Entropy in the context of aperiodic order

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Abstract

Entropy is a well-studied concept and the literature contains a vast amount of material on this concept in the context of actions of countable discrete amenable groups. In this thesis we extend several statements about entropy and topological pressure to the context of unimodular amenable groups. This allows us to study a notion of complexity of aperiodic ordered structures, called patch counting entropy.

One of the main tools in order to study topological and measure theoretical entropy and topological pressure is the so called Ornstein-Weiss Lemma. This statement can be considered as a generalization of Fekete's lemma. In Chapter 3 we provide two proofs of the statement. The first proof uses that the statement is shown in the discrete context and assumes the structure of a cut and project scheme in order to provide the tool for all groups that are relevant in the study of aperiodic order. The second proof is more technical, based on the Ornstein-Weiss quasi-tiling machinery and provides the statement in the more general context of all unimodular amenable groups.

In Chapter 4 we present that some classical ideas can be used in order to define relative topological and measure theoretical entropy for actions of non-discrete groups and discuss why some other ideas cannot be used. We furthermore discuss several non-equivalent definitions of entropy in the literature and relate our approach to them.

In Chapter 5 we see that entropy can already be calculated by using only the knowledge of how elements of a Delone subset act. We present that this allows to extrapolate certain results about entropy from the discrete to the general setting. In particular, we obtain from this method the Rokhlin-Abramov Theorem and Bowen's formula. However, we also discuss that this extrapolation causes difficulties at several points. We show that it is not straightforward to extrapolate statements like the variational principle or common sufficient conditions for the upper semi-continuity of the entropy map and discuss how to resolve some of these problems.

In Chapter 6 we discuss patch counting entropy of Delone sets. We see that this quantity equals the topological entropy of an associated dynamical system whenever the acting group is a compactly generated and locally compact Abelian group. We present examples of Delone sets of p-adic numbers of finite local complexity where this result and others fail. To construct these Delone sets, we also discuss a geometrical approach to topological entropy that is inspired by patch counting entropy.

In Chapter 7 we see how to define topological pressure for actions of unimodular amenable groups. In particular, we present that the Ornstein-Weiss Lemma allows to define and study topological pressure in this context. We show that the discrete restriction, as mentioned above in the context of entropy, also works for topological pressure

whenever one appropriately modifies the potential. However, the corresponding extrapolation from a uniform lattice does not allow to conclude the variational principle for non-discrete groups directly. In order to obtain the variational principle for topological pressure for groups in the context of aperiodic order, we provide a proof of this statement for actions of σ-compact locally compact Abelian groups. In particular, we obtain the variational principle for the topological entropy of such actions.

Zusammenfassung

Entropie ist ein vielseitig studiertes Konzept und die Literatur enthält eine umfassende Menge an Resultaten im Kontext von Wirkungen abzählbarer diskreter und mittelbarer Gruppen. In dieser Arbeit verallgemeinern wir verschiedene Aussagen über Entropie und über topologischen Druck in den Kontext von Wirkungen unimodularer mittelbarer Gruppen. Dies erlaubt uns insbesondere einen Begriff von Komplexität aperiodischer Strukturen, die „Patchzahlentropie" (Patch counting entropy), zu untersuchen.

Eines der Hauptwerkzeuge zur Definition und zum Studium topologischer Entropie, maßtheoretischer Entropie, sowie des topologischen Drucks ist das Ornstein-Weiss-Lemma, welches als eine Verallgemeinerung von Feketes Lemma betrachtet werden kann. In Kapitel 3 stellen wir zwei Beweise für das Ornstein-Weiss-Lemma bereit. Der erste Beweis nutzt die Struktur von „Cut and project schemes" und die Tatsache, dass die entsprechende Aussage im diskreten Kontext wohlbekannt und bewiesen ist. Dieser Beweis liefert die Aussage für alle Gruppen, welche im Studium aperiodisch geordneter Strukturen relevant sind. Der zweite Beweis ist technischer, basiert auf der Ornstein-Weiss-quasi-tiling-Maschinerie und liefert die Aussage im allgemeineren Kontext von unimodularen mittelbaren Gruppen.

In Kapitel 4 präsentieren wir, dass einige klassische Ideen genutzt werden können um relative topologische und maßtheoretische Entropie für Wirkungen nicht diskreter Gruppen zu definieren und diskutieren warum andere Ideen nicht genutzt werden können. Wir sehen weiterhin, dass es mehrere nicht äquivalente Definitionen von Entropie in der Literatur gibt und verknüpfen diese mit unserem Zugang.

In Kapitel 5 sehen wir, dass die Entropie einer Wirkung bereits berechnet werden kann, sobald man weiß, wie Elemente einer Delone Teilmenge wirken. Wir diskutieren, dass dies erlaubt, einige Resultate aus dem diskreten Kontext auch für nicht diskrete Gruppen zu erhalten. Auf diese Weise erhalten wir das Rokhlin-Abramov-Theorem und Bowens Formel. Diese Herangehensweise ist allerdings nicht für alle Resultate möglich, wie wir am Variationsprinzip sowie einer wohlbekannten hinreichenden Bedingung für die Oberhalbstetigkeit der Entropieabbildung verdeutlichen. Strategien zur Lösung dieser Probleme werden diskutiert.

In Kapitel 6 wird die Patchzahlentropie von Delone Mengen diskutiert. Wir zeigen, dass diese der topologischen Entropie entspricht, sobald die wirkende Gruppe eine kompakt erzeugte und lokalkompakte Abelsche Gruppe ist. Weiterhin präsentieren

wir Beispiele für Delone Mengen p-adischer Zahlen von endlicher lokaler Komplexität, für welche dieses und andere Resultate nicht gelten. Um diese Mengen zu konstruieren stellen wir außerdem einen geometrischen Zugang zur topologischen Entropie vor, welcher von der Patchzahlentropie inspiriert ist.

In Kapitel 7 sehen wir, wie topologischer Druck für die Wirkungen unimodularer mittelbarer Gruppen definiert werden kann. Wir diskutieren insbesondere, wie das Ornstein-Weiss-Lemma genutzt werden kann, um den topologischen Druck zu definieren und zu studieren. Wir zeigen, dass man auch den topologischen Druck mittels bestimmter diskreter Teilmengen berechnen kann sobald man das Potential anpasst. Wie im Falle der topologischen Entropie kann man auch hier nicht einfach das Variationsprinzip des Druckes aus dem diskreten Kontext erhalten. Um dieses wichtige Resultat für das Studium aperiodischer Ordnung zur Hand zu haben, werden wir deshalb einen Beweis des Variationsprinzips für Wirkungen von σ-kompakten lokalkompakten Abelschen Gruppen geben. Insbesondere erhalten wir damit das Variationsprinzip für die topologische Entropie von Wirkungen solcher Gruppen.

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1 Introduction

Walking through a gallery filled with pictures of Maurits Cornelis Escher, such as 'Horsemen', 'Sun and moon' or 'Ghosts' [Esc75], one might wonder about the mathematics behind these beautiful works. In fact, the artist had a correspondence with several mathematicians such as Roger Penrose. These mathematicians influenced and in turn were influenced by his work in their studies [CEPT86, Pen86]. The mathematical structure behind some of the graphical works of M. C. Escher is called nowadays a tiling. Roughly speaking, tilings are infinite partitions of the plane (or other Euclidean spaces) with a finite number of shapes that appear. Most often these shapes are polyhedra, but as richly illustrated by M. C. Escher, these shapes can take very interesting and well-known other forms, like the form of the silhouette of a horseman or fish.

Mathematically, one often studies the strongly related concept of a Delone set instead of the tiling itself [BG13]. Delone sets ω are subsets of $\mathbb{R}^d$ such that two points in ω cannot be arbitrarily close and that there are not arbitrarily large gaps. To be more precise about the notion of a Delone set, note that these sets are characterized by the existence of an open neighbourhood U of 0 and a compact subset K of $\mathbb{R}^d$ such that the family of translates $\{U + g;\, g \in \mathbb{R}^d\}$ is a disjoint family and that for any $g \in \mathbb{R}^d$ there exist $k \in K$ and $v \in \omega$ such that $g = k + v$.

Beyond the interpretation in the context of tilings there is a physical interpretation of Delone sets as sets of atoms, which underlies the research about crystals and quasicrystals. These investigations have received a lot of attention since the discovery of aperiodic ordered structures as real solids by Dan Shechtman [SBGC84] and the related Nobel prize.

As the Euclidean structure of $\mathbb{R}^d$ seems to be important only for some aspects of the study of aperiodic order, one often encounters that Delone sets are studied in the context of locally compact Abelian groups (LCA groups). This leads to a natural generalization of the shift spaces known for sequences, as discussed for example in [BL04]. These associated *Delone dynamical systems* yield a source of actions of more exotic groups, like the additive group of the p-adic numbers, and they are thus also of interest from the dynamical perspective.

In order to study the complexity of Delone sets, one often considers the notion of patch counting entropy [Lag99, LP03, BLR07, HR15, BH15][1]. An A-*patch* of a Delone set ω is a set of the form $(\omega - g) \cap A$, where A is a compact subset of G and $g \in \omega$. The set of all A-patches of a Delone set ω is denoted by $\mathrm{Pat}_\omega(A)$. Clearly, an arbitrary

[1] Note that patch counting entropy is called *configurational entropy* in [Lag99, LP03].

Delone set can have infinitely many A-patches for a given compact subset A. It is thus natural to restrict to Delone sets for which $\mathrm{Pat}_\omega(A)$ is finite for all compact subsets $A \subseteq G$. Such Delone sets are said to be of *finite local complexity* or *FLC* for short.

Patch counting entropy of FLC Delone sets is studied for σ-compact and metrizable LCA groups, for example in [HR15]. In order to define this concept, one needs the notion of a Van Hove sequence[2]. This notion will be defined below, but for the moment it is sufficient to know that Van Hove sequences are sequences of compact subsets of G that can be used for averaging similar to the sequence of sets $(\{0, \cdots, n\})_{n\in\mathbb{N}}$ in $\mathbb{Z}$. The most prominent examples of Van Hove sequences in $\mathbb{R}^d$ are the sequence $(\overline{B}_n(0))_{n\in\mathbb{N}}$ of closed centred balls and the sequence $(C_n)_{n\in\mathbb{N}}$ of closed centred cubes. Considering a Van Hove sequence $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ one can compute the *patch counting entropy along* $\mathcal{A}$ as

$$\limsup_{n\to\infty} \frac{\log |\mathrm{Pat}_\omega(A_n)|}{\theta(A_n)}, \tag{1.1}$$

where here (and below) θ denotes the Haar measure of G. In [LP03], it was claimed (for FLC Delone sets in $\mathbb{R}^d$) that one can use a subadditivity argument to show that the limit superior in this formula is always a limit, whenever we consider the Van Hove sequence of closed centred balls. Aside from asking for a proof of this statement, it is thus natural to ask the following questions.

Question 1.1. *For which Van Hove sequences is the limit superior in (1.1) a limit?*

Question 1.2. *Is the formula for the patch counting entropy (1.1) independent of the choice of the Van Hove sequence?*

We will see in Example 6.30 that Question 1.2 can easily be answered. We will see that (1.1) depends on the choice of the Van Hove sequence already for FLC Delone sets in $\mathbb{R}$. Note that this observation also gives a partial answer to Question 1.1, as the dependence of (1.1) implies that this limit cannot exist for all Van Hove sequences. The problem seems to be that the Van Hove sequences need to be 'centred' in order to guarantee the existence of the limit. To make this precise we will introduce the notion of being 'compactly connected to 0'. We will give the precise definition in Chapter 6, but for the moment it is sufficient to know that any Van Hove sequence in $\mathbb{R}^d$ that consists of connected sets containing 0 is compactly connected to 0. In particular, we obtain the sequence of closed centred balls and the sequence of closed centred cubes in $\mathbb{R}^d$ as examples. The notion of compact connectedness to 0 can also be applied in the context of non-connected groups such as $\mathbb{Z}^d$ and allows to give the following partial answer to the Questions 1.1 and 1.2 in Theorem 6.4.

[2] In this introduction we assume all topological groups to be metrizable and σ-compact. This allows to avoid the concept of nets. Nevertheless, we will henceforth avoid these countability assumptions and work with nets. See Chapter 2 for the definition of a net and note that every sequence is a net.

Theorem. *Whenever ω is a FLC Delone set in a compactly generated LCA group (for example $\mathbb{R}^d$ or $\mathbb{Z}^d$), then the limit in (1.1) exists and is independent of the choice of a Van Hove sequence that is compactly connected to* 0.

We will discuss the strategy to prove the theorem in a second. For the moment note that

$$A \mapsto \log |\operatorname{Pat}_\omega(A)| \tag{1.2}$$

is subadditive[3] for all LCA groups and not only for $\mathbb{R}^d$. Unfortunately the sequence of closed and centred balls is not always a Van Hove sequence for all LCA groups[4], but it still seems natural to ask the following.

Question 1.3. *Does the limit in (1.1) exist whenever we consider the sequence of closed and centred balls (instead of $(A_n)_{n\in\mathbb{N}}$) and whenever this sequence is a Van Hove sequence?*

We will see in Theorem 6.51 that this is not the case. Considering FLC Delone sets in the metrizable and σ-compact LCA group $\mathbb{Q}_2$ of 2-adic numbers we will see the following.

Theorem. *There exists a FLC Delone set ω in $\mathbb{Q}_2$ such that*

$$\frac{\log \left|\operatorname{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta\left(\overline{B}_n(0)\right)}$$

does not converge as n tends to ∞.

Let us next discuss the strategies to achieve these results. In order to fix ideas, consider the case of Delone sets in $\mathbb{Z}$. A brief look into the literature, for example [BS02], yields that Question 1.1 is indeed answered by a subadditivity argument which can be formalized as Fekete's lemma[5]. In Chapter 3 we will thus discuss various versions of this technique in the context of aperiodic order. We will see that the versions from [Oll85, PS16] are not suitable for our purposes. The somehow most suitable version is the Ornstein-Weiss Lemma. This technique is based on the quasi-tiling technique developed in [OW87] and a sketch of a proof is presented in [Gro99]. In the context of discrete amenable groups, there are various publications, like for example [WZ92, LW00, Kri07, Buf11, CSCK14], presenting a proof based on the ideas from [Gro99]. However, beyond this context there seems no work on the statement in the context of

[3] We denote by $\mathcal{K}(G)$ the set of all compact subsets of G. A map $f\colon \mathcal{K}(G) \to \mathbb{R}$ is called *subadditive*, whenever for all $A, B \in \mathcal{K}(G)$ there holds $f(A \cup B) \leq f(A) + f(B)$.

[4] Consider for example the closed ball of radius 1 in any infinite countable abelian group equipped with the discrete metric d with $d(g, g') = 1$ for $g \neq g'$.

[5] See Chapter 3 for the statement of Fekete's lemma.

unimodular amenable groups except for the aforementioned sketch of a proof in [Gro99]. In Chapter 3 we thus give two proofs for the Ornstein-Weiss Lemma for all groups that appear in the study of aperiodic order. The first proof uses the discrete result and the structure of a cut and project scheme[6]. The second proof is more technical, based on a slightly modified version of the Ornstein-Weiss quasi-tiling results as well as the ideas of [Gro99], and completely self-contained.

Theorem (Ornstein-Weiss Lemma)**.** *Let $f\colon \mathcal{K}(G) \to \mathbb{R}$ be a monotone, right invariant and subadditive mapping.*[7] *Then whenever $(A_n)_{n\in\mathbb{N}}$ is a Van Hove sequence in G, then the limit*

$$\lim_{n\to\infty} \frac{f(A_n)}{\theta(A_n)}$$

exists, is finite and does not depend on the choice of the Van Hove sequence.

As remarked above, the patch counting map (1.2) is subadditive and it is furthermore straightforward to see that this map is monotone. Nevertheless, we will see in Example 6.1 that this function is not right invariant. A straightforward application of the Ornstein-Weiss Lemma to our problem is thus not possible. However, in [BLR07] it is discussed that the topological entropy and the patch counting entropy of a FLC Delone set (in $\mathbb{R}^d$) are equal. To be more precise, one can associate an action with a Delone set ω (in an LCA group) as follows. One first considers the set of all translates of ω and then takes the closure X_ω of this set with respect to a suitable topology. X_ω is then a compact Hausdorff space that consists of Delone sets. Acting by translation on X_ω we obtain the associated G-action. For details see Chapter 2.

As well-known in the discrete context and for example presented in [CSCK14], the Ornstein-Weiss Lemma can be applied in order to define topological entropy independently of a particular Van Hove sequence[8]. We will thus present in Chapter 4 that one can use the Ornstein-Weiss Lemma in order to define topological entropy also for actions of unimodular amenable groups. This detailed treatment of the matter is motivated by the lack of a detailed treatment of entropy theory for actions of unimodular amenable groups that are not necessarily discrete in the literature. Based on our current knowledge the only references in this direction are [Fel80, OW87, TZ91, Sch15, Sin16] and unfortunately in none of these the required techniques are discussed. We thus present in Chapter 4 the straightforward generalizations of some well-known techniques of entropy theory and furthermore discuss why other parts of the theory cannot easily be generalized. In particular, we discuss several non-equivalent notions of topological entropy. Having the right concept[9] of topological entropy at hand, the following question has to

[6] For a definition see Chapter 2.

[7] The function f is called *monotone*, whenever $f(A) \leq f(B)$ for $A, B \in \mathcal{K}(G)$ with $A \subseteq B$. f is called *right invariant*, whenever $f(A) = f(\{ag;\, a \in A\})$ for all $A \in \mathcal{K}(G)$ and all $g \in G$.

[8] In the discrete context the concepts of Følner and Van Hove sequences are equivalent.

[9] Note that the differences in the approaches to topological entropy that we will discuss in Chapter 4 already appear for actions of $\mathbb{R}^d$, where Question 1.4 is answered affirmatively by [BLR07].

be raised.

Question 1.4. *Does the patch counting entropy (along a specified Van Hove sequence, for example a Van Hove sequence of closed centred balls) of a FLC Delone set always equal the topological entropy of the corresponding action?*

This question is of particular interest in the following context. In [BLR07] the statement is shown and used (in the context of FLC Delone sets in $\mathbb{R}^d$) in order to show that pure point diffraction implies zero patch counting entropy. As recent interest in diffraction in LCA groups beyond $\mathbb{R}^d$ [BHP18] and other techniques in this direction are also generalizable [FGL18], a positive answer to Question 1.4 would shed light on the general situation. In Corollary 6.22 we will present the following.

Theorem. *Whenever ω is a FLC Delone set in a (non-compact) compactly generated LCA group (for example $\mathbb{R}^d$ or $\mathbb{Z}^d$) and whenever $\mathcal{A}$ is a Van Hove sequence that is compactly connected to 0, then the patch counting entropy of ω (along $\mathcal{A}$) equals the topological entropy of the associated dynamical system.*

As already considered, a natural choice for an averaging sequence in a metrizable LCA group is the sequence of closed and centred balls whenever this sequence is a Van Hove sequence. One would thus expect that patch counting entropy along such a Van Hove sequence always yields the associated topological entropy. It is thus surprising that one can construct the following FLC Delone sets of 2-adic numbers. For reference see Theorem 6.47.

Theorem. *There exists a FLC Delone set ω in $\mathbb{Q}_2$ such that the topological entropy of the dynamical system associated with ω is 0, but for which the following limit exists and satisfies*

$$\lim_{n\to\infty} \frac{\log \left|\mathrm{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta\left(\overline{B}_n(0)\right)} = \log(2).$$

As the study of Delone sets uses certain parts of entropy theory, like the variational principle [BLR07] or Bowens formula [JLO16], it is natural to ask for proofs of these statements in the context of actions of LCA groups and in particular for actions of $\mathbb{R}^d$. Unfortunately the literature seems to contain no such proofs, which seems to be justified by the following observation. Whenever one wants to consider an action π of $\mathbb{R}^d$, then one can also consider the restricted action $\pi|_{\mathbb{Z}^d}$ of $\mathbb{Z}^d$ and obtains the topological entropy and the measure theoretical entropy of π as the topological and measure theoretical entropy of $\pi|_{\mathbb{Z}^d}$ respectively. A similar statement can also be given more generally. To formulate this, observe that $\mathbb{Z}^d$ is a Delone subgroup in $\mathbb{R}^d$. Delone subgroups are also called uniform lattices, which allows us to formulate the result as follows. For reference see Theorem 5.21 and Lemma 3.6.

Theorem. *If Λ is a uniform lattice in a unimodular amenable group G, then*

$$\mathrm{E}(\pi) = \mathrm{dens}(\Lambda)\,\mathrm{E}\left(\pi|_\Lambda\right).$$

Furthermore, any G-invariant Borel probability measure μ on X is also Λ-invariant and we obtain

$$\mathrm{E}_\mu(\pi) = \mathrm{dens}(\Lambda)\,\mathrm{E}_\mu\left(\pi|_\Lambda\right).$$

Here $\mathrm{dens}(\Lambda) := \lim_{n\to\infty} |\Lambda \cap A_n|/\theta(A_n)$ *denotes the uniform density of Λ, which can be computed independently from the choice of a Van Hove sequence $(A_n)_{n\in\mathbb{N}}$.*

The theory of entropy is well-developed in the context of discrete amenable groups. Statements like the Rokhlin-Abramov Theorem, the Kolmogorov-Sinai generator theorem, Bowens formula, the variational principle, and sufficient conditions for the upper semi-continuity[10] of the entropy map $\mu \mapsto \mathrm{E}_\mu(\pi)$ are given in the literature. We could thus hope that one can easily obtain all these results for LCA groups by restricting to a uniform lattice [TZ91][11]. Nevertheless, one encounters problems as we will discuss in more detail in Chapter 5. As already remarked by Y. Meyer in [Mey72] there are (metrizable and σ-compact) LCA groups, like the additive group $\mathbb{Q}_p$ of p-adic numbers that contain no uniform lattices. We will discuss strategies to partially solve this problem in a second. Compactly generated LCA groups like $\mathbb{R}^d$, however, contain uniform lattices. We will discuss in Chapter 5 that the Bowen formula or the Rokhlin-Abramov Theorem can be obtained with the discussed strategy for such groups. Nevertheless, another problem arises whenever one wants to know whether the variational principle holds for actions of $\mathbb{R}^d$ or whenever one asks for conditions to ensure the upper semi-continuity of the entropy map. Before we discuss strategies to solve these problems, let us shed some light on the situation.

The variational principle relates the topological entropy and the measure theoretical entropy of an action. Considering the supremum over all invariant Borel probability measures μ, we can formulate the variational principle as

$$\mathrm{E}(\pi) = \sup_\mu \mathrm{E}_\mu(\pi).$$

The problem is that, whenever one restricts the action to a uniform lattice Λ, the set of invariant Borel probability measures can enlarge. Considering an LCA group G and a uniform lattice Λ in G, we thus only obtain

$$\mathrm{E}(\pi) = \mathrm{E}(\pi|_\Lambda) = \sup_\nu \mathrm{E}_\nu(\pi|_\Lambda) \geq \sup_\mu \mathrm{E}_\mu(\pi|_\Lambda) = \sup_\mu \mathrm{E}_\mu(\pi),$$

[10] Note that we equip the set of all invariant Borel probability measures with the weak-* topology induced by the Riesz-Markov-Kakutani representation theorem.

[11] In [TZ91] it was suggested that the variational principle can be obtained by this technique for actions of $\mathbb{R}^d$.

where the suprema are taken over all Λ- and G-invariant Borel probability measures ν and μ respectively and it remains open how to obtain the second half of the variational principle for actions of arbitrary unimodular amenable groups.

A second problem appears whenever one wants to formulate sufficient conditions for the upper semi-continuity of the entropy map. This property is important in the study of equilibrium states and equivalent to the converse variational principle for the topological pressure, as we will see in Chapter 7. It is thus natural to also ask for such sufficient conditions for actions of $\mathbb{R}^d$. As we will discuss in more detail in Chapter 5, one obtains for actions of discrete amenable groups from the Kolmogorov-Sinai generator theorem that for the upper semi-continuity in a measure μ of the entropy map it is sufficient to find a generating partition that has almost no boundary with respect to μ. Here and below we call a (Borel measurable) finite partition α of the phase space *generating*, whenever the Borel σ-algebra is the σ-algebra generated by all intersections of the form $\bigcap_{g\in F}\{ag; a \in A_g\}$ for finite subsets F of G and $A_g \in \alpha$. It is thus natural to ask the following.

Question 1.5. *Let μ be an invariant Borel probability measure. Does the existence of a finite generating partition α with almost no boundary with respect to μ imply the upper semi-continuity in μ for the entropy map of an action of an LCA group?*

In Proposition 5.52 we present, that all Delone sets in $\mathbb{R}^d$ allow the construction of a finite generating partition that has almost no boundary with respect to a given invariant Borel probability measure μ on the phase space X_ω of the associated dynamical system. Unfortunately we will see the following in Example 6.37, which answers Question 1.5 negatively.

Example. *There exists a Delone set in $\mathbb{R}$ such that the entropy map of the associated dynamical system is not upper semi-continuous.*

It remains open to give a sufficient condition on the upper semi-continuity for actions of $\mathbb{R}^d$ and in particular whether FLC Delone sets in $\mathbb{R}^d$ always have an upper semi-continuous entropy map. Unfortunately these questions will not be solved in this thesis.

Let us now return to the question of how to treat actions of groups that contain no uniform lattice. In fact for non-discrete groups Van Hove sequences never consist of finite sets and some important approaches to entropy are not at hand. In the non-discrete setting one needs to consider the infinite refinement of a finite measurable partition and cannot expect to obtain a finite partition. As various techniques for entropy depend on the finiteness of Van Hove sets, one needs to find an analogue. For groups that contain uniform lattices the idea is to compute the entropy of the action restricted to the uniform lattice, as presented above. Now recall that uniform lattices are precisely the Delone subgroups. We will see in Chapter 5 that the group structure is not necessary. In fact we will see in Theorem 5.14 and Theorem 5.5 that one can compute entropy even if one only has the knowledge how a Delone subset of the acting group acts. With this observation we will see that one can replace the Van Hove sets by the finite intersections of a Van Hove set with a Delone set. This allows us to use various approaches to entropy that depend on the finiteness of Van Hove sets, like the classical approaches from [Kol58, Sin59, AKM65]. Having these approaches to measure theoretical entropy at hand, we will then be able to show the following half of the variational principle for the topological pressure in full generality.

Theorem (Goodwyn's theorem)**.** *Whenever π is an action of a unimodular amenable group G and whenever μ is an invariant Borel probability measure on the phase space of π, then there holds $\mathrm{E}_\mu(\pi) \leq \mathrm{E}(\pi)$. Furthermore, for any $f \in C(X)$ there holds $\mathrm{E}_\mu(\pi) + \mu(f) \leq \mathrm{p}_f(\pi)$, where $\mathrm{p}_f(\pi)$ denotes the topological pressure of the potential f with respect to π.*

Unfortunately, it remains open how to show the variational principle for actions of general unimodular amenable groups with the methods developed. However, we present a proof of the variational principle for all groups that occur in the study of aperiodic order.

