We are able to achieve the asymptotic stabilization of holonomic systems, but we can only guarantee partial convergence (to the desired state) of nonholonomic ones [3]. This is true even in the fully-actuated case (see Example 6.2) and is a consequence of the Brockett's necessary condition [209]. In this chapter, with the aim of asymptotic stability, we validate our results on three underactuated benchmark examples with holonomic constraints: the portal crane (Section 7.1), the cart-pole (Section 7.2), and the PVTOL aircraft (Section 7.3). For better visibility, values presented in this chapter have been rounded to four decimals.

## 7.1 Portal Crane System

The portal (or overhead) crane, shown in Figure 7.1, is a classic example of an UMS [65, 201, 230]. It consists of a **bridge** sliding on **parallel runways**, a **trolley** to provide horizontal motion of a **winch** through the bridge, and a **payload** that hangs from the winch's (wire) **rope**. The bridge of mass $m_{\mathrm{b}}$ moves along the $x$-axis and is actuated by the force $\tau_x$ while the trolley of mass $m_{\mathrm{t}}$ does so in the $y$-axis and is actuated by the force $\tau_y$. The winch lifts the payload of mass $m_{\mathrm{p}}$ by exerting a force $\tau_l$ on the rope of length $l > 0$. Figure 7.2 depicts the portal crane schematic diagram, where $(x_{\mathrm{t}}, y_{\mathrm{t}})$ denotes the trolley position (on the plane $z = 0$), which is also one end of the rope, $(x_{\mathrm{p}}, y_{\mathrm{p}}, z_{\mathrm{p}})$ is the payload position relative to $(x_{\mathrm{t}}, y_{\mathrm{t}})$, and $\alpha$ and $\beta$ are the angles of the wire rope with respect to the $x$-axis and $\mathrm{vec}(0, \cos\alpha, \sin\alpha)$, respectively.

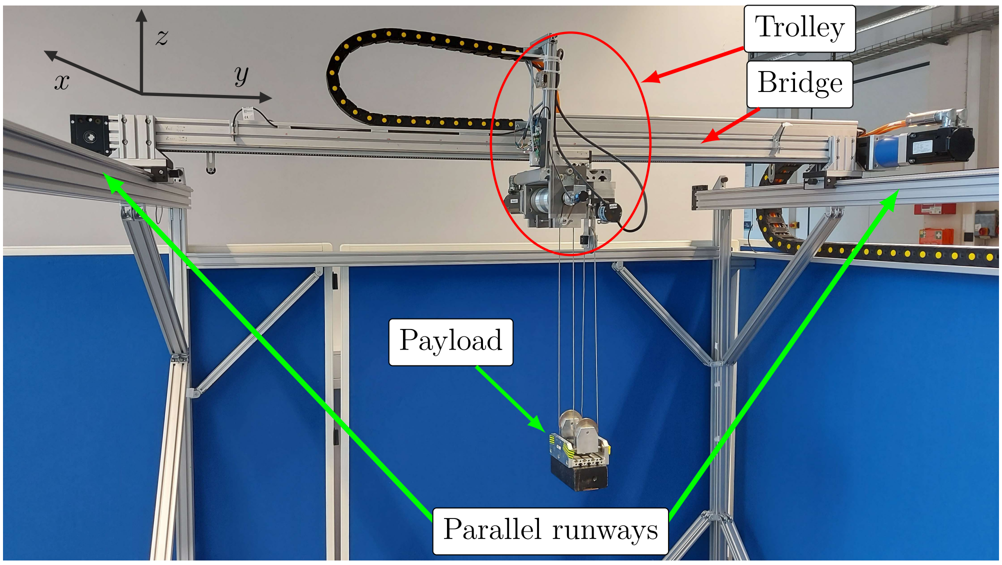


**Figure 7.1.** – Portal crane of the Control Engineering Group at TU Ilmenau.

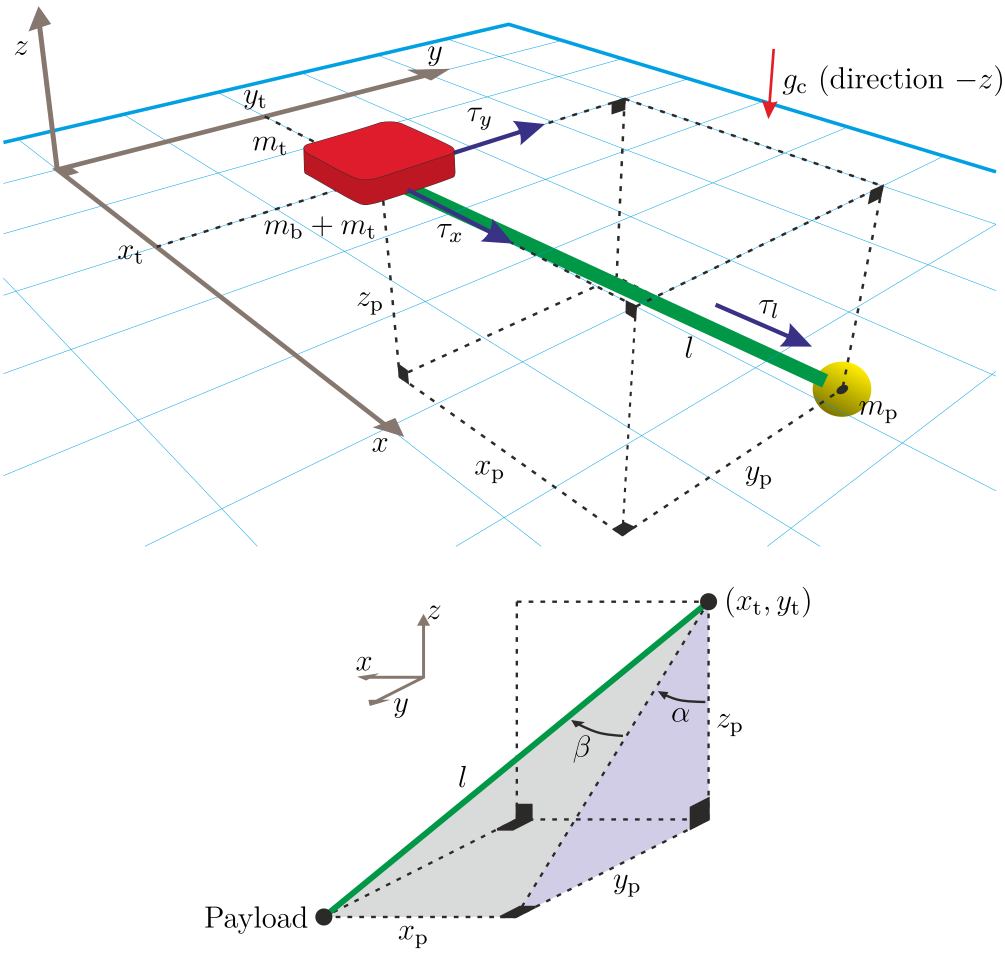


**Figure 7.2.** – Schematic diagram of the portal crane.

Our goal is to design implicit IDA-PBC controllers that asymptotically stabilize the crane in the desired payload's position. For this task we device a 4-DoF model in implicit representation under Assumption 7.1 and test Algorithms 6.1 and 6.2 together with Proposition 6.6 in the synthesis of three controllers: two full-state feedback and a position feedback. We consider the 4-DoF model because it is commonly used in the PBC literature when given in the explicit representation [20, 231–233];[62] however, this model requires a fixed rope length $l$, which is a restrictive description of the crane. To avoid this limitation, we also devise a 5-DoF implicit model and synthesize a full-state feedback with Algorithm 6.2. We recall that in either case (with and without fixed rope length), two DoF remain underactuated. All (four) controllers are designed with a constant target inertia matrix (of the implicit representation), and the asymptotic stability is demonstrated with Lyapunov's indirect method on the reduced (explicit) systems, see Section 6.3. The results are compared in simulation and the best controller is implemented in the test-bench of Figure 7.1, which is located at the laboratory of the Control Engineering Group (Fachgebiet Regelungstechnik), Department of Computer Science and Automation, Technische Universität Ilmenau.

**Assumption 7.1.** *i) The rope's mass is negligible. ii) The payload is a point mass and the air friction is negligible. iii) The gravity of magnitude $g_{\mathrm{c}}$ points downwards (direction $-z$). iv) The initial conditions are consistent (Assumption 5.7 holds).*

The physical system is equipped with five encoders to measure $x_{\mathrm{t}}$, $y_{\mathrm{t}}$, $l$, $\alpha$ and $\beta$. In addition, it possesses three modes to operate its servomotors: **current, velocity and position tracking**. Similar to the cart-pole system of Section 4.4, we use an integrator in the input of the velocity tracking mode to have an approximate representation of the portal crane in Partial Feedback Linearization (PFL) with new input $u^* = \mathrm{vec}(\ddot{x}_{\mathrm{t}}^*, \ddot{y}_{\mathrm{t}}^*, \ddot{l}^*) \approx \mathrm{vec}(\ddot{x}_{\mathrm{t}}, \ddot{y}_{\mathrm{t}}, \ddot{l})$, see Figure 7.3. This scheme avoids the identification of masses, inertia matrices and others parameters on the implicit and explicit representations of the portal crane, see [102] for the case with fix length $l$. We remark that *the approaches of Chapters 5 and 6 do not require a system in PFL* (see e.g., [100, 101]), but we use such a representation to streamline design and implementation.

### 7.1.1 4-DoF Implicit Model

If the length of $l$ is constant, the system has four DoF and $\mathbb{R}^2 \times \mathbb{S}^2$ represents its configuration space. Therefore, we can choose the coordinates

$$r = \mathrm{vec}(x_{\mathrm{p}}, y_{\mathrm{p}}, z_{\mathrm{p}}, x_{\mathrm{t}}, y_{\mathrm{t}}) \in \mathcal{R} = \mathbb{R}^5$$

[62] The works [232, 233] actually employ the 2-DoF model (restriction to a plane) since two planar models (one for $x = 0$ and the other for $y = 0$) are a local approximation of the one with 4-DoF [102].

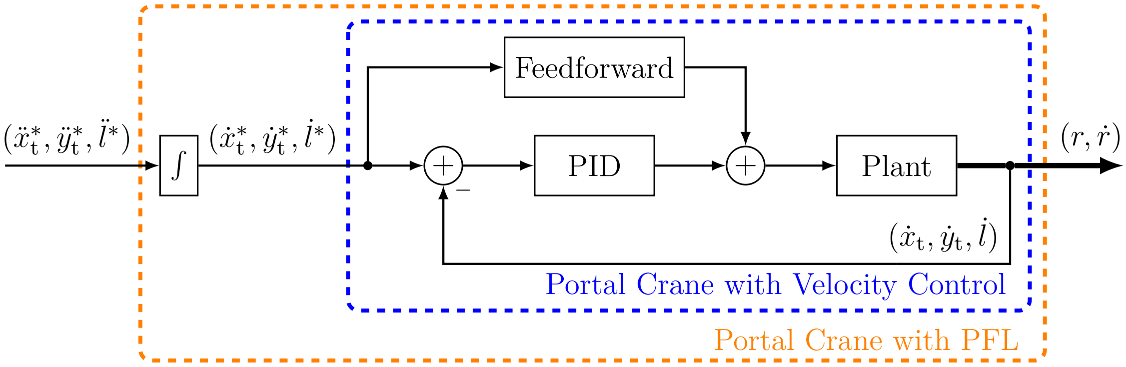


**Figure 7.3.** – Approximate portal crane with PFL.

satisfying the holonomic constraint

$$\Phi(r) := \frac{1}{2}\left(x_p^2 + y_p^2 + z_p^2 - l^2\right) = 0. \tag{7.1}$$

Then, by Assumption 7.1, the kinetic and potential energies are calculated as

$$\mathbf{E}_k(r, \dot{r}) = \frac{m_b + m_t}{2}\dot{x}_t^2 + \frac{m_t}{2}\dot{y}_t^2 + \frac{m_p}{2}(\dot{x}_t + \dot{x}_p)^2 + \frac{m_p}{2}(\dot{y}_t + \dot{y}_p)^2 + \frac{m_p}{2}\dot{z}_p^2, \tag{7.2a}$$

$$\hat{\mathbf{V}}(r) = g_c m_p z_p. \tag{7.2b}$$

Now, the Lagrange equations of motion (first kind) with Rayleigh dissipation $\mathbf{D}(r, \dot{r}) = \frac{1}{2}c_1\dot{x}_t^2 + \frac{1}{2}c_2\dot{y}_t^2 \geq 0$ (viscous friction in actuators) are

$$\hat{\mathbf{M}}\ddot{r} + \hat{\mathbf{R}}\dot{r} + \frac{\partial^\top \hat{\mathbf{V}}}{\partial r} = \hat{\mathbf{G}}\tau + \frac{\partial^\top \Phi}{\partial r}\hat{\lambda}, \tag{7.3}$$

where $\tau = \mathrm{vec}(\tau_x, \tau_y)$, $\hat{\mathbf{R}} = \mathrm{diag}(0, 0, 0, c_1, c_2)$,

$$\hat{\mathbf{M}} = \begin{bmatrix} m_p & 0 & 0 & m_p & 0 \\ 0 & m_p & 0 & 0 & m_p \\ 0 & 0 & m_p & 0 & 0 \\ m_p & 0 & 0 & m_p + m_t + m_b & 0 \\ 0 & m_p & 0 & 0 & m_p + m_b \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{G}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

### Partial Feedback Linearization

To derive the partially linearized model from (7.3), see [200, 201] for the unconstrained situation, we write (7.3) together with the hidden constraints—second derivative of (7.1) with respect to time—as

$$\begin{bmatrix} \hat{\mathbf{M}} & -\frac{\partial^\top \Phi}{\partial r} \\ \frac{\partial^\top \Phi}{\partial r} & 0 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \hat{\lambda} \end{bmatrix} + \begin{bmatrix} \frac{\partial^\top \hat{\mathbf{V}}}{\partial r} + \mathbf{R}\dot{r} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^\top \Phi}{\partial r}\right)\dot{r} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{G}} \\ 0 \end{bmatrix} \tau, \tag{7.4}$$

where $\begin{bmatrix} \hat{\mathbf{M}} & -\frac{\partial^\top \Phi}{\partial r} \\ \frac{\partial^\top \Phi}{\partial r} & 0 \end{bmatrix}$ is nonsingular for all $r \in \mathcal{R}_\Phi := \{r \in \mathcal{R} \mid \Phi(r) = 0\}$, meaning that $\hat{\lambda}$ has a unique solution (well-posed problem). Define the output $y := h(r)$ and functions

$$\Lambda(r) := \begin{bmatrix} \frac{\partial h}{\partial r} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{M}} & -\frac{\partial^\top \Phi}{\partial r} \\ \frac{\partial^\top \Phi}{\partial r} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\mathbf{G}} \\ 0 \end{bmatrix}, \qquad \bar{\Lambda}(r) := \begin{bmatrix} \begin{bmatrix} \frac{\partial h}{\partial r} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{M}} & -\frac{\partial^\top \Phi}{\partial r} \\ \frac{\partial^\top \Phi}{\partial r} & 0 \end{bmatrix}^{-1} \\ \begin{array}{c:c} \hat{\mathbf{G}}_\perp & 0 \\ \hdashline 0 & 1 \end{array} \end{bmatrix}$$

with $h : \mathcal{R}_\Phi \to \mathbb{R}^{n_u}$. Left multiplying (7.4) by $\bar{\Lambda}$, which is nonsingular if and only if so is $\Lambda$ (see Lemma 5.2), yields the equivalent expressions

$$\begin{aligned} \frac{\partial h}{\partial r}\ddot{r} + \begin{bmatrix} \frac{\partial h}{\partial r} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{M}} & -\frac{\partial^\top \Phi}{\partial r} \\ \frac{\partial^\top \Phi}{\partial r} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^\top \hat{\mathbf{V}}}{\partial r} + \mathbf{R}\dot{r} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^\top \Phi}{\partial r}\right)\dot{r} \end{bmatrix} &= \Lambda(r)\tau, \\ \hat{\mathbf{G}}_\perp \hat{\mathbf{M}}\ddot{r} + \hat{\mathbf{G}}_\perp \left(\frac{\partial^\top \hat{\mathbf{V}}}{\partial r} + \mathbf{R}\dot{r}\right) &= \hat{\mathbf{G}}_\perp \frac{\partial^\top \Phi}{\partial r}\hat{\lambda}. \end{aligned} \tag{7.5}$$

Consequently, if $\Lambda$ is invertible, the state feedback law

$$\tau = \Lambda^{-1}(r) \begin{bmatrix} \frac{\partial h}{\partial r} & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{M}} & -\frac{\partial^\top \Phi}{\partial r} \\ \frac{\partial^\top \Phi}{\partial r} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial^\top \hat{\mathbf{V}}}{\partial r} + \mathbf{R}\dot{r} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^\top \Phi}{\partial r}\right)\dot{r} \end{bmatrix} - \Lambda^{-1}(r)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h}{\partial r}\right)\dot{r} + \Lambda^{-1}(r)u \tag{7.6}$$

transforms (7.3) into (7.5) and

$$\ddot{y} = \frac{\partial h}{\partial r}\ddot{r} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h}{\partial r}\right)\dot{r} = u. \tag{7.7}$$

By setting $h(r) = \mathrm{vec}(x_\mathrm{t}, y_\mathrm{t})$ and $\hat{\mathbf{G}}_\perp = m_\mathrm{p}^{-1}\begin{bmatrix} I_3 & 0_{3\times 2} \end{bmatrix}$, we have

$$\det \Lambda(r) = \frac{1}{(m_\mathrm{t} + m_\mathrm{b})m_\mathrm{b}l^2 + m_\mathrm{p}m_\mathrm{t}x_\mathrm{p}^2 + m_\mathrm{p}(m_\mathrm{t} + m_\mathrm{b})y_\mathrm{p}^2} > 0 \qquad \forall r \in \mathcal{R}_\Phi$$

and the resulting equations of motion can now be expressed as

$$\mathbf{M}\ddot{r} + \frac{\partial^\top \mathbf{V}}{\partial r} = \frac{\partial^\top \Phi}{\partial r}\lambda + \mathbf{G}u$$

or equivalently as (see Section 2.5.3)

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & I_5 \\ -I_5 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}}{\partial r} \\ \frac{\partial^\top \mathbf{H}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial^\top \Phi}{\partial r} \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ \mathbf{G} \end{bmatrix} u, \tag{7.8}$$

where $\mathbf{H}(r,\rho) = \frac{1}{2}\rho^\top \mathbf{M}^{-1}\rho + \mathbf{V}(r)$, $\mathbf{M} = I_5$, $\mathbf{V} = g_c z_p$ and $\mathbf{G}^\top = \left[\begin{smallmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{smallmatrix}\right]$. We have the following observation regarding (7.8) with the holonomic constraint (7.1).

- The accelerations of $x_t$ and $y_t$ are the new inputs of the 4-DoF model with PFL, i.e., $u = \mathrm{vec}(\ddot{x}_t, \ddot{y}_t)$.
- The model is independent of the masses ($m_b$, $m_t$, $m_p$) and the frictions ($c_1$, $c_2$).
- The set $\mathcal{R}_\Phi := \{r \in \mathcal{R} \mid 0 = \Phi(r)\} \cong \mathbb{S}^2$ is a smooth manifold (see Example 2.4), and

$$\Delta(r) := x_p^2 + y_p^2 + z_p^2 = l^2 > 0 \qquad \forall r \in \mathcal{R}_\Phi,$$

  that is, Assumption 5.6 is verified.
- $\mathrm{rank}\,\mathbf{N}(r) = 3 > n_\lambda = 1$ for all $r \in \mathcal{R}_\Phi$ (Assumption 5.8 is verified), where

$$\mathbf{N}(r) := \begin{bmatrix} \mathbf{G} & \bar{\mathbf{B}}(r) \end{bmatrix} = \begin{bmatrix} \mathbf{G} & \frac{\partial^\top \Phi}{\partial r} \end{bmatrix} = \begin{bmatrix} -1 & 0 & x_p \\ 0 & -1 & y_p \\ 0 & 0 & z_p \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

- $\mathbf{M}$, $\mathbf{V}$, $\mathbf{B}$ and $\mathbf{N}$ are all polynomial.

### Reduction and Linearization

Let us consider system (7.1), (7.8) in closed-loop with $u = \mathbf{u}_{\mathrm{ida}}(r,\rho)$. Then, the reduction of Corollary 6.2 yields

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_4 \\ -I_4 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^\top H}{\partial q} \\ \frac{\partial^\top H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u_{\mathrm{ida}}(q,p), \tag{7.9}$$
$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q)$$

where $q = \mathrm{vec}(\beta, \alpha, x_t, y_t)$, $M(q) = \frac{\partial^\top \xi}{\partial q}\frac{\partial \xi}{\partial q} = \mathrm{diag}(l^2, l^2\cos^2\beta, 1, 1)$, $V(q) = \mathbf{V}(\xi(q)) = -g_c l \cos\beta\cos\alpha$, $p = \frac{\partial^\top \xi}{\partial q}\rho$, $G(q) = \frac{\partial^\top \xi}{\partial q}\mathbf{G}$, $L(q) = \frac{\partial \xi}{\partial q} M^{-1}(q)$, $u_{\mathrm{ida}}(q,p) = \mathbf{u}_{\mathrm{ida}}(\xi(q), L(q)p)$ and $r = \xi(q)$ follows from the geometric relation of Figure 7.2, i.e.,

$$x_p = l \sin\beta, \qquad y_p = l\cos\beta\sin\alpha, \qquad z_p = -l\cos\beta\cos\alpha. \tag{7.10}$$

Note that $\xi$ is a diffeomorphism for all $\alpha, \beta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, and the implicit inertia matrix $\mathbf{M}$ is constant while the explicit one, $M$, is not.

Linearizing the closed-loop (7.9) about the equilibrium $x_\mathrm{d} = \mathrm{vec}(q_\mathrm{d}, 0)$, with $q_\mathrm{d} = \xi^{-1}(r_\mathrm{d})$, yields

$$\dot{\tilde{x}} = \underbrace{\left( \begin{bmatrix} 0 & M^{-1}(q_\mathrm{d}) \\ -\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_\mathrm{d}} & 0 \end{bmatrix} + \left.\begin{bmatrix} 0 \\ \frac{\partial G u}{\partial x} \end{bmatrix}\right|_{\substack{x=x_\mathrm{d} \\ u=u_\mathrm{ida}(q_\mathrm{d},0)}} \right)}_{=:A_\mathrm{r}} \tilde{x} + \underbrace{\begin{bmatrix} 0_{4\times 2} \\ G(q_\mathrm{d}) \end{bmatrix}}_{=:B_\mathrm{r}} \underbrace{\left.\frac{\partial u_\mathrm{ida}}{\partial x}\right|_{x=x_\mathrm{d}}}_{=:K_\mathrm{r}} \tilde{x}, \tag{7.11}$$

where $\tilde{x} = x - x_\mathrm{d}$, $x = \mathrm{vec}(q, p)$.

### 7.1.2 4-DoF System: Heuristic Solution

We shall determine an IDA-PBC stabilizing controller for the 4-DoF portal crane (fixed length $l$) with PFL. Since the implicit model is polynomial and verifies Assumptions 5.6 to 5.8, we test the heuristic solution of Algorithm 6.1 whereby the controller synthesis is recast as an SOS program that we solve with SOSTOOLS and **SDPT3**.[63]

**Step 1:** We select

$$\mathbf{N}_\perp(r) = \begin{bmatrix} -z_\mathrm{p} & 0 & x_\mathrm{p} & -z_\mathrm{p} & 0 \\ 0 & -z_\mathrm{p} & y_\mathrm{p} & 0 & -z_\mathrm{p} \end{bmatrix}$$

and $r_\mathrm{d} = \mathrm{vec}(0, 0, -l, x_\mathrm{t}^\star, y_\mathrm{t}^\star) \in \mathcal{R}_a = \{r \in \mathcal{R} \mid 0 = \Phi(r),\, 0 = x_\mathrm{p},\, 0 = y_\mathrm{p}\}$, where $(x_\mathrm{t}^\star, y_\mathrm{t}^\star)$ is the desired trolley position.

**Step 2:** To have a low polynomial order on $\mathbf{V}_\mathrm{d}$, $\mathbf{M}_\mathrm{d}^{-1}$ and $\bar{\mathbf{\Gamma}}_1$, we choose $\mathbf{V}_\mathrm{d}$ as a quadratic function of $r - r_\mathrm{d}$, $\mathbf{M}_\mathrm{d}^{-1}$ to be constant, and $\bar{\mathbf{\Gamma}}_1(r, \rho) = 0$.

**Step 3:** Since $\mathbf{R}(r) = 0$, condition (6.10) is already verified. Furthermore, imposing (6.14) satisfies (6.7a) and $0 = \mathbf{K}_\mathrm{e}(r)\mathbf{J}^{-\top}(r)\bar{\mathbf{G}}(r)$, and reduces (6.7b) to

$$0 = \mathbf{K}_\mathrm{e}(r)\left(\mathbf{J}^{-\top}(r)\frac{\partial^\top \mathbf{V}}{\partial r} - \frac{\partial^\top \mathbf{V}_\mathrm{d}}{\partial r}\right) \qquad \forall r \in \mathcal{R}_\Phi,$$

which is a polynomial equation if $\mathbf{K}_\mathrm{e}$ is polynomial. Now, according to Option 1, we can select $\mathbf{K}_\mathrm{e}(r) = \mathbf{N}_\perp(r)$.

[63] SDPT3 is a Matlab software package for SDP [185].

**Step 4:** We select $\epsilon = 10^{-5}$, $g_c = 9.81$

$$\left(\frac{\partial^\top \Phi}{\partial r}\right)_\perp = \mathbf{B}_\perp(r) = \begin{bmatrix} -z_p & 0 & x_p & -z_p & 0 \\ 0 & -z_p & y_p & 0 & -z_p \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and proceed to solve the SOS Program 6.1 on Matlab with SOSTOOLS and SDPT3. After setting $l = 1$ in the stabilizing conditions (5.33), we obtain $\mu_d = 4.243$,

$$\mathbf{V}_d(r) = 4.243(l + z_p) + 1.727(x_p - x_t + x_t^\star)^2 + 1.727(y_p - y_t + y_t^\star)^2,$$

$$\mathbf{M}_d^{-1} = \begin{bmatrix} 2.628 & 0 & 0 & -2.196 & 0 \\ 0 & 2.628 & 0 & 0 & -2.196 \\ 0 & 0 & 0.4325 & 0 & 0 \\ -2.196 & 0 & 0 & 2.628 & 0 \\ 0 & -2.196 & 0 & 0 & 2.628 \end{bmatrix} \succ 0.$$

We remark that the solver converges for every $l > 0$, but we take $l = 1$ for simplicity and because it is compatible with the test-bench of Figure 7.1.

**Step 5:** From the full-rank feature of $\bar{\mathbf{G}}$ and

$$\hat{\mathbf{G}}(r) := \mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_d^{-1}(r)\bar{\mathbf{G}}(r) = \frac{1}{l^2}\begin{bmatrix} 0 & 0 & 4.824l^2 - 2.196x_p^2 & -2.196x_p y_p \\ 0 & 0 & -2.196x_p y_p & 4.824l^2 - 2.196y_p^2 \end{bmatrix}^\top,$$

we have $\bar{\mathbf{G}}_\perp(r)\nu = 0$ and can select

$$\hat{\mathbf{G}}^g(r) = \frac{1}{23.28l^2 + 10.59z_p^2 - 10.59}\begin{bmatrix} 0 & 0 & 4.824l^2 - 2.196y_p^2 & 2.196x_p y_p \\ 0 & 0 & 2.196x_p y_p & 4.824l^2 - 2.196x_p^2 \end{bmatrix},$$

which is also the Moore–Penrose inverse of $\hat{\mathbf{G}}$. Since $\mathbf{R} = 0$, we pick $\bar{\mathbf{\Gamma}}_2(r) = \mathbf{K}_v \bar{\mathbf{G}}^\top(r)\mathbf{J}^{-1}(r)$ with $\mathbf{K}_v = \operatorname{diag}(k_{v1}, k_{v2}) \succ 0$. Hence, the IDA-PBC controller (6.8) reads

$$\mathbf{u}_{ida}(r,\dot{r}) = \frac{1}{u_3(r)}\begin{bmatrix} u_1(r) \\ u_2(r) \end{bmatrix} + \frac{103.9z_p - 10.59(\dot{x}_p^2 + \dot{y}_p^2 + \dot{z}_p^2)}{u_3(r)}\begin{bmatrix} x_p \\ y_p \end{bmatrix} - \bar{\mathbf{\Gamma}}_2(r)\dot{r}, \tag{7.12}$$

$$u_1(r) = 16.66l^2(x_p - x_t + x_t^\star) + 7.583y_p(x_t y_p - x_t^\star y_p - x_p y_t + x_p y_t^\star),$$

$$u_2(r) = 16.66l^2(y_p - y_t + y_t^\star) + 7.583x_p(y_t x_p - x_p y_t^\star - x_t y_p + x_t^\star y_p),$$

$$u_3(r) = 23.28l^2 + 10.59(z_p^2 - 1),$$

$$\bar{\mathbf{\Gamma}}_2(r)\dot{r} = \frac{1}{l^2}\mathbf{K}_v\begin{bmatrix} 4.825(\dot{x}_t - \dot{x}_p) - 2.196x_p(\dot{x}_t x_p + \dot{y}_t y_p + \dot{z}_p z_p) \\ 4.825(\dot{y}_t - \dot{y}_p) - 2.196y_p(\dot{x}_t x_p + \dot{y}_t y_p + \dot{z}_p z_p) \end{bmatrix},$$

where $\dot{r} = \rho$. We used $\frac{\partial \Phi}{\partial r}\dot{r} = 0$ to reduce the expressions in $\bar{\mathbf{\Gamma}}_2(r)\dot{r}$.

