Quantum integrability of the geodesic flow for c-projectively equivalent metrics

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Zusammenfassung

Zwei (pseudo-)Riemannsche Metriken heißen projektiv äquivalent, wenn sie die selben Geodätischen (als unparametrisierte Kurven betrachtet) besitzen. C-projektive Äquivalenz ist eine natürliche Übertragung des Begriffs projektiver Äquivalenz von Riemannschen Mannigfaltigkeiten auf Kählersche Mannigfaltigkeiten. Eine reguläre Kurve γ auf einer Kählersche Mannigfaltigkeit $(\mathcal{M}, g, J)$ heißt J-planar, wenn ihr Beschleunigungsvektor in der Ebene liegt, die durch den Tangentialvektor der Kurve und den Vektor, der durch Anwendung der komplexen Struktur J auf den Tangentialvektor hervorgeht, aufgespannt wird. Zwei Kählersche Metriken auf einer Mannigfaltigkeit von reeller dimension echt grösser als zwei heißen c-projektiv äquivalent, wenn jede J-planare Kurve der einen Metrik auch bezüglich der anderen Metrik J-planar ist.

Es werden zunächst grundlegende Eigenschaften c-projektiv äquivalenter Metriken besprochen, sowie die Integrale des geodätischen Flusses, die aus der Existenz einer c-projektiv äquivalenten Metrik hervorgehen.

Im Hauptteil der Arbeit werden die folgenden Probleme behandelt:

Erstens: Kommutieren die den Integralen durch eine Quantisierungsvorschrift zugewiesenen Differentialoperatoren? Diese Frage konnte mit Ja beantwortet werden.

Zweitens: Ist eine Verallgemeinerung zu natürlichen Hamiltonschen Systemen möglich? Das heißt: Gibt es Potentialfunktionen, die sich zu den Integralen der Bewegung hinzuaddieren lassen, sodass die neu entstandenen Funktionen auf dem Kotangentialbündel weiterhin bezüglich der Poisson Klammer kommutieren? Weiterhin: Kommutieren auch die durch Quantisierung aus diesen Funktionen hervorgehenden Differentialoperatoren? Die Antwort ist Ja, und sowohl auf der Ebene klassischer Integrabilität als auch auf der Quantenebene sind die selben Potentiale zulässig. Für gewisse Fälle konnten sämtliche zulässige Potentiale gefunden werden. In den anderen Fällen geben wir eine Schar zulässiger Potentiale an, es kann aber sein, dass neben diesen noch mehr existieren.

Drittens: Für Kählersche Metriken, die nichttrivial c-projektiv äquivalente Metriken besitzen, wurde die Separation der Variablen für die Schrödingergleichung untersucht. Es wurde gezeigt, dass sich die Suche nach Funktionen, die gleichzeitige Eigenfunktionen der in den vorigen Abschnitten betrachteten Operatoren sind, auf das Lösen von Differentialgleichungen in niedrigeren Dimensionen zurückführen lässt. Als besondere Anwendung der erzielten Resultate lässt sich im Fall, dass die Anzahl der so konstruierten unabhängingen Integrale maximal ist, unter Voraussetzung einer positiv definiten Metrik die Konstruktion einer Orthonormalbasis im Raum der quadratintegrablen Funktionen auf das Lösen gewöhnlicher Differentialgleichungen zurückführen.

Abstract

Two (pseudo-) Riemannian metrics are called *projectively equivalent*, if they possess the same geodesics (considered as unparametrized curves). *C-projective eqivalence* is a natural translation of projective equivalence to Kähler manifolds: A regular curve γ on a Kähler manifold $(\mathcal{M}, g, J)$ is called J-planar, if the acceleration lies within the span of the tangent vector and J applied to the tangent vector. Two Kähler metrics on a complex manifold of real dimension larger than four are called *c-projectively equivalent*, if every J-planar curve of one metric is also J-planar with respect to the other metric.

In the first section we will introduce c-projectively equivalent metrics and fundamental properties. We will also introduce the integrals of the geodesic flow that arise as a consequence of the existence of a c-projectively equivalent structure.

In the main part we will tackle the following questions:

Firstly: Do the integrals of the geodesic flow commute as quantum operators? This could successfully be answered with yes.

Secondly: Is a generalization to natural Hamiltonian systems possible? This means: is it possible to add potentials to the integrals of the geodesic flow such that the resulting functions still commute with respect to the Poisson bracket? Furthermore do the assigned quantum operators still commute? The answer is yes. The admissible potentials are the same at the level of classical integrability as well as at the quantum level. For certain cases all admissible potentials could be described. For other cases a family of admissible potentials will be given, but there may be more admissible potentials.

Thirdly: For Kähler metrics that possess c-projectively equivalent metrics the separation of variables for Schrödinger's equation was investigated: We showed that the search for simultaneous eigenfunctions of the previously constructed quantum operators can be reduced to differential equations in lower dimension, in the case of maximal integrability a reduction to ordinary differential equations is possible. In the maximally integrable case this provides a possibility to construct an orthonormal basis of eigenfunctions of the Laplace-Beltrami operator in the space of square-integrable functions by solving ordinary differential equations.

Contents

1 Foundations

1.1 C-projective equivalence and the canonical Killing tensors

In all our work we assume that the manifold we are working on is connected.

Definition 1 (Kähler manifold) *A Kähler manifold (of arbitrary signature) is a manifold $\mathcal{M}^{2n}$ of real dimension $2n$ endowed with the following objects and properties:*

- *a (pseudo-)Riemannian metric g and its associated Levi-Civita connection ∇*
- *a complex structure J, i.e. an endomorphism on the space of vector fields with $J^2 = -Id$*
- *g and J must be compatible in the sense that $g(JX, Y) = -g(X, JY)$ and $\nabla J = 0$*
- *We denote by Ω the two-form $\Omega(X, Y) = g(JX, Y)$ which is called the Kähler form.*

We do not need to dig overly deep into the theory of Kähler manifolds, but two properties of the Riemann and Ricci tensor will be required:

Proposition 1 (Symmetries of the Riemann and Ricci tensor on Kähler manifolds) *In addition to the usual symmetries of the Riemann and Ricci tensor on a Kähler manifold $(\mathcal{M}, g, J)$ the relations*

$$\begin{gathered} R^i_{jkl} = -J^i_s R^s_{tkl} J^t_j = R^i_{jab} J^a_k J^b_l \\ J^i_s R^s_j = R^i_s J^s_j \end{gathered} \tag{1}$$

are true. Here and throughout the rest of the work we use the Einstein sum convention.

Proof of proposition 1: Because J is ∇-parallel:

$$0 = (\nabla_k \nabla_l - \nabla_l \nabla_k) J^i_j = R^i{}_{skl} J^s_j - R^s{}_{jkl} J^i_s$$

Using the block symmetry $R_{abcd} = R_{cdab}$ of the Riemann tensor gives the second equality. Consequently $R^i{}_{jkl} = J^i_p R^p{}_{qrs} J^q_j J^r_k J^s_l$ and contraction of i with k shows that the Ricci tensor is Hermitian, concluding the proof.

Definition 2 (J-planar curves) *A regular curve $\gamma : I \to \mathcal{M}$ is called* J-planar *if there exist functions $\alpha, \beta : I \to \mathbb{R}$ such that*

$$\nabla_{\dot\gamma} \dot\gamma = \alpha(t) \dot\gamma + \beta(t) J(\dot\gamma) \tag{2}$$

is fulfilled on I where $\dot\gamma$ denotes the tangent vector to γ.

This is a natural generalization of geodesics on (pseudo)-Riemannian manifolds that in arbitrary parametrization are solutions of the equation $\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma$. Similarly, the property of a curve to be J-planar survives under reparametrization.

Definition 3 (C-projective equivalence) *Let $g, \tilde{g}$ be two Kähler metrics (of arbitrary signature) on $(\mathcal{M}, J)$ with* $\dim_{\mathbb{R}} \mathcal{M} \geq 4$ *They are called* c-projectively equivalent *if and only if every J-planar curve of g is also a J-planar curve of $\tilde{g}$.*

It is clear that if all J-planar curves of g are J-planar curves of $\tilde{g}$ then every J-planar curve of $\tilde{g}$ is also a J-planar curve of g, justifying the word equivalence in the definition. C-projective equivalence is the Kähler analogue of *projective equivalence* on (pseudo-) Riemannian manifolds and was proposed by T. Otsuki and Y. Tashiro [14]. For a thorough introduction to c-projective geometry see [4]. For completeness, we recall the definition of *projective equivalence*, since the theorem 12 is the projective analogue of theorem 11. Because their proofs run in parallel, all statements about the projective setting will be phrased as remarks placed after their c-projective counterparts.

Definition 4 (Projective equivalence) *Let $g, \tilde{g}$ be two (pseudo-)Riemannian metrics on a manifold $\mathcal{M}$. They are called* projectively equivalent *if and only if every unparametrized geodesic of g is also an unparametrized geodesic of $\tilde{g}$.*

In the definition of c-projective equivalence we made two restrictions:
Firstly we imposed that the real dimension should be at least 4. We excluded the case of dimension $2n = 2$, because in two dimensions any curve is J-planar, so in that case c-projective equivalence would be a tautological property of any two Kähler structures and there is no information that can be extracted from this and it is not interesting to us.
Secondly we imposed that the two metrics shall be Kähler with respect to the same complex structure J. This is actually not a restriction: While in dimension $2n = 2$ there exist Kähler metrics $(g, \tilde{g})$ with distinct complex structures $(J, \tilde{J})$ and they trivially have the same J-planar or $\tilde{J}$-planar curves this cannot occur in higher dimensions, as the following proposition shows:

Proposition 2 *Let (g, J) and $(\tilde{g}, \tilde{J})$ be two Kähler structures on $\mathcal{M}$ of real dimension $2n \geq 4$. Then if every J-planar curve of g is a $\tilde{J}$-planar curve of $\tilde{g}$ and vice versa then $\tilde{J} = \pm J$. The signature of $g, \tilde{g}$ may be arbitrary.*

Proof of proposition 2: From the definition 2 it is clear that any J-planar curve γ is uniquely determined by a starting point $\gamma(0) = p$, a tangent vector at this point $\dot{\gamma}(0) = X$ and functions $\alpha(t), \beta(t)$. At the same time any J-planar curve that goes through p may be reparametrized such that the tangent vector at p is rescaled by an arbitrary non-zero factor. We now choose an arbitrary point p, an arbitrary non-zero vector X and arbitrary functions $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t)$. We consider the two J-planar curves γ_1, γ_2 of g, given by the data

$$\begin{aligned} \gamma_1 : \quad & \gamma_1(0) = p, \quad \dot{\gamma}_1(0) = X, \quad \alpha_1(t), \quad \beta_1(t) \\ \gamma_2 : \quad & \gamma_2(0) = p, \quad \dot{\gamma}_2(0) = X, \quad \alpha_2(t), \quad \beta_2(t) \end{aligned}$$

Then by our assumption there exist functions $\tilde{\alpha}_1(t), \tilde{\alpha}_2(t), \tilde{\beta}_1(t), \tilde{\beta}_2(t)$ such that

$$\begin{aligned}
\ddot{\gamma}_1^i + \Gamma^i_{jk}\dot{\gamma}_1^j\dot{\gamma}_1^k =&\alpha_1\dot{\gamma}_1^i + \beta_1 J^i_k\dot{\gamma}_1^k && (3)\\
\ddot{\gamma}_1^i + \tilde{\Gamma}^i_{jk}\dot{\gamma}_1^j\dot{\gamma}_1^k =&\tilde{\alpha}_1\dot{\gamma}_1^i + \tilde{\beta}_1 \tilde{J}^i_k\dot{\gamma}_1^k && (4)\\
\ddot{\gamma}_2^i + \Gamma^i_{jk}\dot{\gamma}_2^j\dot{\gamma}_2^k =&\alpha_2\dot{\gamma}_2^i + \beta_2 J^i_k\dot{\gamma}_2^k && (5)\\
\ddot{\gamma}_2^i + \tilde{\Gamma}^i_{jk}\dot{\gamma}_2^j\dot{\gamma}_2^k =&\tilde{\alpha}_2\dot{\gamma}_2^i + \tilde{\beta}_2 \tilde{J}^i_k\dot{\gamma}_2^k && (6)
\end{aligned}$$

Subtracting (4) from (3) and (6) from (5) we get

$$\begin{aligned}
(\Gamma - \tilde{\Gamma})^i_{jk}\dot{\gamma}_1^j\dot{\gamma}_1^k =&(\alpha_1 - \tilde{\alpha}_1)\dot{\gamma}_1^i + \beta_1 J^i_k\dot{\gamma}_1^k - \tilde{\beta}_1\tilde{J}^i_k\dot{\gamma}_1^k && (7)\\
(\Gamma - \tilde{\Gamma})^i_{jk}\dot{\gamma}_2^j\dot{\gamma}_2^k =&(\alpha_2 - \tilde{\alpha}_2)\dot{\gamma}_2^i + \beta_2 J^i_k\dot{\gamma}_2^k - \tilde{\beta}_2\tilde{J}^i_k\dot{\gamma}_2^k && (8)
\end{aligned}$$

We evaluate these equations at the parameter $t = 0$. Because $\gamma_1(0) = \gamma_2(0) = p$, equations (7) and (8) are equations on the tangent space at p and we can subtract (8) from (7). Because we assumed that $\dot{\gamma}_1(0) = \dot{\gamma}_2(0) = X$ the left hand side becomes zero and we get:

$$0 = (\alpha_1 - \tilde{\alpha}_1 - \alpha_2 + \tilde{\alpha}_2)_{|t=0}X + (\beta_1 - \beta_2)_{|t=0}J_{|p}X - (\tilde{\beta}_1 - \tilde{\beta}_2)_{|t=0}\tilde{J}_{|p}X \qquad (9)$$

This equation tells us that $\tilde{J}_{|p}X$ lies in the span of $\{X, J_{|p}X\}$. But X could be chosen arbitrarily, so for any tangent vector Z at p its image under $\tilde{J}_{|p}$ lies in the span of $\{Z, J_{|p}Z\}$.
We now choose a basis $(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ of $T_p\mathcal{M}$, such that for $i = 1, \ldots, n$ we have $JX_i = Y_i$. Then because of the above there exist coefficients $(\zeta_i, \eta_i, \varkappa_i, \lambda_i | i = 1, \ldots, n)$ such that for $i = 1, \ldots, n$:

$$\begin{aligned}
\tilde{J}_{|p}X_i =&\zeta_i X_i + \eta_i Y_i\\
\tilde{J}_{|p}Y_i =&\varkappa_i Y_i - \lambda_i X_i
\end{aligned} \qquad (10)$$

Consider an arbitrary tangent vector Z at p and decompose it according to the chosen basis:

$$Z = \sum_{i=1}^{n}(\mu_i X_i + \nu_i Y_i)$$

We shall denote by μ the tuple $(\mu_1, \ldots, \mu_n)$, likewise $\nu = (\nu_1, \ldots, \nu_n)$. From the discussion above we know that the image of Z under $\tilde{J}_{|p}$ lies in the span of $\{Z, J_{|p}Z\}$. Thus there must be $\varphi(\mu, \nu), \psi(\mu, \nu)$ such that

$$\tilde{J}_{|p}Z = \varphi(\mu, \nu)Z + \psi(\mu, \nu)J_{|p}Z \qquad (11)$$

Because we chose our basis such that $J_{|p}X_i = Y_i$ we have $J_{|p}Y_i = -X_i$ and thus we have:

$$\begin{aligned}
\tilde{J}_{|p}Z =&\sum_{i=1}^{n}(\varphi(\mu,\nu)\mu_i X_i + \varphi(\mu,\nu)\nu_i Y_i + \psi(\mu,\nu)\mu_i J_{|p}X_i + \psi(\mu,\nu)\nu_i J_{|p}Y_i)\\
=&\sum_{i=1}^{n}((\varphi(\mu,\nu)\mu_i - \psi(\mu,\nu)\nu_i)X_i + (\varphi(\mu,\nu)\nu_i + \psi(\mu,\nu)\mu_i)Y_i)
\end{aligned} \qquad (12)$$

But by linearity of $\tilde{J}_{|p}$ we also have:

$$\begin{aligned}\tilde{J}_{|p}Z &= \sum_{i=1}^{n}(\mu_i\tilde{J}_{|p}X_i + \nu_i\tilde{J}_{|p}Y_i) \\ &= \sum_{i=1}^{n}(\mu_i\zeta_iX_i + \mu_i\eta_iY_i + \nu_i\varkappa_iY_i - \nu_i\lambda_iX_i) \\ &= \sum_{i=1}^{n}((\mu_i\zeta_i - \nu_i\lambda_i)X_i + (\mu_i\eta_i + \nu_i\varkappa_i)Y_i)\end{aligned} \tag{13}$$

Equating the last lines of (12) and (13) gives:

$$\begin{aligned}&\sum_{i=1}^{n}((\varphi(\mu,\nu)\mu_i - \psi(\mu,\nu)\nu_i)X_i + (\varphi(\mu,\nu)\nu_i + \psi(\mu,\nu)\mu_i)Y_i) \\ &\qquad = \sum_{i=1}^{n}((\mu_i\zeta_i - \nu_i\lambda_i)X_i + (\mu_i\eta_i + \nu_i\varkappa_i)Y_i)\end{aligned} \tag{14}$$

Because $(X_i, Y_i|i = 1, \ldots, n)$ is a basis and thus linearly independent we must have that:

$$\left.\begin{aligned}\varphi(\mu,\nu)\mu_i - \psi(\mu,\nu)\nu_i &= \mu_i\zeta_i - \nu_i\lambda_i \\ \varphi(\mu,\nu)\nu_i + \psi(\mu,\nu)\mu_i &= \mu_i\eta_i + \nu_i\varkappa_i\end{aligned}\right\} \forall i = 1, \ldots, n \tag{15}$$

We look at these equations for particular choices of (μ, ν): those that have only one or two entries equal to one and all others are zero. The positions where the ones are will be put in the upper indices

$$\begin{aligned}\mu_i^{i_1} &= \begin{cases} 1 & i = i_1 \\ 0 & \text{else}\end{cases} \\ \nu_i^{i_1} &= \begin{cases} 1 & i = i_1 \\ 0 & \text{else}\end{cases} \\ \mu_i^{i_1,i_2} &= \begin{cases} 1 & i = i_1 \text{ or } i = i_2 \\ 0 & \text{else}\end{cases} \\ \nu_i^{i_1,i_2} &= \begin{cases} 1 & i = i_1 \text{ or } i = i_2 \\ 0 & \text{else}\end{cases}\end{aligned}$$

By μ^{i_1} we mean the collection $(\mu_1^{i_1}, \ldots, \mu_n^{i_1})$ etc.
Now consider equations (15) for $\mu = \mu^{i_1,i_2}, \nu = 0$. Then the only nontrivial equations are those for $i = i_1$ and $i = i_2$:

$$\begin{aligned}\varphi(\mu^{i_1,i_2}, 0) &= \zeta_{i_1} \\ \psi(\mu^{i_1,i_2}, 0) &= \eta_{i_1} \\ \varphi(\mu^{i_1,i_2}, 0) &= \zeta_{i_2} \\ \psi(\mu^{i_1,i_2}, 0) &= \eta_{i_2}\end{aligned}$$

But this implies that for all $i_1, i_2 \in \{1, \ldots, n\}$ the constants ζ_{i_1} and ζ_{i_2} must be equal. We also see that for any i_1, i_2 we have $\eta_{i_1} = \eta_{i_2}$.
Next we consider equations (15) for $(\mu = 0, \nu = \nu^{i_1,i_2})$. In the same way as before we can conclude that for all $i_1, i_2 \in \{1, \ldots, n\}$ the equalities $\lambda_{i_1} = \lambda_{i_2}$ and $\varkappa_{i_1} = \varkappa_{i_2}$ must hold.
Then we look at (15) with $\mu = \mu^{i_1}, \nu = \nu^{i_2}$ with $i_1 \neq i_2$. This is the moment where we use that the dimension of $\mathcal{M}$ is at least 4.

$$
\begin{aligned}
\varphi(\mu^{i_1}, \nu^{i_2}) =& \zeta_{i_1} \\
\varphi(\mu^{i_1}, \nu^{i_2}) =& \varkappa_{i_2} \\
\psi(\mu^{i_1}, \nu^{i_2}) =& \lambda_{i_2} \\
\psi(\mu^{i_1}, \nu^{i_2}) =& \eta_{i_1}
\end{aligned}
$$

Thus each of the ζ_i is equal to one of the $\varkappa_j$. But the $\varkappa_j$ are all the same. Thus $\zeta_1 = \ldots = \zeta_n = \ldots = \varkappa_1 = \ldots = \varkappa_n$.
In the same fashion we get that $\lambda_1 = \ldots = \lambda_n = \eta_1 = \ldots = \eta_n$. Inserting this in equation (10) and using $J^2_{|p} = -Id_{|p}$ we get

$$
\begin{aligned}
\tilde{J}_{|p} X_i =& \zeta_1 X_i + \eta_1 J_{|p} X_i \\
\tilde{J}_{|p} Y_i =& \zeta_1 Y_i + \eta_1 J_{|p} Y_i
\end{aligned}
$$

Because a linear operator is uniquely determined by its action on the basis vectors we must have that

$$
\tilde{J}_{|p} = \zeta_1 Id_{|p} + \eta_1 J_{|p}
$$

Because $\tilde{J}^2_{|p} = -Id_{|p}$ it followst that

$$
-Id_{|p} = (\zeta_1^2 - \eta_1^2) Id_{|p} + 2\zeta_1 \eta_1 J_{|p}
$$

Since $Id_{|p}$ and $J_{|p}$ are linearly independent the product $\zeta_1 \eta_1$ must be zero. But working in real coordinates, η_1 cannot be zero because otherwise $(\zeta_1^2 - \eta_1^2) \geq 0$ and the equation cannot hold. Thus $\zeta_1 = 0$. Inserting this into the equation we get that $\eta_1^2 = 1$. Thus $\tilde{J}_{|p} = \pm J_{|p}$.
As the point p was chosen arbitrarily, the equation must be true on all of $\mathcal{M}$. It follows from the smoothness of $\tilde{J}$ that the sign is the same at every point and the proof is complete.

1.1.1 The tensor A

Proposition 3 *[6], see also section 5 of [4]. Two Kähler metrics (of arbitrary signature) on a manifold $(\mathcal{M}, J)$ are c-projectively equivalent if and only if the tensor*

$$
A^i_j \stackrel{\text{def}}{=} \left| \frac{\det \tilde{g}}{\det g} \right|^{\frac{1}{2(n+1)}} \tilde{g}^{il} g_{lj}, \qquad \textit{where} \qquad \tilde{g}^{il} \tilde{g}_{lm} = \delta^i_m \tag{16}
$$

satisfies the equation

$$\nabla_k A_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \bar{\lambda}_i \Omega_{jk} + \bar{\lambda}_j \Omega_{ik} \tag{17}$$

where

$$\lambda \stackrel{\text{def}}{=} \frac{1}{4} \operatorname{tr} A, \qquad \lambda_i \stackrel{\text{def}}{=} \nabla_i \lambda \qquad \textit{and} \qquad \bar{\lambda}_i \stackrel{\text{def}}{=} J_i^j \lambda_j \tag{18}$$

Raising and lowering indices is always by means of g: $\lambda^i = g^{ij}\lambda_j$, where $g^{is}g_{sj} = \delta^i_j$. The sole exception to this is when we speak about a covariant *(c-)projectively equivalent metric* $\tilde{g}$, for which we will define the symbol with upper indices via $\tilde{g}^{is}\tilde{g}_{sj} = \delta^i_j$.

Corollary 3.1 *Let A be a nondegenerate Hermitian solution of* (17) *then we can construct a metric $\tilde{g}$ that is* c-projectively equivalent *to g via $\tilde{g}^{ij} = A^i_s g^{sj}\sqrt{\det A}$. By $\sqrt{\det A}$ we mean taking the eigenvalues of A and multiplying them with powers equal to half of their algebraic mulitplicities(Because $J^2 = -Id$ and A commutes with J all multiplicities are even).*

Remark 1 (for the projective case) *[2, theorem 2] Two (pseudo-)Riemannian metrics $g, \tilde{g}$ on a manifold $\mathcal{M}$ are projectively equivalent if and only if the tensor*

$$A^i_j \stackrel{\text{def}}{=} \left|\frac{\det \tilde{g}}{\det g}\right|^{\frac{1}{n+1}} \tilde{g}^{il} g_{lj}, \qquad \textit{where} \qquad \tilde{g}^{il}\tilde{g}_{lm} = \delta^i_m \tag{19}$$

satisfies the equation

$$\nabla_k A_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \qquad \textit{with} \qquad \lambda \stackrel{\text{def}}{=} \frac{1}{2} \operatorname{tr} A \qquad \lambda_i \stackrel{\text{def}}{=} \nabla_i \lambda \tag{20}$$

If A is a nondegenerate g-self-adjoint solution of (20) *then a metric that is projectively equivalent to g can be constructed via the formula $\tilde{g}^{ij} = A^i_s g^{sj} \det A$*

Definition 5 (C-compatibility) *We shall call Hermitian (g-self-adjoint and J-commuting) solutions of* (17) *and* (18) c-projectively compatible *or simply* c-compatible *with (g, J). As in the definition of c-projective equivalence we shall require that the real dimension of the manifold be at least 4. Likewise symmetric solutions of* (20) *shall be called* projectively compatible *or simply* compatible *with g.*

Remark 2 *It is also possible to define c-compatibility in the following sense: a tensor A is called c-compatible with (g, J) if there exists a covector λ_i, such that the triplet $(A, \lambda_i, \bar{\lambda}_i = J^j_i \lambda_i)$ satisfies* (17). *This may seem like a less restrictive definition but taking the trace of* (17) *shows that $\lambda_i = \frac{1}{4}\nabla_i \operatorname{tr} A$, making this definition equivalent to the one given above.*
Similarly one could call a tensor A projectively compatible, if there exists a covector field λ_i, such that $\nabla_k A_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}$ is satisfied. But taking the trace of this equation reveals that then $\lambda_i = \frac{1}{2}\nabla_i \operatorname{tr} A$.

Before we procede to prove proposition 3 we shall first establish some properties of Hermitian solutions of (17) that will be used at several points later on.