Theorem. *Whenever π is an action of a σ-compact LCA group and whenever $f \in C(X)$, then there holds*

$$\mathrm{p}_f(\pi) = \sup_\mu \mathrm{E}_\mu(\pi) + \mu(f),$$

where we consider the supremum over all invariant Borel probability measures μ. In particular, there holds

$$\mathrm{E}(\pi) = \sup_\mu \mathrm{E}_\mu(\pi).$$

Both proofs are presented in Chapter 7. It remains open whether the variational principle holds for all actions of all unimodular amenable groups.

We would like to mention that the material of this thesis was published so far only in parts. The less general proof of the Ornstein-Weiss Lemma involving cut and project schemes appeared in [Hau20c]. [Hau20c] also contains the parts of the content of Sections 4.1 and 4.2 about topological entropy as well as most of the results about restriction to Delone subsets of the acting group for topological entropy of Chapter 5. The article was published in the Journal of Dynamics and Differential Equations. Furthermore, [Hau20a] contains the results of the Sections 6.1, 6.2 and partly of Section 6.3 about the patch counting entropy are contained . [Hau20a] has been accepted for publication in Mathematische Nachrichten and will be published soon. The results about FLC Delone sets of p-adic numbers of Section 6.4 are contained in the preprint [Hau20b]. The general proof of the Ornstein-Weiss Lemma, the original results about measure theoretical entropy and topological pressure from Chapters 4 and 7 and in particular the considerations about the variational principle in Section 7.4 and about generating partitions of Section 5.3 have not been published yet. The unpublished results seem to provide material for at least one further publication. We would furthermore like to mention that the content of the joint work of the author with his supervisor [HJ19] was not included in this thesis as recommended by the supervisor and as it seemed slightly off-topic.

2 Preliminaries

In this chapter we introduce notation and background that we will need in the following and present some proofs, whenever the statements seem to appear nowhere in the literature. Note that Lemma 2.6, Proposition 2.26 and Proposition 2.45 are original and key results to later theorems. These results are presented in this chapter as they seem to fit best here. This chapter contains furthermore a detailed discussion of the concepts of ergodic, Følner and Van Hove nets continuing the discussion by [Tem92, PS16] and presents some results concerning such nets that seem to be new.

We start by introducing some set theoretic notions and denote by $\mathbb{N}$ the set of all positive integers without 0 and by $\mathbb{N}_0$ the set of positive integers with 0. We furthermore write $|M|$ for the cardinality of a set M. We abbreviate by $\mathcal{P}(M)$ the *power set*, i.e. the set of all subsets and by $\mathcal{F}(M)$ the set of all finite subsets of M respectively. We furthermore denote by χ_M the *characteristic function* of M. For a map $f\colon A \to B$ and $M \subseteq A$ we write $f|_M$ for the restriction $f|_M : M \to B\colon a \mapsto f(a)$.

2.1 Topological spaces

2.1.1 Basic notions

Let X be a topological space. It is called *locally compact*, whenever each point has a compact neighbourhood. It is called *metrizable*, whenever there exists a metric that induces the topology of X. We denote by $\mathcal{A}(X)$ the set of all closed subsets of X. For the set of all compact subsets we furthermore write $\mathcal{K}(X)$. We denote by $\mathcal{D}(X)$ the set if all discrete subsets. For a subset $A \subseteq G$ we denote $\overline{A}$ for its *closure*. The *interior* will be denoted by $\operatorname{int}(A)$. We write ∂A for the *topological boundary* and A^c for the *complement* (w.r.t. X) of A. A subset $M \subseteq X$ is called *precompact*, whenever $\overline{M}$ is compact. For further reference on topological spaces we recommend [Kel55].

We denote by $C(X)$ the Banach space of all real valued continuous functions on X equipped with the supremum norm. We order $C(X)$ with the pointwise order, i.e. $f \leq g$, if and only if $f(x) \leq g(x)$ for all $x \in X$. We denote the constant 0 function also by 0 and call $f \in C(X)$ *positive*, whenever $f \geq 0$. Let furthermore $C(X)^*$ be the space of all linear bounded functionals on $C(X)$ equipped with the weak-* topology.

For $M \subseteq X$ a family $\mathcal{U}$ of subsets of X is called a *cover of* M and said *to cover* M, if $\bigcup_{U\in\mathcal{U}} U \supseteq M$. A family of subsets of X consisting of open sets is called *open*. A cover of X consisting of disjoint Borel measurable sets is called a *partition* of X.

2.1.2 Nets and convergence

A partially ordered set $(I, \geq)$ is said to be *directed*, if I is not empty and if every finite subset of I has an upper bound. A map f from a directed set I to a set X is called a *net* in X. We also write x_i for $f(i)$ and $(x_i)_{i\in I}$ for f. Another net $(y_j)_{j\in J}$ is called a *subnet* of $(x_i)_{i\in I}$, whenever there exists a map $\phi\colon J \to I$, such that $y_j = x_{\phi(j)}$ for all $j \in J$ and such that for each $i \in I$ there exists $j \in J$ such that $\phi(j') \geq i$ for all $j' \geq j$. In this case we also write $\left(x_{\phi(j)}\right)_{j\in J}$ for the subnet.

A net $(x_i)_{i\in I}$ in a topological space X is said to *converge to* $x \in X$, if for every open neighbourhood U of x, there exists $j \in I$ such that $x_i \in U$ for all $i \geq j$. In this case we also write $\lim_{i\in I} x_i = x$. A *cluster point* of a net $(x_i)_{i\in I}$ in X is a convergence point of a subnet. It can be shown that $(x_i)_{i\in I}$ converges to $x \in X$, if and only if every subnet converges to x.

For a net $(x_i)_{i\in I}$ in $\mathbb{R} \cup \{-\infty, \infty\}$, we define $\limsup_{i\in I} x_i := \inf_{i\in I} \sup_{j\geq i} x_j$ and similarly $\liminf_{i\in I} x_i$. Then, if finite, $\limsup_{i\in I} x_i$ is the maximum of all cluster points, i.e. there exists a subnet of $(x_i)_{i\in I}$ that converges to $\limsup_{i\in I} x_i$. Note that $(x_i)_{i\in I}$ converges to $x \in \mathbb{R}\cup\{-\infty, \infty\}$, if and only if there holds $\limsup_{i\in I} x_i = x = \liminf_{i\in I} x_i$. For more details on these notions see [DS88] and [Kel55].

2.1.3 Upper semi-continuous functions

Let X be a topological space, $x \in X$ and consider a function $f\colon X \to \mathbb{R}$. The function f is called *upper semi-continuous in* x, whenever for all $\epsilon > 0$ there exists a neighbourhood U of x such that for all $y \in U$ there holds $f(y) \leq f(x) + \epsilon$ and easily obtain that f is upper semi continuous in x if and only if $f(x) = \limsup_{y\to x} f(y) := \inf_U \sup_{y\in U} f(y)$, where the infimum is taken over all neighbourhoods U of x. The function f is called *upper semi-continuous* (*u.s.c.*), whenever it is upper semi-continuous in all $x \in X$, which is equivalent to $[f \geq t] := \{x \in X;\ f(x) \geq t\}$ being closed for all $t \in \mathbb{R}$.

Note that a upper semi-continuous map is bounded on a compact set and furthermore attains a maximum. The point wise infimum of functions that are upper semi-continuous in a point $x \in X$ is upper semi-continuous in x (whenever this infimum is a real valued function). For reference and further details on upper semi-continuous functions we recommend [Dow11, Appendix A.1.4].

2.2 Compact Hausdorff spaces

In this section let X be a compact Hausdorff space.

2.2.1 The uniformity of a compact Hausdorff space

Denote by $\mathbb{U}_X$ the *uniformity* (of X), i.e. the set of all neighbourhoods $\eta \subseteq X \times X$ of the diagonal $\Delta_X := \{(x, x);\ x \in X\}$. Note that one can define general "uniform spaces", but as we are only interested in compact Hausdorff spaces, this definition works for us. For details and the general definition we recommend [Kel55]. Note that we obtain our definition to be a restriction of the general definition from [Kel55, Theorem 6.22] and [Mun00, Theorem 32.3].

We call the elements of $\mathbb{U}_X$ *entourages* and denote $\eta^{-1} := \{(y,x);\, (x,y) \in \eta\}$ and $\eta\kappa := \{(x,z);\, \exists y \in X : (x,y) \in \eta \text{ and } (y,z) \in \kappa\}$ for entourages $\eta, \kappa \in \mathbb{U}_X$. An entourage η is called *symmetric*, whenever $\eta = \eta^{-1}$. It is furthermore called *open*, whenever it is open with respect to the product topology of $X \times X$ and one similarly defines the notion of a *closed* entourage. From [Kel55] we obtain that every $\eta \in \mathbb{U}_X$ contains a $\kappa \in \mathbb{U}_X$ such that $\kappa\kappa \subseteq \eta$.

2.2.2 Bases of the uniformity

A subfamily $\mathbb{B}_X \subseteq \mathbb{U}_X$ is called a *base of* $\mathbb{U}_X$, if every $\eta \in \mathbb{U}_X$ contains a member of $\mathbb{B}_X$.

Example 2.1. *The set of all symmetric and open entourages is a base of* $\mathbb{U}_X$. *Another base is the set of all symmetric and closed entourages. For reference see [Kel55, Theorem 6.6] and [Kel55, Theorem 6.8].*

Example 2.2. *When* d *is a metric that is compatible with the topology of* X*, then* $\{[d < \epsilon];\, \epsilon > 0\}$ *and* $\{[d \leq \epsilon];\, \epsilon > 0\}$ *are bases of* $\mathbb{U}_X$*, where we define* $[d < \epsilon] := \{(x,y) \in X^2;\, d(x,y) < \epsilon\}$ *and similarly* $[d \leq \epsilon]$.

Example 2.3. *The family* $\{\langle \mathcal{U} \rangle;\, \mathcal{U} \text{ finite open cover of } X\}$ *is a base of* $\mathbb{U}_X$*, where we denote* $\langle \mathcal{U} \rangle := \bigcup_{U \in \mathcal{U}} U^2$.

Example 2.4. *Let* P *be the set of all continuous* pseudometrics *on* X*, i.e. the set of all continuous maps* $d\colon X \times X \to [0,\infty)$ *that satisfy* $d(x,z) \leq d(x,y) + d(y,z)$ *and* $d(x,y) = d(y,x)$ *for all* $x,y,z \in X$*. Then the family that consists of all finite intersections* $\bigcap_{n=1}^{N} [d_n < \epsilon_n]$ *with* $d_n \in P$ *and* $\epsilon_n > 0$ *is a base of* $\mathbb{U}_X$*. Here we define* $[d < \epsilon]$ *similarly as above. For reference see [Kel55, Theorem 6.14] and [Kel55, Theorem 6.29].*

2.2.3 Some geometric definitions

For $x, y \in X$ we denote $[x]\eta := \{y' \in X;\, (x,y') \in \eta$ and $\eta[y] := \{x' \in X;\, (x',y) \in \eta\}$. If η is symmetric we call $B_\eta(x) := \eta[x]$ the *ball* with *radius* η and *centre* x. From [Kel55, Theorem 6.5] we obtain that for any open subset $U \subseteq X$ and any $x \in X$ there exists an $\eta \in \mathbb{U}_X$ such that $B_\eta(x) \subseteq U$. To obtain some further geometric intuition we say that x is η-*close* to y, whenever $(x,y) \in \eta$ and think of two elements to be "very close", whenever the pair is η-close for "many" entourages η. Note that if x is η-close to y and y is κ-close to z, then x is $\eta\kappa$-close to z.

Whenever d is a continuous pseudometric on X we denote by $B_\epsilon^d(x) := B_{[d<\epsilon]}(x) = \{y \in X;\, d(x,y) < \epsilon\}$ the *open ball* of *radius* ϵ and *centre* x, where $\epsilon > 0$ and $x \in X$ and define similarly the *closed ball* $\overline{B}^d_\epsilon(x)$. We omit the d in these definitions whenever the metric is understood implicitely.

2.2.4 Uniform continuity

If $f\colon X \to Y$ is a continuous mapping between compact Hausdorff spaces X and Y, then f is *uniformly continuous*, i.e. for all entourages $\eta \in \mathbb{U}_Y$ there holds $\{(x, x') \in X^2;\ (f(x), f(x')) \in \eta\} \in \mathbb{U}_Y$. For reference see [Kel55, Theorem 6.31].

2.2.5 Lebesgue entourages

If $\mathcal{U}$ is an open cover of a compact Hausdorff space, then we call a symmetric $\eta \in \mathbb{U}_X$ a *Lebesgue entourage of* $\mathcal{U}$, whenever for any $x \in X$ there is $U \in \mathcal{U}$ such that $B_\eta(x) \subseteq U$. As we do not know of any reference for this concept we include the short proof of the existence of Lebesgue entourages for the convenience of the reader.

Lemma 2.5. *For every open cover $\mathcal{U}$ of a compact Hausdorff space X there exists a Lebesgue entourage.*

Proof. For each $x \in X$ choose a symmetric and open entourage $\eta_x \in \mathbb{U}_X$ such that $\eta_x\eta_x[x] \subseteq U$ for some $U \in \mathcal{U}$. As $\{\eta_x[x];\ x \in X\}$ is an open cover of X there exists a finite $F \subseteq X$ such that $\{\eta_x[x];\ x \in F\}$ covers X. Setting $\eta := \bigcap_{x\in F} \eta_x$ one easily shows that η is a Lebesgue entourage for $\mathcal{U}$. □

2.2.6 Borel measures on X

By $\mathcal{B}_X$ we denote the Borel σ-algebra of the compact Hausdorff space X. Furthermore, we denote the set of all regular Borel probability measures by $\mathcal{M}(X)$. By the Riesz-Markov theorem we can identify $\mathcal{M}(X)$ with the set of all positive (and continuous) functionals on $C(X)$ that map the unit $(X \to \mathbb{R}; x \mapsto 1)$ to 1. We equip $\mathcal{M}(X)$ with the restricted weak-* topology and obtain a compact topological space from the Banach-Alaoglu theorem. For further reference see [EFHN15, Theorem E.11].

For a Borel measure μ on X and a continuous map $p\colon X \to Y$ we define $p_*\mu(A) := \mu(p^{-1}(A))$ for all Borel sets $A \subseteq X$. Note that $p_*\mu$ is a Borel measure on Y. We call $p_*\mu$ the *push forward* of μ. We furthermore define the *support* of a Borel measure μ on a Hausdorff space as

$$\operatorname{supp}(\mu) := \{x \in X;\ \mu(U) > 0 \text{ for any open neighbourhood } U \text{ of } x\}.$$

2.2.7 On neighbourhoods with boundary of measure 0

It is well-known that from Froda's theorem[1] it follows that whenever X is metrizable and μ is a Borel measure on X we can find for all $x \in X$ and all open neighbourhoods U of x some radius $r > 0$ such that the open ball $B_r(x)$ is contained in U and has almost no boundary with respect to μ. In fact we next show that one can use Froda's theorem to obtain a similar statement for general compact Hausdorff spaces.

Lemma 2.6. *Let μ be a finite Borel measure of a compact Hausdorff space X and $K \subseteq X$ be compact. Then for any neighbourhood U of K there exists a compact neighbourhood M of K that is contained in U and satisfies $\mu(\partial M) = 0$.*

Proof. We first show the lemma in the case where $K = \{x\}$ is a singleton. Denote by P the set of all continuous pseudometrics d on X and recall from Example 2.4 that the family $\mathbb{B}_P$ that consists of all finite intersections $\bigcap_{n=1}^{N} [d_n < \epsilon_n]$ with $d_n \in P$ and $\epsilon_n > 0$ is a base of $\mathbb{U}_X$. Thus, for $x \in X$ and a neighbourhood U of x there are continuous pseudometrics $d_1, \cdots, d_N$ on X and $\epsilon > 0$ such that $\eta := \bigcap_{n=1}^{N} [d_n < \epsilon]$ satisfies $B_\eta(x) \subseteq U$.

For $n \in \{1, \cdots, N\}$ the pseudometric d_n is continuous and we obtain that the closed ball $\overline{B}_r^{d_n}(x)$ is indeed a closed set and in particular compact. Now consider the map

$$(0, \epsilon) \ni r \mapsto \mu\left(\overline{B}_r^{d_n}(x)\right).$$

From Froda's theorem we obtain that this map has at most countably many discontinuities and thus there is a $r_n \in (0, \epsilon)$ such that

$$\sup_{s \in (0, r_n)} \mu\left(\overline{B}_s^{d_n}(x)\right) = \mu\left(\overline{B}_{r_n}^{d_n}(x)\right).$$

We know that $\overline{B}_s^{d_n}(x) \subseteq B_{r_n}^{d_n}(x)$ for all $s \in (0, r_n)$ and in particular that

$$\mu\left(B_{r_n}^{d_n}(x)\right) \geq \sup_{s \in (0, r_n)} \mu\left(\overline{B}_s^{d_n}(x)\right) = \mu\left(\overline{B}_{r_n}^{d_n}(x)\right),$$

which shows

$$0 \leq \mu\left(\partial \overline{B}_{r_n}^{d_n}(x)\right) \leq \mu\left(\overline{B}_{r_n}^{d_n}(x)\right) - \mu\left(B_{r_n}^{d_n}(x)\right) \leq 0.$$

Thus, also $M := \bigcap_{n=1}^{N} \overline{B}_{r_n}^{d_n}(x)$ has almost no boundary with respect to μ. As $r_n < \epsilon$ we furthermore see that there holds $M \subseteq \bigcap_{n=1}^{N} B_\epsilon^{d_n}(x) \subseteq U$. From $\bigcap_{n=1}^{N} [d_n < r_n] \in \mathbb{B}_P$ we obtain furthermore that M is a compact neighbourhood of x. This shows the statement whenever K is a singleton.

[1] *Froda's theorem* can be stated as follows. Whenever $f\colon I \to \mathbb{R}$ is a monotone map on an interval I, then f has at most countably many discontinuities [Rud76, Theorem 4.30].

For general K and $x \in K$ let K_x be a compact neighbourhood of x with $\mu(\partial K_x) = 0$ and $K_x \subseteq U$. As K is compact there exists a finite subset $F \subseteq K$ such that $K \subseteq M := \bigcup_{x \in F} K_x \subseteq U$ holds. Clearly, M is compact and $\mu(\partial M) = 0$. □

2.3 Topological groups

2.3.1 Basic notions

Consider a group G. We write e_G for the neutral element in G. For subsets $A, B \subseteq G$ the *Minkowski product* is defined as $AB := \{ab;\ (a,b) \in A \times B\}$. For $A \subseteq G$ and $g \in G$ we denote $Ag := A\{g\}$, $gA := \{g\}A$, $A^c := G \setminus A$ and $A^{-1} := \{a^{-1};\ a \in A\}$. We call $A \subseteq G$ *symmetric*, if $A = A^{-1}$. In order to omit brackets, we will use the convention, that the inverse and the complement are stronger binding than the Minkowski product, which is stronger binding than the remaining set theoretic operations. Note that the complement and the inverse commute, i.e. $(A^c)^{-1} = (A^{-1})^c$.

A *topological group* is a group G equipped with a Hausdorff topology, such that the multiplication $\cdot\colon G \times G \to G$ and the inverse function $(\cdot)^{-1}\colon G \to G$ are continuous. With our definition every topological group is regular, as shown in [HR79, Theorem 4.8]. An *isomorphism of topological groups* is a homeomorphism that is a group homomorphism as well. We denote $\mathcal{N}(G)$ for the set of all open neighbourhoods of e_G.

A topological group is called *σ-compact*, whenever G can be written as the countable union of compact sets. A topological group is called *compactly generated*, whenever there exists a compact subset $K \subseteq G$, such that $G = \bigcup_{n \in \mathbb{N}} K^n$, where we abbreviate $K^1 := K$ and $K^{n+1} := KK^n$.

2.3.2 Locally compact Abelian groups

We abbreviate the term locally compact abelian group by *LCA* group and usually denote the operation of an LCA groups G by $+$, the inverse of $g \in G$ by $-g$ and the neutral element by 0. Therefore we write $A+B$ for the Minkowski "product" of subsets $A, B \subseteq G$ and similar notation. From [DE14, Theorem 4.2.2] we cite the following structural result.

Proposition 2.7. *For every compactly generated LCA group G there are $a, b \in \mathbb{N}_0$ and a compact group H, such that G is isomorphic as a topological group to $\mathbb{R}^a \times \mathbb{Z}^b \times H$.*

For further details and results on LCA groups we recommend [DE14].

2.3.3 Haar measure and unimodular groups

If G is a locally compact group, a *left Haar measure* on G is a non-zero regular Borel measure θ on G, which satisfies $\theta(gA) = \theta(A)$ for all $g \in G$ and all Borel sets $A \subseteq G$. Similarly one defines a right Haar measure. A measure that is a left Haar measure and also a right Haar measure will be called a *Haar measure*. A topological group is called *unimodular*, whenever it admits a Haar measure. This terminology will be suitable for us, as most of the groups we deal with will be unimodular. A Borel measurable subset $M \subseteq G$ will be called *regular*, whenever $\theta(\partial A) = 0$.

For a Haar measure θ there holds $\theta(U) > 0$ for all non-empty open subsets $U \subseteq G$ and for subsets $A \subseteq G$ there furthermore holds $\theta(A) < \infty$, whenever A is precompact. A Haar measure is unique up to scaling, i.e. if θ and ν are Haar measures on G, then there is $c > 0$ such that $\theta(A) = c\nu(A)$ for all Borel measurable sets $A \subseteq G$. If nothing else is mentioned, we denote a Haar measure of a topological group G by θ. All LCA groups are unimodular. If G is discrete, then the cardinality $|\cdot|$ is a Haar measure and we equip G with this choice of a Haar measure. For further reference see [Fol99].

Note that there holds $\theta(A) = \theta(A^{-1})$ for all Borel measurable subsets A of a unimodular group G. Indeed we obtain that $A \mapsto \theta'(A) := \theta(A^{-1})$ is also a Haar measure and as $\theta(K) = \theta'(K)$ holds for any compact and symmetric K we obtain that there holds $\theta = \theta'$.

2.3.4 The p-adic numbers

We next introduce the structure of the p-adic numbers, which is actually much richer than the structure of a topological group and should be better discussed in a section called "topological fields". Nevertheless, the additive group of this structure will often suit us as an interesting example of a metrizable and σ-compact LCA group and we thus introduce the p-adic numbers in this section.

To do this recall first that $\mathbb{Q}$ is a field and usually equipped with the standard absolute value $|\cdot|$, which allows to turn $\mathbb{Q}$ into a metric space. Realizing that this metric space is incomplete one then performs a completion to reach the set of real numbers $\mathbb{R}$ and observes that one can equip $\mathbb{R}$ with the structure of a field and an absolute value $|\cdot|$ to obtain $\mathbb{Q}$ as a dense subfield of $\mathbb{R}$ and the absolute value on $\mathbb{Q}$ as a restriction of the absolute value on $\mathbb{R}$. The standard absolute value satisfies the following properties for x, y in $\mathbb{R}$.

(i) $|xy| = |x||y|$.

(ii) $|xy| \leq |x| + |y|$.

(iii) $|x| = 0$ if and only if $x = 0$.

Replacing the field $\mathbb{R}$ above by any other field $\mathbb{K}$, we call any map $|\cdot|: \mathbb{K} \to [0, \infty)$ an *absolute value*, whenever it satisfies (i), (ii) and (iii) for $x, y \in \mathbb{K}$. As usual an absolute value can be used to construct a metric and thus a topology on $\mathbb{K}$. We say that two absolute values on a field are *equivalent*, whenever they introduce the same topology. In particular, one can ask for the possible absolute values on $\mathbb{Q}$ up to equivalence and obtain from the Ostrowski theorem [Gou97, Theorem 3.1.3] that the only absolute values on $\mathbb{Q}$ up to equivalence are the trivial absolute value[2], the standard absolute value and the *p-adic absolute values* for prime numbers p. The latter are defined as follows. For a prime number p we can write any rational number uniquely in the form $x = p^n a/b$ with $a, b, n \in \mathbb{Z}$ and such that p does not divide a and b. We then define $|x|_p := p^{-n}$ and obtain an absolute value $|\cdot|_p$ on $\mathbb{Q}$. In fact one can now perform the standard construction of $\mathbb{R}$ via equivalence classes of Cauchy sequences also for the absolute values $|\cdot|_p$ for any prime p and obtains the field of *p-adic numbers*, which we denote by $\mathbb{Q}_p$. As for $\mathbb{R}$ we consider $\mathbb{Q}$ as a subset of $\mathbb{Q}_p$. To be more precise we quote from [Gou97, Theorem 3.2.13] that for each prime p there exists a field $\mathbb{Q}_p$ with an absolute value $|\cdot|_p$, such that the following statements are valid.

(i) $\mathbb{Q} \subseteq \mathbb{Q}_p$ is a dense subfield and $|\cdot|_p$ extends the p-adic absolute value on $\mathbb{Q}$.

(ii) $\mathbb{Q}_p$ is complete with respect to the metric introduced by $|\cdot|_p$.

Note that one can also show that $\mathbb{Q}_p$ with the mentioned properties is unique up to a unique isomorphism of fields with absolute values. We define the *p-adic integers* $\mathbb{Z}_p$ as the closed ball of radius 1 in $\mathbb{Q}_p$. It can be shown that this subgroup is compact and the closure of $\mathbb{Z}$.

Now recall that the standard absolute value is *Archimedian*, i.e. for x, y and $x \neq 0$ there is a $n \in \mathbb{N}$ such that $|nx| > |y|$. Clearly, the Archimedian property can be defined for any absolute value on any field $\mathbb{K}$ and it can be shown that an absolute value is non-Archimedian, whenever there holds $|x+y| \leq \max\{|x|, |y|\}$ for any $x, y \in \mathbb{K}$ [Gou97, Theorem 2.2.2]. The trivial absolute value on $\mathbb{Q}$ and the p-adic absolute values on $\mathbb{Q}$ and $\mathbb{Q}_p$ are non-Archimedian.

Let us denote the smallest subring of $\mathbb{Q}$ that contains $\mathbb{Z}$ and p^{-1} by $\mathbb{Z}[p^{-1}]$. This subring consists of all rational numbers that have a finite expansion with base p, i.e. which can be written in the from $\sum_{i=-n}^{m} x_i p^i$ for a finite sequence $(x_i)_{i=-n}^{m}$ in $\{0, \cdots, p-1\}$. Using the topological structure given by the standard absolute value on $\mathbb{R}$ one can extend this expansion to all elements of $\mathbb{R}$ by considering series that converge with respect to the topological structure of $\mathbb{R}$. From [Gou97, Corollary 3.3.12] we obtain the following analogue for $\mathbb{Q}_p$.