**Step 6:** By Proposition 5.2, system (7.1), (7.8) in feedback with (7.12) is stable in $(r_{\rm d}, 0)$ with $l = 1$ for every $\mathbf{K}_{\rm v} \succ 0$. Asymptotic stability can be demonstrated if $(r_{\rm d}, 0)$ is an isolated point of $\Omega_{\rm inv}$ (defined in Proposition 5.2), or by using Lyapunov's indirect method (Theorem 2.2) on the reduced system (7.9), see Section 6.3.

Following the second approach, we linearize (7.9) about the equilibrium $x_{\rm d} = \mathrm{vec}(q_{\rm d}, 0)$, $q_{\rm d} = \mathrm{vec}(0, 0, x_{\rm t}^\star, y_{\rm t}^\star) = \xi^{-1}(r_{\rm d})$, obtaining (7.11) with

$$A_{\rm r} = \begin{bmatrix} 0 & I_4 \\ -9.81\,\mathrm{diag}(1,1,0,0) & 0 \end{bmatrix}, \qquad B_{\rm r} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}^\top,$$

$$K_{\rm r} = \begin{bmatrix} -3.749 & 0 & -0.7158 & 0 & 4.825k_{\rm v1} & 0 & -4.825k_{\rm v1} & 0 \\ 0 & -3.749 & 0 & -0.7158 & 0 & 4.825k_{\rm v2} & 0 & -4.825k_{\rm v2} \end{bmatrix}.$$

Let e.g., $k_{\rm v1} = k_{\rm v2} = 0.21$. Hence, asymptotic stability is a direct consequence of the eigenvalues of $(A_{\rm r} + B_{\rm r}K_{\rm r})$ being in the left half plane: $\lambda_{1,2} = \lambda_{3,4} = -0.0678 \pm 2.2703i$, $\lambda_{5,6} = \lambda_{7,8} = -0.9454 \pm 0.6836i$.

### 7.1.3 4-DoF System: Constructive Solution

In this section, we test the constructive perspective of Algorithm 6.2 with a constant target inertia matrix to synthesize an IDA-PBC feedback for the portal crane described by (7.1), (7.8). We omit Step 1 because it is equal to the one from the previous section.

**Step 2:** Since $\mathbf{M}$ is constant, we can satisfy the kinetic matching (5.16a) by using a constant target inertia matrix $\mathbf{M}_d$, setting $\mathbf{\Gamma}_1(r, \rho) = 0$ and imposing (6.19), see Section 6.2.2. Hence,

$$\mathbf{M}_{\rm d} = \begin{bmatrix} a_1 & e_1 & 0 & d - a_1 & -e_1 \\ e_1 & a_2 & 0 & -e_1 & d - a_2 \\ 0 & 0 & d & 0 & 0 \\ d - a_1 & -e_1 & 0 & b_1 & e_2 \\ -e_1 & d - a_2 & 0 & e_2 & b_2 \end{bmatrix}$$

where $a_i$, $b_i$, $e_i$ and $d$ are arbitrary constants.

**Step 3:** We recall that $\mathbf{B}_{\rm d}(r) = \mathbf{M}_{\rm d}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$, $\Delta_{\rm d}(r) = \mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\mathbf{B}_{\rm d}(r)$ and $\mathbf{Z}_{\rm d}(r) = I_5 - \mathbf{B}_{\rm d}(r)\Delta_{\rm d}^{-1}(r)\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)$. Then,

$$\begin{aligned}\mathbf{Q}(r) &:= \begin{bmatrix} \mathbf{N}_\perp(r)\mathbf{Z}_{\rm d}(r)\mathbf{M}_{\rm d}(r)\mathbf{M}^{-1}(r) & -\mathbf{N}_\perp(r)\frac{\partial \mathbf{V}}{\partial r} \end{bmatrix} \\ &= \begin{bmatrix} -dz_{\rm p} & 0 & dx_{\rm p} & -z_{\rm p}(d - a_1 + b_1) & 0 & -g_{\rm c}x_{\rm p} \\ 0 & -dz_{\rm p} & dy_{\rm p} & 0 & -z_{\rm p}(d - a_2 + b_2) & -g_{\rm c}y_{\rm p} \end{bmatrix},\end{aligned}$$

where

$$e_i = 0 \tag{7.13}$$

is chosen to streamline analysis. From $\mathbf{Q}_{\perp}(r) = \begin{bmatrix} \frac{\partial^\top \Phi}{\partial r} & \bar{\mathbf{S}}(r) & \hat{\mathbf{S}}(r) \\ 0 & 0 & 1 \end{bmatrix}$, we can select

$$\bar{\mathbf{S}} = \begin{bmatrix} a_1 - d - b_1 & 0 & 0 & d & 0 \\ 0 & a_2 - d - b_2 & 0 & 0 & d \end{bmatrix}^\top, \qquad \hat{\mathbf{S}} = \mathrm{vec}(0, 0, \frac{g_\mathrm{c}}{d}, 0, 0),$$

which are constants matrices, meaning that their columns are gradient vectors.

**Step 4:** The dissipation condition (5.36b) is trivially satisfied by $\mathbf{R} = 0$. For the sake of simplicity we pick

$$\beta(\gamma) := \frac{1}{2}(\gamma - \gamma^\star)^\top \mathbf{K}_\gamma (\gamma - \gamma^\star) + \beta_\mathrm{c},$$

where $\gamma^\star = \mathrm{vec}(\gamma_1^\star, \gamma_2^\star) \in \mathbb{R}^2$, $\mathbf{K}_\gamma = \mathrm{diag}(k_{\gamma 1}, k_{\gamma 2}) \in \mathbb{R}^{2\times 2}$ and $\beta_\mathrm{c} \in \mathbb{R}$ are constants, and

$$\gamma(r) = \bar{\mathbf{S}}^\top r.$$

From (5.33a)–(5.33b), or equivalently (6.33a)–(6.33c), $\mathbf{V}_\mathrm{d}|_{\mathcal{R}_\Phi}$ has a strict minimum in $r_\mathrm{d}$ if

$$\gamma_1^\star = d x_\mathrm{t}^\star, \qquad \gamma_2^\star = d y_\mathrm{t}^\star, \qquad \mu^\star = \frac{g_\mathrm{c}}{l d}, \qquad d > 0, \qquad k_{\gamma 1} > 0, \qquad k_{\gamma 2} > 0. \tag{7.14}$$

Given that $\bar{\Delta}_\mathrm{d}(r_\mathrm{d}) = l^2 d > 0$, (5.33c) is verified if and only if so is (5.35), see Lemma 5.4. Hence, by setting

$$\bar{\mathbf{B}}_\perp(r) = \mathbf{B}_\perp(r) = \begin{bmatrix} -z_\mathrm{p} & 0 & x_\mathrm{p} & -z_\mathrm{p} & 0 \\ 0 & -z_\mathrm{p} & y_\mathrm{p} & 0 & -z_\mathrm{p} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

condition (5.35) reduces to

$$b_1 > 0, \qquad b_2 > 0, \qquad (2d - a_1 + b_1)a_1 > d^2, \qquad (2d - a_2 + b_2)a_2 > d^2. \tag{7.15}$$

**Steps 5:** From $\mathbf{R} = 0$ and

$$\bar{\mathbf{G}}(r) = \frac{1}{l^2}\begin{bmatrix} x_\mathrm{p}^2 - l^2 & x_\mathrm{p} y_\mathrm{p} & x_\mathrm{p} z_\mathrm{p} & l^2 & 0 \\ x_\mathrm{p} y_\mathrm{p} & y_\mathrm{p}^2 - l^2 & y_\mathrm{p} z_\mathrm{p} & 0 & l^2 \end{bmatrix}^\top,$$

we have $\bar{\mathbf{G}}_{\perp}\nu = 0$ and can select

$$\bar{\mathbf{G}}^\mathrm{g} = \begin{bmatrix} 0_{2\times 3} & I_2 \end{bmatrix}, \qquad \bar{\mathbf{\Gamma}}_2(r) = \bar{\mathbf{G}}^\mathrm{g}(r)\mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\mathbf{K}_\mathrm{v}(r)\bar{\mathbf{S}}^\top(r),$$

with

$$\mathbf{K}_{\mathrm{v}} = \mathrm{diag}(k_{\mathrm{v}1}, k_{\mathrm{v}2}) \succ 0. \tag{7.16}$$

Note that $\bar{\mathbf{G}}^{\mathrm{g}}$ is not the Moore–Penrose inverse of $\bar{\mathbf{G}}$. Hence, the controller (5.18) with

$$\mathbf{V}_{\mathrm{d}}(r) := \hat{\mathbf{S}}^{\top} r + \beta(\gamma(r)) = \frac{g_{\mathrm{c}}(z_{\mathrm{p}}+l)}{d} + \frac{1}{2}(\gamma(r)-\gamma^{\star})^{\top}\mathbf{K}_{\gamma}(\gamma(r)-\gamma^{\star})$$

renders the system (7.8) stable in $(r_{\mathrm{d}}, 0)$ whenever (7.13)–(7.16) are verified. The constant $\beta_{\mathrm{c}} = g_{\mathrm{c}} l d^{-1}$ assures $\mathbf{V}_{\mathrm{d}}(r_{\mathrm{d}}) = 0$.

**Steps 6:** In view of conditions (7.13)–(7.16), we see a great flexibility to select the controller parameters that we tackle by assigning an optimal local performance. For this we use the linearized system (7.11) about $x_{\mathrm{d}} = \mathrm{vec}(q_{\mathrm{d}}, 0)$, with $q_{\mathrm{d}} = \mathrm{vec}(0, 0, x_{\mathrm{t}}^{\star}, y_{\mathrm{t}}^{\star}) = \xi^{-1}(r_{\mathrm{d}})$, obtaining

$$A_{\mathrm{r}} = \begin{bmatrix} 0 & \mathrm{diag}(l^{-2}, l^{-2}, 1, 1) \\ -g_{\mathrm{c}} l\, \mathrm{diag}(1,1,0,0) & 0 \end{bmatrix}, \qquad B_{\mathrm{r}} = \begin{bmatrix} 0 & 0 & 0 & 0 & -l & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -l & 0 & 1 \end{bmatrix}^{\top},$$

$$K_{\mathrm{r}} = \begin{bmatrix} k_{11} & 0 & k_{12} & 0 & k_{13} & 0 & k_{14} & 0 \\ 0 & k_{21} & 0 & k_{22} & 0 & k_{23} & 0 & k_{24} \end{bmatrix},$$

$$k_{i1} = \frac{g_{\mathrm{c}} a_i}{d} + k_{\gamma i} l (d - a_i + b_i)\eta_i - g_{\mathrm{c}}, \qquad k_{i2} = -d k_{\gamma i}\eta_i,$$

$$k_{i3} = \frac{1}{l} k_{\mathrm{v}i}(d - a_i + b_i)\eta_i, \qquad k_{i4} = -d k_{\mathrm{v}i}\eta_i, \qquad \eta_i = (2a_i d + a_i b_i - a_i^2 - d^2),$$

where the pair $(A_{\mathrm{r}}, B_{\mathrm{r}})$ is controllable for any $l > 0$. Consequently, we can easily set a LQR local optimal performance to (5.18) by matching the linearized IDA-PBC feedback $K_{\mathrm{r}}\tilde{x}$ with the LQR feedback $u_{\mathrm{lqr}}(x) = -K_{\mathrm{lqr}}\tilde{x}$, i.e., we impose $-K_{\mathrm{lqr}} = K_{\mathrm{r}}$.

Setting the weighting matrices $Q_{\mathrm{lqr}}$ and $R_{\mathrm{lqr}}$ to be diagonal, produces an optimal gain matrix $K_{\mathrm{lqr}}$ with the same structure (position of zeros) as $K_{\mathrm{r}}$. As a result, the equation $-K_{\mathrm{lqr}} = K_{\mathrm{r}}$ always provides a solution to $a_i$, $b_i$, $\mathbf{K}_{\mathrm{v}}$ and $\mathbf{K}_{\gamma}$ with an arbitrary constant $d$. Fix $l = 1$, $d = 10$, $e_1 = e_2 = 0$, $Q_{\mathrm{lqr}} = \mathrm{diag}(10I_4, I_4)$ and $R_{\mathrm{lqr}} = I_2$, then

$$a_1 = a_2 = 12.8055,\ b_1 = b_2 = 5.493,\ k_{\gamma 1} = k_{\gamma 2} = 0.00506,\ k_{\mathrm{v}1} = k_{\mathrm{v}2} = 0.00497, \tag{7.17}$$

which are all consistent with (7.13)–(7.16) and yield a positive definite target inertia matrix $\mathbf{M}_{\mathrm{d}}$. Now, the control law (5.18) reads

$$\mathbf{u}_{\mathrm{ida}}(r, \dot{r}) = \frac{1}{10l^2 + 2.806x_{\mathrm{p}}^2 + 2.806y_{\mathrm{p}}^2}\begin{bmatrix} u_1(r,\dot{r}) \\ u_2(r,\dot{r}) \end{bmatrix}, \tag{7.18}$$

$$\begin{aligned} u_1(r,\dot{r}) = {} & 2.806x_{\mathrm{p}}(\dot{x}_{\mathrm{p}}^2 + \dot{y}_{\mathrm{p}}^2 + \dot{z}_{\mathrm{p}}^2) + (y_{\mathrm{p}}^2 + 3.564l^2)(2.342\dot{x}_{\mathrm{p}} - 8.714\dot{x}_{\mathrm{t}} + 8.872(x_{\mathrm{t}}^{\star} - x_{\mathrm{t}})) \\ & - 27.52x_{\mathrm{p}}z_{\mathrm{p}} + 8.499l^2 x_{\mathrm{p}} - x_{\mathrm{p}}y_{\mathrm{p}}(2.342\dot{y}_{\mathrm{p}} - 8.714\dot{y}_{\mathrm{t}} + 8.872(y_{\mathrm{t}}^{\star} - y_{\mathrm{t}})), \end{aligned}$$

$$u_2(r,\dot r) = 2.806y_{\rm p}(\dot x_{\rm p}^2+\dot y_{\rm p}^2+\dot z_{\rm p}^2) + (x_{\rm p}^2+3.564l^2)(2.342\dot y_{\rm p} - 8.714\dot y_{\rm t} + 8.872(y_{\rm t}^\star - y_{\rm t}))$$
$$- 27.52y_{\rm p}z_{\rm p} + 8.499l^2y_{\rm p} - x_{\rm p}y_{\rm p}(2.342\dot x_{\rm p} - 8.714\dot x_{\rm t} + 8.872(x_{\rm t}^\star - x_{\rm t})),$$

where $\dot r = \rho$. Asymptotic stability is a direct consequence of Lyapunov's indirect method.

### 7.1.4 4-DoF System: Position Feedback

We now test Proposition 6.6 in the synthesize of a position feedback for the portal crane system (7.1), (7.8). In this regard, we employ the solutions of $\mathbf{V}_{\rm d}$ and $\mathbf{M}_{\rm d}$ from the previous section (constructive solution) because they already meet the conditions of Propositions 6.3 and 6.5. Furthermore, since $\mathbf{M}$ and $\mathbf{M}_{\rm d}$ are constant and $\mathbf{R} = 0$, we impose

$$0 = \bar{\mathbf{B}}_\perp(r)\mathbf{B}_{\rm d}(r), \tag{7.19}$$

or equivalently $a_1 = a_2 = d$, to fulfill all the conditions of Proposition 6.6 (see Remark 6.3). It follows that the position feedback (6.35) stabilizes the system (7.1), (7.8) in $(r_{\rm d}, 0)$, with $r_{\rm d} = \mathrm{vec}(0,0,-l,x_{\rm t}^\star,y_{\rm t}^\star) \in \mathcal{R}_a$, whenever (7.13)–(7.14), $b_1 > 0$, $b_2 > 0$, $\bar{\mathbf{K}}_{\rm u} \succ 0$ and $\mathbf{\Lambda}_\zeta(r) + \mathbf{\Lambda}_\zeta^\top(r) \succ 0$ are fulfilled. Let us define $\bar{\mathbf{K}}_{\rm u} = \tau_\gamma^{-1}\mathbf{K}_{\rm v}$ and $\mathbf{\Lambda}_\zeta = \mathbf{K}_{\rm v}^{-1}$ with $\tau_\gamma > 0$ and $\mathbf{K}_{\rm v}$ from (7.16), then the controller (6.35) can be interpreted as replacing $\dot\gamma$ from the term

$$\bar{\mathbf{\Gamma}}_2(r)\frac{\partial^\top \mathbf{H}}{\partial \rho} = \bar{\mathbf{G}}^{\rm g}(r)\mathbf{Z}_{\rm d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\mathbf{K}_{\rm v}(r)\bar{\mathbf{S}}^\top(r)\dot r = \bar{\mathbf{G}}^{\rm g}(r)\mathbf{Z}_{\rm d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\mathbf{K}_{\rm v}(r)\dot\gamma,$$

in the full-state feedback (5.18), with the dirty derivative of $\gamma$ given by $\dfrac{\mathrm{D}_t}{\tau_\gamma \mathrm{D}_t + 1}\gamma$, $\mathrm{D}_t = \frac{\mathrm{d}}{\mathrm{d}t}$. After setting the controller parameters from (7.17) with $\tau_\gamma = 0.1$, feedback (6.35) reads

$$\mathbf{u}_{\rm of}(r,\zeta) = \begin{bmatrix} 21.17x_{\rm p} - 49.34x_{\rm t} + 4.56x_{\rm t}^\star - 3.497\zeta_1 \\ 21.17y_{\rm p} - 49.34y_{\rm t} + 4.56y_{\rm t}^\star - 3.497\zeta_2 \end{bmatrix}, \tag{7.20a}$$
$$\dot\zeta_1 = 54.93x_{\rm p} - 128.1x_{\rm t} - 10\zeta_1, \tag{7.20b}$$
$$\dot\zeta_2 = 54.93y_{\rm p} - 128.1y_{\rm t} - 10\zeta_2, \tag{7.20c}$$

which is linear in the coordinates $r$ but nonlinear in $q = \mathrm{vec}(\beta,\alpha,x_{\rm t},y_{\rm t})$.

### 7.1.5 5-DoF Implicit Model

The controllers of Sections 7.1.2 to 7.1.4 guarantee the (asymptotic) stability of the crane in $r_{\rm d}$ for all values of $l$ in a neighborhood of $l = 1$. In fact, the constructive solutions (full-state feedback and position feedback) assure stability for any length $l > 0$. However, $l$ must be fixed during operation; otherwise, we risk losing performance and even unstable behavior. To overcome this problem, we consider a non-fixed length $l$, meaning that our system has five DoF and the configuration space is given by $\mathbb{R}^3 \times \mathbb{S}^2$.[64] Then, we can select the coordinates $r = \mathrm{vec}(x_\mathrm{p}, y_\mathrm{p}, z_\mathrm{p}, x_\mathrm{t}, y_\mathrm{t}, l) \in \mathcal{R} = \mathbb{R}^6$ satisfying the holonomic constraint (7.1). From Assumption 7.1 and the chosen coordinates, the kinetic and potential energy remains as in (7.2), meaning that the Lagrange equations with Rayleigh dissipation $\frac{1}{2}c_1\dot{x}_\mathrm{t}^2 + \frac{1}{2}c_2\dot{y}_\mathrm{t}^2 + \frac{1}{2}c_3\dot{l}^2 \geq 0$ (viscous friction in actuators) are

$$\hat{\mathbf{M}}\ddot{r} + \hat{\mathbf{R}}\dot{r} + \frac{\partial^\top \hat{\mathbf{V}}}{\partial r} = \hat{\mathbf{G}}(r)\tau + \frac{\partial^\top \Phi}{\partial r}\hat{\lambda}, \tag{7.21}$$

where $\tau = \mathrm{vec}(\tau_x, \tau_y, \tau_l)$, $\hat{\mathbf{R}} = \mathrm{diag}(0, 0, 0, c_1, c_2, c_3)$,

$$\hat{\mathbf{M}} = \begin{bmatrix} m_\mathrm{p} & 0 & 0 & m_\mathrm{p} & 0 & 0 \\ 0 & m_\mathrm{p} & 0 & 0 & m_\mathrm{p} & 0 \\ 0 & 0 & m_\mathrm{p} & 0 & 0 & 0 \\ m_\mathrm{p} & 0 & 0 & m_\mathrm{p} + m_\mathrm{t} + m_\mathrm{b} & 0 & 0 \\ 0 & m_\mathrm{p} & 0 & 0 & m_\mathrm{p} + m_\mathrm{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{G}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\begin{bmatrix} \hat{\mathbf{M}} & -\frac{\partial^\top \Phi}{\partial r} \\ \frac{\partial^\top \Phi}{\partial r} & 0 \end{bmatrix}$ is nonsingular for all $r \in \mathcal{R}_\Phi := \{r \in \mathcal{R} \mid \Phi(r) = 0\}$, system (7.21) can be rewritten as $\ddot{r} = f(r, \dot{r}, \hat{\lambda}, \tau)$ for some function $f$, where $\hat{\lambda}$ has a unique solution that guarantees $\mathrm{vec}(\dot{r}, f)$ being a vector field on the constrained state space, see Section 2.4.2. In other words, (7.21) is well-posed despite $\hat{\mathbf{M}}$ being singular.

### Partial Feedback Linearization

The partially linearized model of (7.21) is obtained with the procedure of Section 7.1.1 but with $h(r) = \mathrm{vec}(x_\mathrm{t}, y_\mathrm{t}, l)$ and $\hat{\mathbf{G}}_\perp = m_\mathrm{p}^{-1}\begin{bmatrix} I_3 & 0_{3\times 3}\end{bmatrix}$. The resulting system is now

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & I_6 \\ -I_6 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}}{\partial r} \\ \frac{\partial^\top \mathbf{H}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{\mathbf{B}} \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ \mathbf{G} \end{bmatrix} u, \tag{7.22}$$

where $\mathbf{H}(r, \rho) = \frac{1}{2}\rho^\top \mathbf{M}^{-1}\rho + \mathbf{V}(r)$, $\mathbf{M} = I_6$, $\mathbf{V} = g_\mathrm{c} z_\mathrm{p}$, $\bar{\mathbf{B}}(r) = \mathrm{vec}(x_\mathrm{p}, y_\mathrm{p}, z_\mathrm{p}, 0, 0, 0)$ and

$$\mathbf{G} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

[64]Similar to the 4-DoF crane, the model with 5-DoF can be approximate with two or more lower-order models but at the price of confining the length of $l$ to a tiny neighborhood. Recall that two 2-DoF models are a local approximation of the 4-DoF model.

Note that the system only requires the gravity constant $g_\mathrm{c}$ as a parameter, simplifying the implementation task. Besides, Colsp $\bar{\mathbf{B}}(r) \neq$ Colsp $\frac{\partial^\top \Phi}{\partial r}$, i.e., the constraint forces $\bar{\mathbf{B}}\lambda$ do not satisfy the Lagrange-d'Alembert principle, meaning that they are not workless. We leave the verification of Assumptions 5.6 and 5.8 to the reader.

#### Reduction and Linearization

Reducing system (7.1), (7.22), in feedback with $u = \mathbf{u}_\mathrm{ida}(r, \rho)$, to the explicit representation (see Corollary 6.2) results in

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} M^{-1}(q)p \\ -f_1(q) - f_2(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ G(q) \end{bmatrix} u_\mathrm{ida}(q,p), \tag{7.23}$$

where $q = \mathrm{vec}(\beta, \alpha, x_\mathrm{t}, y_\mathrm{t}, l)$, $M(q) = \frac{\partial^\top \xi}{\partial q}\frac{\partial \xi}{\partial q} = \mathrm{diag}(l^2, l^2\cos^2\beta, 1, 1, 2)$, $r = \xi(q)$ follows from (7.10), $u_\mathrm{ida}(q,p) = \mathbf{u}_\mathrm{ida}(\xi(q), L(q)p)$, $p := \mathrm{vec}(p_1, p_2, p_3, p_4, p_5) = \frac{\partial^\top \xi}{\partial q}\rho$, $G(q) = \frac{\partial^\top \xi}{\partial q}\mathbf{G}$, $L(q) = \frac{\partial \xi}{\partial q} M^{-1}(q)$,

$$\begin{aligned} f_1(q) &= g_\mathrm{c} l \begin{bmatrix} \cos(\alpha)\sin(\beta) & \cos(\beta)\sin(\alpha) & 0 & 0 & 0 \end{bmatrix}^\top, \quad \text{and} \\ f_2(q,p) &= \frac{1}{l^2\cos^3(\beta)} \begin{bmatrix} p_2^2 \sin(\beta) & 0 & 0 & 0 & 0 \end{bmatrix}^\top. \end{aligned}$$

Observe that the inertia matrix $M$ is non-constant, and the reduction is well-defined for $\alpha, \beta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. Besides, since the system (7.23) is not port-Hamiltonian and $M$ depends on the actuated coordinate $l$, the geometric-PBC of [231] and the PID-PBC of [20, 24] that were applied to the explicit model of 4-DoF, cannot work with the one of 5-DoF.

The linearization of (7.23) about the equilibrium $x_\mathrm{d} = \mathrm{vec}(q_\mathrm{d}, 0)$, with $q_\mathrm{d} = \xi^{-1}(r_\mathrm{d})$, yields

$$\dot{\tilde{x}} = \underbrace{\left( \begin{bmatrix} 0 & M^{-1}(q_\mathrm{d}) \\ \frac{\partial f_1}{\partial q}\Big|_{q=q_\mathrm{d}} & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial G u}{\partial x} \end{bmatrix}\Bigg|_{\substack{x=x_\mathrm{d} \\ u=u_\mathrm{ida}(q_\mathrm{d},0)}} \right)}_{=:A_\mathrm{r}} \tilde{x} + \underbrace{\begin{bmatrix} 0_{5\times 3} \\ G(q_\mathrm{d}) \end{bmatrix}}_{=:B_\mathrm{r}} \underbrace{\frac{\partial u_\mathrm{ida}}{\partial x}\Bigg|_{x=x_\mathrm{d}}}_{=:K_\mathrm{r}} \tilde{x}, \tag{7.24}$$

where $\tilde{x} = x - x_\mathrm{d}$, $x = \mathrm{vec}(q, p)$.

### 7.1.6 5-DoF System: Constructive Solution

In this section, we test Algorithm 6.2 with a constant target inertial matrix and an optimal local assignment for the asymptotic stabilization of the 5-DoF crane with PFL. Although such a model is not port-Hamiltonian in the sense that it includes non-workless constraint forces, the procedure is fairly similar to the one discussed for the 4-DoF model, see Section 7.1.3.

**Step 1:** We select

$$\mathbf{N}_\perp(r) = \begin{bmatrix} -z_\mathrm{p} & 0 & x_\mathrm{p} & -z_\mathrm{p} & 0 & 0 \\ 0 & -z_\mathrm{p} & y_\mathrm{p} & 0 & -z_\mathrm{p} & 0 \end{bmatrix}$$

and $r_\mathrm{d} = \mathrm{vec}(0, 0, -l^\star, x_\mathrm{t}^\star, y_\mathrm{t}^\star, l^\star) \in \mathcal{R}_a = \{r \in \mathcal{R} \mid 0 = \Phi(r),\ 0 = x_\mathrm{p},\ 0 = y_\mathrm{p}\}$, where $(x_\mathrm{t}^\star, y_\mathrm{t}^\star)$ is the desired trolley position and $l^\star > 0$ is the desired rope length.

