Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 21/06

On ReH-matrices and corresponding topics

Hildenbrandt, Regina

Oktober 2021

URN: urn:nbn:de:gbv:ilm1-2021200134

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
https://www.tu-ilmenau.de/mathematik/

On ReH-matrices and corresponding topics

R. Hildenbrandt, Department of Mathematics,
Ilmenau Technical University

1 Introduction

We are interested in ReH-matrices because they help us to solve a complex operations research problem (see [5], [10] and also Appendix C). However, these matrices themselves represent an interesting combinatorial structure. ReH-matrices can be initially computed by means of a simple enumeration (however a laborious method). Unfortunately, no formulas for explicit computation are known for most elements of ReH-matrices.

ReH-matrices are defined by three positive integers n, k_0, su. The number of rows and columns of such a matrix is equivalent to the number of restricted (unordered) partitions of su into at most n parts with summands not greater than k_0. The elements of a row (which corresponds to a partition (s)) are computed as numbers of "balanced" covers of certain "vectors w" by partitions, which differ from the partition s at least as possible (divided by a normalizing factor) if the components of these vectors are discrete uniformly distributed (and in an analogous way for other distributions of the vectors).

In Section 3 we will introduce "perturbed permutations". They can be used to compute the elements of ReH-matrices more effectively than enumeration. A polynomial and sometimes an exponential dependence of the elements on the variables n and k_0 in the case of discrete uniformly distributed components of the vectors w (and similar relationships for other distributions) can also be shown by means of perturbed permutations.

Limits of ReH-matrices are also significant. In Section 4 we will compute such limits if corresponding sequences consist of matrices of the same type.

So-called Poisson equations (Section 5) are important for applications. If their solutions are "monotone" then ReH-matrices correspond to optimal solutions of the above mentioned operations research problem. But proofs for monotone solutions are very difficult, since ReH-matrices do not fulfil a "dominance property", in general. We give an overview on this topic and present a new result using limits from Section 4.

1.1 Notation and terminology

Let su, n and k_0 be integers with $n \geq 2$ and $1 \leq k_0 \leq su < nk_0$. $S_{n;su;k_0}$ denotes the *set of the restricted (unordered) partitions* of su into at most n parts with summands not greater than k_0. r is the number of partitions of such a set. We write the elements s of $S_{n;su;k_0}$ as n-dimensional vectors. That means $s = (s_1, s_2, \cdots, s_n)$ with $s_1 + s_2 + \cdots + s_n = su$ and (w.l.o.g.) $s_1 \geq s_2 \geq \cdots \geq s_n$ [1].

Furthermore, the box $B_{n;k_0} := \left\{ w \in \mathbb{Z}_+^n | 0 \leq w_i \leq k_0, i = 1, \cdots, n \right\}$ is called the *set of requirements.* We assume that the requirements w are random vectors [2] and that their components w_i, $(i = 1, \cdots, n)$ are independent and identically distributed. Let

$$q(w) := \prod_{i=1}^{n} q_0(w_i) \tag{1}$$

be a corresponding *probability function* where the *marginal or single probabilities* $q_0(w_i)$ are such that

$$q_0(w_i) > 0 \text{ if and only if } w_i \in \{0, 1, \cdots, k_0\} \quad \text{and} \quad \sum_{j=0}^{k_0} q_0(j) = 1. \tag{2}$$

If, in particular, w_i are discrete uniformly distributed then $q(w) = \left(\frac{1}{k_0+1}\right)^n$ for all $w \in B_{n;k_0}$ follows.

Principally, two cases in relation to the requirements have to distinguished: $\sum_{i=1}^{n} w_i \leq su$ (surplus-situation) and $\sum_{i=1}^{n} w_i \geq su$ (scarcity-situation). [3]

$B^1_{n;k_0} = B_{n;k_0} \cap \left\{ w \in B_{n;k_0} | \sum_{i=1}^{n} w_i \leq su \right\}$ and
$B^2_{n;k_0} = B_{n;k_0} \cap \left\{ w \in B_{n;k_0} | \sum_{i=1}^{n} w_i \geq su \right\}$ are corresponding subsets of $B_{n;k_0}$.

For given $s \in S_{n;su;k_0}$ and $w \in B_{n;k_0}$ a *feasible partition* s' is defined by:
1) w is covered by a permutation s'_π of s' in the case of the surplus-situation.
2) w should be met as much as possible in case of the scarcity-situation.
3) The "difference" $(\sum_{i=1}^{n} |s_i - s'_{\pi_i}|)$ between s and s'_π is as a small as possible.

[1] However, each permutation of $(s_1, s_2, \cdots, s_n)$ represent the same partition s.

[2] We use the same notations for the random vectors and their realizations.

[3] The equal sign in both cases is useful for the following and does not lead to contradictions.

$$\text{Thus : } s'(= s'(s,w)) \in \hat{A}_{n;su;k_0}(s,w)$$

$$= \left\{ s' \in S_{n;su;k_0} \;\middle|\; \begin{array}{l} \exists\ s'_\pi \text{ permutation of } s' : \\ w_i \le s'_{\pi_i} \le max\{s_i, w_i\},\ i = 1,\cdots,n \text{ if } \sum_{i=1}^{n} w_i \le su \\ min\{s_i, w_i\} \le s'_{\pi_i} \le w_i,\ i = 1,\cdots,n \text{ if } \sum_{i=1}^{n} w_i \ge su \end{array} \right\} \tag{3}$$

and

$$\frac{1}{2}\sum_{i=1}^{n} |s_i - s'_{\pi_i}| = \begin{cases} \sum_i \max\{0, w_i - s_i\} & \text{if } \sum_{i=1}^{n} w_i \le su, \\ \sum_i \max\{0, s_i - w_i\} & \text{if } \sum_{i=1}^{n} w_i \ge su \end{cases}.$$

We denote $s^* = s^*(s,w) \in \hat{A}_{n;su;k_0}(s,w)$ as a *feasible balanced partition* with regard to s and w if $\sum_{i=1}^{n}(s_i^*)^2$ is as small as possible. The fewer the coordinates s_i^* differ from one another, the smaller is $\sum_{i=1}^{n}(s_i^*)^2$, since

$$(s_i^*)^2 + (s_j^*)^2 < (s_i^* - x)^2 + (s_j^* + x)^2 \text{ for } 0 < x \le s_i^* \le s_j^*. \tag{4}$$

In addition, we define $B^*_{n;k_0}(s,s^*) = \{w \in B_{n;k_0} \mid s^* = s^*(s,w)\}$ as the *set of balancing requirements* and $B^{*1}_{n;k_0}(s,s^*) = B^*_{n;k_0}(s,s^*) \cap B^1_{n;k_0}$, $B^{*2}_{n;k_0}(s,s^*) = B^*_{n;k_0}(s,s^*) \cap B^2_{n;k_0}$.

2 Definition of ReH-matrices and their iterative computation

At first, we want to introduced ReH-matrices. For this, let $S_{n;su;k_0} = \{s^1, s^2, \cdots, s^r\}$, $B_{n;k_0}$ and q as above.

ReH-matrices are matrices $P^* = P^*_{n;su;k_0} = (p^*_{fl})_{\substack{f=1,\cdots,r,\\ l=1,\cdots,r}}$ with elements

$$p^*_{fl} = p^*(s^l|s^f) = \sum_{w: s^l = s^*(s^f,w)} q(w) = \sum_{w \in B^*_{n;k_0}(s^f,s^l)} q(w). \tag{5}$$

(For an example, see Appendix A, a).) Thereby,

$$p^*_{fl} > 0 \text{ for } f = 1,\cdots,r,\ l = 1,\cdots,r, \tag{6}$$

because of $p^*_{fl} \ge q(w = s^l)$ and (2).

Thus, the definition of ReH-matrices includes the usage of (balanced) transitions $s^*(s^f, w)$, $f = 1, \cdots, r, w \in B_{n;k_0}$.

Using property (4), feasible balanced partitions for given $s \in S_{n;su;k_0}$ and $w \in B_{n;k_0}$ can be computed by the following iterative method. (This is easy to prove.):

$$\textit{Set } s_i^* = \begin{cases} \sum\limits_i \max\{s_i, w_i\} \ \textit{if} \ \sum\limits_{i=1}^n w_i \le su, \\ \sum\limits_i \min\{s_i, w_i\} \ \ \textit{if} \ \ \sum\limits_{i=1}^n w_i \ge su \end{cases},$$

$$d = \begin{cases} \sum\limits_i \max\{0, w_i - s_i\} \ \ \textit{if} \ \ \sum\limits_{i=1}^n w_i \le su, \\ \sum\limits_i \max\{0, s_i - w_i\} \ \ \textit{if} \ \ \sum\limits_{i=1}^n w_i \ge su \end{cases}.$$

(*) If $d = 0$, then s^* is the desired partition (end) else:
Determine a component s_j^* of s^*

$$\textit{such that } s_j^* = \begin{cases} max\{s_i^* \mid s_i^* > w_i\} \ \ \textit{if} \ \sum\limits_{i=1}^n w_i \le su, \\ min\{s_i^* \mid s_i^* < w_i\} \ \ \textit{if} \ \sum\limits_{i=1}^n w_i \ge su \end{cases}.$$

$$\textit{Set } s_j^* = \begin{cases} s_j^* - 1 \ \ \textit{if} \ \sum\limits_{i=1}^n w_i \le su \\ s_j^* + 1 \ \ \textit{if} \ \sum\limits_{i=1}^n w_i \ge su \end{cases} \quad \textit{and } d = d - 1.$$

Go to (*).

If the feasible balanced partitions (with regard to s^f and all $w \in B_{n;k_0}$) have been computed then the elements $p^*_{fl}, l = 1, \cdots, r$ of a ReH-matrix can be calculated as sums of the probabilities of the requirements w, for which feasible balanced transitions from s^f to s^l can be found. The computation of ReH-matrices by means of such enumerations is a laborious method.

An important question is wether we can determine sets $B^*_{n;k_0}(s^f, s^l)$ for given s^f and s^l by a more effective method than the above enumeration.

For this, an idea could be to partition $B^*_{n;k_0}(s^f, s^l)$ in relation to the permutations of s^l. In more detail, that means
$B^*_{n;k_0}(s^f, s^l) = \bigcup\limits_{s^{l(j)} \ \textit{permutation of } s^l} B^{*\pi}_{n;k_0}(s^f, s_\pi^{l(j)})$, where

$$B^{*\pi}_{n;k_0}(s^f, s_\pi^{l(j)}) = \left\{ w \in B^*_{n;k_0}(s^f, s^l) \,\middle|\, w, s_\pi^{l(j)} \ \textit{fulfill } (3) \right\}. \tag{7}$$

Such sets have a simple structure:

$B^{*\pi}_{n;k_0}(s^f, s^l_\pi) =$

$$\left\{ w \in \mathbb{Z}^n_+ | w_i \left\{ \begin{array}{ll} = s^l_{\pi_{i_a}} & \text{for } \ i_a \in I_a \\ \in \left\{ 0, 1, \cdots, s^l_{\pi_{i_b}} \right\} & \text{for } \ i_b \in I_b \end{array} \right. , I_a \cup I_b = \{1, 2, \cdots, n\} , \sum_{i=1}^{n} w_i \leq su \right\} \bigcup$$

$$\left\{ w \in \mathbb{Z}^n_+ | w_i \left\{ \begin{array}{ll} = s^l_{\pi_{i_a}} & \text{for } \ i_a \in \bar{I}_a \\ \in \left\{ s^l_{\pi_{i_b}}, s^l_{\pi_{i_b}} + 1, \cdots, k_0 \right\} & \text{for } \ i_b \in \bar{I}_b \end{array} \right. , \bar{I}_a \cup \bar{I}_b = \{1, 2, \cdots, n\} , \sum_{i=1}^{n} w_i \geq su \right\} ,$$

where $I_a, I_b, \bar{I}_a, \bar{I}_b$ must be determined.

(8)

However in general, it must be stated that

$$B^{*\pi}_{n;k_0}(s^f, s^{l_{(j)}}_\pi) \bigcap B^{*\pi}_{n;k_0}(s^f, s^{l_{(i)}}_\pi) \neq \emptyset. \tag{9}$$

If we use perturbed permutations (which will be define in the following section) instead of the permutations themselves, then the intersection (9) is empty.

3 Computation of ReH-matrices by means of perturbed permutations

A ReH-matrices (and especially, single elements of it) can be determined more effectively if we use perturbed permutations.

Perturbed permutations $\hat{s}^l_\pi$ are particularly characterized by J_o, j_0, j_1 in the case of the surplus-situation and by $\bar{J}_o, \bar{j}_0, \bar{j}_1$ in case of the scarcity-situation, respectively. Based on perturbed permutations $\hat{s}^l_\pi$ and corresponding J_o, j_0, j_1 or $\bar{J}_o, \bar{j}_0, \bar{j}_1$, we define sets of requirements $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$ and $\hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^l_\pi)$, which are subsets of $B^{*}_{n;k_0}(s^f, s^l)$. The sets $\hat{B}^{*m}_{n;k_0}(s^f, \hat{s}^l_\pi)$ $(m = 1, 2)$ are pairwise disjoint (regarding $\hat{s}^l_\pi$) and their union (over all perturbed permutations) is equal to $B^{*m}_{n;k_0}(s^f, s^l)$. Thus, an element p^*_{fl} of a ReH-matrices can be calculated as sum of the probabilities of requirements which are elements from sets $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$ or $\hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^l_\pi)$ (minus a simple term). An example for such a calculation can be found in Appendix A, b).

Let us assume in this section that (without loss of generality) the components of the partitions $s^f \in S_{n;su;k_0}$ and $s^l \in S_{n;su;k_0}$ are initially ordered monotonically decreasing:

$$s_1^f \geq s_2^f \geq \cdots \geq s_n^f \text{ and } s_1^l \geq s_2^l \geq \cdots \geq s_n^l.$$

Furthermore, we use the the following additional notation and symbols:

$$\begin{aligned} &\text{F: the number of components of } s^f \text{ which are not equal to 0,} \\ &\text{L: the number of components of } s^l \text{ which are not equal to 0.} \end{aligned} \tag{10}$$

$$\begin{aligned} \mathbf{s^f}:\ & s_1^f = \cdots = s_{F_1}^f > s_{F_1+1}^f = \cdots = s_{F_2}^f > \cdots > s_{F_{z-1}+1}^f = \cdots = s_{F_z}^f > 0 \\ & \left(s_i^f = 0 \text{ for } i \geq F_z + 1 \text{ if } F_z < n\right) \\ & (\text{with } F_1 < F_2 < \cdots < F_z = F(< F_{z+1} = n \ for \ F_z < n)), \end{aligned} \tag{11}$$

$$\begin{aligned} \mathbf{s^l}:\ & s_1^l = \cdots = s_{L_1}^l > s_{L_1+1}^l = \cdots = s_{L_2}^l > \cdots \\ & \boldsymbol{> s_{L_{J_o-2}+1}^l = \cdots = s_{L_{J_o-1}}^l > s_{L_{J_o-1}+1}^l = \cdots = s_{L_{J_o}}^l > s_{L_{J_o}+1}^l =} \\ & \boldsymbol{\cdots = s_{L_{J_o+1}}^l >} \cdots > s_{L_{y-1}+1}^l = \cdots = s_{L_y}^l > 0 \\ & (0 = s_{L_y+1}^l = \cdots = s_{L_{y+1}}^l = s_n^l \text{ if } L_y < n) \end{aligned} \tag{12}$$

(with $L_1 < L_2 < \cdots < L_y = L(< L_{y+1} = n \ for \ L_y < n)$, furthermore $L_0 := 0$). Moreover, we define

$$\begin{aligned} & \sigma_J^l := s_{L_J}^l \text{ for } J = 1, 2, \ldots, y \ (or \ y+1 \ for \ L_y < n) \text{ and so} \\ & \sigma_1^l > \sigma_2^l > \boldsymbol{\cdots > \sigma_{J_o-1}^l > \sigma_{J_o}^l > \sigma_{J_o+1}^l >} \cdots > \sigma_y^l \ (> \sigma_{y+1}^l = 0 \\ & \text{for } L_y < n) \end{aligned} \tag{13}$$

follows.

Lastly let

$$\delta(s^f, s^l) = \delta_{fl} := \begin{cases} 1 & \text{if } s^f = s^l, \\ 0 & \text{if } s^f \neq s^l. \end{cases} \tag{14}$$

Below we will show how to compute the requirements $w \in B_{n;k_0}^*(s^f, s^l)$ in the cases $\sum_{i=1}^n w_i \leq su$ and $\sum_{i=1}^n w_i \geq su$ by means of sets of perturbed permutations.

The elements p_{fl}^* of the ReH-matrices can then be calculated in the following way

$p^*_{fl} = p^{*1}_{fl} + p^{*2}_{fl} - p^{*1,2}_{fl}$

with $p^{*1}_{fl} = \sum_{w \in B^{*1}_{n;k_0}(s^f,s^l)} q(w),\ p^{*2}_{fl} = \sum_{w \in B^{*2}_{n;k_0}(s^f,s^l)} q(w)$

$$\text{and } p^{*1,2}_{fl} = \sum_{w \in B^{*1}_{n;k_0}(s^f,s^l) \bigcap B^{*2}_{n;k_0}(s^f,s^l)} q(w) = \sum_{s^l_\pi:\ permutation\ of\ s^l} q(s^l_\pi). \quad (15)$$

At first we consider **Case** $\sum_{i=1}^{n} w_i \leq su$ **(***the requirements can be completely fulfilled***)**:

Given a partition $s^f \in S_{n;su;k_0}$ and a permutation s^l_π of a partition $s^l \in S_{n;su;k_0}$, we then compare the components of s^f with the components of s^l_π in order of increasing $s^l_{\pi_i}$. More formally we state:

Definition 3.1 *Let* $J_o \in \{1, 2, \dots, y \ (or\ y+1\ for\ L_y < n)\}$ *(see (12)) and* $j_0 \in \{1, 2, \dots, L_{J_o} - L_{J_o - 1}\}$.

(i) If

$$s^f_i \leq s^l_{\pi_i} \qquad for\ any\ s^l_{\pi_i} \leq \sigma^l_{J_o+1},^4 \qquad (a1)$$

$$s^f_i \leq s^l_{\pi_i} \qquad for\ L_{J_o} - L_{J_o-1} - j_0\ elements\ of\ the\ set\ \left\{ s^l_{\pi_1}, \cdots, s^l_{\pi_n} | s^l_{\pi_i} = \sigma^l_{J_o} \right\} \qquad (a2)$$

$$and\ s^f_i > s^l_{\pi_i} \qquad for\ j_0\ elements\ of\ the\ set\ \left\{ s^l_{\pi_1}, \cdots, s^l_{\pi_n} | s^l_{\pi_i} = \sigma^l_{J_o} \right\}, \qquad (a3)$$

we then refer to a $(\mathbf{J_o}, \mathbf{j_0})$**-perturbation of the relation "≤" between** $\mathbf{s^f}$ **and** $\mathbf{s^l_\pi}$.

(ii) $\hat{s}^l_\pi$ *with*

$$\hat{s}^l_{\pi_i} = \begin{cases} s^l_{\pi_i} + 1 & for \quad s^f_i > s^l_{\pi_i} = \sigma^l_{J_o} \quad (a4) \\ & (see\ (a3)\ from\ Definition\ 3.1), \\ s^l_{\pi_i} & otherwise \quad (a5) \end{cases}$$

is called a $(\mathbf{J_o}, \mathbf{j_0})$**-perturbed permutation with respect to** $\mathbf{s^f}$.