Proposition 2.8. *For every $g \in \mathbb{Q}_p$ there exists $n \in \mathbb{N}$ and a unique sequence $(g_i)_{i=-n}^{\infty}$ in $\{0, \cdots, p-1\}$ such that the following series converges in $\mathbb{Q}_p$ and such that $g = \sum_{i=-n}^{\infty} g_i p^i$.*

Remark 2.9. *Note that for $x \in \mathbb{Q} \setminus \mathbb{Z}[p^{-1}] \subseteq \mathbb{R} \cap \mathbb{Q}_p$ the expansion in base p depends on the chosen topology. For example the binary expansion ($p = 2$) of $1/3$ with respect to the standard absolute value is $0.\overline{01}$. Nevertheless, there holds $1/3 = p^0 + \sum_{i=0}^{\infty} p^{2i+1}$, which can be denoted by $\overline{01}1.0$ in $\mathbb{Q}_2$. Note that the addition of $\mathbb{Z}[p^{-1}]$ is given by the classical carry over rule and this rule can also be applied to infinite terms as considered above to be used to define the addition of $\mathbb{Q}_p$.*

[2] The *trivial absolute value* is defined by $|x| = 1$ for $x \neq 0$ and $|0| = 0$.

Let us next collect some properties about the structure of $\mathbb{Q}_p$, which are shown for example in [Gou97].

Proposition 2.10. *The absolute value* $|\cdot|_p$ *(and thus also the metric) on* $\mathbb{Q}_p$ *takes values in* $\{p^k;\, k \in \mathbb{Z}\}$*. Thus, for* $n \in \mathbb{Z}$*,* $r \in [p^n, p^{n+1})$ *and* $x \in \mathbb{Q}_p$ *there holds* $\overline{B}_r(x) = \overline{B}_{p^n}(x)$*.*

Proposition 2.11. *For* $r > 0$ *and* $g \in \mathbb{Q}_p$ *the closed ball* $\overline{B}_r(g)$ *is topologically open and compact. Furthermore, whenever* B *and* B' *are two closed balls that intersect then there holds* $B \subseteq B'$ *or* $B' \subseteq B$*. For* $n \in \mathbb{Z}$ *and* $g \in \mathbb{Q}_p$ *the closed ball* $\overline{B}_{p^{n+1}}(g)$ *contains* p *disjoint closed balls of radius* p^n*. Similar statements hold for open balls.*

Proposition 2.12. $\mathbb{Q}_p$ *is metrizable,* σ*-compact, totally disconnected and locally compact. Every closed ball and every open ball in* $\mathbb{Q}_p$ *which contains* 0 *is a compact and open subgroup of* $\mathbb{Q}_p$*.*

Let us close this little introduction of the p-adic numbers with some geometric intuition. It is well-known that $\mathbb{R}$ can be imagined as a line and for our purposes it will be useful to have a similar image for $\mathbb{Q}_p$ at hand. The following image is inspired by the well-known picture of the odometers given for example in [Dow05] but as we are not aware of any reference that interprets $\mathbb{Q}_p$ geometrically we chose to include this image here.

We will present the following for $p = 2$, but note that the involved ideas can easily also be given for any prime p. Let us first consider the closed and centred 1-ball $\overline{B}_1(0) = \mathbb{Z}_2$ in $\mathbb{Q}_2$. It contains 2 closed and centred balls of radius 2^{-1}. Each of these balls contains two closed and centred 2^{-2}-balls and so on. We thus imagine $\overline{B}_1(0)$ as the well-known middle third Cantor set C that arises by removing the middle third from the unit interval $[0, 1]$, then removing the middle thirds from the remaining two intervals and proceeding inductively. We identify the points contained in the remaining of the left interval on the first level of this construction with the elements $\overline{B}_{2^{-1}}(0)$ and the remaining of the right interval is identified with $\overline{B}_{2^{-1}}(1)$. One can also localize the integers in this image. In fact they are precisely the end points of intervals at a certain level of this construction. If these end points are on the left of the respective interval, then they are identified with positive integers (or 0) and right end points are identified with the negative integers. The structure of all of $\mathbb{Q}_2$ can now be imagined as follows. The closed and centred ball $\overline{B}_2(0)$ consists of two copies of $\overline{B}_1(0)$. So one simply draws two copies of C beside each other. The new image just looks as C, but the interpretation changed. Now the end points of the intervals are given by $1/2\mathbb{Z}$. The end points of the left copy of C are given by $\mathbb{Z}$ as just discussed and the end points in the right are given by $\mathbb{Z} + 1/2$. Copying our image again we imagine $\overline{B}_4(0)$ as four copies of C and interpret the end points of the intervals of this image as the elements of $1/4\mathbb{Z}$. We continue with this construction and obtain a image of $\mathbb{Q}_p$ that similarly to the infinite line of $\mathbb{R}$ does not fit on our paper, but which actually gives a lot of geometrical intuition for the object. Note however that the field operations are difficult to trace in this image. Nevertheless, one has some properties, which can be seen. Note for example that for any $g \in \mathbb{Q}_2$ the mapping $g' \mapsto g' + g$ is a bijective isometry and thus maps closed balls of a certain size bijectively to closed balls of the same size. We thus obtain that the intervals at a certain level of the construction of the infinitely many copies of C are actually only permuted by such an operation. Nevertheless, what happens inside them is somehow not encoded in our image.

2.4 Amenable groups

We only define amenability in the context of unimodular groups and use the approach of [Føl55]. We refer to [Pie84] for the general notion. In this section we assume that G is a unimodular group and denote a Haar measure on G by θ.

2.4.1 Van Hove boundary and symmetric difference

For $K, A \subseteq G$ we define the *K-boundary of A* (*Van Hove boundary*) as

$$\partial_K A := \left(K\overline{A}\right) \cap \left(K\overline{A^c}\right).$$

Note that $K\overline{A}$ is the set of all $g \in G$ such that $(K^{-1}g) \cap \overline{A}$ is not empty. Thus, $\partial_K A$ is the set of all elements $g \in G$ such that $K^{-1}g$ intersects both $\overline{A}$ and $\overline{A^c}$. We use the convention, that the Minkowski product is stronger binding than the operation of taking the K-boundary and that the set theoretic operations, except from complementation, are weaker binding. Let us summarize some well-known properties of the Van Hove boundary in the following lemma. For reference see for example [Kri07, Kri10, PS16].

Lemma 2.13. *Let G be a locally compact group. Then for $g \in G$, subsets $A, B \subseteq G$, a family of subsets $(A_i)_{i\in I}$ of G, and all compact subsets $K, M \subseteq G$ there holds*

(a) $\partial_M A \subseteq \partial_K A$, *whenever* $M \subseteq K$.

(b) $\partial_K A = \partial_K A^c$.

(c) $\partial_K \left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I} \partial_K A_i$.

(d) $\partial_K \left(\bigcap_{i\in I} A_i\right) \subseteq \bigcup_{i\in I} \partial_K A_i$.

(e) $\partial_K (A \setminus B) \subseteq \partial_K A \cup \partial_K B$.

(f) $g\partial_K A = \partial_{gK} A$ *and* $M\partial_K A \subseteq \partial_{MK} A$.

(g) $\partial_K(gA) = \partial_{Kg} A$ *and* $\partial_K(MA) \subseteq \partial_{KM} A$.

(h) $\partial_K(Ag) = (\partial_K A)g$ *and* $\partial_K(AM) \subseteq (\partial_K A)M$.

(i) $K\overline{A} \subseteq kA \cup \partial_K A$ *for all* $k \in K$.

Remark 2.14. *Note that for a family of precompact sets* $(A_i)_{i\in I}$ *and a precompact subset* $A \subseteq G$ *there holds* $A\left(\bigcup_{i\in I} A_i\right) \subseteq \bigcup_{i\in I}(AA_i)$ *and* $A\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I}(AA_i)$*, which can be seen from a straightforward argument.*

Proof. One obtains (a) and (b) directly from the definition of the Van Hove boundary. As $\overline{(\bigcup_{i\in I} A_i)^c} = \overline{\bigcap_{i\in I} A_i^c} \subseteq \bigcap_{i\in I} \overline{A_i^c}$ and $\overline{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}$ a straightforward application of the first inclusion mentioned in Remark 2.14 shows (c). From (c) and (b) one easily obtains (d). Furthermore, (e) is a direct consequence of (b) and (d). A straightforward argument shows (f) and the first statements of (g) and of (h) respectively. The second statement of (g) is trivial whenever M is empty and otherwise one considers $m \in M$ and obtains $\overline{(MA)^c} \subseteq \overline{(mA)^c} = m\overline{A^c} \subseteq M\overline{A^c}$. With this observation it is straightforward to show the second statement of (g). The second statement of (h) can be obtained by combining (c) with the first statement of (h). To see (i) note that (f) allows to restrict to the the case $k = e_G \in K$. In this case assume for $g \in K\overline{A}$ that $g \notin A$. Then $g \in A^c \subseteq K\overline{A^c}$ and we obtain $g \in \partial_K A$. □

For $K, A \subseteq G$ we define the *symmetric difference* as $A\Delta K := (A \setminus K) \cup (K \setminus A)$. Similarly as for the Van Hove boundary we will use the convention that the symmetric difference is weaker binding than the Minkowski operations and taking the complement, but stronger binding than all the other set theoretic operations. We summarize some ideas contained in [Pie84] in the following lemma.

Lemma 2.15. *Let* A *be a precompact subset of* G *and* K *be a compact subset of* G*. Then there holds*

(a) $KA\Delta A \subseteq \partial_K A$*, whenever* $e_G \in K$*.*

(b) $\theta(KA\Delta A) \leq 2\theta(KA \setminus A)$*, whenever* K *is non-empty.*

Proof. Whenever $e_G \in K$, then there holds $A \subseteq KA$ and thus

$$KA\Delta A = KA \setminus A = KA \cap A^c \subseteq K\overline{A} \cap K\overline{A^c} = \partial_K A$$

and we obtain (a). Now whenever K is non-empty there exists $k \in K$ and we compute

$$\theta(A \cap KA) + \theta(A \setminus KA) = \theta(A) = \theta(kA) \leq \theta(KA) = \theta(KA \cap A) + \theta(KA \setminus A)$$

and thus $\theta(A \setminus KA) \leq \theta(KA \setminus A)$, which implies $\theta(KA\Delta A) = \theta(KA \setminus A) + \theta(A \setminus KA) \leq 2\theta(KA \setminus A)$ and we have shown (b). □

For $K \in \mathcal{K}(G)$ and $\epsilon > 0$ we say that a Borel measurable subset $A \subseteq G$ of positive Haar measure is (ϵ, K)*-invariant*, whenever

$$\alpha(A, K) := \frac{\theta(\partial_K A)}{\theta(A)} < \epsilon.$$

Lemma 2.16. *Let K be a compact subset of G and $\epsilon \in (0,1)$. Then any (ϵ, K)-invariant subset $A \subseteq G$ satisfies $\theta(K) \leq \theta(A)$.*

Proof. The statement is trivial whenever $K = \emptyset$ and we assume without lost of generality that there exists $k \in K^{-1}$. Then there holds $\theta(A \setminus \partial_{kK} A) \geq \theta(A) - \theta(\partial_{kK} A) = \theta(A) - \theta(\partial_K A) \geq \theta(A)(1-\epsilon) > 0$ and in particular there is $g \in A \setminus \partial_{kK} A$. Such g satisfy $(kK)^{-1} g \subseteq A$ and we conclude $\theta(K) = \theta((kK)^{-1} g) \leq \theta(A)$. □

2.4.2 Ergodic, Følner and Van Hove nets

Let $(A_i)_{i\in I}$ be a net of compact subsets of G such that for sufficiently large $i \in I$ there holds $\theta(A_i) > 0$. The net $(A_i)_{i\in I}$ is called *(left) ergodic*, whenever there holds $\lim_{i\in I} \theta((gA_i)\Delta A_i)/\theta(A_i) = 0$ for all $g \in G$. It is called *(left) Følner*, whenever there holds $\lim_{i\in I} \theta((KA_i)\Delta A_i)/\theta(A_i) = 0$ for all non-empty and compact subsets $K \subseteq G$. It is furthermore said to be *(left) Van Hove*, whenever there holds $\lim_{i\in I} \theta(\partial_K A_i)/\theta(A_i) = 0$ for all compact subsets $K \subseteq G$. As statements about ergodic, Følner and Van Hove nets are somehow scattered over the literature, we next present some characterizations of these concepts and furthermore relate them. We include the short and straightforward proof for the convenience of the reader and as we are not aware of a reference for some of the statements.

Proposition 2.17. *Let $(A_i)_{i\in I}$ be a net of compact subsets of G such that for large $i \in I$ there holds $\theta(A_i) > 0$.*

(a) *The net $(A_i)_{i\in I}$ is ergodic, if and only if there holds $\lim_{i\in I} \theta((gA_i) \setminus A_i)/\theta(A_i) = 0$ for all $g \in G$.*

(b) *The following statements are equivalent.*

- *(i)* *$(A_i)_{i\in I}$ is Følner.*
- *(ii)* *$\lim_{i\in I} \theta(KA_i \Delta A_i)/\theta(A_i) = 0$ for all compact and symmetric $K \subseteq G$ that satisfy $e_G \in K$.*
- *(iii)* *$\lim_{i\in I} \theta(KA_i \setminus A_i)/\theta(A_i) = 0$ for all compact $K \subseteq G$.*
- *(iv)* *$\lim_{i\in I} \theta(KA_i \setminus A_i)/\theta(A_i) = 0$ for all compact and symmetric $K \subseteq G$ that satisfy $e_G \in K$.*

(c) *The following statements are equivalent.*

- *(i)* *$(A_i)_{i\in I}$ is Van Hove.*
- *(ii)* *$\lim_{i\in I} \theta(\partial_K A_i)/\theta(A_i) = 0$ for all compact and symmetric subsets $K \subseteq G$ that contain e_G.*
- *(iii)* *$(A_i)_{i\in I}$ is ergodic and satisfies $\lim_{i\in I} \theta(\partial_W A_i)/\theta(A_i) = 0$ for some precompact neighbourhood W of e_G.*

(d) *Every Følner net is ergodic and every Van Hove net is Følner.*

(e) *Whenever G is discrete, then every ergodic net is Van Hove.*

Proof. The statement (a) follows from Lemma 2.15 as singletons are compact in G. To show (b) note first that (i) trivially implies (ii). As $KA_i \setminus A_i \subseteq KA_i \Delta A_i$ we obtain that (ii) implies (iv). Considering $K \cup K^{-1} \cup \{e_G\}$ for a compact subset one obtains that (iv) implies (iii) and from Lemma 2.15 one obtains that (iii) implies (i). This shows (b).

The equivalence between (i) and (iii) of (c) is shown in [Tem92, Appendix 3.K] and considering $K \cup K^{-1} \cup \{e_G\}$ for a compact subset K one easily obtains the equivalence of (i) and (ii).

As singletons are compact in G we obtain that any ergodic net is Følner. Furthermore, considering Lemma 2.15 one sees that (c)(ii) implies (b)(ii) and thus every Van Hove net is Følner. This shows (d). As $\{e_G\}$ is a precompact neighbourhood of e_G in a discrete group we obtain from $\partial_{\{e_G\}} A = \overline{A} \cap \overline{A^c} = A \cap A^c = \emptyset$ and (d) that any ergodic net is Van Hove in a discrete group. This shows (e). □

We will see in Proposition 2.33 below that a unimodular (and non-compact) group G is discrete if and only if the concepts of Følner and Van Hove nets coincide for G. Proposition 2.34 will furthermore give a similar characterization with respect to ergodic and Følner nets respectively. As a foretaste we consider the following examples, which are special cases of the constructions presented below. The example that separates the notions of Følner nets and Van Hove nets is inspired by [Tem92, Appendix; (3.4)].

Example 2.18. *Consider $A_n := [0, n] \cup ([n+1, n+1+n^2] \cap \mathbb{Z})$ and $A'_n := [0, n] \setminus (B_{n^{-2}}(0) + \mathbb{Z})$ for $n \in \mathbb{N}$. Then $(A_n)_{n\in\mathbb{N}}$ is a Følner sequence in $\mathbb{R}$ that is not Van Hove; and $(A'_n)_{n\in\mathbb{N}}$ is an ergodic sequence in $\mathbb{R}$ that is not Følner.*

2.4.3 Amenability

We define G to be *amenable*, whenever G contains a Van Hove net.

Remark 2.19. *Whenever G is a σ-compact unimodular amenable group, then G contains a Van Hove sequence. Indeed, as every Van Hove net is ergodic we obtain the statement from [Tem92, Appendix (3.H)] and [Tem92, Appendix (3.L)].*

The mentioned statements furthermore allow to conclude that the existence of Van Hove nets is equivalent to the existence of ergodic nets whenever G is assumed to be σ-compact and unimodular. In Proposition 2.35 below we will see that this statement is also true without the assumption of σ-compactness and thus in particular shows that our definition of amenability is equivalent to the definition given in [Tem92] and in [Pie84]. As we do not know of a reference of the statement we include a full proof for the convenience of the reader. From [Tem92, Appendix (3.A)] we cite the following, but note that this statement can also be deduced from Proposition 3.4 below, for which we present a full proof.

Proposition 2.20. *Every LCA group is amenable.*

2.4.4 Van Hove nets in the literature

The following lemma connects different approaches to the K-boundary given in the literature and was found in a discussion with C. Oertel. In particular, it shows in combination with Proposition 2.17 that our notion of Van Hove nets is equivalent to the notions in [Tem92], in [Sch00] and in [FGJO18].

Lemma 2.21. *Let K, A be compact subsets of G. Then there holds $\partial_K A = \overline{KA} \setminus \left(\operatorname{int}\left(\bigcap_{k\in K} kA\right)\right)$. Furthermore, whenever K is assumed to be symmetric and to contain e_G, then there holds $\partial_K A = \left(\left(K\overline{A}\right) \setminus \operatorname{int}(A)\right) \cup \left(\left(K^{-1}\overline{A^c}\right) \setminus \operatorname{int}(A^c)\right)$.*

Proof. We have $\left(\operatorname{int}\left(\bigcap_{k\in K} kA\right)\right)^c = \overline{\bigcup_{k\in K}(kA)^c} = \overline{KA^c}$. Thus, as K is compact, there holds $\partial_K A = K\overline{A} \cap K\overline{A^c} = \overline{KA} \cap \overline{KA^c} = \overline{KA} \setminus \left(\operatorname{int}\left(\bigcap_{k\in K} kA\right)\right)$. To see the second equality note $\overline{A} \subseteq K\overline{A}$ and $\overline{A^c} \subseteq K\overline{A^c}$ and calculate

$$\begin{aligned}
\partial_K A &= K\overline{A} \cap K\overline{A^c} \\
&= G \cap \left(K\overline{A} \cup \overline{A}\right) \cap \left(\overline{A^c} \cup K\overline{A^c}\right) \cap G \\
&= \left(K\overline{A} \cup K\overline{A^c}\right) \cap \left(K\overline{A} \cup \overline{A}\right) \cap \left(\overline{A^c} \cup K\overline{A^c}\right) \cap \left(\overline{A^c} \cup \overline{A}\right) \\
&= \left(K\overline{A} \cap \overline{A^c}\right) \cup \left(K\overline{A^c} \cap \overline{A}\right) \\
&= \left((K\overline{A}) \setminus \operatorname{int}(A)\right) \cup \left((K\overline{A^c}) \setminus \operatorname{int}(A^c)\right) \\
&= \left((K\overline{A}) \setminus \operatorname{int}(A)\right) \cup \left((K^{-1}\overline{A^c}) \setminus \operatorname{int}(A^c)\right).
\end{aligned}$$

□

2.4.5 Stability properties

Stability of ergodic nets

Proposition 2.22. *Whenever $(A_i)_{i\in I}$ is a ergodic net and $(F_i)_{i\in I}$ is a net (with the same index set) of compact subsets of G such that $\lim_{i\in I} \theta(F_i)/\theta(A_i) = 0$, then $(A_i \cup F_i)_{i\in I}$ is also ergodic.*

Remark 2.23. *This construction in particular applies, whenever $(F_i)_{i\in I}$ is a net of finite sets in any non-discrete unimodular amenable group, as finite sets in such groups always have Haar measure 0.*

Proof of Proposition 2.22. For $g \in G$ a straightforward computation shows

$$(g(A_i \cup F_i))\Delta(A_i \cup F_i) \subseteq F \cup (gF_i) \cup ((gA_i)\Delta A_i).$$

As $(A_i)_{i\in I}$ is ergodic we thus conclude

$$0 \leq \frac{\theta((g(A_i \cup F_i))\Delta(A_i \cup F_i))}{\theta(A_i \cup F_i)} \leq \frac{2\theta(F_i) + \theta(gA_i\Delta A_i)}{\theta(A_i)} \to 0.$$

□

Stability of Følner nets

Proposition 2.24. *Let $(A_i)_{i\in I}$ be a Følner net and $M \subseteq G$ be non-empty and compact. Then $(MA_i)_{i\in I}$ is also Følner and satisfies $\lim_{i\in I} \theta(MA_i)/\theta(A_i) = 1$.*

Proof. Let $m \in M$ and set $N := m^{-1}M$. From

$$\begin{aligned} A_i \cup ((NA_i)\Delta A_i) &= A_i \cup ((NA_i) \setminus A_i) \cup (A_i \setminus (NA_i)) \\ &= A_i \cup ((NA_i) \setminus A_i) \supseteq NA_i \end{aligned}$$

for all $i \in I$ we obtain

$$1 \leq \frac{\theta(MA_i)}{\theta(A_i)} = \frac{\theta(NA_i)}{\theta(A_i)} \leq 1 + \frac{\theta((NA_i)\Delta A_i)}{\theta(A_i)} \to 1,$$

which shows $\lim_{i\in I} \theta(MA_i)/\theta(A_i) = 1$. To show that $(MA_i)_{i\in I}$ is Følner let $K \subseteq G$ be a non-empty and compact subset. Then by Lemma 2.15 there holds

$$\begin{aligned} \theta((KMA_i)\Delta(MA_i)) &\leq 2\theta((KMA_i) \setminus (MA_i)) \leq 2\theta((KMA_i) \setminus (mA_i)) \\ &= 2\theta((m^{-1}KMA_i) \setminus A_i) \leq 2\theta((m^{-1}KMA_i)\Delta A_i) \end{aligned}$$

and as $\lim_{i\in I} \theta(MA_i)/\theta(A_i) = 1$ there holds

$$0 \leq \limsup_{i\in I} \frac{\theta((KMA_i)\Delta(MA_i))}{\theta(MA_i)} \leq 2 \lim_{i\in I} \frac{\theta((m^{-1}KMA_i)\Delta A_i)}{\theta(A_i)} = 0.$$

□

Stability of Van Hove nets

Proposition 2.25. *Let $(A_i)_{i\in I}$ be a Van Hove net and $M \subseteq G$ be non-empty and compact. Then $(MA_i)_{i\in I}$ is also Van Hove and satisfies $\lim_{i\in I} \theta(MA_i)/\theta(A_i) = 1$.*

Proof. As every Van Hove net is Følner we obtain $\lim_{i\in I} \theta(MA_i)/\theta(A_i) = 1$ from Proposition 2.24. To show that $(MA_i)_{i\in I}$ is Van Hove let $K \subseteq G$ be compact. Then for large i there holds $\theta(A_i) > 0$ and in particular $A_i \neq \emptyset$. Thus, $\theta(A_i) \leq \theta(MA_i)$ and we obtain from KM being compact and $\partial_K MA_i \subseteq \partial_{KM} A_i$ that

$$0 \leq \frac{\theta(\partial_K MA_i)}{\theta(MA_i)} \leq \frac{\theta(\partial_{KM} A_i)}{\theta(A_i)} \stackrel{i\in I}{\to} 0,$$

which proofs $(MA_i)_{i\in I}$ to be a Van Hove net. ☐

The following reverse of Proposition 2.25 holds and allows to shrink the elements of a Van Hove net without loosing the limiting behaviour. This possibility will be the key to the results in Chapter 5 and furthermore to some results in Chapter 3.

Proposition 2.26. *Whenever $(A_i)_{i\in I}$ is a Van Hove net and $M \subseteq G$ is a compact and non-empty subset of G, then there exists a Van Hove net $(B_i)_{i\in I}$ with the same index set such that there holds $MB_i \subseteq A_i$ and $B_i^c \subseteq M^{-1}A_i^c$ for all $i \in I$ and which satisfies* $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$.

Proof. As M is non-empty there exists a $m \in M$. Then $M' := Mm^{-1}$ contains e_G and for a Van Hove net $(B_i')_{i\in I}$ that satisfies $M'B_i' \subseteq A_i$ and $(B_i')^c \subseteq (M')^{-1}A_i^c$ for all $i \in I$, and $\lim_{i\in I} \theta(B_i')/\theta(A_i) = 1$, we can set $B_i := m^{-1}B_i'$ to obtain a Van Hove net $(B_i)_{i\in I}$ with the required properties. We thus assume without lost of generality that $e_G \in M$ and set $B_i := \overline{\{g \in G;\, Mg \subseteq A_i\}}$ for $i \in I$. As A_i is closed, we obtain $MB_i \subseteq A_i$ for all $i \in I$. Furthermore, for $g \in B_i^c$ there holds $Mg \not\subseteq A_i$, i.e. $Mg \cap A_i^c \neq \emptyset$. As $Mg \cap A_i^c \neq \emptyset$ is equivalent to $g \in M^{-1}A_i^c$ we obtain $B_i^c \subseteq M^{-1}A_i^c$ and it remains to show that $(B_i)_{i\in I}$ is a Van Hove net that satisfies $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$.

As $e_G \in M$ and A_i is closed we easily obtain $B_i \subseteq A_i$. Now assume that $g \in A_i$ and observe that there holds $g \in M^{-1}\overline{A_i}$. Hence, whenever $g \notin \partial_{M^{-1}} A_i = M^{-1}\overline{A_i} \cap M^{-1}\overline{A_i^c}$, then $g \notin M^{-1}\overline{A_i^c} \supseteq M^{-1}A_i^c$. As $g \notin M^{-1}A_i^c$ is equivalent to $Mg \cap A_i^c = \emptyset$ we obtain $Mg \subseteq A_i$, which implies $g \in B_i$. We have shown that for all $i \in I$ there holds $B_i \subseteq A_i \subseteq B_i \cup \partial_{M^{-1}} A_i$. As $(A_i)_{i\in I}$ is assumed to be Van Hove we know that for sufficiently large i there holds $\theta(A_i) > 0$ and compute

$$1 \geq \frac{\theta(B_i)}{\theta(A_i)} \geq \frac{\theta(A_i) - \theta(\partial_{M^{-1}} A_i)}{\theta(A_i)} \geq 1 - \frac{\theta(\partial_{M^{-1}} A_i)}{\theta(A_i)} \to 1$$

for large i. Hence, there holds $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$ and we obtain in particular that $\theta(B_i) > 0$ for sufficiently large i.