**Step 2:** Since $\mathbf{M}$ is constant, we can satisfy the kinetic matching (5.16a) by using a constant target inertia matrix $\mathbf{M}_d$, setting $\mathbf{\Gamma}_1(r, \rho) = 0$ and imposing (6.19), see Section 6.2.2. Then,

$$\mathbf{M}_\mathrm{d} = \begin{bmatrix} a_1 & -e_1 & 0 & d_1 - a_1 & e_1 & -e_3 \\ -e_1 & a_2 & 0 & e_1 & d_1 - a_2 & -e_4 \\ 0 & 0 & d_1 & 0 & 0 & 0 \\ d_1 - a_1 & e_1 & 0 & b_1 & e_2 & e_3 \\ e_1 & d_1 - a_2 & 0 & e_2 & b_2 & e_4 \\ -e_3 & -e_4 & 0 & e_3 & e_4 & d_2 \end{bmatrix},$$

where $a_i$, $b_i$, $d_i$ and $e_i$ are arbitrary constants.

**Step 3:** To simplify the analysis, we set

$$e_i = 0. \tag{7.25}$$

Then,

$$\mathbf{Q}(r) = \begin{bmatrix} -d_1 z_\mathrm{p} & 0 & d_1 x_\mathrm{p} & -z_\mathrm{p}(d_1 - a_1 + b_1) & 0 & 0 & -g_\mathrm{c} x_\mathrm{p} \\ 0 & -d_1 z_\mathrm{p} & d_1 y_\mathrm{p} & 0 & -z_\mathrm{p}(d_1 - a_2 + b_2) & 0 & -g_\mathrm{c} y_\mathrm{p} \end{bmatrix},$$

and we can select

$$\bar{\mathbf{S}} = \begin{bmatrix} a_1 - d_1 - b_1 & 0 & 0 & d_1 & 0 & 0 \\ 0 & a_2 - d_1 - b_2 & 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top, \qquad \hat{\mathbf{S}} = \mathrm{vec}(0, 0, \frac{g_\mathrm{c}}{d_1}, 0, 0, 0).$$

Clearly, the columns of $\bar{\mathbf{S}}$ and $\hat{\mathbf{S}}$ are gradient vectors.

**Step 4:** Since $\mathbf{R} = 0$, (5.36b) holds. Now, define

$$\beta(\gamma) := \frac{1}{2}(\gamma - \gamma^\star)^\top \mathbf{K}_\gamma (\gamma - \gamma^\star) + \beta_\mathrm{c},$$

where $\gamma^\star = \mathrm{vec}(\gamma_1^\star, \gamma_2^\star, \gamma_3^\star) \in \mathbb{R}^3$, $\mathbf{K}_\gamma = \mathrm{diag}(k_{\gamma 1}, k_{\gamma 2}, k_{\gamma 3}) \in \mathbb{R}^{3\times 3}$ and $\beta_\mathrm{c} \in \mathbb{R}$ are constants, $\gamma(r) = \bar{\mathbf{S}}^\top r$ and $\beta_\mathrm{c}$ is selected such that $\mathbf{V}_\mathrm{d}(r_\mathrm{d}) = 0$. Then, from (6.33a)–(6.33c), it follows that

$$\begin{bmatrix}\gamma_1^\star\\ \gamma_2^\star\\ \gamma_3^\star\end{bmatrix} = \begin{bmatrix} d_1 x_\mathrm{t}^\star \\ d_1 y_\mathrm{t}^\star \\ l^\star - g_\mathrm{c}/(k_{\gamma 3} d_1)\end{bmatrix}, \quad \mu^\star = \frac{g_\mathrm{c}}{l^\star d_1}, \quad d_1 > 0, \quad k_{\gamma 1} > 0, \quad k_{\gamma 2} > 0, \quad k_{\gamma 3} > 0. \tag{7.26}$$

Besides, from Lemma 5.4, (5.33c) is verified if and only if so is (5.35) and $\bar{\Delta}_\mathrm{d}(r_\mathrm{d}) := \mathbf{B}^\top(r_\mathrm{d})\mathbf{M}^{-1}\mathbf{M}_\mathrm{d}\mathbf{M}^{-1}\mathbf{B}(r_\mathrm{d}) = (d_1 + d_2)(l^\star)^2$ is nonsingular. Hence, by setting

$$\bar{\mathbf{B}}_\perp(r) = \begin{bmatrix} -z_\mathrm{p} & 0 & x_\mathrm{p} & 0 & 0 & 0\\ 0 & -z_\mathrm{p} & y_\mathrm{p} & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix},$$

the previous conditions reduces to

$$b_1 > 0, \quad b_2 > 0, \quad d_2 > 0, \quad (2d_1 - a_1 + b_1)a_1 > d_1^2, \quad (2d_1 - a_2 + b_2)a_2 > d_1^2. \tag{7.27}$$

**Step 5:** Given that

$$\bar{\mathbf{G}}(r) = \frac{1}{l^2}\begin{bmatrix} x_\mathrm{p}^2 - l^2 & x_\mathrm{p}y_\mathrm{p} & x_\mathrm{p}z_\mathrm{p} & l^2 & 0 & 0\\ x_\mathrm{p}y_\mathrm{p} & y_\mathrm{p}^2 - l^2 & y_\mathrm{p}z_\mathrm{p} & 0 & l^2 & 0\\ l x_\mathrm{p} & l y_\mathrm{p} & l z_\mathrm{p} & 0 & 0 & l^2\end{bmatrix}^\top$$

and $\mathbf{R} = 0$, we have $\bar{\mathbf{G}}_\perp \nu = 0$ and can select

$$\bar{\mathbf{G}}^\mathrm{g} = \begin{bmatrix}0_{3\times 3} & I_3\end{bmatrix}, \qquad \bar{\mathbf{\Gamma}}_2(r) = \bar{\mathbf{G}}^\mathrm{g}(r)\mathbf{Z}_\mathrm{d}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\mathbf{K}_\mathrm{v}(r)\bar{\mathbf{S}}^\top(r),$$

with

$$\mathbf{K}_\mathrm{v} = \mathrm{diag}(k_{\mathrm{v}1}, k_{\mathrm{v}2}, k_{\mathrm{v}3}) \succ 0. \tag{7.28}$$

Hence, the controller (5.18) with

$$\mathbf{V}_\mathrm{d}(r) := \hat{\mathbf{S}}^\top r + \beta(\gamma(r)) = \frac{g_\mathrm{c}(z_\mathrm{p} + l^\star)}{d_1} + \frac{1}{2}(\gamma(r) - \gamma^\star)^\top \mathbf{K}_\gamma(\gamma(r) - \gamma^\star) - \frac{g_\mathrm{c}^2}{2d_1^2 k_{\gamma 3}}$$

renders the system (7.8) stable in $(r_\mathrm{d}, 0)$ whenever (7.25)–(7.28) are verified.

**Steps 6:** To select the controller parameters with an optimal local performance, we use the linearized system (7.23) about $x_{\rm d} = {\rm vec}(q_{\rm d}, 0)$, $q_{\rm d} = {\rm vec}(0, 0, x_{\rm t}^\star, y_{\rm t}^\star, l^\star) = \xi^{-1}(r_{\rm d})$. Then,

$$A_{\rm r} = \begin{bmatrix} 0 & {\rm diag}((l^\star)^{-2}, (l^\star)^{-2}, 1, 1, \frac{1}{2}) \\ -g_{\rm c} l^\star\, {\rm diag}(1,1,0,0,0) & 0 \end{bmatrix},$$

$$B_{\rm r} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -l^\star & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -l^\star & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}^\top,$$

$$K_{\rm r} = \begin{bmatrix} k_{11} & 0 & k_{12} & 0 & 0 & k_{13} & 0 & k_{14} & 0 & 0 \\ 0 & k_{21} & 0 & k_{22} & 0 & 0 & k_{23} & 0 & k_{24} & 0 \\ 0 & 0 & 0 & 0 & k_{31} & 0 & 0 & 0 & 0 & k_{32} \end{bmatrix}$$

$$k_{i1} = \frac{g_{\rm c} a_i}{d_1} + k_{\gamma i} l^\star (d_1 - a_i + b_i)\eta_i - g_{\rm c}, \qquad k_{i2} = -d_1 k_{\gamma i}\eta_i,$$

$$k_{i3} = \frac{1}{l^\star} k_{{\rm v}i}(d_1 - a_i + b_i)\eta_i, \qquad k_{i4} = -d_1 k_{{\rm v}i}\eta_i, \qquad \eta_i = (2a_i d_1 + a_i b_i - a_i^2 - d_1^2),$$

$$k_{31} = -\frac{d_1 d_2}{d_1 + d_2} k_{\gamma 3}, \qquad k_{32} = -\frac{d_1 d_2}{2(d_1 + d_2)} k_{{\rm v}3},$$

with $i \in \{1, 2\}$. Since the pair $(A_{\rm r}, B_{\rm r})$ is controllable for any $l^\star > 0$ and the optimal LQR gain $K_{\rm lqr}$ has the same structure (position of zeros) as $K_{\rm r}$ for any diagonal weighting matrices $Q_{\rm lqr}$ and $R_{\rm lqr}$, then the local optimal assignment can be obtained from $K_{\rm r} = -K_{\rm lqr}$.[65] This equation produces a solution of $a_i$, $b_i$, $K_{\rm v}$ and $K_\gamma$ with arbitrary constants $d_1$ and $d_2$. Fix $l^\star = 1$, $d_1 = d_2 = 10$, $e_1 = e_2 = e_3 = e_4 = 0$, $Q_{\rm lqr} = {\rm diag}(10 I_5, I_4, 5)$ and $R_{\rm lqr} = I_3$, then

$$\begin{aligned} &a_1 = a_2 = 12.8055, && k_{\gamma 1} = k_{\gamma 2} = 0.00506, && k_{\gamma 3} = 0.6325, \\ &b_1 = b_2 = 5.493, && k_{{\rm v}1} = k_{{\rm v}2} = 0.00497, && k_{{\rm v}3} = 1.0261, \end{aligned} \tag{7.29}$$

which are all consistent with (7.25)–(7.28) and yield a positive definite target inertia matrix $\mathbf{M}_{\rm d}$. At this point, the controller (5.18) is

$$\mathbf{u}_{\rm ida}(r, \dot r) = \frac{1}{20 l^2 + 2.806 x_{\rm p}^2 + 2.806 y_{\rm p}^2} \begin{bmatrix} u_1(r, \dot r) \\ u_2(r, \dot r) \\ u_3(r, \dot r) \end{bmatrix}, \tag{7.30}$$

$$\begin{aligned} u_1(r, \dot r) = {} & 2.806 x_{\rm p}(\dot x_{\rm p}^2 - \dot l^2 + \dot y_{\rm p}^2 + \dot z_{\rm p}^2) + 27.52 x_{\rm p}(l - z_{\rm p}) + l x_{\rm p}(28.79 \dot l - 17.74 l^\star) \\ & + (y_{\rm p}^2 + 7.129 l^2)(2.342 \dot x_{\rm p} - 8.714 \dot x_{\rm t} + 8.872 x_{\rm t}^\star - 8.872 x_{\rm t}) + 34.74 l^2 x_{\rm p} \\ & - x_{\rm p} y_{\rm p}(2.342 \dot y_{\rm p} - 8.714 \dot y_{\rm t} + 8.872 y_{\rm t}^\star - 8.872 y_{\rm t}), \end{aligned}$$

[65] Matching the linearized IDA-PBC controller $K_{\rm r}\tilde x$ with the LQR feedback $-K_{\rm lqr}\tilde x$ gives $K_{\rm r} = -K_{\rm lqr}$.

$$
\begin{aligned}
u_2(r,\dot r) = {} & 2.806 y_\mathrm{p}(\dot x_\mathrm{p}^2 - \dot l^2 + \dot y_\mathrm{p}^2 + \dot z_\mathrm{p}^2) + 27.52 y_\mathrm{p}(l - z_\mathrm{p}) + l y_\mathrm{p}(28.79\dot l - 17.74 l^\star) \\
& + (x_\mathrm{p}^2 + 7.129 l^2)(2.342\dot y_\mathrm{p} - 8.714\dot y_\mathrm{t} + 8.872 y_\mathrm{t}^\star - 8.872 y_\mathrm{t}) + 34.74 l^2 y_\mathrm{p} \\
& - x_\mathrm{p} y_\mathrm{p}(2.342\dot x_\mathrm{p} - 8.714\dot x_\mathrm{t} + 8.872 x_\mathrm{t}^\star - 8.872 x_\mathrm{t}), \\
u_3(r,\dot r) = {} & 10 l(\dot x_\mathrm{p}^2 - \dot l^2 + \dot y_\mathrm{p}^2 + \dot z_\mathrm{p}^2) - (x_\mathrm{p}^2 + y_\mathrm{p}^2)(28.79\dot l - 17.74 l^\star + 26.24 l + 27.52) \\
& - l x_\mathrm{p}(8.347\dot x_\mathrm{p} - 31.06\dot x_\mathrm{t} + 31.62 x_\mathrm{t}^\star - 31.62 x_\mathrm{t}) + 63.25 l^2(l - l^\star) \\
& - l y_\mathrm{p}(8.347\dot y_\mathrm{p} - 31.06\dot y_\mathrm{t} + 31.62 y_\mathrm{t}^\star - 31.62 y_\mathrm{t}) - 98.1 l(l + z_\mathrm{p}) - 102.6 l^2\dot l,
\end{aligned}
$$

where $\dot r = \rho$. Asymptotic stability is a direct consequence of Lyapunov's indirect method on the reduced closed-loop (7.23).

Unlike the 4-DoF case, we cannot impose the condition $0 = \bar{\mathbf{B}}_\perp(r)\mathbf{B}_\mathrm{d}(r)$ to build a position feedback controller (see Section 7.1.4) because it implies that $a_1 = a_2 = d_1$ and $d_2 = 0$, which contradicts the condition (7.27).

### 7.1.7 Simulations and Real-system Implementation

In this section, we compare the controllers (7.12), (7.18), (7.20), (7.30) and

$$
u_\mathrm{e}(q,\dot q) = \begin{bmatrix} 3.162(x_\mathrm{t}^\star - x_\mathrm{t}) - 3.106\dot x_\mathrm{t} + 0.3672\frac{\sin\beta}{l^2}(l\dot\beta^2 + 9.81\cos\beta + \dot\beta\dot x_\mathrm{t}\cos\beta) \\ 3.162(y_\mathrm{t}^\star - y_\mathrm{t}) - 3.106\dot y_\mathrm{t} + 0.3672\frac{\sin\alpha}{l^2}(l\dot\alpha^2 + 9.81\cos\alpha + \dot\alpha\dot y_\mathrm{t}\cos\alpha) \end{bmatrix}. \quad (7.31)
$$

The latter, is obtained from the simplified IDA-PBC of Ryalat and Laila [57] that we briefly discussed in Example 6.3. This controller does not admit an arbitrary local assignment [102], so we tuned it to behave locally as close as possible to the implicit controller (7.18).

Figures 7.4 to 7.6 show the simulation results for the portal crane with the controllers of Table 7.1 and initial conditions $x_\mathrm{p}(0) = y_\mathrm{p}(0) = x_\mathrm{t}(0) = y_\mathrm{t}(0) = 0\,\mathrm{m}$, $l(0) = 1\,\mathrm{m}$ and $\rho(0) = 0$. We restrict the time span to 15 s, where the desired position is set to

- $x_\mathrm{p}^\star = y_\mathrm{p}^\star = 0\,\mathrm{m}$ for all time $t$,
- $x_\mathrm{t}^\star = y_\mathrm{t}^\star = 0\,\mathrm{m}$ and $l^\star = 1\,\mathrm{m}$ for $t \in [0, 1[$ s, and
- $x_\mathrm{t}^\star = y_\mathrm{t}^\star = 1\,\mathrm{m}$ and $l^\star = 0.5\,\mathrm{m}$ for $t \in [1, 15[$ s.

Figure 7.4 depicts the positions of $x_\mathrm{t}$, $y_\mathrm{t}$ and $l$ while Figure 7.5 shows the position of $x_\mathrm{p}$ and $y_\mathrm{p}$ as well as the Hamiltonian's value. Clearly, all controllers achieve the asymptotic stabilization of $x_\mathrm{p}^\star = y_\mathrm{p}^\star = 0\,\mathrm{m}$ and $x_\mathrm{t}^\star = y_\mathrm{t}^\star = 1\,\mathrm{m}$ but only (7.30) can do so for $l^\star = 0.5\,\mathrm{m}$. Besides, it can be seen that the Hamiltonians are monotonically decreasing functions, which is consistent with the design, namely $\dot{\mathbf{H}}_\mathrm{d} \leq 0$ and $\dot H_\mathrm{d} \leq 0$. Figure 7.6 displays the velocities $\dot x_\mathrm{t}$, $\dot y_\mathrm{t}$ and $\dot l$, which in the real experiment would correspond to the input of the velocity tracking system, see Figure 7.3.

| Name | Description | Eq. |
|---|---|---|
| Im-4DoF-Heu | Full-state feedback IDA-PBC synthesized from the heuristic method (SOS programs) with a constant $\mathbf{M}_d$, a quadratic $\mathbf{V}_\mathrm{d}$ and $k_{\mathrm{v}1} = k_{\mathrm{v}2} = 0.21$ for the 4-DoF implicit model, (Section 7.1.2) | (7.12) |
| Im-4DoF-Cons | Full-state feedback IDA-PBC synthesized from the constructive method with a constant $\mathbf{M}_d$ for the 4-DoF implicit model (Section 7.1.3) | (7.18) |
| Im-4DoF-PosF | Position feedback IDA-PBC synthesized from the constructive method with a constant $\mathbf{M}_d$ for the 4-DoF implicit model (Section 7.1.4) | (7.20) |
| Im-5DoF-Cons | Full-state feedback IDA-PBC synthesized from the constructive method with a constant $\mathbf{M}_d$ for the 5-DoF implicit model (Section 7.1.6) | (7.30) |
| Ex-2DoF-Simp | Full-state feedback IDA-PBC synthesized from the simplified method [102] for the 2-DoF explicit model | (7.31) |

**Table 7.1.** – IDA-PBC controllers for the portal crane.

Comparing the controllers, we observe that the optimal local assignment allows a faster settling time and smaller overshoot of the full-state feedbacks with the constructive approach, namely (7.18) and (7.30). We remark that if $l^\star$ is set to remain equal to 1 m, then there is no significant difference between them. The simplified IDA-PBC has a slightly inferior performance (settling time and overshoot) than (7.18) and (7.30). This a consequence of not allowing an arbitrary local assignment together with the simplicity of its model (2-DoF). The IDA-PBC obtained from the heuristic approach has the largest settling time because the solution of the SDP solver was obtained without any minimization objective or local assignment. We will improve this aspect in cart-pole example of Section 7.2. The position feedback has almost the same parameters as the full-state feedback from the constructive method with 4-DoF, but it requires the additional restriction $a_1 = a_2 = d$, so it yields also a slower settling time, but avoids any velocity measurement.

The physical setup is composed of a PC with Matlab/Simulink, an RTI-toolbox and a dSPACE controller [234] that is connected to the servo drives of the portal crane as well as the sensors (encoders, limit switches, etc.). In summary we take the fast prototyping perspective, meaning that our control algorithm is written in Simulink, which compiles and exports the algorithm to the dSPACE. For simplicity, we restrict our attention to the controller that yields the best results, namely (7.30). Figure 7.7 illustrates real system behavior, where the experiments are very close to the simulation results of Figures 7.4 to 7.6.