1.1.2 Basic properties of (c-)compatible structures

Lemma 4 *Let (g, J, A) be c-compatible. Then*

1. $\nabla_k \det(A) = 4\det(A){A^{-1}}^s_k\lambda_s$
2. *A is self-adjoint w.r.t. the Hessian of λ: $A^i_j\lambda_{i,k} = \lambda_{j,s}A^s_k$ [4, lemma 5.13]. For real numbers t the $(1,1)$-tensors $(tId - A), A^{-1}$ and $(tId - A)^{-1}$ are also self-adjoint w.r.t. $\nabla^2\lambda$ (the latter of course assuming that A and $tId - A$ are invertible).*
3. *Additionally let S be an endomorphism on the space of vector fields on $\mathcal{M}$ with the following properties: $J \circ S = S \circ J$, $A \circ S = S \circ A$, $\nabla^2\lambda(S\cdot, \cdot) = \nabla^2\lambda(\cdot, S\cdot)$ and $g(S\cdot, \cdot) = g(\cdot, S\cdot)$. Then the formula*
$$R^r{}_{ijk}A_{rl}S^{ij} - R^r{}_{ijl}A_{rk}S^{ij} = 0 \qquad (21)$$
is valid. In particular (taking $S^{ij} = g^{ij}$) A commutes with the Ricci tensor: $R^i_sA^s_k = A^i_sR^s_k$.
4. *The Hessian of λ is Hermitian: $\lambda_{i,j} = \lambda_{j,i} = J^s_j\lambda_{s,t}J^t_i$*

Here and further on, an index preceded by a comma is meant to indicate a covariant derivative. This lemma contains by no means new wisdom. Item 1 will be proven in a straightforward computation, the others can be found in [6, 3, 4]. The proofs that will be given also recyle the ideas of the cited sources. Only the proof of item 2 appears to be a new one, although fairly simple.

Corollary 4.1 *Linear algebra applied to (21) provides the formula:*
$$(tId - A)^{-1}{}^{lr}R^k{}_{jir}S^{ij} - (tId - A)^{-1}{}^{kr}R^l{}_{jir}S^{ij} = 0 \qquad (22)$$

Proof: Consider the endomorphism $R^r{}_{ijl}S^{ij}$ on the space of vector fields. Raising and lowering indices in (21) shows that it commutes with A. Consequently it also commutes with $(tId - A)$ and thus with $(tId - A)^{-1}$. Standard index manipulations using the symmetries of the curvature tensor imply the result. We will see this argument recurring in the proof of item 2 of lemma 4.

Remark 3 (for the projective case) *Let (g, A) be compatible. Then*

1. $\nabla_k \det(A) = 2\det(A){A^{-1}}^s_k\lambda_s$
2. *A is self-adjoint w.r.t. the Hessian of λ: $A^i_j\lambda_{i,k} = \lambda_{j,s}A^s_k$. The same is true for $(tId - A), A^{-1}$ and $(tId - A)^{-1}$ (the latter of course assuming that A and $tId - A$ are invertible).*
3. *Additionally let S be an endomorphism on the space of vector fields on $\mathcal{M}$ with the following properties: $J \circ S = S \circ J$, $A \circ S = S \circ A$, $\nabla^2\lambda(S\cdot, \cdot) = \nabla^2\lambda(\cdot, S\cdot)$ and $g(S\cdot, \cdot) = g(\cdot, S\cdot)$. Then the formula*
$$R^r{}_{ijk}A_{rl}S^{ij} - R^r{}_{ijl}A_{rk}S^{ij} = 0 \qquad (23)$$
is valid.

4. Equation (22) *is true in the projective case as well.*

Proof of lemma 4, item 1: We use Jacobi's formula $\mathrm{d}\det M = \mathrm{tr}(\mathrm{Ad}(M)\,\mathrm{d}\,M)$:

$$\begin{aligned}\nabla_k \det(A) =& \det(A) {A^{-1}}^q_r \nabla_k A^r_q \\ =& \det(A) {A^{-1}}^q_r \left[\lambda^r g_{qk} + \lambda_q \delta^r_k + g^{rs}\bar{\lambda}_s \Omega_{qk} + \bar{\lambda}_q g^{rs} \Omega_{sk}\right] \\ =& 4\det(A) {A^{-1}}^s_k \lambda_s\end{aligned} \tag{24}$$

Equation (17) was used to expand the covariant derivative of A and then the properties of J interacting with g were exploited to obtain this.

Remark 4 (for the projective case) *The procedure is the same in the projective case. In the second line of* (24) *the terms involving* $\bar{\lambda}$ *do not appear, resulting in a factor of* 2 *rather than* 4 *in the third line.*

Proof of lemma 4, item 2: To the author's knowledge the proof that we present here is a new one. The advantage of the proof given here is that it uses only the statements we have already established in the earlier sections of this document.
Without loss of generality we may assume that A is invertible. Otherwise we may choose ε, such that $\varepsilon Id - A$ is invertible. Then we can apply the same procedure to $\ln\det(\varepsilon Id - A)$ instead of $\ln\det A$. This will show that $(\varepsilon Id - A)^{-1}$ is self-adjoint with respect to $\nabla^2\lambda$. And thus by linear algebra $\varepsilon Id - A$ and consequentially A are also $\nabla^2\lambda$-self-adjoint.
Now to the main argument: We compute the second covariant derivative of $\ln\det A$ using (24), (17) as well as the general identity $\mathrm{d}\,A^{-1} = -A^{-1}\cdot(\mathrm{d}\,A)\cdot A^{-1}$ and the antisymmetry of J with respect to g:

$$\begin{aligned}\frac{1}{4}\nabla_k\nabla_l \ln\det A =& -\lambda_j {A^{-1}}^j_p g^{ps}\lambda_s {A^{-1}}^q_l g_{qk} - \lambda_j {A^{-1}}^j_k {A^{-1}}^q_l \lambda_q \\ & -\lambda_j {A^{-1}}^j_p g^{ps}\bar{\lambda}_s {A^{-1}}^q_l \Omega_{qk} + \lambda_j {A^{-1}}^j_p J^p_k \bar{\lambda}_q {A^{-1}}^q_l + {A^{-1}}^j_l \lambda_{j,k}\end{aligned} \tag{25}$$

The left hand side is symmetric with respect to $(k \leftrightarrow l)$. The first, second and fourth term on the right hand side are symmetric as well. The third term vanishes. Consequently the last term must be symmetric as well. By means of linear algebra, the self-adjointness with respect to $\lambda_{j,k}$ is also true for $(tId - A)$ or $(tId - A)^{-1}$. The latter of course is only true, provided that t is not chosen to be within the spectrum of A. This concludes the proof of item 2.

Remark 5 (for the projective setting) *Performing the same computations as above gives the intermediate result*

$$\frac{1}{2}\nabla_k\nabla_l \ln\det A = -\lambda_j {A^{-1}}^j_p g^{ps}\lambda_s {A^{-1}}^q_l g_{qk} - \lambda_j {A^{-1}}^j_k {A^{-1}}^q_l \lambda_q + {A^{-1}}^j_l \lambda_{j,k}$$

to which the same logic is applied as in the c-projective case.

Proof of lemma 4, item 3: In the proof we reuse and extend the ideas used in the proof of equation (12) in [10]. We inspect the second derivative of A:

$$\begin{aligned} R^r{}_{jkl}A_{ir} + R^r{}_{ikl}A_{rj} =& (\nabla_l\nabla_k - \nabla_k\nabla_l)A_{ij} \\ =& \lambda_{i,l}g_{jk} - \lambda_{i,k}g_{jl} + \lambda_{j,l}g_{ik} - \lambda_{j,k}g_{il} \\ & + \bar{\lambda}_{i,l}\Omega_{jk} - \bar{\lambda}_{i,k}\Omega_{jl} + \bar{\lambda}_{j,l}\Omega_{ik} - \bar{\lambda}_{j,k}\Omega_{il} \end{aligned} \tag{26}$$

The first equality is the Ricci identity and is true for any $(0,2)$-tensor. The second equality comes from differentiating (17). We continue by adding the equation with itself three times after performing cyclic permutations of (j,k,l). The three terms rising from the first term on the left hand side of (26) vanish due to the Bianchi identity. On the right hand side only terms involving $\bar{\lambda}$ remain:

$$\begin{aligned} R^r{}_{ikl}A_{rj} + R^r{}_{ijk}A_{rl} + R^r{}_{ilj}A_{rk} =& (+\bar{\lambda}_{i,l}\Omega_{jk} + \bar{\lambda}_{i,k}\Omega_{lj} + \bar{\lambda}_{i,j}\Omega_{kl} \\ & - \bar{\lambda}_{i,k}\Omega_{jl} - \bar{\lambda}_{i,l}\Omega_{kj} - \bar{\lambda}_{i,j}\Omega_{lk}) + (i \leftrightarrow j) \end{aligned} \tag{27}$$

$+(i \leftrightarrow j)$ means that the preceding term is to be added again, but with indices i and j interchanged. We now multiply this equation with S^{ij}. Since S is g-self-adjoint and commutes with J we have $\bar{\lambda}_{i,j}S^{ij} = 0$. Using this and using again that S commutes with J the right hand side simplifies as follows:

$$R^r{}_{ikl}A_{rj}S^{ij} + R^r{}_{ijk}A_{rl}S^{ij} + R^r{}_{ilj}A_{rk}S^{ij} = 4(S^j_k\lambda_{j,l} - S^j_l\lambda_{j,k})$$

$S \circ A$ is g-self-adjoint because S and A are g-self-adjoint and commute. As a consequence, we know that $R^r{}_{ikl}A_{rj}S^{ij}$ vanishes on the left hand side. Further utilizing the symmetry of the curvature tensor as well as the self-adjointness of S with respect to $\nabla^2\lambda$ we reach the desired result:

$$R^r{}_{ijk}A_{rl}S^{ij} - R^r{}_{ijl}A_{rk}S^{ij} = 0$$

Remark 6 (for the projective case) *To prove item 3 of remark 3 we perform the same steps as in the proof of item 3 of lemma 4. Removing the terms involving* Ω *from* (26) *gives the intermediate step for the projective case. Equation* (27) *trivially simplifies to* $R^r{}_{ikl}A_{rj} + R^r{}_{ijk}A_{rl} + R^r{}_{ilj}A_{rk} = 0$*. After multiplication with* S^{ij} *the first term vanishes with the same argument as in the c-projective case and the result is obtained. For the proof of item 4 of remark 3 we observe that the proof of corollary 4.1 only involves linear algebra. Thus the proof is exactly the same in the projective and the c-projective case.*

Proof of lemma 4, item 4: We covariantly differentiate (17) and use the Ricci identity:

$$\begin{aligned} R^r{}_{jkl}A_{ir} + R^r{}_{ikl}A_{rj} =& (\nabla_l\nabla_k - \nabla_k\nabla_l)A_{ij} \\ =& \lambda_{i,l}g_{jk} - \lambda_{i,k}g_{jl} + \lambda_{j,l}g_{ik} - \lambda_{j,k}g_{il} \\ & + \bar{\lambda}_{i,l}\Omega_{jk} - \bar{\lambda}_{i,k}\Omega_{jl} + \bar{\lambda}_{j,l}\Omega_{ik} - \bar{\lambda}_{j,k}\Omega_{il} \end{aligned}$$

Contracting with g^{jk} and rearranging terms gives

$$-R_l^r A_{ir} + R^r{}_{ikl} A_{rj} g^{jk} + g^{jk} \lambda_{j,k} g_{il} - J_i^s \lambda_{s,t} J_l^t = (2n-1)\lambda_{i,l}$$

We subtract $\lambda_{i,l}$ on both sides:

$$-R_l^r A_{ir} + R^r{}_{ikl} A_{rj} g^{jk} + g^{jk} \lambda_{j,k} g_{il} - \lambda_{i,l} - J_i^s \lambda_{s,t} J_l^t = (2n-2)\lambda_{i,l}$$

Because A commutes with J and R (see item 3 with S^{ij} and $S^{ij} = A^{ij}$) and because of the way J interacts with g and the Riemann and Ricci tensor (1) we see that the left hand side is Hermitian. Thus $\lambda_{i,j}$ must be Hermitian as well (remember we excluded dimension two in the definition of c-projective equivalence, so $(2n-2)$ is non-zero), concluding the proof.

Proof of proposition 3 The idea of this proof is based on the ideas in section 40 of [8]. We first show that if two metrics $g, \tilde{g}$ are c-projectively equivalent then A formed as in (16) satisfies (17). We start off with the equation for J-planar curves:

$$\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k = \alpha(t)\dot{\gamma}^i + \beta(t) J^i_k \dot{\gamma}^k$$

We multiply with $\dot{\gamma}^l$, then alternate $(l \leftrightarrow i)$ and subtract both equations:

$$\dot{\gamma}^l\ddot{\gamma}^i - \dot{\gamma}^i\ddot{\gamma}^l + \left(\Gamma^i_{jk}\dot{\gamma}^l - \Gamma^l_{jk}\dot{\gamma}^i\right)\dot{\gamma}^j\dot{\gamma}^k = \beta\left(\dot{\gamma}^l J^i_k\dot{\gamma}^k - \dot{\gamma}^i J^l_k \dot{\gamma}^k\right) \tag{28}$$

Because we assumed that $g, \tilde{g}$ are c-projectively equivalent, for any curve γ there exist $\tilde{\alpha}, \tilde{\beta}$ such that the same operations can be performed, putting a tilde on Γ, α, β without compromising the validity of the equations. Taking the "$\sim$" analogue of (28) and subtracting it from (28) leaves us with

$$P^i_{jk}\dot{\gamma}^l\dot{\gamma}^j\dot{\gamma}^k - P^l_{jk}\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k = (\beta - \tilde{\beta})\dot{\gamma}^l J^i_k\dot{\gamma}^k - (\beta - \tilde{\beta})\dot{\gamma}^i J^l_k\dot{\gamma}^k \tag{29}$$

Here we have introduced P^i_{jk} as a shorthand for $\Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}$.
At the same time we can multiply (28) with $J^m_l J^n_i$:

$$\begin{aligned} J^m_l\dot{\gamma}^l\ddot{\gamma}^i J^n_i - J^m_l\ddot{\gamma}^l\dot{\gamma}^i J^n_i + J^m_l\dot{\gamma}^l\Gamma^i_{jk}J^n_i\dot{\gamma}^j\dot{\gamma}^k - J^m_l\Gamma^l_{jk}\dot{\gamma}^i J^n_i\dot{\gamma}^j\dot{\gamma}^k& \\ = \beta(-J^m_l\dot{\gamma}^l\dot{\gamma}^n + J^n_i\dot{\gamma}^i\dot{\gamma}^m)& \end{aligned} \tag{30}$$

From this we subtract its "$\sim$" equivalent. After renaming some indices we obtain

$$J^i_n P^n_{jk}\dot{\gamma}^j\dot{\gamma}^k J^l_m\dot{\gamma}^m - J^l_n P^n_{jk}\dot{\gamma}^j\dot{\gamma}^k J^i_m\dot{\gamma}^m = (\beta - \tilde{\beta})J^i_m\dot{\gamma}^m\dot{\gamma}^l - (\beta - \tilde{\beta})J^l_m\dot{\gamma}^m\dot{\gamma}^i \tag{31}$$

from (28). Subtracting (31) from (29) gives:

$$\dot{\gamma}^s\dot{\gamma}^j\dot{\gamma}^k\left(\delta^l_s P^i_{jk} - \delta^i_s P^l_{jk} - J^i_n P^n_{jk} J^l_s + J^l_n P^n_{jk} J^i_s\right) = 0 \tag{32}$$

This equation is valid for any J-planar curve γ and thus $\dot{\gamma}$ may be considered to be arbitrary. This implies that the symmetric part of the expression in the parentheses must vanish. Because the expression in the parentheses is symmetric when exchanging $(j \leftrightarrow k)$ this is equivalent to the following condition:

$$\begin{aligned} &\delta^l_s P^i_{jk} - \delta^i_s P^l_{jk} - J^i_m P^n_{jk} J^l_s + J^l_n P^n_{jk} J^i_s \\ +&\delta^l_k P^i_{sj} - \delta^i_k P^l_{sj} - J^i_m P^n_{sj} J^l_k + J^l_n P^n_{sj} J^i_k \\ +&\delta^l_j P^i_{ks} - \delta^i_j P^l_{ks} - J^i_m P^n_{ks} J^l_j + J^l_n P^n_{ks} J^i_j = 0 \end{aligned} \tag{33}$$

i.e. we took the expresion in parentheses, took cyclic permutations of (s, j, k) and added them together. Contracting l with k and using that P is symmetric in the lower indices it follows that

$$0 = 2(n+1)P^i_{js} - \delta^i_s P^k_{jk} - \delta^i_j P^k_{ks} + J^i_s J^m_j P^k_{km} + J^i_j J^m_s P^k_{km}$$

Because $P^i_{jk} = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}$ and $\Gamma^k_{kl} = 1/2\partial_l \ln|\det g|, \tilde{\Gamma}^k_{kl} = 1/2\partial_l \ln|\det \tilde{g}|$ we have

$$P^i_{js} = \delta^j_s \varphi_j + \delta^i_j \varphi_s - J^i_s J^m_j \varphi_m - J^i_j J^m_s \varphi_m \tag{34}$$

where we have introduced

$$\varphi = \frac{1}{4(n+1)} \ln \left| \frac{\det g}{\det \tilde{g}} \right|, \qquad \varphi_s = \nabla_s \varphi \tag{35}$$

We can then use equation (34) to work out $\nabla_j \tilde{g}^{il}$ in terms of $\tilde{g}$ and the derivatives of φ via

$$0 = \tilde{\nabla}_j \tilde{g}^{il} = \nabla_j \tilde{g}^{il} - P^i_{js} \tilde{g}^{sl} - P^l_{js} \tilde{g}^{is}$$

and use the result to compute

$$\begin{aligned} \nabla_j e^{-2\varphi} \tilde{g}^{il} g_{lp} = \delta^i_j e^{-2\varphi} \varphi_s \tilde{g}^{sl} g_{lp} + e^{-2\varphi} \varphi_s \tilde{g}^{is} g_{jp} \\ - e^{-2\varphi} J^i_j J^m_s \varphi_m \tilde{g}^{sl} g_{lp} - e^{-2\varphi} J^l_j g_{lp} \tilde{g}^{is} J^m_s \varphi_m \end{aligned}$$

We see that the left hand side is just $\nabla_j A^i_p$. So by contracting i with p in this equation and defining $\lambda = 1/4 \operatorname{tr} A$ and $\lambda_j = \nabla_j \lambda$ we get $\lambda_j = e^{-2\varphi} \varphi_s \tilde{g}^{sl} g_{lj}$. Reinserting this into the equation above, introducing the shorthand $\bar{\lambda}_j = J^s_j \lambda_s$ and renaming indices gives

$$\nabla_k A_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \bar{\lambda}_i \Omega_{jk} + \bar{\lambda}_j \Omega_{ik}$$

as we have claimed.

Now we show the other direction: we assume that two metrics $g, \tilde{g}$ are given, such that A formed with (16) fulfills (17) and then show that $g, \tilde{g}$ are c-projectively equivalent. The computation is straightforward, starting with the well known formula for the Christoffel symbols of $\tilde{g}$:

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2} \tilde{g}^{il} (\partial_j \tilde{g}_{lk} + \partial_k \tilde{g}_{jl} - \partial_l \tilde{g}_{jk})$$

We express the partial derivatives of $\tilde{g}$ in terms of the covariant derivatives of the Levi-Civita connection of g: $\partial_j \tilde{g}_{jk} = \nabla_j \tilde{g}_{jk} + \Gamma^s_{jl}\tilde{g}_{sk} + \Gamma^s_{jk}\tilde{g}_{ls}$. The result simplifies to

$$\Gamma^i_{jk} - \tilde{\Gamma}^i_{jk} = -\frac{1}{2}\tilde{g}^{il}(\nabla_j \tilde{g}_{lk} + \nabla_k \tilde{g}_{jl} - \nabla_l \tilde{g}_{jk})$$

We can express $\tilde{g}$ on the right hand side via $\tilde{g}^{ij} = A^i_s g^{sj}\sqrt{\det A}$ and use the properies (17) and (24) of A as well as the general formula $\nabla A^{-1} = A^{-1}\cdot(\nabla A)\cdot A^{-1}$. Then we use that A is Hermitian to reduce the terms to

$$\Gamma^i_{jk} - \tilde{\Gamma}^i_{jk} = \delta^i_k (A^{-1})^\alpha_j \lambda_\alpha + \delta^i_j (A^{-1})^\alpha_k \lambda_\alpha - (A^{-1})^\alpha_j \bar{\lambda}_\alpha J^i_k - (A^{-1})^\alpha_k \bar{\lambda}_\alpha J^i_j$$

We then use the fact that A commutes with J, the previously established formula $(A^{-1})^\alpha_j \lambda_\alpha = 1/4\partial_j \ \ln\det A$ and that $\det A = |\det\tilde{g}/\det g|^{-1/(n+1)}$. In terms of the quantity $\varphi = 1/(4(n+1))\ln(\det g/\det\tilde{g})$ the above becomes

$$\Gamma^i_{jk} - \tilde{\Gamma}^i_{jk} = \delta^i_k \partial_j\varphi + \delta^i_j \partial_k\varphi - J^i_j J^s_k \partial_s\varphi - J^i_k J^s_j \partial_s\varphi$$

With this formula we show that all J-planar curves of g are also J-planar curves of $\tilde{g}$. Suppose γ satisfies

$$\ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\ddot{\gamma}^k = \alpha(t)\ddot{\gamma}^i + \beta(t)J^i_k\dot{\gamma}^k$$

for some functions $\alpha(t), \beta(t)$. We can now compute $\ddot{\gamma}^i + \tilde{\Gamma}^i_{jk}\dot{\gamma}^j\dot{\gamma}^k$:

$$\begin{aligned}
\ddot{\gamma}^i + \tilde{\Gamma}^i_{jk}\dot{\gamma}^j\dot{\gamma}^k =& \ddot{\gamma}^i + \Gamma^i_{jk}\dot{\gamma}^j\dot{\gamma}^k - (\Gamma^i_{jk} - \tilde{\Gamma}^i_{jk})\dot{\gamma}^j\dot{\gamma}^k \\
=& \alpha(t)\dot{\gamma}^i + \beta(t)J^i_s\dot{\gamma}^s - 2\dot{\gamma}^i\dot{\gamma}^j\partial_j\varphi - 2J^i_j\dot{\gamma}^j\dot{\gamma}^k J^s_k\partial_s\varphi
\end{aligned}$$

Thus γ satisfies $\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = \tilde{\alpha}(t)\dot{\gamma} + \tilde{\beta}(t)J(\dot{\gamma})$ for $\tilde{\alpha}(t) = \alpha(t) - 2\dot{\gamma}^j\partial_j\varphi$ and $\tilde{\beta}(t) = \beta(t) - 2\dot{\gamma}^k J^s_k\partial_s\varphi$ and γ is also a J-planar curve of $\tilde{g}$. The fact that any J-planar curve of $\tilde{g}$ is also a J-planar curve of g is obvious and proposition 3 is proven.

1.1.3 Conserved quantities of the geodesic flow

Throughout this work we shall canonically identify symmetric covariant tensors with polynomials on $T^*\mathcal{M}$ via the isomorphisms $^\flat$ and $^\sharp$:

$$\begin{aligned}
\flat :& T^{a_1\dots a_l} \mapsto T^{a_1\dots a_l}p_{a_1}\dots p_{a_l} \\
\sharp :& T^{a_1\dots a_l}p_{a_1}\dots p_{a_l} \mapsto T^{(a_1\dots a_l)}
\end{aligned}$$

By the parentheses we mean symmetrisation with the appropriate combinatorial factor: $T^{(a_1\dots a_l)} = 1/l!\sum_{(b_1,\dots,b_l)=\pi(a_1,\dots,a_l)} T^{b_1\dots b_l}$ (π means permutation here).
Let (g, J, A) be c-compatible on $\mathcal{M}$ and consider the one-parameter family

$$\overset{t}{K}{}^{ij} \overset{\text{def}}{=} \sqrt{\det(tId - A)}\ (tId - A)^{-1}{}^i_l g^{lj} \tag{36}$$

Throughout this work the root is to be taken in such a way that we simply halve the powers of the eigenvalues in $\det(tId - A)$. This is well defined because A is g-self-adjoint and commutes with J and thus all eigenvalues of A are of even multiplicity.

In particular $\sqrt{\det(tId - A)}$ can be negative and it is smooth also near points where $\det(tId - A) = 0$. With the tensors $\overset{t}{K}$ we associate the functions $\overset{t}{I}\colon T^*\mathcal{M} \to \mathbb{R}$

$$\overset{t}{I} \overset{\text{def}}{=} \overset{t}{K}{}^{ij} p_i p_j \tag{37}$$

Theorem 5 *P. Topalov[17], see also [4, §5]. Let (g, J, A) be c-compatible. Then for any pair of real numbers (v, w) the quantities $\overset{v}{I}$ and $\overset{w}{I}$ are Poisson commuting integrals of the geodesic flow i.e. $\{\overset{v}{I}, \overset{w}{I}\} = 0$ and $\{\overset{t}{I}, g^{ij} p_i p_j\} = 0$*

The tensors $\overset{t}{K}$ defined above are called the *canonical Killing tensors* for the c-compatible structure (g, J, A). Killing tensors and integrals of the geodesic flow that are homogeneous in momenta are in one to one correspondence.

Remark 7 *The quantities $\overset{t}{K}$ are well-defined for all values of t: if we denote by $2n$ the dimension of the manifold, then $\overset{t}{K}$ is a polynomial of degree $n-1$. This is a consequence of $J^2 = -Id$, the antisymmetry of J with respect to g, the commutativity of A with J and the construction of $\overset{t}{K}$ from A.*

Remark 8 (for the projective case) *Let (g, A) be compatible. We shall define $\overset{t}{K}{}^{ij} \overset{\text{def}}{=} \det(tId - A)\ (tId - A)^{-1}{}^i_l g^{lj}$ for the projective case. Then for any pair $s, t \in \mathbb{R}$ the quantities $\overset{s}{I} \overset{\text{def}}{=} \overset{s}{K}{}^{ij} p_i p_j$ and $\overset{t}{I} \overset{\text{def}}{=} \overset{t}{K}{}^{ij} p_i p_j$ are commuting integrals of the geodesic flow for g [2, 13]. The projective and the c-projective case differ merely by the power of the determinant.*

Theorem 6 *See [4, theorem 5.11] and [3, lemma 2.2]. Let (g, J, A) be c-compatible. Then the quantities $\overset{t}{L}$ given by*

$$\overset{t}{L} = \overset{t}{V}{}^j p_j, \qquad \overset{t}{V}{}^j = J^j_k g^{ki} \nabla_i \sqrt{\det(tId - A)} \tag{38}$$

form a one-parameter family of commuting integrals of the geodesic flow: for all values of $s, t \in \mathbb{R}$ the following relation is true:

$$\{\overset{s}{L}, \overset{t}{L}\} = 0 \qquad \textit{and} \qquad \{\overset{s}{L}, g^{ij} p_i p_j\} = 0 \tag{39}$$

The vector fields $\overset{t}{V}$ are called the *canonical Killing vector fields* for the c-compatible structure. Killing vector fields and integrals of the geodesic flow that are linear in momenta are in one to one correspondence.