(iii) $\hat{\mathbf{S}}^{\mathbf{f,l}}_\pi(\mathbf{J_o}, \mathbf{j_0})$ *is the set of all* (J_o, j_0)*-perturbed permutations* $\hat{s}^l_\pi$ *of permutations* s^l_π *of* s^l*, for which a* (J_o, j_0)*-perturbation of the relation "≤" between* s^l_π *and* s^f *is present.*

(iv) $\hat{\mathbf{S}}^{\mathbf{f,l}}_\pi = \bigcup_{(J_o, j_0)} \hat{S}^{f,l}_\pi(J_o, j_0)$ *is the set of perturbed permutations in the case of the surplus-situation.*

Thus, $\sum\limits_{i=1}^{n} \hat{s}^l_{\pi_i} = su + j_0$ follows for $\hat{s}^l_\pi \in \hat{S}^{f,l}_\pi(J_o, j_0)$.

Definition 3.2 $\mathbf{\hat{S}^{f,l}_\pi(J_o, j_0, j_1)}$ *denotes the subset of* $\hat{S}^{f,l}_\pi(J_o, j_0)$*, where* $j_1 + j_0$ *is the number of* i*'s with:*

$$s^f_i \geq \hat{s}^l_{\pi_i} = \sigma^l_{J_o} + 1 \qquad (a6)$$

for all elements $\hat{s}^l_\pi$ *of this subset.*

(Obviously, $j_1 \in \{0, 1, \cdots, L_{J_o-1} - L_{J_o-2}\}$ *if* $\sigma^l_{J_o} = \sigma^l_{J_o-1} - 1$ *and*

$$j_1 = 0 \qquad \text{if } \sigma^l_{J_o} < \sigma^l_{J_o-1} - 1.) \qquad (a7)$$

Example: Let $n = 8, su = 25, k_o = 6$,
$s^f = (6, 4, 4, 4, 3, 2, 2, 0)^T, s^l = (4, 4, 4, 3, 3, 3, 2, 2)^T$. Then for instance
$s^{l(1)}_\pi = (4, 4, \underline{3}, \mathbf{2}, \underline{3}, 4, 3, 2)^T$, where $J_o = 3, j_0 = 1, j_1 = 2$
and $s^{l(2)}_\pi = (4, \underline{3}, \underline{3}, \mathbf{2}, \underline{3}, 4, 4, 2)^T$, where $J_o = 3, j_0 = 1, j_1 = 3$ are elements of $\hat{S}^{f,l}_\pi$.

Lemma 3.1 *Given a set* $\hat{S}^{f,l}_\pi(J_o, j_0, j_1)$ *and let* $\hat{s}^l \in \mathbb{Z}^n$

$$\text{with} \begin{cases} \hat{s}^l_j = s^l_j & \text{for } j \in \{1, 2, \cdots, L_{J_o-1}\} \text{ or} \\ & \phantom{\text{for }} j \in \{L_{J_o-1} + j_0 + 1, \cdots, n\} \qquad (a8) \\ \hat{s}^l_j = s^l_j + 1 (= \sigma^l_{J_o} + 1) & \text{for } j \in \{L_{J_o-1} + 1, \cdots, L_{J_o-1} + j_0\}. \quad (a9) \end{cases}$$

Then $\hat{s} \in \mathbb{Z}^n$ *is an element of the set* $\hat{S}^{f,l}_\pi(J_o, j_0, j_1)$ *if and only if* $\hat{s}$ *fulfils the following conditions regarding* s^f *and* $\hat{s}^l$*:*

(i) $\hat{s}$ *is a permutation of* $\hat{s}^l$,

(ii) $s^f_i \leq \hat{s}_i$ *if* $\hat{s}_i \leq \sigma^l_{J_o}$, $\qquad (a10)$

(iii) $s^f_i < \hat{s}_i$ *for* $L_{J_o-1} - L_{J_o-2} - j_1$ *elements of the set* $\{\hat{s}_1, \cdots, \hat{s}_n | \hat{s}_i = \sigma^l_{J_o} + 1\}$ *if* $\sigma^l_{J_o} + 1 = \sigma^l_{J_o-1}$, $\qquad (a11)$

(iv) $s^f_i \geq \hat{s}_i$ *for* $j_0 + j_1$ *elements of the set* $\{\hat{s}_1, \cdots, \hat{s}_n | \hat{s}_i = \sigma^l_{J_o} + 1\}$. $\qquad (a12)$

PROOF. Obviously, (i) is a necessary condition for $\hat{s} \in \hat{S}^{f,l}_\pi(J_o, j_0, j_1)$.
1. ($\Rightarrow$): Let $\hat{s}^l_\pi \in \hat{S}^{f,l}_\pi(J_o, j_o, j_1)$ be given.

Condition (a10) is valid for $\hat{s}^l_{\pi_i} \leq \sigma^l_{J_o+1}$ according to (a1) and for $\hat{s}^l_{\pi_i} = \sigma^l_{J_o}$ according to (a2) and (a4).

The condition (a11) is fulfilled for the remaining $L_{J_o-1} - L_{J_o-2} - j_1$ components $\hat{s}^l_{\pi_i}$ equal to $\sigma^l_{J_o} + 1$ according to the definition of j_1 (see (a6)).

Condition (a12) is valid according to (a6).

---

[4]Clearly, if $J_o = y$ (or $= y + 1$ for $L_y < n$) this case does not exist.

2. ($\Leftarrow$): Now, let $\hat{s}$ be a permutation of $\hat{s}^l$ satisfying (a10), (a11) and (a12).

A permutation s^l_π of s^l may then be constructed in the following way:

$$s^l_{\pi_i} = \begin{cases} \hat{s}_i - 1 & \text{for} \quad j_o \text{ components } \hat{s}_i = \sigma^l_{J_o} + 1 \leq s^f_i \\ & \quad\text{(which thus also satisfies (a12))}, \\ \hat{s}_i & \text{otherwise.} \end{cases} \tag{16}$$

We show that s^l_π fulfils the conditions from Definition 3.1:

Condition (a1): This follows from (a10) (specifically for $\hat{s}^l_{\pi_i} \leq \sigma^l_{J_o+1} (< \sigma^l_{J_o})$).

Condition (a2): According to the definition of $\hat{s}^l$ (and (12)) $\hat{s}_i = \sigma^l_{J_o}$ is valid for $L_{J_o} - L_{J_o - 1} - j_0$ components. Only (a10) can be present in Lemma 3.1 for these components, which means $s^f_i \leq \hat{s}_i$ (see also (16)). (a2) then follows.

Then (a12) (and (12)), the definition of $\hat{s}^l$ (see (a9)) and (16) yield (a3).

Vice versa, the permutation s^l_π leads to $\hat{s}$, according to (a4) and (a5). Reconsidering (a12) and (a11) we can conclude that, $\hat{s}^l_\pi$ is an element of the set $\hat{S}^{f,l}_\pi(J_o, j_o, j_1)$. ■

Together Definitions 3.1, 3.2 and Lemma 3.1 obviously yield:

Lemma 3.2 *Let $\hat{S}^{f,l}_\pi(J^1_o, j^1_0, j^1_1)$ and $\hat{S}^{f,l}_\pi(J^2_o, j^2_0, j^2_1)$ (with respect to s^f) be given and assuming $J^1_o \neq J^2_o$ or $j^1_0 \neq j^2_0$ or $j^1_1 \neq j^2_1$.*

Then, $\hat{S}^{f,l}_\pi(J^1_o, j^1_0, j^1_1) \bigcap \hat{S}^{f,l}_\pi(J^2_o, j^2_0, j^2_1) = \varnothing$ follows.

PROOF.

Case $J^1_o \neq J^2_o$: For perturbed permutations from $\hat{S}^{f,l}_\pi(J^1_o, j^1_o, j^1_1)$ or $\hat{S}^{f,l}_\pi(J^2_o, j^2_o, j^2_1)$, respectively different $\sigma^l_{J^1_o}$ and $\sigma^l_{J^2_o}$ have been increased by 1 (see Definition 3.1, (a4) and (a5)).

Case $J^1_o = J^2_o$ and $j^1_o \neq j^2_o$: Perturbed permutations of $\hat{S}^{f,l}_\pi(J^1_o, j^1_o, j^1_1)$ and $\hat{S}^{f,l}_\pi(J^2_o, j^2_o, j^2_1)$ have then different numbers of components with value $\sigma^l_{J^{1=2}_o}$, since different numbers of components have been increased by 1 (see Definition 3.1).

Case $J^1_o = J^2_o$, $j^1_o = j^2_o$ and $j^1_1 \neq j^2_1$: Perturbed permutations of $\hat{S}^{f,l}_\pi(J^1_o, j^1_o, j^1_1)$ and $\hat{S}^{f,l}_\pi(J^2_o, j^2_o, j^2_1)$ have different numbers of components for which (a11) or (a12) from Lemma 3.1 is valid. ■

Definition 3.3 *Let an element $\hat{s}^l_\pi$ from a set $\hat{S}^{f,l}_\pi(J_o, j_0, j_1)$ be given.*

Then the set of all $w \in B_{n;k_0}$ which fulfil the properties:

(i) $w_i \in \{0, 1, \ldots, \hat{s}^l_{\pi_i}\}$ *if* $s^f_i = \hat{s}^l_{\pi_i} \leq \sigma^l_{J_o}$ *(a13)*

(ii) $w_i \in \{0, 1, \ldots, \hat{s}^l_{\pi_i} = \sigma^l_{J_o} + 1\}$ *and* w *with at most* $\mathbf{j_1}$ *coordinates* $w_i = \sigma^l_{J_o} + 1$

if $s^f_i \geq \hat{s}^l_{\pi_i} = \sigma^l_{J_o} + 1,$ *(a14)*

(iii) $w_i = \hat{s}^l_{\pi_i}$ *otherwise* *(a15)*

is denoted by $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$.

Remark 3.1 *Properties (a14) and (a13) show that the increase of components of value* $\sigma^l_{J_o}$ *by 1 in order to determine* (J_o, j_0)*-perturbed permutations for* $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$ *is in fact not necessary in the case* $\sigma^l_{J_o} + 1 < \sigma^l_{J_o-1}$*, since the last inequality implies that* $\mathbf{j_1} = 0$ *because of (a7). However, this method leads to clearer and more uniform representations of the Definitions 3.1, 3.2, 3.3 and so on, so that distinctions in certain cases in the representations are not necessary.*

Lemma 3.3 *Let an element* $\hat{s}^l_\pi$ *from a set* $\hat{S}^{f,l}_\pi(J_o, j_0, j_1)$ *be given. In addition, let* $w \in \hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$.

Then, the case "otherwise" in Definition 3.3 is valid if

$$s^f_i < \hat{s}^l_{\pi_i} \text{ or} \tag{a16}$$

$$s^f_i \geq \hat{s}^l_{\pi_i} > \begin{cases} \sigma^l_{J_o-1} & \text{if } \sigma^l_{J_o} + 1 = \sigma^l_{J_o-1}, \\ \sigma^l_{J_o} & \text{if } \sigma^l_{J_o} + 1 < \sigma^l_{J_o-1}. \end{cases} \tag{a17}$$

PROOF.

Case $\hat{s}^l_{\pi_i} \leq \sigma^l_{J_o}$: The inequality $s^f_i \leq \hat{s}^l_{\pi_i}$ follows according to (a10) from Lemma 3.1.

$s^f_i = \hat{s}^l_{\pi_i} (\leq \sigma^l_{J_o})$ can be found in (a13) of Definition 3.3.

$s^f_i < \hat{s}^l_{\pi_i}$ belongs to "otherwise" in this definition.

Case $\hat{s}^l_{\pi_i} = \sigma^l_{J_o} + 1$: $s^f_i \leq \hat{s}^l_{\pi_i} (= \sigma^l_{J_o} + 1)$ can be found in (a14) of Definition 3.3.

$s^f_i < \hat{s}^l_{\pi_i}$ belongs to "otherwise" in this definition.

Case $\hat{s}^l_{\pi_i} > \sigma^l_{J_o} + 1$: This means $\hat{s}^l_{\pi_i} > \begin{cases} \sigma^l_{J_o-1} & \text{if } \sigma^l_{J_o} + 1 = \sigma^l_{J_o-1}, \\ \sigma^l_{J_o} & \text{if } \sigma^l_{J_o} + 1 < \sigma^l_{J_o-1}. \end{cases}$

If we have in addition $s^f_i \geq \hat{s}^l_{\pi_i}$, then (a17) is valid and if we have $s^f_i < \hat{s}^l_{\pi_i}$, then (a16) is valid. ■

Lemma 3.4 *Let a set* $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$ *be defined according to Definition 3.3.*

Then there exist exactly $\binom{j_o + j_1}{j_o}$ *requirements* $w \in \hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$*, which satisfy the cases* $\sum_{i=1}^n w_i \leq su$ *and* $\sum_{i=1}^n w_i \geq su$ *simultaneously.*

PROOF.

Case $\sigma^l_{J_o}+1=\sigma^l_{J_o-1}$: ($j_1>0$ is possible in this case, see Definition 3.2).

If the components of s^l_π and the (J_o, j_o)-perturbed permutation $\hat{s}^l_\pi$ from $\hat{S}^{f,l}_\pi(J_o, j_o, j_1)$ (with respect to s^f) are compared, then

$$\hat{s}^l_{\pi_i} > s^l_{\pi_i} \text{ may only be possible if } s^f_i \geq \hat{s}^l_{\pi_i} = \sigma^l_{J_o}+1 \qquad (*1)$$

(see also Definition 3.3)

is valid according to (i) and (ii) of Definition 3.1.

The condition (a14) from Definition 3.3

$$s^f_i \geq \hat{s}^l_{\pi_i} = \sigma^l_{J_o}+1 \text{ is valid for exactly } j_o+j_1 \text{ components } \hat{s}^l_{\pi_i} \qquad (*2)$$

according to Lemma 3.1, (a12).

In relation to (a14) from Definition 3.3 let

$$w_i = \sigma^l_{J_o}+1 \text{ for } j_2 \text{ coordinates } w_i \text{ where } j_2 \leq j_1. \qquad (*3)$$

From (*3) and (*1) (refer also to (*2)) and Definition 3.3 it follows that

$$\sum_{i=1}^n w_i \leq j_2(\sigma^l_{J_o}+1) + [j_o+(j_1-j_2)]\sigma^l_{J_o} + \sum_{i:\ if\ not\ s^f_i \geq \hat{s}^l_{\pi_i} = \sigma^l_{J_o}+1} s^l_{\pi_i}$$

$$= \sum_{i=1}^n s^l_{\pi_i} + (j_2-j_1) = su + (j_2-j_1) \leq su. \qquad (*4)$$

The equation $\sum_{i=1}^n w_i = su$ is only correct, if $j_2=j_1$ in (*4) and all w_i are as large as possible, according to Definition 3.3. This means that, in relation to (a14), j_1 coordinates w_i are equal to $\sigma^l_{J_o}+1$ and j_0 coordinates w_i are equal to $\sigma^l_{J_o}$ (see also (*2)).

Thus, exactly $\binom{j_o+j_1}{j_o}$ different requirements w satisfy the cases $\sum_{i=1}^n w_i \leq su$ and $\sum_{i=1}^n w_i \geq su$ simultaneously.

Case $\sigma^l_{J_o}+1<\sigma^l_{J_o-1}$: $j_1=0$ follows according to Definition 3.2.

Hence, in relation to (a14), it is impossible that $w_i=\sigma^l_{J_o}+1$ (for any i with $s^f_i \geq \hat{s}^l_{\pi_i} = \sigma^l_{J_o}+1$).

Since $\binom{j_o+0=j_o}{j_o}=1$, there is only one possibility, in which all coordinates w_i from Definition 3.3 are as large as possible, which then implies $\sum_{i=1}^n w_i = su$. ■

Remark 3.2 *Probabilities of w from Lemma 3.4 are added in order to compute p^{*1}_{fl} and also p^{*2}_{fl}. Therefore, these probabilities must be subtracted once from $p^{*1}_{fl}+p^{*2}_{fl}$ for the determination of p^*_{fl} in (15).*

Theorem 3.5 *Let $s^f \in S_{n;su;k_0}$ and $s^l \in S_{n;su;k_0}$ be given. Then we have:*

$$B^{*1}_{n;k_0}(s^f, s^l) = \bigcup_{\hat{s}^l_\pi \,\in\, \hat{S}^{f,l}_\pi} \hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi) \text{ if } s^f \neq s^l \text{ and}$$

$$B^{*1}_{n;k_0}(s^f, s^l) = \bigcup_{\hat{s}^l_\pi \,\in\, \hat{S}^{f,l}_\pi} \hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi) \bigcup \{w \in B_{n;k_0} | \; w_i \leq s^f_i, \; i = 1, ..., n\}$$

$$\text{if } s^f = s^l.$$

PROOF.

1. **(⇒)**: Let a requirement $w \in B^{*1}_{n;k_0}(s^f, s^l)$ with $\sum_{i=1}^{n} w_i < su$ be given.

The easy case in which $w_i \leq s^f_i$ for $i = 1, ..., n$, so that a permutation s^l_π of s^l, s^f and w satisfy (3), is only possible if $s^f_i = s^l_{\pi_i}$ for $i = 1, ..., n$. In consequence we have to add $\{w \in B_{n;k_0} | \; w_i \leq s^f_i, \; i = 1, ..., n\}$ in the second set-equality.

Now, let $s^l_\pi \neq s^f$ be a permutation of s^l such that s^l_π, s^f and w satisfy (3). The implication

$$w_i \geq s^f_i \;\Rightarrow\; s^l_{\pi_i} = w_i \;\; (\text{thus also } s^l_{\pi_i} \geq s^f_i) \qquad (*1)$$

follows from (3) (case $\sum_{i=1}^{n} w_i \leq su$).

Next we consider the iterative method from Section 2 in case $\sum_{i=1}^{n} w_i \leq su$:

Since $s^*_j = s^*_j - 1$ for $s^*_j = \max\{s^*_i \mid s^*_i > w_i\}$, components of different permutations of s^l (which, together with s^f and w, satisfy (3)), differ by at most 1. (Different components are only possible if j with $s^*_j = \max\{s^*_i \mid s^*_i > w_i\}$ is not unique in the final iteration steps). In more detail, $s^l_{\pi_{i_0}}$ can be different if:

$$s^l_{\pi_{i_0}} = \min\{s^l_{\pi_i} \mid s^f_i > s^l_{\pi_i} \geq w_i\} \qquad (*2)$$

and if i_1 exists so that

$$s^f_{i_1} \geq s^l_{\pi_{i_1}} = s^l_{\pi_{i_0}} + 1 > w_{i_1}. \qquad (*3)$$

(Then $s^l_{\pi_{i_1}}$ could be reduced by 1 instead of $s^l_{\pi_{i_0}} + 1 (= s^*_j)$ in the final iteration steps if the iterative method from Section 2 is used.)

Possible relationships between $s^l_{\pi_{i_0}}$ (in (*2)) and certain $s^f_i, w_i, s^l_{\pi_i}$ can be:

$$s^f_i = s^l_{\pi_i} = s^l_{\pi_{i_0}} > w_i \text{ or} \qquad (*4)$$

$$s^f_i > s^l_{\pi_i} = s^l_{\pi_{i_0}} + 1 = w_i \text{ or} \qquad (*5)$$

$$s^f_i > s^l_{\pi_i} = w_i > s^l_{\pi_{i_0}} + 1. \qquad (*6)$$

With regard to Definition 3.1, we now use $s^l_{\pi_{i_0}}$ (from (*2)) as $\sigma^l_{J_o}$, and the number of i_0, for which (*2) is satisfied, as j_0.