To show that $(B_i)_{i\in I}$ is a Van Hove net, let $K \subseteq G$ be a compact subset. From $e_G \in M$ and A_i being closed we get that there holds $B_i \subseteq A_i \subseteq M^{-1}\overline{A_i}$ and above it was presented that $B_i^c \subseteq M^{-1}A_i^c$. As M^{-1} is compact we thus obtain $\overline{B_i^c} \subseteq M^{-1}\overline{A_i^c}$ and compute

$$\partial_K B_i = KB_i \cap K\overline{B_i^c} \subseteq KM^{-1}A_i \cap KM^{-1}\overline{A_i^c} = \partial_{KM^{-1}} A_i.$$

From $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$ and as $(A_i)_{i\in I}$ is Van Hove we finally obtain

$$0 \leq \limsup_{i\in I} \frac{\theta(\partial_K B_i)}{\theta(B_i)} \leq \lim_{i\in I} \frac{\theta(\partial_{KM^{-1}} A_i)}{\theta(A_i)} = 0,$$

which shows $(B_i)_{i\in I}$ to be a Van Hove net. □

Counterexamples

It is natural to ask, whether the presented statements can be achieved in greater generality. The next example shows that the stability statement about ergodic nets in Proposition 2.22 cannot be drawn for Følner or Van Hove nets respectively.

Example 2.27. *Consider the Van Hove net* $([0,n]_{n\in\mathbb{N}})$ *in* $\mathbb{R}$ *and furthermore the finite sets* $F_n := [n+1, n+1+n^2]\cap\mathbb{Z}$. *Then* $([0,n]\cup F_n)_{n\in\mathbb{N}}$ *is not Følner. Indeed for the compact set* $K=[0,1]$ *we obtain*

$$\frac{\theta(K([0,n]\cup F_n)\setminus([0,n]\cup F_n))}{\theta([0.n]\cup F_n)} = \frac{\theta([n,2+n+n^2])}{\theta([0,n])} \to \infty.$$

The following example shows that one cannot enlarge ergodic nets similar to Følner or Van Hove nets as presented in the Propositions 2.24 and 2.25 respectively.

Example 2.28. *Consider* $A_n := [0,n]\cup([n+1,n+1+n^2]\cap\mathbb{Z})$. *From Proposition 2.22 we obtain that* $(A_n)_{n\in\mathbb{N}}$ *is an ergodic sequence in* $\mathbb{R}$. *Nevertheless, for the compact neighbourhood* $M=[-1/4,1/4]$ *of* 0 *the net* $(M+A_n)_{n\in\mathbb{N}}$ *is not ergodic, as*

$$\frac{\theta\left((1/2+M+A_n)\,\Delta\,(M+A_n)\right)}{\theta(M+A_n)} \to 2.$$

From Proposition 2.26 the question arises, whether a similar construction can be done for Følner nets, i.e. whether for all Følner nets $(A_i)_{i\in I}$ and for all compact $M\subseteq G$ there exists a Følner net $(B_i)_{i\in I}$ with $MB_i\subseteq A_i$ and $\lim_{i\in I}\theta(B_i)/\theta(A_i)=1$. The next example shows that this is not the case.

Example 2.29. *Let* $(A_i)_{i\in I}$ *be a Følner net in* $\mathbb{R}$ *such that* A_i *is disjoint from* $\mathbb{Z}$ *for all* $i\in I$. *One could for example consider* $A_n=[0,n]\setminus(\mathbb{Z}+B_{1/n}(0))$ *and* $I:=\mathbb{N}$. *Then for all compact neighbourhoods* M *of* 0 *there does not exist a Følner net* $(B_i)_{i\in I}$ *such that* $M+B_i\subseteq A_i$ *and such that* $\lim_{i\in I}\theta(B_i)/\theta(A_i)=1$. *Note that this implies that* $(A_i)_{i\in I}$ *is not Van Hove.*

Indeed, if there would exist such a sequence $(B_i)_{i\in I}$, *then* $M+B_i$ *is disjoint from* $\mathbb{Z}$ *for all* i. *Letting* $K:=[0,1]$ *we thus obtain* $\overline{(M+B_i)^c}\supseteq(M+B_i)^c\supseteq\mathbb{Z}$, *hence* $K\overline{(M+B_i)^c}=\mathbb{R}$. *Therefore* $\partial_K(M+B_i)=K(M+B_i)\supseteq(M+B_i)$ *and* $(M+B_i)_{i\in I}$ *cannot be Van Hove in contradiction to Proposition 2.30, which we will show next.*

2.4.6 Interplay of these notions

Van Hove nets can be seen as "fat" Følner nets. Whenever $(A_i)_{i\in I}$ is a Følner net, then one can thicken it an arbitrary tiny bit to turn it into a Van Hove net. The following statement makes this idea precise and was found in a discussion with Gabriel Fuhrmann.

Proposition 2.30. *Whenever $(A_i)_{i\in I}$ is a Følner net in G and $M \subseteq G$ is a compact neighbourhood of e_G, then $(MA_i)_{i\in I}$ is a Van Hove net.*

Proof. By Proposition 2.24 we know that $(MA_i)_{i\in I}$ is Følner and in particular ergodic and satisfies $\lim_{i\in I} \theta(MA_i)/\theta(A_i) = 1$. Thus, by Proposition 2.17 it is sufficient to show that there exists a compact neighbourhood W of 0 such that $\lim_{i\in I} \theta(\partial_W(MA_i))\theta(MA_i) = 0$. Choose a compact and symmetric neighbourhood W of e_G such that $WW \subseteq M$. Then for $g \in \partial_W(MA_i)$ there holds $g \in WMA_i$ and $g \in W\overline{(MA_i)^c}$ and thus there is $u \in W = W^{-1}$ such that $ug \in \overline{(MA_i)^c}$. As W is a neighbourhood of e_G there is $v \in W$ such that $vug \in (MA_i)^c$ and thus $g \notin (vu)^{-1}MA_i \supseteq A_i$. This shows $g \in (WMA_i) \setminus A_i$ and we obtain $\partial_W(MA_i) \subseteq (WMA_i) \setminus A_i = (WMA_i)\Delta A_i$. Thus, the statement follows from $(A_i)_{i\in I}$ being Følner. □

As a corollary we obtain the following.

Corollary 2.31. *For every Følner net $(A_i)_{i\in I}$ there holds $\lim_{i\in I} \theta(A_i) = \theta(G)$. In particular, if G is non-compact, there holds $\lim_{i\in I} \theta(A_i) = \infty$.*

Proof. Let $M \subseteq G$ be a compact neighbourhood of e_G. Then $(MA_i)_{i\in I}$ is a Van Hove net and we obtain $\theta(MA_i) \to \theta(G)$ from the regularity of θ and Lemma 2.16. As Proposition 2.24 furthermore implies $\lim_{i\in I} \theta(MA_i)/\theta(A_i) = 1$ we obtain the statement. □

Remark 2.32. *It remains open, whether a similar statement holds for ergodic nets.*

Whenever G is compact also the reverse statement of Corollary 2.31 holds, i.e. $(A_i)_{i\in I}$ is Følner, if and only if $\theta(A_i) \to \theta(G)$. For compact G one can furthermore show that $(A_i)_{i\in I}$ is Van Hove, if and only if $A_i = G$ for sufficiently large i.

2.4.7 Ergodic, Følner and Van Hove nets in non-discrete groups

As presented above the concepts of ergodic, Følner and Van Hove nets coincide for discrete groups. We will see next that this coincidence actually characterizes discrete groups. We first present that a unimodular group is discrete if an only if the concepts of Van Hove and Følner nets coincide. Note that the idea of the following construction originates from [Tem92, Appendix Example 3.4] where Van Hove nets in $\mathbb{R}^d$ are discussed.

Proposition 2.33. *Every non-discrete unimodular amenable group G contains a Følner net that is not Van Hove. If G is in addition σ-compact and and first countable[3], then G contains a Følner sequence that is not Van Hove.*

[3] A topological space is called *first countable*, whenever every point has a countable neighbourhood basis.

Proof. Let ω be a discrete subset of G such that there exists a compact set C and an open neighbourhood W of e_G such that $C\omega = G$ and such that $\{Wv;\, v \in \omega\}$ is disjoint. We will see in Remark 2.37 below that such "Delone sets" exist in all locally compact groups and assume without lost of generality that W and C are symmetric and contain e_G. As G is assumed to be unimodular amenable we obtain that there exists a Van Hove net $(B_j)_{j\in J}$ in G. As $\theta(B_j) > 0$ for large j and as we can translate each B_j we assume without lost of generality that there holds $\theta(B_j) > 0$ and that ω and B_j intersect for all $j \in J$.

To construct the index set of our net let N be the set of all open and symmetric neighbourhoods of e_G that are contained in W. We order N by the reversed set inclusion and obtain from the regularity of the Haar measure θ and as G is non-discrete that there holds

$$0 = \theta(\{e_G\}) = \inf_{V\in N} \theta(V) = \lim_{V\in N} \theta(V). \tag{2.1}$$

We furthermore define I as the set of all tuples $(j, V) \in J \times N$ that satisfy

$$\theta(V) \leq \frac{\theta(B_j)}{2|\omega \cap WB_j|}. \tag{2.2}$$

Recall that for $j \in J$ the sets ω and $B_j \subseteq WB_j$ intersect and thus $|\omega \cap WB_j| \geq 1$. Furthermore, there holds $\theta(B_j) > 0$ and thus we obtain from (2.1) the existence of $V \in N$ such that $(j, V) \in I$. This shows that I is non-empty. Furthermore, for $(j, V), (j', V') \in I$ we obtain from the directedness of J that there exists $\tilde{j} \in J$ such that $j, j' \leq \tilde{j}$ and obtain with similar arguments as above that there exists $\widetilde{V} \in N$ such that $(\tilde{j}, \widetilde{V} \cap V \cap V') \in I$. Thus, we can equip I with the component wise order of $J \times N$ to obtain a directed set. Setting $A_{(j,V)} := B_j \setminus V\omega$ for all $(j, V) \in I$ we then obtain a net of compact sets.

Claim 1: $(A_i)_{i\in I}$ is a Følner net.

Let K be a compact subset of G that contains e_G and consider $(j, V) \in N$. For $g \in B_j \cap V\omega$ there exist $v \in V$ and $w \in \omega$ such that $g = vw$. Thus, $w = v^{-1}g \in VB_j$ and hence $g = vw \in V(VB_j \cap \omega)$. We thus obtain $B_j \cap V\omega \subseteq V(VB_j \cap \omega)$. As $V \subseteq W$ and as $\{Vg;\, g \in \omega\}$ is disjoint we get

$$\theta(B_j \cap V\omega) \leq \theta(V(VB_j \cap \omega)) = \theta(V)|VB_j \cap \omega| \leq \theta(V)|WB_j \cap \omega|. \tag{2.3}$$

Thus, the definition of I implies

$$\begin{aligned}\theta(A_{(j,V)}) &\geq \theta(B_j) - \theta(B_j \cap V\omega) \\ &\geq \theta(B_j) - \theta(V)|WB_j \cap \omega| \\ &\geq \theta(B_j) - \frac{1}{2}\theta(B_j) \\ &= \frac{1}{2}\theta(B_j).\end{aligned}$$

Furthermore, as $\{Wv;\, v \in \omega\}$ is disjoint we obtain from (2.3) that there holds

$$\theta(B_j \cap V\omega) \leq \theta(V)|WB_j \cap \omega| = \frac{\theta(V)}{\theta(W)}\theta(W(WB_j \cap \omega)) \leq \frac{\theta(V)}{\theta(W)}\theta(WWB_j).$$

As K contains e_G there holds $B_j \subseteq KB_j$ and hence $KB_j \setminus A_{(j,V)} = (KB_j \setminus B_j) \cup (B_j \cap V\omega)$ and we compute

$$\begin{aligned}\theta(KA_{(j,V)} \setminus A_{(j,V)}) &\leq \theta(KB_j \setminus A_{(j,V)}) \\ &\leq \theta(KB_j \setminus B_j) + \theta(B_j \cap V\omega) \\ &\leq \theta(KB_j \setminus B_j) + \frac{\theta(V)}{\theta(W)}\theta(WWB_j).\end{aligned}$$

As $(B_j)_{j\in J}$ is a Følner net we observe $\lim_{i\in I} \theta(WWB_j)/\theta(B_j) = 1$ from Proposition 2.24. Thus, (2.1) implies

$$0 \leq \frac{\theta(KA_{(j,V)} \setminus A_{(j,V)})}{\theta(A_{(j,V)})} \leq 2\frac{\theta(KB_j \setminus B_j)}{\theta(B_j)} + 2\frac{\theta(V)}{\theta(W)}\frac{\theta(WWB_j)}{\theta(B_j)} \to 2 \cdot 0 + 2 \cdot 0 \cdot 1 = 0.$$

and we obtain that $(A_i)_{i\in I}$ is indeed a Følner net.

Claim 2: $(A_i)_{i\in I}$ is not Van Hove.

Let $(j, V) \in I$ and consider $g \in A_{(j,V)}$. As $C\omega = G$ there exists $v \in \omega$ such that $g \in Cv$. Note that $e_G \in V$ and thus ω and $A_{(j,V)} = B_j \setminus V\omega$ are disjoint. Thus, $v \in \overline{A^c_{(j,V)}}$ and we obtain $g \in Cv \subseteq C\overline{A^c_{(j,V)}}$. As $e_G \in C$ this implies that $g \in CA_{(j,V)} \cap C\overline{A^c_{(j,V)}} = \partial_C A_{(j,V)}$. This shows that $A_{(j,V)} \subseteq \partial_C A_{(j,V)}$ and we obtain that

$$\frac{\theta(\partial_C A_{(j,V)})}{\theta(A_{(j,V)})} \geq 1$$

for all $(j, V) \in I$. Thus, $(A_i)_{i\in I}$ is not Van Hove.

If G is in addition assumed to be σ-compact and first countable, then we can choose $(B_j)_{j\in J}$ to be a sequence, i.e. $J = \mathbb{N}$ and furthermore a sequence $(V_j)_{j\in\mathbb{N}}$ in N such that $(j, V_j) \in I$ and such that for every $V \in N$ there holds $V_j \subseteq V$ for sufficiently large

j. Then the diagonal sequence $(A_{(j,V_j)})_{j\in\mathbb{N}}$ is a subnet of $(A_i)_{i\in I}$ and thus in particular Følner. This sequence still satisfies $\theta(\partial_C A_{(j,V_j)})/\theta(A_{(j,V_j)}) \geq 1$ for all $j \in \mathbb{N}$ and can thus not be Van Hove. □

The next result shows that a (non-compact) unimodular group is discrete if an only if the concepts of Følner and ergodic nets coincide.

Proposition 2.34. *Every non-discrete and non-compact unimodular group G contains an ergodic net that is not Følner. If G is in addition σ-compact, then G contains an ergodic sequence that is not Følner.*

Proof. Let ω a discrete subset of G such that there exist a compact subset K and an open neighbourhood V of e_G such that $K\omega = G$ and such that $\{Vv;\, v \in \omega\}$ is disjoint. Such "Delone sets" exist as we will discuss in Remark 2.37 below and without lost of generality we assume that $V \subseteq K$. As G is assumed to be amenable there exists a Van Hove net $(B_i)_{i\in I}$ in G and this net can be chosen such that there holds $\theta(B_i) > 0$ for all $i \in I$. For $i \in I$ the compactness of B_i implies that $\omega \cap B_i$ is finite and as G is not compact we obtain that $\omega \setminus B_i$ is infinite. Thus, for $i \in I$ we can choose a finite subset $F_i \subseteq \omega \setminus B_i$ that satisfies $|F_i| \geq \theta(B_i)^2$. We set $A_i := B_i \cup F_i$ and obtain from Proposition 2.22 that $(A_i)_{i\in I}$ is an ergodic net.

As $\{Vv;\, v \in \omega\}$ is disjoint and as $V \subseteq K$ it follows that $\theta(KF_i) \geq \theta(VF_i) = \theta(V)|F_i| \geq \theta(V)\theta(B_i)^2$. The non-discreteness of G furthermore implies that $\theta(A_i) = \theta(B_i)$ and we obtain from Corollary 2.31 that there holds

$$\frac{\theta(KA_i \setminus A_i)}{\theta(A_i)} \geq \frac{\theta(KF_i \setminus A_i)}{\theta(A_i)} \geq \frac{\theta(KF_i)}{\theta(A_i)} - 1 \geq \theta(V)\frac{\theta(B_i)^2}{\theta(B_i)} - 1 \to \infty.$$

Thus, $(A_i)_{i\in I}$ cannot be Følner. If G is in addition σ-compact, then $(B_i)_{i\in I}$ can be assumed to be a sequence, i.e. $I = \mathbb{N}$ and the construction yields a sequence $(A_i)_{i\in\mathbb{N}}$. □

2.4.8 Existence of ergodic, Følner and Van Hove nets

We have seen that for non-discrete groups the concepts of Van Hove, Følner and ergodic nets differ and it is natural to ask, whether every amenable group, i.e. every group that contains a Van Hove net also contains Følner or even ergodic nets. From Proposition 2.30 we obtain that G contains a Van Hove net (sequence) if and only if G contains a Følner net (sequence). This statement can also be found in [PS16, Lemma 2.7.], where Følner sequences are called "weak Følner sequences" and Van Hove sequences are called "strong Følner sequences". In [Tem92, Appendix 3.L] it is presented that the existence of a Van Hove sequence is equivalent to the existence of an ergodic sequence, whenever G is σ-compact. As we do not know of a reference for the general case we include a proof without countability assumptions for the convenience of the reader. Note that Proposition 2.35 holds for sequences if G is assumed to be σ-compact [Tem92, Appendix 3.L].

Proposition 2.35. *For a unimodular group G the following statements are equivalent.*

(i) G is amenable, i.e. G contains a Van Hove net.

(ii) G contains a Følner net.

(iii) G contains an ergodic net.

(iv) For all $\epsilon > 0$ and all finite $F \subseteq G$ there exists a compact set $A \subseteq G$ such that $\theta(gA \setminus A)/\theta(A) < \epsilon$ for all $g \in F$.

(v) For all $\epsilon > 0$ and all compact $K \subseteq G$ with $e_G \in K$ there exists a compact set $A \subseteq G$ such that $\theta(KA \setminus A)/\theta(A) < \epsilon$.

Remark 2.36. *In particular, this shows that our definition is equivalent to the definition of amenability in the monograph [Pie84]. In order to see this compare (iv) with [Pie84, Theorem 7.3(F*)] in combimantion with [Pie84, Proposition 7.4]. Furthermore, it implies that the notion of "left-amenability" in [Tem92] is equivalent to our notion of amenability.*

Proof. Clearly, (i) implies (ii), (ii) implies (iii), and (iii) implies (iv). The equivalence of (iv) and (v) can be found in [Pie84] by combining [Pie84, Theorem 7.3 (F^*)] , [Pie84, Proposition 7.4], [Pie84, Theorem 7.9 (SF_e)] and [Pie84, Proposition 7.11]. It thus remains to show that (v) implies (i).

If G is compact, then $(G)_{n\in\mathbb{N}}$ is a Van Hove net in G. We can thus assume G to be not compact. Let W be a compact and symmetric neighbourhood W of e_G and I be the set of all finite subsets of G containing e_G, ordered by set inclusion. For $i \in I$ define $K_i := \bigcup_{g\in i} Wg$. Note that there hold $\theta(K_i) \geq \theta(W) > 0$ and $e_G \in K_i$ for all $i \in I$. Thus, by (v) for every $i \in I$ there exists a compact set $A_i \subseteq G$ such that $\theta(K_iA_i \setminus A_i)/\theta(A_i) < 1/\theta(K_i)$. We now show that $(A_i)_{i\in I}$ is a Følner net, which implies $(WA_i)_{i\in I}$ to be a Van Hove net in G by Proposition 2.30. To show this let $K \subseteq G$ compact and note that $\{Wg;\, g \in G\}$ is an open cover of K. Thus, the compactness of K implies the existence a $j \in I$ with $K \subseteq \bigcup_{g\in j} Wg = K_j$. Note that $K_j \subseteq K_i$ for all $i \geq j$ and hence

$$0 \leq \frac{\theta((KA_i)\Delta A_i)}{\theta(A_i)} \leq 2\frac{\theta(KA_i \setminus A_i)}{\theta(A_i)} \leq 2\frac{\theta(K_iA_i \setminus A_i)}{\theta(A_i)} < \frac{2}{\theta(K_i)}$$

for all $i \geq j$. As θ is regular there holds $\lim_{i\in I} \theta(K_i) = \infty$ and we obtain $(A_i)_{i\in I}$ to be Følner. □

2.5 Discrete substructures of topological groups

2.5.1 Delone sets

Let G be a locally compact group. A subset $\omega \subseteq G$ is called *locally finite*, whenever $\omega \cap K$ is finite for all compact subsets $K \subseteq G$. For $M \subseteq G$ a subset $\omega \subseteq G$ is called *M-dense* in G, if $M\omega = G$. It is furthermore called *M-discrete*, if $\{Mg;\, g \in \omega\}$ is a disjoint family. The set ω is called *relatively dense*, if ω is K-dense for some compact $K \subseteq G$. It is called *uniformly discrete*, if it is V-discrete for some open neighbourhood V of e_G. If ω is relatively dense and uniformly discrete, we call ω a *Delone set*. For a compact subset $K \subseteq G$ and an open neighbourhood V of e_G we denote by $\mathcal{D}_V(G)$ the set of all V-discrete subsets of G and by $\mathcal{D}_{K,V}(G)$ the set of all K-dense and V-discrete subsets of G. Note that $\mathcal{D}_{K,V}(G) \subseteq \mathcal{D}_V(G) \subseteq \mathcal{A}(G)$.

Remark 2.37. *Whenever G is a locally compact group and V is a compact neighbourhood of e_G, then there exists a Delone set ω in G that is V-discrete and $V^{-1}V$-dense. Indeed, let us order $\mathcal{D}_V(G)$ by set inclusion. To apply Zorn's lemma, we consider a* chain *$\mathcal{C}$ in $\mathcal{D}_V(G)$, i.e. a subset $\mathcal{C} \subseteq \mathcal{D}_V(X)$ that satisfies that two elements $\omega, \omega' \in \mathcal{C}$ satisfy $\omega \subseteq \omega'$ or $\omega' \subseteq \omega$. We consider furthermore the union $\xi := \bigcup_{\omega \in \mathcal{C}} \omega$. If $\xi \notin \mathcal{D}_V(X)$, then there are distinct $g, g' \in \xi$ such that Vg and Vg' intersect. As $\mathcal{C}$ is a chain we obtain that there is $\omega \in \mathcal{C}$ such that $g, g' \in \omega$, which contradicts the fact that ω is V-discrete. Thus, $\xi \in \mathcal{D}_V(G)$ and as $\omega \subseteq \xi$ for all $\omega \in \mathcal{C}$ we have found that each chain in $\mathcal{D}_V(G)$ has an upper bound. Thus, Zorn's lemma yields a maximal element in $\mathcal{D}_V(G)$, i.e. a V-discrete set ω that is maximal with respect to set inclusion. For $g \in G$ we then obtain that there is $g' \in \omega$ such that Vg and Vg' intersect. Thus, there holds $g \in V^{-1}Vg' \subseteq V^{-1}V\omega$ and we obtain ω to be $V^{-1}V$-dense.*

Uniform density

A Delone set $\omega \subseteq G$ is said to have a *uniform density*, whenever for every Van Hove net $(A_i)_{i \in I}$ the limit $\lim_{i \in I} |\omega \cap A_i| / \theta(A_i)$ exists in $(0, \infty)$ and whenever this limit is independent of the choice of a Van Hove net. In this case we define the *uniform density* of ω as this limit and denote it by $\mathrm{dens}(\omega)$.

Delone sets in LCA groups

Let G be an LCA group. For $A \subseteq G$ compact and $g \in \omega$, we call $(\omega - g) \cap A$ an *A-patch* of $\omega \subseteq G$ and denote the set of all A-patches by $\mathrm{Pat}_\omega(A)$. A Delone set is said to have *finite local complexity* (*FLC*), if $\mathrm{Pat}_\omega(A)$ is finite for every compact set $A \subseteq G$.

2.5.2 Uniform lattices

Let G be a locally compact topological group. A discrete subgroup $\Lambda \subseteq G$ is called a *uniform lattice*, whenever it is *cocompact*, i.e. whenever the quotient G/Λ is a compact topological space. Every locally compact group that contains a uniform lattice is unimodular, i.e. there exists a (left and right invariant) Haar measure on G [DE14, Theorem 9.1.6]. A Borel measurable subset $C \subseteq G$ is called a *fundamental domain* (of Λ), whenever every $g \in G$ has a unique representation as $g = cl$ with $c \in C$ and $l \in \Lambda$. In [Mor15, Lemma 4.1.1] it is stated that every uniform lattice has a fundamental domain and we obtain from [Mor15, Proposition 4.1.3] that $\theta(C)$ does not depend on the choice of a fundamental domain.

Remark 2.38. *Assume that C is a fundamental domain of a uniform lattice Λ. If $F \subseteq \Lambda$ is finite, then the finite union $\bigcup_{z\in F} Cz$ is disjoint and measurable. Thus, by the right invariance of the Haar measure there holds*

$$\theta(CF) = \sum_{z\in F} \theta(Cz) = \theta(C)|F|.$$

We next present examples of uniform lattices. For further details see [CdlH16, Chapter 5.C].

Example 2.39. *$\mathbb{Z}^d$ is a uniform lattice in $\mathbb{R}^d$ with fundamental domain $[0, 1)^d$.*

Example 2.40. *The multiplicative group $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$ contains the uniform lattice $\{2^n; n \in \mathbb{Z}\}$ with fundamental domain $(-2, -1] \cup [1, 2)$.*

Example 2.41. *Every compactly generated LCA group contains a uniform lattice. Indeed, such groups are topologically and algebraically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times H$ for some $a, b \in \mathbb{N}_0$ and a compact group H, which contains the countable uniform lattice $\mathbb{Z}^{a+b} \times \{e_H\}$.*

The following example[4] shows that there are metrizable σ-compact LCA groups that contain no uniform lattice.

Example 2.42. *The additive group of the p-adic numbers $\mathbb{Q}_p$ contains no uniform lattice.*

Proof. Consider a discrete subgroup Λ of G and $g \in \Lambda$. Now recall that $\mathbb{Q}_p$ actually carries the structure of a field and that $|p^n \cdot g|_p = p^{-n}|g|_p$. Thus, $p^n \cdot g \to 0$ as $n \to \infty$. As furthermore $p^n \cdot g = \sum_{j=1}^{p^n} g \in \Lambda$ the discreteness of Λ implies $g = 0$. This shows that $\{0\}$ is the only discrete subgroup of $\mathbb{Q}_p$ and in particular that $\mathbb{Q}_p$ contains no uniform lattice. □

Example 2.43. *The multiplicative group $\mathbb{Q}_p^\times$ contains the uniform lattice $\{p^n; n \in \mathbb{Z}\}$.*

[4] This example was presented by Yves de Cornulier in a correspondence.

Example 2.44. *The discrete Heisenberg group $H_3(\mathbb{Z})$ is a uniform lattice in $H_3(\mathbb{R})$ with fundamental domain $H_3([0,1))$, where we denote*

$$H_3(M) := \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ; \, a, b, c \in M \right\}$$

for $M \subseteq \mathbb{R}$ and equip $H_3(\mathbb{Z})$ and $H_3(\mathbb{R})$ with the matrix multiplication.