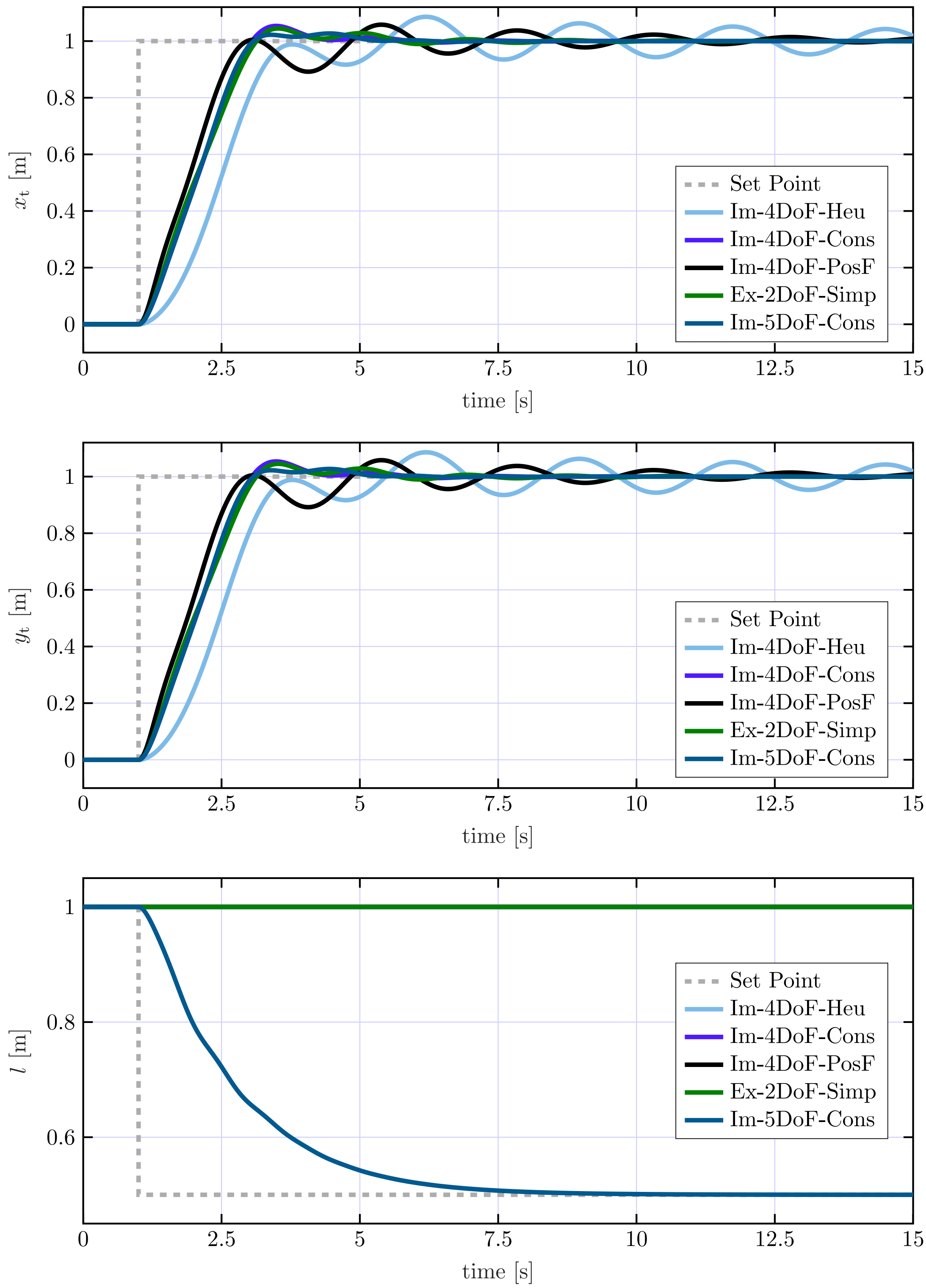


**Figure 7.4.** – Trolley position and length $l$.

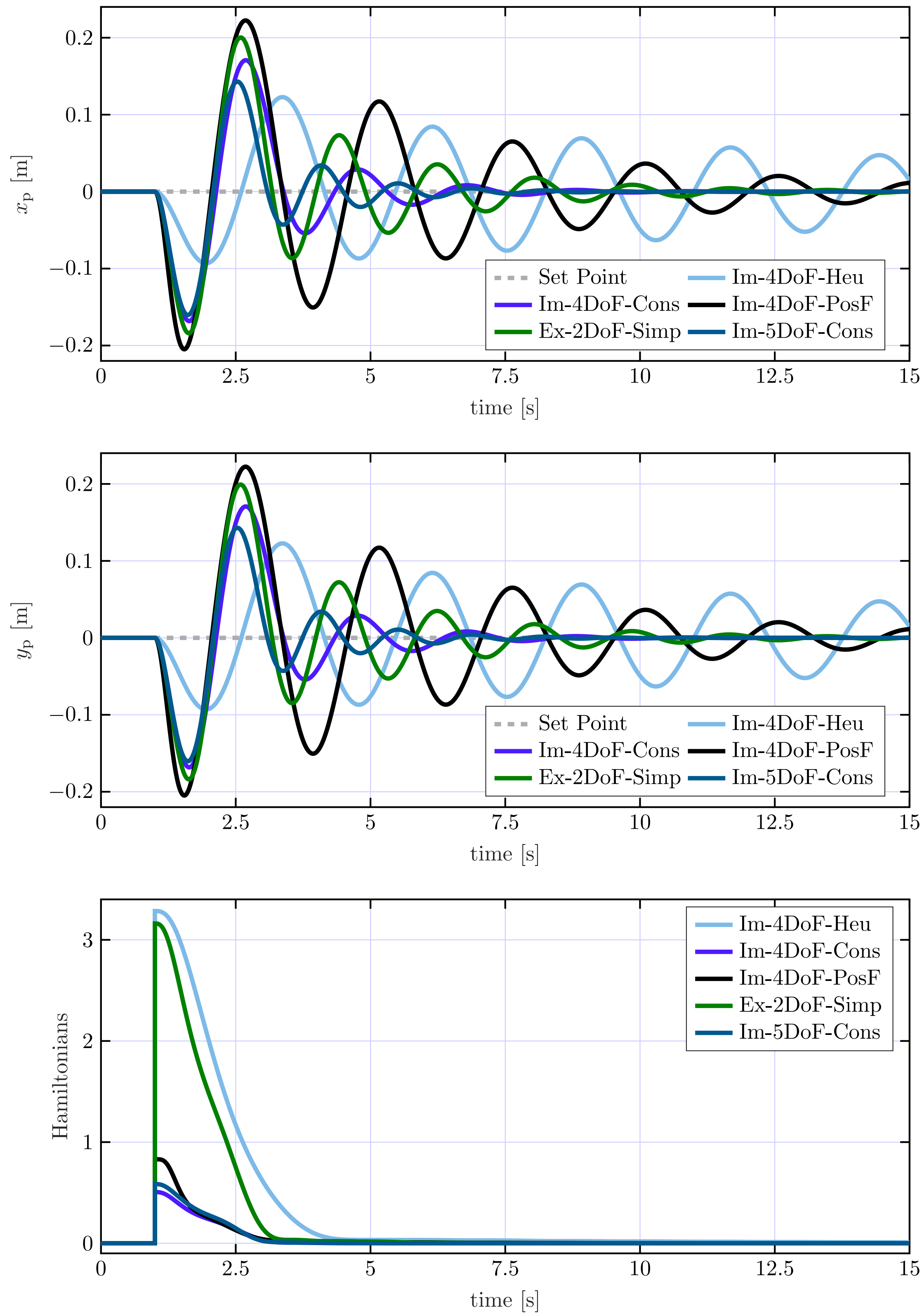


**Figure 7.5.** – Payload position relative to trolley.

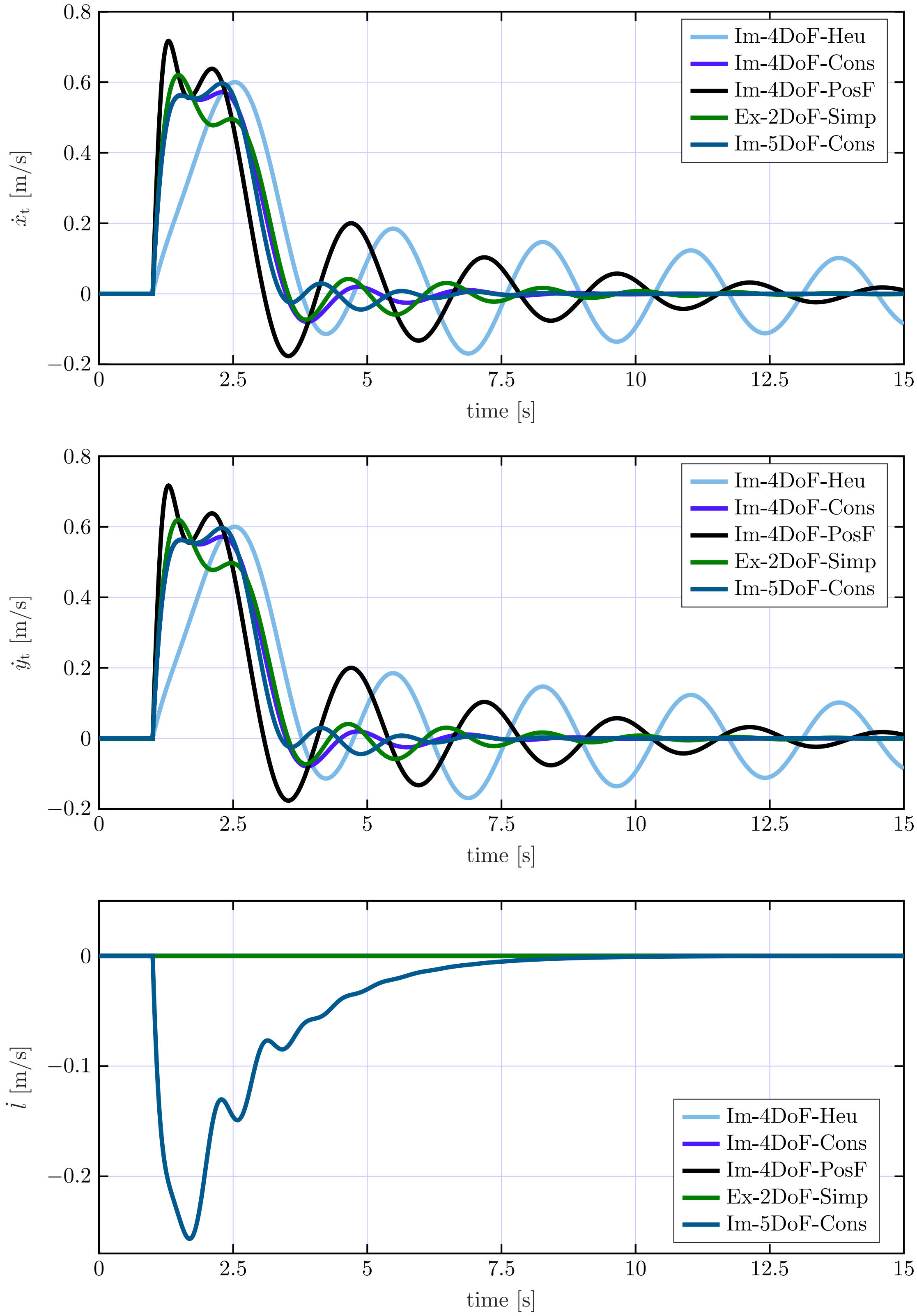


**Figure 7.6.** – Input of the crane with PFL.

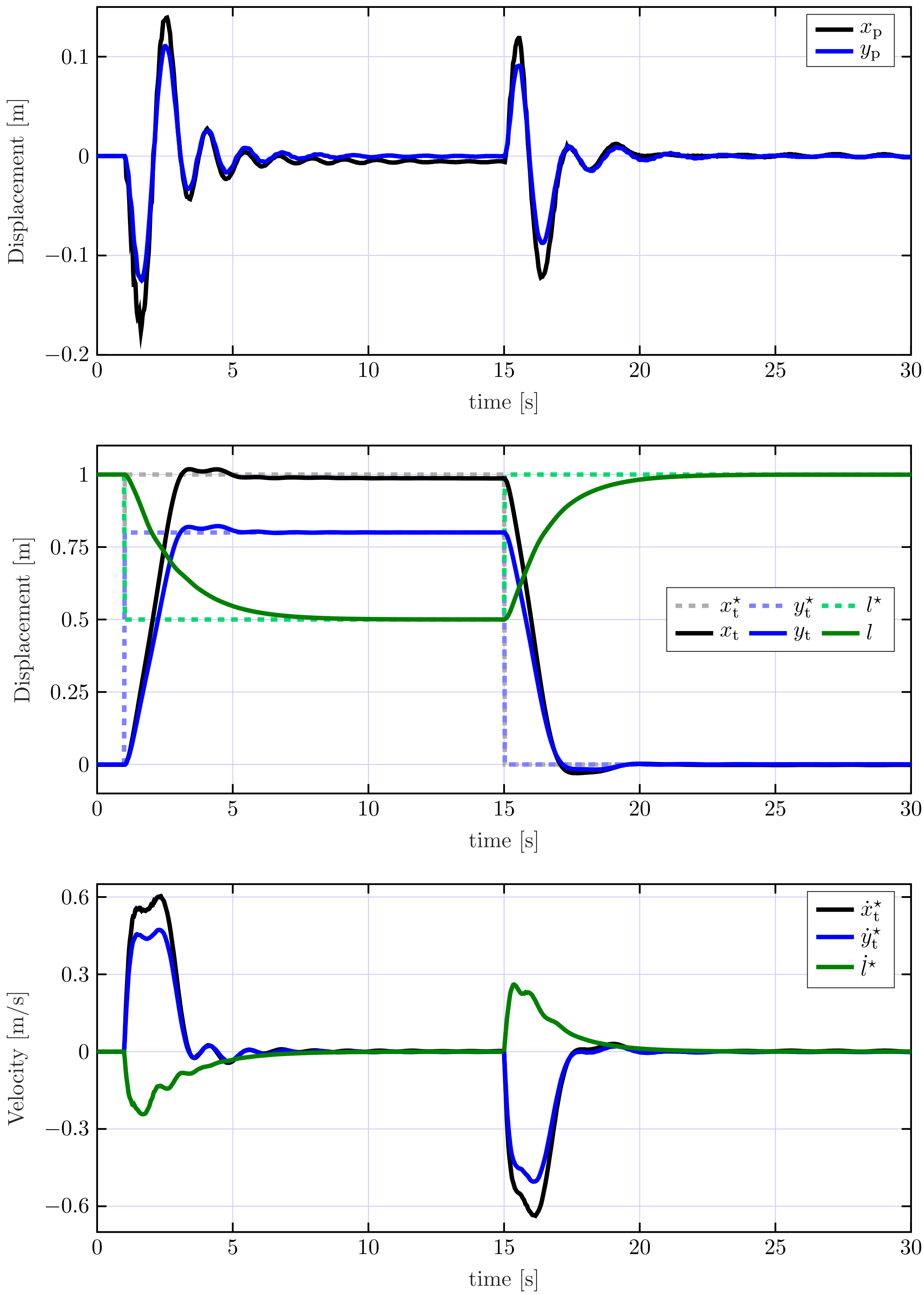


**Figure 7.7.** – Implementation of the IDA-PBC controller (7.30) on the portal crane test-bench of Figure 7.1.

## 7.2 Cart-pole System

When designing the IDA-PBC for the portal crane (in implicit representation) with Algorithm 6.1 or 6.2 (see Section 7.1), we always have a positive definite target inertia matrix $\mathbf{M}_\mathrm{d}$. As for other systems, having a sign-indefinite target inertia matrix can certainly simplify the analysis or even improve the results. This is the case of the stabilization of the cart-pole upright pendulum position. In this regard, we consider the cart-pole system of Sections 4.4 and 5.2.1 and synthesize a full-state feedback controller with Algorithm 6.2 under a constant target inertia matrix. Then, to enlarge the region of convergence, we design two additional full-state feedback controllers under state-dependent target inertia matrices: one with Algorithm 6.1 and the other with Algorithm 6.2. All (three) controllers are tuned to have the same optimal local performance. The results are compared in simulation and the best controller is implemented in the test-bench of Figure 4.7.

### 7.2.1 Implicit Model with PFL

The implicit port-Hamiltonian model of the cart-pole with negligible friction in the pendulum and Rayleigh dissipation $\frac{1}{2}c_\mathrm{c}\dot{x}_\mathrm{c}^2 \geq 0$ is given by (5.10). As described in Section 4.4, for the real system implementation, we employ the velocity tracking mode (of the servomotor) with an integrator in its input to have an approximate representation of the cart-pole in PFL with new input $u^* = \ddot{x}_\mathrm{c}^* \approx \ddot{x}_\mathrm{c}$, see Figure 4.9. Therefore, to derive such a model, it is used the method of Section 7.1.1 with

$$h(r) = x_\mathrm{c} \qquad \text{and} \qquad \hat{\mathbf{G}}_\perp = m_\mathrm{p}^{-1}\begin{bmatrix} I_2 & 0_{2\times 1}\end{bmatrix}.$$

Hence,

$$\Lambda(r) = \frac{l^2}{m_\mathrm{p}x_\mathrm{p}^2 + m_\mathrm{c}l^2} > 0 \qquad\qquad \forall r \in \mathcal{R}_\Phi,$$

and the complete system may be written as

$$\mathbf{M}\ddot{r} + \frac{\partial^\top \mathbf{V}}{\partial r} = \frac{\partial^\top \Phi}{\partial r}\lambda + \mathbf{G}u,$$

or equivalently (see Section 2.5.3)

$$\begin{bmatrix} \dot{r} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}}{\partial r} \\ \frac{\partial^\top \mathbf{H}}{\partial \rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial^\top \Phi}{\partial r} \end{bmatrix}\lambda + \begin{bmatrix} 0 \\ \mathbf{G} \end{bmatrix} u \tag{7.32a}$$

with holonomic constraints

$$\Phi(r) := \tfrac{1}{2}\left(x_\mathrm{p}^2 + y_\mathrm{p}^2 - l^2\right) = 0, \tag{7.32b}$$

where $\mathbf{M} = I_3$, $\mathbf{V} = g_c y_p$, $\mathbf{H}(r,\rho) = \frac{1}{2}\rho^\top \mathbf{M}^{-1}\rho + \mathbf{V}(r)$, $\mathbf{G} = \mathrm{vec}(-1,0,1)$ and $\mathcal{R} = \mathbb{R}^3$, meaning that the system is polynomial (see Definition 6.1). We leave the verification of Assumptions 5.6 and 5.8 to the reader.

### Reduction and Linearization

Reducing system (7.32), in feedback with $u = \mathbf{u}_{\mathrm{ida}}(r,\rho)$, to the explicit representation (see Corollary 6.2) results in

$$\begin{bmatrix}\dot q\\ \dot p\end{bmatrix} = \begin{bmatrix}0 & I_2\\ -I_2 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial^\top H}{\partial q}\\ \frac{\partial^\top H}{\partial p}\end{bmatrix} + \begin{bmatrix}0\\ G(q)\end{bmatrix} u_{\mathrm{ida}}(q,p),$$
$$H(q,p) = \tfrac{1}{2}p^\top M^{-1}(q)p + V(q) \tag{7.33}$$

where $q = \mathrm{vec}(\theta, x_c)$, $M(q) = \frac{\partial^\top \xi}{\partial q}\frac{\partial \xi}{\partial q} = \mathrm{diag}(l^2,1)$, $V(q) = \mathbf{V}(\xi(q)) = g_c l\cos\theta$, $u_{\mathrm{ida}}(q,p) = \mathbf{u}_{\mathrm{ida}}(\xi(q), L(q)p)$, $p = \frac{\partial^\top \xi}{\partial q}\rho$, $G(q) = \frac{\partial^\top \xi}{\partial q}\mathbf{G}$, $L(q) = \frac{\partial \xi}{\partial q}M^{-1}(q)$ and $r = \xi(q)$ follows from the geometric relation of Figure 4.8, i.e.,

$$x_p = l\sin\theta, \qquad\qquad y_p = l\cos\theta. \tag{7.34}$$

The linearization of the reduced system (7.33) about the equilibrium $x_d = \mathrm{vec}(q_d, 0)$, with $q_d = \xi^{-1}(r_d)$, yields

$$\dot{\tilde x} = \underbrace{\left(\begin{bmatrix}0 & \mathrm{diag}(l^{-2},1)\\ -\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q_d} & 0\end{bmatrix} + \left.\begin{bmatrix}0\\ \frac{\partial G u}{\partial x}\end{bmatrix}\right|_{\substack{x=x_d\\ u=u_{\mathrm{ida}}(q_d,0)}}\right)}_{=:A_r}\tilde x + \underbrace{\begin{bmatrix}0_{2\times 1}\\ G(q_d)\end{bmatrix}}_{=:B_r}\underbrace{\left.\frac{\partial u_{\mathrm{ida}}}{\partial x}\right|_{x=x_d}}_{=:K_r}\tilde x, \tag{7.35}$$

where $\tilde x = x - x_d$, $x = \mathrm{vec}(q,p)$.

## 7.2.2 Constructive Solution with Constant $\mathbf{M}_d$

We test Algorithm 6.2 with a constant $\mathbf{M}_d$ on the cart-pole system (7.32).

**Step 1:** We build

$$\mathbf{N}(r) := \begin{bmatrix}\mathbf{G} & \bar{\mathbf{B}}(r)\end{bmatrix} = \begin{bmatrix}\mathbf{G} & \frac{\partial^\top \Phi}{\partial r}\end{bmatrix} = \begin{bmatrix}-1 & x_p\\ 0 & y_p\\ 1 & 0\end{bmatrix},$$

and select $\mathbf{N}_\perp = \begin{bmatrix}y_p & -x_p & y_p\end{bmatrix}$ and $r_d = \mathrm{vec}(0, l, x_c^\star) \in \mathcal{R}_a = \{r \in \mathcal{R} \mid 0 = \Phi(r),\, 0 = x_p\}$, (upward pendulum position), where $x_c^\star$ is the desired cart position.

**Step 2:** Since $\mathbf{M}$ is constant, we can satisfy the kinetic matching (5.16a) by using a constant target inertia matrix $\mathbf{M}_d$, setting $\mathbf{\Gamma}_1(r,\rho)=0$ and imposing (6.19), see Section 6.2.2. Hence,

$$\mathbf{M}_{\mathrm{d}} = \begin{bmatrix} a & 0 & d-a \\ 0 & d & 0 \\ d-a & 0 & b \end{bmatrix}$$

for some arbitrary constants $a$, $b$ and $d$.

**Step 3:** We recall that $\mathbf{B}_{\mathrm{d}}(r) = \mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)$, $\Delta_{\mathrm{d}}(r) = \mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\mathbf{B}_{\mathrm{d}}(r)$ and $\mathbf{Z}_{\mathrm{d}}(r) = I_3 - \mathbf{B}_{\mathrm{d}}(r)\Delta_{\mathrm{d}}^{-1}(r)\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)$. Then,

$$\mathbf{Q}(r) := \begin{bmatrix}\mathbf{N}_{\perp}(r)\mathbf{Z}_{\mathrm{d}}(r)\mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r) & -\mathbf{N}_{\perp}(r)\frac{\partial \mathbf{V}}{\partial r}\end{bmatrix} = \begin{bmatrix} dy_{\mathrm{p}} & -dx_{\mathrm{p}} & y_{\mathrm{p}}(d-a+b) & g_{\mathrm{c}}x_{\mathrm{p}}\end{bmatrix}.$$

From $\mathbf{Q}_{\mathrm{I}}(r) = \begin{bmatrix} \frac{\partial^{\top}\Phi}{\partial r} & \bar{\mathbf{S}}(r) & \hat{\mathbf{S}}(r) \\ 0 & 0 & 1 \end{bmatrix}$, we can choose the gradient vectors

$$\bar{\mathbf{S}} = \begin{bmatrix} a-d-b \\ 0 \\ d \end{bmatrix}, \qquad \hat{\mathbf{S}} = \frac{1}{d}\begin{bmatrix} 0 \\ g_{\mathrm{c}} \\ 0 \end{bmatrix}.$$

**Step 4:** The dissipation condition (5.36b) is trivially satisfied by $\mathbf{R}=0$. For the sake of simplicity, we pick

$$\beta(\gamma) := \frac{1}{2}\mathbf{K}_{\gamma}(\gamma-\gamma^{\star})^2 + \beta_{\mathrm{c}},$$

where $\gamma^{\star}, \mathbf{K}_{\gamma}, \beta_{\mathrm{c}} \in \mathbb{R}$ are constants, $\beta_{\mathrm{c}}$ is selected such that $\mathbf{V}_{\mathrm{d}}(r_{\mathrm{d}})=0$, and $\gamma(r) = \bar{\mathbf{S}}^{\top}r$. From (6.33a)–(6.33c), $\mathbf{V}_{\mathrm{d}}|_{\mathcal{R}_{\Phi}}$ has a strict minimum in $r_{\mathrm{d}}$ if

$$\gamma^{\star} = dx_{\mathrm{c}}^{\star}, \qquad \mu^{\star} = -\frac{g_{\mathrm{c}}}{dl}, \qquad d<0, \qquad \mathbf{K}_{\gamma}>0. \tag{7.36}$$

Note that stabilizing $r_{\mathrm{d}} = \mathrm{vec}(0,l,x_{\mathrm{c}}^{\star})$ requires $\mathbf{M}_{\mathrm{d}} \neq \mathbf{M}$; however, if our goal were the downward pendulum position, i.e., $r_{\mathrm{d}} = \mathrm{vec}(0,-l,x_{\mathrm{c}}^{\star})$, then $d>0$ and we can actually take $\mathbf{M}_{\mathrm{d}} = \mathbf{M}$. Given that $\bar{\Delta}_{\mathrm{d}}(r_{\mathrm{d}}) = l^2 d > 0$, (5.33c) is verified if and only if so is (5.35), see Lemma 5.4. Hence, by setting

$$\bar{\mathbf{B}}_{\perp}(r) = \mathbf{B}_{\perp}(r) = \begin{bmatrix} -y_{\mathrm{p}} & x_{\mathrm{p}} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

condition (5.35) leads to

$$b>0, \qquad (2d+b-a)\,a > d^2. \tag{7.37}$$

**Step 5:** Since $\mathbf{R} = 0$, we pick $\bar{\mathbf{\Gamma}}_2(r) = \bar{\mathbf{G}}^{\mathrm{g}}(r)\mathbf{Z}_{\mathrm{d}}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\mathbf{K}_{\mathrm{v}}(r)\bar{\mathbf{S}}^\top(r)$ with

$$\mathbf{K}_{\mathrm{v}} > 0. \tag{7.38}$$

From $\bar{\mathbf{G}} = \mathrm{vec}(-y_{\mathrm{p}}^2 l^{-2}, x_{\mathrm{p}} y_{\mathrm{p}} l^{-2}, 1)$, we have $\bar{\mathbf{G}}_{\perp}\nu = 0$ and can select

$$\bar{\mathbf{G}}^{\mathrm{g}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

which is not the Moore–Penrose inverse of $\bar{\mathbf{G}}$. Hence, the controller (5.18) with

$$\mathbf{V}_{\mathrm{d}}(r) := \int_0^1 \hat{\mathbf{S}}^\top r \mathrm{d}v + \beta(\gamma(r)) = \frac{g_{\mathrm{c}}(y_{\mathrm{p}} - l)}{d} + \frac{1}{2}(\gamma(r) - \gamma^\star)^2 \mathbf{K}_\gamma$$

renders the system (7.32a) stable in $(r_{\mathrm{d}}, 0)$ whenever (7.36)–(7.38) are verified. Note that the stability is achieved despite the fact that $\mathbf{M}_{\mathrm{d}}$ is sign-indefinite from (7.36)–(7.37).

**Steps 6:** In view of conditions (7.36)–(7.38), we observe a great flexibility to pick the controller parameters. This problem is tackled by assigning an optimal local performance. For this we use the length $l = 0.4840\,\mathrm{m}$, measured from the test-bench of Figure 4.7 (see also [199]), and the linearized system (7.35) about $x_{\mathrm{d}} = \mathrm{vec}(q_{\mathrm{d}}, 0)$, with $q_{\mathrm{d}} = \mathrm{vec}(0, x_{\mathrm{c}}^\star) = \xi^{-1}(r_{\mathrm{d}})$, obtaining

$$A_{\mathrm{r}} = \begin{bmatrix} 0 & \mathrm{diag}(l^{-2}, 1) \\ g_{\mathrm{c}} l \,\mathrm{diag}(1, 0) & 0 \end{bmatrix}, \qquad B_{\mathrm{r}} = \begin{bmatrix} 0 & 0 & -l & 1 \end{bmatrix}^\top, \qquad K_{\mathrm{r}} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix},$$

$$k_1 = -\frac{g_{\mathrm{c}} a}{d} + \mathbf{K}_\gamma l(d - a + b)\eta + g_{\mathrm{c}}, \qquad k_2 = -d\mathbf{K}_\gamma \eta,$$

$$k_3 = \frac{1}{l}\mathbf{K}_{\mathrm{v}}(d - a + b)\eta, \qquad k_4 = -d\mathbf{K}_{\mathrm{v}}\eta, \qquad \eta = (2ad + ab - a^2 - d^2),$$

where the pair $(A_{\mathrm{r}}, B_{\mathrm{r}})$ is controllable. We set an LQR local optimal performance to the controller (5.18) by matching the linearized IDA-PBC feedback $K_{\mathrm{r}}\tilde{x}$ with the LQR feedback $u_{\mathrm{lqr}}(x) = -K_{\mathrm{lqr}}\tilde{x}$, i.e., we impose

$$-K_{\mathrm{lqr}} = K_{\mathrm{r}},$$

resulting in a solution to $a$, $b$, $\mathbf{K}_{\mathrm{v}}$ and $\mathbf{K}_\gamma$ with an arbitrary constant $d$. Fix $d = -1$, $Q_{\mathrm{lqr}} = \mathrm{diag}(10I_2, I_2)$ and $R_{\mathrm{lqr}} = 1$, then

$$a = 1.9483, \qquad b = 6.7953, \qquad \mathbf{K}_\gamma = 0.6955, \qquad \mathbf{K}_{\mathrm{v}} = 0.9099, \tag{7.39}$$

which are all consistent with (7.36)–(7.38). Now, the control law (5.18) reads

$$\mathbf{u}_{\mathrm{ida}}(r, \dot{r}) = \frac{1.308\dot{x}_{\mathrm{c}} + 5.033\dot{x}_{\mathrm{p}} + x_{\mathrm{c}} - x_{\mathrm{c}}^\star + 3.847x_{\mathrm{p}} + 39.04x_{\mathrm{p}}y_{\mathrm{p}} - 3.98x_{\mathrm{p}}(\dot{x}_{\mathrm{p}}^2 + \dot{y}_{\mathrm{p}}^2)}{3.98y_{\mathrm{p}}^2 - 0.6161}. \tag{7.40}$$

Asymptotic stability is a direct consequence of Lyapunov's indirect method.

### Position Feedback

We cannot impose the condition $0 = \bar{\mathbf{B}}_\perp(r)\mathbf{B}_\mathrm{d}(r)$ to build a position feedback controller (see Proposition 6.6 and Remark 6.3) because it implies $a = d$, contradicting the condition (7.36).

### Equivalence with the PID-PBC

Using Corollary 6.3, the closed-loop can be written in explicit representation as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & J(q) \\ -J^\top(q) & \Gamma_1(q,p) + \Gamma_2(q) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top H_\mathrm{d}}{\partial q} \\ \frac{\partial^\top H_\mathrm{d}}{\partial p} \end{bmatrix}, \tag{7.41}$$

where $H_\mathrm{d}(q,p) = \frac{1}{2}p^\top M_\mathrm{d}^{-1}(q)p + V_\mathrm{d}(q)$, $J(q) = M^{-1}M_\mathrm{d}(q)$, $T^\top(q) = \mathbf{J}^{-1}(\xi(q))\frac{\partial \xi}{\partial q}J(q)$,

$$\begin{aligned}
V_\mathrm{d}(q) &= \mathbf{V}_\mathrm{d}(\xi(q)) = g_\mathrm{c}k_\mathrm{e}k_\mathrm{u}l(\cos\theta - 1) + K_\mathrm{I}(k_\mathrm{a}x_\mathrm{c} - k_\mathrm{u}l\sin(\theta))^2, \\
M_\mathrm{d}^{-1}(q) &= M^{-1}\frac{\partial^\top \xi}{\partial q}\mathbf{M}_\mathrm{d}^{-1}\frac{\partial \xi}{\partial q}M^{-1} = \begin{bmatrix} \frac{k_\mathrm{e}k_\mathrm{u}}{l^2} + \frac{K_\mathrm{k}k_\mathrm{u}^2\cos^2\theta}{l^2} & -\frac{K_\mathrm{k}k_\mathrm{a}k_\mathrm{u}\cos\theta}{l} \\ -\frac{K_\mathrm{k}k_\mathrm{a}k_\mathrm{u}\cos\theta}{l} & k_\mathrm{a}k_\mathrm{e} + K_\mathrm{k}k_\mathrm{a}^2 \end{bmatrix}, \\
\Gamma_1(q,p) &= \left( \frac{\partial T\rho}{\partial q}J(q) - J^\top(q)\frac{\partial^\top T\rho}{\partial q} \right)\Bigg|_{\rho = \frac{\partial \xi}{\partial q}M^{-1}(q)p}, \\
\Gamma_2(q) &= T(q)\mathbf{\Gamma}_2(\xi(q))T^\top(q) = J^\top(q)\frac{\partial^\top \xi}{\partial q}\bar{\mathbf{S}}\mathbf{K}_\mathrm{v}\bar{\mathbf{S}}^\top\frac{\partial \xi}{\partial q}J(q),
\end{aligned}$$

$$k_\mathrm{u} = \frac{k_\mathrm{a}(d-a+b)}{d}, \quad k_\mathrm{e} = \frac{1}{k_\mathrm{a}(d-a+b)}, \quad K_\mathrm{k} = \frac{-d^2(a-d)}{k_\mathrm{a}^2(d-a+b)\det\mathbf{M}_\mathrm{d}}, \quad K_\mathrm{I} = \mathbf{K}_\gamma\frac{d^2}{k_\mathrm{a}^2}$$

and $k_\mathrm{a} \in \mathbb{R}$ is arbitrary. From the reduced system (7.41), we observe that the target energy function (Hamiltonian $H_\mathrm{d}$) and its time derivative $\dot{H}_\mathrm{d}(q,p) = K_p(\dot{x}_\mathrm{c}k_\mathrm{a} - \dot{\theta}k_\mathrm{u}l\cos\theta)^2$, with $K_\mathrm{p} = \mathbf{K}_\mathrm{v}\frac{d^2}{k_\mathrm{a}^2}$, are equal to the ones constructed in the PID-PBC of [20, 24], see also [18]. In other words, the control algorithms are equivalent.

### Region of Convergence

A central requirement in the IDA-PBC for implicit systems is the nonsingularity of $\Delta_\mathrm{d}$. This property assures that the trajectories of the closed-loop are consistent with the constrained state-space $\mathcal{X}_\mathrm{c}$. Besides, from Lemma 5.2, the stabilizing condition (5.31a) does not hold in $\{r \in \mathcal{R}_\Phi \mid \det\Delta_\mathrm{d}(r) = 0\}$. Consequently, we may focus on regions of convergence that are bounded by $\Delta_\mathrm{d}(r) = 0$. Figure 7.8 shows this bound for the previous controller where

$y_p > 0.3934\,\mathrm{m}$, meaning that such a region is a subset of $y_p > 0.3934\,\mathrm{m}$, or equivalently $\theta \in \,]-35.63°, 35.63°[$.

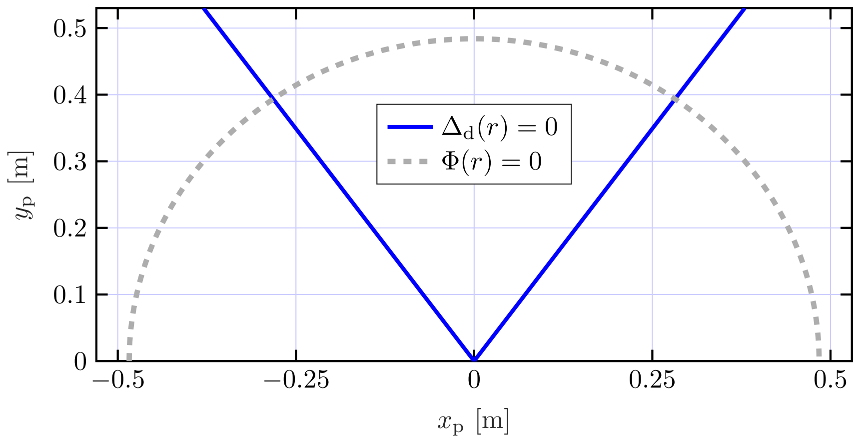


**Figure 7.8.** – Zero level sets of $\Delta_d$ and $\Phi$ for the constructive solution with constant $\mathbf{M}_d$.

### 7.2.3 Heuristic Solution with State-dependent $\mathbf{M}_d$

To increase the region of convergence of the previous controller, we have two main options: change the controller parameters or redesign the controller with a state-depended $\mathbf{M}_d$. The former is clearly the simplest option but it has a negative impact in the local performance assignment, whereas the latter may keep its local behavior but increases the design complexity. In this section, we choose the second option and solve the IDA-PBC problem for the cart-pole system (7.32) with the heuristic solution of Algorithm 6.1. We omit Step 1 because is identical to the one of the previous section.