Proposition 7 *Let (g, J, A) be c-compatible. Then for all values of $s, t \in \mathbb{R}$ the integrals $\overset{t}{I}$ and $\overset{t}{L}$ of the geodesic flow (see (36), (37) and (38) for the definition) Poisson commute:*

$$\{\overset{t}{L}, \overset{s}{I}\} = 0 \tag{40}$$

Corollary 7.1

- *Let (g, J, A) be c-compatible. Then the flow of the canonical Killing vector fields $\overset{t}{V}$ in* (38) *preserves A.*
- *Let $g, \tilde{g}$ be c-projectively equivalent on $(\mathcal{M}, J)$. Then the canonical Killing vector fields* (38) *of g are also Killing vector fields for $\tilde{g}$.*

This is of course nothing new and can be found in [4, 3]. In lemma 2.2 of [3] a particularly elegant proof of the second statement of the corollary is given which then implies the first item and proposition 7.

Definition 6 (Functional independence) *Let $(f_1, \dots, f_k)$ be a set of functions on the cotangent bundle. They are called functionally independent near a point p on the cotangent bundle if their differentials are linearly independent at p. They are called functionally independent, if their differentials are linearly independent almost everywhere on the cotangent bundle.*

Theorem 8 *[4, theorem 5.18] Let (g, J, A) be c-compatible and consider the integrals of the geodesic flow $\overset{t}{I}$ and $\overset{s}{L}$ from* (36), (37) *and* (38). *Then*

1. *The number of functionally independent integrals in the family $\overset{s}{L}$ is equal to the number of non-constant Eigenvalues of A.*
2. *The number of functionally independent integrals within the family $\overset{t}{I}$ is equal to the degree of the the minimal polynomial of A.*
3. *The integrals $\overset{s}{L}$ are functionally independent of the integrals $\overset{t}{K}$, so the total number of independent integrals within the two families is equal to the number of non-constant eigenvalues of A plus the degree of the minimal polynomial of A.*

Remark 9 *Let (g, A) be compatible. Then the number of independent integrals in the family $\overset{t}{I}$ is equal to the degree of the minimal polynomial of A [16].*

Proofs

Definition 7 (Schouten-Nijenhuis bracket) *Let (x, p) denote a set of canonical coordinates on the cotangent bundle of a smooth manifold $\mathcal{M}$. Let P, Q be homogeneous polynomials in the momentum variables of degrees k, l. Then we define the Schouten-Nijenhuis bracket of their associated tensors $P^\sharp, Q^\sharp$ as*

$$[P^\sharp, Q^\sharp]_S = \{P, Q\}^\sharp \tag{41}$$

Proposition 9 *Let (x, p) denote a set of canonical coordinates on the cotangent bundle of a smooth manifold. Then two homogeneous polynomials $P = P^{i_1 \cdots i_k} p_{i_1} \cdots p_{i_k}$ and $Q = Q^{j_1 \cdots j_l} p_{j_1} \cdots p_{j_l}$ ($P^{i_1 \cdots i_k}, Q^{j_1 \cdots j_l}$ are symmetric) have a vanishing Poisson bracket* *if and only if the Schouten-Nijenhuis bracket of $P^\sharp$ and $Q^\sharp$ vanishes. The Schouten-Nijenhuis bracket can be expressed via the formula*

$$[P^\sharp, Q^\sharp]_S^{i_2\cdots i_k j_2\cdots j_l b} = -lP^{a(i_2\cdots i_k}\nabla_a Q^{j_2\cdots j_l b)} + kQ^{a(j_2\cdots j_l}\nabla_a P^{i_2\cdots i_k b)} \tag{42}$$

The parentheses indicate the symmetrization of the indices with the appropriate combinatorial factor. Proposition 9 is a well known fact, see e.g. [7]. A proof is given for completeness.

Proof of proposition 9: We compute:

$$\begin{aligned}\{P, Q\} =&(kQ^{aj_2\cdots j_l}\nabla_a P^{j_2\cdots j_k b} - lP^{aj_2\cdots j_k}\nabla_a Q^{j_2\cdots j_l b})\\ &\times p_b p_{i_2}\cdots p_{i_k} p_{j_2}\cdots p_{j_l}\\ =&(kQ^{a(j_2\cdots j_l}\nabla_a P^{j_2\cdots j_k b)} - lP^{a(j_2\cdots j_k}\nabla_a Q^{j_2\cdots j_l b)})\\ &\times p_b p_{i_2}\cdots p_{i_k} p_{j_2}\cdots p_{j_l}\end{aligned}$$

In the first step we used the definition of the Poisson bracket and the symmetry of $P^{i_1\cdots i_k}, Q^{j_1\cdots j_l}$. The Christoffel symbols all cancel out, ther is no difference in writing the first step with partial or covariant derivatives. In the second step we can symmetrize all upper indices except a because the product of the p's is obviously symmetric with respect to index permutation. Because homogeneous polynomials on $T^*\mathcal{M}$ are in one to one correspondence with symmetric contravariant tensors we see that $\{P, Q\}^\sharp$ must be equal to the term in parentheses, proving formula (42). Now because symmetric contravariant tensors and polynomials in momenta on $T^*\mathcal{M}$ are isomorphic it is clear from the definition of the Schouten-Nijenhuis bracket that $[P^\sharp, Q^\sharp]$ vanishes if and only if $\{P, Q\}$ vanishes as well. Proposition 9 is proven.

Proof of theorem 5: We will prove the theorem by direct calculation, using the relations that have already been proven on the previous pages. When looking at the proof of the main theorem 11, the reader will already be familiar with much of the technique after this proof. Without loss of generality we may prove it for the cases where s and t in $\overset{s}{K}$ and $\overset{t}{K}$ are chosen such that they are not in the spectrum of A. Otherwise we recall the fact that $\overset{s}{K}$ and $\overset{t}{K}$ are polynomials and therefore smooth. Then we can consider sequences $(s_i), (t_i), i \in \mathbb{N}$ that have elements outside the spectrum of A and converge towards the values s and t that we want to and take the limit. We introduce a shorthand notation:

$$\Lambda = \operatorname{grad}\lambda = (g^{ij}\lambda_j), \qquad \bar{\Lambda} = J\operatorname{grad}\lambda, \qquad \overset{t}{M} = (tId - A)^{-1} \tag{43}$$

for any t outside the spectrum of A. $(\overset{s}{M}\Lambda)^i$ denotes the i^{th} component of $\overset{s}{M}\Lambda$ etc. We get

$$\nabla_k \det(tId - A) = -4\det(tId - A)\ \overset{t}{M}{}^s_k\lambda_s \tag{44}$$

in the same way as we have obtained formula (24). Using this, the general matrix identity $\mathrm{d}\,C^{-1} = -C^{-1}\cdot \mathrm{d}\,C\cdot C^{-1}$ and (17), the covariant derivative of the Killing tensor $\overset{t}{K}$ evaluates to:

$$\begin{aligned}\nabla_k \overset{t}{K}{}^{jl} = & \sqrt{\det(tId - A)}[-2 \overset{t}{M}{}^s_k \lambda_s \ \overset{t}{M}{}^j_r g^{rl} \\ & + \overset{t}{M}{}^j_p \lambda^p \overset{t}{M}{}^l_k + \overset{t}{M}{}^j_k \ \overset{t}{M}{}^l_q \lambda^q - \overset{t}{M}{}^j_p g^{ps} \bar{\lambda}_s \overset{t}{M}{}^l_q J^q_k - \overset{t}{M}{}^l_r g^{rq} \bar{\lambda}_q \overset{t}{M}{}^j_p J^p_k]\end{aligned} \tag{45}$$

With this we form $\overset{s}{K}{}^{aj} \nabla_a \overset{t}{K}{}^{jb} - \overset{t}{K}{}^{aj} \nabla_a \overset{s}{K}{}^{jb}$:

$$\begin{aligned}\overset{s}{K}{}^{ai} \nabla_a \overset{t}{K}{}^{jb} - \overset{t}{K}{}^{ai} \nabla_a \overset{s}{K}{}^{jb} = & \sqrt{\det(tId - A)} \sqrt{\det(sId - A)} \\ & \times \Big[-2(\overset{s}{M} \ \overset{t}{M} \Lambda)^i \ \overset{t}{M}{}^j_r g^{rb} \\ & \quad + (\overset{t}{M} \Lambda)^j \ \overset{s}{M}{}^i_m \ \overset{t}{M}{}^m_a g^{ab} + (\overset{t}{M} \Lambda)^b \ \overset{s}{M}{}^i_m \ \overset{t}{M}{}^m_a g^{ai} \\ & \quad + (\overset{t}{M} \bar{\Lambda})^j \ \overset{s}{M}{}^i_m \ \overset{t}{M}{}^m_a g^{ac} J^b_c + (\overset{t}{M} \bar{\Lambda})^b \ \overset{s}{M}{}^i_m \ \overset{t}{M}{}^m_a g^{ac} J^j_c \Big] \\ & - (s \leftrightarrow t)\end{aligned}$$

The parentheses $-(s \leftrightarrow t)$ mean that the previous term is to be subtracted with s and t interchanged. We use the identity

$$(t - s)(sId - A)^{-1}(tId - A)^{-1}(sId - A)^{-1} - (tId - A)^{-1} \tag{46}$$

This is true for any matrix A and numbers s, t for which $(sId - A)$ and $(tId - A)$ are invertible and can be verified by elementary computation. In our notation this is

$$(t - s) \ \overset{s}{M} \ \overset{t}{M} = \overset{s}{M} - \overset{t}{M} \tag{47}$$

We apply it to each term and its $(s \leftrightarrow t)$-conjugate to combine them:

$$\begin{aligned}\overset{s}{K}{}^{ai} \nabla_a \overset{t}{K}{}^{jb} - \overset{t}{K}{}^{ai} \nabla_a \overset{s}{K}{}^{jb} = & \sqrt{\det(tId - A)} \sqrt{\det(sId - A)} (s - t) \\ & \times \Big[-2(\overset{s}{M} \ \overset{t}{M} \Lambda)^i \ \overset{t}{M}{}^j_r \ \overset{s}{M}{}^r_l g^{lb} \\ & \quad + (\overset{t}{M} \ \overset{s}{M} \Lambda)^j \ \overset{s}{M}{}^i_m \ \overset{t}{M}{}^m_a g^{ab} \\ & \quad + (\overset{t}{M} \ \overset{s}{M} \Lambda)^b \ \overset{s}{M}{}^i_m \ \overset{t}{M}{}^m_a g^{ai} \\ & \quad + (\overset{t}{M} \ \overset{s}{M} \bar{\Lambda})^j \ \overset{s}{M}{}^i_m \ \overset{t}{M}{}^m_a g^{ac} J^b_c \\ & \quad + (\overset{t}{M} \ \overset{s}{M} \bar{\Lambda})^b \ \overset{s}{M}{}^i_m \ \overset{t}{M}{}^m_a g^{ac} J^j_c \Big] \\ & - (s \leftrightarrow t)\end{aligned}$$

Now we look at what happens upon symmetrization of the free indices (i, j, b): The first three terms produce into the same summands, but with different signs and prefactors, such that they cancel out. The fourth term is antisymmetric in $(i \leftrightarrow b)$ and the fifth term is antisymmetric in $(i \leftrightarrow j)$, so they vanish when symmetrizing. Thus the

Schouten-Nijenhuis bracket of $\overset{s}{K}$ and $\overset{t}{K}$ vanishes and so does the Poisson bracket $\{\overset{s}{I},\overset{t}{I}\}$. To show that the $\overset{s}{I}$ are integrals of the geodesic flow we observe that g^{ij} is the coefficient of t^{n-1} in $\overset{t}{K}{}^{ij}$ ($2n$ is the dimension of the manifold). Thus g^{ij} is in the span of the canonical Killing tensors and must therefore have a vanishin Schouten-Nijenhuis bracket with them. This implies $\{\overset{t}{I}, g^{ij}p_ip_j\} = 0$. Theorem 5 is proven.

Proof of theorem 6: We use a direct computation, using the formulae that were proven earlier in this document. We first show that $\overset{s}{L}$ is an integral of the geodesic flow. This is equivalent to $\overset{s}{V}$ having a vanishing poisson bracktet with g, as we have established already. The Schouten-Nijenhuis bracket of $\overset{s}{V}$ with g^{ij} is simply the Lie derivative $\mathcal{L}$ of g^{ij} along the flow of $\overset{s}{V}$. If A is c-compatible then $(sId - A)$ is also c-compatible. Thus it suffices to show that $J\,\mathrm{grad}\sqrt{\det A}$ is a Killing vector field. Otherwise we can simply rename $(sId - A)$. We may further restrict ourselves to sets where A is invertible: on non-empty open sets where $\det A = 0$ there would be nothing to show. If $\det A = 0$ on a set of volume zero, then we can prove the statement everywhere else and infer it everywhere by smoothness.

We are ready to compute:

$$\begin{aligned}\mathcal{L}_{J\,\mathrm{grad}\sqrt{\det A}}g_{jk} =& g_{ij}\nabla_j \overset{s}{V}{}^i + g_{ji}\nabla_k \overset{s}{V}{}^i \\ =& -J^l_k\nabla_j\nabla_l\sqrt{\det A} - J^l_j\nabla_k\nabla_l\sqrt{\det A}\end{aligned} \tag{48}$$

Here we have used that J is parallel and antisymmetric with respect to g. From equations (24) and (17) we get

$$\begin{aligned}\nabla_k\nabla_l\sqrt{\det A} = 2\sqrt{\det A}[&{A^{-1}}^s_l\lambda_s\lambda_r{A^{-1}}^r_k - g_{ks}{A^{-1}}^s_l\lambda_a g^{ab}{A^{-1}}^c_b\lambda_c \\ &+ {A^{-1}}^s_l J^r_s\lambda_s\lambda_p{A^{-1}}^p_m J^m_k + {A^{-1}}^s_l\lambda_{s,k}]\end{aligned} \tag{49}$$

by similar means as we have obtained (25). We insert this into (48) and use the antisymmetry of J with respect to g, the Hermitian nature of A and $J^2 = -Id$. Then all terms except those with second derivatives of λ cancel out. We are left with

$$\mathcal{L}_{J\,\mathrm{grad}\sqrt{\det A}}g_{jk} = -2\sqrt{\det A}[J^l_j{A^{-1}}^s_l\lambda_{s,k} + J^l_k{A^{-1}}^s_l\lambda_{s,j}] \tag{50}$$

But J commutes with A and item 4 of lemma 4 tells us that the Hessian of λ is Hermitian. Thus both terms cancel each other and $J\,\mathrm{grad}\sqrt{\det A}$ is Killing. As explained above this immediately proves that $\overset{s}{V}$ is Killing for all values of $s \in \mathbb{R}$.

We now show that for any $s, t \in \mathbb{R}$ the relation $\{\overset{s}{L},\overset{t}{L}\} = 0$ holds. We do so by computing the Schouten-Nijenhuis bracket of $\overset{s}{V}$ and $\overset{t}{V}$, which is simply the commutator of the two vector fields (with the same argument that we made when showing that the $\overset{t}{V}$ are Killing vector fields, we may assume without loss of generality that $(sId - A)$ and $(tId - A)$ are invertible):

$$\begin{aligned}[\overset{s}{V},\overset{t}{V}]^i =& \overset{s}{V}{}^a\nabla_a \overset{t}{V}{}^i - \overset{t}{V}{}^a\nabla_a \overset{s}{V}{}^i \\ =& J^a_b g^{bc}(\nabla_c\sqrt{\det(sId - A)}\,)\nabla_a J^i_d g^{dc}\nabla_c\sqrt{\det(tId - A)} - (s \leftrightarrow t)\end{aligned} \tag{51}$$

We use the properties (44) and (17) of A and the general identity $\nabla_X A^{-1} = -A^{-1} \cdot (\nabla_X A) \cdot A^{-1}$ which is true for any $(1,1)$ tensor A and any vector field X. We sort the resulting terms and cancel out those that vanish due to the Hermitian nature of A and the antisymmetry of J w.r.t. g. The result is expressed in the previously introduced shorthand (43):

$$\begin{aligned}[\overset{s}{V}, \overset{t}{V}]^i =& 4\sqrt{\det(sId - A)}\sqrt{\det(tId - A)} \\ &\times J^i_d g^{de}[- \overset{t}{M}{}^m_e \lambda_m \lambda_n \overset{t}{M}{}^n_a + \overset{t}{M}{}^r_e g_{ra}\, g(\Lambda, \overset{t}{M}\Lambda) \\ &\quad - \overset{t}{M}{}^q_e J^m_q \lambda_m \lambda_n \overset{t}{M}{}^n_p J^p_a + \overset{t}{M}{}^m_e \lambda_{m,a}] J^a_b g^{bc} \overset{s}{M}{}^k_c \lambda_k - (s \leftrightarrow t)\end{aligned}$$

With the way that g, J, A interact with one another the result further refines to

$$\begin{aligned}[\overset{s}{V}, \overset{t}{V}]^i =& -4\sqrt{\det(sId - A)}\ \sqrt{\det(tId - A)} \\ &\times [(\overset{s}{M}\ \overset{t}{M}\Lambda)^i g(\overset{t}{M}\Lambda, \Lambda) - (\overset{t}{M}\Lambda)^i\ g(\overset{s}{M}\ \overset{t}{M}\Lambda, \Lambda) \\ &g^{id} J^e_d \overset{t}{M}{}^s_e \lambda_{s,a} J^a_b g^{bc} \overset{s}{M}{}^m_c \lambda_m] - (s \leftrightarrow t)\end{aligned} \tag{52}$$

When pairing each term on the second line with its "$-(s \leftrightarrow t)$" counterpart we can apply the matrix identity $\overset{t}{M} - \overset{s}{M} = (s-t)\ \overset{s}{M}\ \overset{t}{M}$ and the terms cancel out each other. The last term is symmetric with respect to $(s \leftrightarrow t)$ because of lemma 4, item 2. Thus it is eliminated by the $(s \leftrightarrow t)$-antisymmetrization. Theorem 6 is proven.

Proof of proposition 7: Note that both the $\overset{s}{L}$ and the $\overset{t}{I}$ are polynomials of degree $n-1$ in s, t respectively (where the dimension of the manifold is $2n$). Then $\{\overset{s}{L}, \overset{t}{I}\}$ is also a polynomial of degree $n-1$ in s and t. Let $(s_i|i = 1, \ldots, n), (t_j|j = 1, \ldots, n)$ be two tuples and let $s_i \neq s_j$ and $t_i \neq t_j$ for every $i \neq j$. Then by application of the Lagrange interpolation formula with respect to s and t it is clear that this polynomial in two variables is uniquely determined by its values at the n^2 points $\{(s_i, t_j)|i, j = 1, \ldots n\}$. At any arbitrary point p we can find a sufficiently small neighbourhood where we can choose n pairwise different values for s and n pairwise different values for t such that $(sId - A)$ and $(tId - A)$ are nondegenerate in that neighbourhood of p, because the eigenvalues of A are smooth. Thus it is sufficient to consider a sufficiently small neighbourhood of an arbitrary point and assume that $(tId - A)$ and $(sId - A)$ are invertible and prove the statement for this case.

We compute the Schouten-Nijenhuis bracket of $[\overset{s}{V}, \overset{t}{K}]_S$ which is the Lie derivative of

$\overset{t}{K}$ in the direction of $\overset{s}{V}$. With the shorthand notation (43) we get:

$$
\begin{aligned}
[\overset{s}{V},\overset{t}{K}]^{ij}_S =& \overset{s}{V}{}^a\nabla_a \overset{t}{K}{}^{ij} - \overset{t}{K}{}^{aj}\nabla_a \overset{s}{V}{}^i - \overset{t}{K}{}^{ia}\nabla_a \overset{s}{V}{}^j \\
=& -2\sqrt{\det(sId-A)}\sqrt{\det(tId-A)} \\
& \times [(\overset{s}{M}\bar{\Lambda})^a[-2\overset{t}{M}{}^p_a\lambda_p \overset{t}{M}{}^j_r g^{ri} + (\overset{t}{M}\Lambda)^j \overset{t}{M}{}^i_a + (\overset{t}{M}\Lambda)^i \overset{t}{M}{}^j_a \\
& \qquad + (\overset{t}{M}\bar{\Lambda})^j \overset{t}{M}{}^i_q J^q_a + (\overset{t}{M}\bar{\Lambda})^i \overset{t}{M}{}^j_q J^q_a] \\
& + [-\overset{t}{M}{}^a_b g^{bj}[-\overset{s}{M}{}^p_a\lambda_p(\overset{s}{M}\bar{\Lambda})^i + J^i_l \overset{s}{M}{}^l_a\, g(\overset{s}{M}\Lambda,\Lambda) \\
& \qquad - (\overset{s}{M}\Lambda)^i J^q_a \overset{s}{M}{}^m_q\lambda_q + J^i_l \overset{s}{M}{}^l_m g^{mn}\lambda_{n,a}]] + (i\leftrightarrow j)]
\end{aligned}
$$

There is no new technique in this, we proceeded exactly as we did in the proof of theorem 6: we used the properties (44) and (17) of A and the general identity $\nabla_X A^{-1} = A\cdot(\nabla_X A)\cdot A$ which is true for any $(1,1)$ tensor A and any vector field X. We sorted the resulting terms and cancel out those that vanish due to the Hermitian nature of A and the antisymmetry of J w.r.t. g. Expanding the brackets and using the way how g, J, A interact with one another this simplifies to

$$
\begin{aligned}
[\overset{s}{V},\overset{t}{K}]^{ij}_S =& -2\sqrt{\det(sId-A)}\sqrt{\det(tId-A)} \\
& \times [(\overset{s}{M}\,\overset{t}{M}\bar{\Lambda})^i((\overset{t}{M}-\overset{s}{M})\Lambda)^j - (\overset{s}{M}\,\overset{t}{M}\Lambda)^i((\overset{t}{M}-\overset{s}{M})\bar{\Lambda})^j \\
& + J^i_l \overset{s}{M}{}^l_m g^{mn}\lambda_{n,a}\overset{t}{M}{}^a_b g^{bj}] + (i\leftrightarrow j)
\end{aligned}
$$

Now we see that after the application of $\overset{t}{M}-\overset{s}{M} = (s-t)\,\overset{t}{M}\,\overset{s}{M}$ to the first two terms they cancel each other out after the addition of the $(i\leftrightarrow j)$ terms. Because the Hessian of λ is Hermitian (lemma 4, item 4) and because $\overset{t}{M}$ and $\overset{s}{M}$ are self-adjoint with respect to it (lemma 4, item 2 we see that the last term is antisymmetric in $(i\leftrightarrow j)$ and consequentially we have $[\overset{s}{V},\overset{t}{K}]_S = 0$, concluding the proof.

Before we prove theorem 8 we will establish some properties c-compatible structures that will be used at later times as well.

Definition 8 (Regular point) *Let (g,J,A) be c-compatible. A point $x\in\mathcal{M}$ is called regular with respect to A if in a neighbourhood of x the number of different eigenvalues of A is constant and for each eigenvalue ϱ either $\mathrm{d}\,\varrho\neq 0$ or ϱ is constant in a neighbourhood of x. The set of regular points shall be denoted $\mathcal{M}^0$.*

Lemma 10 (Eigenvectors and eigenvalues of A) *See lemma 2.2 of [3], see also [4]. Let (g,J,A) be c-compatible on $\mathcal{M}$. Then*

1. *Suppose for a smooth function ϱ on an open subset $U\subseteq\mathcal{M}$ and for any point $p\in U$ the number $\varrho(p)$ is an eigenvalue of A at p of algebraic multiplicity ≥ 4. Then this function ϱ is a constant on U. Moreover, for any point of the manifold the constant ϱ is an eigenvalue of A.*

2. *Let ϱ be an eigenvalue of A. Then at all points where ϱ is smooth the vectors* $\operatorname{grad}\varrho$ *and* $J\operatorname{grad}\varrho$ *are eigenvectors of A with eigenvalue ϱ.*

3. *At a generic point, the number of linearly independent canonical Killing vector fields coincides with the number of non-constant eigenvalues of A.*

4. *At each regular point the number of eigenvalues ϱ with $\mathrm{d}\,\varrho \neq 0$ is the same.*

Proof of lemma 10: We did not reinvent the wheel, the proofs given here are taken from [3] and listed here for convenience of the reader. We prove item 1 first: because A is smooth, so are the coefficients of its characteristic polynomial and its roots and almost any point has a neighbourhood in which the algebraic multiplicities of the eigenvalues of A do not change. We will work in such a neighbourhood of a suitable point p.

We have shown that for a c-compatible tensor A the Hessian of its trace is Hermititan, cf. lemma 4, item 4. By equation (49) we see that the Hessian of $\sqrt{\det A}$ is Hermitian as well. This implies that the Hessian of $\sqrt{\det(tId - A)}$ is Hermitian as well because for any c-compatible tensor A the tensor $tId - A$ is c-compatible as well.