The presented examples have fundamental domains that satisfy further properties, such as precompactness or regularity, i.e. vanishing boundary with respect to the Haar measure. These properties will be useful in the following and we will see next that for all uniform lattices in a locally compact group such a choice of a fundamental is possible. Using ideas from the proof in [Mor15] we will present next that there always exist fundamental domains with these additional properties. As we do not know of any reference for this statement we include a full proof.

Proposition 2.45. *Let G be a locally compact topological group. Then for any uniform lattice in G there exists a regular and precompact fundamental domain that is also a neighbourhood of e_G.*

Remark 2.46. *In particular, we obtain that there holds $\theta(C) \in (0, \infty)$ for all fundamental domains C of Λ, as all fundamental domains have the same Haar measure.*

Proof of Proposition 2.45. As we assume that G is locally compact and as Λ is discrete there exists a compact neighbourhood $\widetilde{M}$ of e_G such that $\widetilde{M}^{-1}\widetilde{M} \cap \Lambda = \{e_G\}$. The Haar measure restricted to $\widetilde{M}$ is a finite Borel measure on a compact Hausdorff space and we thus obtain from Lemma 2.6 that there exists a compact neighbourhood M of e_G that is contained in $\tilde{M}$ and satisfies $\theta(\partial M) = 0$. This M in particular satisfies $M^{-1}M \cap \Lambda = \{e_G\}$.

Now $U := \operatorname{int}(M)$ is a precompact neighbourhood of e_G and $gU\Lambda$ is open in G for all $g \in G$. Denoting the quotient map by q we thus obtain that $\{q(gU);\, g \in G\}$ is an open cover of the compact space G/Λ. Thus, there exists a finite sequence $(g_n)_{n=1}^N$ in G such that $\bigcup_{n=1}^N q(g_n U) = G/\Lambda$. This sequence then in particular satisfies

$$\bigcup_{n=1}^{N} g_n M\Lambda = \bigcup_{n=1}^{N} g_n U\Lambda = G.$$

Setting $g_0 := e_G$ we define

$$C := \bigcup_{n=0}^{N} \left(g_n M \setminus \left(\bigcup_{0 \leq i < n} g_i M\Lambda \right) \right).$$

We will first show that every $g \in G$ can be written uniquely as a product $g = cl$ with $c \in C$ and $l \in \Lambda$. Indeed for $g \in G = \bigcup_{n=0}^{N} g_n M\Lambda$ there is a minimal $n \in \{0, \cdots, N\}$ such that $g \in g_n M\Lambda$. We choose $l \in \Lambda$ such that $g \in g_n Ml$. As $g \notin g_i M\Lambda = g_i M\Lambda l$ for all $i < n$ we obtain that $gl^{-1} \in g_n M \setminus \left(\bigcup_{0\leq i<n} g_i M\Lambda\right) \subseteq C$ and thus in particular that $g \in Cl \subseteq C\Lambda$. To show that this representation is unique let $c, c' \in C$ and $l, l' \in \Lambda$ be such that $cl = c'l'$. As $c, c' \in C$ there are unique $n, m \in \{0, \cdots, N\}$ such that $c \in g_n M \setminus \left(\bigcup_{0\leq i<n} g_i M\Lambda\right)$ and $c' \in g_m M \setminus \left(\bigcup_{0\leq i<m} g_i M\Lambda\right)$. If $m \neq n$ we assume without lost of generality that $m < n$. Then $c = c'l'l^{-1} \in a_m M\Lambda$ establishes a contradiction to $c \in g_n M \setminus \left(\bigcup_{0\leq i<n} g_i M\Lambda\right)$. Thus, there holds $n = m$ and we obtain

$$c^{-1}c' = l(l')^{-1} \in \Lambda \cap M^{-1}g_n^{-1}g_m M = \Lambda \cap M^{-1}M = \{e_G\}.$$

It follows that $c = c'$ and $l = l'$. This shows that every $g \in G$ has a unique representation. Setting furthermore $L := \Lambda \cap \left(\bigcup_{n=0}^{N} g_n M\right)^{-1} \left(\bigcup_{n=0}^{N} g_n M\right)$ we obtain a finite subset and one easily sees that

$$C = \bigcup_{n=0}^{N} \left(g_n M \setminus \left(\bigcup_{0\leq i<n} g_i ML \right) \right).$$

As $\theta(\partial M) = 0$ we thus obtain that C is a Borel measurable set that also satisfies $\theta(\partial C) = 0$. From $C \subseteq \bigcup_{n=0}^{N} g_n M$ we see that C is precompact. As $M = a_0 M \subseteq C$ is a neighbourhood of e_G we have shown that the constructed fundamental domain C satisfies the additional properties. □

2.5.3 Cut and project schemes

Consider locally compact groups G and H and a uniform lattice Λ in $G \times H$. Then (G, H, Λ) is called a *cut and project scheme* (*CPS*), whenever the projections π_G and π_H satisfy the following properties. The restriction $\pi_G|_\Lambda$ is injective and $\pi_H(\Lambda)$ is dense in H. In this context we call G the *physical space* and H the *internal space* of (G, H, Λ).

Given a relatively compact subset $W \subseteq H$, usually called a *window* in this context, such a CPS produces a subset of G via $\Lambda(W) := \pi_G(\Lambda \cap (G \times W))$. Subsets of G that arise by this construction are called *(uniform) weak model sets*. It is called a *(uniform) model set*, whenever W has non-empty interior. A model set is said to be *regular*, whenever the window is regular, i.e. the window has a topological boundary of Haar measure 0. Note that every uniform lattice Λ in a locally compact group G is a regular model set, as one can consider the CPS $(G, \{0\}, \Lambda \times \{0\})$ to obtain $\Lambda = \Lambda(\{0\})$. A relatively dense subset of a model set is called a *Meyer set*. Note that we assume that Λ is a uniform lattice for all CPS. Thus, our definition implies that G and H are unimodular and in particular differs slightly from the notions developed in [BHP18]. Model sets are usually studied in the context of commutative groups. For further reference on commutative CPS we recommend [Mey72] and [BG13, Chapter 7]. For a reference in the non-commutative case see [BHP18, BH18].

2.5.4 Uniform approximate lattices

Let G be a locally compact group. A *uniform approximate lattice* in G is a symmetric Delone set $\omega \subseteq G$ that contains e_G and for which there exists a finite set $F \subseteq G$ that satisfies $\omega\omega \subseteq F\omega$. Note that whenever ζ is a K-dense and symmetric subset of a uniform lattice ω, then one easily shows $\zeta\zeta \subseteq (K \cap \zeta\zeta\zeta)\zeta$ and obtains that ζ is also a uniform approximate lattice. Thus, we obtain in particular all symmetric Meyer sets that contain e_G to be uniform approximate lattices from [BH18, Proposition 2.13]. In particular, any locally compact group that is the physical space of a CPS contains uniform approximate lattices. The reverse of this statement is shown for LCA groups in [Mey72, Chapter II]. It is shown that for every LCA group G that contains a uniform approximate lattice one can construct a CPS that has G as a physical space (and $\mathbb{R}^d$ as an internal space), but it seems open, whether a similar statement holds for general locally compact groups [BH18, Problem 1]. A further partial result can be found in [Mac20].

There are examples such as $\mathbb{Q}_p$ that contain no uniform lattices but uniform approximate lattices. An example of a metrizable and separable LCA group G that contains no uniform approximate lattices and therefore is not a physical space of a CPS is given in [Mey72, Chapter II.11]. For further reference on uniform approximate lattices we recommend [Mey72, BHP18, BH18, Mac20].

2.6 Convex geometry

Let $\mathcal{X}$ be a *topological vector space*, i.e. a (real) vector space equipped with a topology, such that $\mathcal{X}$ equipped with addition is a topological group and such that the scalar multiplication $\mathbb{R} \times \mathcal{X} \to \mathcal{X}$ is continuous. A subset $K \subseteq \mathcal{X}$ is called *convex*, whenever $\lambda x + (1-\lambda)y \in K$ for all $x, y \in K$ and $\lambda \in [0,1]$.

A subset F of a convex set K is called a *face* of K, if for any $x, y \in K$ there holds $x, y \in F$ as soon as there is $\lambda \in (0,1)$ with $\lambda x + (1-\lambda)y \in F$. Furthermore, $x \in K$ is called an *extreme point* of K, whenever $\{x\}$ is a face of K. We denote the set of all extreme points of K by $\mathrm{ex}(K)$.

Let K be a convex subset of $\mathcal{X}$. A mapping $f\colon K \to (-\infty, \infty]$ is said to be *affine*, whenever f preserves convex combinations, i.e. whenever $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in K$ and $\lambda \in [0,1]$. Here we use the convention that $r + \infty := \infty$ for $r \in (-\infty, \infty]$. Let μ be a Borel probability measure on the Borel subsets of K. Then there is a unique $x_\mu \in K$ such that $f(x_\mu) = \mu(f) (= \int f(x) d\mu(x))$ for every continuous and affine real function f on K [Phe01]. We write $x_\mu = \int_K x d\mu(x)$ and call x_μ the *barycenter* of the measure μ.

A *Choquet simplex* is a compact convex set K such that for every $x \in K$ there exists a unique Borel probability measure $\hat{x}$ supported on $\mathrm{ex}(K)$ such that x is the barycenter of $\hat{x}$.

2.7 Dynamical systems

2.7.1 Basic notions

Let G be a topological group and X be a topological space. A continuous map $\pi\colon G \times X \to X$ is called an *action of* G *on* X (also *dynamical system* or *flow*), whenever for all $x \in X$ and all $g, g' \in G$ there holds $\pi(e_G, x) = x$ and $\pi(gg', x) = \pi(g, \pi(g', x))$. We write $\pi^g := \pi(g, \cdot)\colon X \to X$ for all $g \in G$ and furthermore $g.x := \pi(g, x)$ whenever the action π is understood implicitely. In this context X is called the *phase space* of the action. A subset $A \subseteq X$ is called *invariant*, whenever $\pi^g(A) = A$ for all $g \in G$. If π and ϕ are actions of a topological group G on topological spaces X and Y respectively, we call a surjective continuous map $p\colon X \to Y$ a *factor map*, if $p \circ \pi^g = \phi^g \circ p$ for all $g \in G$. We then refer to ϕ as a *factor* of π and write $\pi \xrightarrow{p} \phi$. If p is in addition a homeomorphism, then p is called a *topological conjugacy* and we call π and ϕ *topologically conjugated*.

2.7.2 Invariant Borel measures

Let π be an action of a topological group G on a compact Hausdorff space X. We call a Borel probability measure μ on X *invariant* (with respect to π), whenever for all $g \in G$ and all Borel measurable subsets $A \subseteq X$ there holds $\mu(g.A) = \mu(A)$. An invariant Borel probability measure μ is furthermore called *ergodic*, whenever $\mu(A) \in \{0, 1\}$ for all invariant Borel sets $A \subseteq X$. The action π is called *uniquely ergodic*, if there exists exactly one ergodic measure on X, which is equivalent to the existence of exactly one invariant measure.

The set of all invariant Borel probability measures is denoted by $\mathcal{M}_G(X)$. This set is a convex subset of $\mathcal{M}(X)$ and the extreme points of $\mathcal{M}_G(X)$ are exactly the ergodic measures of π. Recall furthermore that we equip $\mathcal{M}(X)$ with the weak-* topology. Equipping $\mathcal{M}_G(X) \subseteq \mathcal{M}(X)$ with the induced weak-* topology we obtain $\mathcal{M}_G(X)$ to be a Choquet simplex. For reference of the statements see for example [Phe01, Chapter 12].

2.7.3 Delone actions

Consider an LCA group G and recall that we denote by $\mathcal{A}(G)$ the closed subsets of G. For $K \subseteq G$ compact, an open neighbourhood V of 0 and $\xi, \zeta \in \mathcal{A}(G)$ we write

$$\xi \overset{K,V}{\approx} \zeta,$$

whenever there is $\xi \cap K \subseteq \zeta + V$ and $\zeta \cap K \subseteq \xi + V$. Furthermore, we define

$$\epsilon(K, V) := \left\{(\xi, \zeta) \in \mathcal{A}(G)^2;\, \xi \overset{K,V}{\approx} \zeta\right\}.$$

Then there exists a compact Hausdorff topology, called the *local rubber topology*, such that

$$\mathbb{B}_{lr} := \{\epsilon(K,V);\ (K,V) \in \mathcal{K}(G) \times \mathcal{N}(G)\}$$

is a base for the corresponding uniformity $\mathbb{U}_{\mathcal{A}(G)}$ on $\mathcal{A}(G)$. See [BL04, Theorem 3] for reference. We call this base the *local rubber base*. The uniformity is called the *local rubber uniformity*. Note that $\mathcal{D}_V(G)$ and $\mathcal{D}_{K,V}(G)$ are compact subsets of $\mathcal{A}(G)$ for every compact subset $K \subseteq G$ and every open neighbourhood V of 0.

Consider now the action $\pi\colon G \times \mathcal{A}(G) \to \mathcal{A}(G)$ with $\pi(\omega, x) := \omega + x$, which we call the *full shift on* G. For a proof of the continuity of this action see for example [BL04]. For every compact invariant set $X \subseteq \mathcal{A}(G)$ we obtain an action by restricting the action of π to an action on X. We refer to this action as *the shift on* X. For a Delone set $\omega \subseteq G$ we denote X_ω for the closure of $D_\omega := \{\omega + g;\ g \in G\}$ with respect to the local rubber topology. The shift on X_ω is also called the *Delone dynamical system* or *Delone action* of ω and denoted by π_ω. We denote $\epsilon_X(K,V) := \epsilon(K,V) \cap X^2$ for invariant compact subsets $X \subseteq \mathcal{A}(G)$ and $\epsilon_\omega(K,V) := \epsilon_{X_\omega}(K,V)$ for Delone sets $\omega \subseteq G$.

If $\omega \subseteq G$ is a FLC Delone set there is another base of $\mathbb{U}_{X_\omega}$ that allows more control over the considered sets. For $K \subseteq G$ compact and any open neighbourhood V of 0 let

$$\eta_\omega(K,V) := \{(\xi,\zeta) \in X_\omega^2;\ \exists x, z \in V\colon (\xi + x) \cap K = (\zeta + z) \cap K\}.$$

If $\omega \subseteq G$ is a FLC Delone set, then

$$\mathbb{B}_{\mathrm{lm}}(\omega) := \{\eta_\omega(K,V);\ (K,V) \in \mathcal{K}(G) \times \mathcal{N}(G)\}$$

is a base of $\mathbb{U}_{X_\omega}$. This easily follows from [BL04, Prop. 4.5]. We will refer to this base as the *local matching base* of $\mathbb{U}_{X_\omega}$.

3 Generalizations of Fekete's lemma

In order to define the measure theoretical entropy of an action of $\mathbb{Z}$ it is common to use a subadditivity argument, which is often referred to as Fekete's lemma [Wal82] and which can be stated as follows.

Lemma 3.1 (Fekete's lemma)**.** *Let* $(a_n)_{n\in\mathbb{N}}$ *be a* subadditive sequence *in* $[0,\infty)$*, i.e. a sequence that satisfies* $a_{n+m} \leq a_n + a_m$ *for all* $n, m \in \mathbb{N}$*. Then the limit* $\lim_{n\to\infty} a_n/n$ *exists and equals* $\inf_{n\in\mathbb{N}} a_n/n$*.*

If one is also interested in defining measure theoretical entropy of actions of more general groups one needs to generalize this technique. This generalization can be done in different directions, which we will discuss in this chapter. To do this we will need the following notions. Recall that we denote by $\mathcal{K}(G)$ the set of all compact subsets of a locally compact group G. A map $f\colon \mathcal{K}(G) \to \mathbb{R} \cup \{\infty\}$ is called

positive, whenever $f(A) \geq 0$ for all $A \in \mathcal{K}(G)$.

monotone, if $f(A) \leq f(B)$ holds for all $A, B \in \mathcal{K}(G)$ with $A \subseteq B$.

right invariant, if $f(Ag) = f(A)$ holds for all $A \in \mathcal{K}(G)$ and all $g \in G$.

subadditive[1], if $f(A \cup B) \leq f(A) + f(B)$ holds for all $A, B \in \mathcal{K}(G)$.

strongly subadditive[2], if $f(A\cup B)+f(A\cap B) \leq f(A)+f(B)$ holds for $A, B \in \mathcal{K}(G)$.

amenable along a Van Hove net $(A_i)_{i\in I}$, if the limit $\lim_{i\in I} f(A_i)/\theta(A_i)$ exists in $\mathbb{R}$.

bounded, if $\sup_{A\in\mathcal{K}(G)} f(A) < \infty$.

We define $x/0 := \infty$ for $x \in [0,\infty)$ and abbreviate $f/\theta\colon \mathcal{K}(G) \to [0,\infty]$ with $f/\theta(A) := f(A)/\theta(A)$ for all functions $f\colon \mathcal{K}(G) \to [0,\infty)$. The main question of this chapter will be under which assumptions on a function $f\colon \mathcal{K}(G) \to \mathbb{R}\cup\{\infty\}$ and on a Van Hove net $(A_i)_{i\in I}$ we obtain that f is amenable along $(A_i)_{i\in I}$.

The first and probably the most direct answer is due to B. Weiss and given in [Wei03, Theorem 5.9] in the context of countable and discrete amenable groups. It is presented

[1] A subadditive function is automatically positive. For a compact subset $K \subseteq G$ the function $\mathcal{K}(G) \ni A \mapsto \theta(\partial_K A)$ can be seen to be subadditive and right invariant. This function however is not monotone.

[2] A strongly subadditive function is automatically subadditive.

that any monotone, right invariant and subadditive map is amenable along any Van Hove net that consists of tiles. A subset $A \subseteq G$ is said to be a *tile in* G, or *to tile* G, whenever there exists a subset $\omega \subseteq G$, such that $A\omega = G$ and such that $\theta(Au\Delta Av) = 0$ holds for all $u, v \in \omega$ with $u \neq v$. In particular, if G is a countable discrete group A tiles, if and only if $\{Av;\, v \in \omega\}$ is a partition of G. We will present the generalization of this technique to unimodular amenable groups in Section 3.1. This generalization is straightforward, but we include the short proof into this thesis for the convenience of the reader and as the proof gives a good intuition for the more involved quasi-tiling techniques that we will discuss later. In Theorem 3.3 below we present the following.

Theorem (Weiss Lemma)**.** *Let G be a unimodular amenable group. If $f\colon \mathcal{K}(G) \to \mathbb{R}$ is a subadditive, right invariant and monotone function and $(A_i)_{i\in I}$ is a Van Hove net that consists of tiling sets in G, then there holds*

$$\lim_{i\in I} \frac{f(A_i)}{\theta(A_i)} = \inf_{i\in I} \frac{f(A_i)}{\theta(A_i)} = \inf_{A} \frac{f(A)}{\theta(A)},$$

where the last infimum is taken over the set of all compact sets A that tile G.

Note that Weiss' technique also generalizes the feature of Fekete's lemma, that the limit can actually be seen as an infimum. This property is important in studying upper semi-continuity of the measure theoretical entropy map.

Naturally the question arises whether all unimodular amenable groups contain Van Hove nets that consist of tiling sets. Unfortunately this question seems open even if one restricts to countable discrete amenable groups [DHZ19]. Nevertheless, in the context of aperiodic order we are most often interested in Abelian groups. In Section 3.1 we will present that all LCA groups contain such a net.

A second generalization of Fekete's lemma is given by J. M. Ollagnier in [Oll85]. The statement of Ollagniers technique is similar to Weiss' statement. Restricting from the assumption of subadditivity to strong subadditivity we obtain the convergence for all Van Hove nets, i.e. also without the assumption that the net has to consist of tiling sets. This statement can be used to define and study measure theoretical entropy [Oll85], but it is not suitable for the context of topological entropy and pressure [DFR16]. In [Oll85] it is presented in the context of discrete amenable groups. As we will not need this statement beyond this setting we state Ollagniers technique without generalization to the non-discrete setting and it remains open, whether this generalization can be proven. A proof in the discrete setting can be found in [Oll85].

Theorem 3.2 (Ollagnier lemma)**.** *Let G be a discrete amenable group. If $f\colon \mathcal{K}(G) \to \mathbb{R}$ is a strongly subadditive[3], right invariant and monotone function and $(A_i)_{i\in I}$ is a Van Hove net in G, then there holds*

$$\lim_{i\in I} \frac{f(A_i)}{\theta(A_i)} = \inf_{i\in I} \frac{f(A_i)}{\theta(A_i)}.$$

A third generalization fo Fekete's lemma is given in [Pog13, PS16], where the authors consider also unimodular amenable groups but Banach spaces instead of $\mathbb{R}$. We will not present the precise statement but shortly discuss why this result cannot be used to define entropy of continuous groups such as $\mathbb{R}$. In fact a simplification of the statement to the Banach space $\mathbb{R}$ and right invariant functions yields that a map $f\colon \mathcal{K}(G) \to \mathbb{R}$ that is right invariant, "almost subadditive" and for which f/θ is bounded is amenable along any Van Hove net. If G is a discrete group, then positivity, subadditivity and right invariance can be used to show that f/θ is bounded but this is not possible for non-discrete groups. Consider for example a map that is constant on $\mathcal{K}(\mathbb{R}) \setminus \{\emptyset\}$ and satisfies $f(\emptyset) = 0$. If f is positive and not constant 0, then $f([0,\epsilon])/\theta([0,\epsilon])$ tends to infinity as ϵ tends to 0 and thus f/θ is not bounded. Note that we want to consider such maps in order to define topological entropy of actions of $\mathbb{R}$ as we will see in Remark 4.17 and thus cannot use this generalization of Fekete's Lemma.

The for our purposes most suitable generalization of Fekete's lemma that we encountred is based on the quasi-tiling machinery developed by D. Ornstein and B. Weiss in [OW87]. This version can be stated as follows. Every positive, monotone, right invariant and subadditive map is amenable along any Van Hove sequence. In this formulation the statement was first phrased in [LW00] in the context of countable discrete amenable groups. In [Gro99] M. Gromov claims that the statement can be strengthened in the following way. He claims, considering a metrizable unimodular amenable group, that any right invariant and subadditive map is amenable along any Van Hove sequence. Note that in [Gro99] a short sketch for a proof is given. Working out the details of this sketch, it is presented in [Kri07, Kri10, HYZ11, CSCK14] that the statement is valid whenever we consider discrete amenable groups. In particular, these proofs use the fact that f/θ is bounded whenever one considers discrete amenable groups and right invariant and subadditive f. This boundedness cannot be assumed for non-discrete unimodular amenable groups, such as $\mathbb{R}$ as we have seen above and it remained open, whether the sketch also works in the non-discrete setting. We will thus present two proofs of the statement in Section 3.2. We did not succeed in showing the claim of M. Gromov in full generality and need to add the assumption of monotonicity to our assumptions on f. However, the statement that we will present is sufficient in order to define and study measure theoretical entropy, topological entropy and topological pressure of a measure theoretical or topological action respectively as discussed for discrete

[3] Note that this statement was improved in [DFR16], where the strong additivity assumption could be replaced by a strictly weaker assumption called "Shearers inequality".

amenable groups for example in [WZ92, LW00, Buf11, CSCK14]. The precise statement of the Ornstein-Weiss Lemma can be stated as follows.

Theorem (Ornstein-Weiss Lemma)**.** *Let G be a unimodular amenable group. If the function $f\colon \mathcal{K}(G) \to \mathbb{R}$ is subadditive, right invariant and monotone and if $(A_i)_{i\in I}$ is a Van Hove net in G, then the limit*

$$\lim_{i\in I} \frac{f(A_i)}{\theta(A_i)} \tag{3.1}$$

exists, is finite and independent of the choice of a Van Hove net.

In fact we will give two proofs of the Ornstein-Weiss Lemma for groups that arise in the study of aperiodic order. The first proof will be presented in Subsection 3.2.1 and yields the Ornstein-Weiss Lemma for unimodular amenable groups that allow the construction of a cut-and-project scheme (CPS), such that the group is a physical space of this group. Using the classical quasi-tiling machinery and the ideas of M. Gromov one can show that the Ornstein-Weiss Lemma holds for discrete amenable groups [Kri07, Kri10, HYZ11, CSCK14]. One then shows that the Ornstein-Weiss Lemma holds in a group with a uniform lattice by extrapolation from the lattice. Thus, whenever (G, H, Λ) is a CPS, one obtains that the Ornstein-Weiss Lemma holds in $G \times H$ and a careful projection argument yields that the statement of the Ornstein-Weiss Lemma also holds in G. The advantage of this proof is that it is simpler than the second proof.

Nevertheless, as shown by Y. Meyer [Mey72], there are metrizable and σ-compact LCA groups that do not allow the construction of a CPS, i.e. which are not the physical space of a CPS. In order to get the Ornstein-Weiss Lemma for all unimodular amenable groups we thus present a second proof in Subsection 3.2.2. This proof follows ideas of M. Gromov [Gro99]. In order to perform this proof we will need to strengthen the quasi-tiling result from [OW87] and thus study quasi-tilings. Note that our proof actually shows the Ornstein-Weiss Lemma without any countability assumption on G and in particular without the assumption of metrizability, which M. Gromov used to define Van Hove nets.

3.1 The Weiss Lemma

We next show the Weiss Lemma as presented above and that every LCA group contains a Van Hove net that consists of tiling sets. Recall that a subset A of a unimodular amenable group G is said to be a *tile* in G or *to tile* G, whenever there exists a subset $\omega \subseteq G$, such that $A\omega = G$ and such that there holds $\theta(Au\Delta Av) = 0$ for all $u, v \in \omega$ with $u \neq v$. In this context we call ω a set of *tiling centres* for A. For finite subsets

$F \subseteq \omega$ we obtain the following relation between the cardinality and the Haar measure

$$\theta(AF) = \theta\left(\bigcup_{f\in F} Af\right) = \sum_{f\in F} \theta(Af) = \theta(A)|F|.$$

The following proof of the Weiss Lemma follows closely the arguments from [Wei03, Theorem 5.9], which are given there in the context of discrete amenable groups. We restated the Weiss Lemma and included the short proof to give the reader the possibility to gain some intuition before we discuss the more involved quasi-tilings.

Theorem 3.3 (Weiss Lemma). *Let G be a unimodular amenable group. If $f\colon \mathcal{K}(G) \to \mathbb{R}$ is a subadditive, right invariant and monotone function and $(A_i)_{i\in I}$ is a Van Hove net that consists of tiles in G, then there holds*

$$\lim_{i\in I} \frac{f(A_i)}{\theta(A_i)} = \inf_{i\in I} \frac{f(A_i)}{\theta(A_i)} = \inf_{A} \frac{f(A)}{\theta(A)},$$

where the last infimum is taken over the set of all compact sets A that tile G.