With the help of (*1), ..., (*5) we can show that (J_o, j_o) is a perturbation of the relation "$\leq$" between s^f and s^l_π:

(a1): $s^l_{\pi_i} \leq \sigma^l_{J_o+1} (< \sigma^l_{J_o})$ can only be possible if (*1) is valid,

from which $s^l_{\pi_i} \geq s^f_i$ follows,

(a2) and (a3): $s^l_{\pi_i} = \sigma^l_{J_o}$ is only valid if (*2) and (*4) are valid,

then (a3) follows from (*2) and (a2) from (*4).

So, $\hat{s}^l_\pi$ with

$$\hat{s}^l_{\pi_i} = \begin{cases} s^l_{\pi_i} + 1 & \text{if (*2) is satified for } i = i_0, \\ s^l_{\pi_i} & \text{otherwise} \end{cases}$$

is a (J_o, j_o)-perturbed permutation of s^l_π with respect to s^f.

With regard to Definition 3.2, we set j_1 equal to the number of i_2's for which (*5) is satisfied. Thus, $\hat{s}^l_\pi \in \hat{S}^{f,l}_\pi(J_o, j_o, j_1)$.

Finally, we show that w is an element of $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$ (see Definition 3.3):

(a13): mainly follows from (*4) and (*1) with $s^l_{\pi_i} = s^f_i \leq s^l_{\pi_{i_0}}$,

(a14): mainly follows from (*2), (*3) and (*5)

(considering the previous determination of $\hat{s}^l_\pi$),

(a15): mainly follows from (*1) and (*6).

2. **($\Leftarrow$):** Let $w \in \hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$, $\hat{s}^l_\pi \in \hat{S}^{f,l}_\pi(J_o, j_0, j_1)$.

If $w_i \leq s^f_i$ for $i = 1, ..., n$, then $s^*(s^f, w) = s^f (= s^l)$ follows immediately.

Now, let i exist with $w_i > s^f_i$.

We will show that the iterative method from Section 2 (case $\sum_{i=1}^{n} w_i \leq su$), initially leads to $\hat{s}^l_\pi$ and further to a s^l_π (as in (16)). This means that $s^*(s^f, w) = s^l$.

At first, we note that

$$\sum_{i: s^f_i < w_i} (w_i - s^f_i) = \sum_{i: s^l_{\pi_i} < s^f_i} (s^f_i - s^l_{\pi_i}) \qquad (*7)$$

is a necessary condition for $s^*(s^f, w)(= s^l_\pi) = s^l$. According to the iterative method from Section 2, differences between s_i^f and w_i, in the cases that $s_i^f < w_i$, are used in order to reduce s_j^f to certain $s_{\pi_j}^l$ in the cases that $s_j^f > w_j$ (where s_π^l is a permutation of $s^l = s^*(s^f, w)$).

We prove that (*7) is valid:

$s_i^f < w_i$ is only possible in the case (a15) of Definition 3.3 where $w_i = \hat{s}_{\pi_i}^l$. Thus,

$$\sum_{i:s_i^f < w_i} (w_i - s_i^f) = \sum_{i:s_i^f < \hat{s}_{\pi_i}^l} (\hat{s}_{\pi_i}^l - s_i^f) \qquad (*8)$$

follows.

According to (a4) (together with (a3)), and since $\sum_{i=1}^n s_i^f = \sum_{i=1}^n s_i^l = su$,

$$\sum_{i:s_i^f < \hat{s}_{\pi_i}^l} (\hat{s}_{\pi_i}^l - s_i^f) = \sum_{i:\hat{s}_{\pi_i}^l < s_i^f} (s_i^f - \hat{s}_{\pi_i}^l) + j_o \qquad (*9)$$

is valid and

$$\sum_{i:\hat{s}_{\pi_i}^l < s_i^f} (s_i^f - \hat{s}_{\pi_i}^l) + j_o = \sum_{i:s_{\pi_i}^l < s_i^f} (s_i^f - s_{\pi_i}^l) \qquad (*10)$$

follows for s_π^l as in (16).

So, (*8), (*9) and (*10) imply (*7).

Lastly, the consideration of the following cases show that the iterative method from Section 2, case $\sum_{i=1}^n w_i \leq su$, initially leads to $\hat{s}_\pi^l$ (from this theorem) and then to a s_π^l (as in (16)):

- Case $s_i^f < w_i$:
  According to the iterative method (and also according to (3)) it follows in this case that $w_i = \hat{s}_{\pi_i}^l (= s_{\pi_i}^l)$, which corresponds to (a15).

- Case $s_i^f = w_i$:
  In this case the iterative method leads to $s_i^f = s_{\pi_i}^l$, which corresponds to the relevant cases of Definition 3.3 ($s_i^f = s_{\pi_i}^l = \hat{s}_{\pi_i}^l$).

- Case $s_i^f > w_i > \begin{cases} \sigma_{J_o-1}^l & \text{if } \sigma_{J_o}^l + 1 = \sigma_{J_o-1}^l, \\ \sigma_{J_o}^l & \text{if } \sigma_{J_o}^l + 1 < \sigma_{J_o-1}^l \end{cases}$:
  This means $s_{\pi_i}^l = w_i (= \hat{s}_{\pi_i}^l)$ which corresponds to (a17) (partial case of (a15)). Because the values s_i^f are reduced to w_i using the iterative method.

- Case $s_i^f > w_i$ and $w_i \leq \begin{cases} \sigma^l_{J_o-1} & \text{if } \sigma^l_{J_o}+1 = \sigma^l_{J_o-1}, \\ \sigma^l_{J_o} & \text{if } \sigma^l_{J_o}+1 < \sigma^l_{J_o-1} \end{cases} \leq s_i^f$:
  Initially, s_i^f are reduced to $\sigma^l_{J_o}+1$ using the iterative method, which corresponds to $\hat{s}^l_{\pi_i}$ from (a14). So j_0 units remain, which can be used to further reduce the j_0 parts of the value $\sigma^l_{J_o}+1$ by 1 (if $w_i \leq \sigma^l_{J_o}$) (which then corresponds to $\hat{s}^l_{\pi_i}$ from Definition 3.3, (a13) with $s_i^f = \sigma^l_{J_o}$).

- Case $s_i^f > w_i$ and $w_i < s_i^f < \begin{cases} \sigma^l_{J_o-1} & \text{if } \sigma^l_{J_o}+1 = \sigma^l_{J_o-1}, \\ \sigma^l_{J_o} & \text{if } \sigma^l_{J_o}+1 < \sigma^l_{J_o-1} \end{cases}$:
  In this case the iterative method can not reduce components s_i^f (see also (*7), (*8), (*9) and (*10)). So we have $s^l_{\pi_i} = s_i^f (= \hat{s}^l_{\pi_i})$ as also in (a13).

■

Theorem 3.6

$$\textit{Let } \hat{s}^{l,1}_\pi \in \hat{S}^{f,l}_\pi \textit{ and } \hat{s}^{l,2}_\pi \in \hat{S}^{f,l}_\pi \textit{ be given with } \hat{s}^{l,1}_\pi \neq \hat{s}^{l,2}_\pi. \tag{*1}$$

Then, $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^{l,1}_\pi) \bigcap \hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^{l,2}_\pi) = \varnothing$.

(Furthermore, in the case $s^f = s^l$ *we have*
$\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi) \bigcap \{w \in B_{n;k_0} | \ w_i \leq s_i, \ i = 1, ..., n\} = \varnothing$.*)*

PROOF. Let w^1 be an element of the set $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^{l,1}_\pi)$, w^2 an element of $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^{l,2}_\pi)$ and $\hat{s}^{l,1}_{\pi_{i_o}} \neq \hat{s}^{l,2}_{\pi_{i_o}}$ according to (*1). In order to show that $w^1 \neq w^2$ we have to consider 3 cases.

Case 1: $\hat{s}^{l,1}_{\pi_{i_o}} > s^f_{i_o}$

From Definition 3.3 it follows that

$w^1_{\pi_{i_o}} = \hat{s}^{l,1}_{\pi_{i_o}}$ (in particular, see (a15) and (a10)),

$$w^2_{\pi_{i_o}} \begin{cases} = \hat{s}^{l,2}_{\pi_{i_o}} & \text{if } \hat{s}^{l,2}_{\pi_{i_o}} > s^f_{i_o}, \\ \leq \hat{s}^{l,2}_{\pi_{i_o}} & \text{if } \hat{s}^{l,2}_{\pi_{i_o}} \leq s^f_{i_o} \end{cases}$$

Thus, $w^1_{\pi_{i_o}} \neq w^2_{\pi_{i_o}}$.

Case 2:

$$s^f_{i_o} \geq \hat{s}^{l,2}_{\pi_{i_o}} > \hat{s}^{l,1}_{\pi_{i_o}} \text{ and } \sigma^l_{J^1_o} \geq \sigma^l_{J^2_o} \tag{*2}$$

The relationship $\hat{s}^{l,1}_{\pi_{i_o}} \leq \sigma^l_{J^1_o}$ is not possible according to (*2), Definition 3.3 and in particular (a10), Lemma 3.1

Hence, it remains to consider the case $\hat{s}^{l,1}_{\pi_{i_o}} \geq \sigma_{J^1_o} + 1$. For requirements $w^1 \in \hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^{l,1}_\pi)$ we have $w^1_{\pi_{i_o}} \leq \hat{s}^{l,1}_{\pi_{i_o}}$ and with regard to (*2) we get

$\hat{s}^{l,2}_{\pi_{i_o}} > (\hat{s}^{l,1}_{\pi_{i_o}} \geq \sigma_{J^1_o} + 1 \geq) \ \sigma_{J^2_o} + 1$.

Thus, $w^2_{\pi_{i_o}} = \hat{s}^{l,2}_{\pi_{i_o}}$ is valid according to Definition 3.3 (see (a15) together with (a17)) and so $w^1_{\pi_{i_o}} \neq w^2_{\pi_{i_o}}$.

Case 3:

$$s^f_{i_o} \geq \hat{s}^{l,2}_{\pi_{i_o}} > \hat{s}^{l,1}_{\pi_{i_o}} \text{ and } \sigma^l_{J^1_o} < \sigma^l_{J^2_o} \qquad (*3)$$

In this case there exists i_1 with $s^f_{i_1} > s^{l,1}_{\pi_{i_1}} = \hat{s}^{l,1}_{\pi_{i_1}} - 1 = \sigma^l_{J^1_o}$ (*4) according to (a3) and (a4).

Regarding $\hat{s}^{l,2}_{\pi}$ and i_1 either

$s^{l,2}_{\pi_{i_1}} \geq \sigma^l_{J^2_o}$ (*5a)

or

$\sigma^l_{J^2_o} > s^{l,2}_{\pi_{i_1}} \geq s^f_{i_1}$ (see also (a1)) (*5b)

is valid.

Relations (*4),(*5a) and $\sigma^l_{J^1_o} < \sigma^l_{J^2_o}$ lead to

$s^{l,2}_{\pi_{i_1}} \geq \sigma^l_{J^2_o} > \sigma^l_{J^1_o} = \hat{s}^{l,1}_{\pi_{i_1}} - 1 = s^{l,1}_{\pi_{i_1}}$ (*6a)

and (*4) and (*5b), respectively to

$s^{l,2}_{\pi_{i_1}} \geq s^f_{i_1} > \hat{s}^{l,1}_{\pi_{i_1}} - 1 = s^{l,1}_{\pi_{i_1}} = \sigma^l_{J^1_o}$. (*6b)

Because of (*6a) and (*6b) it follows that

$s^{l,2}_{\pi_{i_1}} > s^{l,1}_{\pi_{i_1}} = \sigma^l_{J^1_o}$ (*7)

(where $i_o = i_1$ is possible).

Since $s^{l,2}_{\pi}$ is a permutation of $s^{l,1}_{\pi}$

there exists i_2 ($i_2 \neq i_1$) with $s^{l,2}_{\pi_{i_2}} = s^{l,1}_{\pi_{i_1}} (= \sigma^l_{J^1_o} \leq \sigma^l_{J^2_o} - 1)$. (*8)

Furthermore, $s^f_{i_2} \leq \hat{s}^{l,2}_{\pi_{i_2}} = s^{l,2}_{\pi_{i_2}}$ is valid (see also (a1)). (*9)

If $\hat{s}^{l,1}_{\pi_{i_2}} > \hat{s}^{l,2}_{\pi_{i_2}} (\geq s^f_{i_2})$ then $w^1_{i_2} = \hat{s}^{l,1}_{\pi_{i_2}} > \hat{s}^{l,2}_{\pi_{i_2}} \geq w^2_{i_2}$ follows according to Definition 3.3 and Lemma 3.3. Thus

$w^1_{i_2} \neq w^2_{i_2}$. (*10)

In addition, if $\hat{s}^{l,1}_{\pi_{i_2}} \leq \hat{s}^{l,2}_{\pi_{i_2}} (= s^{l,2}_{\pi_{i_2}} = s^{l,1}_{\pi_{i_1}})$ (see also (*8) and (*9)), we can conclude again in a similar way for the two possible subcases: $s^{l,1}_{\pi_{i_2}} = s^{l,2}_{\pi_{i_2}}$ and $s^{l,1}_{\pi_{i_2}} < s^{l,2}_{\pi_{i_2}}$, with $s^{l,2}_{\pi}$ as a permutation of $s^{l,1}_{\pi}$.

There exists i_3 ($i_3 \neq i_2 \wedge i_3 \neq i_1$) with $s^{l,2}_{\pi_{i_3}} = s^{l,1}_{\pi_{i_2}} (\leq \sigma^l_{J^1_o} \leq \sigma^l_{J^2_o} - 1)$ and so on. Since the numbers of parts of $s^{l,2}_{\pi}$ and $s^{l,1}_{\pi}$ are finite, we can conclude that $w^1_{i_m} \neq w^2_{i_m}$ for a certain m analogous to (*10).

(Finally, the Definition 3.3 of $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)$ and Lemma 3.3 directly yields $\hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi) \bigcap \{w \in B_{n;k_0} | \, w_i \leq s_i, \; i = 1, ..., n\} = \varnothing$ if $s^f = s^l$.) ∎

We now compute the probability of requirements $w \in B^*_{n;k_0}(s^f, s^l)$ in the case $\sum_{i=1}^{n} w_i \leq su$ using Definition 3.3, Theorem 3.5 and Theorem 3.6:

$$p^{*1}_{fl} = \sum_{\hat{s}^l_\pi \in \hat{S}^{f,l}_\pi} \sum_{w \in \hat{B}^{*1}_{n;k_0}(s^f, s^l_\pi)} q(w)$$

$$+ \ \delta_{fl} \ * \sum_{w \in \{w \in B_{n;k_0} \mid w_i \le s^f_i,\ i=1,\ldots,n\}} q(w). \tag{17}$$

In the case of discrete uniformly distributed requirements, Definition 3.3 yields:

$$\sum_{w \in \hat{B}^{*1}_{n;k_0}(s^f, \hat{s}^l_\pi)} q(w) \ = \ \frac{1}{(k_0+1)^n} \Bigg[\prod_{i:\hat{s}^l_{\pi_i} = s^f_i \le \sigma^l_{J_o}} (\hat{s}^l_{\pi_i} + 1)$$

$$\left((\sigma^l_{J_o} + 2)^{j_0+j_1} - \binom{j_0+j_1}{j_1+1} (\sigma^l_{J_o} + 1)^{j_0-1} - \cdots - \binom{j_0+j_1}{j_0+j_1} (\sigma^l_{J_o} + 1)^0 \right) \Bigg]$$

$$\text{and } \delta_{fl} \ * \sum_{w \in \{w \in B_{n;k_0} \mid w_i \le s^f_i,\ i=1,\ldots,n\}} q(w) \ = \ \delta_{fl} \ * \ \frac{1}{(k_0+1)^n} \prod_{i=1}^{n} (s^f_i + 1). \tag{18}$$

It remains to consider the **case** $\sum_{i=1}^{n} w_i \ge su$ (*the requirements cannot be completely fulfilled*):

The considerations are analogous to the case $\sum_{i=1}^{n} w_i \le su$. Therefore, we only present definitions, lemmas and theorems in this part of the section but no corresponding proofs, which would be very similar to the proofs in the other case.

Given a partition $s^f \in S_{n;su;k_0}$ and a permutation s^l_π of a partition $s^l \in S_{n;su;k_0}$, we then compare the components of s^f with the components of s^l_π in order of decreasing $s^l_{\pi_i}$. More formally we state:

Definition 3.4 *Let $\bar{J}_o \in \{1, 2, \ldots, y\}$ and $\bar{j}_o \in \{1, 2, \ldots, L_{\bar{J}_o} - L_{\bar{J}_o - 1}\}$.*

(i) If

$$s^f_i \ge s^l_{\pi_i} \qquad \text{for any } s^l_{\pi_i} \ge \sigma^l_{\bar{J}_o - 1},^{5} \tag{b1}$$

$$s^f_i \ge s^l_{\pi_i} \qquad \text{for } L_{\bar{J}_o} - L_{\bar{J}_o-1} - \bar{j}_o \text{ elements of the set } \left\{ s^l_{\pi_1}, \cdots, s^l_{\pi_n} | s^l_{\pi_i} = \sigma^l_{\bar{J}_o} \right\} \tag{b2}$$

$$\text{and } s^f_i < s^l_{\pi_i} \qquad \text{for } \bar{j}_o \text{ elements of the set } \left\{ s^l_{\pi_1}, \cdots, s^l_{\pi_n} | s^l_{\pi_i} = \sigma^l_{\bar{J}_o} \right\}, \tag{b3}$$

we then refer to a $(\bar{\mathbf{J}}_{\mathbf{o}}, \bar{\mathbf{j}}_{\mathbf{o}})$**-perturbation of the relation "$\geq$" between $\mathbf{s^f}$ and $\mathbf{s^l_\pi}$**.

(ii) $\hat{s}^l_\pi$ *with*

$$\hat{s}^l_{\pi_i} = \begin{cases} s^l_{\pi_i} - 1 & \text{for} \quad s^f_i < s^l_{\pi_i} = \sigma^l_{\bar{J}_o} \quad (b4) \\ & \text{(see (b3) from Definition 3.4),} \\ s^l_{\pi_i} & \text{otherwise} \quad (b5) \end{cases}$$

is called a $(\bar{\mathbf{J}}_{\mathbf{o}}, \bar{\mathbf{j}}_{\mathbf{o}})$**-perturbed permutation of the** $(\bar{\mathbf{J}}_{\mathbf{o}}, \bar{\mathbf{j}}_{\mathbf{o}})$**-perturbed partition $\hat{\mathbf{s}}^{\mathbf{l}}$ with respect to $\mathbf{s^f}$**.

(iii) $\hat{\mathbf{S}}^{\mathbf{f,l}}_\pi(\bar{\mathbf{J}}_{\mathbf{o}}, \bar{\mathbf{j}}_{\mathbf{0}})$ *is the set of all* $(\bar{J}_o, \bar{j}_0)$*-perturbed permutations* $\hat{s}^l_\pi$ *of permutations* s^l_π *of* s^l*, for which a* $(\bar{J}_o, \bar{j}_0)$*-perturbation of the relation "$\geq$" between* s^l_π *and* s^f *is present.*

(iv) $\hat{\mathbf{S}}^{\mathbf{f,l}}_\pi = \bigcup_{(\bar{J}_o, \bar{j}_0)} \hat{S}^{f,l}_\pi(\bar{J}_o, \bar{j}_0)$ *is the set of perturbed permutations in case of the scarcity-situation.*

Thus, $\sum_{i=1}^{n} \hat{s}^l_{\pi_i} = su - \bar{j}_0$ follows for $\hat{s}^l_\pi \in \hat{S}^{f,l}_\pi(\bar{J}_o, \bar{j}_0)$.