**Step 2:** To avoid undesirable behavior and reduce the computational cost, we aim at a low polynomial order on $\mathbf{V}_d$, $\mathbf{M}_d^{-1}$ and $\bar{\mathbf{\Gamma}}_1$. Therefore, we choose $\mathbf{V}_d$ as a quadratic function of $r - \mathrm{vec}(0, 0, x_c^\star)$, $\mathbf{M}_d^{-1}$ quadratic in $(x_p, y_p)$ and $\bar{\mathbf{\Gamma}}_1(r, \rho)$ linear in $\rho$ but independent of $r$.

**Step 3:** Since $\mathbf{R}(r) = 0$, $\mathbf{B}(r) = \bar{\mathbf{B}}(r)$ and $\mathbf{M}$ is a constant, the dissipation condition (6.10) is trivially verified. Besides, we can impose (6.14) to satisfy $0 = \mathbf{K}_e(r)\mathbf{J}^{-\top}(r)\bar{\mathbf{G}}(r)$ and reduce (6.7a)–(6.7b) to

$$0 = \mathbf{K}_e(r)\left(-\frac{1}{2}\frac{\partial^\top \rho^\top \mathbf{M}_d^{-1} \rho}{\partial r} + \bar{\mathbf{\Gamma}}_1(r, \rho)\mathbf{M}^{-1}(r)\rho\right) \qquad \forall (r, \rho) \in \mathcal{X}_c,$$

$$0 = \mathbf{K}_e(r)\left(\mathbf{J}^{-\top}(r)\frac{\partial^\top \mathbf{V}}{\partial r} - \frac{\partial^\top \mathbf{V}_d}{\partial r}\right) \qquad \forall r \in \mathcal{R}_\Phi,$$

which are polynomial equations if $\mathbf{K}_e$ is polynomial. Now, we may employ Option 1, i.e., $\mathbf{K}_e(r) = \mathbf{N}_\perp(r)$, but the SDP solver will not converge, meaning that there is no solution for the upright position of the pendulum.[66] Consequently, we resort to Option 2, and employ $\hat{\mathbf{V}}_\mathrm{d}$ and $\hat{\mathbf{M}}_\mathrm{d}$ from the constructive solution. To avoid confusion, we use the accent $\hat{(\ )}$ to denote functions obtained from the constructive solution (see the previous section). Since $\hat{\mathbf{V}}_\mathrm{d}$ is polynomial and has a strict local minimum in $r_\mathrm{d}$, we can now select $\mathbf{K}_\mathrm{a} = 1$ and

$$\mathbf{K}_\mathrm{e}(r) = \mathbf{N}_\perp(r)\hat{\mathbf{Z}}_\mathrm{d}(r)\hat{\mathbf{M}}_\mathrm{d}\mathbf{M}^{-1}(r) = \begin{bmatrix} dy_\mathrm{p} & -dx_\mathrm{p} & y_\mathrm{p}(d-a+b)\end{bmatrix},$$

to satisfy the conditions (6.15), where $\hat{\mathbf{Z}}_\mathrm{d}(r) = I_3 - \hat{\mathbf{B}}_\mathrm{d}(r)\hat{\Delta}_\mathrm{d}^{-1}(r)\mathbf{B}^\top(r)\mathbf{M}^{-1}(r)$, $\hat{\Delta}_\mathrm{d}(r) = \mathbf{B}^\top(r)\mathbf{M}^{-1}(r)\hat{\mathbf{B}}_\mathrm{d}(r)$, $\hat{\mathbf{B}}_\mathrm{d}(r) = \hat{\mathbf{M}}_\mathrm{d}\mathbf{M}^{-1}(r)\mathbf{B}(r)$ and $(a, b, d)$ satisfies (7.36)–(7.37).

**Step 4:** We select $\epsilon = 10^{-5}$, $g_\mathrm{c} = 9.81$,

$$\left(\frac{\partial^\top \Phi}{\partial r}\right)_\perp = \mathbf{B}_\perp(r) = \begin{bmatrix} -y_\mathrm{p} & x_\mathrm{p} & 0 \\ 0 & 0 & 1\end{bmatrix},$$

and replace (5.33c) with the stronger condition

$$\mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r) - \psi(r)Q(r) - \epsilon I_2 \succeq 0, \qquad Q^\top(r) = Q(r) \succeq 0 \tag{7.42}$$

for some functions $Q : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ and $\psi : \mathcal{R} \to \mathbb{R}$ such that $r_\mathrm{d} \in \mathcal{R}_\psi := \{r \in \mathcal{R} \mid \psi(r) \geq 0\}$, $\mathcal{R}_\psi$ is compact and $\psi$ is user defined. Since $\epsilon > 0$, condition (7.42) guarantees

$$\mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r) \succ 0 \qquad \forall r \in \mathcal{R}_\psi, \tag{7.43}$$

which is necessary for $\Delta_\mathrm{d}$ being nonsingular in $\mathcal{R}_\psi$ (see Lemma 5.2), but sufficient for for (5.33c). Hence, by setting $\psi$ and verifying that $\mathbf{M}_\mathrm{d}$ is nonsingular in $\mathcal{R}_\psi$, we can change the bound $\Delta_\mathrm{d}(r) = 0$ (of the region of convergence). Note that $\mathbf{M}_\mathrm{d}$ can be sign indefinite, and therefore, we do not include any constraint on it, but analyze the nonsingularity later.

Now, let us consider the closed-loop from the constructive solution, By using Corollary 6.3, it is not difficult to see, that the local behavior, or equivalently the eigenvalues, are completely specified from

$$M\hat{M}_\mathrm{d}^{-1}(q_\mathrm{d})M = \begin{bmatrix} 0.3501 & 0.3138 \\ 0.3138 & 0.4285\end{bmatrix}, \qquad \left.\frac{\partial^2 \hat{V}_\mathrm{d}}{\partial q^2}\right|_{q=q_\mathrm{d}} = \begin{bmatrix} 7.159 & 1.295 \\ 1.295 & 0.6955\end{bmatrix}, \qquad \hat{\mathbf{K}}_\mathrm{v} = 0.9099.$$

Therefore, we can impose the following minimization objective to ensure convergence to the desired $M\hat{M}_\mathrm{d}^{-1}(q_\mathrm{d})M$ and $\left.\frac{\partial^2 \hat{V}_\mathrm{d}}{\partial q^2}\right|_{q=q_\mathrm{d}}$ as $\alpha \to 0$.

[66] Using Option 1 it is possible to find a solution for the downward position of the pendulum.

**Optimization 7.1** (Local performance assignment).

$$\begin{aligned} \text{minimize} \quad & \alpha \\ \text{subject to} \quad & \begin{bmatrix} \alpha I_4 & F \\ F^\top & I_4 \end{bmatrix} \succeq 0, \\ & F = \left[\begin{array}{c:c} M\hat{M}_\mathrm{d}^{-1}(q_\mathrm{d})M - \frac{\partial^\top \xi}{\partial q}\mathbf{M}_\mathrm{d}^{-1}(r_\mathrm{d})\frac{\partial \xi}{\partial q} & 0 \\ \hdashline 0 & \frac{\partial^2 \hat{V}_\mathrm{d}}{\partial q^2}\Big|_{q=q_\mathrm{d}} - \frac{\partial^2 \mathbf{V}_\mathrm{d}(\xi(q))}{\partial q^2}\Big|_{q=q_\mathrm{d}} \end{array}\right]. \end{aligned}$$

To be consistent with the local assignment, we also replace the values of $(a, b, d)$ from (7.39), obtaining $\mathbf{K}_\mathrm{e}(r) = \begin{bmatrix} -y_\mathrm{p} & x_\mathrm{p} & 3.847y_\mathrm{p} \end{bmatrix}$. For avoiding excessively large values on $\mathbf{M}_\mathrm{d}^{-1}$, we constrain the monomial coefficients of order 2 in $\mathrm{elem}_{1,1}(\mathbf{M}_\mathrm{d}^{-1})$ to be smaller than 20.

Let us set $l = 0.4840$ (obtained from the test-bench of Figure 4.7) and select $\psi(r) = 1 - \frac{1}{0.4^2}x_\mathrm{p}^2 - \frac{1}{0.3^2}(y_\mathrm{p} - l^2)$. The solution of the SOS Program 6.1 and Optimization 7.1 with SOSTOOLS and SDPT3 is $\mu_\mathrm{d} = 7.879$, $\alpha = 0$,

$$\mathbf{V}_\mathrm{d}(r) = 2.676(x_\mathrm{c} - x_\mathrm{c}^\star)x_\mathrm{p} + 0.3477(x_\mathrm{c} - x_\mathrm{c}^\star)^2 - 9.81y_\mathrm{p} + 9.876x_\mathrm{p}^2 + 4.73y_\mathrm{p}^2 + 3.64$$

$$\mathbf{M}_\mathrm{d}^{-1}(r) = \begin{bmatrix} 17.89(x_\mathrm{p}^2 + y_\mathrm{p}^2) - 24.94y_\mathrm{p} + 9.375 & 0 & 4.649(x_\mathrm{p}^2 + y_\mathrm{p}^2) - 6.483y_\mathrm{p} + 2.697 \\ 0 & -1 & 0 \\ 4.649(x_\mathrm{p}^2 + y_\mathrm{p}^2) - 6.483y_\mathrm{p} + 2.697 & 0 & 1.209(x_\mathrm{p}^2 + y_\mathrm{p}^2) - 1.685y_\mathrm{p} + 0.961 \end{bmatrix},$$

$$\bar{\mathbf{\Gamma}}_1(r, \rho) = \begin{bmatrix} 0 & 12.47\rho_1 + 3.241\rho_3 & 0 \\ -12.47\rho_1 - 3.241\rho_3 & 0 & -3.241\rho_1 - 0.8426\rho_3 \\ 0 & 3.241\rho_1 + 0.8426\rho_3 & 0 \end{bmatrix}.$$

Observe that $\alpha = 0$ and $\mathbf{M}_\mathrm{d}$ is sign-indefinite but nonsingular:

$$\det \mathbf{M}_\mathrm{d}^{-1}(r) = -3.441x_\mathrm{p}^2 - 3.441y_\mathrm{p}^2 + 4.798y_\mathrm{p} - 1.736 < 0.$$

Figure 7.9 illustrates the zero level sets of $\Delta_\mathrm{d}$, $\psi$ and $\Phi$, where the points of $\Delta_\mathrm{d}(r) = 0$ do not belong to the set $\mathcal{R}_\psi$ because $\mathbf{M}_\mathrm{d}^{-1}$ is nonsingular. Besides, we can conclude that the region of convergence is a subset of $y_\mathrm{p} > 0.3081\,\mathrm{m}$, or equivalently $\theta \in\, ]-50.46°, 50.46°[$.

**Step 5:** From the full-rank feature of $\bar{\mathbf{G}}$ and

$$\hat{\mathbf{G}}(r) := \mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\bar{\mathbf{G}}(r) = \begin{bmatrix} 6.483y_\mathrm{p}^2 - 4.786y_\mathrm{p} + 62.18y_\mathrm{p}^3 - 106.5y_\mathrm{p}^4 \\ 27.67y_\mathrm{p}^3 - 16.16y_\mathrm{p}^2 - 1.685y_\mathrm{p} + 1.244 \end{bmatrix},$$

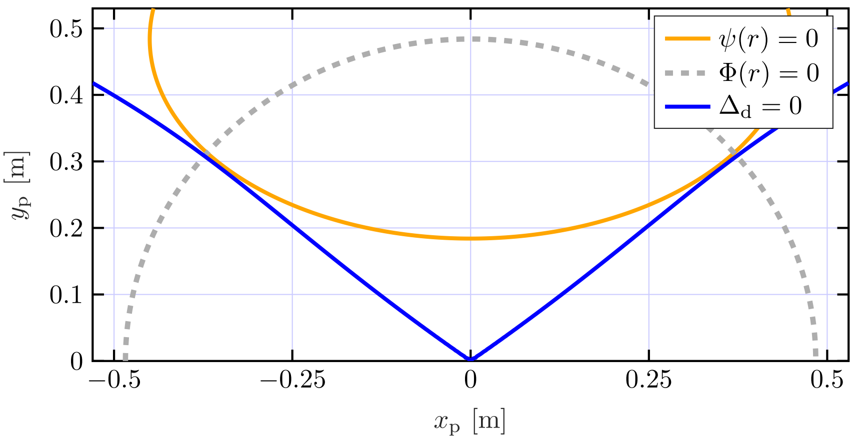


**Figure 7.9.** – Zero level sets of $\Delta_d$, $\psi$ and $\Phi$ for the heuristic solution with state-dependent inertia matrix $\mathbf{M}_d$.

we have $\bar{\mathbf{G}}_{\perp}(r)\nu = 0$ and can select

$$\hat{\mathbf{G}}^{g}(r) = \frac{1}{27.67y_p^3 - 16.16y_p^2 - 1.685y_p + 1.244}\begin{bmatrix}0 & 1\end{bmatrix},$$

Since $\mathbf{R} = 0$ and $\alpha = 0$, we pick $\bar{\mathbf{\Gamma}}_2(r) = \mathbf{K}_v\bar{\mathbf{G}}^{\top}(r)\mathbf{J}^{-1}(r)$ with $\mathbf{K}_v = 18.812$, which is calculated from the linearized system with $\hat{\mathbf{K}}_v$. Hence, the IDA-PBC controller (6.8) reads

$$\begin{aligned}
\mathbf{u}_{ida}(r,\dot{r}) &= \frac{1}{27.67y_p^3 - 16.16y_p^2 - 1.685y_p + 1.244}u_1(r,\dot{r}) - \mathbf{K}_v u_2(r,\dot{r}), \qquad (7.44)\\
u_1(r,\dot{r}) &= 0.6955(x_c^{\star} - x_c) - 2.676x_p + 0.8426\dot{x}_c\dot{y}_p + 3.241\dot{x}_p\dot{y}_p - 158.6x_py_p\\
&\quad + 16.16(\dot{x}_p^2 + \dot{y}_p^2)x_p + 271.5x_py_p^2 - 27.67(\dot{x}_p^2 + \dot{y}_p^2)x_py_p,\\
u_2(r,\dot{r}) &= 0.961\dot{x}_c + 2.697\dot{x}_p - 1.685\dot{x}_cy_p - 6.483\dot{x}_py_p + 1.209\dot{x}_cx_p^2 - 10.3\dot{x}_cy_p^2\\
&\quad + 27.67\dot{x}_cy_p^3 - 19.85\dot{x}_cy_p^4 - 35.37\dot{x}_py_p^2 + 106.5\dot{x}_py_p^3 - 76.35\dot{x}_py_p^4\\
&\quad + 76.35\dot{y}_px_py_p^3 - 19.85\dot{x}_cx_p^2y_p^2 - 8.918\dot{y}_px_py_p,
\end{aligned}$$

where used $\frac{\partial\Phi}{\partial r}\dot{r} = 0$ to reduce the expressions in $\bar{\mathbf{\Gamma}}_2(r)\dot{r}$. In summary, by using a state-dependent $\mathbf{M}_d$, we achieve the desired local performance and successfully enlarge the boundaries of the region of convergence.

**Step 6:** Asymptotic stability is a direct consequence of Lyapunov's indirect method.

### 7.2.4 Constructive Solution with State-dependent $\mathbf{M}_d$

Similar to the heuristic solution of Section 7.2.3 we may intent to enlarge the region of converge by synthesizing the cart-pole controller with Algorithm 6.1 under a state-dependent inertia matrix $\mathbf{M}_{\rm d}$ obtained from the characterization of Corollary 6.1. We omit the Step 1 because is equal to the one of Section 7.2.2.

**Step 2:** From Corollary 6.1, there exists a solution of $\mathbf{M}_{\rm d}$ characterized by (6.25) that satisfies the kinetic matching condition (5.16a) whenever (5.31a), (6.19) and (6.26) hold. Given that we verify (5.31a) or equivalently (5.33c) in Step 4, a solution to (6.26) is

$$\mathbf{M}_{\rm d4}(r) = \begin{bmatrix} -y_{\rm p} & x_{\rm p} & 0 \\ 0 & 0 & 1 \end{bmatrix}^\top,$$

and (6.19) results in

$$\mathbf{M}_{\rm d} = \begin{bmatrix} a_1 & 0 & a_2 - a_1 \\ 0 & a_2 & 0 \\ a_2 - a_1 & 0 & a_3 \end{bmatrix} + \mathbf{M}_{\rm d4}(r) \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix} \mathbf{M}_{\rm d4}^\top(r) + \mathbf{N}(r) \begin{bmatrix} b_1(r) & b_2(r) \\ b_2(r) & b_3(r) \end{bmatrix} \mathbf{N}^\top(r),$$

where $a_i$ and $d_i$ are arbitrary constants, and $b_i : \mathcal{R}_\Phi \to \mathbb{R}$ are also arbitrary.

**Step 3:** For simplicity, let us define $a_{\rm x} := a_2 - a_1 + a_3 + d_3$ and set

$$a_2 = -d_1 l^2, \qquad b_1(r) = b_2(r) = 0, \tag{7.45}$$

then

$$\mathbf{Q}(r) = \begin{bmatrix} -d_2 y_{\rm p}^2 & d_2 x_{\rm p} y_{\rm p} & a_{\rm x} y_{\rm p} - d_2 l^2 & g_{\rm c} x_{\rm p} \end{bmatrix}.$$

Assume that $d_2 \neq 0$ and $y_{\rm p} > 0$. By $x_{\rm p}^2 + y_{\rm p}^2 = l^2$, we can choose the gradient vectors

$$\bar{\mathbf{S}}(r) = \begin{bmatrix} -\dfrac{l^2}{l^2 - x_{\rm p}^2} + \dfrac{a_{\rm x}}{d_2\sqrt{l^2 - x_{\rm p}^2}} \\ 0 \\ 1 \end{bmatrix}, \qquad \hat{\mathbf{S}}(r) = \frac{1}{d_2 y_{\rm p}} \begin{bmatrix} 0 \\ -g_{\rm c} \\ 0 \end{bmatrix},$$

which are state-dependent in comparison with all the previous examples where they are just constants.

**Step 4:** Since $\mathbf{R} = 0$, (5.36b) holds. Now, define

$$\beta(\gamma) := \frac{1}{2}\mathbf{K}_\gamma(\gamma - \gamma^\star)^2 + \beta_{\rm c},$$

where $\gamma^\star, \mathbf{K}_\gamma, \beta_{\rm c} \in \mathbb{R}$ are constants,

$$\gamma(r) := \int_0^1 \bar{\mathbf{S}}^\top(vr)\, r dv = x_{\rm c} - l\,\text{arctanh}(\frac{x_{\rm p}}{l}) + \frac{a_{\rm x}}{d_2}\arcsin(\frac{x_{\rm p}}{l}).$$

and $\beta_c$ is selected such that $\mathbf{V}_d(r_d) = 0$. Then, from (6.33a)–(6.33c), it follows that

$$\gamma^\star = x_c^\star, \qquad \mu^\star = \frac{g_c}{d_2 l^2}, \qquad d_2 > 0, \qquad \mathbf{K}_\gamma > 0. \tag{7.46}$$

From Lemma 5.4, a local condition to satisfy (5.31a) is (5.35) with $\bar{\Delta}_d(r_d) = (b_3(r_d) - d_1)l^4$ being nonsingular. Hence, by setting

$$\bar{\mathbf{B}}_\perp(r) = \mathbf{B}_\perp(r) = \begin{bmatrix} -y_p & x_p & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the previous conditions reduce to

$$b_3(r_d) \neq d_1, \quad a_1 + d_1 l^2 > 0, \quad a_1 a_x l^2 - d_2^2 l^4 - 2a_1 d_2 l^3 + a_x d_1 l^4 - 2d_1 d_2 l^5 > 0. \tag{7.47}$$

**Step 5:** $\bar{\mathbf{\Gamma}}_2$, $\bar{\mathbf{G}}^g$ and $\bar{\mathbf{G}}_\perp \nu = 0$ remain as in Step 5 of Section 7.2.2. Hence, the controller (6.21) with

$$\mathbf{V}_d(r) := \int_0^1 \hat{\mathbf{S}}^\top(vr)\, r\mathrm{d}v + \beta(\gamma(r)) = -\frac{g_c}{d_2}\ln(\frac{y_p}{l}) + \frac{1}{2}\mathbf{K}_\gamma(\gamma(r) - \gamma^\star)^2$$

renders the system (7.32a) stable in $(r_d, 0)$ whenever (7.45)–(7.47) are verified.

**Steps 6:** To select the controller parameters, we follow the same procedure of Step 6 in Section 7.2.2 but with $a_1 = 0.03$, $d_1 = 0.6$ and $b_3(r) = 0$ ($l$, $Q_{lqr}$ and $R_{lqr}$ remain equal), then

$$d_2 = 0.1809, \qquad a_x = 0.4243, \qquad \mathbf{K}_\gamma = 7.9447, \qquad \mathbf{K}_v = 10.3945,$$

which are all consistent with (7.46)–(7.47). Note that under these parameters $\mathbf{M}_d$ is sign-indefinite. Now, the control law (6.21) reads

$$\mathbf{u}_{ida}(r,\dot{r}) = \frac{u_1(r)}{10.7y_p^3 - 0.441y_p} + \frac{u_2(r,\dot{r})}{u_4(r)} + \frac{u_3(r,\dot{r})}{45.69y_p^5 - 1.882y_p^3}, \tag{7.48}$$

$$\begin{aligned}
u_1(r) &= (x_c - x_c^\star)(0.1484 + 5.599y_p + 24.43y_p^2) - 4.326x_p + 81.59x_p y_p + 105.0x_p y_p^2 \\
&\quad + \arcsin(2.066x_p)(0.3482 - 13.14y_p + 57.32y_p^2) \\
&\quad - \operatorname{arctanh}(2.066x_p)(0.0718 - 2.71y_p + 11.83y_p^2), \\
u_2(r,\dot{r}) &= \dot{y}_p^2\left(20.2x_p y_p^2 - 1.877x_p y_p - 0.3567x_p + 4.438x_p y_p^3 - 89.56x_p y_p^4\right) \\
&\quad + \dot{x}_c\dot{y}_p\left(0.7607y_p + 0.9788y_p^2 - 18.46y_p^3 - 0.0403\right) \\
&\quad - \dot{x}_p\dot{y}_p\left(2.648y_p^2 - 3.062y_p + 41.18y_p^3 - 6.703y_p^4 - 120.1y_p^5 + 0.2104\right) \\
&\quad + (\dot{y}_p^2 + \dot{x}_p^2)(47.56 + 130.5y_p)x_p y_p^5, \\
u_4(r) &= 0.2482y_p^2 - 0.0807y_p + 3.915y_p^3 - 15.83y_p^4 - 47.51y_p^5 + 207.3y_p^6 + 0.0021,
\end{aligned}$$

$$u_3(r, \dot{r}) = 9.271\dot{x}_\mathrm{p}y_\mathrm{p} - 0.1942\dot{x}_\mathrm{p} + 0.8291\dot{x}_\mathrm{c}y_\mathrm{p}^2 - 31.27\dot{x}_\mathrm{c}y_\mathrm{p}^3 + 136.5\dot{x}_\mathrm{c}y_\mathrm{p}^4 - 105.3\dot{x}_\mathrm{p}y_\mathrm{p}^2 + 320.2\dot{x}_\mathrm{p}y_\mathrm{p}^3,$$

Asymptotic stability is a direct consequence of Lyapunov's indirect method.

Figure 7.10 depicts the zero level sets of $\Delta_\mathrm{d}$, $\det \mathbf{M}_\mathrm{d}^{-1}$ and $\Phi$, where we can deduce that region of convergence is a subset of $y_\mathrm{p} > 0.203\,\mathrm{m}$, or equivalently $\theta \in \,]-65.20°, 65.20°[$. In other words, we can achieve the desired local behavior and, at the same time, have the largest bound for the region of convergence.

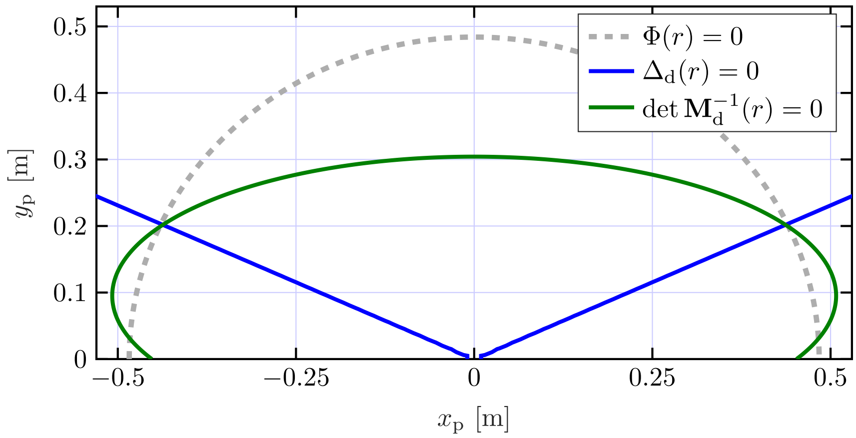


**Figure 7.10.** – Zero level sets of $\Delta_\mathrm{d}$, $\Phi$ and $\det \mathbf{M}_\mathrm{d}^{-1}$ for the constructive solution with state-dependent and sign-indefinite $\mathbf{M}_\mathrm{d}$.

As discussed in Section 5.2.3, forcing $\mathbf{M}_\mathrm{d}$ to be positive definite may hinder its solution. This can be seen e.g., by setting $b_3 > d_1$. Then, $\mathbf{M}_\mathrm{d}(r_\mathrm{d}) \succ 0$, but the zero level set of $\det \mathbf{M}_\mathrm{d}^{-1}$ significantly reduces the maximum bound for the region of convergence, see Figure 7.11 where $b_3 = 1$.

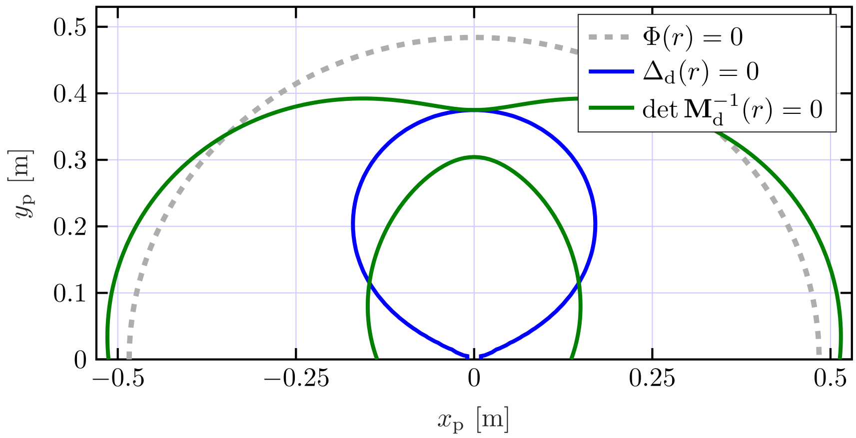


**Figure 7.11.** – Zero level sets of $\Delta_\mathrm{d}$, $\Phi$ and $\det \mathbf{M}_\mathrm{d}^{-1}$ for the constructive solution with state-dependent and locally positive definite $\mathbf{M}_\mathrm{d}$.

### 7.2.5 Simulations and Real-system Implementation

Figures 7.12 to 7.14 illustrates the simulation results for the cart-pole with the controllers of Table 7.2 and initial conditions

*i)* $\theta(0) = 30°$, $x_c(0) = -0.9\,\text{m}$, $\dot{\theta}(0) = -120°/\text{s}$ and $\dot{x}_c(0) = 0\,\text{m/s}$,

*ii)* $\theta(0) = 45°$, $x_c(0) = -0.9\,\text{m}$, $\dot{\theta}(0) = -120°/\text{s}$ and $\dot{x}_c(0) = 0\,\text{m/s}$, and

*iii)* $\theta(0) = 60°$, $x_c(0) = -0.87\,\text{m}$, $\dot{\theta}(0) = -134.7°/\text{s}$ and $\dot{x}_c(0) = -0.09\,\text{m/s}$,

respectively. We restrict the time span of $t$ to 10 s, where the desired cart position is set to

- $x_c^\star = -0.9\,\text{m}$ for $t \in [0, 5[$ s and
- $x_c^\star = 1\,\text{m}$ for $t \in [5, 10[$ s.

| Name | Description | Eq. |
|---|---|---|
| Im-Cons-C | Full-state feedback IDA-PBC synthesized from the constructive method with constant $\mathbf{M}_d$ for the implicit model (Section 7.2.2) | (7.40) |
| Im-Heu | Full-state feedback IDA-PBC synthesized from the heuristic method (SOS programs) with state-dependent $\mathbf{M}_d$ for the implicit model (Section 7.2.3) | (7.44) |
| Im-Cons-SD | Full-state feedback IDA-PBC synthesized from the constructive method with state-dependent $\mathbf{M}_d$ for the implicit model (Section 7.2.4) | (7.48) |
| Ex-NL | Full-state feedback IDA-PBC synthesized from the algebraic method with SOS programs for the polynomial explicit model (Section 4.4) | (4.24) |

**Table 7.2.** – IDA-PBC controllers for the cart-pole.

Clearly, from the initial condition *i*, all controllers achieve the asymptotic stabilization in $x_c^\star$ and $\theta^\star = 0$. However, the controller from constructive method and constant $\mathbf{M}_d$, which has an equivalence with the PID-PBC, fails under the initial conditions *ii* and *iii*. Similarly, the controller from the heuristic solution enlarges the region of convergence of the aforementioned controller, but it also fails for the initial conditions *iii*. These results are consistent with the bound determined from the zero level set of $\Delta_d$, see Figures 7.8 to 7.10.