This also makes sense for complex values of t: if by $2n$ we denote the dimension of the manifold then $\sqrt{\det(tId - A)}$ is a polynomial of degree $n-1$ in t. Because the Hessian of the polynomial $\sqrt{\det(tId - A)}$ is Hermitian for all real values of t all its coefficients must be Hermitian. For complex values of t the real and imaginary parts of $\sqrt{\det(tId - A)}$ are linear combinations of these coefficients and thus for complex values of t the Hessian of $\sqrt{\det(tId - A)}$ is a complex valued matrix that satisfies

$$(\nabla^2 \sqrt{\det(tId - A)})(J\cdot, J\cdot) = (\nabla^2 \sqrt{\det(tId - A)})(\cdot,\cdot) \tag{53}$$

The number of distinct eigenvalues ϱ_k of A shall be denoted by r and the algebraic multiplicity of the eigenvalue ϱ_k shall be denoted by $m(\varrho_k)$. Then in the neighbourhood of the point p we have

$$\sqrt{\det(tId - A)} = \prod_{k=1}^{r} (t - \varrho_k)^{m(\varrho_k)/2} \tag{54}$$

and

$$\mathrm{d}\sqrt{\det(tId - A)} = \sum_{i=1}^{r} \left(\prod_{k=1, k\neq i}^{r} (t - \varrho_k)^{m(\varrho_k)/2} \right) (t - \varrho_i)^{m(\varrho_i)/2 - 1}\, \mathrm{d}\,\varrho_i \tag{55}$$

Now without loss of generality we may assume that it is the first eigenvalue ϱ_1 which has algebraic multiplicity $m(\varrho_1) \geq 4$. For the value of t we choose the value of ϱ_1 at p: $t = \varrho_1(p)$. Because we have assumed that $m(\varrho_1) \geq 4$ we see that $\mathrm{d}\sqrt{\det(\varrho_1(p)Id - A)}$ is zero at p. Then from the two formulae above we see that at p the Hessian is

$$\nabla^2 \sqrt{\det(tId - A)}\Big|_{p, t=\varrho_1(p)} = \left(\prod_{k=2}^{r} (t - \varrho_k)^{m(\varrho_k)/2} \right) (t - \varrho_1)^{m(\varrho_1)/2 - 2}\, \mathrm{d}\,\varrho_1 \otimes \mathrm{d}\,\varrho_1 \Big|_{p, t=\varrho_1(p)}$$

All other terms except the one proportional to $\mathrm{d}\,\varrho_1 \otimes \mathrm{d}\,\varrho_1$ disappear because they contain a factor 0 at p.
If ϱ_1 is of algebraic multiplicity ≥ 6 then the expression above is zero because the prefactor of $\mathrm{d}\,\varrho_1 \otimes \mathrm{d}\,\varrho_1$ becomes zero at p.
If the algebraic multiplicity of ϱ_1 is 4 then we have to distinguish two cases: if ϱ_1 is real-valued then $\mathrm{d}\,\varrho_1 \otimes \mathrm{d}\,\varrho_1$ has either rank 1 or 0. But it cannot have rank 1 because $\nabla^2\sqrt{\det(tId - A)}$ is Hermitian. Thus $\mathrm{d}\,\varrho_1$ is zero at p.
If ϱ_1 is complex and we write it as $\varrho_1 = \alpha + i\beta$ then

$$\mathrm{d}\,\varrho_1 \otimes \mathrm{d}\,\varrho_1 = \mathrm{d}\,\alpha \otimes \mathrm{d}\,\alpha - \mathrm{d}\,\beta \otimes \mathrm{d}\,\beta + i(\mathrm{d}\,\alpha \otimes \mathrm{d}\,\beta + \mathrm{d}\,\beta \otimes \mathrm{d}\,\alpha)$$

If $\mathrm{d}\,\alpha$ and $\mathrm{d}\,\beta$ are linearly dependent then the rank of $\mathrm{d}\,\alpha \otimes \mathrm{d}\,\alpha - \mathrm{d}\,\beta \otimes \mathrm{d}\,\beta$ and $\mathrm{d}\,\alpha \otimes \mathrm{d}\,\beta + \mathrm{d}\,\beta \otimes \mathrm{d}\,\alpha$ must be either one or zero. But rank one cannot occur because $\nabla^2\sqrt{\det(tId - A)}$ is Hermitian.
The case where $\mathrm{d}\,\alpha$ and $\mathrm{d}\,\beta$ are linearly independent cannot occur: In this case both the real and imaginary part of $\mathrm{d}\,\varrho_1 \otimes \mathrm{d}\,\varrho_1$ have signature $(1, 1, 2n-2)$ which contradicts that they are Hermitian. We have proven that $\nabla^2\sqrt{\det(\varrho_1(p)Id - A)} = 0$ at p.
Remember that for any value of t the vector field $J\,\mathrm{grad}\sqrt{\det(tId - A)}$ is a Killing vector field (even in the case of complex t in the sense that its real and imaginary parts are Killing vector fields) and that if at one point both the value and the covariant derivative of a Killing vector field are known, then it is uniquely determined on all the manifold (provided of course the manifold is connected). But by our endeavours above we have proven that for $t = \varrho_1(p)$ the Killing vector field $J\,\mathrm{grad}\sqrt{\det(\varrho_1(p)Id - A)}$ and its covariant derivative vanish at p. Thus it vanishes on the whole manifold, implying that the function $\sqrt{\det(\varrho_1(p)Id - A)}$ is a constant on the whole manifold. But because the function is zero at p it is zero everywhere. Thus the constant $\varrho_1(p)$ is an eigenvalue of A on the whole manifold and item 1 is proven.
We now prove item 2: if ϱ is constant then there is nothing to prove. So let X be an eigenvector of A with non-constant eigenvalue ϱ. Then we covariantly differentiate the equation $(A - \varrho Id)X = 0$. We use (17) and rearrange terms to obtain

$$(A - \varrho Id)\nabla_Y X = \mathrm{d}\,\varrho(Y)X - g(Y,X)\Lambda - g(\Lambda, X)Y - g(JY, X)J\Lambda - g(J\Lambda, X)JY$$

We choose Y to be orthogonal to X and JX. Then the right hand side is a linear combination of X, Y and JY. The left hand side is a vector orthogonal to the kernel of $(A - \varrho Id)$. As a consequence the coefficient of X must vanish. Thus ϱ is constant in all directions orthogonal to X, JX. But because ϱ is non-constant the algebraic and geometric mulitplicity of ϱ is 2 by the first statement. The ϱ-eigenspace of A is spanned by X, JX. The orthogonal complement of the orthogonal complement of $\mathrm{span}\{X, JX\}$ is $\mathrm{span}\{X, JX\}$ itself and thus $\mathrm{grad}\,\varrho$ and $J\,\mathrm{grad}\,\varrho$ must be eigenvectors of A with eigenvalue ϱ. Statement 2 is proven.
For the third statement: We let the non-constant eigenvalues be denoted by $(\varrho_1, \ldots, \varrho_r)$ and collect the constant eigenvalues in $\overset{\mathrm{c}}{E}$. Denote the algebraic multiplicity of an eigenvalue ϱ by $m(\varrho)$. Then at any point where there are no bifurcations and all non-constant eigenvalues have non-vanishing differentials we can write $\sqrt{\det(tId - A)}$

as

$$\sqrt{\det(tId - A)} = \prod_{\varrho \in \overset{c}{E}} (t - \varrho)^{m(\varrho)/2} \prod_{k=1}^{r} (t - \varrho_k)$$

because all non-constant eigenvalues are of multiplicity 2 (again r is the number of non-constant eigenvalues of A and $\overset{c}{E}$ is the set of constan eigenvalues). So for all values of t $\sqrt{\det(tId - A)}$ is proportional by a constant to $\prod_{k=1}^{r}(t - \varrho_k)$. This is a polynomial of degree r in t with leading coefficient 1. So there are at most r independent Killing vector fields. But in view of equation (55) and the proof of the first statement we know that at every point p for each non-constant eigenvalue ϱ there is a Killing vector field that is parallel to $J \operatorname{grad} \varrho$ at p. So at all points the canonical Killing vector fields have the same span as the vector fields $J \operatorname{grad} \varrho$ of the non-constant eigenvalues. But because for non-constant eigenvalues ϱ_i the vector fields $J \operatorname{grad} \varrho_i$ belong to the respective ϱ_i-eigenspaces of A they are linearly independent. Thus at each point the dimesion of their span is equal to the number of eigenvalues with non-vanishing differential. Consequentially at generic points the span of the canonical Killing vector fields has dimension equal to the number of non-constant eigenvalues and our claim is proven.
We prove item 4 by contradiction: Assume that there exist two regular points p_1, p_2 and the number of non-constant eigenvalues at p_1 is k_1 and at p_2 it is k_2 (w.l.o.g. $k_1 > k_2$). Then by the previous statement in a neighbourhood of p_1 the number of independent Killing vector fields is k_1. Likewise, in a neighbourhood of p_2 the number of independent Killing vector fields is k_2. But if a Killing vector field vanishes on an open set then it vanishes everywhere. This contradicts our assumption $k_1 > k_2$ and the claim is proven.

Proof of theorem 8: We show that the number of functionally independent integrals in the family $\overset{s}{L}$ is at least the number of non-constant eigenvalues of A. With canonical coordinates (x, p) on T^*M their differentials are

$$\mathrm{d} \overset{s}{L} = (\partial_x \overset{s}{V}{}^i p_i, \overset{s}{V}) \tag{56}$$

We see that if for a tuple of numbers $(s_1, \ldots, s_k)$ the tuple $(\overset{s_i}{V} | i = 1, \ldots, k)$ is linearly independent at almost every point on $\mathcal{M}$, then the differentials $(\mathrm{d} \overset{s_i}{L} | i = 1, \ldots, k)$ are linearly independent almost everywhere on $T^*\mathcal{M}$. The number of independent Killing vector fields is the same at each regular point and is equal to the number of non-constant eigenvalues. Thus there are at least as many functionally independent integrals in $\overset{s}{L}$ as there are non-constant eigenvalues. If we denote by $\overset{c}{E}$ the set of constant eigenvalues of A, by $(\varrho_1, \ldots, \varrho_r)$ the non-constant eigenvalues of A, and by

$m(\varrho)$ the algebraic multiplicity of the eigenvalue ϱ then we have:

$$
\begin{aligned}
\overset{s}{L} =& p_i J^i_j g^{jk} \partial_k \sqrt{\det sId - A} \\
=& -p_i J^i_j g^{jk} \prod_{\varrho \in \overset{c}{E}} (s-\varrho)^{m(\varrho)/2} \sum_{p=1}^{r} \prod_{\substack{q=1 \\ q \neq p}}^{r} (s-\varrho_q) \partial_k \varrho_p \\
=& \prod_{\varrho \in \overset{c}{E}} (t-\varrho)^{m(\varrho)/2} \overset{s}{\tilde{L}}
\end{aligned}
$$

where we have introduced

$$
\overset{s}{\tilde{L}} = p_i J^i_j g^{jk} \partial_k \prod_{q=1}^{r} (s-\varrho_q)
$$

We see that for all values of s the integral $\overset{s}{L}$ is a constant multiple of $\overset{s}{\tilde{L}}$. $\overset{s}{\tilde{L}}$ is a polynomial whose degree is $r-1$, where r is the number of non-constant eigenvalues of A. For each value of s, the polynomial $\overset{s}{\tilde{L}}$ is a linear combination of its coefficients. Thus the number of functionally independent integrals is at most equal to the number of non-constant eigenvalues and item 1 is proven.
To prove item 2 we show first that the number of functionally independent integrals is at most equal to the degree of the minimal polynomial of A: for each eigenvalue ϱ of A denote by $m(\varrho)$ its algebraic multiplicity and by $\varepsilon(\varrho)$ its index. The index $\varepsilon(\varrho)$ is the size of the largest Jordan block corresponding to ϱ or equivalently the multiplicity of ϱ as a root of the minimal polynomial of A. We shall further denote by E the set of distinct eigenvalues of A. If we look at the explicit formula for the inverse matrix of a Jordan block, which is given by

$$
\begin{pmatrix}
\varrho & 1 & & & \\
 & \varrho & 1 & & \\
 & & \ddots & \ddots & \\
 & & & \varrho & 1 \\
 & & & & \varrho
\end{pmatrix}^{-1}
=
\begin{pmatrix}
\varrho^{-1} & -\varrho^{-2} & \varrho^{-3} & \ddots & \ddots \\
 & \varrho^{-1} & -\varrho^{-2} & \ddots & \ddots \\
 & & \ddots & \ddots & \varrho^{-3} \\
 & & & \ddots & -\varrho^{-2} \\
 & & & & \varrho^{-1}
\end{pmatrix}
\tag{57}
$$

we see that (because A is Hermitian) the quantity $\overset{t}{\tilde{K}}$ defined by

$$
\overset{t}{\tilde{K}}{}^{ij} \overset{\text{def}}{=} \prod_{\varrho \in E} (t-\varrho)^{m(\varrho)/2-\varepsilon(\varrho)} (tId - A)^{-1}{}^i_k g^{kj}
$$

is a polynomial in t of degree equal to the degree of the minimal polynomial of A. Because eigenvalues of A of algebraic multiplicity ≥ 4 are constants we know that for

any value of t the family $\overset{t}{\tilde{K}}$ is a family of Killing tensors as well. We also see that for any value t_1 the Killing tensor $\overset{t_1}{K}$ is a linear combination of the coefficients of the polynomial $\overset{t}{\tilde{K}}$. This implies that the number of functionally independent integrals in the family $\overset{t}{I}$ is at most equal to the degree of the minimal polynomial of A.
Now we will show that the number of functionally independent integrals is at least equal to the number of linearly independent Killing tensors. The differentials of the quadratic integrals $\overset{t}{I}$ are

$$\mathrm{d}\overset{t}{I} = (\partial_x \overset{t}{K}{}^{ij} p_i p_j, 2 \overset{t}{K}{}^{ij} p_j) \tag{58}$$

It thus suffices to show that if for some $(t_1, \ldots, t_k)$ the Killing tensors $(\overset{t_i}{K} | i = 1, \ldots, k)$ are linearly independent, then for almost any cotangent vector p at almost any point x the covectors $(\overset{t_i}{K}{}^i_j p_i | i = 1, \ldots, k)$ are linearly independent.
Consider a set of one-forms $\vartheta_{i,j}$ such that A (considered as a map from one-forms to one-forms) is in Jordan normal form. Here we make an exception to our notation: the comma is not a covariant derivative, it merely separates two indices. We denote by ϱ_l the eigenvalue of the l^{th} Jordan block (it is possible that several blocks have the same eigenvalue), by κ_l the size of the l^{th} Jordan block. The one-forms $\vartheta_{i,j}$ satisfy

$$(A - \varrho_k Id)\vartheta_{k,j} = \vartheta_{k,j-1}$$

The first index of the $\vartheta_{i,j}$ identifies the Jordan block to which it belongs and the second index denotes the rank of the generalized eigenvector, i.e.

$$(A - \varrho_k Id)^j \vartheta_{k,j} = 0, \qquad (A - \varrho_k Id)^{j-1}\vartheta_{k,j} \neq 0$$

Here the superscript j denotes a power, not a component index. Because A is Hermitian, the number of Jordan blocks is even and we shall call it $2q$. It also implies that every Jordan block appears twice and allows that we can choose the ϑ in such a way that

$$\vartheta_{i,j} \circ J = \begin{cases} \vartheta_{i+q,j} & \forall \quad 1 \leq i \leq q \\ -\vartheta_{i-q,j} & \forall \quad q+1 \leq i \leq 2q \end{cases}$$

It follows by direct computation that the Killing tensors (as (1,1) tensors) act on thes one-forms by:

$$\overset{t_s}{K} \vartheta_{i,j} = \sum_{l=0}^{j-1} \prod_{\substack{m=1 \\ m \neq i \bmod n}}^{q} (t_s - \varrho_m)^{\kappa_m} (t_s - \varrho_i)^{\kappa_i - 1 - l} \vartheta_{i,j-l}$$

We decompose a covector p as

$$p = \sum_{i=1}^{2q} \sum_{j=1}^{\kappa_i} p_{i,j} \vartheta_{i,j}$$

For almost any covector p all coefficients $p_{i,j}$ are non-zero. We will show that for such a covector and linearly independent $(\overset{t_i}{K} | i = 1, \dots, k)$ (considered as (1,1)-tensors) the covectors $(\overset{t_i}{K} p | i = 1, \dots, k)$ are linearly independent. Suppose a linear combination of $(\overset{t_s}{K} p | s = 1, \dots, k)$ with coefficients (α_s) is zero. Then by direct calculation we have

$$\sum_{s=1}^{k} \alpha_s \sum_{i=1}^{2q} \sum_{j=1}^{\kappa_i} \sum_{l=0}^{\kappa_i - j} \prod_{\substack{m=1 \\ m \neq i \bmod q}}^{q} (t_s - \varrho_m)^{\kappa_m} (t_s - \varrho_i)^{\kappa_i - 1 - l} p_{i,j+l} \vartheta_{i,j} = 0$$

But the $\vartheta_{i,j}$ are linearly independent, thus for all possible combinations of i, j we must have:

$$\sum_{s=1}^{k} \alpha_s \sum_{l=0}^{\kappa_i - j} \prod_{\substack{m=1 \\ m \neq i \bmod q}}^{q} (t_s - \varrho_m)^{\kappa_m} (t_s - \varrho_i)^{\kappa_i - 1 - l} p_{i,j+l} = 0$$

For each i we consider the equations for $j = \kappa_i, \kappa_i - 1, \dots, 1$ in that order and find that the equations above are equivalent to

$$\sum_{s=1}^{k} \alpha_s \prod_{\substack{m=1 \\ m \neq i \bmod q}}^{q} (t_s - \varrho_m)^{\kappa_m} (t_s - \varrho_i)^{\kappa_i - 1 - l} = 0$$

because we assumed all the p's to be non-zero. This would imply that

$$\sum_{s=1}^{k} \alpha_s \overset{t_s}{K} = 0$$

But because we assumed the $\overset{t_s}{K}$ to be linearly independent it implies that all the α's are zero and the covectors $\overset{t_s}{K} p$ are linearly independent, as we wanted to prove. Thus there are at least as many functionally independent integrals in the family $\overset{t}{I}$ as there are linearly independent Killing tensors in the family $\overset{s}{K}$.

It remains to show that the number of linearly independent Killing tensors in the family $\overset{t}{K}$ is equal to the degree of the minimal polynomial of A. The number of linearly independent Killing tensors in the family $\overset{t}{K}$ is of course the same as the number of linearly independent (1,1) tensors in the family $(tId - A)^{-1}$. Consider any regular point x and consider for appropriate integers l, j the operators

$$B_{l,j} = \frac{1}{2\pi i} \int_{C(\varrho_l(x))} (w - \varrho_l(x))^j (wId - A)^{-1} \, \mathrm{d}\, w$$

where $C(\varrho_l(x))$ is a small circle in the complex plane around $\varrho_l(x)$ such that it contains no other eigenvalues of A. Clearly these tensors lie in the span of the family $(tId - A)$. Considered as (1,1)-tensors in the basis of the ϑ that we have introduced the matrices

of the $B_{l,j}$ can be visualized as follows: choose t_0, such that $(t_0 Id - A)$ is invertible. Take the Jordan normal form of $(t_0 Id - A)$ and invert it. Remove all entries that do not belong to blocks corresponding to Jordan blocks with eigenvalue $t_0 - \varrho_l$ and make these entries 0. Further set to zero all entries except those on the j^{th} superdiagonal (the 0^{th} superdiagonal means the diagonal). Set all remaining non-zero entries to 1. From this it follows that the number of linearly independent Killing tensors is equal to the minimal polynomial of A.
By similar arguments as before it is clear that for $(t_1, \ldots, t_k)$ and $(s_1, \ldots, s_m)$ for which the Killing tensors $(\overset{t_i}{K} | i = 1, \ldots, k)$ are linearly independent and for which the Killing vector fields $(\overset{s_j}{V} | j = 1, \ldots, m)$ are linearly independent the equation

$$\sum_{i=1}^{k} \alpha_i \overset{t_i}{K}{}^{ab} p_b = \sum_{j=1}^{m} \beta_j \overset{s_j}{V}{}^{a}$$

has only trivial solutions (α_i, β_j) for almost any covector p. Thus the integrals $(\overset{t_i}{I} | i = 1, \ldots, k)$ and $(\overset{s_j}{L} | j = 1, \ldots, m)$ are functionally independent.

1.2 Quantization rules

We adopt the quantization rules introduced by B. Carter [5] and C. Duval and G. Valent [7, §3], see these and their references for a reasoning and more details. It is sufficient for our purposes to recall the quantization formulae they give: for a homogeneous polynomial $P_m : T^*\mathcal{M} \to \mathbb{R}$ of degree m we construct its symmetric contravariant tensor via $\sharp$ and compose with the covariant derivative:

$$\begin{aligned}
P_0 &\mapsto \hat{P}_0 \overset{\text{def}}{=} P_0^\sharp Id \\
P_1 &\mapsto \hat{P}_1 \overset{\text{def}}{=} \frac{i}{2}((P_1^\sharp)^j \circ \nabla_j + \nabla_j \circ (P_1^\sharp)^j) \\
P_2 &\mapsto \hat{P}_2 \overset{\text{def}}{=} -\nabla_j \circ (P_2^\sharp)^{jk} \circ \nabla_k \\
P_3 &\mapsto \hat{P}_3 \overset{\text{def}}{=} -\frac{i}{2}(\nabla_j \circ (P_3^\sharp)^{jkl} \circ \nabla_k \circ \nabla_l + \nabla_j \circ \nabla_k \circ (P_3^\sharp)^{jkl} \circ \nabla_l)
\end{aligned} \tag{59}$$

For polynomials that are not homogeneous the quantization shall be done by quantizing the homogeneous parts and adding the results.
In the previous sections we have been considering polynomials of degree two on the cotangent bundle of degree at most two and covariant tensors of valence at most $(2, 0)$. These correspond to differential operators of degree at most two. But the commutator of two such second order operators generally is an operator of order three. Later on we can facilitate the expression for the commutator of the quantum operators of two polynomials of degree two by using the quantum operator of the Poisson bracket of the two polynomials of degree two.

2 Results

Most of the results and proofs that are presented in this work mostly have been prepublished in [15].
Whenever we are dealing with a c-compatible structure we shall denote the real dimension of the manifold by $2n$.

2.1 Quantum integrals of the geodesic flow

The main result of this work is a quantum version of theorem 5: using the quantization rules (59) from [5] and [7] we construct differential operators from symmetric covariant tensors and show that these differential operators commute.

Let (g, J, A) be c-compatible and $\overset{t}{K}, \overset{t}{I}$ denote the associated Killing tensors and integrals of the geodesic flow. By (59) their associated quantum operators are:

$$\overset{t}{\hat{I}}(f) \stackrel{\text{def}}{=} -\nabla_j \circ \overset{t}{K}{}^{jk} \circ \nabla_k f$$

(The differing letters I and K must not confuse the reader, for $\overset{t}{I}{}^{\sharp} = \overset{t}{K}$.)

Theorem 11 *Let (g, J, A) be* c-compatible*. Then for any pair (v, w) the operators $\overset{v}{\hat{I}}$ and $\overset{w}{\hat{I}}$ commute, i.e. $[\overset{v}{\hat{I}}, \overset{w}{\hat{I}}] = 0$*

This is a new result for both the Riemannian as well as the pseudo-Riemmanian case. We remind the reader that for c-compatible structures there also exist integrals of the geodesic flow that are linear in momenta. We will see that their corresponding quantum operators also commute with each other and with the quantum operators of the integrals that are quadratic in momenta, see theorem 13 and its proof, where the potential is zero.

Theorem 12 *Let (g, A) be* projectively compatible*. Then for any pair (v, w) the operators $\overset{v}{\hat{I}}$ and $\overset{w}{\hat{I}}$ commute, i.e. $[\overset{v}{\hat{I}}, \overset{w}{\hat{I}}] = 0$*

Remark 10 *Theorem 12 was already proven by V. Matveev [11, 12]. The proof that we give however only uses C^3-smoothness whereas the original proof used C^8. The proof that will be given here runs in parallel with the proof of theorem 11. A series of remarks to the proof of theorem 11 will thus provide the proof of theorem 12, giving the intermediate steps for the projective case and pointing out the analogues and differences.*

2.2 Addition of potential | Natural Hamiltonian systems

We improve the result of theorem 11 by adding potential terms to these second order differential operators, finding commuting quantum observables for certain natural Hamiltonian systems.

Theorem 13 *Let (g, J, A) be c-compatible. Let*

$$\overset{t}{\hat{I}} \stackrel{\text{def}}{=} -\nabla_j \circ \overset{t}{K}{}^{jk} \circ \nabla_k, \qquad \overset{t}{K}{}^{ij} \stackrel{\text{def}}{=} \sqrt{\det(tId - A)}\ (tId - A)^{-1}{}^i_l g^{lj}$$

be as in theorem 11.
Let

$$\overset{t}{\hat{L}} = \frac{i}{2}(\nabla_j \circ \overset{t}{V}{}^j + \overset{t}{V}{}^j \circ \nabla_j), \qquad \overset{t}{V}{}^j = J^j_k g^{ki} \nabla_i \sqrt{\det(tId - A)} \tag{60}$$

be the differential operators associated with the canonical Killing vector fields of g. Let $\overset{\text{nc}}{E} = \{\varrho_1, \dots, \varrho_r\}$ be the set of non-constant eigenvalues of A. Let $\overset{\text{c}}{E} = \{\varrho_{r+1}, \dots, \varrho_{r+R}\}$ be the set of constant eigenvalues and $E = \overset{\text{nc}}{E} \cup \overset{\text{c}}{E}$. Denote by $m(\varrho_i)$ the algebraic multiplicity of ϱ_i. Let the family of potentials $\overset{t}{U}$, parametrized by t, be given by

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2 - 1} f_i \tag{61}$$

with $\mathrm{d} f_l \circ A = \varrho_l \, \mathrm{d} f_l$ for all $l = 1, \dots, r + R$ and with $\mathrm{d} f_l$ proportional to $\mathrm{d}\varrho_l$ for all l for which ϱ_l is non-constant. Let associated operators $\overset{t}{\hat{U}}$ act on functions by mere multiplication, i.e. for any point p we have $(\overset{t}{\hat{U}}(f))(p) = \overset{t}{U}(p) f(p)$. Then the operators

$$\overset{t}{\hat{Q}} \stackrel{\text{def}}{=} \overset{t}{\hat{I}} + \overset{t}{\hat{U}}, \qquad \overset{t}{\hat{L}} \tag{62}$$

commute within the one-parameter-families as well as crosswise, i.e. for all values of $t, s \in \mathbb{R}$:

$$[\overset{t}{\hat{Q}}, \overset{s}{\hat{Q}}] = [\overset{t}{\hat{Q}}, \overset{s}{\hat{L}}] = [\overset{t}{\hat{L}}, \overset{s}{\hat{L}}] = 0 \tag{63}$$

Remark 11 *Since $\overset{t}{V}{}^j = J^j_k g^{ki} \nabla_i \sqrt{\det(tId - A)}$ is a Killing vector field for any choice of the real parameter t [4, §5], we have $\overset{t}{\hat{L}} = \frac{i}{2}(\nabla_j \circ \overset{t}{V}{}^j + \overset{t}{V}{}^j \circ \nabla_j) = i\, \overset{t}{V}{}^j \circ \nabla_j$.*

Remark 12

1. *We do not discuss whether the $\overset{t}{U}$ are smooth at all points of the manifold. Smoothness is guaranteed at points that have a neighbourhood in which the number of different eigenvalues is constant (see the definition 8 of regular points below), provided that the f_i are smooth.*

2. *Formula (61) generally allows $\overset{t}{U}$ to be complex-valued. The conditions under which $\overset{t}{U}$ is real for any choice of $t \in \mathbb{R}$ are the following: for any real eigenvalue ϱ_i of A the corresponding function f_i must be real-valued. For all pairs $(\varrho_i, \varrho_j = \bar{\varrho}_i)$ of complex-conjugate eigenvalues of A the corresponding functions f_i and f_j must be complex conjugate to each other: $f_i = \bar{f}_j$.*

3. *The potentials that are admissible to be added to the quantum operators are the same that may be added to the Poisson commuting integrals. In the proof we show that the quantization imposes no stronger conditions on the potential than classical integrability and then use the conditions imposed by the Poisson brackets to find the allowed potentials.*

Theorem 14 *Let* (g, J, A) *be c-compatible and* A *semi-simple. Let* $\overset{t}{\hat{I}}$, $\overset{t}{\hat{L}}$ *be as in theorem 13. Then, for the operators*

$$\overset{t}{\hat{Q}} \overset{\text{def}}{=} \overset{t}{\hat{I}} + \overset{t}{\hat{U}} \qquad \textit{and} \qquad \overset{t}{\hat{L}} \tag{64}$$

the commutation relations $[\overset{t}{\hat{Q}}, \overset{s}{\hat{Q}}] = [\overset{t}{\hat{Q}}, \overset{s}{\hat{L}}] = 0$ *are satisfied if and only if the potentials are of the form* (61) *with the sole exception that a function of* t *alone may be added to* $\overset{t}{U}$.