Definition 3.5 $\hat{\mathbf{S}}^{\mathbf{f,l}}_\pi(\bar{\mathbf{J}}_{\mathbf{o}}, \bar{\mathbf{j}}_{\mathbf{0}}, \bar{\mathbf{j}}_{\mathbf{1}})$ *denotes the subset of* $\hat{S}^{f,l}_\pi(\bar{J}_o, \bar{j}_0)$*, where* $\bar{j}_1 + \bar{j}_0$ *is the number of* i*'s with:*

$$s^f_i \leq \hat{s}^l_{\pi_i} = \sigma^l_{\bar{J}_o} - 1 \qquad (b6)$$

for all elements $\hat{s}^l_\pi$ *of this subset.*

(Obviously, $\bar{j}_1 \in \{0, 1, \cdots, L_{\bar{J}_o+1} - L_{\bar{J}_o}\}$ *if* $\sigma^l_{\bar{J}_o} - 1 = \sigma^l_{\bar{J}_o+1}$ *and*

$$\bar{j}_1 = 0 \qquad \text{if } \sigma^l_{\bar{J}_o} - 1 > \sigma^l_{\bar{J}_o+1}.) \qquad (b7)$$

Lemma 3.7 *Given a set* $\hat{S}^{f,l}_\pi(\bar{J}_o, \bar{j}_0, \bar{j}_1)$ *and let* $\hat{s}^l \in \mathbb{Z}^n$

$$\text{with} \begin{cases} \hat{s}^l_j = s^l_j & \text{for} \quad j \in \{1, 2, \cdots, L_{\bar{J}_o} - \bar{j}_o\} \text{ or} \\ & \qquad j \in \{L_{\bar{J}_o} + 1, \cdots, n\} \quad (b8) \\ \hat{s}^l_j = s^l_j - 1 (= \sigma^l_{\bar{J}_o} - 1) & \text{for} \quad j \in \{L_{\bar{J}_o} - \bar{j}_o + 1, \cdots, L_{\bar{J}_o}\}. \quad (b9) \end{cases}$$

Then $\hat{s} \in \mathbb{Z}^n$ *is an element of the set* $\hat{S}^{f,l}_\pi(\bar{J}_o, \bar{j}_0, \bar{j}_1)$ *if and only if* $\hat{s}$ *fulfils the following conditions regarding* s^f *and* $\hat{s}^l$*:*

---

[5] Clearly, if $J_o = 1$ this case does not exist.

(i) $\hat{s}$ *is a permutation of* $\hat{s}^l$,

(ii) $s_i^f \geq \hat{s}_{\pi_i}^l$ *if* $\hat{s}_i \geq \sigma_{\bar{J}_o}^l$, $(b10)$

(iii) $s_i^f > \hat{s}_i$ *for* $L_{\bar{J}_o+1} - L_{\bar{J}_o} - \bar{j}_1$ *elements of the set* $\left\{\hat{s}_1, \cdots, \hat{s}_n | \hat{s}_i = \sigma_{\bar{J}_o}^l - 1\right\}$

if $\sigma_{\bar{J}_o}^l - 1 = \sigma_{\bar{J}_o+1}^l$, $(b11)$

(iv) $s_i^f \leq \hat{s}_i$ *for* $\bar{j}_o + \bar{j}_1$ *elements of the set* $\left\{\hat{s}_1, \cdots, \hat{s}_n | \hat{s}_i = \sigma_{\bar{J}_o}^l - 1\right\}$. $(b12)$

Together Definitions 3.4, 3.5 and Lemma 3.7 obviously yield:

Lemma 3.8 . *Let* $\hat{\hat{S}}_\pi^{f,l}(\bar{J}_o^1, \bar{j}_o^1, \bar{j}_1^1)$ *and* $\hat{\hat{S}}_\pi^{f,l}(\bar{J}_o^2, \bar{j}_o^2, \bar{j}_1^2)$ *(with respect to* s^f*) be given and assuming* $\bar{J}_o^1 \neq \bar{J}_o^2$ *or* $\bar{j}_o^1 \neq \bar{j}_o^2$ *or* $\bar{j}_1^1 \neq \bar{j}_1^2$.

Then, $\hat{\hat{S}}_\pi^{f,l}(\bar{J}_o^1, \bar{j}_o^1, \bar{j}_1^1) \bigcap \hat{\hat{S}}_\pi^{f,l}(\bar{J}_o^2, \bar{j}_o^2, \bar{j}_1^2) = \varnothing$ *follows.*

Definition 3.6 *Let an element* $\hat{s}_\pi^l$ *from the set* $\hat{\hat{S}}_\pi^{f,l}(\bar{J}_o, \bar{j}_o, \bar{j}_1)$ *be given.*

Then the set of all $w \in B_{n;k_0}$ *which fulfil the properties:*

(i) $w_i \in \{\hat{s}_{\pi_i}^l, \hat{s}_{\pi_i}^l + 1, \ldots, k_o\}$ *if* $s_i^f = \hat{s}_{\pi_i}^l \geq \sigma_{\bar{J}_o}^l$ $(b13)$

(ii) $w_i \in \{\hat{s}_{\pi i}^l = \sigma_{\bar{J}_o}^l - 1, \hat{s}_{\pi_i}^l + 1, \ldots, k_0\}$ *and* w *with at most* $\boldsymbol{j_1}$ *coordinates* $w_i = \sigma_{\bar{J}_o}^l - 1$,

if $s_i^f \leq \hat{s}_{\pi_i}^l = \sigma_{\bar{J}_o}^l - 1$, $(b14)$

(iii) $w_i = \hat{s}_{\pi_i}^l$ *otherwise* $(b15)$

is denoted by $\hat{B}_{n;k_0}^{*2}(s^f, \hat{s}_\pi^l)$.

Remark 3.3 *Properties (b14) and (b13) show that the reduction of components of value* $\sigma_{\bar{J}_o}^l$ *by 1 in order to determine* $(\bar{J}_o, \bar{j}_o)$*-perturbed permutations for* $\hat{B}_{n;k_0}^{*2}(s^f, \hat{s}_\pi^l)$ *is in fact not necessary in the case* $\sigma_{\bar{J}_o}^l - 1 > \sigma_{\bar{J}_o+1}^l$*, since the last inequality implies that* $\bar{\mathbf{j}}_\mathbf{1} = 0$ *because of (b7). However, this method leads to a clearer and more uniform representation of the Definitions 3.4, 3.5, 3.6 and so on, so that distinctions in certain cases for the representations are not necessary.*

Lemma 3.9 *Let an element* $\hat{s}_\pi^l$ *from a set* $\hat{\hat{S}}_\pi^{f,l}(\bar{J}_o, \bar{j}_o, \bar{j}_1)$ *be given. In addition, let* $w \in \hat{B}_{n;k_0}^{*2}(s^f, \hat{s}_\pi^l)$.

Then, the case "otherwise" in Definition 3.6 is valid if

$$s_i^f > \hat{s}_{\pi_i}^l \text{ or} \qquad (b16)$$

$$s_i^f \leq \hat{s}_{\pi_i}^l < \begin{cases} \sigma_{\bar{J}_o+1}^l & \text{if } \sigma_{\bar{J}_o}^l - 1 = \sigma_{\bar{J}_o+1}^l, \\ \sigma_{\bar{J}_o}^l & \text{if } \sigma_{\bar{J}_o}^l - 1 > \sigma_{\bar{J}_o+1}^l. \end{cases} \qquad (b17)$$

Theorem 3.10 *Let $s^f \in S_{n;su;k_0}$ and $s^l \in S_{n;su;k_0}$ be given. Then we have:*

$$B^{*2}_{n;k_0}(s^f, s^l) = \bigcup_{\hat{s}^l_\pi \in \hat{\hat{S}}^{f,l}_\pi} \hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^l_\pi) \text{ if } s^f \neq s^l \text{ and}$$

$$B^{*2}_{n;k_0}(s^f, s^l) = \bigcup_{\hat{s}^l_\pi \in \hat{\hat{S}}^{f,l}_\pi} \hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^l_\pi) \bigcup \{w \in B_{n;k_0} | \, w_i \leq s^f_i, \; i = 1, ..., n\}$$

$$\text{if } s^f = s^l.$$

Theorem 3.11

Let $\hat{s}^{l,1}_\pi \in \hat{\hat{S}}^{f,l}_\pi$ and $\hat{s}^{l,2}_\pi \in \hat{\hat{S}}^{f,l}_\pi$ be given with $\hat{s}^{l,1}_\pi \neq \hat{s}^{l,2}_\pi$.

Then, $\hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^{l,1}_\pi) \bigcap \hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^{l,2}_\pi) = \varnothing$.

(Furthermore, in the case $s^f = s^l$ we have $\hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^l_\pi) \bigcap \{w \in B_{n;k_0} | \, w_i \geq s^f_i, \; i = 1, ..., n\} = \varnothing$.)

According to Definition 3.6, Theorem 3.10 and Theorem 3.11

$$p^{*2}_{fl} = \sum_{\hat{s}^l_\pi \in \hat{\hat{S}}^{f,l}_\pi} \; \sum_{w \in \hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^l_\pi)} q(w) + \delta_{fl} \; * \sum_{w \in \{w \in B_{n;k_0} | \, w_i \geq s^f_i, \; i=1,...,n\}} q(w) \tag{19}$$

follows.

In the case of discrete uniformly distributed requirements, Definition 3.6 yields:

$$\sum_{w \in \hat{B}^{*2}_{n;k_0}(s^f, \hat{s}^l_\pi)} q(w) = \tfrac{1}{(k_0+1)^n} \Bigg[\prod_{i: \hat{s}^l_{\pi_i} = s^f_i \geq \sigma^l_{\bar{J}_o}} (k_0 + 1 - \hat{s}^l_{\pi_i})$$

$$\left((k_0 - \sigma^l_{\bar{J}_o} + 2)^{\bar{j}_o + \bar{j}_i} - \binom{\bar{j}_o + \bar{j}_1}{\bar{j}_1 + 1} (k_0 - \sigma^l_{\bar{J}_o} + 1)^{\bar{j}_o - 1} - \cdots - \binom{\bar{j}_o + \bar{j}_1}{\bar{j}_o + \bar{j}_1} (k_o - \sigma^l_{\bar{J}_o} + 1)^0 \right) \Bigg]$$

$$\text{and } \delta_{fl} \; * \sum_{w \in \{w \in B_{n;k_0} | \, w_i \geq s^f_i, \; i=1,...,n\}} q(w) = \delta_{fl} \; * \; \tfrac{1}{(k_0+1)^n} \prod_{i=1}^{n} (k_0 + 1 - s^f_i). \tag{20}$$

Theorem 3.12 *Elements p^*_{fl} of ReH-matrices can be calculated by:*

$$p^*_{fl} = p^{*1}_{fl} + p^{*2}_{fl} - p^{*1,2}_{fl},$$

where p^{*1}_{fl} *is computed as in (17),* p^{*2}_{fl} *as in (19) and* $p^{*1,2}_{fl} = \sum_{s^l_\pi:\ permutations\ of\ s^l} q(s^l_\pi)$.

In particular, in the case of discrete uniformly distributed requirements (18) and (20) can be used to compute elements p^*_{fl}.

In addition, we would like to note the following. (20) (together with (18) and Theorems 3.6, 3.10) implies that the numbers of elements in the sets $B^*_{n,k_0}(s^f, s^l)$ are polynomials in k_0 as well as elements p^*_{fl} of the corresponding ReH-matrices multiplied by $(k_0+1)^n$ in the case of discrete uniformly distributed requirements. It is more difficult to show that the numbers of elements in the sets $B^*_{n,k_0}(s^f, s^l)$ are either polynomials or sums of exponential functions and polynomials in n as well as elements p^*_{fl} of the corresponding ReH-matrices multiplied by $(k_0+1)^n$ in the case of discrete uniformly distributed requirements. As you can see in [10], Theorem 4.5.1.

4 Limits of ReH-matrices

If we want to determine limits of elements from ReH-matrices, there are two approaches:

a) Let s^f and s^l be partitions of a given su. Then we could consider $\lim_{n\to\infty} p^*(s^l(n)|s^f(n))$ for arbitrary but fixed $k_0 \geq \max_i \left\{s^f_i, s^l_i\right\}$, where

$$s^f(n), s^l(n) \in S_{n;su;k_0} \text{ and all positive components of } s^f(n)\ (s^l(n)) \text{ are equal to the positive components of } s^f\ (s^l). \tag{21}$$

If $n \geq su$, then all $S_{n;su;k_0}$ $(n = su, su+1, \cdots)$ have the same number of elements and the corresponding ReH-matrices all have the same numbers of rows and columns. Sets $S_{n;su;k_0}$ with $n > su$ are called *sets of sparse partitions.*

b) If $s \in S_{n;su;k'_0}$ with $k'_0 = \max_i \{s_i\}$, then let

$$\bar{s} = (k_0, \cdots, k_0)^T - s \text{ for a given } k_0 \geq k'_0. \tag{22}$$

(Hence, $\bar{s}$ is a partition of $\bar{su} = n\, k_0 - su$.)

Now, let n, $\bar{su}$, $\bar{s}^f \in S_{n;\bar{su};k'^f_0}$, $\bar{s}^l \in S_{n;\bar{su};k'^l_0}$ be given and let $s^f(k_0) = (k_0, \cdots, k_0)^T - \bar{s}^f$, $s^l(k_0) = (k_0, \cdots, k_0)^T - \bar{s}^l$, where $k_0 \geq \max\left\{k'^f_0, k'^l_0\right\}$. Then we could consider $\lim_{k_0\to\infty} p^*(s^l(k_0)|s^f(k_0))$.

If $k_0 \geq \bar{su}$ then all $S_{n;su;k_0}$ with $su = n\ k_0 - \bar{su}$ ($k_0 = \bar{su}, \bar{su}+1, \cdots$) have the same number of elements and the corresponding ReH-matrices all have the same numbers of rows and columns. Sets $S_{n;su;k_0}$ with $k_0 > \bar{su}$ are called *sets of heavy partitions*.

In more detail,
– Sets $S_{n;su;k_0}$ with $k_0 > \bar{su}$ and $n > \bar{su}$ are called *sets of non-truncated heavy partitions*,
– Sets $S_{n;su;k_0}$ with $k_0 > \bar{su}$ and $n \leq \bar{su}$ are called *sets of truncated heavy partitions*.
[6]

Please note that limits with only $su \to \infty$ are not possible since su is limited by nk_0, see Section 1.1.

Now, let us consider the two cases in more detail. a) Let $\eta = |\{i \mid s_i > 1\}|$ for given $s \in S_{n;su;k_0}$ where $n \underset{(=)}{>} su$. Furthermore, let s^f and s^l be partitions of given su with (w.l.o.g.) $s_1^f \geq s_2^f \geq \cdots \geq s_{\eta^f}^f$ and $s_1^l \geq s_2^l \geq \cdots \geq s_{\eta^l}^l$. Then s^l is called a *monotone successor* of s^f if $\eta_f \geq \eta_l$ and $s_i^f \geq s_i^l$ for $i = 1, \cdots, \eta_l(\eta_f)$. If $s^f(n), s^l(n)$ are given as in (21) then,

$$\lim_{n\to\infty} p^*(s^l(n)|s^f(n)) = \begin{cases} 0 & \text{if } s^l \text{ is not a monotone successor of } s^f, \\ (q_0(0)+q_0(1))^{\eta_f-\eta_l} \sum\limits_{s_\pi^l \in S_{\eta_f}^l} \prod\limits_{i: s_i^f > s_{\pi_i}^l \geq 2} q_0(s_{\pi_i}^l) \prod\limits_{i: s_i^f = s_{\pi_i}^l \geq 2} (q_0(s_i^f) + \cdots + q_0(k_0)) & \\ & \text{if } s^l \text{ is a monotone successor of } s^f \end{cases}$$

where $S_{\eta_f}^l = \big\{ s_\pi \in \mathbb{Z}_+^{\eta_f} \mid s_\pi$ is a permutation of $(s_1^l, s_2^l, \cdots, s_{\eta_l}^l, 0, \cdots, 0)^T \in \mathbb{Z}_+^{\eta_f}$ with $s_i^f \geq s_{\pi_i}$ for $i = 1, \cdots, \eta_f \big\}$.
(See [8], [11] or[10], Section 4.4.2)

b) This case requires additional properties of the probability functions q. Let q^{k_0} denote probability functions corresponding to $B_{n;k_0}$ (where n is fixed). Then we assume that

$$\lim_{k_0\to\infty} q_0^{k_0}(k_0 - \bar{w}_i) = 0 \text{ for } \bar{w}_i = 0, 1, \ldots \text{ and} \tag{23}$$

$$\exists\, c(k_0) \text{ (with } 1 > c(k_0) > 0) : \exists \lim_{k_0\to\infty} \frac{q_0^{k_0}(k_0 - \bar{w}_i)}{c(k_0)} \neq 0 \ \text{ for } \ \bar{w}_i = 0, 1, \ldots. \tag{24}$$

(The limits $\lim_{k_0\to\infty} \frac{q_0^{k_0}(\cdot)}{c(k_0)}$ are unique. But they can differ by a constant multiple in relation to $c(k_0)$.)

[6] A more detailed classification of sets of restricted partitions can be found in [6].

Definition 4.1 *Let $s^f \in S_{n;su;k_0}$ be a heavy partition. Then $s^l \in S_{n;su;k_0}$ is called an essential partition respecting s^f if $s^l = s^f$ or if $s^l \neq s^f$ and a permutation s^l_π of s^l exists such that $s^f_{j_0} < s^l_{\pi_{j_0}}$ for exactly one j_0 and $s^l = s^*(s^f, w)$, where $w_i = \begin{cases} s^l_{\pi_i} & if\ i = j_0 \\ 0 & if\ i \neq j_0\ (i \in \{1, 2, \cdots, n\}) \end{cases}$. Such a corresponding permutation s^l_π is also called essential.*

If

$$\begin{array}{l} B^{*1\pi}_{n,k_0}(s^f, s^l_\pi) = B^1_{n,k_0} \cap B^{*\pi}_{n,k_0}(s^f, s^l_\pi) \text{ and} \\ B^{*2\pi}_{n,k_0}(s^f, s^l_\pi) = B^2_{n,k_0} \cap B^{*\pi}_{n,k_0}(s^f, s^l_\pi) \text{ (see(7))} \end{array} \tag{25}$$

then Definition 4.1 and (8) yield the following lemma.