Comparing the controllers with the largest region of convergence (relative to the pendulum inclination), i.e., (4.24) and (7.48), we observe that the controller from the constructive method and state-dependent $\mathbf{M}_d$ has a faster settling time, but it also yields a bigger overshoot whenever the initial conditions are more demanding. For the algebraic IDA-PBC controller, we notice that the initial conditions *iii* are slightly outside of the conservative

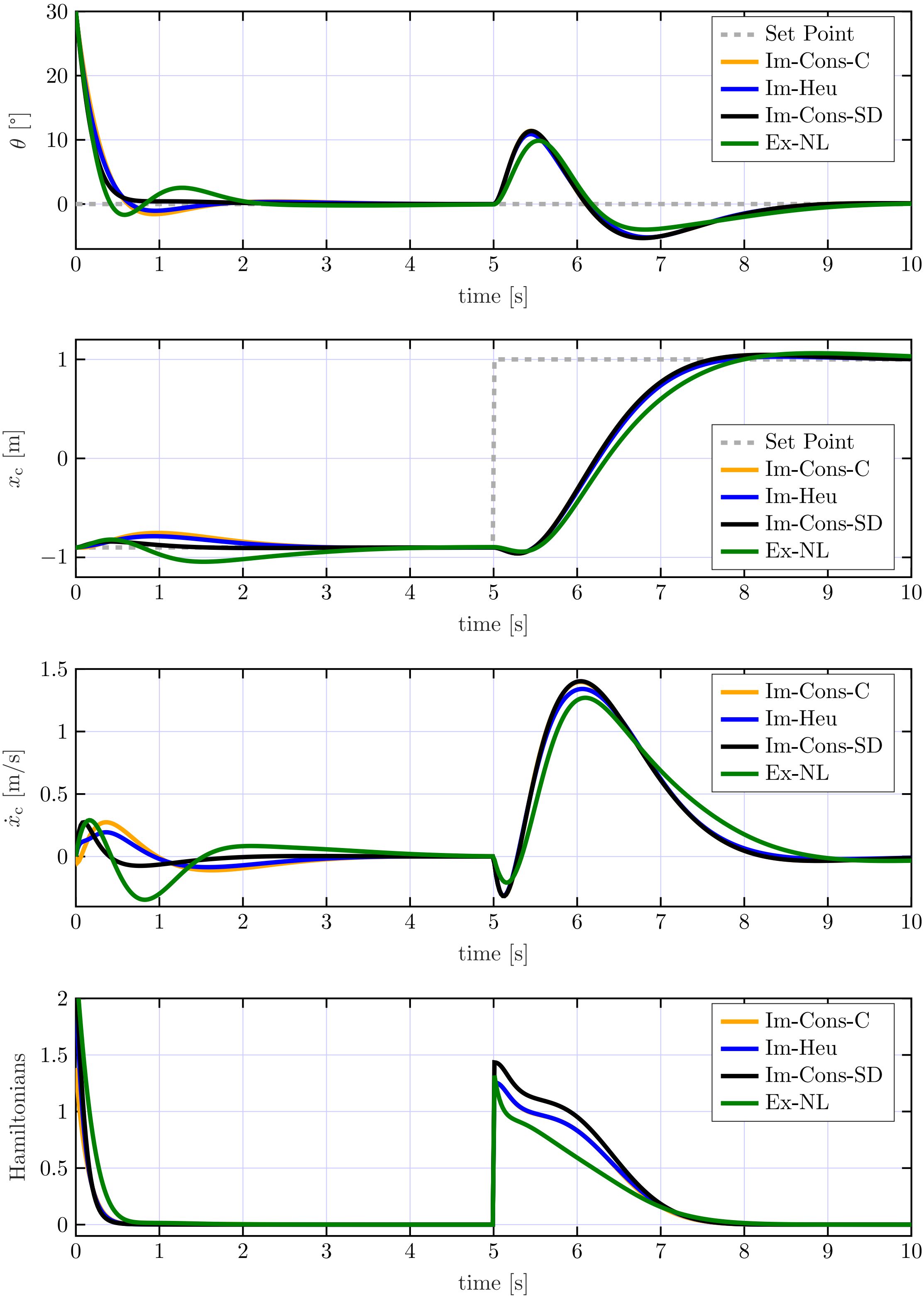


**Figure 7.12.** – Cart-pole simulation with initial conditions *i*.

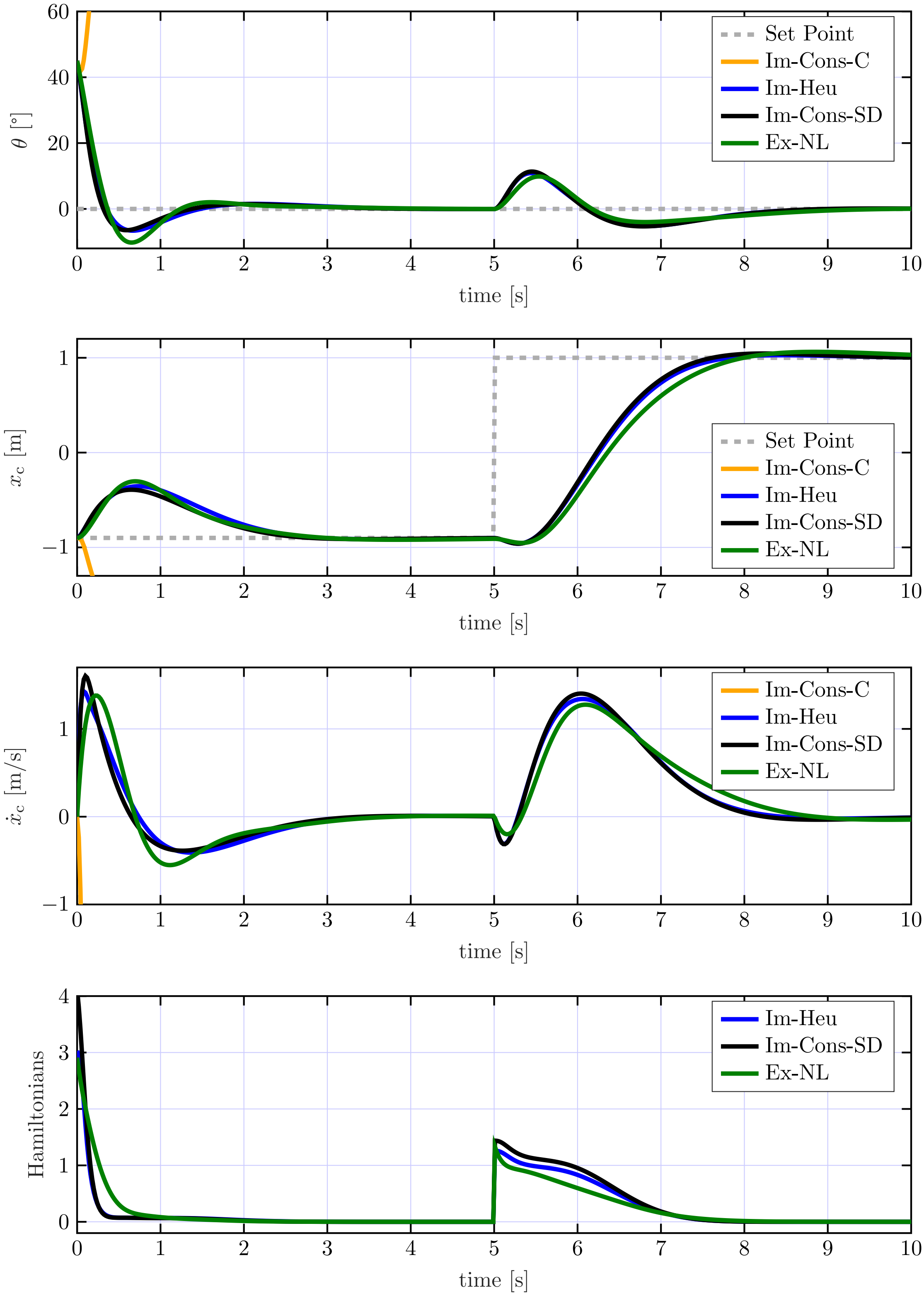


**Figure 7.13.** – Cart-pole simulation with initial conditions *ii*.

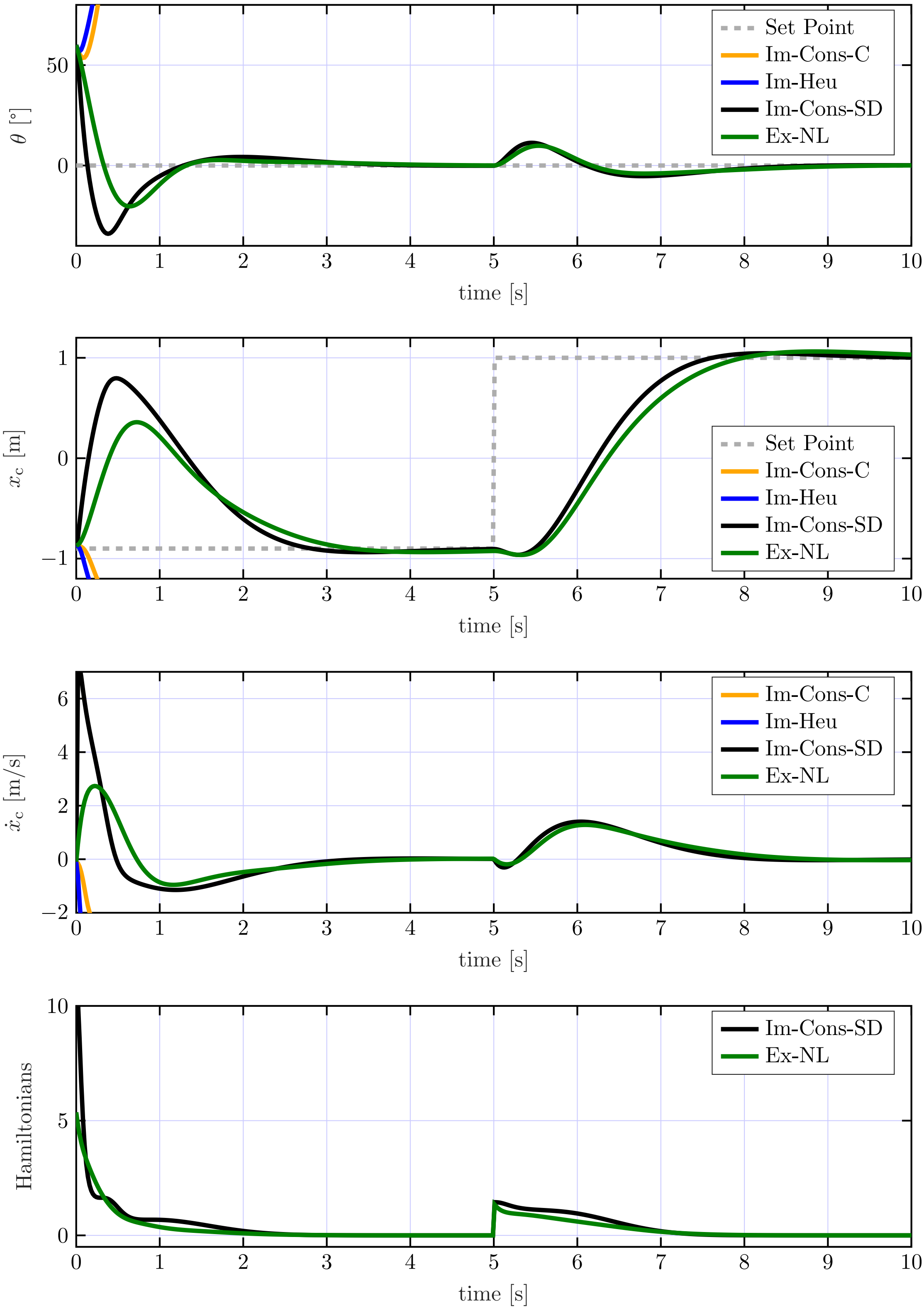


**Figure 7.14.** – Cart-pole simulation with initial conditions *iii*.

region of attraction $\mathcal{A}_{0.5}$ without affecting the stability result, see Figure 4.10. Besides, this controller does not posses exactly the same LQR local optimal performance but the best one it can achieve with the same weighting matrices, namely $Q_{\text{lqr}} = \text{diag}(10I_2, I_2)$ and $R_{\text{lqr}} = 1$. For visualization, the Hamiltonians of (7.48) and (4.24) are scaled by 10 and $10^{-1}$, respectively. Besides, they are monotonically decreasing functions, which is consistent with $\dot{\mathbf{H}}_{\text{d}} \leq 0$ and $\dot{H}_{\text{d}} \leq 0$.

The physical setup is analogous to the one for the portal crane, it consist of a PC with Matlab/Simulink, an RTI-toolbox and a dSPACE controller [234] that is connected to the servo drive of the cart-pole as well as the sensors (encoders, limit switches, etc.) we recall that $\dot{x}_{\text{c}}$ is the input of the velocity tracking system, see Figure 4.9. The control strategy is carried out in two stages: swing-up of the pendulum and set point stabilization with IDA-PBC. The swing-up controller is taken from [103, 199] and it follows a passivity based design for implicit systems that does not preserve the port-Hamiltonian form of the closed-loop. This controller is employed whenever $\theta \notin\ ]-60°, 60°[$. For the second stage, given by the IDA-PBC, we test the controllers with the largest region of convergence, i.e., (4.24) and (7.48). Figure 7.15 portraits the experimental results in the test bench of Figure 4.7 with initial conditions *iii*. Both controllers yield similar results to the simulation (see Figure 7.14), but there is a remarkable difference in the first 2 seconds of the $x_{\text{c}}$, which corresponds to the saturation in $\dot{x}_{\text{c}}$ and $\ddot{x}_{\text{c}}$ of the real experiment.

## 7.3 PVTOL Aircraft

When addressing the portal crane or cart-pole systems with the techniques of Chapters 5 and 6, we design the controllers under the assumption of negligible friction in the underactuated coordinates, meaning that the dissipation condition (5.36b) was trivial. As our last example, we consider the simplified model of a Planar Vertical Takeoff-and-landing (PVTOL) aircraft, introduced by [235]. Since the effect of drag forces for laminar or turbulent flow (dissipation) is not usually included in the controller design, see e.g., [63, 235–239], our objective is to design an IDA-PBC controller with the constructive solution of Algorithm 6.2 under the presence of drag for a laminar flow (low velocities) that renders the closed-loop system asymptotically stable in a desired spatial position. For simplicity, we omit the drag for turbulent flow in our discussion, i.e., dissipation forces that are proportional to the squared velocity, because they would modify the kinetic matching (5.16a) rather than in the dissipation terms of (5.17). Figure 7.16 illustrates the PVTOL aircraft diagram, where $(x_{\text{a}}, y_{\text{a}})$ describes the planar position of the aircraft's center of mass, $\theta$ is roll angle, $\varepsilon > 0$ denotes the relationship between rolling torque and lateral acceleration, $u_1$ and $u_2$ are control inputs, and $g_{\text{c}}$ is the gravity constant.

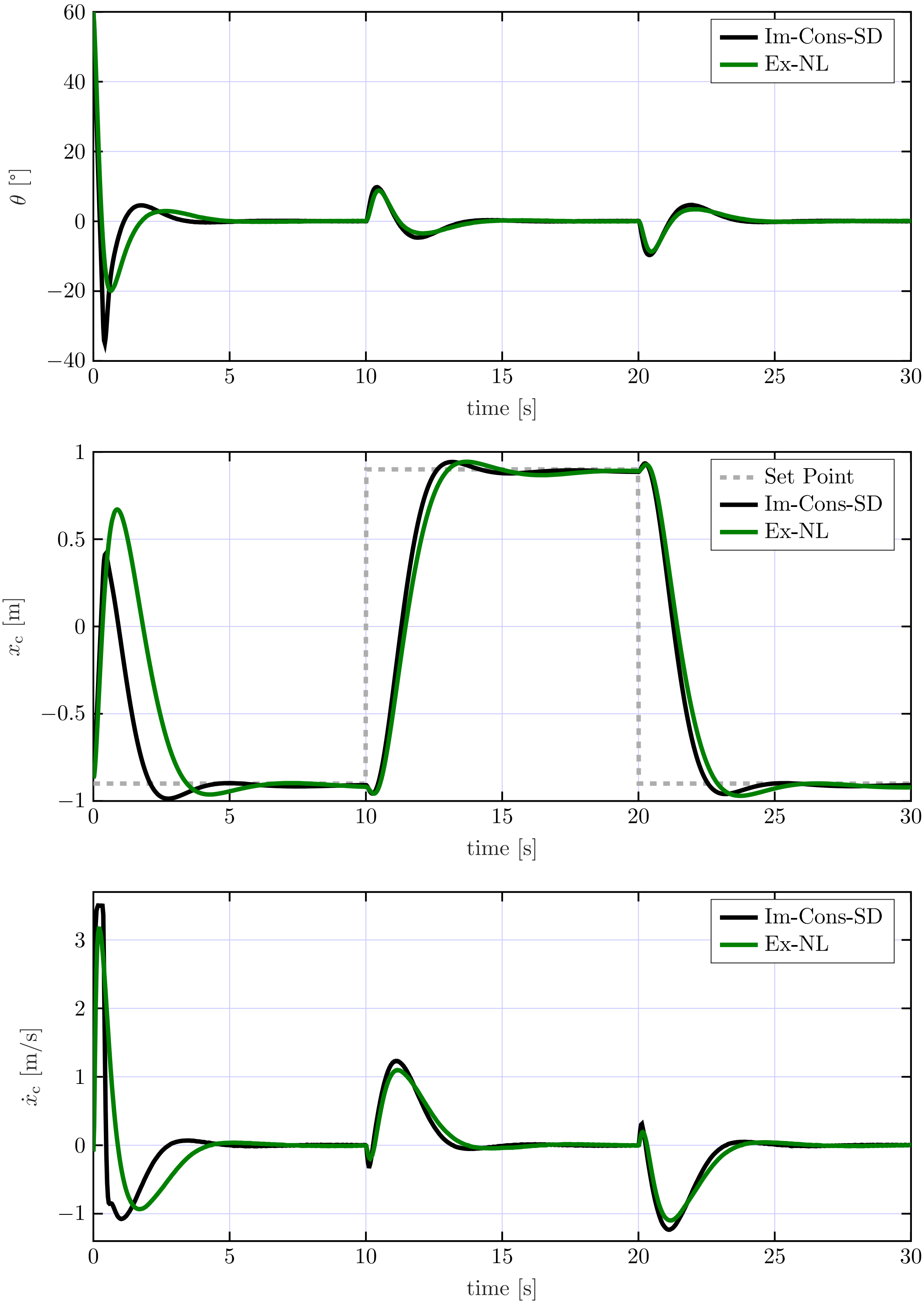


**Figure 7.15.** – Cart-pole real experiment with initial conditions *iii*.

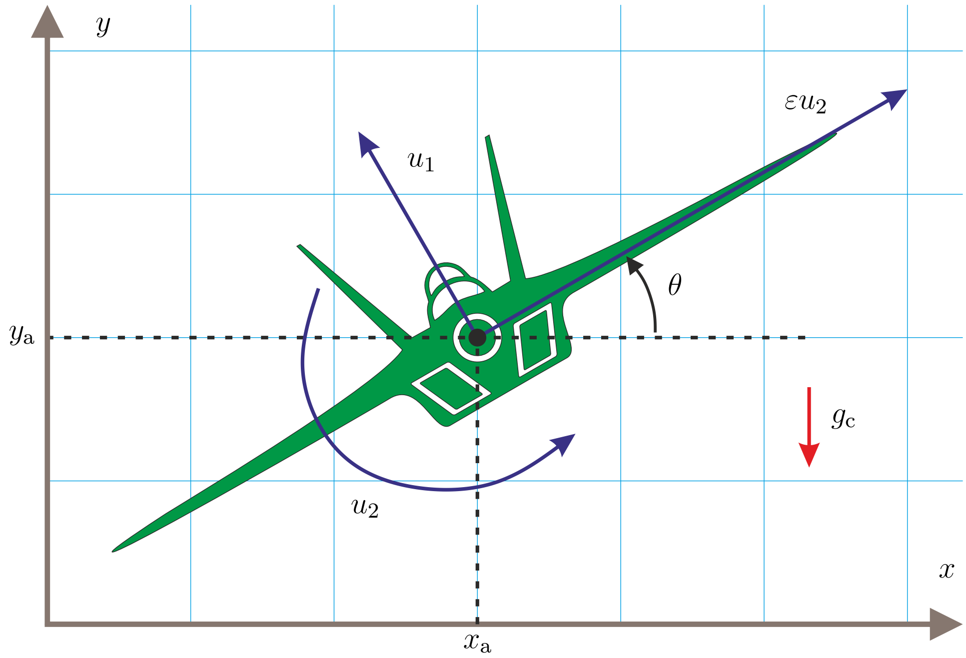


**Figure 7.16.** – PVTOL aircraft diagram.

### 7.3.1 Explicit Model

In summary, the PVTOL aircraft has a configuration space given by $\mathbb{R}^2 \times \mathbb{S}^1$ and its dynamical model with Rayleigh dissipation

$$D(q,\dot{q}) = \frac{1}{2}c_1\left(\dot{y}_\mathrm{a}\cos(\theta) - \dot{x}_\mathrm{a}\sin(\theta)\right)^2 + \frac{1}{2}c_2\left(\dot{x}_\mathrm{a}\cos(\theta) + \dot{y}_\mathrm{a}\sin(\theta)\right)^2 + \frac{1}{2}c_\theta\dot{\theta}^2$$

can be written as

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I_3\\ -I_3 & -R(q)\end{bmatrix}\begin{bmatrix}\frac{\partial^\top H}{\partial q}(q,p)\\ \frac{\partial^\top H}{\partial p}(q,p)\end{bmatrix} + \begin{bmatrix}0\\ G(q)\end{bmatrix}u, \tag{7.49}$$

$$H(q,p) = \tfrac{1}{2}p^\top M^{-1}p + g_\mathrm{c}m_\mathrm{a}y_\mathrm{a} = \tfrac{1}{2}\dot{q}^\top M\dot{q} + g_\mathrm{c}m_\mathrm{a}y_\mathrm{a},$$

$$M = \begin{bmatrix}m_\mathrm{a} & 0 & 0\\ 0 & m_\mathrm{a} & 0\\ 0 & 0 & J_\mathrm{a}\end{bmatrix}, \qquad G(q) = \begin{bmatrix}-\sin\theta & \varepsilon\cos\theta\\ \cos\theta & \varepsilon\sin\theta\\ 0 & 1\end{bmatrix},$$

$$R(q) = \begin{bmatrix}c_2\cos(\theta)^2 + c_1\sin(\theta)^2 & \frac{c_2-c_1}{2}\sin(2\theta) & 0\\ \frac{c_2-c_1}{2}\sin(2\theta) & c_1\cos(\theta)^2 + c_2\sin(\theta)^2 & 0\\ 0 & 0 & c_\theta\end{bmatrix},$$

where $q = \mathrm{vec}(x_\mathrm{a}, y_\mathrm{a}, \theta) \in \mathcal{Q} \subset \mathbb{R}^3$, $m_\mathrm{a}$ is the mass of the aircraft, $J_\mathrm{a}$ is the moment of inertia from the center of mass, and $(c_1, c_2, c_\theta)$ are the dissipation coefficients for a movement in the direction of $u_1$, $\varepsilon u_2$ and $\theta$, respectively.

We remark that this system cannot be addressed with the well-known PID-PBC because the distribution spanned by the columns of the input matrix is not involutive. Consequently, there are no change of coordinates such that the input matrix $G$ can be written as $G = \begin{bmatrix} 0_2 & I_2 \end{bmatrix}^\top$, see [19].

### 7.3.2 Implicit Model

Let us choose the coordinates $r = \mathrm{vec}(x_\mathrm{a}, y_\mathrm{a}, x_\theta, y_\theta) \in \mathcal{R} \subset \mathbb{R}^4$ subject to the holonomic constraint

$$0 = \Phi(r) := \frac{1}{2}\left(x_\theta^2 + y_\theta^2 - 1\right), \tag{7.50a}$$

with $x_\theta = \cos\theta$ and $y_\theta = \sin\theta$, then the system (in implicit representation) takes the form

$$\begin{bmatrix} \dot r \\ \dot\rho \end{bmatrix} = \begin{bmatrix} 0 & I_4 \\ -I_4 & -\mathbf{R}(r) \end{bmatrix} \begin{bmatrix} \frac{\partial^\top \mathbf{H}}{\partial r}(r,\rho) \\ \frac{\partial^\top \mathbf{H}}{\partial \rho}(r,\rho) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\partial^\top \Phi}{\partial r}(r) \end{bmatrix}\lambda + \begin{bmatrix} 0 \\ \mathbf{G}(r) \end{bmatrix} u, \tag{7.50b}$$

$$\mathbf{H}(r,\rho) = \tfrac{1}{2}\rho^\top \mathbf{M}^{-1}\rho + g_\mathrm{c} m_\mathrm{a} y_\mathrm{a} = \tfrac{1}{2}\dot r^\top \mathbf{M}\dot r + g_\mathrm{c} m_\mathrm{a} y_\mathrm{a},$$

$$\mathbf{M} = \begin{bmatrix} m_\mathrm{a} & 0 & 0 & 0 \\ 0 & m_\mathrm{a} & 0 & 0 \\ 0 & 0 & I_\mathrm{a} & 0 \\ 0 & 0 & 0 & I_\mathrm{a} \end{bmatrix}, \quad \mathbf{G}(r) = \begin{bmatrix} -y_\theta & \varepsilon x_\theta \\ x_\theta & \varepsilon y_\theta \\ 0 & -y_\theta \\ 0 & x_\theta \end{bmatrix}, \quad \frac{\partial^\top \Phi}{\partial r}(r) = \begin{bmatrix} 0 \\ 0 \\ x_\theta \\ y_\theta \end{bmatrix},$$

$$\mathbf{R}(r) = \begin{bmatrix} c_2 x_\theta^2 + c_1 y_\theta^2 & (c_2 - c_1)\, x_\theta y_\theta & 0 & 0 \\ (c_2 - c_1)\, x_\theta y_\theta & c_1 x_\theta^2 + c_2 y_\theta^2 & 0 & 0 \\ 0 & 0 & c_\theta & 0 \\ 0 & 0 & 0 & c_\theta \end{bmatrix},$$

where $\mathbf{G}$ and $\mathbf{R}$ are obtained from $\dot r^\top \mathbf{G}(r)u = \dot q^\top G(q)u$, the Rayleigh dissipation function

$$\mathbf{D}(r,\dot r) = \frac{1}{2}c_1\left(\dot y_\mathrm{a} x_\theta - \dot x_\mathrm{a} y_\theta\right)^2 + \frac{1}{2}c_2\left(\dot x_\mathrm{a} x_\theta + \dot y_\mathrm{a} y_\theta\right)^2 + \frac{1}{2}c_\theta(\dot x_\theta^2 + \dot y_\theta^2)$$

and $\dot\theta^2 = \dot x_\theta^2 + \dot y_\theta^2$. In addition, we assume that the initial conditions are consistent (Assumption 5.7) and leave the verification of Assumptions 5.6–5.8 to the reader.