Proof. As $\liminf_{i\in I} f(A_i)/\theta(A_i) \geq \inf_{i\in I} f(A_i)/\theta(A_i) \geq \inf_A f(A)/\theta(A)$ it is sufficient to show that for all compact and tiling subsets A of G there holds $\limsup_{i\in I} f(A_i)/\theta(A_i) \leq f(A)/\theta(A)$. To do this let $\epsilon > 0$ and consider a set of tiling centers ω with respect to A. Set $\hat{F}_i := \{g \in \omega;\ Ag \cap A_i \neq \emptyset\}$ and $\check{F}_i := \{g \in \omega;\ Ag \subseteq A_i\}$. It is then straightforward to obtain that

$$\begin{aligned}
\partial_{A^{-1}} A_i &= A^{-1}A_i \cap A^{-1}\overline{A_i^c} \\
&= \{g \in G;\ Ag \cap A_i \neq \emptyset, Ag \cap \overline{A_i^c} \neq \emptyset\} \\
&\supseteq \{g \in \omega;\ Ag \cap A_i \neq \emptyset, Ag \cap A_i^c \neq \emptyset\} \\
&= \hat{F}_i \setminus \check{F}_i.
\end{aligned}$$

As ω is a set of tiling centers and as $\check{F}_i \subseteq \hat{F}_i$ it follows that

$$\theta\Big(A\Big(\hat{F}_i \setminus \check{F}_i\Big)\Big) = \theta(A)\Big|\hat{F}_i \setminus \check{F}_i\Big| = \theta(A)\Big(\Big|\hat{F}_i\Big| - \Big|\check{F}_i\Big|\Big).$$

Since AA^{-1} is compact and since $A\partial_{A^{-1}}A_i \subseteq \partial_{AA^{-1}}A_i$ one thus obtains from the Van Hove property of $(A_i)_{\in I}$ for large $i \in I$ that

$$\Big|\hat{F}_i\Big| - \Big|\check{F}_i\Big| \leq \frac{\theta\left(A\partial_{A^{-1}}A_i\right)}{\theta(A)} \leq \frac{\theta\left(\partial_{AA^{-1}}A_i\right)}{\theta(A_i)}\frac{\theta(A_i)}{\theta(A)} \leq \epsilon\frac{\theta(A_i)}{\theta(A)}. \tag{3.2}$$

As clearly $A\check{F}_i \subseteq A_i$ we obtain $\left|\check{F}_i\right| \leq \theta(A_i)/\theta(A)$. Using (3.2) we get for large i that

$$\left|\hat{F}_i\right| \leq \left|\check{F}_i\right| + \epsilon\frac{\theta(A_i)}{\theta(A)} \leq (1+\epsilon)\frac{\theta(A_i)}{\theta(A)}.$$

As ω is A-dense for $a \in A_i$ there is $g \in \omega$ such that $a \in Ag$. For such g there holds $a \in Ag \cap A_i$ and thus it follows that $g \in \hat{F}_i$. Hence, $a \in Ag \subseteq A\hat{F}_i$ and we have shown that there holds $A_i \subseteq A\hat{F}_i$. Using the assumptions on f we thus obtain for large i that

$$\frac{f(A_i)}{\theta(A_i)} \leq \frac{f\left(A\hat{F}_i\right)}{\theta(A_i)} \leq \sum_{g\in\hat{F}_i}\frac{f(Ag)}{\theta(A_i)} = \left|\hat{F}_i\right|\frac{f(A)}{\theta(A_i)} \leq (1+\epsilon)\frac{f(A)}{\theta(A)}.$$

Thus, $\limsup_{i\in I} f(A_i)/\theta(A_i) \leq (1+\epsilon)f(A)/\theta(A)$ and we obtain the statement since $\epsilon > 0$ was arbitrary. □

Naturally the question arises whether all unimodular amenable groups contain Van Hove nets of tiling sets. Unfortunately even in the case of discrete groups this question seems open [DHZ19]. In [Wei01] it is shown that residually finite groups contain Van Hove nets of tiling sets and it is noted that discrete abelian groups always contain Van Hove nets of tiling sets. If G is a compactly generated LCA group then G is isomorphic as topological groups to $\mathbb{R}^a \times \mathbb{Z}^b \times C$ for some $a, b \in \mathbb{N}_0$ and a compact group C. As $([-n,n]^a \times \{-n,\cdots,n\}^b \times C)_{n\in\mathbb{N}}$ is a Van Hove sequence that consists of symmetric and tiling sets, we obtain that all compactly generated LCA groups contain Van Hove sequences that consist of symmetric and tiling sets. The next proposition yields that we can apply the Weiss Lemma in all LCA groups.

Proposition 3.4. *Every LCA group contains a Van Hove net of symmetric and tiling sets. If the group is assumed to be σ-compact, then there exists a Van Hove sequence of symmetric and tiling sets.*

Proof. Assume that G is an LCA group and order $I := \mathcal{K}(G) \times (0,1)$ such that $(K,\epsilon) \leq (K',\epsilon')$ whenever $K \subseteq K'$ and $\epsilon \geq \epsilon'$. Clearly, I is directed. For $(K,\epsilon) \in I$ denote by $\langle K\rangle$ the subgroup of G generated by K. Then $\langle K\rangle$ is a compactly generated LCA group and thus contains a Van Hove sequence of symmetric and tiling sets. In particular, we can choose a symmetric subset $A_{(K,\epsilon)} \subseteq \langle K\rangle$ that is (ϵ, K)-invariant and a tiling with respect to $\langle K\rangle$. Furthermore, as $\langle K\rangle$ is a subgroup of G we can choose ω' such that $G = \bigcup_{g\in\omega'}\langle K\rangle + g$ is a disjoint union. Considering tiling centres $\omega \subseteq \langle K\rangle$ we obtain $\omega + \omega'$ to be a set of tiling centres for $A_{(K,\epsilon)}$. This shows that A_i tiles G for all $i \in I$. To show that $(A_i)_{i\in I}$ is a Van Hove net let $\epsilon > 0$ and $K \in \mathcal{K}(G)$. Then for $(K',\epsilon') \geq (K,\epsilon)$ there holds

$$\alpha(A_{(K',\epsilon')}, K) \leq \alpha(A_{(K',\epsilon')}, K') \leq \epsilon' \leq \epsilon$$

and we have shown that $(A_i)_{i\in I}$ is indeed a Van Hove net of symmetric and tiling sets.

If additionally G is assumed to be σ-compact, then there is a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} K_n = G$. Without lost of generality we assume $K_n \subseteq K_{n+1}$ and choose $B_n := A_{(K_n,1/n)}$. Then $(B_n)_{n\in\mathbb{N}}$ is a sequence and a subnet of $(A_i)_{i\in I}$ and thus in particular a Van Hove sequence that consists of symmetric and tiling sets. □

3.2 The Ornstein-Weiss Lemma

In this section we present two proofs of the Ornstein-Weiss Lemma. As discussed in the introduction of this chapter the first one will be simpler and uses the structure of a CPS and the discrete version of the statement which can be found in [Kri07, CSCK14]. The second one gives a more general statement, is based on ideas of [Gro99] and involves an improved version of the quasi-tiling machinery of [OW87].

3.2.1 A proof in the context of cut and project schemes

For this proof we use that all discrete groups satisfy the Ornstein-Weiss Lemma, which we cite from [Kri07, Kri10, HYZ11, CSCK14].

Proposition 3.5 (Ornstein-Weiss Lemma - discrete version)**.** *Let G be a discrete amenable group and let $f\colon \mathcal{K}(G) \to \mathbb{R}$ be a monotone, right invariant and subadditive map. Then for any Van Hove net $(A_i)_{i\in I}$ the limit $\lim_{i\in I} f(A_i)/|A_i|$ exists, is finite and furthermore independent of the choice of the Van Hove net.*

As uniform lattices in unimodular amenable groups are discrete amenable groups, we can use that the statement holds within the uniform lattice to obtain the Ornstein-Weiss Lemma for the surrounding group. In order to do this, we relate Van Hove nets in the group and the uniform lattice by the following lemma. Note that this lemma will be also important in later chapters, which motivated us to include the statements about the fundamental domains here.

Lemma 3.6. *Let Λ be a uniform lattice in a unimodular amenable group G. Let $(A_i)_{i\in I}$ be a Van Hove net and set $F_i := A_i \cap \Lambda$. Then Λ has a uniform density and $(F_i)_{i\in I}$ is a Van Hove net in Λ that satisfies $\mathrm{dens}(\Lambda) = \lim_{i\in I} |F_i|/\theta(A_i)$. For regular and precompact fundamental domains C of Λ there furthermore holds $\theta(C) = \mathrm{dens}(\Lambda)^{-1} \in (0,\infty)$.*

Remark 3.7. *In [Moo02] it is shown that whenever (G, H, Λ) is a CPS (and G and H are LCA groups), then every regular model set is a Delone set and posesses a uniform density. In particular, if $W \subseteq H$ is a precompact and regular window with non-empty interior, then $\Lambda(W)$ has uniform density $\theta_H(W)$. Note that this in particular implies uniform lattices Λ in an LCA group G to have a uniform density as one can consider the trivial CPS $(G, \{0\}, \Lambda \times \{0\})$.*

To have the result also in the non-commutative case at hand and for the convenience of the reader we include a full proof.

Proof of Lemma 3.6. From Proposition 2.45 we know that there exists a regular and precompact fundamental domain C of Λ and the regularity of C and Remark 2.46 yield $\theta\left(\overline{C}\right) = \theta(C) \in (0,\infty)$. By Proposition 2.26 there exists a Van Hove net $(B_i)_{i\in I}$ such that $C^{-1}B_i \subseteq \overline{C}^{-1}B_i \subseteq A_i$ and $\lim_{i\in I}\theta(B_i)/\theta(A_i) = 1$. It is straightforward to show that $C^{-1}B_i \subseteq A_i$ and $C\Lambda = G \supseteq B_i$ imply that there holds $B_i \subseteq CF_i$. Furthermore, by Proposition 2.25 there holds $\lim_{i\in I}\theta\left(\overline{C}A_i\right)/\theta(A_i) = 1$. From $B_i \subseteq CF_i \subseteq \overline{C}A_i$ one thus obtains $\lim_{i\in I}\theta(CF_i)/\theta(A_i) = 1$. As $\theta(C)|F_i| = \theta(CF_i)$ by Remark 2.38 we obtain that $\lim_{i\in I}|F_i|/\theta(A_i) = 1/\theta(C)$ is independent of the choice of a Van Hove net and thus that Λ has a uniform density which satisfies $\mathrm{dens}(\Lambda) = 1/\theta(C)$.

To show that $(F_i)_{i\in I}$ is a Van Hove net in Λ let $F \subseteq \Lambda$ be a compact subset. We denote by $\partial_F^\Lambda F_i$ the F-boundary of F_i with respect to Λ and compute

$$\begin{aligned}
C\partial_F^\Lambda F_i &\subseteq (CFF_i) \cap \left(CF\overline{\Lambda\setminus F_i}\right)\\
&= (CFF_i) \cap \left(CF\overline{(\Lambda\cap(G\setminus A_i))}\right)\\
&\subseteq \left(\overline{C}FA_i\right) \cap \left(\overline{C}F\overline{G\setminus A_i}\right)\\
&= \partial^G_{\overline{CF}}A_i,
\end{aligned}$$

where we furthermore denote by ∂^G the Van Hove boundary with respect to G. Recall that there holds $B_i \subseteq CF_i$ and $\lim_{i\in I}\theta(B_i)/\theta(A_i) = 1$. Thus,

$$\begin{aligned}
0 \leq \limsup_{i\in I}\frac{\left|\partial_F^\Lambda F_i\right|}{|F_i|} &= \limsup_{i\in I}\frac{\theta\left(C\partial_F^\Lambda F_i\right)}{\theta(CF_i)}\\
&\leq \limsup_{i\in I}\frac{\theta\left(\partial^G_{\overline{CF}}A_i\right)}{\theta(B_i)} = \lim_{i\in I}\frac{\theta\left(\partial^G_{\overline{CF}}A_i\right)}{\theta(A_i)} = 0
\end{aligned}$$

and we obtain $(F_i)_{i\in I}$ to be Van Hove in Λ from the arbitrary choice of $F \in \mathcal{K}(\Lambda)$. □

We can now show that one can extrapolate the Ornstein-Weiss Lemma from a uniform lattice.

Theorem 3.8 (Ornstein-Weiss Lemma - uniform lattice version)**.** *Let G be a unimodular amenable group that contains a uniform lattice and let $f\colon \mathcal{K}(G) \to \mathbb{R}$ be a monotone, right invariant and subadditive map. Then for any Van Hove net $(A_i)_{i\in I}$ the limit $\lim_{i\in I} f(A_i)/\theta(A_i)$ exists, is finite and independent of the choice of the Van Hove net.*

Proof. Let K be a compact and symmetric subset of G such that Λ is K-dense. From Proposition 2.26 we obtain the existence of a Van Hove net $(B_i)_{i\in I}$ such that $KB_i \subseteq A_i$ and such that $\lim_{i\in I}\theta(B_i)/\theta(A_i) = 1$. Furthermore, by Proposition 2.25 $(KA_i)_{i\in I}$ is a Van Hove net that satisfies $\lim_{i\in I}\theta(KA_i)/\theta(A_i) = 1$. Thus, setting $\check{F}_i := B_i \cap \Lambda$ and $\hat{F}_i := (KA_i)\cap\Lambda$ for all $i \in I$ we obtain by Lemma 3.6 Van Hove nets $(\check{F}_i)_{i\in I}$ and $(\hat{F}_i)_{i\in I}$ in Λ that satisfy $\mathrm{dens}(\Lambda) = \lim_{i\in I}\left|\check{F}_i\right|/\theta(B_i) = \lim_{i\in I}\left|\check{F}_i\right|/\theta(A_i)$ and $\mathrm{dens}(\Lambda) =$

$\lim_{i\in I}\left|\hat{F}_i\right|/\theta(KA_i) = \lim_{i\in I}\left|\hat{F}_i\right|/\theta(A_i)$. For $i \in I$ there holds $K\check{F}_i \subseteq KB_i \subseteq A_i$. Since $G = K\Lambda$ for $a \in A_i$ there are $k \in K$ and $l \in \Lambda$ such that $a = kl$. We therefore obtain from the symmetry of K that $l = k^{-1}a \in KA_i$. Hence, $l \in \hat{F}_i$ and we have shown $A_i \subseteq K\hat{F}_i$.

In order to use that every discrete amenable group satisfies the Ornstein-Weiss Lemma, we define $f^\Lambda\colon \mathcal{K}(\Lambda) \to \mathbb{R}; F \mapsto f\left(KF\right)$. It is straightforward to see, that f^Λ is right invariant and monotone. In order to show, that f^Λ is subadditive let $F, E \in \mathcal{K}(\Lambda)$. As $K(F \cup E) \subseteq KF \cup KE$ we obtain from the monotonicity and the subadditivity of f that $f^\Lambda(F \cup E) \leq f(KF \cup KE) \leq f^\Lambda(F) + f^\Lambda(E)$.

As Λ is a discrete amenable group it satisfies the Ornstein-Weiss Lemma, as shown in [Kri07, Kri10, HYZ11, CSCK14]. This implies the existence of the following limits and their equality

$$\lim_{i\in I} \frac{f^\Lambda\left(\check{F}_i\right)}{\left|\check{F}_i\right|} = \lim_{i\in I} \frac{f^\Lambda\left(\hat{F}_i\right)}{\left|\hat{F}_i\right|}. \tag{3.3}$$

In particular, the Ornstein-Weiss Lemma for discrete amenable groups implies that the value of (3.3) can also obtained by replacing $\left(\check{F}_i\right)_{i\in I}$ by any other Van Hove net in Λ. It is thus in particular independent of the choice of $(A_i)_{i\in I}$. Using $\mathrm{dens}(\Lambda) = \lim_{i\in I} |\check{F}_i|/\theta(A_i) = \lim_{i\in I} |\hat{F}_i|/\theta(A_i)$ and $K\check{F}_i \subseteq A_i \subseteq K\hat{F}_i$ we obtain

$$\mathrm{dens}(\Lambda) \lim_{i\in I} \frac{f^\Lambda\left(\check{F}_i\right)}{\left|\check{F}_i\right|} \leq \liminf_{i\in I} \frac{f(A_i)}{\theta(A_i)} \leq \limsup_{i\in I} \frac{f(A_i)}{\theta(A_i)} \leq \mathrm{dens}(\Lambda) \lim_{i\in I} \frac{f^\Lambda\left(\hat{F}_i\right)}{\left|\hat{F}_i\right|}$$

and thus the limit $\lim_{i\in I} f(A_i)/\theta(A_i)$ exists, is finite and independent of the choice of the Van Hove net $(A_i)_{i\in I}$. □

Theorem 3.8 is sufficient whenever one considers compactly generated LCA groups. In the context of aperiodic order one is among other structures interested in model sets in LCA groups and one can ask, whether all physical spaces G of a CPS (G, H, Λ) satisfy the Ornstein-Weiss Lemma. As $G \times H$ contains the uniform lattice Λ one could hope to somehow project this lattice in order to obtain a uniform lattice in G and thus the Ornstein-Weiss Lemma from Theorem 3.8. This is not possible as we will see in the next example. This example was already studied by Yves Meyer in [Mey72, Chapter II.10] and can be found in [CdlH16, Example 5.C.10(2)]. We include a short proof of the claims for the convenience of the reader.

Example 3.9. *Consider the additive group of the p-adic numbers $\mathbb{Q}_p$ for some prime p. As presented in Chapter 2 this group is a metrizable σ-compact LCA group and we have seen in Example 2.42 that this group contains no uniform lattice. Nevertheless, it is the physical space of the following CPS. Denote by $\mathbb{Z}[p^{-1}]$ the smallest subring of $\mathbb{Q}_p$*

(or $\mathbb{R}$) that contains $\mathbb{Z}$ and p^{-1}. Then $(\mathbb{Q}_p, \mathbb{R}, \Lambda)$ with $\Lambda := \{(x,x);\, x \in \mathbb{Z}[p^{-1}]\}$ is a cut and project scheme.

Proof. As $\pi_{\mathbb{R}}(\Lambda) = \mathbb{Z}[p^{-1}]$ is dense in $\mathbb{R}$ and as $\pi_{\mathbb{Q}_p}|_{\Lambda}$ is clearly injective it remains to show that Λ is a uniform lattice in $\mathbb{Q}_p \times \mathbb{R}$. Note first that Λ is a subgroup of $\mathbb{Q}_p \times \mathbb{R}$. To show that this subgroup is discrete let $x \in \mathbb{Z}[p^{-1}]$ such that $|x|_p < 1$ and such that $|x|_{\mathbb{R}} < 1$. Now recall that $x \in \mathbb{Z}[p^{-1}]$ and thus there is a finite sequence $(x_i)_{i=-n}^m$ such that $x = \sum_{-n}^m x_i p^i = \sum_{-m}^n x_{-i} p^{-i}$. From $|x|_p < 1$ we obtain that $x_i = 0$ for $i \geq 0$ and from $|x|_{\mathbb{R}} < 1$ we obtain $x_{-i} = 0$ for $i \geq 0$. Thus, $x = 0$ and we have shown Λ to be $B_{1/2}(0) \times (-1/2, 1/2)$-discrete. Here we denote the open ball in $\mathbb{Q}_p$ by the usual notion and use the interval notion for $\mathbb{R}$. It remains to show that Λ is co-compact. To see this consider the closed centred ball $\overline{B}_1(0)$ in $\mathbb{Q}_p$ and $C := \overline{B}_1(0) \times [0,1]$. As $\mathbb{Z} \subseteq \overline{B}_1(0)$ and as $\overline{B}_1(0)$ is a subgroup of $\mathbb{Q}_p$ we obtain $C + \{(n,n);\, n \in \mathbb{Z}\} = \overline{B}_1(0) \times \mathbb{R}$. As $\mathbb{Z}[p^{-1}]$ is dense in $\mathbb{Q}_p$ we thus obtain that $C + \Lambda = \mathbb{Q}_p \times \mathbb{R}$. Being the continuous projection of the compact set C we then observe that $(\mathbb{Q}_p \times \mathbb{R})/\Lambda$ is compact. This shows that Λ is a uniform lattice. □

We next show that even if one cannot project the uniform lattice, one can use that the Ornstein-Weiss Lemma is valid in $G \times H$ to show the statement also for G. To do this we need to relate Van Hove nets in G with some Van Hove nets in the product space $G \times H$.

Lemma 3.10. *Let G and H be unimodular groups and assume that $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ are Van Hove nets in G and H respectively. Then $(A_i \times B_j)_{(i\times j)\in I\times J}$ is a Van Hove net in $G \times H$, where $I \times J$ is ordered component wise.*

Proof. Consider a compact subset $M \subseteq G \times H$ and let $K := \pi_G(M)$ and $C := \pi_H(M)$ the projections. A straightforward computation shows

$$\partial_M(A_i \times B_j) \subseteq (\partial_K A_i \times \partial_C B_j) \cup (KA_i \times \partial_C B_j) \cup (\partial_K A_i \times CB_j).$$

for all $i \in I$ and $j \in J$ and we compute

$$\begin{aligned} 0 &\leq \frac{\theta_G \times \theta_H(\partial_M(A_i \times B_j))}{\theta_G \times \theta_H(A_i \times B_j)} \\ &\leq \frac{\theta_G(\partial_K A_i)\theta_H(\partial_C B_j)}{\theta_G(A_i)\theta_H(B_j)} + \frac{\theta_G(KA_i)\theta_H(\partial_C B_j)}{\theta_G(A_i)\theta_H(B_j)} + \frac{\theta_G(\partial_K A_i)\theta_H(CB_j)}{\theta_G(A_i)\theta_H(B_j)} \overset{(i,j)\in I\times J}{\longrightarrow} 0 \end{aligned}$$

□

We can now show that all physical spaces of CPS satisfy the Ornstein-Weiss Lemma.

Theorem 3.11 (Ornstein-Weiss Lemma - CPS version)**.** *If (G, H, Λ) is a cut and project scheme and $f\colon \mathcal{K}(G) \to \mathbb{R}$ is a monotone, right invariant and subadditive map, then for any Van Hove net $(A_i)_{i\in I}$ in G the limit $\lim_{i\in I} f(A_i)/\theta(A_i)$ exists, is finite and independent of the choice of a Van Hove net.*

Proof. Consider a monotone right-invariant and subadditive mapping $f\colon \mathcal{K}(G) \to \mathbb{R}$ and a Van Hove net $(A_i)_{i\in I}$. Denote by θ_H the Haar measure of H and choose the Haar measure $\theta_{G\times H}$ as the product measure $\theta_G \times \theta_H$. Let $(B_j)_{j\in J}$ be any Van Hove net in H. It is easy to see that $h\colon \mathcal{K}(G\times H) \to \mathbb{R}$ with

$$h(Q) := \inf\left\{\sum_{n=1}^{N} f(C_n)\theta_H(D_n);\ N\in\mathbb{N}, C_n\in\mathcal{K}(G), D_n\in\mathcal{K}(H), Q\subseteq \bigcup_{n=1}^{N} C_n\times D_n\right\}$$

is monotone and right invariant. To see that it is also subadditive let Q and R be compact subsets of $G\times H$. Then whenever $Q\subseteq\bigcup_{n=1}^{N} C_n\times D_n$ and $R\subseteq\bigcup_{m=1}^{M} E_m\times F_m$ for $N, M\in\mathbb{N}$ and C_n, D_n, E_m, F_m in $\mathcal{K}(G)$ and $\mathcal{K}(H)$ respectively there holds $Q\cup R\subseteq \bigcup_{n=1}^{N} C_n\times D_n \cup \bigcup_{m=1}^{M} E_m\times F_m$ and we obtain

$$h(R\cup Q) \leq \sum_{n=1}^{N} f(C_n)\theta_H(D_n) + \sum_{m=1}^{M} f(E_m)\theta_H(F_m).$$

Taking the infimum over the considered families we see $h(Q\cup R)\leq h(Q)+h(R)$ and we have shown the subadditivity of h. We next show that for compact subsets $A\subseteq G$ and $B\subseteq H$ there holds $h(A\times B) = f(A)\theta_H(B)$. Clearly there holds $h(A\times B)\leq f(A)\theta_H(B)$. To show the other inequality let $C_n\in\mathcal{K}(G)$ and $D_n\in\mathcal{K}(H)$ such that $A\times B\subseteq\bigcup_{n=1}^{N} C_n\times D_n$. As θ_H is monotone we assume without lost of generality that there holds $D_n\subseteq B$ for $n\in\mathcal{N} := \{1,\cdots,N\}$. Let furthermore $\{E_1,\cdots,E_M\}$ be a finite Borel partition of B s.t. $\bigcup_{m\in\mathcal{M}; E_m\subseteq D_n} E_m = D_n$ for all $n\in\mathcal{N}$, where we denote $\mathcal{M} := \{1,\cdots,M\}$. We denote $\mathcal{M}_n := \{m'\in\mathcal{M};\ E_{m'}\subseteq D_n\}$ and $\mathcal{N}_m := \{n'\in\mathcal{N};\ E_m\subseteq D_{n'}\}$ for $n\in\mathcal{N}$ and $m\in\mathcal{M}$ and obtain

$$A\times E_m \subseteq \bigcup_{n\in\mathcal{N}_m} C_n\times D_n.$$

Thus, there holds $A\subseteq\bigcup_{n\in\mathcal{N}_m} C_n$ for all $m\in\mathcal{M}$. As f is subadditive we get $f(A)\leq\sum_{n\in\mathcal{N}_m} f(C_n)$ for all $m\in\mathcal{M}$ and hence there holds

$$\begin{aligned}\sum_{n\in\mathcal{N}} f(C_n)\theta_H(D_n) &= \sum_{n\in\mathcal{N}}\sum_{m\in\mathcal{M}_n} f(C_n)\theta_H(E_m) = \sum_{\substack{n\in\mathcal{N}, m\in\mathcal{M},\\ E_m\subseteq D_n}} f(C_n)\theta_H(E_m)\\ &= \sum_{m\in\mathcal{M}}\sum_{n\in\mathcal{N}_m} f(C_n)\theta_H(E_m) \geq \sum_{m\in\mathcal{M}} f(A)\theta_H(E_m) = f(A)\theta_H(B).\end{aligned}$$

This shows $h(A\times B) = f(A)\theta_H(B)$ for compact subsets $A\subseteq G$ and $B\subseteq H$. By Lemma 3.10 we obtain that $(A_i\times B_j)_{(i,j)\in I\times J}$ is a Van Hove net in $G\times H$. As $G\times H$ satisfies the Ornstein-Weiss Lemma this implies that

$$\frac{f(A_i)}{\theta_G(A_i)} = \frac{f(A_i)\theta_H(B_j)}{\theta_G(A_i)\theta_H(B_j)} = \frac{h(A_i\times B_j)}{\theta_{G\times H}(A_i\times B_j)}$$

converges to a finite limit which is independent of $(A_i)_{i\in I}$. □

Remark 3.12. *The proof of Theorem 3.11 also shows that whenever G and H are two unimodular amenable groups and $G \times H$ satisfies the statement of the Ornstein-Weiss Lemma, then so do G and H. Thus, we can also deduce from the arguments presented in this section that all internal spaces of CPS satisfy the Ornstein-Weiss Lemma.*

3.2.2 A proof using quasi-tilings

We next present a proof for general unimodular amenable groups, which is self-contained and independent of the results in Subsection 3.2.1.