Lemma 4.1 *Let $S_{n;su;k_0}$ be a set of heavy partitions. Furthermore, let s^l_π be an essential permutation respecting $s^f \in S_{n;su;k_0}$ with $s^f_{j_0} < s^l_{\pi_{j_0}}$. Then,*

$$B^{*1\pi}_{n,k_0}(s^f, s^l_\pi) = \left\{ w \in B^1_{n;k_0} | w_i \begin{cases} = s^l_{\pi_i} & if\ i = j_0 \\ \in \left\{0, 1, \cdots, s^l_{\pi_i}\right\} & if\ i \neq j_0\ (i \in \{1, 2, \cdots, n\}) \end{cases} \right\}.$$

Theorem 4.2 *(Limits of ReH-matrices with regard to sets of heavy partitions) Let $S_{n;su;k_0}$ be sets of heavy partition where su is represented by $su = nk_0 - \bar{su}$, $s \in S_{n;su;k_0}$ by $s = (k_0, \cdots, k_0)^T - \bar{s}$ with fixed $n(\geq 2)$ and $\bar{su}(\geq 2)$. Finally, let given probability functions q^{k_0} fulfill (23) and (24) for certain $c(k_0)$ and let $q^0_0(\bar{w}_i) := \lim_{k_0 \to \infty} \frac{q^{k_0}_0(k_0 - \bar{w}_i)}{c(k_0)}$.*

Then, $\lim_{k_0 \to \infty} \frac{1}{c(k_0)} (p^*(s^l(k_0)|s^f(k_0)) - \delta(s^l(k_0), s^f(k_0))) =$

$$\begin{cases} 0 & \textit{if } s^l(k_0) \textit{ is not an essential partition respecting } s^f(k_0),\ ^7 \\ -\sum\limits_{i:\bar{s}^f_i \geq 1}{}^8 (q^0_0(\bar{s}^f_i - 1) + \cdots + q^0_0(0)) + \sum\limits_{s^l_\pi(k_0) \in S^l_{\tilde{e}}} q^0_0(\bar{s}^l_{\pi_{j_0}}) & \textit{if } s^f(k_0) = s^l(k_0), \\ \sum\limits_{s^l_\pi(k_0) \in S^l_{\tilde{e}}} q^0_0(\bar{s}^l_{\pi_{j_0}}) & \textit{if } s^l(k_0) \textit{ is an essential partition respecting } s^f(k_0) \\ & \textit{and } s^l(k_0) \neq s^f(k_0) \end{cases}$$

where $\delta(s^l(k_0), s^f(k_0)) = \begin{cases} 1 & if\ s^f(k_0) = s^l(k_0) \\ 0 & if\ s^f(k_0) \neq s^l(k_0) \end{cases}$ *and*

$S^l_{\tilde{e}}$ is the maximal set of essential permutations $s^l_\pi(k_0)$ such that the following conditions are valid for all pairs $s^{l_a}_\pi(k_0)(\neq s^f(k_0)) \in S^l_{\tilde{e}}$, $s^{l_b}_\pi(k_0)(\neq$

[7] Clearly, $s^l(k_0)$ is a essential partition respecting $s^f(k_0)$ for all $k_0 = \bar{su}+1, \bar{su}+2, \cdots$ or for no k_0.

[8] This is a finite number of i since $\bar{su}$ is fixed.

$s^f(k_0)) \in S^l_{\tilde{e}}$: $j_{0a} \neq j_{0b}$, *(where* $s^f(k_0)_{j_{0a}} < s^{l_a}_\pi(k_0)_{j_{0a}}$ *and* $s^f(k_0)_{j_{0b}} < s^{l_b}_\pi(k_0)_{j_{0b}}$*) or* $s^{l_a}_\pi(k_0)_{j_{0a}} \neq s^{l_b}_\pi(k_0)_{j_{0b}}$ *if* $j_{0a} = j_{0b}$. [9]

Proof:
Let us denote
$p^*(s^l(k_0)|s^f(k_0)) = \sum_{w \in B^*_{n,k_0}(s^f(k_0),s^l(k_0))} q^{k_0}(w) = \sum_{w \in B^{*1}} q^{k_0}(w) + \sum_{w \in B^{*2}} q^{k_0}(w)$,
where the following abbreviations $B^{*1} = B^{*1}_{n,k_0}(s^f(k_0), s^l(k_0))$ and $B^{*2} = B^{*2}_{n,k_0}(s^f(k_0), s^l(k_0))$ are used.
At first, we consider $w \in B^{*2}$: The scarcity-situation implies (see (3)) that a permutation $s^l_\pi(k_0)$ of $s^l(k_0)$ exists with $w \geq s^l_\pi(k_0)$.
Hence, $B^{*2} \subseteq \{w \in B_{n;k_0} \mid w_i \geq k_0 - \bar{su}$ for $i = 1, \cdots, n\}$, which can be used to conclude the following:

$$\sum_{w \in B^{*2}} q^{k_0}(w) \leq \left[q^{k_0}_0(k_0 - \bar{su}) + q^{k_0}_0(k_0 - \bar{su} + 1) + \cdots + q^{k_0}_0(k_0)\right]^n. \quad (*1)$$

Furthermore,

$$\begin{aligned} &\lim_{k_0 \to \infty} \frac{1}{c(k_0)} \sum_{w \in B^{*2}} q^{k_0}(w) \\ &\leq \lim_{k_0 \to \infty} \left[\frac{q^{k_0}_0(k_0 - \bar{su})}{c(k_0)} + \cdots + \frac{q^{k_0}_0(k_0)}{c(k_0)}\right] \left[q^{k_0}_0(k_0 - \bar{su}) + \cdots + q^{k_0}_0(k_0)\right]^{n-1} \\ &= 0, \end{aligned} \quad (*2)$$

since the limit of the first factor (a finite sum) exists according to (24) and the limit of the second factor (a power of a finite sum) is equal to 0 according to (23). Thus, it remains to consider
$\lim_{k_0 \to \infty} \frac{1}{c(k_0)} p^*(s^l(k_0)|s^f(k_0)) = \lim_{k_0 \to \infty} \frac{1}{c(k_0)} \sum_{w \in B^{*1}} q^{k_0}(w)$.

Case 1: Let $s^l(k_0)$ not be an essential partition respecting $s^f(k_0)$.

If $B^{*1\pi}_{n,k_0}(s^f(k_0), s^l_\pi(k_0)) \neq \emptyset$ then $B^{*1\pi}_{n,k_0}(s^f(k_0), s^l_\pi(k_0)) =$

$$\left\{ w \in \mathbb{Z}^n_+ | w_i \left\{ \begin{array}{l} = s^l_\pi(k_0)_{i_a} \text{ for } i_a \in I_a \\ \in \{0, 1, \cdots, s^l_\pi(k_0)_{i_b}\} \text{ for } i_b \in I_b \end{array} \right. , I_a \cup I_b = \{1, 2, \cdots, n\}, \sum_{i=1}^n w_i \leq su \right\}$$

where $|I_a| \geq 2$ (for any permutation $s^l_\pi(k_0)$ of $s^l(k_0)$) according to Lemma 4.1 and (8). Thus,
$\lim_{k_0 \to \infty} \frac{1}{c(k_0)} \sum_{w \in B^{*1\pi}_{n,k_0}(s^f(k_0),s^l_\pi(k_0))} q^{k_0}(w)$

[9] A corresponding statement, but only regarding sets of non-truncated heavy partitions, can be found in [8], [11] or[10], Section 4.4.3. In this case, the term "s^l is a restricted monotone successor of s^f" is equivalent to "s^l is essential respecting s^f". Unfortunately, the statement in the references above includes a representation, which is not fully correct.

$$= \lim_{k_0\to\infty} \frac{q_0^{k_0}(s_\pi^l(k_0)_{i_{a1}})}{c(k_0)} \prod_{i_a\in I_a\setminus i_{a1}} q_0^{k_0}(s_\pi^l(k_0)_{i_a}) \prod_{i_b\in I_b} \Big[q_0^{k_0}(0)+q_0^{k_0}(1)+\cdots+q_0^{k_0}(s_\pi^l(k_0)_{i_b})\Big]$$
$$= \lim_{k_0\to\infty} \frac{q_0^{k_0}(k_0-\bar{s}^l_{\pi_{i_{a1}}})}{c(k_0)} \prod_{i_a\in I_a\setminus i_{a1}} q_0^{k_0}(k_0-\bar{s}^l_{\pi_{i_a}}) \prod_{i_b\in I_b} \Big[q_0^{k_0}(0)+q_0^{k_0}(1)+\cdots+q_0^{k_0}(s_\pi^l(k_0)_{i_b})\Big]$$
$$= 0$$

according to (24), (23) and since $q_0^{k_0}(0)+q_0^{k_0}(1)+\cdots+q_0^{k_0}(s_\pi^l(k_0)_{i_b}) \le 1$.

$$\lim_{k_0\to\infty} \frac{1}{c(k_0)} \sum_{w\in B^{*1}} q^{k_0}(w)$$
$$\le \lim_{k_0\to\infty} \frac{1}{c(k_0)} \sum_{s_\pi^l(k_0):B^{*1\pi}_{n,k_0}(s^f(k_0),s_\pi^l(k_0))\neq\emptyset} \ \sum_{w\in B^{*1\pi}_{n,k_0}(s^f(k_0),s_\pi^l(k_0))} q^{k_0}(w) = 0$$

follows.

Case 2a: Let $s^l(k_0)$ be an essential partition respecting $s^f(k_0)$ and $s^f(k_0)\neq s^l(k_0)$.

We partition $B^{*1} = B^{*1a}\cup B^{*1b}$ with

$B^{*1a} = \{w\in B^{*1} \mid$ all permutations $s_\pi^l(k_0)$ of $s^l(k_0)$ satisfying (3) are not essential$\}$,

$B^{*1b} = \{w\in B^{*1} \mid$ permutation $s_\pi^l(k_0)$ of $s^l(k_0)$ satisfying (3) exists which is essential$\}$.

Analogously to Case 1, it follows, that $\lim_{k_0\to\infty} \frac{1}{c(k_0)} \sum_{w\in B^{*1a}} q^{k_0}(w) = 0$.

Now, let $s_\pi^l(k_0)$ be an essential permutation respecting $s^f(k_0)\in S_{n;su;k_0}$ with $s^f(k_0)_{j_0} < s_\pi^l(k_0)_{j_0}$ and $\Delta = s_\pi^l(k_0)_{j_0} - s^f(k_0)_{j_0} = \bar{s}^f_{j_0} - \bar{s}^l_{\pi_{j_0}}$. Then,

$$\lim_{k_0\to\infty} \frac{q_0^{k_0}(s_\pi^l(k_0)_{j_0})}{c(k_0)} \prod_{i\in\{1,2,\cdots,n\}\setminus\{j_0\}} \Big[q_0^{k_0}(0)+q_0^{k_0}(1)+\cdots+q_0^{k_0}(\max\big\{s_\pi^l(k_0)_i-\Delta,0\big\})\Big]$$
$$\le \lim_{k_0\to\infty} \frac{1}{c(k_0)} \sum_{w\in B^{*1b}\cap B^{*\pi}_{n,k_0}(s^f(k_0),s_\pi^l(k_0))} q^{k_0}(w)$$
$$\le \lim_{k_0\to\infty} \frac{q_0^{k_0}(s_\pi^l(k_0)_{j_0})}{c(k_0)} \prod_{i\in\{1,2,\cdots,n\}\setminus\{j_0\}} \Big[q_0^{k_0}(0)+q_0^{k_0}(1)+\cdots+q_0^{k_0}(s_\pi^l(k_0)_i)\Big].$$

Hence,

$$\lim_{k_0\to\infty} \frac{q_0^{k_0}(s_\pi^l(k_0)_{j_0})}{c(k_0)} \prod_{i\in\{1,2,\cdots,n\}\setminus\{j_0\}} \Big[1-q_0^{k_0}(k_0-\min\big\{\bar{s}^l_{\pi_i}+\Delta,k_0\big\}+1)-\cdots-q_0^{k_0}(k_0)\Big]$$
$$\le \lim_{k_0\to\infty} \frac{1}{c(k_0)} \sum_{w\in B^{*1b}\cap B^{*\pi}_{n,k_0}(s^f(k_0),s_\pi^l(k_0))} q^{k_0}(w)$$
$$\le \lim_{k_0\to\infty} \frac{q_0^{k_0}(s_\pi^l(k_0)_{j_0})}{c(k_0)} \prod_{i\in\{1,2,\cdots,n\}\setminus\{j_0\}} \Big[1-q_0^{k_0}(k_0-\bar{s}^l_{\pi_i}+1)-\cdots-q_0^{k_0}(k_0)\Big].$$

The limit of the product is equal to 1, since finite sums are subtracted from 1 in the factors (see also (23)). Then,

$$\lim_{k_0\to\infty} \frac{1}{c(k_0)} \sum_{w\in B^{*1b}\cap B^{*\pi}_{n,k_0}(s^f(k_0),s_\pi^l(k_0))} q^{k_0}(w) = \lim_{k_0\to\infty} \frac{q_0^{k_0}(s_\pi^l(k_0)_{j_0})}{c(k_0)} = q_0^0(\bar{s}^l_{\pi_{j_0}}) \tag{*3}$$

follows according to (24). Finally, $\lim_{k_0\to\infty} \frac{1}{c(k_0)} \sum_{w\in B^{*1b}} q^{k_0}(w) = \sum_{s^l_\pi(k_0)\in S^l_{\tilde{e}}} q^0_0(\bar{s}^l_{\pi_{j_0}})$ is right by reason of the following facts. On the one hand, we have $\left[B^{*1b} \cap B^{*\pi}_{n,k_0}(s^f(k_0), s^{l_a}_\pi(k_0))\right] \cap \left[B^{*1b} \cap B^{*\pi}_{n,k_0}(s^f(k_0), s^{l_b}_\pi(k_0))\right] = \emptyset$ for $s^{l_a}_\pi(k_0) \in S^l_{\tilde{e}}$, $s^{l_b}_\pi(k_0) \in S^l_{\tilde{e}}$. On the other hand, if $s^{l_a}_\pi(k_0)_{j_{0a}} = s^{l_b}_\pi(k_0)_{j_{0b}}$ and $j_{0a} = j_{0b}$ then the equivalence $w \in B^{*1b} \cap B^{*\pi}_{n,k_0}(s^f(k_0), s^{l_a}_\pi(k_0)) \Leftrightarrow w \in B^{*1b} \cap B^{*\pi}_{n,k_0}(s^f(k_0), s^{l_b}_\pi(k_0))$ is valid, except for certain w with a finite number of w_i with $s^{l_a}_\pi(k_0)_i < w_i \leq s^{l_b}_\pi(k_0)_i$ or $s^{l_a}_\pi(k_0)_i \geq w_i > s^{l_b}_\pi(k_0)_i$, respectively.

Case 2b: Let $s^f(k_0) = s^l(k_0)$.

Similar considerations as in Case 2a lead to

$$\lim_{k_0\to\infty} \frac{1}{c(k_0)} \sum_{\substack{s^f_\pi(k_0):\text{ essential perm. resp. } s^f(k_0),\\ s^f_\pi(k_0)\neq s^f(k_0)}} \sum_{w\in B^{*1\pi}_{n,k_0}(s^f(k_0), s^f_\pi(k_0))} q^{k_0}(w) = \sum_{s^f_\pi(k_0)\in S^l_{\tilde{e}}} q^0_0(\bar{s}^f_{\pi_{j_0}}). \tag{*4}$$

If $s^f_\pi(k_0) = s^f(k_0)$ then we have $B^{*1\pi}_{n,k_0}(s^f(k_0), s^f(k_0)) = \{w \in B^1_{n;k_0} \mid w_i \leq s^f(k_0)_i, i = 1, 2, \cdots, n\}$. Hence,

$$\begin{aligned}
&\lim_{k_0\to\infty} \frac{1}{c(k_0)} \left[\sum_{w\in B^{*1\pi}_{n,k_0}(s^f(k_0), s^f(k_0))} q^{k_0}(w) - 1\right] \\
&= \lim_{k_0\to\infty} \frac{1}{c(k_0)} \left[\prod_{i\in\{1,2,\cdots,n\}} (q^{k_0}_0(0) + q^{k_0}_0(1) + \cdots + q^{k_0}_0(s^f(k_0)_i = k_0 - \bar{s}^f_i)) - 1\right] \\
&= \lim_{k_0\to\infty} \frac{1}{c(k_0)} \left[\prod_{i:\bar{s}^f_i\geq 1} (1 - q^{k_0}_0(k_0 - \bar{s}^f_i + 1) - \cdots - q^{k_0}_0(k_0)) - 1\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{k_0\to\infty} \tfrac{1}{c(k_0)} \Bigg[- \sum_{i:\bar{s}_i^f \geq 1} (q_0^{k_0}(k_0 - \bar{s}_i^f + 1) + \cdots + q_0^{k_0}(k_0)) \\
&+ \sum_{\substack{i_1,i_2:i_1\neq i_2,\\ \bar{s}^f_{i_1}\geq 1, \bar{s}^f_{i_1}\geq 1}} (q_0^{k_0}(k_0 - \bar{s}_{i_1}^f + 1) + \cdots + q_0^{k_0}(k_0))(q_0^{k_0}(k_0 - \bar{s}_{i_2}^f + 1) + \cdots + q_0^{k_0}(k_0)) \\
&- \sum_{\substack{i_1,i_2,i_3:i_1\neq i_2\neq i_3\neq i_1,\\ \bar{s}^f_{i_1}\geq 1,\bar{s}^f_{i_2}\geq 1,\bar{s}^f_{i_1}\geq 1}} \prod_{j\in\{1,2,3\}} (q_0^{k_0}(k_0 - \bar{s}_{i_j}^f + 1) + \cdots + q_0^{k_0}(k_0)) \\
&+ - \cdots (-1)^{|\{i \mid \bar{s}_i^f \geq 1\}|} \prod_{i:\bar{s}_i^f\geq 1} (q_0^{k_0}(k_0 - \bar{s}_i^f + 1) + \cdots + q_0^{k_0}(k_0)) \Bigg] \\
&= \lim_{k_0\to\infty} \tfrac{1}{c(k_0)} (- \sum_{i:\bar{s}_i^f\geq 1} (q_0^{k_0}(k_0 - \bar{s}_i^f + 1) + \cdots + q_0^{k_0}(k_0)) \text{ (using (23) and (24))} \\
&= - \sum_{i:\bar{s}_i^f\geq 1} {}^{10} (q_0^0(\bar{s}_i^f - 1) + \cdots + q_0^0(0)).
\end{aligned}
$$

(The limits of the products are equal to 0, since the factors are finite sums and using (23) and (24).)
Together with (*4) this proves Case 2b and therefore also the theorem. ■
[10]

Remark 4.1 *Take the same assumption as in Theorem 4.2.*

(i) In addition, let $n > \bar{su}$. Then,

$$
\begin{aligned}
\lim_{k_0\to\infty} & \tfrac{1}{c(k_0)} (p^*(s^l(k_0)|s^f(k_0)) - \delta(s^l(k_0), s^f(k_0))) \\
&= - \sum_{i:\bar{s}_i^f\geq \mathbf{2}} (q_0^0(\bar{s}_i^f - 1) + \cdots + q_0^0(0)) \quad \text{if } s^f(k_0) = s^l(k_0).
\end{aligned} \tag{26}
$$

(ii) If $n = \bar{su}$ then (26) is valid for $s^f(k_0) \neq (k_0 - 1, \cdots, k_0 - 1)^T$ and $s^f(k_0) \neq (k_0, k_0 - 1, \cdots, k_0 - 1, k_0 - 2)^T$.

(iii) For $y \in \mathbb{Z}_+^n$ we define the vector $y[i_1; i_2]$ by

$$
y_i[i_1; i_2] := \begin{cases} y_i + 1 & \text{for } i = i_1, \\ y_i - 1 & \text{for } i = i_2, \\ y_i & \text{otherwise.} \end{cases} \tag{27}
$$

Let the partitions of the sets $S_{n;su;k_0} = \{s^1(k_0), \cdots, s^r(k_0)\}$ be numbered according to (partial) dominance (or in other words majorization) ordering. That implies

$s^f \in S_{n;su;k_0}$ is a direct predecessor of $s^l \in S_{n;su;k_0}$ if and only if

[10]An example can be found in Appendix B.

$\exists\, i_1, i_2 : \; (s^l_{i_1} \geq s^l_{i_2}) \wedge \; (s^l = s^f[i_2; i_1])$ (where $s^y_1 \geq s^y_2 \geq \cdots \geq s^y_n$ for $y = 1, 2$).[11]

If $n > \bar{su}$ then the matrix $\lim\limits_{k_0 \to \infty} \frac{1}{c(k_0)}(P^*(k_0) - I)$ is a triangular matrix.

These statements can be proved directly. Alternatively, (i) and (iii) follows from results for non-truncated heavy partitions.
Please note that (i) means $\sum\limits_{s^l_\pi(k_0) \in S^l_{\bar{e}}} q^0_0(\bar{s}^l_{\pi_{j_0}}) = \sum\limits_{i : \bar{s}^f_i = 1} q^0_0(0)$.