### 7.3.3 Constructive Solution

**Step 1:** We build

$$\mathbf{N}(r) := \begin{bmatrix} \mathbf{G} & \bar{\mathbf{B}}(r) \end{bmatrix} = \begin{bmatrix} \mathbf{G} & \frac{\partial^\top \Phi}{\partial r} \end{bmatrix} = \begin{bmatrix} -y_\theta & \varepsilon x_\theta & 0 \\ x_\theta & \varepsilon y_\theta & 0 \\ 0 & -y_\theta & x_\theta \\ 0 & x_\theta & y_\theta \end{bmatrix},$$

and select

$$\mathbf{N}_\perp \begin{bmatrix} -x_\theta & -y_\theta & -\varepsilon y_\theta & \varepsilon x_\theta \end{bmatrix}$$

and $r_\mathrm{d} = \mathrm{vec}(x_\mathrm{a}^\star, y_\mathrm{a}^\star, 1, 0) \in \mathcal{R}_a = \{r \in \mathcal{R} \mid 0 = \Phi(r),\, 0 = y_\theta\}$, where $(x_\mathrm{a}^\star, y_\mathrm{a}^\star)$ is the desired position of the center of mass.

**Step 2:** By Corollary 6.1, there is a solution of $\mathbf{M}_\mathrm{d}$, characterized by (6.25), verifying the kinetic matching condition (5.16a) whenever (5.31a), (6.19) and (6.26) hold. Hence, we verify (5.31a) or equivalently (5.33c) in Step 4, compute a solution to (6.26) as

$$\mathbf{M}_{\mathrm{d}4}(r) = \begin{bmatrix} x_\theta & y_\theta & 0 & 0 \\ 0 & 0 & -y_\theta & x_\theta \end{bmatrix}^\top,$$

and satisfy (6.19) with

$$\mathbf{M}_\mathrm{d}(r) = \begin{bmatrix} a_1 & a_7 & a_6\varepsilon & a_4\varepsilon - a_3\varepsilon - a_5 \\ a_7 & a_2 & a_5 & -a_6\varepsilon \\ a_6\varepsilon & a_5 & a_3 & a_6 \\ a_4\varepsilon - a_3\varepsilon - a_5 & -a_6\varepsilon & a_6 & a_4 \end{bmatrix} + \mathbf{M}_{\mathrm{d}4}(r) \begin{bmatrix} d_1 & d_2 \\ d_2 & d_3 \end{bmatrix} \mathbf{M}_{\mathrm{d}4}^\top(r)$$
$$+ \mathbf{N}(r)\,\mathrm{diag}(b_1(r), b_2(r), b_3(r))\mathbf{N}^\top(r)$$

for some arbitrary constants $a_i$, $d_i$ and functions $b_i : \mathcal{R}_\Phi \to \mathbb{R}$.

**Step 3:** Define

$$e_1 := \frac{a_1 + d_1 + a_5\varepsilon - d_2\varepsilon + a_3\varepsilon^2 - a_4\varepsilon^2}{m_\mathrm{a}}, \qquad e_2 := \frac{a_2 + d_1 + a_5\varepsilon - d_2\varepsilon}{m_\mathrm{a}},$$
$$e_3 := \frac{a_5 - d_2 + a_3\varepsilon + d_3\varepsilon}{J_\mathrm{a}}.$$

and set

$$a_7 = -a_6\varepsilon^2. \tag{7.51}$$

From

$$\mathbf{Q}(r) = \begin{bmatrix} -x_\theta e_1 & -y_\theta e_2 & -y_\theta e_3 & x_\theta e_3 & g_\mathrm{c} m_\mathrm{a} y_\theta \end{bmatrix},$$

we can select the constant matrices

$$\bar{\mathbf{S}} = \begin{bmatrix} e_3 & 0 & 0 & e_1 \\ 0 & e_3 & -e_2 & 0 \end{bmatrix}^\top, \qquad \hat{\mathbf{S}} = \mathrm{vec}(0, 0, \frac{g_\mathrm{c} m_\mathrm{a}}{e_2}, 0).$$

**Step 4:** The dissipation condition (5.36b) reduces to

$$c_2(e_1 x_\theta^2 + e_2 y_\theta^2) + c_\theta e_3 \varepsilon \geq 0. \tag{7.52}$$

Define

$$\beta(\gamma) := \frac{1}{2}(\gamma - \gamma^\star)^\top \mathbf{K}_\gamma (\gamma - \gamma^\star) + \beta_{\mathrm{c}},$$

where $\gamma^\star = \mathrm{vec}(\gamma_1^\star, \gamma_2^\star) \in \mathbb{R}^2$, $\mathbf{K}_\gamma = \mathrm{diag}(k_{\gamma 1}, k_{\gamma 2}) \in \mathbb{R}^{2\times 2}$ and $\beta_{\mathrm{c}} \in \mathbb{R}$ are constants, and $\gamma(r) = \bar{\mathbf{S}}^\top r$. Then, $\mathbf{V}_{\mathrm{d}}|_{\mathcal{R}_\Phi}$ has a strict local minimum in $r_{\mathrm{d}}$ whenever (6.33a)–(6.33c) hold, i.e.,

$$\gamma^\star = \begin{bmatrix} e_3 x_{\mathrm{a}}^\star \\ e_3 y_{\mathrm{a}}^\star - e_2 \end{bmatrix}, \qquad \mu^\star = -\frac{g_{\mathrm{c}} m_{\mathrm{a}}}{e_3}, \qquad e_3 < 0, \qquad \mathbf{K}_\gamma = \mathrm{diag}(k_{\gamma 1}, k_{\gamma 2}) \succ 0. \tag{7.53}$$

Condition (5.33c) is verified if and only if so is (5.35) and $\bar{\Delta}_{\mathrm{d}}(r_{\mathrm{d}}) = \frac{a_3 + b_3(r_{\mathrm{d}})}{J_{\mathrm{a}}^2} \neq 0$. Hence, by setting

$$\bar{\mathbf{B}}_\perp(r) = \mathbf{B}_\perp(r) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -y_\theta & x_\theta \end{bmatrix},$$

the previous conditions yields $b_3(r_{\mathrm{d}}) \neq -a_3$,

$$\begin{bmatrix} w_1 & -a_6\varepsilon^2 - \frac{a_5 a_6 \varepsilon}{a_3 + b_3(r_{\mathrm{d}})} & w_2 \\ -a_6\varepsilon^2 - \frac{a_5 a_6 \varepsilon}{a_3 + b_3(r_{\mathrm{d}})} & a_2 + b_1 - \frac{a_5^2}{a_3 + b_3(r_{\mathrm{d}})} & -a_6\varepsilon - \frac{a_5 a_6}{a_3 + b_3(r_{\mathrm{d}})} \\ w_2 & -a_6\varepsilon - \frac{a_5 a_6}{a_3 + b_3(r_{\mathrm{d}})} & a_4 + d_3 + b_2(r_{\mathrm{d}}) - \frac{a_6^2}{a_3 + b_3(r_{\mathrm{d}})} \end{bmatrix} \succ 0, \tag{7.54}$$

where $w_1 = a_1 + d_1 + \varepsilon^2 b_2(r_{\mathrm{d}}) - \frac{a_6^2 \varepsilon^2}{a_3 + b_3(r_{\mathrm{d}})}$, $w_2 = d_2 - a_5 - a_3\varepsilon + a_4\varepsilon + \varepsilon b_2(r_{\mathrm{d}}) - \frac{a_6^2 \varepsilon}{a_3 + b_3(r_{\mathrm{d}})}$.

**Step 5:** From $\bar{\mathbf{G}}(r) = \mathbf{G}(r)$, we have $\bar{\mathbf{G}}_\perp \nu = 0$ and can select

$$\bar{\mathbf{G}}^{\mathrm{g}} = \begin{bmatrix} -y_\theta & x_\theta & -x_\theta \frac{a_5}{a_3} & -y_\theta \frac{a_5}{a_3} \\ 0 & 0 & -y_\theta & x_\theta \end{bmatrix},$$

which is not the Moore–Penrose inverse of $\bar{\mathbf{G}}$. From the dissipation condition (5.36b), there exists a function $\bar{\mathbf{\Gamma}}_2$ such that the controller (6.21) with

$$\mathbf{V}_{\mathrm{d}}(r) = \hat{\mathbf{S}}^\top r + \beta(\gamma(r)) = \frac{g_{\mathrm{c}} m_{\mathrm{a}}(x_\theta - 1)}{e_3} + \frac{1}{2}(\gamma(r) - \gamma^\star)^\top \mathbf{K}_\gamma (\gamma(r) - \gamma^\star)$$

renders the system (7.50) stable in $(r_{\mathrm{d}}, 0)$ whenever (7.51)–(7.54) are verified. Observe that we do not consider any assumption on the coupling parameter $\varepsilon$ as it is usually imposed in the literature [5, 235, 236]. For simplicity, suppose $\bar{\mathbf{\Gamma}}_2(r) = \bar{\mathbf{G}}^{\mathrm{g}}(r)\mathbf{Z}_{\mathrm{d}}(r)\mathbf{J}^\top(r)\bar{\mathbf{S}}(r)\mathbf{K}_{\mathrm{v}}(r)\bar{\mathbf{S}}^\top(r)$

with

$$\mathbf{K}_{\mathrm{v}} = \mathrm{diag}(k_{\mathrm{v}1}, k_{\mathrm{v}2}) \succ 0 \tag{7.55}$$

is sufficient to guarantee the desired result.

**Steps 6:** To select the controller parameters we assign an optimal local performance. For this, we reduce the system with Corollary 6.2, linearized it about $x_{\mathrm{d}} = \mathrm{vec}(q_{\mathrm{d}}, 0)$, with $q_{\mathrm{d}} = \mathrm{vec}(0, 0, x_{\mathrm{t}}^{\star}, y_{\mathrm{t}}^{\star}) = \xi^{-1}(r_{\mathrm{d}})$, and match the linearized IDA-PBC feedback $K_{\mathrm{r}}\tilde{x}$ with the LQR feedback $u_{\mathrm{lqr}}(x) = -K_{\mathrm{lqr}}\tilde{x}$, as implement in Section 7.1.3, 7.1.6, or 7.2.2. For illustration, let us select $m_{\mathrm{a}} = J_{\mathrm{a}} = c_{\theta} = \varepsilon = 1$, $c_1 = c_2 = 0.5$, $a_1 = a_2$, $a_3 = a_4 = 10$, $a_6 = d_i = b_i = 0$, $Q_{\mathrm{lqr}} = \mathrm{diag}(10I_5, 1)$ and $R_{\mathrm{lqr}} = \mathrm{diag}(5, 10)$, then

$$\begin{aligned}
a_1 &= 40.014, & k_{\gamma 1} &= 12.277 \times 10^{-4}, & k_{\mathrm{v}1} &= 35.812 \times 10^{-4},\\
a_5 &= -16.6785, & k_{\gamma 2} &= 25.997 \times 10^{-4}, & k_{\mathrm{v}2} &= 50.616 \times 10^{-4},
\end{aligned}$$

which are all consistent with (7.51)–(7.55). Now, the control law (5.18) reads

$$\begin{aligned}
\mathbf{u}_{\mathrm{ida}}(r, \dot{r}) &= u_1(r) + u_2(r, \dot{r}) + u_3(r, \dot{r}), \qquad (7.56)\\
u_1(r) &= \begin{bmatrix} 0.7873(x_{\mathrm{a}} - x_{\mathrm{a}}^{\star})y_{\theta} + 1.414x_{\theta}(y_{\mathrm{a}}^{\star} - y_{\mathrm{a}}) + 2.323y_{\theta}^2 + 15.05x_{\theta} - 5.24 \\ x_{\theta}(x_{\mathrm{a}} - x_{\mathrm{a}}^{\star}) + 2.95x_{\theta}y_{\theta} + 1.796y_{\theta}(y_{\mathrm{a}} - y_{\mathrm{a}}^{\star}) - 19.11y_{\theta} \end{bmatrix},\\
u_2(r, \dot{r}) &= \begin{bmatrix} 0.466(\dot{x}_{\mathrm{a}}\dot{y}_{\theta} - \dot{x}_{\theta}\dot{y}_{\mathrm{a}}) + 0.0607(\dot{x}_{\theta}^2 + \dot{y}_{\theta}^2) \\ 0.5919(\dot{y}_{\theta}\dot{y}_{\mathrm{a}} + \dot{x}_{\theta}\dot{x}_{\mathrm{a}}) \end{bmatrix},\\
u_3(r, \dot{r}) &= \begin{bmatrix} y_{\theta}(1.632\dot{x}_{\mathrm{a}} - 6.046\dot{y}_{\theta}) - x_{\theta}(8.141\dot{x}_{\theta} + 2.197\dot{y}_{\mathrm{a}}) \\ y_{\theta}(10.34\dot{x}_{\theta} + 2.791\dot{y}_{\mathrm{a}}) + x_{\theta}(2.073\dot{x}_{\mathrm{a}} - 7.679\dot{y}_{\theta}) \end{bmatrix}.
\end{aligned}$$

To proof that the selected $\bar{\mathbf{\Gamma}}_2$ verifies (5.31b), we recall that $\mathbf{\Gamma}_1$ is skew-symmetric (see Corollary 6.1) while $\mathbf{B}_{\mathrm{d}}(r)\mathbf{M}_{\mathrm{d}}(r)\rho = 0$ holds along the system trajectories. Then, by the extended Finsler Lemma 3.3, we can write (5.31b) as

$$(\mathbf{B}_{\mathrm{d}}(r))_{\perp}\left(\mathbf{Z}(r)\mathbf{R}(r)\mathbf{J}(r) + \mathbf{J}^{\top}(r)\mathbf{R}(r)\mathbf{Z}^{\top}(r) + \mathbf{J}^{\top}(r)\bar{\mathbf{S}}(r)\mathbf{K}_{\mathrm{v}}\bar{\mathbf{S}}^{\top}(r)\mathbf{J}(r)\right)(\mathbf{B}_{\mathrm{d}}(r))_{\perp}^{\top} \succeq 0,$$

which is satisfied by replacing all the previous parameters. This implies that the closed-loop system including dissipation (drag for a laminar flow) is stable in $(r_{\mathrm{d}}, 0)$. Finally, asymptotic stability is a direct consequence of Lyapunov's indirect method.

### Equivalence with the Controlled Lagrangians

Set $a_3 = a_4$, $d_i = b_2 = b_3 = a_6 = 0$ and $b_1 = \frac{a_5^2}{a_3}$, then from Corollary 6.3 and the equivalence between IDA-PBC and Controlled Lagrangians, see [17, 67], we can deduce that the resulting closed-loop is equivalent to the one of [239], which has shown outstanding performance in comparison with the controllers of [58, 240] but does not analyze dissipation. In other words, our controller is a generalization of [239], where the additional parameters can be used to have a better specification of the transient response.

### 7.3.4 Simulations

As in [58, 239, 240], the simulation experiments of the PVTOL aircraft with feedback (7.56) are divided in two parts.

*i)* Lateral motion near the ground: initial conditions $x_{\mathrm{a}}(0) = -5\,\mathrm{m}$, $y_{\mathrm{a}}(0) = 0\,\mathrm{m}$, $\theta(0) = 5.7296°$, $\dot{x}_{\mathrm{a}}(0) = -0.1\,\mathrm{m/s}$, $\dot{y}_{\mathrm{a}}(0) = -0.1\,\mathrm{m/s}$, $\dot{\theta}(0) = 5.7296\,°/\mathrm{s}$, and target position $x_{\mathrm{a}}^{\star} = 5\,\mathrm{m}$ and $y_{\mathrm{a}}^{\star} = 0\,\mathrm{m}$.

*ii)* Aggressive maneuver (from the upside down position): initial conditions $x_{\mathrm{a}}(0) = 5\,\mathrm{m}$, $y_{\mathrm{a}}(0) = -5\,\mathrm{m}$, $\theta(0) = 180°$, $\dot{x}_{\mathrm{a}}(0) = 0.1\,\mathrm{m/s}$, $\dot{y}_{\mathrm{a}}(0) = -0.1\,\mathrm{m/s}$, $\dot{\theta}(0) = 5.7296\,°/\mathrm{s}$, and target position $x_{\mathrm{a}}^{\star} = -5\,\mathrm{m}$ and $y_{\mathrm{a}}^{\star} = 5\,\mathrm{m}$.

Figures 7.17 and 7.18 illustrates the results for the Parts *i* and *ii*, respectively, where the controller achieves the asymptotic stabilization for both objectives under the presence of drag ($c_1 = c_2 = 1$ and $c_\theta = 0.5$) and the Hamiltonians are monotonically decreasing functions, which is consistent with $\dot{\mathbf{H}}_{\mathrm{d}} \leq 0$.

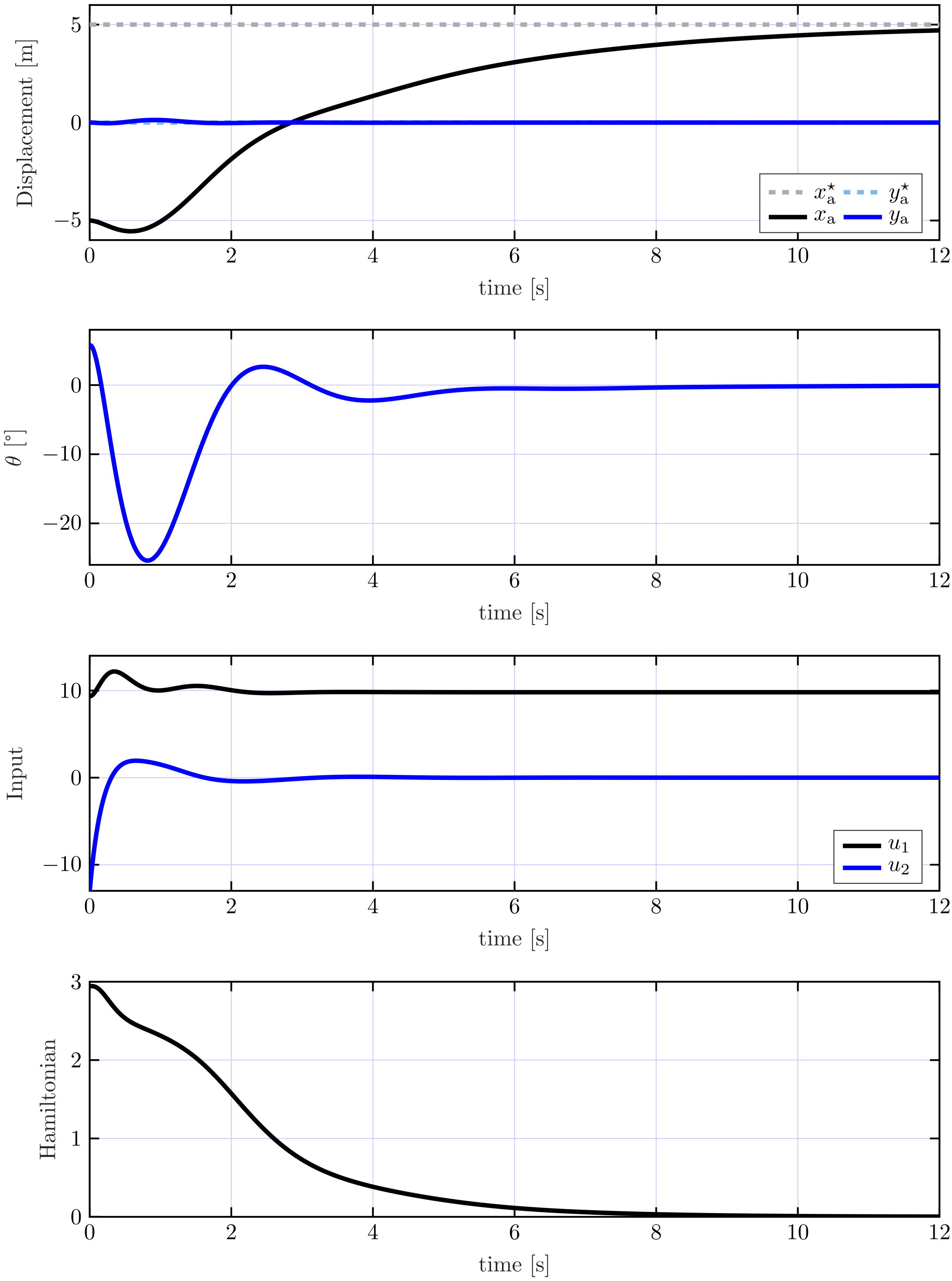


**Figure 7.17.** – PVTOL aircraft simulation from Part *i*.

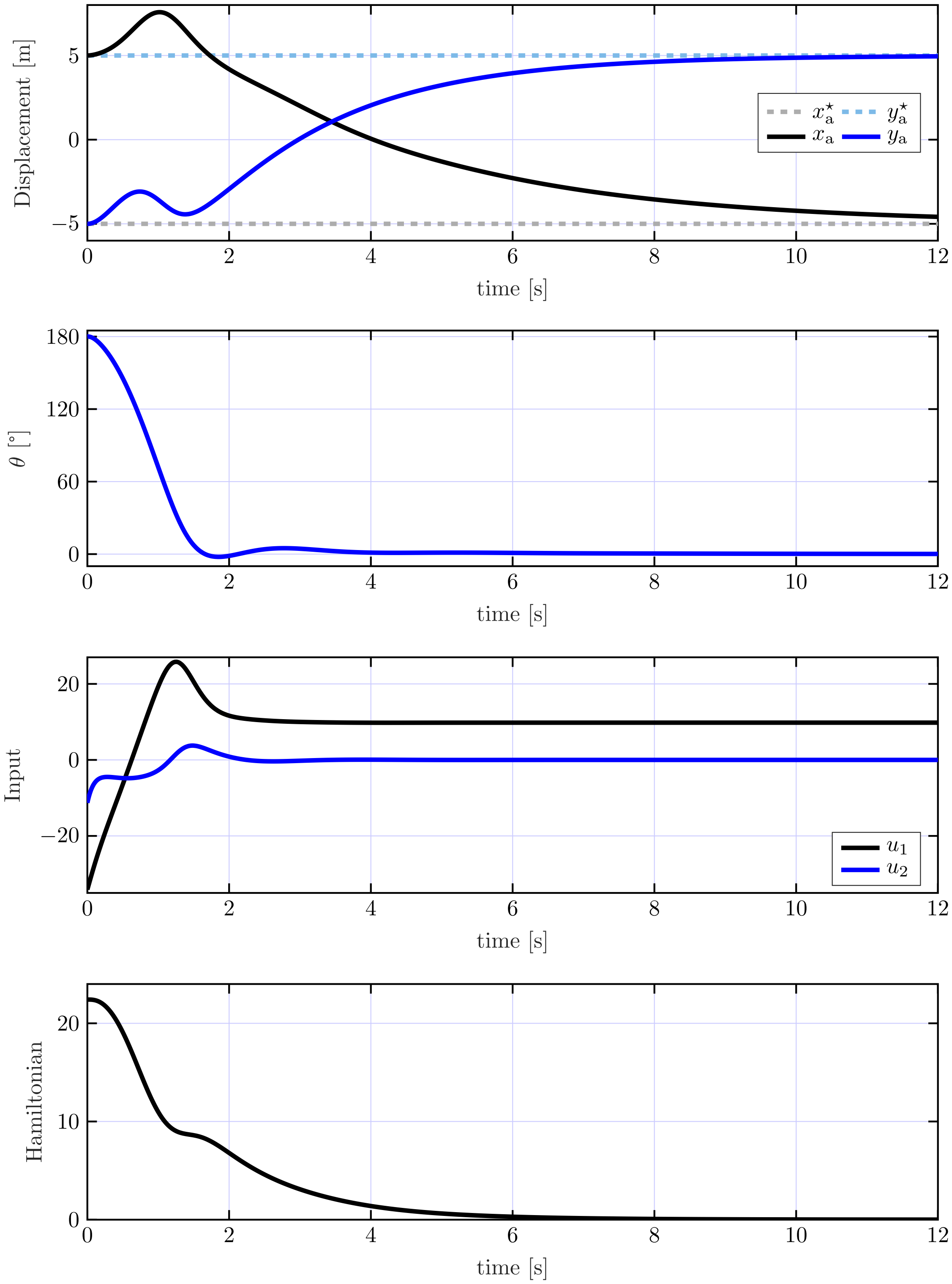


**Figure 7.18.** – PVTOL aircraft simulation from Part *ii*.

# IV

# Concluding Remarks and Appendices

# Chapter 8

# Conclusions and Future work

## 8.1 General Conclusions

In this thesis, we present algebraic solutions for IDA-PBC in a class of affine systems, where the matching and stabilizing conditions remain polynomial. The algebraic essence of the method avoids the solution of PDEs at the expense of an adequate parametrization and selection of the Hamiltonian. Besides, the resulting conditions can be reformulated as SOS programs, simplifying algebraic analysis from a computational perspective. The approach can be seen as a generalization of IDA-PBC for LTI systems [187] to polynomial systems; however, it is not restricted to them, increasing the scope of application in comparison with other controller design methods with SOS programs, see [92, 93] for example. Since the matching and stabilization conditions are solved simultaneously while considering dissipation, we avoid the dissipation condition problem and the loss of generality of a two-stage scheme.

By introducing supplementary constraints that can also be included in the SOS program, we are able to impose restrictions in the control action and a minimum size in the region of convergence. This constraints are then fundamental to incorporate input saturation successfully. To address the great flexibility in the controller parameter selection, we also include four minimization objectives in the SOS programs: volume maximization of the region of convergence, volume minimization of the controller restriction, and local performance assignment with the standard and generalized $\mathcal{H}_2$ optimal controls.

The SOS programs are solved with SOSTOOLS and SDPT3. The algebraic approach is validated on two second-order polynomial systems, a third-order rational system with two inputs and the well-known cart-pole with upright pendulum position. A real experiment with a dSPACE is carried out in a cart-pole test bench, obtaining positive results. We remark that the cart-pole is an UMS whose standard IDA-PBC design cannot satisfy the dissipation condition [71] whenever pendulum friction is considered, but it can be solved with the algebraic method.

The second main contribution is the extension of the total energy shaping IDA-PBC to mechanical systems in the implicit representation, namely full-actuated and underactuated mechanical systems with kinematic constraints that can be holonomic, nonholonomic or both. The approach yields comparable matching and stabilizing conditions, which can be seen as generalizations of the explicit case. Besides, it extends the results of [74, 75, 77] by removing assumptions that hinder the energy shaping, such as *i)* equal directions in the constraint forces of the target and nominal systems, *ii)* positive definite inertia matrices, and *iii)* not modifying the interconnection and dissipation matrix.

Our formulation includes systems that may not preserve the port-Hamiltonian structure and whose constraint forces are allowed to perform work, i.e., they do not necessarily satisfy the Lagrange-d'Alembert principle. To our knowledge, such implicit systems have not been previously discussed in the PBC literature, and they may result from Lagrangian or Hamiltonian dynamical systems with preliminary feedback and change of coordinates. Later, we take a closer look at the simultaneous IDA-PBC for UMSs in explicit representation, of Donaire et al. [24], observing that it only has an advantage if the stability analysis is local and not global as originally claimed. In the global case, the simultaneous perspective turns out to be equivalent to the standard IDA-PBC. The situation extends to implicit systems and allows to derive the dissipation condition in that framework. Since full-actuated systems are much easier to control than underactuated ones, we test the results directly on the simple pendulum and the vertical rolling disk, which are examples of holonomic and nonholonomic systems, respectively. As expected from Brockett's necessary condition [209], we are unable to achieve asymptotic stability for the vertical rolling disk since the feedback is continuous time-invariant and static-state.

For a class of implicit UMSs, we propose a heuristic solution that, by using a supplementary matrix $\mathbf{K}_{\mathrm{e}}$, reformulates the matching and stabilizing conditions as an SOS program. Two methods for the selection of $\mathbf{K}_{\mathrm{e}}$ are addressed, and the solution is then verified in the portal crane with a constant target inertia matrix and the cart-pole with a state-dependent one.

Alternatively, we discuss the traditional approach of first solving the kinetic matching, then the potential one, and finally taking advantage of the arbitrary functions and free parameters to fulfill the stabilizing conditions. Here, we propose three solutions for the kinetic matching condition and one for the potential matching. The solutions show that the quasilinear PDEs of the kinetic matching can be equivalently written as DAEs providing a characterization of the desired inertia matrix. In the case of potential matching, we demonstrate that by finding a full-rank right annihilator of a known expression and whose columns are gradient vectors, the desired potential energy can be calculated from a simple integral. In summary, we follow a constructive method to solve the matching conditions algebraically. We validate our results on the portal crane with four and five DoF, the cart-pole and the PVTOL aircraft, all of them in implicit representation. The former employs a constant target inertia matrix, whereas the last two, constant and state-dependent ones. The approach can be implemented successfully in the 5-DoF portal crane with Partial Feedback Linearization (PFL), where the constraint forces do not satisfy the Lagrange-d'Alembert principle. We should remark that the stabilization of the cart-pole with a constant target inertia matrix is only feasible whenever it is not positive definite. This, perhaps counter-intuitive, outcome is clarified with the reduction from implicit to explicit, where the target inertia matrix is sign-indefinite in the implicit representation but positive definite in the explicit one. The best controllers of the cart-pole and portal crane are implemented in real experiments, confirming the expected results.