On quasi-tilings

Let G be a unimodular amenable group and $\epsilon > 0$. A finite family $(A_i)_{i\in F}$ of compact subsets of G is called *ϵ-disjoint* if for any $i \in F$ there exists a compact subset $B_i \subseteq A_i$ such that $(B_i)_{i\in F}$ is a disjoint family and such that $\theta(B_i) > (1-\epsilon)\theta(A_i)$. For $A \in \mathcal{K}(G)$ a finite family $(A_i)_{i\in F}$ of compact subsets of G is an *ϵ-quasi-tiling* of A, whenever there exists a family of finite sets $(C_i)_{i\in F}$ such that

(a) $\{A_i g; , g \in C_i\}$ is an ϵ-disjoint family for all i;

(b) $\{A_i C_i;\ i \in F\}$ is a disjoint family; and

(c) $\theta(A \cap \bigcup_{i\in F} A_i C_i) \geq (1-\epsilon)\theta(A)$.

A family $(C_i)_{i\in F}$ satisfying these conditions is referred to as a family of *ϵ-quasi-tiling centres* of $(A_i)_{i\in F}$ (with respect to A). In [OW83, OW87, LW00] it is shown that for any Van Hove sequence $(A_n)_{n\in\mathbb{N}}$ in a metrizable unimodular amenable group and any $\epsilon > 0$ one can find a finite subset $F \subseteq \mathbb{N}$, $\delta > 0$ and a compact subset $D \subseteq G$ such that for any (δ, D)-invariant A we obtain that $(A_i)_{i\in F}$ ϵ-quasi-tiles A. We will follow the ideas that lead to this result and show that one can also construct ϵ-quasi-tiling centres C_i such that $R := A \setminus \bigcup_{i\in F} A_i C_i$ is "relatively invariant", i.e. that allows to control $\theta(\partial_K R)/\theta(A)$ for some given $K \in \mathcal{K}(G)$. We begin this investigation with the following statement about ϵ-disjointness. It appears in [OW87] in the discrete setting and we present the simple proof for the convenience of the reader.

Lemma 3.13. *Let G be a unimodular amenable group and $(A_i)_{i\in F}$ be an ϵ-disjoint family in G. Then there holds*

$$(1-\epsilon)\sum_{i\in F}\theta(A_i) \leq \theta\left(\bigcup_{i\in F} A_i\right).$$

Furthermore, if $A \subseteq G$ is a compact subset that satisfies $\theta\left(A \cap \left(\bigcup_{n=1}^{N} A_n\right)\right) < \epsilon\theta(A)$, then also $\{A\} \cup \{A_n;\ n = 1, \cdots, N\}$ is an ϵ-disjoint family.

Proof. As $(A_i)_{i\in F}$ is assumed to be ϵ-disjoint there exists a family of compact disjoint sets $(B_i)_{i\in F}$ such that $\theta(B_i) \geq (1-\epsilon)\theta(A_i)$ and $B_i \subseteq A_i$ holds for all $i \in F$. We compute

$$(1-\epsilon)\sum_{i\in F}\theta(A_i) \leq \sum_{i\in F}\theta(B_i) = \theta\left(\bigcup_{i\in F} B_i\right) \leq \theta\left(\bigcup_{i\in F} A_i\right).$$

To show the second statement note that the assumption on A implies

$$\theta\left(A \setminus \left(\bigcup_{n=1}^{N} A_n\right)\right) = \theta(A) - \theta\left(A \cap \bigcup_{n=1}^{N} A_n\right) > (1-\epsilon)\theta(A)$$

and thus there is $\rho > 0$ such that there holds $(1-\rho)\theta\left(A \setminus \bigcup_{n=1}^{N} A_n\right) > (1-\epsilon)\theta(A)$. As θ is regular there exists furthermore a compact subset $B \subseteq A \setminus \bigcup_{n=1}^{N} A_n$ such that $\theta(B) \geq (1-\rho)\theta\left(A \setminus \bigcup_{n=1}^{N} A_n\right) \geq (1-\epsilon)\theta(A)$. This B is disjoint from all A_n and in particular from all B_n and we obtain $\{A\} \cup \{A_n;\, n = 1, \cdots, N\}$ to be ϵ-disjoint. □

We next present how invariance properties of sets of an ϵ-disjoint family are inherited by the union over these sets. This can be found in [Kri07, Kri10, CSCK14] in the context of discrete amenable groups and the proof is a straightforward generalization. Nevertheless, we included the proof for the convenience of the reader and to keep the proof of the Ornstein-Weiss Lemma self-contained. Recall that we define $\alpha(A, K) := \theta(\partial_K A)/\theta(A)$ for precompact subsets $A, K \subseteq G$.

Lemma 3.14. *Let $K \in \mathcal{K}(G)$ and $\epsilon \in (0,1)$. If $(A_i)_{i\in F}$ is a finite and ϵ-disjoint family of non-empty and compact subsets of G, then there holds*

$$\alpha\left(\bigcup_{i\in F} A_i, K\right) \leq \frac{\max_{i\in F}\alpha(A_i, K)}{1-\epsilon}.$$

Proof. Abbreviate

$$M := \max_{i\in F}\alpha(A_i, K) = \max_{i\in F}\frac{\theta(\partial_K A_i)}{\theta(A_i)}.$$

Recall from Lemma 2.13 that there holds $\partial_K\left(\bigcup_i A_i\right) \subseteq \bigcup_i \partial_K A_i$ and thus

$$\theta\left(\partial_K\left(\bigcup_{i\in F} A_i\right)\right) \leq \theta\left(\bigcup_{i\in F}\partial_K A_i\right) \leq \sum_{i\in F}\theta(\partial_K A_i) = \sum_{i\in F}\theta(A_i)\frac{\theta(\partial_K A_i)}{\theta(A_i)} \leq M\sum_{i\in F}\theta(A_i).$$

As the considered family is ϵ-disjoint we obtain that $(1-\epsilon)\sum_i\theta(A_i) \leq \theta\left(\bigcup_i A_i\right)$ and conclude

$$\alpha\left(\bigcup_{i\in F} A_i, K\right) = \frac{\theta\left(\partial_K \bigcup_i A_i\right)}{\theta\left(\bigcup_i A_i\right)} \leq \frac{M\sum_i\theta(A_i)}{(1-\epsilon)\sum_i\theta(A_i)} = \frac{1}{1-\epsilon}M.$$

□

We will also need to control the invariance of $R \setminus B$, whenever we know how invariant precompact subsets $R, B \subseteq G$ are. This is possible whenever $\theta(R \setminus B)$ is not negliagable with respect to $\theta(R)$. Again the statement can be found in [Kri07, Kri10, CSCK14] in the context of discrete amenable groups and we include the straightforward generalization to unimodular amenable groups.

Lemma 3.15. *Let $R, B \subseteq G$ be precompact subsets that satisfy $0 < \theta(B) \leq \theta(R)$. If there exists $\epsilon > 0$ such that $\theta(R \setminus B) \geq \epsilon\theta(R)$, then for every compact subset $K \subseteq G$ there holds*

$$\alpha(R \setminus B, K) \leq \frac{\alpha(R, K) + \alpha(B, K)}{\epsilon}.$$

Proof. From Lemma 2.13 we know that $\partial_K (R \setminus B) \subseteq \partial_K R \cup \partial_K B$. Hence,

$$\frac{\theta\left(\partial_K\left(R \setminus B\right)\right)}{\theta(R \setminus B)} \leq \frac{1}{\epsilon}\frac{\theta(\partial_K R) + \theta(\partial_K B)}{\theta(R)} \leq \frac{1}{\epsilon}\left(\frac{\theta(\partial_K R)}{\theta(R)} + \frac{\theta(\partial_K B)}{\theta(B)}\right).$$

□

On fillings

The key to prove the Ornstein-Weiss quasi-tiling result mentioned above is to consider fillings [OW87]. We will thus consider this concept next. Let $A, R \subseteq G$ be precompact subsets with positive Haar measure and $\epsilon > 0$. We call $C \subseteq G$ an (ϵ, A)-*filling of* R, whenever $AC \subseteq R$ and $\{Ag;\, g \in C\}$ is ϵ-disjoint. The ideas of the proof of the next lemma is sketched in [Gro99] and given in detail for discrete groups in [Kri07, Kri10, CSCK14].

Lemma 3.16. *Let $A \subseteq G$ be a compact subset, $R \subseteq G$ be a precompact subset and assume both sets to have positive Haar measure. Let furthermore $\epsilon \in (0, 1)$. Then for every finite (ϵ, A)-filling C of R of maximal cardinality (among all (ϵ, A)-fillings of R) there holds*

$$\theta(AC) \geq \epsilon\left(1 - \alpha(R, A^{-1})\right)\theta(R).$$

Proof. As $\theta(A) > 0$ there is $a \in A^{-1}$ such that Aa contains the identity e_G. Then $\theta(\partial_{(Aa)^{-1}} R) = \theta(a^{-1}\partial_{A^{-1}} R) = \theta(\partial_{A^{-1}} R)$ and by translating also C we can assume without lost of generality that A contains e_G. For $g \in R \setminus \partial_{A^{-1}} R$ there holds $g \in R \subseteq A^{-1}\overline{R}$ and thus $g \notin A^{-1}\overline{R^c}$. In particular, $Ag \cap \overline{R^c}$ is empty and we deduce $Ag \subseteq R$. If $g \notin C$ we obtain $\theta(Ag \cap AC) \geq \epsilon\theta(Ag)$, as otherwise by Lemma 3.13 $\{Ag';\, g' \in C \cup \{g\}\}$ would be an ϵ-disjoint family, a contradiction to the maximal cardinality of C. For $g \in C$ we furthermore obtain $Ag \subseteq AC$ and thus $\theta(Ag \cap AC) = \theta(Ag) \geq \epsilon\theta(A)$ from $\epsilon \in (0, 1)$. This shows that for all $g \in R \setminus \partial_{A^{-1}} R$ there holds

$$\theta(Ag \cap AC) \geq \epsilon\theta(A).$$

Recall that we denote by χ_M the characteristic function of a subset $M \subseteq G$. We compute for any $g' \in G$ that

$$\theta(A) = \theta(A^{-1}) = \int_G \chi_{A^{-1}} d\theta = \int_G \chi_A(g^{-1}) d\theta(g) = \int_G \chi_A(g'g^{-1}) d\theta(g).$$

Thus, Tonelli's theorem implies

$$\begin{aligned}
\theta(A)\theta(AC) &= \int_G \theta(A)\chi_{AC}(g') d\theta(g') \\
&= \int_G \int_G \chi_A(g'g^{-1}) d\theta(g) \chi_{AC}(g') d\theta(g') \\
&= \int_G \int_G \chi_A(g'g^{-1}) \chi_{AC}(g') d\theta(g') d\theta(g) \\
&= \int_G \int_G \chi_{Ag \cap AC}(g') d\theta(g') d\theta(g) \\
&= \int_G \theta(Ag \cap AC) d\theta(g) \\
&\geq \int_{R \setminus \partial_{A^{-1}} R} \epsilon \theta(A) d\theta(g) \\
&= \epsilon \theta(A) \theta(R \setminus \partial_{A^{-1}} R).
\end{aligned}$$

We thus obtain

$$\theta(AC) \geq \epsilon \theta(R \setminus \partial_{A^{-1}} R) \geq \epsilon(\theta(R) - \theta(\partial_{A^{-1}} R)) = \epsilon \left(1 - \frac{\theta(\partial_{A^{-1}} R)}{\theta(R)}\right) \theta(R).$$

□

For $\epsilon > 0$, compact subsets A and precompact subsets R of positive Haar measure it is thus natural to ask, whether there exist finite (ϵ, A)-fillings of R of maximal cardinality. If G is discrete, then every compact set is finite and we obtain the cardinality of every (ϵ, A)-filling C of R to be bounded by $|R|$. The next lemma shows that we can bound the cardinality of C also without the assumption of discreteness to obtain the existence of finite (ϵ, A)-fillings of R of maximal cardinality.

Lemma 3.17. *Let $\epsilon > 0$, $A \subseteq G$ be a compact subset of positive Haar measure and $R \subseteq G$ be a precompact subset. Then every (ϵ, A)-filling C of R satisfies*

$$|C| \leq \frac{\theta(R)}{(1-\epsilon)\theta(A)}.$$

In particular, there are finite (ϵ, A)-fillings of R of maximal cardinality.

Proof. As $\{Ag;\, g \in C\}$ is an ϵ-disjoint family we obtain

$$\theta(R) \geq \theta(AC) = \theta\left(\bigcup_{g\in C} Ag\right) \geq (1-\epsilon)\sum_{g\in C}\theta(Ag) = (1-\epsilon)|C|\theta(A).$$

□

A slightly improved quasi-tiling result

Recall that in [OW87] it is shown that for any small $\epsilon > 0$ and any Van Hove sequence $(A_n)_{n\in\mathbb{N}}$ in a metrizable unimodular group there exists a subset $F \subseteq \mathbb{N}$, $\delta > 0$ and a compact subset $D \subseteq G$ such that any compact and (δ, D)-invariant subset $A \subseteq G$ can be ϵ-quasi-tiled by $(A_i)_{i\in F}$. We are now ready to show that one can construct the corresponding ϵ-quasi-tiling centres C_i such that we can control the Haar measure of the K-boundary of the remaining set $R := A \setminus \bigcup_{i\in F} A_iC_i$. As there holds $\theta(\partial_K R) \leq \theta(K\overline{R})$ for all precompact subsets $R, K \subseteq G$, we formulate our result as follows.

Theorem 3.18. *For any Van Hove net $(A_i)_{i\in I}$, any $\epsilon \in (0, 1/2)$ and any non-empty and compact subset $K \subseteq G$, there exist a finite subset $F \subseteq I$, $\delta > 0$ and a compact subset $D \subseteq G$ with the following property. For any (δ, D)-invariant and compact subset $A \subseteq G$ the finite family $(A_i)_{i\in F}$ is an ϵ-quasi-tiling of A such that the ϵ-quasi-tiling centres $(C_i)_{i\in F}$ can be chosen to satisfy $\bigcup_{i\in F} A_iC_i \subseteq A$ and such that furthermore $R := A \setminus \bigcup_{i\in F} A_iC_i$ satisfies $\theta\left(K\overline{R}\right) \leq \epsilon\theta(A)$.*

Proof. As $\epsilon \in (0, 2)$ there is $N \in \mathbb{N}$ such that there holds $(1-\epsilon/2)^N \leq \epsilon/2$ and we set $\delta := \epsilon^{2N+1}$. As $(A_i)_{i\in I}$ is a Van Hove sequence we can choose inductively i_n for $n = N, \cdots, 1$ such that $\theta(A_n) > 0$, the i_n are pairwise distinct and such that

$$\alpha(A_n, K_n) = \frac{\theta(\partial_{K_n} A_n)}{\theta(A_n)} \leq \epsilon^{2(N-n)+4}, \tag{3.4}$$

where we abbreviate $A_n := A_{i_n}$ and $K_n := K \cup \left(\bigcup_{m=n+1}^{N} A_m^{-1}\right)$ for $n = N, \cdots, 0$. We set $D := K_0$ and $F := \{i_1, \cdots, i_N\}$ and obtain that $D = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_N = K$ and $A_n^{-1} \subseteq K_{n-1}$ for $n = 1, \cdots, N$.

Consider now a compact and (δ, D)-invariant subset A of G. Set $R_0 := A$. Using Lemma 3.17 we now choose inductively for $n = 1, \cdots, M$ finite (ϵ, A_n)-fillings C_n of R_{n-1} of maximal cardinality, where we abbreviate $R_n := R_{n-1} \setminus A_nC_n$. Here $M \leq N$ is the smallest integer where our choices lead to

$$\theta(R_M) = \theta(R_{M-1} \setminus A_MC_M) \leq \epsilon\theta(R_{M-1})$$

and $M = N$ if we never encounter this situation. Thus, in particular for $n = 1, \cdots, M-1$

there holds $\theta(R_n) > \epsilon\theta(R_{n-1})$. Note furthermore that

$$R = R_M \subseteq R_{M-1} \subseteq \cdots \subseteq R_0 = A.$$

For $n \in \{M+1, \cdots, N\}$ we set $C_n := \emptyset$. We will now show that defining $C_{i_n} := C_n$ we get that $(C_i)_{i\in F} = (C_n)_{n=1}^N$ is a family of ϵ-quasi-tiling centres that fulfils the required properties.

We first show inductively that for $n = 0, \cdots, M-1$ we obtain R_n to be $(\epsilon^{2(N-n)+1}, K_n)$-invariant, i.e. that there holds

$$\alpha(R_n, K_n) \leq \epsilon^{2(N-n)+1}. \tag{3.5}$$

This is clearly satisfied for $n = 0$, as $R_0 = A$, $K_0 = D$ and $\epsilon^{2(N-0)+1} = \delta$. To proceed inductively we assume R_n to be $(\epsilon^{2(N-n)+1}, K_n)$-invariant for some $n < M-1$ and as $K_{n+1} \subseteq K_n$ we obtain

$$\alpha(R_n, K_{n+1}) \leq \epsilon^{2(N-n)+1}.$$

Now recall from (3.4) that A_{n+1} is $(\epsilon^{2(N-(n+1))+4}, K_{n+1})$-invariant. As C_{n+1} is an (ϵ, A_{n+1})-filling we obtain that $\{A_{n+1}g;\, g \in C_{n+1}\}$ is ϵ-disjoint. We apply Lemma 3.14 to see

$$\begin{aligned}\alpha(A_{n+1}C_{n+1}, K_{n+1}) &\leq \frac{1}{1-\epsilon} \max_{g\in C_{n+1}} \alpha(A_{n+1}g, K_{n+1}) \\ &= \frac{\alpha(A_{n+1}, K_{n+1})}{1-\epsilon} \\ &\leq \frac{\epsilon^{2(N-(n+1))+4}}{1-\epsilon} \\ &\leq \epsilon^{2(N-n)+1}.\end{aligned}$$

For this we have used that $\epsilon < 1/2$ yields that $\epsilon/(1-\epsilon) < 1$. As we assume $n < M-1$ we obtain $\theta(R_n \setminus A_{n+1}C_{n+1}) = \theta(R_{n+1}) > \epsilon\theta(R_n)$. Thus, Lemma 3.15 yields

$$\begin{aligned}\alpha(R_{n+1}, K_{n+1}) &= \alpha\left(R_n \setminus A_{n+1}C_{n+1}, K_{n+1}\right) \\ &\leq \frac{\alpha(R_n, K_{n+1}) + \alpha(A_{n+1}C_{n+1}, K_{n+1})}{\epsilon} \\ &\leq 2\epsilon^{2(N-n)} \\ &\leq \epsilon^{2(N-n)-1} \\ &= \epsilon^{2(N-(n+1))+1}\end{aligned}$$

and we have completed the induction to show (3.5).

We next show that there holds

$$\theta(R) \leq \frac{\epsilon}{2}\theta(A). \tag{3.6}$$

This statement is satisfied whenever $M < N$, as in this case there holds

$$\theta(R) = \theta(R_M) \leq \epsilon\theta(R_{M-1}) \leq \epsilon\theta(A).$$

In order to show (3.6) we thus assume without lost of generality that $M = N$ and that $\theta(R) > 0$. Then there holds $\theta(R_N) = \theta(R) > 0$ and for $n \in \{1, \cdots, N-1\}$ we obtain

$$\theta(R_n) > \epsilon\theta(R_{n-1}) > \epsilon^n\theta(R_0) = \epsilon^n\theta(A) > 0.$$

For $n \leq M = N$ we have chosen C_n to be an (ϵ, A_n)-filling of R_{n-1} of maximal cardinality. Thus, we obtain from Lemma 3.16, (3.5) and $A_n^{-1} \subseteq K_{n-1}$ that

$$\begin{aligned}\frac{\theta(A_nC_n)}{\theta(R_{n-1})} &\geq \epsilon\left(1 - \alpha(R_{n-1}, A_n^{-1})\right)\\ &\geq \epsilon\left(1 - \alpha(R_{n-1}, K_{n-1})\right)\\ &\geq \epsilon\left(1 - \epsilon^{2(N-(n-1))+1}\right)\\ &\geq \frac{\epsilon}{2}.\end{aligned}$$

Thus, there holds

$$\theta(R_n) = \theta(R_{n-1}) - \theta(A_nC_n) \leq \left(1 - \frac{\epsilon}{2}\right)\theta(R_{n-1})$$

and we obtain from our choice of N that

$$\theta(R) = \theta(R_N) \leq \left(1 - \frac{\epsilon}{2}\right)^N \theta(R_0) \leq \frac{\epsilon}{2}\theta(A).$$

This shows the claimed statement (3.6).

As C_n is an (ϵ, A_n)-filling of R_{n-1} we obtain from the construction $R_n = R_{n-1} \setminus A_nC_n$ that $\{A_ng;\, g \in C_n\}$ is ϵ-disjoint for all $n \leq M$ and that $\{A_nC_n;\, n \leq N\} = \{A_nC_n;\, n \leq M\} \cup \{\emptyset\}$ is a disjoint family. Furthermore, one obtains

$$\bigcup_{n=1}^{N} A_nC_n = \bigcup_{n=1}^{M} A_nC_n \subseteq \bigcup_{n=0}^{M} R_n = R_0 = A.$$

Thus, (3.6) allows to compute

$$\theta\left(A \cap \bigcup_{n=1}^{N} A_nC_n\right) = \theta(A) - \theta(R) \geq (1-\epsilon)\theta(A).$$

This shows that $(A_i)_{i\in F}$ is an ϵ-quasi-tiling of A and that $\bigcup_{i\in F} A_iC_i \subseteq A$. It remains to show that $\theta(KR) \leq \epsilon\theta(A)$. To do this we show next that $R = R_M$ is $(\epsilon/2, K)$-invariant.

Recall from (3.4) that A_M is $(\epsilon^{2(N-M)+4}, K_M)$-invariant. As $K \subseteq K_M$ we obtain

$$\alpha(A_M, K) \leq \epsilon^4.$$

As C_M is an (ϵ, A_M)-filling we obtain that $\{A_M g;\ g \in C_M\}$ is ϵ-disjoint and apply Lemma 3.14 to see

$$\begin{aligned}\alpha(A_M C_M, K) &\leq \frac{1}{1-\epsilon} \max_{g \in C_M} \alpha(A_M g, K) \\ &= \frac{\alpha(A_M, K)}{1-\epsilon} \\ &\leq \frac{\epsilon^4}{1-\epsilon} \\ &\leq \epsilon^3.\end{aligned}$$

Furthermore, we have

$$\partial_K R_M = \partial_K (R_{M-1} \setminus A_M C_M) \subseteq \partial_K R_{M-1} \cup \partial_K (A_M C_M).$$

Since $K \subseteq D$ we obtain that A is (δ, K)-invariant and as $\delta \in (0,1)$ there holds $\theta(K) \leq \theta(A)$. Thus, (3.5) and $K \subseteq K_M \subseteq K_{M-1}$ allow to compute

$$\begin{aligned}\frac{\theta(\partial_K R)}{\theta(A)} &\leq \frac{\theta(\partial_K R_{M-1})}{\theta(A)} + \frac{\theta(\partial_K A_M C_M)}{\theta(A)} \\ &\leq \alpha(R_{M-1}, K) + \alpha(A_M C_M, K) \\ &\leq \alpha(R_{M-1}, K_{M-1}) + \epsilon^3 \\ &\leq \epsilon^3 + \epsilon^3 \leq \frac{\epsilon}{2}.\end{aligned}$$

From $K\overline{R} \subseteq \partial_K R \cup kR$ for any $k \in K$ and (3.6) we thus conclude

$$\theta(K\overline{R}) \leq \theta(\partial_K R) + \theta(kR) \leq \left(\frac{\epsilon}{2} + \frac{\epsilon}{2}\right) \theta(A) = \epsilon\theta(A).$$

□

In the proof of the Ornstein-Weiss Lemma below it will be convenient to consider the following corollary of Theorem 3.18.

Corollary 3.19. *Let $(A_i)_{i \in I}$ be a Van Hove net and $(A_{\phi(j)})_{j \in J}$ be a subnet. Let furthermore $\epsilon \in (0, 1/2)$ and $K \subseteq G$ be a non-empty and compact subset. Then there exist a finite subset $F \subseteq \phi(J)(\subseteq I)$, $\delta > 0$ and a compact subset $D \subseteq G$ with the following property. For any (δ, D)-invariant and compact subset $A \subseteq G$ the family $(A_i)_{i \in F}$ is an ϵ-quasi-tiling of A and the ϵ-quasi-tiling centres $(C_i)_{i \in F}$ can be chosen such that $\bigcup_{i \in F} A_i C_i \subseteq A$ and such that $R := A \setminus \bigcup_{i \in F} A_i C_i$ satisfies $\theta(KR) \leq \epsilon\theta(A)$.*

Proof. As $(A_{\phi(j)})_{j\in J}$ is a Van Hove net we apply Theorem 3.18 to obtain a a finite subset $E \subseteq J$, $\delta > 0$ and a compact and non-empty subset $D \subseteq G$ such that any (δ, D)-invariant and compact subset $A \subseteq G$ can be ϵ-quasi-tiled and the ϵ-quasi-tiling centres can be chosen such that the additional requirements of Theorem 3.18 are satisfied. Now we set $F := \phi(E)$ and $E_i := \{j \in E; \phi(j) = i\}$ for $i \in F$. If then A is a compact and (δ, D)-invariant subset, then A can be ϵ-quasi-tiled by $(A_{\phi(j)})_{j\in E}$ and the tiling centres C'_j can be chosen such that the additional requirements are satisfied. Setting now $C_i := \bigcup_{j\in E_i} C'_j$ for all $i \in F$ one easily shows that A is also ϵ-quasi-tiled by $(A_i)_{i\in F}$ with respect to the ϵ-quasi-tiling centres C_i. As a straightforward argument furthermore yields that these C_i also satisfy the additional properties we obtain the statement of the corollary. □

The Ornstein-Weiss Lemma for general unimodular amenable groups

With Theorem 3.18 and Corollary 3.19 we have now a slightly improved quasi-tiling machinery at hand that allows to show the Ornstein-Weiss Lemma also for non-discrete unimodular amenable groups. As mentioned above we cannot assume that f/θ is bounded, a property that follows from right invariance and subadditivity whenever G is discrete. We will next give a lemma that can serve in combination with the improved quasi-tiling machinery given in Theorem 3.18 as a replacement for the boundedness of f/θ also in the non-discrete setting.