Corollary 14.1 *Let* (g, J, A), $\overset{t}{\hat{I}}$, $\overset{t}{\hat{L}}$, $\overset{t}{\hat{U}}$ *be as in theorem 13. Let* $\hat{I}_{(l)}, \hat{L}_{(l)}, \hat{U}_{(l)}$ *be the coefficients of* t^l *in* $\overset{t}{\hat{I}}, \overset{t}{\hat{L}}, \overset{t}{\hat{U}}$ *respectively. Then the commutation relations*

$$\begin{aligned}
[\overset{t}{\hat{I}} + \overset{t}{\hat{U}}, \hat{I}_{(l)} + \hat{U}_{(l)}] = [\overset{t}{\hat{I}} + \overset{t}{\hat{U}}, \hat{L}_{(l)}] &= [\overset{t}{\hat{L}}, \hat{I}_{(m)} + \hat{U}_{(m)}] \\
&= [\overset{t}{\hat{L}}, \hat{L}_{(l)}] = [\overset{t}{\hat{L}}, \hat{I}_{(m)} + \hat{U}_{(m)}] = 0
\end{aligned} \tag{65}$$

$$[\hat{I}_{(l)} + \hat{U}_{(l)}, \hat{I}_{(m)} + \hat{U}_{(m)}] = [\hat{L}_{(l)}, \hat{L}_{(m)}] = [\hat{I}_{(l)} + \hat{U}_{(l)}, \hat{L}_{(m)}] = 0 \tag{66}$$

hold true for any value of t *and any values* $l, m \in \{1, \ldots, n-1\}$.
Equations (66) *are equivalent to*

$$[\overset{t}{\hat{I}} + \overset{t}{\hat{U}}, \overset{s}{\hat{I}} + \overset{s}{\hat{U}}] = [\overset{t}{\hat{L}}, \overset{s}{\hat{L}}] = [\overset{t}{\hat{I}} + \overset{t}{\hat{U}}, \overset{s}{\hat{L}}] = 0 \tag{67}$$

Equations (65), (66), (67) *remain true if a function of* t *alone is added to* $\overset{t}{\hat{U}}$ *and constants* $c_{(l)}$ *are added to* $\hat{U}_{(l)}$. *If all eigenvalues of* A *are non-constant and* $\overset{t}{\hat{I}}, \overset{t}{\hat{V}}$ *are as in theorem 13 then no other than the described* $\hat{U}_{(l)}$ *can be found such that the commutation relations above hold.*

2.3 Simultaneous eigenfunctions | Separation of variables

Lastly, we shall show how the search for simultaneous eigenfunctions of the operators can be reduced to differential equations in lower dimension in appropriate coordinates around regular points. In particular if all eigenvalues of A are non-constant we can reduce it to ordinary differential equations only. Moreover: the case where all eigenvalues of A are non-constant provide an example of *reduced separability* of Schrödinger's equation as described in [1].

Definition 9 (Local normal coordinates) *Let (g, J, A) be c-compatible on $\mathcal{M}$. A local normal coordinate system for $\mathcal{M}$ is a coordinate system where (g, J, A) assume the form of example 1 (see below). Existence of such coordinates in the neighbourhood of regular points is guaranteed by theorem 1.6 in [3].*

Example 1 (General example for c-compatible structures (g, J, Ω, A)) *[3, Example 5] Let $2n \geq 4$ and consider an open subset W of $\mathbb{R}^{2n}$ of the form $W = U \times V \times S_1 \times \ldots \times S_L \times S_{L+1} \times \ldots \times S_{L+Q}$ for open subsets $V, U \subseteq \mathbb{R}^r$, $S_\gamma \subseteq \mathbb{R}^{4m_{c_\gamma}}$ for $\gamma = 1, \ldots, L$ and $S_\gamma \subseteq \mathbb{R}^{2m_{c_\gamma}}$ for $\gamma = L+1, \ldots, L+Q$. Let the coordinates on U be separated into l complex coordinates $z_1 \ldots z_l$ and q real coordinates $x_{l+1}, \ldots x_{l+q}$ and introduce the tuple $(\chi_1, \ldots \chi_r) = (z_1, \bar{z}_1, \ldots, z_l, \bar{z}_l, x_{l+1}, \ldots x_{l+q})$. Suppose the following data is given on these open subsets*

- *Kähler structures $(g_\gamma, J_\gamma, \Omega_\gamma)$ on S_γ for $\gamma = 1, \ldots, L+Q$*
- *For each $\gamma = 1, \ldots, L+Q$, a parallel hermitian endomorphism $A_\gamma : TS_\gamma \to TS_\gamma$ for (g_γ, J_γ). For $\gamma = 1, \ldots, L$ the endormorphism A_γ has a pair of complex conjugate eigenvalues $c_\gamma, \bar{c}_\gamma$ of equal algebraic multiplicity $m(c_\gamma) = m(\bar{c}_\gamma)$. For $\gamma = L+1 \ldots L+Q$ the endomorphism A_γ has a single real eigenvalue c_γ of algebraic multiplicity $m(c_\gamma)$.*
- *Holomorphic functions $\sigma_j(z_j)$ for $1 \leq j \leq l$ and smooth functions $\sigma_j(x_j)$ for $l+1 \leq j \leq r$.*

Moreover, we choose 1-forms $\alpha_1, \ldots, \alpha_r$ on $S = S_1 \times \cdots \times S_N$ that satisfy

$$\mathrm{d}\,\alpha_i = (-1)^i \sum_{\gamma=1}^{L+Q} \Omega_\gamma(A_\gamma^{r-i} \cdot, \cdot) \tag{68}$$

To facilitate the expressions for the c-compatible structure that will be constructed, the following expressions shall be introduced: the tuple

$$E = (\varrho_1, \ldots, \varrho_n) = (\sigma_1, \bar{\sigma}_1, \ldots, \sigma_l, \bar{\sigma}_l, \sigma_{l+1}, \ldots, \sigma_{l+q}, c_1, \bar{c}_1, \ldots, c_L, \bar{c}_L, c_{L+1}, \ldots, c_{L+Q})$$

contains the designated eigenvalues for A. Their algebraic multiplicities shall be denoted by

$$(m(\varrho_l) | l = 1, \ldots, r+R) = (2, \ldots, 2, m(c_1), m(\bar{c}_1), \ldots)$$

The non-constant eigenvalues shall be collected in order in

$$\overset{\mathrm{nc}}{E} = (\varrho_1, \ldots, \varrho_r) = (\sigma_1, \bar{\sigma}_1, \ldots, \sigma_l, \bar{\sigma}_l, \sigma_{l+1}, \ldots, \sigma_{l+q})$$

and the collection of constant eigenvalues shall be referenced as $\overset{\mathrm{c}}{E} = E \setminus \overset{\mathrm{nc}}{E}$. The quantity Δ_i for $i = 1, \ldots, r$ is given by $\Delta_i = \prod_{\varrho \in \overset{\mathrm{nc}}{E} \setminus \{\varrho_i\}} (\varrho_i - \varrho)$. The function μ_i denotes the elementary symmetric polynomial of degree i in the variables $\overset{\mathrm{nc}}{E}$ and $\mu_i(\hat{\varrho_s})$ denotes the elementary symmetric polynomial of degree i in the variables $\overset{\mathrm{nc}}{E} \setminus \{\varrho_i\}$. We shall further define the one-forms $\vartheta_1, \ldots \vartheta_r$ on W via $\vartheta_i = \mathrm{d}\,t_i + \alpha_i$.

Suppose that at every point of W the elements of $\overset{\mathrm{nc}}{E}$ are mutually different and different from the constants $c_1, \bar{c}_1, \dots, c_R$ and their differentials are non-zero. Then (g, ω, J) given by the formulae

$$
\begin{aligned}
g &= \sum_{i=1}^{r} \varepsilon_i \Delta_i \,\mathrm{d}\,\chi_i^2 + \sum_{i=0}^{r} (-1)^i \mu_i \sum_{\gamma=1}^{L+Q} g_\gamma(A_\gamma^{r-i} \cdot, \cdot) \\
&\quad + \sum_{i,j=1}^{l} \left[\sum_{s=1}^{r} \frac{\mu_{i-1}(\hat{\varrho}_s)\mu_{j-1}(\hat{\varrho}_s)}{\varepsilon_s \Delta_s} \left(\frac{\partial \varrho_s}{\partial \chi_s} \right)^2 \right] \vartheta_i \vartheta_j \\
\Omega &= \sum_{i=1}^{r} \mathrm{d}\,\mu_i \wedge \vartheta_i + \sum_{i=0}^{r} (-1)^i \mu_i \sum_{\gamma=1}^{L+Q} \Omega_\gamma(A_\gamma^{r-i} \cdot, \cdot)
\end{aligned}
\tag{69}
$$

$$
\mathrm{d}\,\chi_i \circ J = -\frac{1}{\varepsilon_i \Delta_i} \frac{\partial \varrho_i}{\partial \chi_i} \sum_{j=1}^{r} \mu_{j-1}(\hat{\varrho}_i) \vartheta_j, \qquad \vartheta_i \circ J = (-1)^{i-1} \sum_{j=1}^{r} \varepsilon_j \varrho_j^{r-i} \left(\frac{\partial \varrho_j}{\partial \chi_j} \right)^{-1} \mathrm{d}\,\chi_j
\tag{70}
$$

is Kähler, where $(\varepsilon_1, \dots, \varepsilon_{2l}, \varepsilon_{2l+1}, \dots, \varepsilon_r) = (-1/4, \dots, -1/4, \pm 1, \dots, \pm 1)$ determine the signature of g.
With local coordinates $\overset{\gamma}{y}$ on S_γ we write $\alpha_i = \sum_{\gamma,q} \overset{\gamma}{\alpha}_{iq} \,\mathrm{d}\, \overset{\gamma}{y}_q$ and $A_\gamma = \sum_{p,q} (A_\gamma)_p^q \,\mathrm{d}\, \overset{\gamma}{y}_p \otimes \partial_{\overset{\gamma}{y}_q}$. Then the endomorphism A given by

$$
\begin{aligned}
A = \sum_{s=1}^{r} \varrho_s \,\mathrm{d}\,\chi_s \otimes \partial_{\chi_s} &+ \sum_{i,j=1}^{r} (\mu_i \delta_{1j} - \delta_{i(j-1)}) \vartheta_i \otimes \partial_{t_j} \\
&+ \sum_{\gamma=1}^{L+Q} \sum_{p,q} (A_\gamma)_p^q \,\mathrm{d}\, \overset{\gamma}{y}_p \otimes \left(\partial_{\overset{\gamma}{y}_q} - \sum_{i=1}^{r} \overset{\gamma}{\alpha}_{iq} \,\partial_{t_i} \right)
\end{aligned}
\tag{71}
$$

is c-compatible with (g, J, Ω).

Theorem 15 (Existence of local normal coordinates) *[3, Theorem 1.6 / Example 5] Suppose (g,J,A) are c-compatible on $\mathcal{M}$ of real dimension $2n$.*
Assume that in a small neighbourhood $W \subseteq \mathcal{M}^0$ of a regular point, A has

- *$r = 2l + q$ non-constant eigenvalues on W which separate into l pairs of complex-conjugate eigenvalues $\varrho_1, \bar{\varrho}_1, \dots, \varrho_l, \bar{\varrho}_l : W \to \mathbb{C}$ and q real eigenvalues $\varrho_{l+1}, \dots, \varrho_{l+q} : W \to \mathbb{R}$,*
- *$R = 2L+Q$ constant eigenvalues which separate into L pairs of complex conjugated eigenvalues $c_1, \bar{c}_1, \dots, c_L, \bar{c}_L$ and Q real eigenvalues $c_{L+1}, \dots, c_{L+Q}$*

then the Kähler structure (g, J, Ω) and A are given on W by the formulae of example 1.

Theorem 16 (Simultaneous eigenfunctions) *Let (g, J, A) be c-compatible on $\mathcal{M}$. Let A be semi-simple and let all constant eigenvalues be real. Let (g, J, Ω, A) be given by the formulae of example 1 and adopt the naming conventions of example 1.*

Let ψ be a simultaneous eigenfunction of

$$\overset{t}{\hat{Q}} \stackrel{\text{def}}{=} -\nabla_j \circ \overset{t}{K}{}^{jk} \circ \nabla_k + \overset{t}{\hat{U}}, \qquad \overset{t}{K}{}^{ij} \stackrel{\text{def}}{=} \sqrt{\det(tId - A)}\ (tId - A)^{-1}{}^i_l g^{lj}$$

and

$$\overset{t}{\hat{L}} = \frac{i}{2}(\nabla_j \circ \overset{t}{V}{}^j + \overset{t}{V}{}^j \circ \nabla_j), \qquad \overset{t}{V}{}^j = J^j_k g^{ki} \nabla_i \sqrt{\det(tId - A)}$$

for all t, where

$$\overset{t}{U} = \sum_{i=1}^{r} \prod_{l=1, l\neq i}^{r} \frac{(t-\varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t-\varrho_i)^{m(\varrho_i)/2-1} f_i \tag{72}$$

with $\mathrm{d} f_i \circ A = \varrho_i \,\mathrm{d} f_i$. Then there exist constants $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{r+R-1}, \omega_1, \ldots, \omega_r$, such that ψ satisfies the following ordinary differential equations:

$$\begin{aligned} \frac{-1}{\varepsilon_k \varrho_k'} \partial_{\chi_k} \varrho_k' \prod_{\varrho_\gamma \in \overset{c}{E}} (\varrho_k - \varrho_\gamma)^{m(\varrho_\gamma)/2} \partial_{\chi_k} \psi & \\ + \sum_{i,j=1}^{r} \frac{\varepsilon_k (-\varrho_k)^{2r-i-j}}{(\varrho_k')^2} \prod_{\varrho_\gamma \in \overset{c}{E}} (\varrho_k - \varrho_\gamma)^{m(\varrho_\gamma)/2} \omega_i \omega_j \psi + f_r \psi &= \sum_{i=0}^{n-1} \lambda_i \varrho_k^i \psi \\ i\partial_{t_k} \psi &= \omega_k \psi \end{aligned} \tag{73}$$

for $k = 1, \ldots, r$, where the λ_i are given by

$$\sum_{i=0}^{n-1} \lambda_i s^i = \prod_{\varrho_\gamma \in \overset{c}{E}} (s - \varrho_\gamma)^{m(\varrho_\gamma)/2-1} \sum_{j=0}^{r+R-1} \tilde{\lambda}_j \tag{74}$$

and ψ also fulfills the partial differential equations:

$$
\begin{aligned}
\sum_{j=0}^{r+R-1} \tilde{\lambda}_j \varrho_\gamma^j \psi = -\prod_{\varrho_c \in \overset{c}{E} \setminus \{\varrho_\gamma\}} (\varrho_\gamma - \varrho_c) \Bigg[& \frac{1}{|\det g_\gamma|^{1/2}} \partial_{\overset{\gamma}{y}_i} g_\gamma^{ij} |\det g_\gamma|^{1/2} \partial_{\overset{\gamma}{y}_j} \psi \\
& - i \sum_{q=1}^{r} \frac{1}{|\det g_\gamma|^{1/2}} \partial_{\overset{\gamma}{y}_i} g_\gamma^{ij} |\det g_\gamma|^{1/2} \overset{\gamma}{\alpha}_{qj}\, \omega_q \psi \\
& - i \sum_{q=1}^{r} g_\gamma^{ij} \overset{\gamma}{\alpha}_{qi}\, \omega_q \partial_{\overset{\gamma}{y}_j} \psi \\
& - \sum_{p,q=1}^{r} g_\gamma^{ij} \overset{\gamma}{\alpha}_{qi} \overset{\gamma}{\alpha}_{pj}\, \omega_q \omega_p \psi \Bigg] \\
+ \frac{1}{\prod_{\varrho_c \in \overset{c}{E} \setminus \{\varrho_\gamma\}} (\varrho_\gamma - \varrho_c)^{m(\varrho_c)/2-1}} f_\gamma \psi &
\end{aligned}
\tag{75}
$$

for $\gamma = r+1, \ldots, r+R$.
The converse is also true: if a function ψ satisfies equations (73) and (75) for some constants $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{r+R-1}, \omega_1, \ldots, \omega_r$, then it is an eigenfunction of $\overset{t}{\hat{Q}}$ and $\overset{t}{\hat{L}}$.

A particular application of theorem 16 is in the treatment of the Laplace-Beltrami operator in the case where the number of integrals is maximal:

Theorem 17 *Let $\mathcal{M}$ be compact and without boundary. Let (g, J, Ω) a Kähler structure on $\mathcal{M}$ and g be positive definite. Let A be c-compatible with (g, J, Ω). Let all eigenvalues of A be non-constant.*
Then there exists a countable basis $(\psi_m, m \in \mathbb{N})$ in the space of square integrable functions $L^2(\mathcal{M})$, such that in any local normal coordinates $(\chi_1, \ldots, \chi_n, t_1, \ldots, t_n)$ the elements ψ_m of the basis can be written as

$$
\psi_m = \prod_{k=1}^{n} \psi_{m,k} \prod_{l=1}^{n} \exp(-i\omega_{m,l} t_l) \tag{76}
$$

where for $k = 1, \ldots, n$, $\psi_{m,k}$ (here the comma only separates two indices and is not a covariant derivative) is a function of the single variable χ_k that satisfies the ordinary differential equation

$$
\frac{-1}{\varepsilon_k \varrho_k'} \partial_{\chi_k} \varrho_k' \partial_{\chi_k} \psi_{m,k} + \sum_{i,j=1}^{r} \frac{\varepsilon_k (-\varrho_k)^{2r-i-j}}{(\varrho_k')^2} \omega_i \omega_j \psi_{m,k} = \sum_{i=0}^{n-1} \lambda_{m,i} \varrho_k^i \psi_{m,k} \tag{77}
$$

and $(\omega_{m,1}, \ldots, \omega_{m,n}, \lambda_{m,0}, \ldots, \lambda_{m,n-1})$ (here as well the comma only separates indices, furthermore the λs here is not related to the trace of A) are real constants. Furthermore *we have that*

$$
\begin{aligned}
\Delta\psi_m =& \lambda_{m,n-1}\psi_m \\
\overset{t}{\hat{I}}\ \psi_m =& \sum_{i=0}^{n-1} t^i \lambda_{m,i} \\
\overset{s}{\hat{L}}\ \psi_m =& \sum_{i=1}^{n} s^{n-i}\omega_{m,i}
\end{aligned}
\tag{78}
$$

Remark 13 (for the projective case) *A similar statement to theorem 17 can be found in [12].*

3 Proof of the results

3.1 Quantum integrals of the geodesic flow | Proof of theorem 11 and 12

We will be working in the c-projective setting. The differences between the projective and the c-projective setting will be pointed out in remarks.

The family $\overset{t}{\hat{I}}$ is a polynomial in t and therefore continuous. It is therefore sufficient to show that the commutator vanishes for all v and w that are are not in the spectrum of A. Otherwise we can consider two sequences $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ that converge to v and w where none of the elements of the sequence are in the spectrum of A. Then for each of the pairs (v_n, w_m) from the sequences the commutator $[\overset{v_n}{\hat{I}}, \overset{w_m}{\hat{I}}]$ will vanish and consequently it will vanish in the limit $(m,n)\to\infty$.
Equations (3.12), (3.13) and (3.14) in [7] give us the general formula for the commutator of two operators formed from arbitrary homogeneous polynomials P_2, Q_2 of degree two in momenta on $T^*\mathcal{M}$:

$$
\left[\hat{P}_2, \hat{Q}_2\right] = i\widehat{\{P_2, Q_2\}} + \frac{2}{3}(\nabla_j B^{jk}_{P_2,Q_2})\nabla_k \tag{79}
$$

Here $\{P_2, Q_2\}$ is the Poisson bracket of the two homogeneous polynomials P_2 and Q_2 (of degree two) on $T^*\mathcal{M}$. $\{P_2, Q_2\}$ is a polynomial of degree three in momenta. We recall that the "hat" over $\{P_2, Q_2\}$ is explained in (59): This polynomial is mapped to a differential operator according to

$$
\hat{\ } : P_3 \mapsto \hat{P}_3 \overset{\text{def}}{=} -\frac{i}{2}(\nabla_j \circ P_3^{jkl} \circ \nabla_k \circ \nabla_l + \nabla_j \circ \nabla_k \circ P_3^{jkl} \circ \nabla_l)
$$

where for a given polynomial P_3 the quantities P_3^{jkl} are chosen such that they are symmetric and $P_3 = P_3^{jkl} p_j p_k p_l$.

The tensor $B^{kl}_{P_2,Q_2}$ is given by the formula

$$B^{jk}_{P_2,Q_2} = P^{l[j}\,\nabla_l\nabla_m Q^{\,k]m} + P^{l[j}\,R^{\,k]}_{\ \ mnl} Q^{mn} - (P \leftrightarrow Q)$$
$$- \nabla_l P^{m[j}\,\nabla_m Q^{\,k]l} - P^{l[j}\,R_{lm} Q^{\,k]m} \quad (80)$$

The brackets around the indices mean taking the antisymmetric part. In the first term on the second line ∇_l only acts on P. The subtraction of $(P \leftrightarrow Q)$ is meant to act upon the two leftmost terms. For the two rightmost terms the antisymmetrization w.r.t. $(j \leftrightarrow k)$ is the same as if one were to antisymmetrize these terms w.r.t. $(P \leftrightarrow Q)$. For formula (80) it is important that the sign of the Riemann tensor is chosen such that $R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{ks}\Gamma^s_{lj} - \Gamma^i_{ls}\Gamma^s_{kj}$. But the reader may forget about it at once because it is not needed for our further investigations, as will be seen in the upcoming lemmata 18 and 19.
We plug the operators $\overset{v}{\hat{I}}$ and $\overset{w}{\hat{I}}$ into formula (79). Using theorem 5 we get:

$$[\overset{v}{\hat{I}}, \overset{w}{\hat{I}}] = \frac{2}{3}(\nabla_j B^{jk}_{\overset{v}{I},\overset{w}{I}})\nabla_k \tag{81}$$

Remark 14 *$i\{\widehat{P_2, Q_2}\}$ in formula (79) is a differential operator of order 3 while the other term on the right hand side is a differential operator of order one. Therefore it is a necessary condition for the quantities $\overset{v}{I}$ and $\overset{w}{I}$ to Poisson commute in order for their associated differential operators to commute. This is of course a long known fact.*

Remark 15 (for the projective case) *The fact that $\overset{t}{K}$ is a family of Killing tensors polynomial of degree $n-1$ in t and that their corresponding quadratic polynomials on $T^*\mathcal{M}$ Poisson commute pairwise can be found in [2]. Employing this instead of theorem 5 brings proof in the projective case to the point where only equation (82) needs to be verified.*

It remains to prove that

$$\nabla_j B^{jk}_{\overset{v}{I},\overset{w}{I}} = \nabla_j \Big((\overset{v}{K}{}^{l[j}\,\nabla_l \nabla_m \overset{w}{K}{}^{k]m} - (v \leftrightarrow w)) - \nabla_l \overset{v}{K}{}^{m[j}\,\nabla_m \overset{w}{K}{}^{k]l}$$
$$+ (\overset{v}{K}{}^{l[j}\,R^{\,k]}_{\ \ mnl} \overset{w}{K}{}^{mn} - (v \leftrightarrow w)) - \overset{v}{K}{}^{l[j}\,R_{lm} \overset{w}{K}{}^{k]m} \Big) = 0 \quad (82)$$

The proof will be split into three steps: first we show that $\overset{v}{K}{}^{l[j}\,R^{\,k]}_{\ \ mnl} \overset{w}{K}{}^{mn} - (v \leftrightarrow w) = 0$. In the second step we show that $\overset{v}{K}{}^{l[j}\,R_{lm} \overset{w}{K}{}^{k]m} = 0$. This will be done in the lemmata 18 and 19. These will reduce (82) to

$$\nabla_j \Big(\overset{v}{K}{}^{l[j}\,\nabla_l \nabla_m \overset{w}{K}{}^{k]m} - (v \leftrightarrow w) - \nabla_l \overset{v}{K}{}^{m[j}\,\nabla_m \overset{w}{K}{}^{k]l} \Big) = 0 \tag{83}$$

which we will show in the last step.

Lemma 18 $\overset{v}{K}{}^{l[j}\,R^{\,k]}_{\ \ mnl} \overset{w}{K}{}^{mn} - (v \leftrightarrow w) = 0$

Proof of lemma 18: It follows from item 2 of lemma 4 that in corollary 4.1 we may take $S = \sqrt{\det(wId - A)}(wId - A)^{-1}$, i.e. $S^{ij} = \overset{w}{K}{}^{ij}$. Plugging this into equation (22) and multiplying with $\sqrt{\det(vId - A)}$ gives:

$$2 \overset{v}{K}{}^{l[j} R^{k]}{}_{mnl} \overset{w}{K}{}^{mn} = 0 \tag{84}$$

Interchanging v and w, subtracting the result from this and dividing by 2 proves lemma 18.

Remark 16 (for the projective case) *Lemma 13 is true in the projective and the c-projective case. The proof for the projective case only requires to take* $S = \det(wId - A)(wId - A)^{-1}$ *and multiplying with* $\det(vId - A)$ *instead of* $\sqrt{\det(vId - A)}$.

Lemma 19 $\overset{v}{K}{}^{l[j} R_{lm} \overset{w}{K}{}^{k]m} = 0$

Proof of lemma 19 If we let $S = Id$ in corollary 4.1, then in formula (22) the multiplication with $S^{ij} = g^{ij}$ means contraction of the Riemann tensor to the negative of the Ricci tensor. Raising and lowering indices yields that $(vId - A)^{-1}$ commutes with the Ricci tensor when both are considered as endomorphisms on the space of vector fields:

$$(vId - A)^{-1}{}^{r}_{l} R^{k}_{r} - (vId - A)^{-1}{}^{k}_{r} R^{r}_{l} = 0 \tag{85}$$

Of course $(wId - A)^{-1}$ commutes with $(vId - A)^{-1}$ and the Ricci tensor as well, so by multiplying (85) with $(wId - A)^{-1}{}^{l}_{j}$ and using the commutativity gives

$$(wId - A)^{-1}{}^{k}_{r} R^{r}_{l} (vId - A)^{-1}{}^{l}_{j} - (vId - A)^{-1}{}^{k}_{r} R^{r}_{l} (wId - A)^{-1}{}^{l}_{j} = 0 \tag{86}$$

After multiplication of (86) with $\sqrt{\det(vId - A)\det(wId - A)}$ and raising and lowering indices, lemma 19 is proven.