On the basis of the constructive solutions and under some mild conditions, it is developed an output feedback law that is implemented in the portal crane at the cost of less parameter tuning. The cart-pole case with upright pendulum position is not tractable with such output feedback. Furthermore, we reduce the closed-loop of the cart-pole system with a constant target inertia matrix to an explicit representation, finding that it is equivalent to the PID-PBC of [20, 24]. However, the heuristic and constructive solutions of the cart-pole with state-dependent inertia matrices are not equivalent to the PID-PBC, and we can use them to enlarge the region of convergence. A similar result is obtained for the PVTOL aircraft, where the reduction shows that the recent controller of [239] is a specific case of our approach.

Finally, we have replaced the Moore-Penrose inverse of the input matrix in the feedback expression with generalized inverses, which are usually non-unique. This relatively small modification does not affect the stabilization result and provides more flexibility in the controller's final expression, which can be used, for example, to reduce computation cost.

## 8.2 Future Work

The algebraic method's major adversity, in the first contribution, is the adequate parametrization and selection of the Hamiltonian given by $z$. This could significantly influence the solution of the matching and stabilizing conditions, and therefore having a method for such a parametrization is desirable. A possible solution would be to invert the interconnection and dissipation matrix, see [241]. This changes the problem of parametrizing the Hamiltonian, to the one of parametrizing the inverse of the dissipation and interconnection matrix, which may be helpful as in the heuristic solution of our second contribution. Another improvement is introducing new constraints and optimizations objectives. For example, imposing local asymptotic disturbance rejection, restricting the eigenvalues of the closed-loop Jacobian linearization to a region, assigning a local $\mathcal{H}_\infty$ performance [197], assigning a non-local optimal performance, robust stabilization of nonlinear systems [198], etc.

For the IDA-PBC in implicit mechanical system, we assume that the differential index of the model, described by DAEs, is two for nonholonomic systems and three otherwise. For example, in the portal crane, this is interpreted as having a positive pendulum length $l$ along its trajectories, which is assumed in all the literature. However, even when the length of $l$ is equal to zero, no laws of mechanics have been broken because the system is just fully-actuated in that set. This situation can be considered if we reformulate the approach to allow higher index DAEs, meaning that the constrained state space is no longer a regular manifold, but may still be an immersed one. On the other hand, by considering kinematic constraints, it is possible to obtain a Lagrangian model that does not meet the regularity (or nondegeneracy) condition of the Lagrangian, i.e., the inertia matrix can be singular. Since regularity is required to obtain the port-Hamiltonian model because, by definition, the Hamiltonian uses the inverse of the inertia matrix, this may be a disadvantage for the IDA-PBC. Therefore, future research can include our developments in the framework of controlled Lagrangians, where we may avoid inertia matrix inversion. To conclude our future work, we remark that most research focuses on explicit systems rather than implicit ones, even though physical system are naturally described by DAEs. In this context, it is desirable to enlarge the scope of application of IDA-PBC to general nonlinear systems with DAEs and to extend other PBC techniques to the implicit framework.

# Appendix A

# Standard Lemmas

In this appendix, we state some standard lemmas employed throughout this thesis.

**Lemma A.1** (Sylvester's inequality [242])**.** *Suppose* $A \in \mathbb{R}^{n\times m}$ *and* $B \in \mathbb{R}^{m\times s}$*, then*

$$\mathrm{rank}(A) + \mathrm{rank}(B) - m \leq \mathrm{rank}(AB) \leq \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}.$$

**Lemma A.2.** *Consider* $A : \mathcal{X} \in \mathbb{R}^n \to \mathbb{R}^{n\times m}$*. Then,* $\sigma_{\max}(A(x)) = \|A(x)\|_2$ *and*

$$\gamma \geq \sigma_{\max}(A(x)) \iff \gamma^2 I_n \succeq A(x)A^\top(x) \iff \gamma^2 I_m \succeq A^\top(x)A(x),$$

*where* $\sigma_{\max}(A(x))$ *is the maximum singular value of* $A(x)$*, and* $\|A(x)\|_2$ *is the spectral norm of* $A(x)$*. If* $A(x)$ *is additionally symmetric and positive semi-definite for all* $x \in \mathcal{X}$*, then*

$$\gamma \geq \sigma_{\max}(A(x)) \iff \gamma I_n \succeq A(x).$$

**Proof.** The equivalence $\sigma_{\max}(A(x)) = \|A(x)\|_2$ is immediate, see [242, Example 5.6.6]. Let $A(x) = U_1(x)\Sigma(x)U_2^\top(x)$ be the singular value decomposition of $A$ at each point $x$, where $U_i$ are unitary matrices. Then, $A(x)A^\top(x) = U_1(x)\Sigma^2(x)U_1^\top(x)$ and $\gamma^2 I_n \succeq A(x)A^\top(x) \iff \gamma^2 I_n \succeq \Sigma^2(x)$. By definition of $\sigma_{\max}(A(x))$ and $\Sigma$ we have $\gamma^2 I_n \succeq \Sigma^2(x) \iff \gamma \geq \sigma_{\max}(A(x))$. The proof of the equivalence with $\gamma^2 I_m \succeq A^\top(x)A(x)$ is identical. Similarly, if $A$ is symmetric and positive semi-definite, we see that $U_2(x) \equiv U_1(x)$ and $\gamma I_n \succeq A(x) \iff \gamma I_n \succeq \Sigma(x) \iff \gamma \geq \sigma_{\max}(A(x))$. □

**Lemma A.3** (Compact image [107, Proposition 2.14])**.** *Let* $f : \mathcal{X} \to \mathbb{R}^n$ *be a continuous function and* $\mathcal{X} \subset \mathbb{R}^n$ *be a compact set. The image of* $\mathcal{X}$ *through* $f$ *is compact.*

**Lemma A.4** ([110, Lemma 4.1])**.** *Consider*

$$\dot{x} = f(x), \qquad x \in \mathcal{X},$$

*where $\mathcal{X}$ is an open and connected subset of $\mathbb{R}^n$, and $f : \mathcal{X} \to \mathbb{R}^n$ is a locally Lipschitz continuous function. Suppose a solution $x(\cdot)$ is bounded, then its positive limit set $L^+$ is a nonempty, compact, connected, and invariant set. Furthermore, $x(t) \to L^+$ as $t \to \infty$.*

**Lemma A.5** ($\mathcal{H}_2$ performance)**.** *Consider a LTI system*

$$\dot{x} = Ax + Bw \tag{A.1a}$$

$$y = Cx \tag{A.1b}$$

*where $x \in \mathbb{R}^{n_x}$ are the states, $w \in \mathbb{R}^{n_w}$ are exogenous disturbances and $z \in \mathbb{R}^{n_z}$ is the output. Let $T_{yw}$ be the closed-loop transfer function from $w$ to $y$. System (A.1) is internally stable with **generalized $\mathcal{H}_2$ performance***

$$J_{\mathcal{H}_2} = \sigma_{\max}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} T_{yw}(j\omega)T_{yw}^*(j\omega)\mathrm{d}\omega\right) < \gamma, \qquad \gamma \in \mathbb{R},\, \gamma > 0,$$

*if and only if there exists a positive definite matrix $Q \in \mathbb{R}^{n_x \times n_x}$ such that*

$$-AQ - QA^\top - BB^\top \succ 0, \tag{A.2a}$$

$$\sigma_{\max}\left(CQC^\top\right) < \gamma \tag{A.2b}$$

*hold. The **standard $\mathcal{H}_2$ performance** is obtained by replacing $\sigma_{\max}$ with* trace *in $J_{\mathcal{H}_2}$ and (A.2b).*

**Proof.** It can be found in [196, 197], see also [195]. □

**Lemma A.6** (Matrix inversion [243, Section 2.3])**.** *Let $M \in \mathbb{R}^{n\times n}$ be partitioned as*

$$M := \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

*where $A$ and $D$ are square matrices. If $D \in \mathbb{R}^{m\times m}_m$, then $M$ can be decomposed as*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_{n-m} & BD^{-1} \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ D^{-1}C & I_m \end{bmatrix}.$$

*Similarly, if $A \in \mathbb{R}^{(n-m)\times(n-m)}_{(n-m)}$, then*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_{n-m} & 0 \\ CA^{-1} & I_m \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_{n-m} & A^{-1}B \\ 0 & I_m \end{bmatrix}.$$

*Suppose, additionally, that $M$ is nonsingular, then*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I_{n-m} & 0 \\ -D^{-1}C & I_m \end{bmatrix} \begin{bmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} I_{n-m} & -BD^{-1} \\ 0 & I_m \end{bmatrix}$$

*and*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} I_{n-m} & -A^{-1}B \\ 0 & I_m \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I_{n-m} & 0 \\ -CA^{-1} & I_m \end{bmatrix}.$$

**Lemma A.7** (Matrix inversion properties)**.** *Let $G \in \mathbb{R}^{n\times m}$. Then:*

*i)* *$GG^{\mathrm{g}}$ and $G^{\mathrm{g}}G$ are idempotent and have the same rank as $G$.*[67]

*ii)* *If $G$ is nonsingular, $G^{\mathrm{g}} = G^{-1}$ uniquely.*

*iii)* $\operatorname{rank} G^{\mathrm{g}} \geq \operatorname{rank} G$.

*iv)* *If $G^{+}$ exists, it is unique.*

*v)* $G^{++} = G$.

*vi)* *If $G$ is full column rank, $G^{\mathrm{g}}G = I_m$ and $G^{+} = \left(G^{\top}G\right)^{-1} G^{\top}$.*

*vii)* *If $G$ is full row rank, $GG^{\mathrm{g}} = I_n$ and $G^{+} = G^{\top}\left(GG^{\top}\right)^{-1}$.*

*viii)* *For some $B \in \mathbb{R}^{n\times e}$,*

$$(I_n - GG^{\mathrm{g}})\, B = 0 \;\Leftrightarrow\; \operatorname{Colsp} B \subset \operatorname{Colsp} G \;\Leftrightarrow\; \exists D \text{ s.t. } B = GD \;\Leftrightarrow\; G_{\perp}B = 0.$$

*ix)* *For some $C \in \mathbb{R}^{d\times m}$,*

$$C\,(I_m - G^{\mathrm{g}}G) = 0 \;\Leftrightarrow\; \operatorname{Rowsp} C \subset \operatorname{Rowsp} G \;\Leftrightarrow\; \exists E \text{ s.t. } C = EG \;\Leftrightarrow\; CG_{\top} = 0.$$

**Proof.** The proof of assertions *i–vii* as well as the first two equivalences of *viii* are given in [169, 177]. For the third equivalence (of *viii*) we left multiply $B = GD$ by $G_{\perp}$ obtaining $G_{\perp}B = 0$. Conversely, if $G_{\perp}B = 0$ holds. Then, from Definition 3.1, we have $\operatorname{Colsp} B \subset \operatorname{Colsp} G$, which completes the proof of *viii*. Assertion *ix* is the dual of *viii*. □

**Lemma A.8** (Schur complements [88, 171])**.** *Let $M \in \mathbb{R}^{n\times n}$ be symmetric and partitioned as*

$$M := \begin{bmatrix} A & B \\ B^{\top} & D \end{bmatrix},$$

[67] A square matrix $A$ is idempotent if $A^2 = A$.

*where $A$ and $D$ are square matrices. Then,*

$$M \succeq 0 \quad \iff \quad \begin{cases} D \succeq 0 \\ M/D \succeq 0 \\ \operatorname{Rowsp} B \subset \operatorname{Rowsp} D \end{cases} \quad \iff \quad \begin{cases} A \succeq 0 \\ M/A \succeq 0 \\ \operatorname{Colsp} B \subset \operatorname{Colsp} A. \end{cases}$$

*Here, $M/D := A - BD^{\mathrm{g}}B^\top$ and $M/A := D - B^\top A^{\mathrm{g}} B$ are the* **generalized Schur complements** *of $D$ and $A$ in $M$, respectively. Similarly,*

$$M \succ 0 \quad \iff \quad \begin{cases} D \succ 0 \\ M/D \succ 0 \end{cases} \quad \iff \quad \begin{cases} A \succ 0 \\ M/A \succ 0, \end{cases}$$

*where $M/D$ and $M/A$ are the standard* **Schur complements** $M/D = A - BD^{-1}B^\top$ *and* $M/A = D - B^\top A^{-1} B$.

**Lemma A.9** (Finsler [244]). *Consider $Q \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $\operatorname{rank} B < n$. The following are equivalent:*

- $x^\top Q x \succ 0$ *for all* $x \in \left\{x \in \mathbb{R}^n \mid B^\top x = 0, x \neq 0\right\}$.
- $B_\perp \left(Q + Q^\top\right) B_\perp^\top \succ 0$.
- *There exists $\mu \in \mathbb{R}$ such that $Q + Q^\top - \mu \left(BB^\top\right) \succ 0$.*
- *There exists $C \in \mathbb{R}^{n\times m}$ such that $Q + Q^\top + CB^\top + BC^\top \succ 0$.*

# Appendix B

# Proofs

## B.1 Proof of Lemma 3.3

($i \Leftrightarrow ii$) Along the same lines of [244, Appendix A].
($ii \Leftrightarrow iii$) If $\operatorname{rank} B = 0$ the equivalence is evident. If $1 < \operatorname{rank} B < n$, we multiply $A^{\mathrm{s}} + BK + K^\top B^\top \succeq 0$ on the left by

$$\begin{bmatrix} B_\perp \\ B_\mathrm{l}^+ \end{bmatrix} \in \mathbb{R}_n^{n\times n}$$

and on the right by its transpose to obtain

$$\begin{bmatrix} B_\perp A^{\mathrm{s}} B_\perp^\top & \left(B_\mathrm{l}^+ A^{\mathrm{s}} B_\perp^\top + \tilde{K}_3\right)^\top \\ B_\mathrm{l}^+ A^{\mathrm{s}} B_\perp^\top + \tilde{K}_3 & \tilde{K}_1^{\mathrm{s}} + B_\mathrm{l}^+ A^{\mathrm{s}} B_\mathrm{l}^{\top +} \end{bmatrix} \succeq 0, \tag{B.1}$$

where $\tilde{K}_1 = B_\mathrm{r} K B_\mathrm{l}^{\top +}$ and $\tilde{K}_3 = B_\mathrm{r} K B_\perp^\top$, that is,[68]

$$B_\mathrm{r} K = \tilde{K}_1 B_\mathrm{l}^\top + \tilde{K}_3 B_\perp^{+\top}. \tag{B.2}$$

Now, from Lemmas A.7 and A.8, the inequality (B.1) can be equivalently written as $ii$,

$$\exists \tilde{K}_2 \quad \text{s.t.} \quad B_\mathrm{l}^+ A^{\mathrm{s}} B_\perp^\top + \tilde{K}_3 = \tilde{K}_2 B_\perp A^{\mathrm{s}} B_\perp^\top, \quad \text{and} \tag{B.3a}$$

$$\tilde{K}_1^{\mathrm{s}} + B_\mathrm{l}^+ A^{\mathrm{s}} B_\mathrm{l}^{\top +} - \left(B_\mathrm{l}^+ A^{\mathrm{s}} B_\perp^\top + \tilde{K}_3\right)\left(B_\perp A^{\mathrm{s}} B_\perp^\top\right)^{\mathrm{g}}\left(B_\mathrm{l}^+ A^{\mathrm{s}} B_\perp^\top + \tilde{K}_3\right)^\top \succeq 0, \tag{B.3b}$$

where we can additionally define (without loss of generality) $\tilde{K}_1 := B_\mathrm{r} K_1 B_\mathrm{r}^\top$ and $\tilde{K}_2 := B_\mathrm{r} K_2$ for some matrices $K_1$ and $K_2$ because $B_\mathrm{r}$ and $B_\mathrm{l}$ are the full rank factors of $B$. Given that $B_\mathrm{r}^+ B_\mathrm{l}^+ = B^+$ and $B_\perp$ is the full-rank right annihilator of $B$ and $B_\mathrm{r}$, we deduce that (B.2)

[68]Note that $E^{\top +} = E^{+\top}$ and $B_\perp B_\mathrm{l} = 0$.

with (B.3a) is (3.1a), whereas (B.3b) with (B.3a) reads (3.1b). The result with strict inequalities follows a similar procedure.

## B.2 Proof of Lemma 3.4

($i \Rightarrow iii$) See e.g. [242, Section 0.5].
($iii \Rightarrow ii$) Straightforward application of Lemma A.1 on $B_\perp(x)A(x)$ and $C_\perp(x)A(x)$.
($ii \Rightarrow i$) Direct computation gives

$$A^{-1}(x)A(x) = \begin{bmatrix} (C_\perp(x)B(x))^{-1} C_\perp(x) \\ (B_\perp(x)C(x))^{-1} B_\perp(x) \end{bmatrix} \begin{bmatrix} B(x) & C(x) \end{bmatrix} = I_n.$$

## B.3 Proof of Lemma 5.2

Define $\tilde{A} := N^\top A \bar{N}$ with

$$N^\top = \begin{bmatrix} \left(B_\perp B_\perp^\top\right)^{-1} B_\perp \\ B^\top \end{bmatrix}^\top \quad \text{and} \quad \bar{N}^\top = \begin{bmatrix} \left(C_\perp C_\perp^\top\right)^{-1} C_\perp \\ C^\top \end{bmatrix}.$$

Observe that $\tilde{A}$ can be partitioned as $\tilde{A} = \left[\begin{smallmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{smallmatrix}\right]$, where $\tilde{A}_{22} = B^\top A C$.
($i \Leftrightarrow ii$) Suppose affirmation $i$ holds, i.e., $\tilde{A}_{22} \in \mathbb{R}_m^{m\times m}$, then $B$ and $C$ are full rank while $N$, $\bar{N}$ and $\tilde{A}$ are square and nonsingular. Thus, from the decomposition of Lemma A.6 we conclude that $\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$ is also nonsingular. Now, using Lemmas 3.4 and A.6 for the inverses of $\tilde{A}$, $\bar{N}$ and $N$ give $C_\perp A^{-1} B_\perp^\top = \left(\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}\right)^{-1}$, which implies affirmation $ii$. The converse is along the same lines.
($i \Leftrightarrow iii$) Observe that

$$\bar{N}^\top \begin{bmatrix} C_\perp^\top & A^\top B \end{bmatrix} = \begin{bmatrix} I_{n-m} & \tilde{A}_{21}^\top \\ 0 & \tilde{A}_{22}^\top \end{bmatrix}. \tag{B.4}$$

If affirmation $i$ is satisfied, then $\bar{N} \in \mathbb{R}_n^{n\times n}$ and the implication is immediate. Conversely, if $iii$ holds, $C$ is full column rank and $\bar{N} \in \mathbb{R}_n^{n\times n}$. Now, the result is a direct consequence of Lemma A.1 with (B.4).
Given that $\begin{bmatrix} C_\perp^\top & A^\top B \end{bmatrix}$ is nonsingular from $i$, $ii$, or $iii$, the identity is obtained from $I_n = \begin{bmatrix} C_\perp^\top & A^\top B \end{bmatrix} \begin{bmatrix} C_\perp^\top & A^\top B \end{bmatrix}^{-1}$ and Lemma 3.4.

## B.4 Proof of Lemma 5.4

By definition $\mathbf{M}(r), \mathbf{M}_\mathrm{d}(r) \in \mathbb{R}_{n_r}^{n_r\times n_r}$. Consequently, $\mathbf{B}_\perp(r)\mathbf{M}(r)\mathbf{M}_\mathrm{d}^{-1}(r)\mathbf{M}(r)\mathbf{B}_\perp^\top(r)$ is nonsingular if and only if $\bar{\Delta}_\mathrm{d}(r)$ is nonsingular, see Lemma 5.2 and Assumption 5.8. Now, we have the following chain of implications

$$
\begin{aligned}
& \mathbf{B}_{\perp}(r)\mathbf{M}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{M}(r)\mathbf{B}_{\perp}^{\top}(r) \succ 0 \\
\iff\quad & \left(\mathbf{B}_{\perp}(r)\mathbf{M}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{M}(r)\mathbf{B}_{\perp}^{\top}(r)\right)^{-1} \succ 0 \\
\iff\quad & \bar{\mathbf{B}}_{\perp}(r)\mathbf{M}(r)\mathbf{B}_{\perp}^{\top}(r)\left(\mathbf{B}_{\perp}(r)\mathbf{M}(r)\mathbf{M}_{\mathrm{d}}^{-1}(r)\mathbf{M}(r)\mathbf{B}_{\perp}^{\top}(r)\right)^{-1}\mathbf{B}_{\perp}(r)\mathbf{M}(r)\bar{\mathbf{B}}_{\perp}^{\top}(r) \succ 0 \\
\iff\quad & \bar{\mathbf{B}}_{\perp}(r)\left(\mathbf{M}_{\mathrm{d}}(r) - \mathbf{M}_{\mathrm{d}}(r)\mathbf{M}^{-1}(r)\mathbf{B}(r)\bar{\Delta}_{\mathrm{d}}^{-1}(r)\mathbf{B}^{\top}(r)\mathbf{M}^{-1}(r)\mathbf{M}_{\mathrm{d}}(r)\right)\bar{\mathbf{B}}_{\perp}^{\top}(r) \succ 0,
\end{aligned}
$$

where the second equivalence is obtained with Lemma 5.2 and Assumption 5.6, and the last one with Lemma 5.2 and $\bar{\Delta}_{\mathrm{d}}$ being nonsingular.

## B.5 Proof of Lemma 5.5

($\Leftarrow$) Multiplying (5.38) on the left by $\left(y^{\top}\bar{e}_k\right)y^{\top}$ and right by $y$, summing over $k$, and replacing $Ay$ by $z$ results in

$$
\begin{aligned}
0 &= \sum_{k=1}^{m}\left(y^{\top}\bar{e}_k\right)y^{\top}\left(\sum_{i=1}^{n}A^{\top}Q_i^{\mathrm{s}}A\left(\bar{e}_k^{\top}A^{\top}e_i\right)+\sum_{i=1}^{n}A^{\top}e_i\bar{e}_k^{\top}A^{\top}Q_i^{\mathrm{s}}A+\sum_{i=1}^{n}A^{\top}Q_i^{\mathrm{s}}A\bar{e}_k e_i^{\top}A\right)y \\
&= \sum_{k=1}^{m}\left(y^{\top}\bar{e}_k\right)\left(\sum_{i=1}^{n}z^{\top}Q_i^{\mathrm{s}}z\left(\bar{e}_k^{\top}A^{\top}e_i\right)+\sum_{i=1}^{n}z^{\top}e_i\bar{e}_k^{\top}A^{\top}Q_i^{\mathrm{s}}z+\sum_{i=1}^{n}z^{\top}Q_i^{\mathrm{s}}A\bar{e}_k e_i^{\top}z\right) \\
&= \sum_{i=1}^{n}z^{\top}Q_i^{\mathrm{s}}z\left(z^{\top}e_i\right)+\sum_{i=1}^{n}\left(z^{\top}e_i\right)z^{\top}Q_i^{\mathrm{s}}z+\sum_{i=1}^{n}z^{\top}Q_i^{\mathrm{s}}z\left(z^{\top}e_i\right),
\end{aligned}
$$

that is, (5.37).

($\Rightarrow$) Set $z := z_1 + z_2$ with $z_j := Ay_j$, then

$$
\begin{aligned}
0 &= \sum_{i=1}^{n}\left(z_1^{\top}+z_2^{\top}\right)e_i\left(z_1^{\top}+z_2^{\top}\right)Q_i\left(z_1+z_2\right) \\
&= \sum_{i=1}^{n}\left(z_1^{\top}+z_2^{\top}\right)e_i\left(z_1^{\top}Q_i z_1 + z_1 Q_i^{\mathrm{s}} z_2 + z_2^{\top}Q_i z_2\right) \\
&= \sum_{i=1}^{n}\left(z_1^{\top}e_i\right)z_1^{\top}Q_i^{\mathrm{s}}z_2+\sum_{i=1}^{n}\left(z_2^{\top}e_i\right)z_1^{\top}Q_i z_1+\sum_{i=1}^{n}\left(z_1^{\top}e_i\right)z_2^{\top}Q_i z_2+\sum_{i=1}^{n}\left(z_2^{\top}e_i\right)z_1^{\top}Q_i^{\mathrm{s}}z_2 \\
&= z_1^{\top}\left(\sum_{i=1}^{n}e_i z_2^{\top}Q_i^{\mathrm{s}}+\left(z_2^{\top}e_i\right)Q_i\right)z_1 + z_2^{\top}\left(\sum_{i=1}^{n}e_i z_1^{\top}Q_i^{\mathrm{s}}+\left(z_1^{\top}e_i\right)Q_i\right)z_2 \\
&= y_1^{\top}A^{\top}\left(\sum_{i=1}^{n}e_i y_2^{\top}A^{\top}Q_i^{\mathrm{s}}+\left(y_2^{\top}A^{\top}e_i\right)Q_i\right)Ay_1+y_2^{\top}A^{\top}\left(\sum_{i=1}^{n}e_i y_1^{\top}A^{\top}Q_i^{\mathrm{s}}+\left(y_1^{\top}A^{\top}e_i\right)Q_i\right)Ay_2.
\end{aligned}
$$

Given that $y_j \in \mathbb{R}^m$ is arbitrary, it follows from the last equation that

$$\bar{A} := A^\top \Big( \sum_{i=1}^{n} e_i y_j^\top A^\top Q_i^{\mathrm{s}} + \big(y_j^\top A^\top e_i\big) Q_i \Big) A$$

is skew-symmetric, or equivalently,

$$\begin{aligned} 0 = \bar{A} + \bar{A}^\top &= \sum_{i=1}^{n} A^\top \Big( e_i y_j^\top A^\top Q_i^{\mathrm{s}} + \big(y_j^\top A^\top e_i\big) Q_i^{\mathrm{s}} + Q_i^{\mathrm{s}} A y_j e_i^\top \Big) A \\ &= \sum_{k=1}^{m} \left(\bar{e}_k^\top y_j\right) \sum_{i=1}^{n} A^\top \left( Q_i^{\mathrm{s}} \left(\bar{e}_k^\top A^\top e_i\right) + e_i \bar{e}_k^\top A^\top Q_i^{\mathrm{s}} + Q_i^{\mathrm{s}} A \bar{e}_k e_i^\top \right) A, \end{aligned} \tag{B.5}$$

wherein the last step we used $\sum_{k=1}^{m} \bar{e}_k \bar{e}_k^\top = I_m$. Equality (5.38) is deduced by using again the arbitrary feature of $y_j$ on (B.5).

## B.6 Proof of Lemma 6.1

By using Lemma 3.2 twice we get

$$\begin{aligned} & 0 = A(x) + G(x) K_1(x) + K_2^\top(x) G^\top(x) \\ \iff & \begin{cases} 0 = G_\perp(x) \Big( A(x) + K_2^\top(x) G^\top(x) \Big) \\ K_1(x) = -G^{\mathrm{g}}(x) \Big( A(x) + K_2^\top(x) G^\top(x) \Big) + G_{\mathrm{I}}(x) \tilde{K}_1(x) \end{cases} \\ \iff & \begin{cases} 0 = G_\perp(x) A^\top(x) G_\perp^\top(x) \\ K_2(x) = -G^{\mathrm{g}}(x) A^\top(x) + \bar{W}(x) G^\top(x) + G_{\mathrm{I}}(x) \hat{K}_2(x) \\ K_1(x) = G^{\mathrm{g}}(x) A(x) \Big( G(x) G^{\mathrm{g}}(x) - I_n \Big)^\top - G^{\mathrm{g}}(x) G(x) \bar{W}^\top(x) G^\top(x) + G_{\mathrm{I}}(x) \tilde{K}_1(x), \end{cases} \end{aligned}$$

where $\tilde{K}_1$ and $\bar{W}$ are arbitrary functions, and $-G^{\mathrm{g}}(x) G(x) \bar{W}^\top(x) G^\top(x) + G_{\mathrm{I}}(x) \tilde{K}_1(x)$ can be equivalently written as $-\bar{W}^\top(x) G^\top(x) + G_{\mathrm{I}}(x) \hat{K}_1(x)$ for an appropriate function $\hat{K}_1(x)$, see Lemma A.7. The proof is completed, by observing that $K$ results from equating $K = K_1 = K_2$ with $A(x)$ being symmetric and $\bar{W}(x) = \frac{1}{2} G^{\mathrm{g}}(x) A(x) G^{\mathrm{g}\top}(x) + W$.

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