5 Poisson equations and the monotonicity of their solutions

In the following we introduce Poisson (vector) equations, where corresponding matrices are ReH-matrices minus identity matrices. Based on the partitions, which characterize the rows of a ReH-matrix, we will compute the right-hand-sides of the equations. Poisson equations are important in connection with application problems (see Appendix C). If the solutions of our Poisson equations are "monotone" then transitions into feasible balanced partitions are optimal decisions for the operations research problem (see [8], [11] or [10], Section 4.6.). We conjecture that the solutions of all Poisson equations with ReH-matrices are monotone. It is easy to prove the conjecture for a small number of Poisson equations where the corresponding ReH-matrices satisfy the conditions of dominance (see Appendix D) and the right-hand-sides of the equations are monotone. However, "most" ReH-matrices do not fulfil these conditions. In this case it is very difficult to show the monotonicity of the solutions. [12] Never the less, we can use limits von ReH-matrices in order to show the monotonicity for certain subsets of Poisson equations. Results regarding sets of sparse partitions or non-truncated heavy partitions are stated in the following. It is possible to expand these results to certain subsets of truncated heavy partitions.

At first we introduce vectors γ: Let a set of restricted partitions $S_{n;su;k_0} = \{s^1, s^2, \cdots, s^r\}$ and a probability function q be given. Then,

$$\begin{aligned} \gamma_l &:= \sum_{i=1}^{n} \sum_{w_i=0}^{s^l_i} (s^l_i - w_i)\, q_0(w_i), \quad l = 1, ..., r \\ \gamma &= (\gamma_1, \cdots, \gamma_r)^T. \end{aligned} \tag{28}$$

[11] This is equivalent to permutations s^f_π of s^f and s^l_π of s^l exist such that $\frac{1}{2}\sum\limits_i \mid s^f_{\pi_i} - s^l_{\pi_i} \mid = 1$ and $\sum\limits_i (s^f_i)^2 > \sum\limits_i (s^l_i)^2$. (See [12], Marshall, A.W. and Olkin, I.,Chapter 1. A., B. and Chapter 5. D. and see also [6].)

[12] Methods as mathematical induction, estimations of the solutions or the use of certain properties (see [10], Lemma 4.6.4 for instance) are so fare not successful to prove the conjecture.

($\gamma_l = \frac{1}{2}\frac{1}{k_0+1}\sum_{i=1}^{n}(s_i^l)^2 + R(n; su; k_0)$ follows in the case of discrete uniformly distributed requirements, where $R(n, su, k_0)$ is independent of s^l. (See [7], 3.3.1, Lemma 3.12 and Remark b).)

Definition 5.1 *Let a set of restricted partitions $S_{n;su;k_0} = \{s^1, s^2, \cdots, s^r\}$ and a probability function q be given. Further on let P^* be the corresponding ReH-matrix and γ as in (28). The vector equations*

$$g(-1, ..., -1)^T + (P^* - I)\nu = -\gamma'$$

(with variables $(g, \nu) \in \mathbb{R} \times \mathbb{R}^r$ and where I is the identity matrix and γ' any affine transformation of γ with $\gamma' = \alpha\ \gamma + \beta(1, ..., 1)^T,\ \alpha > 0$) are called Poisson equations.

Such a equation does have solutions. We have any choice for one ν_l. Then the values of the remaining variables are unique (for instance, see the proof of Theorem 2.4.8 in [13] and Lemma 2.3.2 in [10]).

Definition 5.2 *Let a set of restricted partitions $S_{n;su;k_0} = \{s^1, s^2, \cdots, s^r\}$ and a probability function q be given. The solution of a corresponding Poisson equation is called monotone in ν (in relation to the partial order, see Remark 4.1, (iii)) if:*
$s^f \in S_{n;su;k_0}$ is a (direct) predecessor of $s^l \in S_{n;su;k_0} \Rightarrow \nu_f > \nu_l$.

We known from literature (see [8], [11] or [10], Section 4.6.) that Poisson equations have monotone solutions for ReH-matrices, which are either based on

a) sets $S_{n;su;k_0}$ with 2 or 3 partitions or on sets with 4 partitions for discrete uniformly distributed requirements, respectively

b) m-totally ordered sets $S_{n;su;k_0} = \{s^1, s^2, \cdots, s^r\}$ (that means, only, s^f is a direct predecessor of s^{f+1} for $f = 1, 2, \cdots, r-1$)

   (Corresponding ReH-matrices fulfil the dominance property, see Appendix D)

c) sets $S_{n;su;k_0}$ of sparse partitions with sufficiently large $n > su$ (and where the sets of requirements have the same marginal probability function q_0 for all n, where w_i, $(i = 1, \cdots, n)$ are independent and identically distributed and (2) is additionally assumed)

d) sets of non-truncated heavy partitions $S_{n;su=nk_0-\bar{su};k_0}$ with arbitrary but fixed $\bar{su}$, $n(> \bar{su})$ and sufficiently large $k_0 > \bar{su}$ (where the probability functions q^{k_0} fulfil (23) and (24) for certain $c(k_0)$).

c) and d) can be proved by means of equation systems which are related to the Poisson equations, where limits of ReH-matrices are used. The corresponding solutions are vectors which include generalized harmonic numbers (in relation to the distribution of requirements):

c) $\nu((1,1,\cdots,1,0,0,\cdots,0)^T) := 0, \nu(s) =$
$$\sum_{i:s_i\geq 2}\left(\frac{q_0(1)}{q_0(0)+q_0(1)}+\frac{q_0(1)+q_0(2)}{q_0(0)+q_0(1)+q_0(2)}+\cdots+\frac{q_0(1)+q_0(2)+\cdots+q_0(s_i-1)}{q_0(0)+q_0(1)+\cdots+q_0(s_i-1)}\right),$$
$$s\in S_{n;su;k_0},\ s\neq(1,1,\cdots,1,0,0,\cdots,0)^T$$

d) $\nu((k_0,\cdots,k_0,k_0-1,\cdots,k_0-1)^T) := 0,$
$$\nu(s)=\sum_{i:\bar{s}_i\geq 2}\Big(\frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)}+\frac{q_0^0(1)+q_0^0(2)}{q_0^0(0)+q_0^0(1)+q_0^0(2)}+\cdots$$
$$+\frac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}_i-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}_i-1)}\Big)$$
(where $\bar{s}$ as in (22) and $q_0^0(\bar{s}_i) := \lim\limits_{k_0\to\infty}\frac{q_0^{k_0}(k_0-\bar{s}_i)}{c(k_0)}$).

Now, we want to prove the monotonicity of the solutions of the Poisson equations, which are based on sets of truncated heavy partitions $S_{n;su=nk_0-\bar{su};k_0}$ with $n=\bar{su}$ and sufficiently large $k_0(>\bar{su})$.
For this we use related equation systems which include limits of ReH-matrices:

$$g^0\,(-1,\cdots,-1)^T+\lim_{k_0\to\infty}\frac{1}{c(k_0)}\,(P^*(k_0)-I)\,\nu=-\lim_{k_0\to\infty}\frac{1}{c(k_0)}\,\gamma' \qquad (29)$$

(with $g^0=\lim\limits_{k_0\to\infty}\frac{1}{c(k_0)}\,g,\ g=g(k_0)$).

We will show that the solutions of these equations are qualitatively different from the expressions in d). Please note, that these approach cannot be transferred to the remaining cases of sets of truncated heavy partitions with $n<\bar{su}$.

Firstly, we present an affine transformation of $\gamma(s)$, which straightly depends only on $\bar{s}$ and the distribution of the requirements (but not on s):
$$\gamma(s)=\sum_{i=1}^{n}\sum_{w_i=0}^{s_i}(s_i-w_i)\,q_0(w_i)=\sum_{i=1}^{n}\sum_{w_i=0}^{k_0}(s_i-w_i)\,q_0(w_i)$$
$$-\sum_{i:s_i<k_0-1}\ \sum_{w_i=s_i+1}^{k_0}(s_i-w_i)\,q_0(w_i)+(\bar{su}-\sum_{i:s_i<k_0-1}(k_0-s_i))\,q_0(k_0)$$
$$=\sum_{i=1}^{n}s_i\sum_{w_i=0}^{k_0}q_0(w_i)-\sum_{i=1}^{n}\sum_{w_i=0}^{k_0}w_i\,q_0(w_i)+\sum_{i:s_i<k_0-1}\ \sum_{w_i=s_i+1}^{k_0-1}(w_i-s_i)\,q_0(w_i)$$
$$+\sum_{i:s_i<k_0-1}(k_0-s_i)\,q_0(k_0)+\bar{su}\,q_0(k_0)-\sum_{i:s_i<k_0-1}(k_0-s_i)\,q_0(k_0)$$
$$=su-\sum_{i=1}^{n}\sum_{w_i=0}^{k_0}w_i\,q_0(w_i)+\sum_{i:s_i<k_0-1}\ \sum_{w_i=s_i+1}^{k_0-1}(w_i-s_i)\,q_0(w_i)+\bar{su}\,q_0(k_0)$$

$$= \sum_{i:s_i \le k_0-2} \sum_{w_i=s_i+1}^{k_0-1} (w_i - s_i)\, q_0(w_i) + \bar{R}(n, su, k_0, q), \text{ where}$$

$\bar{R}(n, su, k_0, q) = su - \sum_{i=1}^{n} \sum_{w_i=0}^{k_0} w_i\, q_0(w_i) + \bar{su}\, q_0(k_0)$ is independent of s. We define

$$\gamma'(s) := \begin{cases} \sum_{i:s_i \le k_0-2} \sum_{w_i=s_i+1}^{k_0-1} (w_i - s_i)\, q_0(w_i) = \sum_{i:\bar{s}_i \ge 2} \sum_{\bar{w}_i=1}^{\bar{s}_i \text{ or } (\bar{s}_i-1)} (\bar{s}_i - \bar{w}_i)\, q_0(k_0 - \bar{w}_i), \\ \qquad s \ne (k_0-1, \cdots, k_0-1)^T \\ 0, \qquad s = (k_0-1, \cdots, k_0-1)^T \end{cases} \tag{30}$$

where $\bar{s}_i := k_0 - s_i$. Then

$$\gamma^0(\bar{s}) := \begin{cases} \lim_{k_0 \to \infty} \frac{1}{c(k_0)}\, \gamma'(s) = \sum_{i:\bar{s}_i \ge 2} \sum_{\bar{w}_i=1}^{\bar{s}_i \text{ or } (\bar{s}_i-1)} (\bar{s}_i - \bar{w}_i)\, q_0^0(\bar{w}_i), \\ \qquad \bar{s} \ne (1, \cdots, 1)^T \\ 0, \qquad \bar{s} = (1, \cdots, 1)^T \end{cases} \tag{31}$$

If we use $\gamma^0(\bar{s})$ from (31) in the equation system (29) and fixing $\nu(\bar{s}^r) = 0$, where $s^r = (k_0-1, \cdots, k_0-1)^T$, the following system remains

$$\begin{aligned} g^0\,(-1, \cdots, -1)^T + \sum_{l=1}^{r-1} \lim_{k_0 \to \infty} \frac{1}{c(k_0)} \big(p^*(s^l(k_0)|s^f(k_0)) \\ - \delta(s^l(k_0), s^f(k_0))\big)\ \nu(\bar{s}^l) = -\gamma^0(\bar{s}^f),\ f = 1, \cdots, r-1. \end{aligned} \tag{32}$$

Theorem 5.1 *Let* $S_{n;su=n(k_0-1);k_0}$, $k_0 = n+1, n+2, \cdots$ *be certain sets of truncated heavy partitions with fixed* $n (\ge 2)$ *. Furthermore, let* $B_{n;k_0}$ *be the corresponding sets of requirements, where for any* k_0*, the requirements* w_i, $(i = 1, \cdots, n)$ *are independent and identically distributed and (2) is fulfilled. In addition, let the corresponding given probability functions* q^{k_0} *fulfill (23) and (24) for certain* $c(k_0)$*. Finally, let* $P^*(k_0)$ *be the corresponding ReH-matrices. Then,* $g^0 = \frac{n\, q_0^0(0)\, q_0^0(1)}{n\, q_0^0(0) + q_0^0(1)}$, $\nu((1, \cdots, 1)^T) := 0$,

$$\begin{aligned} \nu(\bar{s}) = \sum_{i:\bar{s}_i \ge 2} \Big(\frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} + \frac{q_0^0(1)+q_0^0(2)}{q_0^0(0)+q_0^0(1)+q_0^0(2)} + \cdots \\ + \frac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}_i-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}_i-1)} \Big) + \frac{(1-(1+\frac{1}{2}+\cdots+\frac{1}{\eta})n)\, q_0^0(0)\, q_0^0(1)}{(q_0^0(0)+q_0^0(1))\, (n\, q_0^0(0)+q_0^0(1))},\ \bar{s} \ne (1, \cdots, 1)^T \end{aligned}$$

(where $\bar{s}$ *is as in (22),* $q_0^0(\bar{s}_i) := \lim_{k_0 \to \infty} \frac{q_0^{k_0}(k_0 - \bar{s}_i)}{c(k_0)}, \eta = |\{i \mid \bar{s}_i > 1\}|$*)*
are solutions of the equation system (32).

Proof:

We use the following notation in relation to partitions:

$s(=s(k_0)) = (k_0, \cdots, k_0, k_0-1, \cdots, k_0-1, k_0-\bar{s}_{n-\eta+1}, \cdots, k_0-\bar{s}_n)^T$

where $2 \le \bar{s}_i$ for $i \in \{n-\eta+1, n-\eta+2, \cdots, n\}$,

$\sum\limits_{h=1}^{\eta} \bar{s}_{n-h+1} \le \bar{su}(=n)$ and

$|\{i \mid s_i = k_0 - 1\}| = \bar{su} - \sum\limits_{h=1}^{\eta} \bar{s}_{n-h+1}, \ |\{i \mid s_i = k_0\}| = \sum\limits_{h=1}^{\eta} \bar{s}_{n-h+1} - \eta.$

(*1)

Let $s^1(k_0) := (k_0, \cdots, k_0, k_0 - \bar{su})^T$,
$s^{r-1}(k_0) := (k_0, k_0-1, \cdots, k_0-1, k_0-2)^T$ and $s^r(k_0) := (k_0-1, \cdots, k_0-1)^T$.
Furthermore, we introduce

$$\nu^i(\bar{s}_i) := \frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} + \frac{q_0^0(1)+q_0^0(2)}{q_0^0(0)+q_0^0(1)+q_0^0(2)} + \cdots + \frac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}_i-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}_i-1)}$$
for $\bar{s}_i \ge 2$,

$$\gamma^0(\bar{s}_i) := \sum_{\bar{w}_i=1}^{\bar{s}_i \text{ or } (\bar{s}_i-1)} (\bar{s}_i - \bar{w}_i)\, q_0^0(\bar{w}_i) \text{ for } \bar{s}_i \ge 2 \text{ (see (31))}$$
$$\gamma^0(0) = \gamma^0(1) := 0.$$

Since $\nu((1, \cdots, 1)^T) = 0$, we have to prove the equation (see (32))

$$-g^0 + \sum_{l=1}^{r-1} \lim_{k_0\to\infty} \tfrac{1}{c(k_0)} \big(p^*(s^l(k_0)|s(k_0)) - \delta(s^l(k_0), s(k_0))\big)\ \nu(\bar{s}^l) = -\gamma^0(\bar{s}) \qquad (*2)$$

(where g^0 and $\nu(\cdot)$ as in Theorem 5.1) for any partitions $s(k_0)$. Please note that $\lim\limits_{k_0\to\infty} \frac{1}{c(k_0)} p^*(s^l(k_0)|s(k_0)) = 0$ if $s^l(k_0)$ is no essential partition respecting $s(k_0)$ (see Theorem 4.2). Hence, we can replace the sum over $l = 1, \cdots, r-1$ in (*2) by a sum over the essential partitions respecting $s(k_0)$.

Case: $s(k_0) \notin \{s^1(k_0), s^{r-1}(k_0), s^r(k_0)\}$

From Definition 4.1 together with the iterative method from Section 2 and (*1) it follows that
$s'(= s'(k_0)) = (k_0, \cdots, k_0, k_0-1, \cdots, k_0-1, k_0-\bar{s}_{n-\eta+1}, k_0-\bar{s}_{j-1}, k_0-\bar{s}'_j, k_0-\bar{s}_{j+1}, k_0-\bar{s}_n)^T$ with $\bar{s}'_j < \bar{s}_j$ are essential respecting $s(k_0)$ and no other essential partitions, different from $s(k_0)$ itself, exist. For $s(k_0)$ the numbers of the components k_0 and k_0-1 are determined by (*1). If $\bar{s}'_j \ge 2$ then $\eta' = \eta$, otherwise $\eta' = \eta - 1$.