Remark 17 (for the projective case) *Replacing the multiplication of* $\sqrt{\det(vId - A)}$ $\sqrt{\det(wId - A)}$ *with* $\det(vId - A)\det(wId - A)$ *is the only difference between the proof of lemma 19 in the projective and c-projective case.*

Having established the lemmata 18 and 19 we now compute the terms $\overset{v}{K}{}^{l[j} \nabla_l \nabla_m \overset{w}{K}{}^{k]m} - (v \leftrightarrow w)$ and $\nabla_m \overset{v}{K}{}^{l[j} \nabla_l \overset{w}{K}{}^{k]m}$ separately and then show that (83) is fulfilled to prove theorem 11.
Using the shorthand introduced in (43) we have already established that the covariant derivative of the Killing-tensor $\overset{t}{K}$ equals (see equation (45)):

$$\begin{aligned}\nabla_k \overset{t}{K}{}^{jl} = \sqrt{\det(tId - A)}[&-2 \overset{t}{M}{}^{s}_{k}\lambda_s \overset{t}{M}{}^{j}_{r} g^{rl} \\ &+ \overset{t}{M}{}^{j}_{p}\lambda^p \overset{t}{M}{}^{l}_{k} + \overset{t}{M}{}^{j}_{k} \overset{t}{M}{}^{l}_{q}\lambda^q - \overset{t}{M}{}^{j}_{p} g^{ps}\bar{\lambda}_s \overset{t}{M}{}^{l}_{q} J^{q}_{k} - \overset{t}{M}{}^{l}_{r} g^{rq}\bar{\lambda}_q \overset{t}{M}{}^{j}_{p} J^{p}_{k}]\end{aligned} \tag{87}$$

Contracting the indices k and l we get:

$$\nabla_k \overset{t}{K}{}^{jk} = \sqrt{\det(tId - A)}[-2 \overset{t}{M}{}^j_l \overset{t}{M}{}^l_s \lambda^s + \overset{t}{M}{}^j_l \lambda^l \operatorname{tr}(\overset{t}{M})] \tag{88}$$

For the derivative of this expression we receive

$$\begin{aligned}
\nabla_l \nabla_m \overset{t}{K}{}^{km} = {} & \sqrt{\det(tId - A)}[2 \overset{t}{M}{}^r_l \lambda_r \lambda_s g^{ps} \overset{t}{M}{}^m_p \overset{t}{M}{}^k_m + 2 \overset{t}{M}{}^q_l \overset{t}{M}{}^r_q \lambda_r \lambda_s g^{ps} \overset{t}{M}{}^k_p \\
& + 2 \overset{t}{M}{}^r_l \bar{\lambda}_r \bar{\lambda}_s g^{ps} \overset{t}{M}{}^m_p \overset{t}{M}{}^k_m + 2 \overset{t}{M}{}^q_l \overset{t}{M}{}^r_q \bar{\lambda}_r \bar{\lambda}_s g^{ps} \overset{t}{M}{}^k_p \\
& - \overset{t}{M}{}^r_l \lambda_r \lambda_s g^{sp} \overset{t}{M}{}^k_p \operatorname{tr}(\overset{t}{M}) - \overset{t}{M}{}^r_l \bar{\lambda}_r \bar{\lambda}_s g^{sp} \overset{t}{M}{}^k_p \operatorname{tr}(\overset{t}{M}) \\
& - 2 \overset{t}{M}{}^k_l \, g(\overset{t}{M}\Lambda, \overset{t}{M}\Lambda) - 2 \overset{t}{M}{}^k_r \overset{t}{M}{}^r_l \, g(\overset{t}{M}\Lambda, \Lambda) + \overset{t}{M}{}^k_l \, g(\overset{t}{M}\Lambda, \Lambda) \operatorname{tr}(\overset{t}{M}) \\
& - 2\lambda_{s,l} g^{rs} \overset{t}{M}{}^p_r \overset{t}{M}{}^k_p + \lambda_{s,l} g^{ps} \overset{t}{M}{}^k_p \operatorname{tr}(\overset{t}{M})]
\end{aligned} \tag{89}$$

We shall again denote by $(\overset{w}{M}\Lambda)^k$ the k-th component of $\overset{w}{M}(\Lambda)$. Now multiplying the previous equation with $\overset{v}{K}{}^{lj}$ and antisymmetrizing with respect to $(j \leftrightarrow k)$ and $(v \leftrightarrow w)$ gives

$$\begin{aligned}
\overset{v}{K}{}^{l[j} \nabla_l \nabla_m \overset{w}{K}{}^{k]m} - (v \leftrightarrow w) = {} & \frac{1}{2} \sqrt{\det(vId - A)} \sqrt{\det(wId - A)} \\
& \cdot [[[2(\overset{v}{M} \overset{w}{M}\Lambda)^j (\overset{w}{M}{}^2 \Lambda)^k + 2(\overset{v}{M} \overset{w}{M}{}^2 \Lambda)^j (\overset{w}{M}\Lambda)^k \\
& - (\overset{v}{M} \overset{w}{M}\Lambda)^j (\overset{w}{M}\Lambda)^k \operatorname{tr}(\overset{w}{M})] \\
& + (\Lambda \leftrightarrow \bar{\Lambda})] - (j \leftrightarrow k)] - (v \leftrightarrow w)
\end{aligned} \tag{90}$$

Here $(\Lambda \leftrightarrow \bar{\Lambda})$ indicates that the previous bracket shall be added with Λ replaced by $\bar{\Lambda}$, $(j \leftrightarrow k)$ indicates antisymmetrization with respect to j and k, likewise for $(v \leftrightarrow w)$. In (90) the terms from (89) involving second derivatives of λ have cancelled out as a consequence of lemma 4, item 2. When forming the right hand side expression of (90) the terms of the second to last row of (89) cancel each other out after the antisymmetrization $(j \leftrightarrow k)$ due to $(vId - A)$ and $(wId - A)$ commuting and being self-adjoint to g.

Remark 18 (for the projective case) *To get the formula for* $\overset{v}{K}{}^{l[j} \nabla_l \nabla_m \overset{w}{K}{}^{k]m} - (v \leftrightarrow w)$ *in the projective case we perform the same steps, using* (20) *instead of the c-projective formula* (17)*. The next three equations give projective analogues of the formulae* (87), (89) *and* (90)*:*

$$\nabla_k \overset{t}{K}{}^{jl} = \det(tId - A)[-2 \overset{t}{M}{}^s_k \lambda_s \overset{t}{M}{}^j_r g^{rl} + \overset{t}{M}{}^j_p \lambda^p \overset{t}{M}{}^l_k + \overset{t}{M}{}^j_k \overset{t}{M}{}^l_q \lambda^q] \tag{91}$$

$$
\begin{aligned}
\nabla_l\nabla_m \overset{t}{K}{}^{km} = \det(tId-A)[&\overset{t}{M}{}^r_l\lambda_r\lambda_s g^{ps} \overset{t}{M}{}^m_p \overset{t}{M}{}^k_m + \overset{t}{M}{}^q_l \overset{t}{M}{}^r_q\lambda_r\lambda_s g^{ps} \overset{t}{M}{}^k_p \\
&- \overset{t}{M}{}^r_l\lambda_r\lambda_s g^{sp} \overset{t}{M}{}^k_p \operatorname{tr}(\overset{t}{M}) \\
&- \overset{t}{M}{}^k_l\ g(\overset{t}{M}\Lambda, \overset{t}{M}\Lambda) - \overset{t}{M}{}^k_r \overset{t}{M}{}^r_l\ g(\overset{t}{M}\Lambda, \Lambda) \\
&+ \overset{t}{M}{}^k_l\ g(\overset{t}{M}\Lambda, \Lambda)\operatorname{tr}(\overset{t}{M}) \\
&- \lambda_{s,l} g^{rs} \overset{t}{M}{}^p_r \overset{t}{M}{}^k_p + \lambda_{s,l} g^{ps} \overset{t}{M}{}^k_p \operatorname{tr}(\overset{t}{M})]
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
\overset{v}{K}{}^{l[j}\nabla_l\nabla_m \overset{w}{K}{}^{k]m} - (v\leftrightarrow w) = &\frac{1}{2}\det(vId-A)\det(wId-A) \\
&\cdot[[(\overset{v}{M}\ \overset{w}{M}\Lambda)^j(\overset{w}{M}{}^2\Lambda)^k + (\overset{v}{M}\ \overset{w}{M}{}^2\Lambda)^j(\overset{w}{M}\Lambda)^k \\
&- (\overset{v}{M}\ \overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\operatorname{tr}(\overset{w}{M})] \\
&- (j\leftrightarrow k)] - (v\leftrightarrow w)
\end{aligned}
\tag{93}
$$

Resuming the proof in the c-projective case we use $\overset{v}{M} - \overset{w}{M} = (w-v)\overset{v}{M}\ \overset{w}{M}$ (see equations (46), (47)), as well as the trace applied to this matrix identity to expand (90):

$$
\begin{aligned}
\overset{v}{K}{}^{l[j}\nabla_l\nabla_m \overset{w}{K}{}^{k]m} - (v\leftrightarrow w) = &\frac{1}{2}\sqrt{\det(vId-A)}\sqrt{(wId-A)}\cdot[[[(w-v)^{-1} \\
&[2(\overset{v}{M}\Lambda)^j(\overset{w}{M}{}^2\Lambda)^k - 2(\overset{w}{M}\Lambda)^j(\overset{w}{M}{}^2\Lambda)^k \\
&+ 2(\overset{v}{M}\ \overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - 2(\overset{w}{M}{}^2\Lambda)^j(\overset{w}{M}\Lambda)^k \\
&- (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\operatorname{tr}(\overset{w}{M}) + (\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\operatorname{tr}(\overset{w}{M})]] \\
&- (v\leftrightarrow w)] - (j\leftrightarrow k)] + (\Lambda\leftrightarrow\bar{\Lambda})
\end{aligned}
$$

We strike out terms that cancel after antisymmetrization w.r.t. $(j\leftrightarrow k)$:

$$
\begin{aligned}
\ldots = &\frac{1}{2}\sqrt{\det(vId-A)}\sqrt{\det(wId-A)}\cdot[[[(w-v)^{-1} \\
&\cdot[2(\overset{v}{M}\Lambda)^j(\overset{w}{M}{}^2\Lambda)^k + 2(\overset{v}{M}\ \overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k \\
&- (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k\operatorname{tr}(\overset{w}{M})] \\
&- (v\leftrightarrow w)] - (j\leftrightarrow k)] + (\Lambda\leftrightarrow\bar{\Lambda})
\end{aligned}
\tag{94}
$$

We expand $(v \leftrightarrow w)$ and $(j \leftrightarrow k)$. The sign of the antisymmetrization is caught in the prefactor $(w-v)^{-1}$:

$$
\begin{aligned}
\ldots = &\frac{1}{2}\sqrt{\det(vId-A)}\sqrt{\det(wId-A)}(w-v)^{-1} \\
&\cdot [2(\overset{v}{M}\Lambda)^j(\overset{w}{M}{}^2\Lambda)^k + 2(\overset{w}{M}\Lambda)^j(\overset{v}{M}{}^2\Lambda)^k \\
&+ 2(\overset{v}{M}\,\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k + 2(\overset{w}{M}\,\overset{v}{M}\Lambda)^j(\overset{v}{M}\Lambda)^k \\
&- 2(\overset{v}{M}\Lambda)^k(\overset{w}{M}{}^2\Lambda)^j - 2(\overset{w}{M}\Lambda)^k(\overset{v}{M}{}^2\Lambda)^j \\
&- 2(\overset{v}{M}\,\overset{w}{M}\Lambda)^k(\overset{w}{M}\Lambda)^j - 2(\overset{w}{M}\,\overset{v}{M}\Lambda)^k(\overset{v}{M}\Lambda)^j \\
&- (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k \operatorname{tr}(\overset{w}{M}) - (\overset{w}{M}\Lambda)^j(\overset{v}{M}\Lambda)^k \operatorname{tr}(\overset{v}{M}) \\
&+ (\overset{v}{M}\Lambda)^k(\overset{w}{M}\Lambda)^j \operatorname{tr}(\overset{w}{M}) + (\overset{w}{M}\Lambda)^k(\overset{v}{M}\Lambda)^j \operatorname{tr}(\overset{v}{M})] \\
&+ (\Lambda \leftrightarrow \bar{\Lambda})
\end{aligned} \tag{95}
$$

We can now apply $\overset{v}{M} - \overset{w}{M} = (w-v)\,\overset{v}{M}\,\overset{w}{M}$ and $\operatorname{tr}(\overset{v}{M}) - \operatorname{tr}(\overset{w}{M}) = (w-v)\operatorname{tr}(\overset{v}{M}\,\overset{w}{M})$ in the opposite direction as before, pairing terms (1,8), (2,7), (3,6), (4,5), (9,12), (10,11) in the bracket:

$$
\overset{v}{K}{}^{l[j}\,\nabla_l\nabla_m\,\overset{w}{K}{}^{k]m} - (v \leftrightarrow w) = \frac{1}{2}\sqrt{\det(vId-A)}\sqrt{\det(wId-A)} \tag{96}
$$

$$
\begin{aligned}
&\cdot [[-2(\overset{v}{M}{}^2\,\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - 2(\overset{v}{M}\Lambda)^j(\overset{v}{M}\,\overset{w}{M}{}^2\Lambda)^k \\
&+ (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k \operatorname{tr}(\overset{v}{M}\cdot\overset{w}{M})] \\
&- (j \leftrightarrow k)] + (\Lambda \leftrightarrow \bar{\Lambda})
\end{aligned} \tag{97}
$$

Remark 19 (for the projective case) *Performing the same steps on* (93) *gives*

$$
\begin{aligned}
\overset{v}{K}{}^{l[j}\,\nabla_l\nabla_m\,\overset{w}{K}{}^{k]m} - (v \leftrightarrow w) = &\frac{1}{2}\det(vId-A)\det(wId-A) \\
&\cdot [[-(\overset{v}{M}{}^2\,\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - (\overset{v}{M}\Lambda)^j(\overset{v}{M}\,\overset{w}{M}{}^2\Lambda)^k \\
&+ (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k \operatorname{tr}(\overset{v}{M}\cdot\overset{w}{M})] - (j \leftrightarrow k)]
\end{aligned} \tag{98}
$$

for the projective scenario.

We have now worked $\overset{v}{K}{}^{l[j}\,\nabla_l\nabla_m\,\overset{w}{K}{}^{k]m} - (v \leftrightarrow w)$ into a suitable form. From (87) we now compute $\nabla_m\,\overset{v}{K}{}^{l[j}\,\nabla_l\,\overset{w}{K}{}^{k]m}$:

$$
\begin{aligned}
\nabla_m\,\overset{v}{K}{}^{l[j}\,\nabla_l\,\overset{w}{K}{}^{k]m} = &\frac{1}{2}\sqrt{\det(vId-A)}\sqrt{\det(wId-A)} \\
&\cdot [[-2(\overset{v}{M}{}^2\,\overset{w}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k - 2(\overset{v}{M}\Lambda)^j(\overset{v}{M}\,\overset{w}{M}{}^2\Lambda)^k \\
&+ (\overset{v}{M}\Lambda)^j(\overset{w}{M}\Lambda)^k \operatorname{tr}(\overset{v}{M}\cdot\overset{w}{M})] \\
&- (j \leftrightarrow k)] - (\Lambda \leftrightarrow \bar{\Lambda})
\end{aligned} \tag{99}
$$

In this equation $(j \leftrightarrow k)$ yields the same result as $(v \leftrightarrow w)$.

Remark 20 (for the projective case) *By means of* (91) *the projective analogue of* (99) *evaluates to*

$$
\begin{aligned}
\nabla_m \overset{v}{K}{}^{l[j} \nabla_l \overset{w}{K}{}^{k]m} =& \frac{1}{2} \det(vId - A) \det(wId - A) \\
& \cdot [[-2(\overset{v}{M}{}^2 \overset{w}{M} \Lambda)^j (\overset{w}{M} \Lambda)^k - 2(\overset{v}{M} \Lambda)^j (\overset{v}{M} \overset{w}{M}{}^2 \Lambda)^k \\
& + (\overset{v}{M} \Lambda)^j (\overset{w}{M} \Lambda)^k \operatorname{tr}(\overset{v}{M} \cdot \overset{w}{M})] - (j \leftrightarrow k)]
\end{aligned} \tag{100}
$$

We see that the right hand side expression is equal to the right hand side expression of (98)*. Thus if we plug* (98) *and* (100) *into* (83) *then both terms cancel each other and* (83) *is satisfied without even having to carry out the differentiation, concluding the proof of theorem 12. The fact that in the projective case* $B^{jk}_{\underset{I,I}{v\ w}}$ *vanishes, whereas in the c-projective case* $\nabla_j B^{jk}_{\underset{I,I}{v\ w}}$ *vanishes but* $B^{jk}_{\underset{I,I}{v\ w}}$ *does not is the most significant difference between the projective and the c-projective case.*

We now compare (94) and (99): they are the same except the first is symmetric with respect to $(\Lambda \leftrightarrow \bar{\Lambda})$ while the latter is antisymmetric. Subtracting both consequently yields:

$$
\begin{aligned}
\overset{v}{K}{}^{l[j} \nabla_l \nabla_m \overset{w}{K}{}^{k]m} & - (v \leftrightarrow w) - \nabla_m \overset{v}{I}{}^{l[j} \nabla_l \overset{w}{I}{}^{k]m} \\
=& \sqrt{\det(vId - A)} \sqrt{\det(wId - A)} \\
& \cdot [-2(\overset{v}{M}{}^2 \overset{w}{M} \bar{\Lambda})^j (\overset{w}{M} \bar{\Lambda})^k - 2(\overset{v}{M} \bar{\Lambda})^j (\overset{v}{M} \overset{w}{M}{}^2 \bar{\Lambda})^k \\
& + (\overset{v}{M} \bar{\Lambda})^j (\overset{w}{M} \bar{\Lambda})^k \operatorname{tr}(\overset{v}{M} \cdot \overset{w}{M})] - (v \leftrightarrow w)
\end{aligned} \tag{101}
$$

It remains to show that (83) is fulfilled, that is to apply ∇_j to this expression and show that this vanishes. In the computation we use

- the compatibility condition (17)
- Jacobi's formula for the derivative of the determinant
- $\mathrm{d}(A^{-1}) = -A^{-1} \cdot \mathrm{d}(A) \cdot A^{-1}$

to expand the left hand side expression of (83). We then immediately strike out terms that vanish individually due to the self-adjointness of A with respect to g and the antisymmetry of J with respect to g:

$$\begin{aligned}
\nabla_j(\overset{v}{K}{}^{l[j}\nabla_l\nabla_m \overset{w}{K}{}^{k]m} - (v \leftrightarrow w) - \nabla_m \overset{v}{K}{}^{l[j}\nabla_l \overset{w}{K}{}^{k]m}) \\
= &- 2\sqrt{\det(vId-A)}\sqrt{\det(wId-A)} \\
&\cdot [2g(\overset{v}{M}\ \overset{w}{M}\bar{\Lambda}, \overset{v}{M}\ \overset{w}{M}\bar{\Lambda})(\overset{w}{M}\Lambda - \overset{v}{M}\Lambda)^k \\
&- 2g(\overset{v}{M}\ \overset{w}{M}\bar{\Lambda}, \overset{w}{M}\bar{\Lambda} - \overset{v}{M}\bar{\Lambda})(\overset{v}{M}\ \overset{w}{M}\Lambda)^k \\
&+ 2g(\bar{\Lambda}, \overset{v}{M}\ \overset{w}{M}\bar{\Lambda})](\overset{v}{M}\ \overset{w}{M}(\overset{w}{M}\Lambda - \overset{v}{M}\Lambda))^k \\
&- 2g(\bar{\Lambda}, \overset{w}{M}\bar{\Lambda} - \overset{v}{M}\bar{\Lambda})(\overset{v}{M}{}^2\ \overset{w}{M}{}^2\Lambda)^k \\
&+ \bar{\lambda}_s g^{ls}\ \overset{v}{M}{}^m_l\ \overset{w}{M}{}^t_m\ \overset{v}{M}{}^j_t\bar{\lambda}_{p,j}g^{pq}\ \overset{w}{M}{}^k_q \\
&- \bar{\lambda}_s g^{ls}\ \overset{w}{M}{}^m_l\ \overset{v}{M}{}^t_m\ \overset{w}{M}{}^j_t\bar{\lambda}_{p,j}g^{pq}\ \overset{v}{M}{}^k_q \\
&+ \bar{\lambda}_s g^{ps}\ \overset{v}{M}{}^j_p\bar{\lambda}_{t,j}g^{tr}\ \overset{w}{M}{}^l_r\ \overset{v}{M}{}^m_l\ \overset{w}{M}{}^k_m \\
&- \bar{\lambda}_s g^{ps}\ \overset{w}{M}{}^j_p\bar{\lambda}_{t,j}g^{tr}\ \overset{v}{M}{}^l_r\ \overset{w}{M}{}^m_l\ \overset{v}{M}{}^k_m \\
&+ \mathrm{tr}(\overset{v}{M}\ \overset{w}{M}) \cdot [g(\bar{\Lambda}, \overset{v}{M}\ \overset{w}{M}\bar{\Lambda})(\overset{w}{M}\Lambda - \overset{v}{M}\Lambda)^k \\
&- g(\bar{\Lambda}, \overset{w}{M}\bar{\Lambda} - \overset{v}{M}\bar{\Lambda})(\overset{v}{M}\ \overset{w}{M}\Lambda)^k \\
&+ \bar{\lambda}_s g^{ps}\ \overset{v}{M}{}^j_p\bar{\lambda}_{t,j}g^{tr}\ \overset{w}{M}{}^k_r - \bar{\lambda}_s g^{ps}\ \overset{w}{M}{}^j_p\bar{\lambda}_{t,j}g^{tr}\ \overset{v}{M}{}^k_r]
\end{aligned} \tag{102}$$

As a consequence of item 2 of lemma 4 the terms involving second derivatives of λ cancel each other out in this expression. The other terms cancel each other out after applying $\overset{v}{M} - \overset{w}{M} = (w-v)\ \overset{v}{M}\ \overset{w}{M}$. Thus theorem 11 is proven.

3.2 Addition of potential | Proof of theorem 13 and 14

Throughout this section we will be working on a c-compatible structure (g, J, A).

3.2.1 Four equivalent problems

Lemma 20 *Let* $K = g^{ij}p_ip_j$,

$$\overset{t}{I} \overset{\text{def}}{=} \overset{t}{K}{}^{jk}p_jp_k, \qquad \overset{t}{K}{}^{ij} \overset{\text{def}}{=} \sqrt{\det(tId-A)}\ (tId-A)^{-1}{}^i_l g^{lj}$$

and

$$\overset{t}{L} = \overset{t}{V}{}^j p_j, \qquad \overset{t}{V}{}^j = J^j_k g^{ki}\nabla_i\sqrt{\det(tId-A)}$$

as well as the corresponding differential operators according to the quantization rules stated earlier.
Then the following four problems are equivalent: describe all functions U, $\overset{t}{U}$, *such that*

1. $\{\overset{s}{I}+\overset{s}{U},K+U\}=0$ *and* $\{\overset{s}{I}+\overset{s}{U},\overset{t}{L}\}=0 \quad \forall t,s\in\mathbb{R}$

2. $[\overset{s}{\hat{I}}+\overset{s}{\hat{U}},\hat{K}+\hat{U}]=0$ *and* $[\overset{s}{\hat{I}}+\overset{s}{\hat{U}},\overset{t}{\hat{L}}]=0 \quad \forall t,s\in\mathbb{R}$

3. $\{\overset{s}{I}+\overset{s}{U},\overset{t}{I}+\overset{t}{U}\}=0$ *and* $\{\overset{s}{I}+\overset{s}{U},\overset{t}{L}\}=0 \quad \forall t,s\in\mathbb{R}$

4. $[\overset{s}{\hat{I}}+\overset{s}{\hat{U}},\overset{t}{\hat{I}}+\overset{t}{\hat{U}}]=0$ *and* $[\overset{s}{\hat{I}}+\overset{s}{\hat{U}},\overset{t}{\hat{L}}]=0 \quad \forall t,s\in\mathbb{R}$

Proof of lemma 20: To do so, we show that

i. $\{\overset{t}{I}+\overset{t}{U},K+U\}=0 \quad \forall t\in\mathbb{R} \quad \Leftrightarrow \quad [\overset{t}{\hat{I}}+\overset{t}{\hat{U}},\hat{K}+\hat{U}]=0 \quad \forall t\in\mathbb{R}$
$\Leftrightarrow \quad \overset{t}{K}{}^i_j\frac{\partial U}{\partial x^i}=\frac{\partial \overset{t}{U}}{\partial x^j} \quad \forall t\in\mathbb{R}$

ii. $\{\overset{t}{I}+\overset{t}{U},K+U\}=0 \quad \forall t\in\mathbb{R} \quad \Leftrightarrow \quad \{\overset{t}{I}+\overset{t}{U},\overset{s}{I}+\overset{s}{U}\}=0 \quad \forall s,t\in\mathbb{R}$

iii. $\{\overset{t}{I}+\overset{t}{U},\overset{s}{I}+\overset{s}{U}\}=0 \quad \forall s,t\in\mathbb{R} \quad \Leftrightarrow \quad [\overset{t}{\hat{I}}+\overset{t}{\hat{U}},\overset{s}{\hat{I}}+\overset{s}{\hat{U}}]=0 \quad \forall s,t\in\mathbb{R}$
$\Leftrightarrow \quad \overset{t}{K}{}^i_j\frac{\partial \overset{s}{U}}{\partial x^i}=\overset{s}{K}{}^i_j\frac{\partial \overset{t}{U}}{\partial x^i}\forall s,t\in\mathbb{R}$

iv. $\{\overset{t}{I}+\overset{t}{U},\overset{s}{L}\}=0 \quad \forall t,s\in\mathbb{R} \quad \Leftrightarrow \quad [\overset{t}{\hat{I}}+\overset{t}{\hat{U}},\overset{s}{\hat{L}}]=0 \quad \forall t,s\in\mathbb{R}$
$\Leftrightarrow \quad \mathrm{d}\,\overset{t}{U}(\overset{s}{V})=0 \quad \forall s,t\in\mathbb{R}$

It is implied that all equations are to hold for all choices of their parameters, we shall not specify it each and every time again.
To iii: We use the linearity of the commutator:

$$[\overset{s}{\hat{I}}+\overset{s}{\hat{U}},\overset{t}{\hat{I}}+\overset{t}{\hat{U}}]=[\overset{s}{\hat{I}},\overset{t}{\hat{I}}]+[\overset{s}{\hat{I}},\overset{t}{\hat{U}}]+[\overset{s}{\hat{U}},\overset{t}{\hat{I}}]+[\overset{s}{\hat{U}},\overset{t}{\hat{U}}] \tag{103}$$