If we additionally apply Remark 4.1 (ii) and recall the definition of $S^l_{\tilde{e}}$ in Theorem 4.2 then we can replace the left side of equation (*2) in the following way

$$-g^0 - \sum_{i:\bar{s}_i \geq 2} (\sum_{\bar{s}'_i=1}^{\bar{s}_i} q_0^0(\bar{s}'_i - 1))\ \nu(\bar{s})$$

$$+ \sum_{j:\bar{s}_j \geq 2} \left\{ \sum_{\bar{s}'_j=2}^{\bar{s}_j-1} q_0^0(\bar{s}'_j)\ \nu((0,\cdots,0,1,\cdots,1,\bar{s}_{n-\eta+1},\bar{s}_{j-1},\bar{s}'_j,\bar{s}_{j+1},\bar{s}_n)^T) \right.$$

$$\left. +\ (q_0^0(1) + q_0^0(0))\ \nu((0,\cdots,0,1,\cdots,1,\bar{s}_{n-\eta+1},\bar{s}_{j-1},\bar{s}_{j+1},\bar{s}_n)^T) \right\}$$

$$= -g^0 - \sum_{j:\bar{s}_j \geq 2} (\sum_{\bar{s}'_j=0}^{\bar{s}_j-1} q_0^0(\bar{s}'_j))\ (\sum_{i:\bar{s}_i \geq 2} \nu^i(\bar{s}_i) + \tfrac{(1-(1+\frac{1}{2}+\cdots+\frac{1}{\eta})n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))})$$

$$+ \sum_{j:\bar{s}_j \geq 2} \left\{ \sum_{\bar{s}'_j=2}^{\bar{s}_j-1} q_0^0(\bar{s}'_j)\ (\sum_{i:\bar{s}_i \geq 2} \nu^i(\bar{s}_i) - \nu^j(\bar{s}_j) + \nu^i(\bar{s}'_j) + \tfrac{(1-(1+\frac{1}{2}+\cdots+\frac{1}{\eta})n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}) \right.$$

$$\left. + (q_0^0(1) + q_0^0(0))\ (\sum_{i:\bar{s}_i \geq 2} \nu^i(\bar{s}_i) - \nu^j(\bar{s}_j) + \tfrac{(1-(1+\frac{1}{2}+\cdots+\frac{1}{\eta-1})n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}) \right\}$$

(reorganizations of the sum:)

$$= -g^0 - \sum_{j:\bar{s}_j \geq 2} (\sum_{\bar{s}'_j=0}^{\bar{s}_j-1} q_0^0(\bar{s}'_j))\ \tfrac{(1-(1+\frac{1}{2}+\cdots+\frac{1}{\eta})n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}$$

$$+ \sum_{j:\bar{s}_j \geq 2} \sum_{\bar{s}'_j=2}^{\bar{s}_j-1} q_0^0(\bar{s}'_i)\ \tfrac{(1-(1+\frac{1}{2}+\cdots+\frac{1}{\eta})n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}$$

$$+ \sum_{j:\bar{s}_j \geq 2} (q_0^0(1) + q_0^0(0))\ \tfrac{(1-(1+\frac{1}{2}+\cdots+\frac{1}{\eta-1})n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))})$$

$$- \sum_{i:\bar{s}_j \geq 2} \left\{ \sum_{\bar{s}'_j=2}^{\bar{s}_j-1} q_0^0(\bar{s}'_j)\ (\nu^j(\bar{s}_j) - \nu^j(\bar{s}'_j)) + (q_0^0(1) + q_0^0(0))\ \nu^j(\bar{s}_j) \right\}$$

$$= -\tfrac{n\ q_0^0(0)\ q_0^0(1)}{n\ q_0^0(0)+q_0^0(1)} + \eta\ (q_0^0(1) + q_0^0(0))\ \tfrac{\frac{1}{\eta}\ n\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}$$

$$- \sum_{i:\bar{s}_j \geq 2} \left\{ \sum_{\bar{s}'_j=2}^{\bar{s}_j-1} q_0^0(\bar{s}'_j)\ (\tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}'_j)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}'_j)} \right.$$

$$+ \tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}'_j+1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}'_j+1)} + \cdots + \tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}_j-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}_j-1)})$$

$$\left. + (q_0^0(1) + q_0^0(0))\ (\tfrac{q_0^0(1)}{q_0^0(0)+q_0^0(1)}) + \cdots + (\tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}_j-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}_j-1)}) \right\}$$

$$= 0 - \sum_{j:\bar{s}_j \geq 2} \sum_{\bar{s}'_j=2}^{\bar{s}_j} (q_0^0(0) + q_0^0(1) + \cdots + q_0^0(\bar{s}'_j - 1))\ \tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}'_j-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}'_j-1)}$$

$$= -\sum_{j:\bar{s}_j \geq 2} \sum_{\bar{s}'_j=2}^{\bar{s}_j} (q_0^0(1) + \cdots + q_0^0(\bar{s}'_j - 1))$$

$$= -\sum_{j:\bar{s}_j \geq 2} \sum_{w_j=1}^{\bar{s}_j - 1} (\bar{s}_j - w_j)\ q_0^0(w_j) = -\gamma^0(\bar{s}).$$

Case: $s(k_0) = s^1(k_0) = (k_0, \cdots, k_0, k_0 - \bar{su})^T$

From Definition 4.1 together with the iterative method from Section 2 and (*1) it follows that
$s'(= s'(k_0)) = (k_0, \cdots, k_0, k_0 - 1, \cdots, k_0 - 1, k_0 - \bar{s}'_n)^T$ with $1 < \bar{s}'_n < \bar{su}$, $(k_0-1, \cdots, k_0-1)^T$ and $(k_0-1, \cdots, k_0-1, k_0-2, k_0)^T$ are essential respecting $s(k_0)$ and no other essential partitions, different from $s(k_0)$ itself, exist. The numbers of the components k_0 and $k_0 - 1$ are determined by (*1).

If we additionally apply Remark 4.1 (ii) and recall to the definition of $S^l_{\tilde{e}}$ in Theorem 4.2 then we can replace the left side of equation (*2) in the following way

$$\begin{aligned}
&-g^0 - (\sum_{\bar{s}'_n=1}^{\bar{su}} q_0^0(\bar{s}'_n - 1))\ \nu(\bar{s}) \\
&\quad + \sum_{\bar{s}'_n=2}^{\bar{su}-1} q_0^0(\bar{s}'_n)\ \nu((0, \cdots, 0, 1, \cdots, 1, \bar{s}'_n)^T) \\
&\quad + q_0^0(1)\ \nu((1, \cdots, 1)^T) + q_0^0(0)\ \nu((0, 1, \cdots, 1, 2)^T)
\end{aligned}$$

$$\begin{aligned}
&= -g^0 - (\sum_{\bar{s}'_n=1}^{\bar{su}} q_0^0(\bar{s}'_n - 1))\ (\nu^n(\bar{su}) + \tfrac{(1-n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}) \\
&\quad + \sum_{\bar{s}'_n=2}^{\bar{su}-1} q_0^0(\bar{s}'_n)\ (\nu^n(\bar{s}'_n) + \tfrac{(1-n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}) \\
&\quad + q_0^0(0)\ (\nu^n(2) + \tfrac{(1-n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))})
\end{aligned}$$

$$\begin{aligned}
&= -g^0 + \tfrac{(1-n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}(-\sum_{\bar{s}'_n=0}^{\bar{su}-1} q_0^0(\bar{s}'_n) + \sum_{\bar{s}'_n=2}^{\bar{su}-1} q_0^0(\bar{s}'_n) + q_0^0(0)) \\
&\quad - \sum_{\bar{s}'_n=2}^{\bar{su}-1} q_0^0(\bar{s}'_n)(\nu^n(\bar{su}) - \nu^n(\bar{s}'_n)) - q_0^0(0)(\nu^n(\bar{su}) - \nu^n(2)) - q_0^0(1)\nu^n(\bar{su})
\end{aligned}$$

$$= -\tfrac{n\ q_0^0(0)\ q_0^0(1)}{n\ q_0^0(0)+q_0^0(1)} + \tfrac{(1-n)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (n\ q_0^0(0)+q_0^0(1))}(-q_0^0(1))$$

$$-\sum_{\bar{s}'_n=2}^{\bar{s}\bar{u}-1} q_0^0(\bar{s}'_n)\ (\tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}'_n)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}'_n)}$$
$$+\tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}'_n+1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}'_n+1)} + \cdots + \tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}\bar{u}-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}\bar{u}-1)})$$
$$-q_0^0(1)\ (\tfrac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} + \cdots + \tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}\bar{u}-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}\bar{u}-1)})$$
$$-q_0^0(0)\ (\tfrac{q_0^0(1)+q_0^0(2)}{q_0^0(0)+q_0^0(1)+q_0^0(2)} + \cdots + \tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}\bar{u}-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}\bar{u}-1)})$$

$$= -\tfrac{q_0^0(0)\ q_0^0(1)}{q_0^0(0)+q_0^0(1)}$$
$$-\sum_{\bar{s}'_n=2}^{\bar{s}\bar{u}} (q_0^0(0) + q_0^0(1) + \cdots + q_0^0(\bar{s}'_n - 1))\ \tfrac{q_0^0(1)+q_0^0(2)+\cdots+q_0^0(\bar{s}'_n-1)}{q_0^0(0)+q_0^0(1)+\cdots+q_0^0(\bar{s}'_n-1)}$$
$$+\ \tfrac{q_0^0(0)\ q_0^0(1)}{q_0^0(0)+q_0^0(1)}$$
$$= -\sum_{\bar{s}'_n=2}^{\bar{s}\bar{u}} (q_0^0(1) + \cdots + q_0^0(\bar{s}'_n - 1)) = -\sum_{w_n=1}^{\bar{s}\bar{u}-1} (\bar{s}\bar{u} - w_n)\ q_0^0(w_n) = -\gamma^0(\bar{s})$$

Case: Let $s(k_0) = s^{r-1}(k_0) = (k_0, k_0 - 1, \cdots, k_0 - 1, k_0 - 2)^T$

In this case, we only have $\lim_{k_0\to\infty} \frac{1}{c(k_0)}(p^*(s^{r-1}(k_0)|s^{r-1}(k_0)) - 1) \neq 0$ and $\lim_{k_0\to\infty} \frac{1}{c(k_0)}(p^*(s^r(k_0)|s^{r-1}(k_0)) \neq 0$ according to Theorem 4.2.

$$S_{\tilde{e}}^{r-1} = \left\{ \begin{pmatrix} k_0-1 \\ k_0 \\ k_0-1 \\ k_0-1 \\ \vdots \\ k_0-1 \\ k_0-2 \end{pmatrix}, \begin{pmatrix} k_0-1 \\ k_0-1 \\ k_0 \\ k_0-1 \\ \vdots \\ k_0-1 \\ k_0-2 \end{pmatrix}, \cdots, \begin{pmatrix} k_0-1 \\ k_0-1 \\ k_0-1 \\ \vdots \\ k_0-1 \\ k_0 \\ k_0-2 \end{pmatrix}, \begin{pmatrix} k_0-2 \\ k_0-1 \\ k_0-1 \\ \vdots \\ k_0-1 \\ k_0-1 \\ k_0 \end{pmatrix} \right\} \text{ for}$$

$s(k_0) = s^{r-1}$ leads to

$$\lim_{k_0\to\infty} \tfrac{1}{c(k_0)}(p^*(s^{r-1}(k_0)|s^{r-1}(k_0)) - 1)$$
$$= -\big((n-2)q_0^0(0) + q_0^0(1) + q_0^0(0)\big) + \big((n-2)q_0^0(0) + q_0^0(0)\big).$$

Then, (*2) has the representation

$$-g^0 - q_0^0(1)\ \nu(\bar{s}^{r-1}) = -\gamma^0(\bar{s}^{r-1}) = -q_0^0(1) \text{ for } \bar{s} = \bar{s}^{r-1}.$$

If we insert the corresponding expressions for g^0 and $\nu(\bar{s}^{r-1})$ then the equation

$-\frac{n\, q_0^0(0)\, q_0^0(1)}{n\, q_0^0(0)+q_0^0(1)} - q_0^0(1)\, \left(\frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} - \frac{(n-1)\, q_0^0(0)\, q_0^0(1)}{(q_0^0(0)+q_0^0(1))\, (n\, q_0^0(0)+q_0^0(1))}\right) = -q_0^0(1)$

is true.

Case: Let $s(k_0) = s^r(k_0) = (k_0 - 1, \cdots, k_0 - 1)^T$

In this case, we only have $\lim\limits_{k_0\to\infty} \frac{1}{c(k_0)}(p^*(s^{r-1}(k_0)|s^r(k_0))) \neq 0$ and $\lim\limits_{k_0\to\infty} \frac{1}{c(k_0)}(p^*(s^r(k_0)|s^r(k_0) - 1) \neq 0$ according to Theorem 4.2.

$$S_{\tilde{e}}^{r-1} = \left\{ \begin{pmatrix} k_0 \\ k_0-1 \\ k_0-1 \\ k_0-1 \\ \vdots \\ k_0-1 \\ k_0-2 \end{pmatrix}, \begin{pmatrix} k_0-1 \\ k_0 \\ k_0-1 \\ k_0-1 \\ \vdots \\ k_0-1 \\ k_0-2 \end{pmatrix}, \cdots, \begin{pmatrix} k_0-1 \\ k_0-1 \\ k_0-1 \\ \vdots \\ k_0-1 \\ k_0 \\ k_0-2 \end{pmatrix}, \begin{pmatrix} k_0-1 \\ k_0-1 \\ k_0-1 \\ \vdots \\ k_0-1 \\ k_0-2 \\ k_0 \end{pmatrix} \right\} \text{ for}$$

$s(k_0) = s^r$ leads to

$\lim\limits_{k_0\to\infty} \frac{1}{c(k_0)}(p^*(s^{r-1}(k_0)|s^r(k_0))) = n\, q_0^0(0)$. Then, (*2) has the representation

$-g^0 + n\, q_0^0(0)\, \nu(\bar{s}^{r-1}) = -\gamma^0(\bar{s}^r) = 0$ for $\bar{s} = \bar{s}^{r-1}$ and

$-\frac{n\, q_0^0(0)\, q_0^0(1)}{n\, q_0^0(0)+q_0^0(1)} + n\, q_0^0(0)\, \left(\frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} - \frac{(n-1)\, q_0^0(0)\, q_0^0(1)}{(q_0^0(0)+q_0^0(1))\, (n\, q_0^0(0)+q_0^0(1))}\right) = 0$

is true. ■

Corollary 5.2 *Let the same assumptions as in Theorem 5.1 be valid. Furthermore, let $s^f(k_0) = (k_0, \cdots, k_0)^T - \bar{s}^f$ be direct predecessors of $s^l(k_0) = (k_0, \cdots, k_0)^T - \bar{s}^l$, $k_0 = n+1, n+2, \cdots$ (see Remark 4.1 (iii)). Then, $\nu(\bar{s}^l) < \nu(\bar{s}^f)$ is valid for all solutions of the equation system (32).*

This statement results from simple computations using the formulas of Theorem 5.1 for $\nu(\bar{s}^f)$ and $\nu(\bar{s}^l)$ (see Appendix F).

Corollary 5.3 *Let the same assumptions as in Theorem 5.1 be valid. Then, the solutions of the Poisson equations (see Definition 5.1) with regard to sets $S_{n;n(k_0-1);k_0}$, are monotone for sufficiently large k_0.*

Main idee of the Proof:

If $(g, \nu) \in \mathbb{R} \times \mathbb{R}^r$ is a solution of a Poisson equation (see Definition 5.1) then $(\tilde{g} = \frac{g}{c(k_0)}, \nu)$ is a solution of a system

$\tilde{g}(-1, ..., -1)^T + \frac{1}{c(k_0)}(P^* - I)\nu = -\frac{1}{c(k_0)}\gamma' \; (c(k_0) \neq 0)$.

The result of Corollary 5.3 follows then from Corollary 5.2 and the solution behavior of a sequence of linear vector equations for which the coefficient matrices and the right-hand-sides each tend towards a limit (there it is important that the inequalities $\nu(\bar{s}^l) < \nu(\bar{s}^f)$ are strict inequalities in Corollary 5.2).

References

[1] A. Borodin, R. El-Yaniv, *Online computation and competitive analysis*, University Press, Cambrigde 1998.

[2] D.J. Daley, Stochastic monotone Markov processes, *Z. Wahrsch. Verw. Gebiete*, Vol. 10 (1968), pp. 305-317.

[3] H.-J. Girlich, *Diskrete stochastische Entscheidungsprozesse*, Teubner Verlagsgesellschaft, Leipzig 1973.

[4] H.-J. Girlich, P. Köchel and H.-U. Küenle, *Steuerung dynamischer Systeme.* Birkhäuser, Basel 1990.

[5] R. Hildenbrandt, *A special stochastic decision problem*, *Optimization*, **28** (1993), 95-110.

[6] R. Hildenbrandt, Eine Verbandsstruktur für Partitionen ganzer Zahlen, $http://www.db-thueringen.de/servlets/DerivateServlet/Derivate-8547/IfM\_Preprint\_M\_94\_14.PDF$.

[7] R. Hildenbrandt, *Methoden aus ganzzahliger Optimierung und Verbandstheorie zur Behandlung eines stochastischen dynamischen Transportproblems*, Habilitation, TU Ilmenau ISBN 3-89811-879-7 (1995).

[8] R. Hildenbrandt, Partitions-requirements-matrices, Symposium on Operations Research, Duisburg 9/2001 (*Proceedings*, Springer, Berlin, Heidelberg, ..., (2002), 303 - 311.

[9] R. Hildenbrandt, Stochastic dynamic programming with random disturbances, *Discussiones Mathematicae Probability and Statistics*, **23** (2003), 5-44.

[10] R. Hildenbrandt, *DA Stochastic Dynamic Programming, SDDP Problems and Partitions-Requirements-Matrices*, research monograph, Cuvillier Verlag, Göttingen 210.

[11] R. Hildenbrandt, Partitions-requirements-matrices as optimal Markov kernels of special stochastic dynamic distance optimal partitioning problems, *International journal of pure and applied mathematics*, Bd. 88 (2013), 2, 183-211.

[12] A.W. Marshall, I. Olkin, *Inequalities: Theory of majorization and its applications*, Mathematics in Science and Engineering, Vol. 143. Academic Press, New York (1979).

[13] P.H. Müller, V. Nollau, *Steuerung stochastischer Prozesse*, Akademie-Verlag, Berlin (1984).

[14] M. Puterman, *Markov decision processes.* Wiley, New York, ... (1994).

[15] W. Szwarc, The Dynamic Transportation Problem, *Mathematica*, **81** (1971), 335-345.

Appendices

A An example of a ReH-matrix and its computation using perturbed permutations

a) Let $n = 4, su = 13, k_0 = 5$ and let the requirements be discrete uniformly distributed. Then, $|B_{4;5}| = 6^4$ and $S_{4;13;5}$ include the elements

$$s^1 = \begin{pmatrix}5\\5\\3\\0\end{pmatrix}, s^2 = \begin{pmatrix}5\\5\\2\\1\end{pmatrix}, s^3 = \begin{pmatrix}5\\4\\4\\0\end{pmatrix}, s^4 = \begin{pmatrix}5\\4\\3\\1\end{pmatrix}, s^5 = \begin{pmatrix}5\\4\\2\\2\end{pmatrix}, s^6 = \begin{pmatrix}5\\3\\3\\2\end{pmatrix},$$

$$s^7 = \begin{pmatrix}4\\4\\4\\1\end{pmatrix}, s^8 = \begin{pmatrix}4\\4\\3\\2\end{pmatrix}, s^9 = \begin{pmatrix}4\\3\\3\\3\end{pmatrix}.$$

For example, let $w^{l_1} = \begin{pmatrix}1\\1\\4\\2\end{pmatrix}, w^{l_2} = \begin{pmatrix}2\\4\\4\\4\end{pmatrix}$. Then, we have:

i) $s^*(s^4, w^{l_1}) = s^8$, where $s_\pi^{8(1)} = \begin{pmatrix}4\\3\\4\\2\end{pmatrix}$ or $s_\pi^{8(2)} = \begin{pmatrix}3\\4\\4\\2\end{pmatrix}$ fulfil (3),

ii) $s^*(s^4, w^{l_2}) = s^8$, where $s_\pi^{8(3)} = \begin{pmatrix}2\\4\\4\\3\end{pmatrix}$ or $s_\pi^{8(4)} = \begin{pmatrix}2\\4\\3\\4\end{pmatrix}$ fulfil (3).

The ReH-matrix

$$P^*_{4;13;5} = 6^{-4}\begin{pmatrix}
180 & 46 & 81 & 258 & 83 & 198 & 39 & 213 & 198\\
16 & 280 & 24 & 196 & 206 & 174 & 64 & 258 & 78\\
21 & 39 & 245 & 144 & 90 & 163 & 163 & 242 & 189\\
12 & 48 & 28 & 428 & 84 & 177 & 72 & 240 & \mathbf{207}\\
12 & 48 & 28 & 72 & 441 & 201 & 24 & 383 & 87\\
12 & 48 & 28 & 72 & 114 & 543 & 24 & 230 & 225\\
12 & 48 & 28 & 192 & 78 & 156 & 316 & 274 & 192\\
12 & 48 & 28 & 60 & 114 & 252 & 36 & 530 & 216\\
12 & 48 & 28 & 60 & 114 & 252 & 36 & 158 & 588
\end{pmatrix}$$

can then be computed using the iterative method from Section 2.

b) Now, we want to determine an element of $P^*_{4;13;5}$ (for example p^*_{49}), with the help of perturbed permutations (see Section 3).

If $f = 4, l = 9$ then (10), ..., (13) yield $F = L = 4$,
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = F_z = 4; z = 4$,

$L_1 = 1, L_2 = L_y = 4; y = 2,$
moreover, $\sigma_1^9 = s_{L_1}^9 = s_1^9 = 4, \sigma_2^9 = s_{L_2}^9 = s_4^9 = 3.$

Case $\sum_{i=1}^{n} w_i \leq su$: Sets of perturbed permutations (see Definitions 3.1, 3.2)

Tabular presentation of the quantities:

s^4	$s_\pi^{9(1)}$		$\hat{s}_\pi^{9(1)}$	$s_\pi^{9(2)}$		$\hat{s}_\pi^{9(2)}$	$s_\pi^{9(3)}$		$\hat{s}_\pi^{9(3)}$	$s_\pi^{9(4)}$		$\hat{s}_\pi^{9(4)}$
5	$\underline{4}$			**3**	+	1	**3**	+	1	**3**	+	1
4	**3**	+	1	$\underline{4}$			**3**	+	1	**3**	+	1
3	3			3			4			3		
1	3			3			3			4		
$\mathbf{J_o}$	2			2			2			2		
$\mathbf{j_o}$	1			1			2			2		
$\underline{j_1}$	1			1			0			0		

Hence, $\hat{S}_\pi^{4,9}(2,1) = \hat{S}_\pi^{4,9}(2,1,1) = \left\{ \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix} (= \hat{s}_\pi^{9(1)} = \hat{s}_\pi^{9(2)}) \right\},$

$$\hat{S}_\pi^{4,9}(2,2) = \hat{S}_\pi^{4,9}(2,2,0) = \left\{ \begin{pmatrix} 4 \\ 4 \\ 4 \\ 3 \end{pmatrix} (= \hat{s}_\pi^{9(3)}), \begin{pmatrix} 4 \\ 4 \\ 3 \\ 4 \end{pmatrix} (= \hat{s}_\pi^{9(4)}) \right\}, \hat{S}_\pi^{4,9} = \left\{ \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Computations of sets $\hat{B}_{4;5}^{*1}(s^4, \hat{s}_\pi^{9(.)})$ (see Definition 3.3):

$$\hat{B}_{4;5}^{*1}(s^4, \hat{s}_\pi^{9(1)}) = \left\{ w \;\middle|\; w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ 3 \end{pmatrix} \text{ with } w_3 \in \{0,1,2,3\} \text{ and } w_j \in \{0,1,...,4\} \text{ for } j = 1,2 \text{ with at most one coordinate } w_j = 4, \right\}$$

and $|\hat{B}_{4;5}^{*1}(s^4, \hat{s}_\pi^{9(1)})| = (5^2 - 1)4 = 96,$

$$\hat{B}_{4;5}^{*1}(s^4, \hat{s}_\pi^{9(3)}) = \left\{ w \;\middle|\; w = \begin{pmatrix} w_1 \\ w_2 \\ 4 \\ 3 \end{pmatrix} \text{ with } w_j \in \{0,1,2,3\} \text{ for } j = 1,2 \right\}$$

and $|\hat{B}_{4,5}^{*1}(s^4, \hat{s}_{\pi)}^{9(3)}| = 4^2 = 16,$

$$\hat{B}_{4,5}^{*1}(s^4, \hat{s}_\pi^{9(4)}) = \left\{ w \;\middle|\; w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ 4 \end{pmatrix} \text{ with } w_j \in \{0,1,2,3\} \text{ for } j = 1,2,3 \right\}$$

and $|\hat{B}^{*1}_{4,5}(s^4, \hat{s}_\pi^{9(4)})| = 4^3 = 64$.

Case $\sum_{i=1}^{n} w_i \geq su$: Sets of perturbed permutations (see Definitions 3.4, 3.5)

Tabular presentation of the quantities:

s^4	$s_\pi^{9(1)}$		$\hat{s}_\pi^{9(1)}$	$s_\pi^{9(2)}$		$\hat{s}_\pi^{9(2)}$	$s_\pi^{9(3)}$		$\hat{s}_\pi^{9(3)}$	$s_\pi^{9(4)}$		$\hat{s}_\pi^{9(4)}$
5	4			3			3			3		
4	3			4			3			3		
3	3			3			**4**	-	1	$\underline{3}$		
1	**3**	-	1	**3**	-	1	3			**4**	-	1
$\bar{\mathbf{J}}_\mathbf{0}$	2			2			1			1		
$\bar{\mathbf{j}}_\mathbf{0}$	1			1			1			1		
$\underline{\bar{j}_1}$	0			0			1			1		

Hence, $\hat{\hat{S}}^{4,9}(2,1) = \hat{\hat{S}}^{4,9}(2,1,0) = \left\{ \begin{pmatrix}4\\3\\3\\2\end{pmatrix} (= \hat{s}_\pi^{9(1)}), \begin{pmatrix}3\\4\\3\\2\end{pmatrix} (= \hat{s}_\pi^{9(2)}) \right\}$,

$$\hat{\hat{S}}^{4,9}(1,1) = \hat{\hat{S}}^{4,9}(1,1,1) = \left\{ \begin{pmatrix}3\\3\\3\\3\end{pmatrix} (= \hat{s}_\pi^{9(3)} = \hat{s}_\pi^{9(4)}) \right\}, \hat{\hat{S}}^{4,9} = \left\{ \begin{pmatrix}4\\3\\3\\2\end{pmatrix}, \begin{pmatrix}3\\4\\3\\2\end{pmatrix}, \begin{pmatrix}3\\3\\3\\3\end{pmatrix} \right\}.$$

Computation of sets $\hat{B}^{*2}_{4;5}(s^4, \hat{s}_\pi^{9(.)})$ (see Definition 3.6):

$$\hat{B}^{*2}_{4,5}(s^4, \hat{s}_\pi^{9(1)}) = \left\{ w \;\middle|\; w = \begin{pmatrix}4\\3\\w_3\\w_4\end{pmatrix} \text{ with } w_j \;\epsilon\; \{3,4,5\} \text{ for } j = 3,4 \right\}$$

$|\hat{B}^{*2}_{4,5}(s^4, \hat{s}_\pi^{9(1)})| = 3^2 = 9$,

$$\hat{B}^{*2}_{4,5}(s^4, \hat{s}_\pi^{9(2)}) = \left\{ w \;\middle|\; w = \begin{pmatrix}3\\w_2\\w_3\\w_4\end{pmatrix} \text{ with } w_2 \;\epsilon\; \{4,5\}, w_j \;\epsilon\; \{3,4,5\} \text{ for } j = 3,4 \right\}$$

$|\hat{B}^{*2}_{4,5}(s^4, \hat{s}_\pi^{9(2)})| = 2 \cdot 3^2 = 18$,

$$\hat{B}^{*2}_{4,5}(s^4, \hat{s}_\pi^{9(3)}) = \left\{ \begin{array}{l} w \;\middle|\; w = \begin{pmatrix}3\\3\\w_3\\w_4\end{pmatrix} \text{ with } w_j \;\epsilon\; \{3,4,5\} \text{ for } j = 3,4 \text{ and} \\ \text{with at most one coordinate } w_j = 4 \end{array} \right\}$$

$|\hat{B}^{*2}_{4,5}(s^4, \hat{s}_\pi^{9(3)})| = 3^2 - 1 = 8$.

Finally, in the case of discrete uniformly distributed requirements, we have

$$\begin{aligned} p^*_{49} &= p^{*1}_{49} + p^{*2}_{49} - p^{*1,2}_{49} \\ &= 6^{-4}(96 + 16 + 64 + 9 + 18 + 8 - 4) \\ &= 6^{-4} \cdot 207 \end{aligned}$$

(see (17), (19), (15) and Lemma 3.4).

B An example for limits of elements in ReH-matrices

We want to consider the limit $\lim\limits_{k_0 \to \infty} \frac{1}{c(k_0)}(p^*(s^f(k_0)|s^f(k_0)) - 1))$ for $s^f(k_0) = \begin{pmatrix} k_0 - 1 \\ k_0 - 2 \\ k_0 - 3 \end{pmatrix}$, where corresponding probability functions q^{k_0} are given and fulfil (23) and (24) for certain $c(k_0)$.

Then $S^f_{\tilde{e}} = \left\{ \begin{pmatrix} k_0 - 2 \\ k_0 - 1 \\ k_0 - 3 \end{pmatrix}, \begin{pmatrix} k_0 - 2 \\ k_0 - 3 \\ k_0 - 1 \end{pmatrix} \right\}$ (or $S^f_{\tilde{e}} = \left\{ \begin{pmatrix} k_0 - 2 \\ k_0 - 1 \\ k_0 - 3 \end{pmatrix}, \begin{pmatrix} k_0 - 3 \\ k_0 - 2 \\ k_0 - 1 \end{pmatrix} \right\}$, respectively).

$$\lim_{k_0 \to \infty} \frac{1}{c(k_0)}(p^*(s^f(k_0)|s^f(k_0)) - 1))$$

$$= -\left[((q^0_0(0)) + (q^0_0(1) + q^0_0(0)) + (q^0_0(2) + q^0_0(1) + q^0_0(0))\right]$$

$$+ q^0_0(1) + q^0_0(1) = -3q^0_0(0) - q^0_0(2)$$

follows according to Theorem 4.2.

C The origin of ReH-matrices

The theory of **S**tochastic **D**ynamic **D**istance Optimal **P**artitioning problems (**SDDP**) (see [5], [7], [10]) can be applied to cost-optimal repeated conversions of machines in successive stages and also to the effective use of manpower in different work places, where probability functions model future requirements.

Optimal solutions of such problems can, for example, be determined by iterativ methods which use Poisson equations: $g\ (-1, \cdots, -1)^T + (P - I)\nu = -\gamma$ with variables $(g, \nu) \in \mathbb{R} \times \mathbb{R}^r)$ and where I is the identity matrix, P a matrix of transition probabilities, γ average one-step reward functions (the last two correspond to a chosen decision, in general). See, for example [13], Section 2.4.2.1 or [14], Section 8.6.

Because such solution methods require an enormous amount of storage space, one is interested in heuristics and useful characteristic properties of the solutions.

If all "distance costs" for SDDP problems are equal (in other words, the costs of converting machines/ all distance costs between two work places are identical, we conjecture that transitions into "balanced" partitions ("states") (see Section 1.1) are optimal. The corresponding matrices P are called ReH-matrices. The conjecture is true if and only if all corresponding Poisson equations have monotone solutions. (See introduction of Section 5.)

D The condition of dominance

Let $P = (p_{fl})_{\substack{f=1,\cdots,r,\\ l=1,\cdots,r}}$ be a stochastic matrix. Then the dominance condition means that
$\sum_{l=1}^{\bar{l}} p_{1l} \geq \sum_{l=1}^{\bar{l}} p_{2l} \geq \cdots \geq \sum_{l=1}^{\bar{l}} p_{rl}$ for $\bar{l} = 1, 2, \cdots, r$, see [2].

E An examples regarding Theorem 5.1

Let $S_{4;su=4(k_0-1);k_0}$

$$= \left\{ s^1 = \begin{pmatrix} k_0 \\ k_0 \\ k_0 \\ k_0 - 4 \end{pmatrix}, s^2 = \begin{pmatrix} k_0 \\ k_0 \\ k_0 - 1 \\ k_0 - 3 \end{pmatrix}, s^3 = \begin{pmatrix} k_0 \\ k_0 \\ k_0 - 2 \\ k_0 - 2 \end{pmatrix}, s^4 = \begin{pmatrix} k_0 \\ k_0 - 1 \\ k_0 - 1 \\ k_0 - 2 \end{pmatrix}, s^5 = \begin{pmatrix} k_0 - 1 \\ k_0 - 1 \\ k_0 - 1 \\ k_0 - 1 \end{pmatrix} \right\},$$

$k_0 = 5, 6, \cdots$ be given.

Using (31) we get the following equation system for (32)

$$\begin{pmatrix} -1 & -\sum_{i=0}^{3} q_0^0(i) & q_0^0(3) & 0 & q_0^0(2) + q_0^0(0) \\ -1 & 0 & -\sum_{i=0}^{2} q_0^0(i) & 0 & q_0^0(2) + q_0^0(0) \\ -1 & 0 & 0 & -2\sum_{i=0}^{1} q_0^0(i) & 2\sum_{i=0}^{1} q_0^0(i) \\ -1 & 0 & 0 & 0 & -q_0^0(1) \\ -1 & 0 & 0 & 0 & 4q_0^0(0) \end{pmatrix} \begin{pmatrix} g^0 \\ \nu^1 \\ \nu^2 \\ \nu^3 \\ \nu^4 \end{pmatrix} = -\gamma^0,$$

$$\gamma^0 = \begin{pmatrix} 3q_0^0(1) + 2q_0^0(2) + q_0^0(3) \\ 2q_0^0(1) + q_0^0(2) \\ 2q_0^0(1) \\ q_0^0(1) \\ 0 \end{pmatrix} \text{ which includes that } \nu^5 (= \nu(\bar{s}^5)) = 0.$$

This system has the solution: $g^0 = \frac{4\, q_0^0(0)\, q_0^0(1)}{4\, q_0^0(0) + q_0^0(1)}$,

$\nu^1 = \frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} + \frac{q_0^0(1)+q_0^0(2)}{q_0^0(0)+q_0^0(1)+q_0^0(2)} + \frac{q_0^0(1)+q_0^0(2)+q_0^0(3)}{q_0^0(0)+q_0^0(1)+q_0^0(2)+q_0^0(3)} + \frac{(-3)\, q_0^0(0)\, q_0^0(1)}{(q_0^0(0)+q_0^0(1))\,(4\, q_0^0(0)+q_0^0(1))}$

$\nu^2 = \frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} + \frac{q_0^0(1)+q_0^0(2)}{q_0^0(0)+q_0^0(1)+q_0^0(2)} + \frac{(-3)\, q_0^0(0)\, q_0^0(1)}{(q_0^0(0)+q_0^0(1))\,(4\, q_0^0(0)+q_0^0(1))}$

$\nu^3 = 2\frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} + \frac{(-5)\, q_0^0(0)\, q_0^0(1)}{(q_0^0(0)+q_0^0(1))\,(4\, q_0^0(0)+q_0^0(1))}$

$\nu^4 = \frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} + \frac{(-3)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))\ (4\ q_0^0(0)+q_0^0(1))} = \frac{q_0^0(1)}{4q_0^0(0)+q_0^0(1)}.$

F Proof of Corollary 5.2

Let
$s^f = (k_0, \cdots, k_0, k_0 - 1, \cdots, k_0 - 1, k_0 - \bar{s}_{n-\eta^f+1}, k_0 - \bar{s}_{n-\eta^f+2}, \cdots, k_0 - \bar{s}_n)^T$
with $\bar{s}_{n-\eta^f+1} \leq \bar{s}_{n-\eta^f+2} \leq \cdots \leq \bar{s}_n$ and $s^l = s^f[j'; i']$ (with $j' > i'$). Then the following cases must be considered (see Remark 4.1 (ii)):

a) $s^f_{i'} = k_0, s^f_{j'} = k_0 - 2$

b) $s^f_{i'} = k_0, s^f_{j'} < k_0 - 2$

c) $s^f_{i'} = k_0 - 1, s^f_{j'} < k_0 - 2$

d) $s^f_{i'} < k_0 - 1, s^f_{j'} < s^f_{i'} - 1 (< k_0 - 3)$

.

If $t(\eta) = -\frac{(1-(1+\frac{1}{2}+\cdots+\frac{1}{\eta})n)\ q_0^0(0)\ q_0^0(-1)}{(q_0^0(0)+q_0^0(-1))\ (n\ q_0^0(0)+q_0^0(-1))}$ then

$$\eta^1 < \eta^2 \ \Leftrightarrow\ t(\eta^1) < t(\eta^2). \tag{*1}$$

Obviously,

$$\bar{s}_i^1 > \bar{s}_i^2 \ \Leftrightarrow\ \nu^i(\bar{s}_i^1) > \nu^i(\bar{s}_i^2),\ \text{where } \bar{s}_i^2 > 0 \tag{*2}$$

and

$$0 < b_1 < b_2, 0 < b_3 \ \Rightarrow\ \frac{b_1}{b_2} < \frac{b_1 + b_3}{b_2 + b_3} \tag{*3}$$

.

Case a): implies that $\bar{s}^f_{i'} = 0, \bar{s}^f_{j'} = 2, \bar{s}^l_{i'} = 1, \bar{s}^l_{j'} = 1$ and $\eta^l = \eta^f - 1$.

Then $\nu(\bar{s}^f) > \nu(\bar{s}^l) \ \Leftrightarrow\ \frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} - t(\eta^f) > -t(\eta^f - 1)$.

Case a1): $\eta^f = 1$:
$\frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} - t(1) > 0 \ \Leftrightarrow\ \frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} > \frac{(n-1)\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))(n\ q_0^0(0)+q_0^0(1))}$
$\Leftrightarrow\ q_0^0(1)(n\ q_0^0(0) + q_0^0(1)) > (n-1)\ q_0^0(0)\ q_0^0(1)$ is true.

Case a2): $\eta^f > 1$: $\frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} - t(\eta^f) > -t(\eta^f - 1)$
$\Leftrightarrow\ \frac{q_0^0(1)}{q_0^0(0)+q_0^0(1)} - \frac{\frac{n}{\eta^f}\ q_0^0(0)\ q_0^0(1)}{(q_0^0(0)+q_0^0(1))(n\ q_0^0(0)+q_0^0(1))} > 0$
$\Leftrightarrow\ q_0^0(1)(n\ q_0^0(0) + q_0^0(1)) > \frac{n}{\eta^f}\ q_0^0(0)\ q_0^0(1)$ is true.

Case b): implies that $\bar{s}^f_{i'} = 0, \bar{s}^f_{j'} > 2, \bar{s}^l_{i'} = 1, \bar{s}^l_{j'} = \bar{s}^f_{j'} - 1$ and $\eta^l = \eta^f$.
Then $\nu(\bar{s}^f) > \nu(\bar{s}^l) \ \Leftrightarrow\ \nu^{j'}(\bar{s}^f_{j'}) > \nu^{j'}(\bar{s}^f_{j'} - 1)$ is true according to (*2).

Case c): implies that $\bar{s}^f_{i'} = 1, \bar{s}^f_{j'} > 2, \bar{s}^l_{i'} = 2, \bar{s}^l_{j'} = \bar{s}^f_{j'} - 1$ and $\eta^l = \eta^f + 1$. Then

$\nu(\bar{s}^f) > \nu(\bar{s}^l) \Leftrightarrow \nu^{j'}(\bar{s}^f_{j'}) - t(\eta^f) > \frac{q^0_0(1)}{q^0_0(0)+q^0_0(1)} + \nu^{j'}(\bar{s}^f_{j'} - 1) - t(\eta^f - 1) \Leftrightarrow \frac{q^0_0(1)+q^0_0(2)+\cdots+q^0_0(\bar{s}^f_{j'}-1)}{q^0_0(0)+q^0_0(1)+\cdots+q^0_0(\bar{s}^f_{j'}-1)} - t(\eta^f) > \frac{q^0_0(1)}{q^0_0(0)+q^0_0(1)} - t(\eta^f - 1)$ is true according to (*2) and (*3).

Case d): implies that $\bar{s}^f_{i'} > 1, \bar{s}^f_{j'} > \bar{s}^f_{i'} + 1, \bar{s}^l_{i'} = \bar{s}^f_{i'} + 1, \bar{s}^l_{j'} = \bar{s}^f_{j'} - 1$ and $\eta^l = \eta^f$.

Then $\nu(\bar{s}^f) > \nu(\bar{s}^l) \Leftrightarrow \nu^{i'}(\bar{s}^f_{i'}) + \nu^{j'}(\bar{s}^f_{j'}) > \nu^{i'}(\bar{s}^f_{i'} + 1) + \nu^{j'}(\bar{s}^f_{j'} - 1) \Leftrightarrow \frac{q^0_0(1)+q^0_0(2)+\cdots+q^0_0(\bar{s}^f_{j'}-1)}{q^0_0(0)+q^0_0(1)+\cdots+q^0_0(\bar{s}^f_{j'}-1)} > \frac{q^0_0(1)+q^0_0(2)+\cdots+q^0_0(\bar{s}^f_{i'})}{q^0_0(0)+q^0_0(1)+\cdots+q^0_0(\bar{s}^f_{i'})}$ is true according to (*3). ■