The term $[\overset{s}{\hat{I}},\overset{t}{\hat{I}}]$ vanishes due to theorem 11 and $[\overset{s}{\hat{U}},\overset{t}{\hat{U}}]$ vanishes trivially since the operators corresponding to the potentials act merely by multiplication. In [7] or by direct computation we have $[\overset{s}{\hat{I}},\overset{t}{\hat{U}}]=\widehat{\{\overset{s}{I},\overset{t}{U}\}}$. Consequently $[\overset{s}{\hat{I}}+\overset{s}{\hat{U}},\overset{t}{\hat{I}}+\overset{t}{\hat{U}}]=\widehat{\{\overset{s}{I},\overset{t}{U}\}}+\widehat{\{\overset{s}{U},\overset{t}{I}\}}$. Since quantization is a linear map and only the zero polynomial is mapped to a vanishing differential operator, we have that $[\overset{s}{\hat{Q}},\overset{t}{\hat{Q}}]=0$ if and only if $\{\overset{s}{I},\overset{t}{U}\}+\{\overset{s}{U},\overset{t}{I}\}=0$. This in turn is true if and only if $\{\overset{s}{I},\overset{t}{U}\}^\sharp+\{\overset{s}{U},\overset{t}{I}\}^\sharp=0$. Expressing this in terms of $\overset{t}{K},\overset{t}{U},\overset{s}{K},\overset{s}{U}$ and lowering an index and rearranging terms yields $\overset{t}{K}{}^i_j\frac{\partial \overset{s}{U}}{\partial x^i}=\overset{s}{K}{}^i_j\frac{\partial \overset{t}{U}}{\partial x^i}$. Likewise using the fact that $\{\overset{s}{I},\overset{t}{I}\}=0$, we have $\{\overset{t}{I}+\overset{t}{U},\overset{s}{I}+\overset{s}{U}\}=\{\overset{s}{I},\overset{t}{U}\}+\{\overset{s}{U},\overset{t}{I}\}$.
Statement i can be seen analogously to iii since K lies in the span of $\overset{t}{K}$.
To ii: It suffices to show the equivalence of the rightmost equations of items i and iii: Fix an arbitrary value for t in $\overset{t}{K}{}^i_j\frac{\partial \overset{s}{U}}{\partial x^i}=\overset{s}{K}{}^i_j\frac{\partial \overset{t}{U}}{\partial x^i}$ and choose pairwise different

values $(s_1,\ldots,s_n)$ for s. Add the resulting equations, weighting the i^{th} equation with $(-1)^{n-1}\mu_{n-1}(\hat{s}_i)/\prod_{i\neq j}(s_i-s_j)$. Here $\mu_{n-1}(\hat{s}_i)$ is the elementary symmetric polynomial of degree $n-1$ in the variables $(s_1,\ldots,s_{i-1},s_{i+1},\ldots s_n)$. On the right hand side this gives the coefficient of s^{n-1} of $\overset{s}{K}$ which is the identity operator (when considered as a (1,1)-tensor) acting on the differential of $\overset{t}{U}$. On the left hand side we identify the sum $\sum_{i=1}^{n}(-1)^{n-1}\mu_{n-1}(\hat{s}_i)/(\prod_{i\neq j}(s_i-s_j))\frac{\partial \overset{s_i}{U}}{\partial x^i}$ with the differential of U and thus arrive at $\overset{t}{K}{}^{i}_{j}\frac{\partial U}{\partial x^i}=\frac{\partial \overset{t}{U}}{\partial x^j}$.
For the other direction, consider two arbitrary values s and t and the equations

$$\overset{t}{K}{}^{i}_{j}\frac{\partial U}{\partial x^i}=\frac{\partial \overset{t}{U}}{\partial x^j},\qquad \overset{s}{K}{}^{i}_{j}\frac{\partial U}{\partial x^i}=\frac{\partial \overset{s}{U}}{\partial x^j} \tag{104}$$

We multiply the first equation with $\overset{s}{K}{}^{j}_{k}$, and use the commutativity of $\overset{t}{K}$ with $\overset{s}{K}$ (again considered as mapping one-forms to one-forms):

$$\overset{t}{K}{}^{j}_{k}\,\overset{s}{K}{}^{i}_{j}\frac{\partial U}{\partial x^i}=\overset{s}{K}{}^{j}_{k}\frac{\partial \overset{t}{U}}{\partial x^j} \tag{105}$$

Now we can use the second equation of (104) to replace $\overset{s}{K}{}^{i}_{j}\frac{\partial U}{\partial x^i}$ with $\frac{\partial \overset{s}{U}}{\partial x^j}$ arriving back at $\overset{t}{K}{}^{i}_{j}\frac{\partial \overset{s}{U}}{\partial x^i}=\overset{s}{K}{}^{i}_{j}\frac{\partial \overset{t}{U}}{\partial x^i}$, as we desired.
To iv: Whenever one applies the quantization rules (59) to a linear polynomial $\overset{s}{L}$ and a polynomial $\overset{t}{I}$ of second degree in the momentum variables on $T^*\mathcal{M}$ and takes the commutator of the operators, then combining equations (3.8) and (3.9) from [7] gives us the formula

$$[\overset{t}{\hat{I}},\overset{s}{\hat{L}}]=i\widehat{\{\overset{t}{I},\overset{s}{L}\}}-\frac{i}{2}\nabla_j(\overset{t}{K}{}^{jk}\nabla_k(\nabla_l\overset{s}{V}{}^{l})) \tag{106}$$

It can be obtained via explicit calculation. The first term on the right hand side vanishes because $\{\overset{t}{I},\overset{s}{L}\}=0$ (theorem 6). The second term on the right hand side of (106) acts on functions by mere multiplication. It vanishes in our case because $\overset{t}{V}$ is a Killing vector field and thus divergence free. Using $[\overset{t}{\hat{I}},\overset{s}{\hat{L}}]=0$ and $\{\overset{t}{I},\overset{s}{L}\}=0$, a direct calculation immediately reveals that both $\{\overset{t}{I}+\overset{t}{U},\overset{s}{L}\}=0$ and $[\overset{t}{\hat{I}}+\overset{t}{\hat{U}},\overset{s}{\hat{L}}]=0$ reduce to the same expression, namely $\mathrm{d}\,\overset{t}{U}(\overset{s}{V})=0\ \forall s,t\in\mathbb{R}$, concluding the proof of lemma 20.

Lemma 21 *Let (g,J,A) be c-compatible and $\overset{t}{K}$ be defined as in* (36). *Consider a simply connected domain where the number of different eigenvalues of A is constant. Let A be semi-simple. Let $\overset{\mathrm{nc}}{E}=\{\varrho_1,\ldots,\varrho_r\}$ be the set of non-constant eigenvalues of A. Let $\overset{\mathrm{c}}{E}=\{\varrho_{r+1},\ldots,\varrho_{r+R}\}$ be the set of constant eigenvalues and $E=\overset{\mathrm{nc}}{E}\cup\overset{\mathrm{c}}{E}$. Denote by*

$m(\varrho_i)$ the algebraic multiplicity of ϱ_i. Let U be a function such that $\overset{t}{K}{}^i_j \frac{\partial U}{\partial x^i}$ is exact for all values of t and let $\overset{t}{U}$ be such that

$$\overset{t}{K}{}^i_j \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^j} \tag{107}$$

is satisfied for all values of t. Then up to addition of a function of the single variable t the family of functions $\overset{t}{U}(t,x)$ may be written as

$$\overset{t}{U} = \prod_{\varrho_l \in \overset{c}{E}} (t-\varrho_l)^{m(\varrho_l)/2-1} \overset{t}{\tilde{U}} \tag{108}$$

where $\overset{t}{\tilde{U}}$ is a polynomial of degree $r-1$ in t. Equally $\overset{t}{U}$ can be written as

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t-\varrho_l)^{m(\varrho_l)/2}}{(\varrho_i-\varrho_l)^{m(\varrho_l)/2}} (t-\varrho_i)^{m(\varrho_i)/2-1} f_i \tag{109}$$

where f_i are functions on $\mathcal{M}$. The functions f_i may however not be chosen arbitrarily.

Proof of lemma 21: Because we assumed that A is semi-simple, we can factorize $\overset{t}{K}$ into $\overset{t}{K} = \prod_{l=1}^{r+R} (t-\varrho_l)^{m(\varrho_l)/2-1} \overset{t}{\tilde{K}}$, with $\overset{t}{\tilde{K}}$ being a polynomial of degree $r+R-1$.

$$\prod_{\varrho_l \in \overset{c}{E}} (t-\varrho_l)^{m(\varrho_l)/2-1} \overset{t}{\tilde{K}}{}^i_j \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^i} \quad \forall t \in \mathbb{R} \tag{110}$$

We used that the non-constant eigenvalues $\varrho_1, \dots, \varrho_r$ all have multiplicity 2(lemma 10, item 1). Thus in the product on the left hand side all factors corresponding to non-constant eigenvalues are equal to 1. We observe that upon addition of a function of the single variable t to $\overset{t}{U}$ the equation above is still satisfied. This allows us to choose an arbitrary point x_0 and an arbitrary function $U_0(t)$ and assume that $\overset{t}{U}(x_0, t) = U_0(t)$. Since the left hand side of (110) is a polynomial in t and is exact for all t, each of the coefficients must be exact. This allows us to integrate the terms of (110) individually:

$$\overset{t}{U}(t,x) = U_0(t) + \int_{x_0}^{x} \overset{t}{K}(\mathrm{d}U) = U_0(t) + \prod_{\varrho_l \in \overset{c}{E}} (t-\varrho_l)^{m(\varrho_l)/2-1} \sum_{i=0}^{r} t^i \int_{x_0}^{x} \tilde{K}_{(i)}(\mathrm{d}U) \tag{111}$$

The last step of this calculation makes use of the fact that if ϱ_l is of multiplicity $m(\varrho_l) \geq 4$ then ϱ_l is constant (lemma 10, item 1). The integral is meant to be taken along any path connecting x_0 and x and the $\tilde{K}_{(i)}$ is the coefficient of t^i in $\overset{t}{\tilde{K}}$. Again

$\overset{t}{\tilde{K}}$ and $\tilde{K}_{(i)}$ are considered as $(1,1)$ tensors mapping 1-forms to 1-forms. So for any value of t the value of $\overset{t}{U}$ at x is uniquely defined by its value at x_0 and the function U. Formula (111) proves the claim that $\overset{t}{U}$ can be written in the form (108) where on the right hand side $U_0(t)$ takes the role of the possible addition of a function of t alone. Evidently, we have $\overset{t}{\tilde{U}}(x) = \sum_{i=0}^{r+R} t^i \int_{x_0}^{x} \tilde{K}_{(i)}(\mathrm{d}\,U)$. Since $\overset{t}{\tilde{U}}$ is a polynomial of degree $r+R-1$ it is uniquely defined by its values at the $r+R$ different eigenvalues of A. Via the Lagrange interpolation formula we have

$$\overset{t}{\tilde{U}} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t-\varrho_l)}{(\varrho_i-\varrho_l)} \tilde{f}_i \tag{112}$$

for some funcions $\tilde{f}_i$. Introducing $f_i = \prod_{\varrho_l \in E\setminus\{\varrho_i\}} (\varrho_i-\varrho_l)^{m(\varrho_l)/2-1} \tilde{f}_i$ the potential U can be written as

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t-\varrho_l)^{m(\varrho_l)/2}}{(\varrho_i-\varrho_l)^{m(\varrho_l)/2}} (t-\varrho_i)^{m(\varrho_i)/2-1} f_i \tag{113}$$

concluding the proof of lemma 21.

Lemma 22 *Let (g,J,A) be c-compatible and $\overset{t}{K}$ as in (36). Let A be semi-simple. Let $\overset{\mathrm{nc}}{E} = \{\varrho_1,\ldots,\varrho_r\}$ be the set of non-constant eigenvalues of A. Let $\overset{\mathrm{c}}{E} = \{\varrho_{r+1},\ldots,\varrho_{r+R}\}$ be the set of constant eigenvalues and $E = \overset{\mathrm{nc}}{E} \cup \overset{\mathrm{c}}{E}$. The multiplicity of ϱ_l is denoted by $m(\varrho_l)$. Let*

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t-\varrho_l)^{m(\varrho_l)/2}}{(\varrho_i-\varrho_l)^{m(\varrho_l)/2}} (t-\varrho_i)^{m(\varrho_i)/2-1} f_i \tag{114}$$

and let

$$\overset{t}{K}{}^i_j \frac{\partial U}{\partial x^i} = \frac{\partial \overset{t}{U}}{\partial x^j} \tag{115}$$

be satisfied for all values of t. Then for all values of i the relation $\mathrm{d}\,f_i \circ A = \varrho_i\,\mathrm{d}\,f_i$ must be satisfied. In other words: the differentials of the functions f_i are eigenvectors of A with eigenvalues ϱ_i, where A is considered as to map one-forms to one-forms.

Proof of lemma 22: We consider $\overset{t}{K}$ as a $(1,1)$ tensor field. Using our assumption that A is semi-simple we rewrite equation (115) in terms of the quantities $\overset{t}{\tilde{U}}$, $\tilde{f}_i$ and $\overset{t}{\tilde{K}}$ defined by

$$\overset{t}{\tilde{U}} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t-\varrho_l)}{(\varrho_i-\varrho_l)} \tilde{f}_i$$

$$f_i = \prod_{\varrho_l \in E\setminus\{\varrho_i\}} (\varrho_i - \varrho_l)^{m(\varrho_l)/2-1} \tilde{f}_i$$

$$\overset{t}{K} = \prod_{\varrho_l \in E\setminus\{\varrho_i\}} (t - \varrho_l)^{m(\varrho_l)/2-1} \overset{t}{\tilde{K}}$$

Then we use the fact that eigenvalues of multiplicity $m(\varrho_l) \geq 4$ are constant (lemma 10, item 1) and equation (115) transforms into

$$\overset{t}{\tilde{K}} (\mathrm{d}\, U) = \mathrm{d}\, \overset{t}{\tilde{U}} \tag{116}$$

by dividing out the common factors.

The right hand side can be rewritten: consider $\overset{t}{\tilde{U}}$ where, rather than choosing a constant value for the parameter t we fill in the l^{th} eigenvalue of A. Then we have $\overset{\varrho_l}{\tilde{U}} = \tilde{f}_l$. Taking the differential and rearranging the terms gives

$$\mathrm{d}\, \tilde{f}_l - \left.\frac{\partial\, \overset{t}{\tilde{U}}}{\partial t}\right|_{t=\varrho_l} \mathrm{d}\, \varrho_l = \mathrm{d}\, \overset{\varrho_l}{\tilde{U}} - \left.\frac{\partial\, \overset{t}{\tilde{U}}}{\partial t}\right|_{t=\varrho_l} \mathrm{d}\, \varrho_l = \mathrm{d}\, \overset{t}{\tilde{U}} \,|_{t=\varrho_l} \tag{117}$$

We evaluate (116) at $t = \varrho_l$ and plug in (117):

$$\overset{\varrho_l}{\tilde{K}} (\mathrm{d}\, U) = \mathrm{d}\, \tilde{f}_l - \left.\frac{\partial\, \overset{t}{\tilde{U}}}{\partial t}\right|_{t=\varrho_l} \mathrm{d}\, \varrho_l \tag{118}$$

Because we assumed A to be semi-simple we can decompose $\mathrm{d}\, U$ into one-forms υ_l such that $\upsilon_l \circ A = \varrho_l \upsilon_l$. From the definition of $\overset{t}{\tilde{K}}$ we have that $\overset{t}{\tilde{K}} (\upsilon_l) = \prod_{\varrho_m \in E\setminus\{\varrho_l\}} (t - \varrho_m)\upsilon_l$, again because we assumed A to be semi-simple. Evaluating at $t = \varrho_k$ yields

$$\overset{\varrho_k}{\tilde{K}} (\upsilon_l) = \left(\prod_{\varrho_m \in E\setminus\{\varrho_l\}} (\varrho_k - \varrho_m) \right) \upsilon_l \tag{119}$$

In particular this means that if $k \neq l$ then $\overset{\varrho_k}{\tilde{K}} (\mathrm{d}\, \varrho_l)$ is zero. Plugging this into (118) we get that on the left hand side $\overset{\varrho_l}{\tilde{K}} (\mathrm{d}\, U) = \overset{\varrho_l}{\tilde{K}} (\upsilon_l)$ holds, which we express via (119):

$$\left(\prod_{\varrho_m \in E\setminus\{\varrho_l\}} (\varrho_l - \varrho_m) \right) \upsilon_l = \mathrm{d}\, \tilde{f}_l - \left.\frac{\partial\, \overset{t}{\tilde{U}}}{\partial t}\right|_{t=\varrho_l} \mathrm{d}\, \varrho_l \tag{120}$$

Since $\mathrm{d}\, \varrho_l \circ A = \varrho_l\, \mathrm{d}\, \varrho_l$ (lemma 10, item 2) and $\upsilon_l \circ A = \varrho_l \upsilon_l$, we have that $\mathrm{d}\, \tilde{f}_l \circ A = \varrho_l\, \mathrm{d}\, \tilde{f}_l$. The way in which $\tilde{f}_l$ was constructed then implies $\mathrm{d}\, f_l \circ A = \varrho_l\, \mathrm{d}\, f_l$, concluding the proof of lemma 22.

Lemma 23 *Let (g, J, A) be c-compatible, $\overset{t}{K}$ as in (36) and $\overset{t}{V}$ as in (38). $\overset{\text{nc}}{E}= \{\varrho_1, \ldots, \varrho_r\}$ denotes the set of non-constant eigenvalues of A. $\overset{\text{c}}{E}= \{\varrho_{r+1}, \ldots, \varrho_{r+R}\}$ denotes the set of constant eigenvalues and $E = \overset{\text{nc}}{E} \cup \overset{\text{c}}{E}$. The multiplicity of ϱ_l is denoted by $m(\varrho_l)$. Let*

$$\overset{t}{U}= \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t-\varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t-\varrho_i)^{m(\varrho_i)/2-1} f_i \tag{121}$$

Let furthermore $\mathrm{d}\, f_l \circ A = \varrho_l \,\mathrm{d}\, f_l$ for all values of $l = 1, \ldots, r+R$. Let $\mathrm{d}\, \overset{t}{U}\, (\overset{s}{V}) = 0$ be satisfied for all values of $s, t \in \mathbb{R}$. Then $\mathrm{d}\, \overset{t}{U}\, (\overset{s}{V}) = 0$ is satisfied for all values of $s, t \in \mathbb{R}$ if and only if for each eigenvalue ϱ_k of A the differential $\mathrm{d}\, f_k$ is proportional to $\mathrm{d}\, \varrho_k$ at all points where $\mathrm{d}\, \varrho_k \neq 0$.

Corollary 23.1 *If ϱ_l is a non-constant real eigenvalue and $\mathrm{d}\, \varrho_l \neq 0$ in the neighbourhood of a given point then locally f_l can be expressed as a smooth function of ϱ_l. Likewise, if ϱ_l is a non-constant complex eigenvalue and $\mathrm{d}\, \varrho_l \neq 0$ in the neighbourhood of a given point, then locally f_l can be expressed as a holomorphic function of ϱ_l.*

Proof of lemma 23: The condition $\mathrm{d}\, \overset{t}{U}\, (\overset{s}{V}) = 0 \;\forall\, s, t \in \mathbb{R}$ is equivalent to $\mathrm{d} \overset{t}{U}\, (\mathrm{span}\{J \,\mathrm{grad}\, \varrho_i | i = 1, \ldots, r\}) = 0$, because

$$\mathrm{span}\{\overset{t}{V}, t \in \mathbb{R}\} = \mathrm{span}\{J \,\mathrm{grad}\, \varrho_i | i = 1, \ldots, r\}$$

From (121) we see that $\mathrm{d}\, \overset{t}{U}$ involves (with some coefficients) only the differentials of the eigenvalues of A and the differentials of the functions f_i. Thus $\mathrm{d}\, \overset{t}{U}\, (J \,\mathrm{grad}\, \varrho_i)$ is a linear combination of $\mathrm{d}\, \varrho_j$ and $\mathrm{d}\, f_j$ applied to $J \,\mathrm{grad}\, \varrho_i$. But $\mathrm{d}\, \varrho_j(J \,\mathrm{grad}\, \varrho_i)$ is zero for all values of i, j. Firstly if $i = j$, then $\mathrm{d}\, \varrho_j(J \,\mathrm{grad}\, \varrho_i) = 0$ due to the fact that J is antisymmetric with respect to g. Secondly if $i \neq j$ then $\mathrm{d}\, \varrho_j(J \,\mathrm{grad}\, \varrho_i) = 0$, because A is g-self-adjoint and $\mathrm{grad}\, \varrho_i$ and $\mathrm{grad}\, \varrho_j$ are eigenvectors of A with different eigenvalues. Thus $\mathrm{d}\, \overset{t}{U}\, (J \,\mathrm{grad}\, \varrho_i)$ is a linear combination of $\{\mathrm{d}\, f_j(J \,\mathrm{grad}\, \varrho_i) | j = 1, \ldots, r{+}R\}$. But because we assumed that $\mathrm{d}\, f_l \circ A = \varrho_l \,\mathrm{d}\, f_l$ for all l and because $A \,\mathrm{grad}\, \varrho_i = \varrho_i \,\mathrm{grad}\, \varrho_i$ (lemma 10, item 2) and A commutes with J and is g-self-adjoint, we get that $\mathrm{d}\, \overset{t}{U}\, (J \,\mathrm{grad}\, \varrho_i)$ is some coefficient times $\mathrm{d}\, f_i(J \,\mathrm{grad}\, \varrho_i)$.
From (121) we see that this coefficient is

$$\prod_{\varrho_l \in E\setminus\{\varrho_i\}} \frac{(t-\varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t-\varrho_i)^{m(\varrho_i)/2-1} \tag{122}$$

But for a given value of t this can only vanish at points on $\mathcal{M}$ where t is equal to an eigenvalue of A. So at each point on the manifold we can choose a value for t such that expression (122) is non-zero. Thus $\mathrm{d}\, f_i(J \,\mathrm{grad}\, \varrho_i)$ must vanish for all values of i. If we consider a value i such that ϱ_i is constant, then $\mathrm{d}\, f_i(J \,\mathrm{grad}\, \varrho_i) = 0$ is trivially satisfied. From lemma 10 we know that if ϱ_i is non-constant, then its multiplicity is 2 and at all points where $\mathrm{d}\,\varrho_i \neq 0$ the set $\{\mathrm{grad}\,\varrho_i, J\,\mathrm{grad}\,\varrho_i\}$ is an orthogonal basis of the ϱ_i-eigenspace of A. It follows that at such points $\mathrm{d}\,f_i$ may be written as a linear combination of $\mathrm{d}\,\varrho_i$ and $\mathrm{d}\,\varrho_i \circ J$. Plugging this decomposition into $\mathrm{d}\,f_i(J\,\mathrm{grad}\,\varrho_i) = 0$ we conclude that $\mathrm{d}\,f_i$ is proportional to $\mathrm{d}\,\varrho_i$ at all points where $\mathrm{d}\,\varrho_i \neq 0$ and lemma 23 is proven.

Lemma 24 *Let (g, J, A) be c-compatible. $\overset{\mathrm{nc}}{E} = \{\varrho_1, \dots, \varrho_r\}$ denotes the set of non-constant eigenvalues of A. $\overset{\mathrm{c}}{E} = \{\varrho_{r+1}, \dots, \varrho_{r+R}\}$ denotes the set of constant eigenvalues and $E = \overset{\mathrm{nc}}{E} \cup \overset{\mathrm{c}}{E}$. The multiplicity of ϱ_l is denoted by $m(\varrho_l)$. Let*

$$\overset{t}{K}{}^{ij} \overset{\mathrm{def}}{=} \sqrt{\det(tId - A)}\ (tId - A)^{-1}{}^i_l g^{lj}, \qquad \overset{t}{V}{}^j \overset{\mathrm{def}}{=} J^j_k g^{ki} \nabla_i \sqrt{\det(tId - A)}$$

and

$$\overset{t}{U} = \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \frac{(t - \varrho_l)^{m(\varrho_l)/2}}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} (t - \varrho_i)^{m(\varrho_i)/2 - 1} f_i \tag{123}$$

with $\mathrm{d}\,f_l \circ A = \varrho_l\,\mathrm{d}\,f_l$ for all $l = 1, \dots, r$ and $\mathrm{d}\,f_l$ proportional to $\mathrm{d}\,\varrho_l$ for all l for which ϱ_l is non-constant.
Then

$$\overset{t}{K}{}^i_j \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}{}^i_j \frac{\partial \overset{t}{U}}{\partial x^i}\ \forall s, t \in \mathbb{R} \quad \textit{and} \quad \mathrm{d}\,\overset{t}{U}\,(\overset{s}{V}) = 0\ \forall s, t \in \mathbb{R} \tag{124}$$

Proof of lemma 24: $\mathrm{d}\,\overset{t}{U}\,(\overset{s}{V}) = 0\ \forall s, t \in \mathbb{R}$ is fulfilled because $\mathrm{d}\,f_i$ and $\mathrm{d}\,\varrho_i$ evaluate to zero when applied to $J\,\mathrm{grad}\,\varrho_j$ for all $i, j = 1, \dots, r + R$.
To see $\overset{t}{K}{}^i_j \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}{}^i_j \frac{\partial \overset{t}{U}}{\partial x^i}$, we compute $\mathrm{d}\,\overset{t}{U}$ using the fact that non-constant eigenvalues of A are of multiplicity 2 (lemma 10, item 1):

$$\begin{aligned}
\mathrm{d}\,\overset{t}{U} = &\sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_i\}} \left(\frac{t - \varrho_l}{\varrho_i - \varrho_l}\right)^{m(\varrho_l)/2} (t - \varrho_i)^{m(\varrho_i)/2 - 1} \left[\mathrm{d}\,f_i - \sum_{\varrho_p \in E \setminus \{\varrho_i\}} \frac{m(\varrho_p)/2}{\varrho_i - \varrho_p} f_i\,\mathrm{d}\,\varrho_i\right] \\
&- \sum_{i=1}^{r+R} \prod_{\varrho_l \in E \setminus \{\varrho_k\}} \frac{t - \varrho_l}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} \sum_{\varrho_p \in \overset{\mathrm{nc}}{E} \setminus \{\varrho_i\}} \prod_{\varrho_l \in E \setminus \{\varrho_p\}} (t - \varrho_l)^{m(\varrho_l)/2} f_i \frac{\mathrm{d}\,\varrho_p}{\varrho_p - \varrho_i}
\end{aligned} \tag{125}$$

Considering $\overset{s}{K}$ as a $(1,1)$-tensor mapping one-forms to one-forms and using that for all $i = 1, \dots, r$: $\mathrm{d}\,\varrho_i \circ A = \varrho_i\,\mathrm{d}\,\varrho_i$ and $\mathrm{d}\,f_i \circ A = \varrho_i\,\mathrm{d}\,f_i$ we have

$$\begin{aligned}
\overset{s}{K}\,(\mathrm{d}\,\varrho_i) &= \left(\prod_{\varrho_l \in E \setminus \{\varrho_i\}} (s - \varrho_l)^{m(\varrho_l)/2}\right)(s - \varrho_i)^{m(\varrho_i)/2 - 1}\,\mathrm{d}\,\varrho_i \\
\overset{s}{K}\,(\mathrm{d}\,f_i) &= \left(\prod_{\varrho_l \in E \setminus \{\varrho_i\}} (s - \varrho_l)^{m(\varrho_l)/2}\right)(s - \varrho_i)^{m(\varrho_i)/2 - 1}\,\mathrm{d}\,f_i
\end{aligned} \tag{126}$$

Again we consider $\overset{s}{K}$ as a $(1,1)$-tensor acting on the differential of $\overset{t}{U}$. By combining (125) and (126) and using that the non-constant eigenvalues have multiplicity 2 we get:

$$\begin{aligned}\overset{s}{K}(\mathrm{d}\,\overset{t}{U}) = & \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_i\}} \left(\frac{(s-\varrho_l)(t-\varrho_l)}{\varrho_i - \varrho_l}\right)^{m(\varrho_l)/2} ((t-\varrho_i)\,(s-\varrho_i))^{m(\varrho_i)/2-1} \\ & \qquad \times \left[\mathrm{d}\,f_i - \sum_{p\neq i} \frac{m(\varrho_p)/2}{\varrho_i - \varrho_p} f_i \,\mathrm{d}\,\varrho_i\right] \\ & - \sum_{i=1}^{r+R} \prod_{\varrho_l \in E\setminus\{\varrho_k\}} \frac{1}{(\varrho_i - \varrho_l)^{m(\varrho_l)/2}} \\ & \qquad \times \sum_{\varrho_p \in \overset{\mathrm{nc}}{E}\setminus\{\varrho_i\}} \prod_{\varrho_l \in E\setminus\{\varrho_p\}} ((s-\varrho_l)(t-\varrho_l))^{m(\varrho_l)/2} f_i \frac{\mathrm{d}\,\varrho_p}{\varrho_p - \varrho_i}\end{aligned} \tag{127}$$

The right hand side is apparently symmetric when exchanging s and t and as a consequence $\overset{t}{K}{}^i_j \frac{\partial \overset{s}{U}}{\partial x^i} = \overset{s}{K}{}^i_j \frac{\partial \overset{t}{U}}{\partial x^i}$ is fulfilled, concluding the proof of lemma 24.
Proof of theorem 13: The theorem results as a combination of the proofs of lemmata 20 and 24.
Proof of theorem 14: combining the proofs of lemmata 20, 21, 22, 23 and 24.

3.3 Simultaneous eigenfunctions | Proof of theorem 16 and 17

Proof of theorem 16: We first show that ψ is an eigenfunction of $\overset{s}{\hat{L}}$ for all values of s, if and only if it is an eigenfunction of ∂_{t_i} for all values of i.
A direct computation from (69), (70), (71) shows:

$$\overset{s}{\hat{L}}\,\psi = i \prod_{\varrho_p \in \overset{\mathrm{c}}{E}} (s-\varrho_p)^{m(\varrho_p)} \sum_{q=1}^{r} s^{r-q} \partial_{t_q} \psi \tag{128}$$

Now suppose that ψ is an eigenfunction of $\overset{s}{\hat{L}}$ for all values of s and denote the eigenvalue by $\overset{s}{\omega}$. Then $\overset{s}{\omega}$ must be a polynomial in s of degree $r+R-1$, because $\overset{s}{\hat{L}}$ is a polynomial in s of degree $r+R-1$. From the equation above we see that $\overset{s}{\omega}$ must have a zero of order $m(\varrho_p)$ at $s=\varrho_p$ for all constant eigenvalues ϱ_p. Thus we can write $\overset{s}{\omega} = \prod_{\varrho_p \in \overset{\mathrm{c}}{E}} (s-\varrho_p)^{m(\varrho_p)} \sum_{q=1}^{r} s^{r-q}\omega_q$. But because polynomials are equal if and only if all their coefficients are equal, we get that $\overset{s}{\hat{L}}\,\psi = \overset{s}{\omega}\ \forall s \in \mathbb{R}$ if and only if $i\partial_{t_q}\psi = \omega_q \psi$ for $q = 1, \ldots, r$.
To obtain the other separated equations we work with the family of second order differential operators $\overset{s}{K}$. Since the metric is not given in terms of the coordinate basis

and the one-forms α and ϑ are not unique it poses an obstruction to using the standard formula for the Laplacian and $\overset{t}{\hat{K}}$. The workaround is quick and simple though.
Let $\{X_i | i = 1, \ldots, n\}$ be a set of n linearly independent differentiable vector fields on a manifold $\mathcal{M}^n$ and denote by $\{\beta^i | i = 1, \ldots, n\}$ its dual basis, i.e. $\beta^i(X_j) = \delta^i_j$. We shall denote by T the matrix relating the coordinate vector fields ∂_i and the vector fields X_i: $T^i_j X_i = \partial_j$. Then for an arbitrary symmetric $(2,0)$-tensor the following formula is easily obtained via the product rule for partial derivatives:

$$\begin{aligned} \frac{1}{\sqrt{|\det g|}} \partial_i \sqrt{|\det g|} \overset{s}{K}{}^{ij} \partial_j = & \frac{\det T}{\sqrt{|\det g|}} X_p \frac{\sqrt{|\det g|}}{\det T} T^p_i \overset{s}{K}{}^{ij} T^k_j X_k \\ & - X_p(T^p_i) \overset{s}{K}{}^{ij} T^k_j X_k + \frac{X_p(\det T)}{\det T} T^p_i \overset{s}{K}{}^{ij} T^k_j X_k \end{aligned} \tag{129}$$

where on the right hand side the X's are to be interpreted as the directional derivative in the sense $X_s = (T^{-1})^i_s \partial_i$ and in the last two terms $X_s(\cdot)$ is meant as to only act on the expression in the parentheses.
The quantity $\frac{\det g}{\det T^2}$ is simply the determinant of the matrix with $(i,j)^{\text{th}}$ component $g(X_i, X_j)$ and $T^s_i \overset{s}{K}{}^{ij} T^k_j$ are the components of $\overset{s}{K}$ in the basis $\{X_i | i = 1, \ldots, n\}$. If $\{X_i | i = 1, \ldots, n\}$ are the coordinate vector fields belonging to some coordinate system then the last two terms cancel out and one arrives at the well known fact that the left hand side expression is independent of the choice of coordinates.
In our case we choose $(X_i) = (\partial_{\chi_i}, \partial_{t_i}, \partial_{\overset{\gamma}{y}_i} - \sum_{p=1}^r \overset{\gamma}{\alpha}_{pi} \partial_{t_p})$. The dual basis consists of the one-forms $(\mathrm{d}\,\chi_i, \mathrm{d}\,t_i + \alpha_i, \mathrm{d}\,\overset{\gamma}{y}_i)$. Taking i as the column index and j as the row index we have the components T^i_j given by

$$\left(\begin{array}{c|c|c} Id & 0 & 0 \\ \hline 0 & Id & 0 \\ \hline 0 & * & Id \end{array}\right)$$

The $*$-block contains the components of α_l as the l^{th} column and the Id-blocks are of dimensions equal to the number of χ-, t- and y-coordinates.
From this we can conclude that $X_s(T^s_i) = (T^{-1})^j_s \partial_j T^s_i = 0$ because the one-forms α do not depend on the t-variables. Furthermore $\det T = 1$ and thus $X_s(\det T) = 0$ for all values of s.
For our specific case (129) simplifies to

$$\frac{1}{\sqrt{|\det g|}} \partial_i \sqrt{|\det g|} \overset{s}{K}{}^{ij} \partial_j = \frac{\det T}{\sqrt{|\det g|}} X_s \frac{\sqrt{|\det g|}}{\det T} T^s_i \overset{s}{K}{}^{ij} T^k_j X_k \tag{130}$$

Here $\frac{\sqrt{|\det g|}}{\det T}$ is the determinant of the matrix of g in the basis $(\mathrm{d}\,\chi_i, \mathrm{d}\,t_i, \mathrm{d}\,\overset{\gamma}{y}_i)$ and $T^s_i K^{ij} T^k_j$ is the matrix of $\overset{s}{K}$ in the basis (X_i). These quantities can be obtained from the formulae (69), (70) and (71). We can then express the vector fields (X_i) in terms of the coordinate basis to get the following result (we use that $A_\gamma = \varrho_\gamma Id$, because we assumed that A is semi-simple and that all constant eigenvalues are real.):

$$
\begin{aligned}
&\nabla_i \overset{s}{K}{}^{ij} \nabla_j = \frac{1}{\sqrt{|\det g|}} \partial_i \sqrt{|\det g|}\, \overset{s}{K}{}^{ij} \partial_j = \\
&\sum_{\varrho_k \in \overset{\text{nc}}{E}} \frac{\prod_{\varrho_l \in E \setminus \{\varrho_k\}} (s-\varrho_l)^{m(\varrho_l)/2}}{\varepsilon_k \Delta_k \prod_{\varrho_\gamma \in \overset{\text{c}}{E}} (\varrho_\gamma - \varrho_k)^{m(\varrho_\gamma)/2} \varrho_k'} \partial_{\chi_k} \varrho_k' \prod_{\varrho_\gamma \in \overset{\text{c}}{E}} (\varrho_\gamma - \varrho_k)^{m(\varrho_\gamma)/2} \partial_{\chi_k} \\
&+ \sum_{i,j=1}^{r} \sum_{\varrho_k \in \overset{\text{nc}}{E}} \frac{\varepsilon_k (-\varrho_k)^{2r-i-j}}{\Delta_k (\varrho_k')^2} \prod_{\varrho_l \in E\setminus\{\varrho_k\}} (s-\varrho_l)^{m(\varrho_l)/2} \partial_{t_i} \partial_{t_j} \\
&+ \sum_{\gamma : \varrho_\gamma \in \overset{\text{c}}{E}} \frac{1}{\sqrt{|\det g_\gamma|}} \partial_{\overset{\gamma}{y}_i} \sqrt{|\det g_\gamma|} \Big(\prod_{\varrho_l \in E} (s-\varrho_l)^{m(\varrho_l)/2} (sId - A_\gamma)^{-1} \\
&\qquad\qquad \prod_{\varrho_k \in \overset{\text{nc}}{E}} (A_\gamma - \varrho_k Id)^{-1} g_\gamma^{-1} \Big)^{ij} \partial_{\overset{\gamma}{y}_j} \qquad (131) \\
&- \sum_{\gamma : \varrho_\gamma \in \overset{\text{c}}{E}} \frac{1}{\sqrt{|\det g_\gamma|}} \partial_{\overset{\gamma}{y}_i} \sqrt{|\det g_\gamma|} \Big(\prod_{\varrho_l \in E} (s-\varrho_l)^{m(\varrho_l)/2} (sId - A_\gamma)^{-1} \\
&\qquad\qquad \prod_{\varrho_k \in \overset{\text{nc}}{E}} (A_\gamma - \varrho_k Id)^{-1} g_\gamma^{-1} \Big)^{ij} \sum_{q=1}^{r} \overset{\gamma}{\alpha}_{qj}\, \partial_{t_q} \\
&- \sum_{\gamma : \varrho_\gamma \in \overset{\text{c}}{E}} \Big(\prod_{\varrho_l \in E} (s-\varrho_l)^{m(\varrho_l)/2} (sId - A_\gamma)^{-1} \prod_{\varrho_k \in \overset{\text{nc}}{E}} (A_\gamma - \varrho_k Id)^{-1} g_\gamma^{-1} \Big)^{ij} \sum_{q=1}^{r} \overset{\gamma}{\alpha}_{qi}\, \partial_{t_q} \partial_{\overset{\gamma}{y}_j} \\
&- \sum_{\gamma : \varrho_\gamma \in \overset{\text{c}}{E}} \Big(\prod_{\varrho_l \in E} (s-\varrho_l)^{m(\varrho_l)/2} (sId - A_\gamma)^{-1} \prod_{\varrho_k \in \overset{\text{nc}}{E}} (A_\gamma - \varrho_k Id)^{-1} g_\gamma^{-1} \Big)^{ij} \sum_{p,q=1}^{r} \overset{\gamma}{\alpha}_{qi} \overset{\gamma}{\alpha}_{pj}\, \partial_{t_q} \partial_{t_p}
\end{aligned}
$$

If now we suppose that ψ simultaneously satisfies the family of eigenvalue equations

$$
-\nabla_i \overset{s}{K}{}^{ij} \nabla_j \psi + \overset{s}{U} \psi = \overset{s}{\lambda} \psi \qquad (132)
$$

then the left hand side is a polynomial in s and for the equations to hold $\overset{s}{\lambda}$ must be a polynomial of degree $n-1$ in s as well and we shall write $\overset{s}{\lambda} = \sum_{l=0}^{n-1} \lambda_l s^l$. Because of our assumption that all constant eigenvalues are real and that A is semi-simple we have $A_\gamma = c_\gamma Id = \varrho_{r+\gamma} Id$ for $\gamma = 1, \ldots, R$. Thus for all values of γ where $m(\varrho_\gamma) > 2$ the left hand side of $-\nabla_j \overset{s}{K}{}^{jk} \nabla_k \psi + \overset{s}{U} \psi = \overset{s}{\lambda} \psi$ has a zero of multiplicity $m(\varrho_\gamma)/2 - 1$ at $s = \varrho_\gamma$. We can thus write $\overset{s}{\lambda} = \prod_{\varrho_\gamma \in \overset{\text{c}}{E}} (s - \varrho_\gamma)^{m(\varrho_\gamma)/2-1} \overset{s}{\tilde{\lambda}}$ where $\overset{s}{\tilde{\lambda}}$ is a polynomial of degree $r + R - 1$ in s. $\overset{s}{\tilde{\lambda}}$ can be written as $\overset{s}{\tilde{\lambda}} = \sum_{j=0}^{r+R-1} s^j \tilde{\lambda}_j$. Because we assumed that the eigenvalue equation (132) is satisfied for all values of s we can split (132) into n equations, one for each coefficient. We choose a non-constant eigenvalue $\varrho_k \in \overset{\mathrm{nc}}{E}$ and multiply the equation for the coefficient of s^l with ϱ_k^l. We do this for all n equations and add them up. If we use $\nabla_i \overset{s}{K}{}^{ij} \nabla_j = \frac{1}{\sqrt{|\det g|}} \partial_i \sqrt{|\det g|}\, \overset{s}{K}{}^{ij} \partial_j$ in (132), then the result of these steps is equal to replacing $\nabla_i \overset{s}{K}{}^{ij} \nabla_j \psi$ via (131) and substituting s with ϱ_k. We end up with

$$\frac{-1}{\varepsilon_k \varrho_k'} \partial_{\chi_k} \varrho_k' \prod_{\varrho_\gamma \in \overset{\mathrm{c}}{E}} (\varrho_k - \varrho_\gamma)^{m(\varrho_\gamma)/2} \partial_{\chi_k} \psi$$
$$- \sum_{i,j=1}^{r} \frac{\varepsilon_k (-\varrho_k)^{2r-i-j}}{(\varrho_k')^2} \prod_{\varrho_\gamma \in \overset{\mathrm{c}}{E}} (\varrho_k - \varrho_\gamma)^{m(\varrho_\gamma)/2} \partial_{t_i} \partial_{t_j} \psi + f_r \psi = \sum_{i=0}^{n-1} \lambda_i \varrho_k^i \psi \tag{133}$$

Because $\partial_{t_q} \psi = -i\omega_q \psi$, ψ satisfies the ordinary differential equations that have been claimed.
To obtain the separated partial differential equations that we have claimed, we divide $-\nabla_j \overset{s}{K}{}^{jk} \nabla_k \psi + \overset{s}{U}\ \psi = \overset{s}{\lambda}\ \psi$ by $\prod_{\varrho_\gamma \in \overset{\mathrm{c}}{E}} (s - \varrho_\gamma)^{m(\varrho_\gamma)/2-1}$. Then we choose a constant eigenvalue ϱ_γ and evaluate the result at $s = \varrho_\gamma \in \overset{\mathrm{c}}{E}$ in the same way as before:

$$\begin{aligned}\sum_{j=0}^{r+R-1} \tilde{\lambda}_j \varrho_\gamma^j \psi = &- \prod_{\varrho_c \in \overset{\mathrm{c}}{E} \setminus \{\varrho_\gamma\}} (\varrho_\gamma - \varrho_c) \Bigg[\frac{1}{|\det g_\gamma|^{1/2}} \partial_{\overset{\gamma}{y}_i} g_\gamma^{ij} |\det g_\gamma|^{1/2} \partial_{\overset{\gamma}{y}_j} \psi \\ &- \sum_{q=1}^{r} \frac{1}{|\det g_\gamma|^{1/2}} \partial_{\overset{\gamma}{y}_i} g_\gamma^{ij} |\det g_\gamma|^{1/2}\ \overset{\gamma}{\alpha}_{qj}\ \partial_{t_q} \psi - \sum_{q=1}^{r} g_\gamma^{ij}\ \overset{\gamma}{\alpha}_{qi}\ \partial_{t_q} \partial_{\overset{\gamma}{y}_j} \psi \\ &+ \sum_{p,q=1}^{r} g_\gamma^{ij}\ \overset{\gamma}{\alpha}_{qi} \overset{\gamma}{\alpha}_{pj}\ \partial_{t_q} \partial_{t_p} \psi \Bigg] + \frac{1}{\prod_{\varrho_c \in \overset{\mathrm{c}}{E} \setminus \{\varrho_\gamma\}} (\varrho_\gamma - \varrho_c)^{m(\varrho_c)/2-1}} f_\gamma \psi \end{aligned} \tag{134}$$

Again: using $\partial_{t_i} \psi = -i\omega_i \psi$ gives us the desired result.
The last thing to do is to show that if ψ fulfills (73) and (75) for some constants $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{r+R-1}, \omega_1, \ldots, \omega_r$ then it is also an eigenfunction of $\overset{s}{\hat{K}}$ for all real values s. To do so, we use $\partial_{t_i} \psi = -i\omega_i \psi$ to obtain (133) and (134) from (73) and (75). Then for all non-constant eigenvalues ϱ_k we multiply the corresponding equation of (73) by

$$\prod_{\varrho_i \in E \setminus \{\varrho_k\}} \left(\frac{s - \varrho_i}{\varrho_k - \varrho_i} \right)^{m(\varrho_i)/2}$$

and for each constant eigenvalue ϱ_γ we multiply the corresponding equation of (134) by

$$\prod_{\varrho_i \in E \setminus \{\varrho_\gamma\}} \left(\frac{s - \varrho_i}{\varrho_k - \varrho_i} \right) \prod_{\varrho_i \in \overset{\mathrm{c}}{E}} (s - \varrho_\gamma)^{m(\varrho_\gamma)/2-1}$$

and add up the results. Then we use the product rule for differentiation and the Lagrange interpolation formula for polynomials and arrive at $\overset{s}{\hat{K}}\psi + \overset{s}{\hat{U}}\psi = \sum_{k=1}^{n-1}\lambda_k s^k \psi$. Theorem 16 is proven.

Proof of theorem 17: Because we assumed $\mathcal{M}$ to be compact (without boundary) and g to be positive definite, there exist a countable basis $(\psi_m | m \in \mathbb{N})$ in $L^2(\mathcal{M})$, such that each element of the basis is an eigenfunction of the Laplace-Beltrami operator:

$$\Delta\psi_m = \xi_m \psi_m$$

see e.g. [9]. In this reference it is also proven that the eigenfunctions of Δ are C^∞-smooth. Furthermore each eigenvalue of the Laplace-Beltrami operator is of finite multiplicity, that is for each $m \in \mathbb{N}$ the set $\{m' | \xi'_m = \xi_m\}$ is finite. Consider the operators $\{\hat{I}_{(j)} | j = 0, \ldots n-1\}$ and $\{\hat{L}_{(j)} | j = 1, \ldots, n-1\}$, as explained in corollary 14.1. In particular we have $\hat{I}_{(n-1)} = -\Delta$. They all commute, and so we can consider them on an eigenspace of Δ which is a finite-dimensional vector space of smooth functions, so we can reduce our problem to finite dimensional linear algebra and skip all trouble with functional analysis: Consider the Hermitian product

$$\langle \varphi, \psi \rangle = \int_{\mathcal{M}} \bar{\varphi}\psi \,\mathrm{d}V \tag{135}$$

on an eigenspace of Δ. Here the bar means complex conjugation. This is clearly a positive definite and we can choose an orthonormal basis. By partial integration it can be verified that for any φ, ψ in this eigenspace of Δ

$$\begin{aligned}\langle \varphi, \hat{I}_{(j)}\psi \rangle =& \langle \hat{I}_{(j)}\varphi, \psi \rangle \quad j = 0, \ldots, n-2 \\ \langle \varphi, \hat{L}_{(j)}\psi \rangle =& \langle \hat{L}_{(j)}\varphi, \psi \rangle \quad j = 0, \ldots, n-1\end{aligned}$$

(remember that each eigenspace of Δ consists of smooth functions and that we assumed that our manifold was without boundary). On such an eigenspace of Δ, in an orthonormal basis we the operators $\{\hat{I}_{(j)} | \hat{L}_{(j)}, j = 1, \ldots, n-1\}$ are given by Hermitian matrices ($\hat{I}_{n-1}$ is given by a real multiple of the identity matrix). But Hermitian matrices are diagonalizable by means of unitary transformations. So we choose a unitary transformation such that $\hat{I}_{(n-2)}$ is diagonal (unitary transformations retain the property of the basis to be orthonormal). Because the operators we consider all commute, we can then consider them on the eigenspaces of $\hat{I}_{(n-2)}$ and choose a unitary transformation such that $\hat{I}_{(n-3)}$ is diagonal as well. Iteration of this procedure tells us: There is a countable orthonormal basis in $L^2(\mathcal{M})$ such that all basis vectors are simultaneous eigenfunctions of all the operators $\{\hat{I}_{(j)}, \hat{L}_{(j)} | j = 0, \ldots, n-1\}$.
By corollary 14.1 a function that is a simultaneous eigenfunction of $\{\hat{I}_{(j)}, \hat{L}_{(j)} | j = 0, \ldots, n-1\}$ if and only if it is a simultaneous eigenfunction of $\{\overset{t}{\hat{I}} | t \in \mathbb{R}\}$ and $\{\overset{s}{\hat{L}} | s \in \mathbb{R}\}$.
This means that we can apply theorem 16 with $U = 0$. Furthermore because we

assumed that all eigenvalues of A are non-constant a local normal coordinate system only has $\chi-$ and $t-$ coordinates. This means that for any element φ_m of our basis there exist constants $\{\lambda_{m,i}|i=0,\ldots,n-1\}$ and $\{\omega_{m,i}|i=1,\ldots,n\}$(here the comma only separates indices), such that φ_m satisfies the ordinary differential equations

$$\frac{-1}{\varepsilon_k \varrho_k'}\partial_{\chi_k}\varrho_k'\partial_{\chi_k}\psi_m + \sum_{i,j=1}^{r}\frac{\varepsilon_k(-\varrho_k)^{2r-i-j}}{(\varrho_k')^2}\omega_i\omega_j\psi_m = \sum_{i=0}^{n-1}\lambda_{m,i}\varrho_k^i\psi_m \tag{136}$$

and

$$i\partial_{t_k}\psi_m = \omega_{m,k}\psi_m \tag{137}$$

for $k=1,\ldots,n$.
This implies that ψ_m is a product:

$$\psi_m = \prod_{k=1}^{n}\psi_{k,m}(\chi_k)\prod_{l=1}^{n}\varphi_{m,l}(t_l) \tag{138}$$

where $\psi_{m,k}$ satisfies the k^{th} equation of (136) and $\varphi_{m,l}$ satisfies the l^{th} equation of (137). But equations (137) can easily be solved, and we have that $\varphi_{m,l}=\exp(-\omega_{m,l}t_l)$. We still have to show that the constants $\lambda_{m,k}$ and $\omega_{m,k}$ are real: firstly the eigenvalues of $I_{n-1}=-\Delta$ are real (even more, nonnegative). Secondly because on the eigenspaces of Δ the operators $\{\hat{I}_{(j)}|j=0,\ldots,n-2\}$ and $\{\hat{L}_{(j)}|j=1,\ldots,n-1\}$ are represented by hermitian matrices, they must have real eigenvalues. An inspection of the proof of theorem 16 reveals that $\lambda_{m,i}$ is the eigenvalue of ψ_m with respect to $\hat{I}_{(i)}$ (in particular $\Delta\psi_m=\lambda_{m,n-1}\psi_m$) and that $\omega_{m,i}$ is the eigenvalue of ψ_m with respect to $\hat{L}_{(n-i)}$. Finally this implies that we have $\overset{t}{\hat{I}}\,\psi_m=\sum_{i=0}^{n-1}t^i\lambda_{m,i}$ and $\overset{s}{\hat{L}}\,\psi_m=\sum_{i=1}^{n}s^{n-i}\omega_{m,i}$. Theorem 17 is proven.

4 Conclusion

We have shown that the integrals of the geodesic flow, quadratic in momenta, that were found by Topalov [17] for the geodesic flow of c-compatible structures also commute as quantum operators. They also commute with the quantum operators of the integrals of the geodesic flow that are linear in momenta.

We have then generalized the result to a class of natural Hamiltonian systems: in the case where the tensor A is semi-simple we have described all potentials that may be added to the kinetic energy term such that the resulting functions on $T^*\mathcal{M}$ still Poisson commute pairwise and their quantum operators commute pairwise as well. The potentials that are admissible in the quantum problem are the same as for the question of classical integrability on the level of Poisson brackets.

In the case that A is not semi-simple, we could present some potentials that may be added to the kinetic energy such that the modified integrals still commute in both the classical and the quantum sense; it is however not clear whether there exist more and this shall be subject to further investigation.

We have tackled the question of the separation of variables for simultaneous eigenfunctions of the constructed differential operators in the case where A is semi-simple and all constant eigenvalues are real. If all eigenvalues of A are non-constant, we get complete reduction to ordinary differential equations. The case where A has constant non-real eigenvalues or Jordan blocks still needs investigation.

In the case of maximal integrability we also have described a way in which the construction of an orthonormal basis of L^2 of eigenfunctions of the Laplace-Beltrami operator can be reduced to solving ordinary differential equations.

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