Lemma 3.20. *Let $f\colon \mathcal{K}(G) \to [0, \infty)$ be a monotone, right invariant and subadditive mapping, K a compact neighbourhood of e_G. Then there exists a constant $c_K > 0$ such that for all non-empty and precompact subsets $R \subseteq G$ there holds*

$$f\left(\overline{R}\right) \leq c_K \theta(KR).$$

Proof. Note first that the subadditivity of f implies that f is positive. Let V be a compact and symmetric neighbourhood of e_G that satisfies $VV \subseteq K$ and set $c_K := f(VV)/\theta(V)$. For a non-empty and precompact subset $R \subseteq G$ we let $F \subseteq \overline{R}$ be a V-discrete subset of maximal cardinality. As such subsets in particular satisfy $\overline{R} \subseteq VVF$ there holds

$$f\left(\overline{R}\right) \leq f(VVF) \leq \sum_{g\in F} f(VVg) = |F| f(VV).$$

As $F \subseteq \overline{R}$ is V-discrete we obtain $VF = \bigcup_{g\in F} Vg$ to be a disjoint union and thus

$$\theta\left(V\overline{R}\right) \geq \theta(VF) = \sum_{g\in F} \theta(Vg) = |F|\theta(V).$$

As $V\overline{R} \subseteq VVR \subseteq KR$ we compute

$$f\left(\overline{R}\right) \leq \frac{f(VV)}{\theta(V)} \theta(V\overline{R}) \leq c_K \theta(KR).$$

□

Theorem 3.21 (Ornstein-Weiss Lemma - general version)**.** *Let G be a unimodular amenable group and let $f\colon \mathcal{K}(G) \to \mathbb{R}$ be a monotone, right invariant and subadditive mapping. Whenever $(A_i)_{i\in I}$ is a Van Hove net in G, then the limit*

$$\lim_{i\in I} \frac{f(A_i)}{\theta(A_i)}$$

exists, it is finite and it does not depend on the choice of the Van Hove net.

Remark 3.22. *In [Gro99] a stronger statement is claimed. In fact it is claimed that it is sufficient to assume that f is right invariant and subadditive. Nevertheless, we need the monotonicity in the proof of Lemma 3.20. Note furthermore that in [GK82] an example[4] of a right invariant but not monotone function $f\colon \mathcal{K}(\mathbb{R}) \to \mathbb{R}$ is presented that satisfies $f(A\cup B) \leq f(A) + f(B)$ for disjoint compact subsets $A, B \subseteq G$ and for which f/θ is not amenable along any Van Hove sequence. Nevertheless, this function is not subadditive and it remains open whether the statement can be shown without the assumption of monotonicity.*

Proof of Theorem 3.21. Let $\epsilon \in (0, 1/2)$ and choose an arbitrary compact neighbourhood K of e_G. Then by Lemma 3.20 there exists a constant $c > 0$ such that $f(\overline{A}) \leq c\theta(KA)$ for all non-empty and precompact subsets $A \subseteq G$. As $(KA_i)_{i\in I}$ is a Van Hove net in G that satisfies $\lim_{i\in I} \theta(KA_i)/\theta(A_i) = 1$, we obtain

$$\lambda := \liminf_{i\in I} \frac{f(A_i)}{\theta(A_i)} = \liminf_{i\in I} \frac{f(A_i)}{\theta(KA_i)} \leq c < \infty.$$

In particular, there exists a subnet $(f(A_{\phi(j)})/\theta(A_{\phi(j)}))_{j\in J}$ of $(f(A_i)/\theta(A_i))_{i\in I}$ that converges to λ. Thus, there is $\iota \in J$ such that

$$\frac{f(A_{\phi(j)})}{\theta(A_{\phi(j)})} \leq \lambda + \epsilon \tag{3.7}$$

for all $j \in J_{\geq\iota} := \{j \in J;\ j \geq \iota\}$. As $(A_{\phi(j)})_{j\in J_{\geq\iota}}$ is a Van Hove net, we apply Corollary 3.19 to obtain a finite subset $F \subseteq \phi(J_{\geq\iota}) \subseteq I$, $\delta > 0$ and a compact and non-empty subset $D \subseteq G$ such that any (δ, D)-invariant and compact subset $A \subseteq G$ can be ϵ-quasi-tiled and such that the corresponding ϵ-quasi-tiling centres can be chosen with the additional properties as in Corollary 3.19.

We now consider a compact and (δ, D)-invariant subset $A \subseteq G$ and choose ϵ-quasi-tiling centres C_i such that these additional properties are satisfied, i.e. such that $\bigcup_{i\in F} A_iC_i \subseteq A$ and such that $R := A \setminus \bigcup_{i\in F} A_iC_i$ satisfies $\theta(KR) \leq \epsilon\theta(A)$. As

[4] The author would like to thank Prof. Aernout van Enter for pointing out this intriguing example to him.

$F \subseteq \phi(J_{\geq \iota})$ we obtain from (3.7) that there holds $f(A_i)/\theta(A_i) \leq \lambda + \epsilon$ for all $i \in F$ and we compute

$$\begin{aligned}\frac{f\left(\bigcup_{i\in F} A_i C_i\right)}{\theta(A)} &\leq \sum_{i\in F}\sum_{g\in C_i}\frac{f(A_i g)}{\theta(A)}\\ &= \sum_{i\in F}\sum_{g\in C_i}\frac{f(A_i)}{\theta(A_i)}\frac{\theta(A_i)}{\theta(A)}\\ &\leq (\lambda+\epsilon)\sum_{i\in F}\sum_{g\in C_i}\frac{\theta(A_i)}{\theta(A)}.\end{aligned}$$

From the properties of ϵ-quasi-tiling centres we obtain that $\{A_i g;\, g \in C_i, i \in F\}$ is an ϵ-disjoint family. Thus, $\bigcup_{i\in F} A_i C_i \subseteq A$ implies that

$$\sum_{i\in F}\sum_{g\in C_i}\frac{\theta(A_i)}{\theta(A)} = \sum_{i\in F}\sum_{g\in C_i}\frac{\theta(A_i g)}{\theta(A)} \leq \frac{1}{1-\epsilon}\frac{\theta(\bigcup_{i\in J} A_i C_i)}{\theta(A)} \leq \frac{1}{1-\epsilon}.$$

We have shown

$$\frac{f\left(\bigcup_{i\in F} A_i C_i\right)}{\theta(A)} \leq \frac{\lambda+\epsilon}{1-\epsilon}. \tag{3.8}$$

As we require the ϵ-quasi-tiling centres to satisfy $\theta(KR) \leq \epsilon\theta(A)$ we obtain from the choice of the constant c at the beginning of the proof that

$$f\left(\overline{R}\right) \leq c\theta(KR) \leq \epsilon c\theta(A).$$

Thus, (3.8) yields

$$\frac{f(A)}{\theta(A)} \leq \frac{f(\bigcup_{i\in F} A_i C_i)}{\theta(A)} + \frac{f\left(\overline{R}\right)}{\theta(A)} \leq \frac{\lambda+\epsilon}{1-\epsilon} + \epsilon c$$

for all (δ, K)-invariant and compact subsets $A \subseteq G$. Thus, considering another Van Hove net $(B_\iota)_{\iota\in\tilde{I}}$, we get

$$\limsup_{\iota\in\tilde{I}}\frac{f(B_\iota)}{\theta(B_\iota)} \leq \frac{\lambda+\epsilon}{1-\epsilon} + \epsilon c.$$

As $\epsilon > 0$ was arbitrary we have shown that for any two Van Hove nets $(A_i)_{i\in I}$ and $(B_\iota)_{\iota\in\tilde{I}}$ there holds

$$\limsup_{\iota\in\tilde{I}}\frac{f(B_\iota)}{\theta(B_\iota)} \leq \lambda = \liminf_{i\in I}\frac{f(A_i)}{\theta(A_i)} \leq c,$$

which clearly implies the statement of the theorem. □

We also obtain the following statement about Følner nets.

Corollary 3.23. *Let* $f\colon \mathcal{K}(G) \to \mathbb{R}$ *be a monotone, right invariant and subadditive mapping. Then whenever* $(A_i)_{i\in I}$ *is a Følner net in* G *and* K *is a compact neighbourhood of* e_G*, then the limit*

$$\lim_{i\in I} \frac{f(KA_i)}{\theta(A_i)}$$

exists, is finite, equals the limit in Theorem 3.21 and does not depend on the choice of the Følner net. Furthermore, there holds

$$\limsup_{i\in I} \frac{f(A_i)}{\theta(A_i)} \leq \lim_{i\in I} \frac{f(KA_i)}{\theta(A_i)}.$$

Proof. From Proposition 2.30 we obtain that $(KA_i)_{i\in I}$ is a Van Hove net in G and Proposition 2.24 yields that $\lim_{i\in I} \theta(KA_i)/\theta(A_i) = 1$. We thus obtain the existence of the limit $\lim_{i\in I} f(KA_i)/\theta(A_i)$ from Theorem 3.21. The claimed inequality follows from the monotonicity of f. □

Remark 3.24. *It remains open, whether the limit* $\lim_{i\in I} f(A_i)/\theta(A_i)$ *exists for all Følner nets.*

4 On relative entropy

One of the most important concepts in the study of complexity of dynamical systems is the concept of entropy. The concept of measure theoretical entropy is due to A. N. Kolmogorov and J. G. Sinai [Kol58, Sin59]. Inspired by the measure theoretical version the concept of topological entropy appeared first in [AKM65]. These concepts were related in [Goo69], where it was shown that the topological entropy always bounds the measure theoretical entropy of an invariant Borel probability measure. Completing the variational principle in [Goo71, Din71] it was shown that the supremum over the measure theoretical entropies with respect to all invariant Borel probability measures yields the topological entropy. The short and elegant proof of the variational principle that can be found nowadays in most of the literature was first given in [Mis76] and already studies the concept of relative entropy. The work [Mis76] also shows the beginning interest into more general (semi-) groups as it already considers actions of $\mathbb{N}_0^d$. In the context of actions of countable discrete amenable groups the concepts of measure theoretical and topological entropy (as well as topological pressure) seem to appear first in [STZ80], where furthermore a general version of the variational principle was shown. Independently from this work a proof of the variational principle for actions of countable discrete amenable groups appeared in [OP82, Oll85].

Even though entropy has been studied extensively in the context of actions of discrete amenable groups ever since as for example in [STZ80, OP82, Oll85, OW87, WZ92, LW00, HYZ11, CSCK14, Yan15, YZ16], it seems that actions of non-discrete groups such as $\mathbb{R}^d$ or more general unimodular amenable groups have received few attention. Up to our knowledge the only references treating these actions are [OW80, TZ91, Sch15] and important concepts like Bowens formula, sufficient conditions for the upper semi-continuity of the entropy map as well as a proof of the variational principle seem to be missing in this context. This somehow surprises as it is for example presented in [Mey72] that there are unimodular amenable groups that contain no discrete subgroups other then the trivial one element group and thus there seems no direct possibillity to obtain the statements from the discrete setting. As statements of entropy theory in the context of non-discrete groups are useful in the study of aperiodic order [BLR07, JLO16] we will thus present in this chapter that some of the techniques of the classical theory can easily be lifted to the non-discrete setting. Others, as for example the standard definitions of topological entropy and measure theoretical entropy as given in [Kol58, Sin59, AKM65], seem not to be directly at hand as we will see next.

In order to see this let us first discuss these well-known approaches for an action π of a discrete amenable groups G on a compact Hausdorff space X. For an open cover

$\mathcal{U}$ of X one defines the shifted open cover $\mathcal{U}_g := \{(\pi^g)^{-1}(U); U \in \mathcal{U}\}$ for $g \in G$. For a finite subset $F \subseteq G$ one furthermore denotes by $\mathcal{U}_F$ the common refinement[1] of the open covers $\mathcal{U}_g$ with $g \in F$ and obtains an open cover of X. Similarly one defines α_g and α_F for a finite partition α of X and a finite set $F \subseteq G$ and obtains finite partitions α_g and α_F of X. As G is assumed to be discrete all compact sets are finite and any Van Hove net $(F_i)_{i\in I}$ consists of finite sets. As one easily shows that[2] $\mathcal{K}(G) \ni F \mapsto H^*(\mathcal{U}_F)$ and $\mathcal{K}(G) \ni F \mapsto H^*_\mu(\alpha_F)$ are monotone, right invariant and subadditive, one uses the Ornstein-Weiss Lemma to define the topological entropy[3] of π as

$$\sup_{\mathcal{U}} \lim_{i\in I} \frac{H^*(\mathcal{U}_{F_i})}{|F_i|}$$

and the measure theoretical entropy[4] of π as

$$\sup_{\alpha} \lim_{i\in I} \frac{H^*_\mu(\alpha_{F_i})}{|F_i|},$$

where the suprema are taken over all open covers $\mathcal{U}$ of X and all finite partitions α of X respectively.

The problem with this approach is that only the finite refinement of translates of an open cover is necessarily an open cover. As clearly a similar statement holds for finite partitions this approach depends on the finiteness of the Van Hove sets. We will see in this chapter and in Chapter 5 strategies to overcome this problem and use less prominent approaches for our definition. As the topology of the phase space of Delone actions is defined by a uniformity and in order to avoid unnecessary countability assumptions on the phase spaces we will follow ideas of [Hoo74] and use the structure of the uniformity of a compact Hausdorff space. We will furthermore use the concept of relative entropy that seems to originate from [Mis76, LW77] in order to state the Bowen entropy formula.

In this chapter we assume that G is a unimodular amenable group and that π and ϕ are actions of G on compact Hausdorff spaces X and Y respectively such that ϕ is a factor of π via a factor map $p\colon X \to Y$.

[1] Whenever $\mathcal{U}$ and $\mathcal{V}$ are two families of subsets of X, then one defines the common refinement of $\mathcal{U}$ and $\mathcal{V}$ as the set of all intersections $U \cap V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Similarly one defines the common refinement of a finite number of families.

[2] For an open cover $\mathcal{U}$ of X we let $H^*(\mathcal{U})$ be the logarithm of the minimal cardinality of a subcover of $\mathcal{U}$. Furthermore, for a finite partition α of X we define $H^*_\mu(\alpha) := -\sum_{A\in\alpha} \mu(A)\log(\mu(A))$, where we use the convention $0 \log 0 = 0$.

[3] We will see in Remark 4.29 below that this approach is equivalent to the general one discussed below whenever we consider discrete amenable groups.

[4] We will see in Corollary 5.17 below that this approach is equivalent to the general one discussed below for actions of discrete amenable groups.

4.1 Static relative entropy

In this section we follow well-known[5] ideas in order to define the static (relative) topological and measure theoretical entropy of finite partitions and finite open covers respectively. We will furthermore present some results from the literature for later use concerning these concepts. However, as discussed above, these concepts cannot directly be used in order to define topological and measure theoretical entropy. We thus follow ideas presented for example in [BS02] in order to define the static (relative) topological entropy at a given scale. Inspired by this approach we also define the static (relative) measure theoretical entropy at a given scale. Recall that X and Y are assumed to be compact Hausdorff spaces and that $p\colon X \to Y$ is assumed to be continuous and surjective.

4.1.1 Some more preliminaries

For families $\mathcal{U}$ and $\mathcal{V}$ that consist of subsets of X we say, that $\mathcal{U}$ *is finer than* $\mathcal{V}$, if for every $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ with $U \subseteq V$. In this case we write $\mathcal{V} \preceq \mathcal{U}$. This defines an order relation on the set of all families of subsets of X. We define furthermore the *common refinement* $\mathcal{U} \vee \mathcal{V}$ as the family consisting of all intersections $U \cap V$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Similarly one defines the common refinement of finitely many families of subsets of X. If these families are partitions; finite; a cover; or open, then also the common refinement is a partition; finite; a cover; or open, respectively.

Note that every finite σ-algebra of Borel sets is generated by a unique finite partition. This allows us to identify finite partitions with finite sub-σ-algebras of $\mathcal{B}_X$, which we will do in the following. With this identification we obtain easily that $\alpha \preceq \beta$ holds if and only if $\alpha \subseteq \beta$ holds in the interpretation as σ-algebras; and that $\alpha \vee \beta$ is the σ-algebra generated by α and β.

Recall that the uniformity $\mathbb{U}_X$ of X is the set of all neighbourhoods of the diagonal in $X \times X$. For $\eta \in \mathbb{U}_X$ we say that $M \subseteq X$ is η-small, if $M^2 \subseteq \eta$. A set $\mathcal{U}$ of subsets of X is said to be *at scale* η, if every $U \in \mathcal{U}$ is η-small. Note that if $\mathcal{U}$ and $\mathcal{V}$ are two families of subsets of X at scale η and κ respectively, then $\mathcal{U} \vee \mathcal{V}$ is at scale $\eta \cap \kappa$. In metric spaces this concept can be reformulated as follows. Whenever X is metric and $\epsilon > 0$, then $\mathcal{U}$ is at scale $[d \leq \epsilon]$, whenever the diameter $\operatorname{diam}(U) := \sup_{(x,y)\in U^2} d(x,y)$ of every $U \in \mathcal{U}$ is less then or equal to ϵ.

Remark 4.1. *Recall from Example 2.3 that for a cover $\mathcal{U}$ of M we define $\langle \mathcal{U} \rangle := \bigcup_{U\in\mathcal{U}} U^2$ and that $\langle \mathcal{U} \rangle \in \mathbb{U}_X$, whenever $\mathcal{U}$ is open. Using this notion we obtain that a cover $\mathcal{U}$ is at scale η, if and only if $\langle \mathcal{U} \rangle \subseteq \eta$. It is thus natural to ask for finite open covers (and finite partitions) $\mathcal{U}$ and $\mathcal{V}$ how $\mathcal{U} \preceq \mathcal{V}$ and $\langle \mathcal{U} \rangle \subseteq \langle \mathcal{V} \rangle$ relate. In fact it is straightforward to show that whenever we consider finite partitions α and β of X, then there holds $\alpha \preceq \beta$, if and only if $\langle \alpha \rangle \subseteq \langle \beta \rangle$. Whenever we consider finite open covers*

[5] See for example [Wal82, Oll85, BS02].

of X, then the matter is more complicated. In fact for all covers $\mathcal{U}$ and $\mathcal{V}$ of X with $\mathcal{V} \preceq \mathcal{U}$ there holds $\langle\mathcal{U}\rangle \subseteq \langle\mathcal{V}\rangle$, but the converse fails in general. To see this consider for example the open cover $\{\{-1,1\},\{0,1\},\{-1,0\}\}$ of $\{-1,0,1\}$ equipped with the discrete topology. Note furthermore that usually $\langle\alpha\rangle \notin \mathbb{U}_X$ for a finite partition α of X.

4.1.2 Static relative topological entropy

For an open cover $\mathcal{U}$ of X and $M \subseteq X$ we define $N_M(\mathcal{U})$ as the minimal cardinality of a subset of $\mathcal{U}$ that covers M. Furthermore, we define $N_p(\mathcal{U}) := \sup_{y\in Y} N_{p^{-1}(y)}(\mathcal{U})$ and $H_p^*(\mathcal{U}) := \log\left(N_p(\mathcal{U})\right)$. For $\eta \in \mathbb{U}_X$ we define the *static topological entropy of p at scale η* as

$$H_p(\eta) := \inf_{\mathcal{U}} H_p^*(\mathcal{U}),$$

where the infimum is taken over all open covers $\mathcal{U}$ of X at scale η. If Y is a singleton, we call $H(\eta) := H_p(\eta)$ the *static topological entropy of X at scale η*.

Remark 4.2. *Let $\eta \in \mathbb{U}_X$. As X is compact there is a finite open cover of X at scale η. Thus, for every $M \subseteq X$ there exists a family of open and η-small subsets of X that covers M. For $M \subseteq X$ and $\eta \in \mathbb{U}_X$ we denote by $\operatorname{cov}_M(\eta)$ the minimal cardinality of such a family. We will see next that there holds*

$$H_p(\eta) = \log\left(\sup_{y\in Y} \operatorname{cov}_{p^{-1}(y)}(\eta)\right).$$

Indeed, whenever $\mathcal{U}$ is an open cover of X at scale η, then every subset of $\mathcal{U}$ consists of open and η-small sets. Thus, $N_{p^{-1}(y)}(\mathcal{U}) \geq \operatorname{cov}_{p^{-1}(y)}(\eta)$ for every $y \in Y$ and we obtain $H_p(\eta) \geq \log(\sup_{y\in Y} \operatorname{cov}_{p^{-1}}(\eta))$. To show the remaining inequality, consider for $y \in Y$ a family $\mathcal{U}^y$ consisting of open and η-small subsets of X that covers $p^{-1}(y)$ and which is of minimal cardinality $\operatorname{cov}_{p^{-1}(y)}(\eta)$. Then $\mathcal{U} := \bigcup_{y\in Y} \mathcal{U}^y$ is an open cover of X. Furthermore, as $\mathcal{U}^y$ is a subset of $\mathcal{U}$ that covers $p^{-1}(y)$, we obtain $N_{p^{-1}(y)}(\mathcal{U}) \leq |\mathcal{U}^y| = \operatorname{cov}_{p^{-1}(y)}(\eta)$ for every $y \in Y$ and it follows that

$$H_p(\eta) \leq H_p^*(\mathcal{U}) \leq \log\left(\sup_{y\in Y} \operatorname{cov}_{p^{-1}(y)}(\eta)\right).$$

4.1.3 Static relative measure theoretical entropy

Consider a Borel probability measure μ on X. Let α a finite partition of X. Let $\mathcal{A}$ be a sub-σ-algebra of the Borel σ-algebra $\mathcal{B}_X$. We denote by $\mathbb{E}_\mu(f|\mathcal{A})$ the conditional expectation of f given $\mathcal{A}$ in $L^1(X,\mu)$. For $\alpha \in \mathcal{P}_X$ we define

$$H_\mu^*(\alpha|\mathcal{A}) := -\sum_{A\in\alpha} \int_X \mathbb{E}_\mu(\chi_A|\mathcal{A}) \log(\mathbb{E}_\mu(\chi_A|\mathcal{A}))d\mu.$$

Furthermore, we define

$$H_\mu^*(\alpha) := H_\mu^*(\alpha|\{\emptyset, X\}) = -\sum_{A\in\alpha} \mu(A)\log(\mu(A)).$$

As $p\colon X \to Y$ is a continuous map $p^{-1}(\mathcal{B}_Y)$ is a sub-σ-algebra of $\mathcal{B}_X$ and we define $H_{\mu,p}^*(\alpha|\mathcal{A}) := H_\mu^*(\alpha|p^{-1}(\mathcal{B}_Y)\vee\mathcal{A})$ and $H_{\mu,p}^*(\alpha) := H_\mu^*(\alpha|p^{-1}(\mathcal{B}_Y))$ for a finite partition α of X and a sub σ-algebra $\mathcal{A}$ of $\mathcal{B}_X$. For $\eta \in \mathbb{U}_X$ we define the *static measure theoretical entropy of X at scale η* as

$$H_\mu(\eta) := \inf_\alpha H_\mu^*(\alpha),$$

where the infimum is taken over all finite partitions α at scale η. Similarly we define the *static measure theoretical entropy of p at scale η* as

$$H_{\mu,p}(\eta) := \inf_\alpha H_{\mu,p}^*(\alpha).$$

4.1.4 Properties of static relative entropy

In this subsection we cite some well-known statements about static entropy.

Basic properties

The statement of the following lemma gives the link between measure theoretical and topological entropy and can be found for example in the proof of [HYZ06, Lemma 2.3.(2)].

Lemma 4.3. *Let μ be a Borel probability measure on X. For every finite open cover $\mathcal{U}$ of X there exists a finite partition α, which is finer than $\mathcal{U}$ and for which there holds $H_{\mu,p}^*(\alpha) \leq H_p^*(\mathcal{U})$.*

The following proposition collects some well-known statements about static relative topological entropy which are straightforward to prove. For reference see [Wal82, Section 7.1].

Proposition 4.4. *For finite open covers $\mathcal{U}$ and $\mathcal{V}$ of X there holds*

(i) $H_p^*(\mathcal{U}) \leq H^*(\mathcal{U})$.

(ii) $H_p^*(\mathcal{U}\vee\mathcal{V}) \leq H_p^*(\mathcal{U}) + H_p^*(\mathcal{V})$.

(iii) *If $\mathcal{V}$ is finer than $\mathcal{U}$, then* $H_p^*(\mathcal{U}) \leq H_p^*(\mathcal{V})$.

The following proposition is the counterpart of Proposition 4.4 for measure theoretical entropy and its statements are also straightforward to proof. For reference for the non-relative versions of the statements of (ii), (iii), and (iv) see [Wal82, Theorem 4.3]. From a remark below [Wal82, Definition 4.8] one then easily deduces the relative statements and (i).

Proposition 4.5. *Let μ be a Borel probability measure. For finite partitions α, β and γ of X there holds*

(i) $H^*_{\mu,p}(\alpha|\beta) \leq H^*_\mu(\alpha|\beta)$, *in particular* $H^*_{\mu,p}(\alpha) \leq H^*_\mu(\alpha)$.

(ii) $H^*_{\mu,p}(\alpha \vee \beta|\gamma) = H^*_{\mu,p}(\alpha|\gamma) + H^*_{\mu,p}(\beta|\alpha \vee \gamma)$, *in particular* $H^*_{\mu,p}(\alpha \vee \beta) = H^*_{\mu,p}(\alpha) + H^*_{\mu,p}(\beta|\alpha)$.

(iii) If β is finer than α, then $H^*_{\mu,p}(\alpha|\gamma) \leq H^*_{\mu,p}(\beta|\gamma)$, *in particular* $H^*_{\mu,p}(\alpha) \leq H^*_{\mu,p}(\beta)$.

(iv) If γ is finer than β, then $H^*_{\mu,p}(\alpha|\beta) \geq H^*_{\mu,p}(\alpha|\gamma)$, *in particular* $H^*_{\mu,p}(\alpha) \geq H^*_{\mu,p}(\alpha|\gamma)$.

Whenever we consider a chain of factor maps we furthermore obtain the following. We omit the straightforward proof.

Proposition 4.6. *Let ψ be a factor of ϕ via a factor map q and note that this yields the situation $\pi \xrightarrow{p} \phi \xrightarrow{q} \psi$. Then for any $\eta \in \mathbb{U}_X$ there holds $H_p(\eta) \leq H_{q\circ p}(\eta)$. Furthermore, for $\eta \in \mathbb{U}_X$ and any invariant Borel probability measure μ on X there holds $H_{\mu,p}(\eta) \leq H_{\mu,q\circ p}(\eta)$.*

The non-relative version of the following is contained in [BS02, Proposition 9.2.2]. Since there holds $H^*_{\mu,p}(\alpha|\beta) \leq H^*_\mu(\alpha|\beta)$ for all finite partitions α, β, as we have seen in Proposition 4.5(i) above, we can also deduce the following relative version from [BS02].

Lemma 4.7. *Let μ be a Borel probability measure on X and $r \geq 0$ be a fixed integer. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that for any two partitions $\alpha = \{A_1, \cdots, A_r\}$ and $\beta = \{B_1, \cdots, B_r\}$ of X in r sets that satisfy $\sum_{i=1}^r \mu(A_i \Delta B_i) < \delta$ there holds $H^*_{\mu,p}(\alpha|\beta) + H^*_{\mu,p}(\beta|\alpha) < \epsilon$.*

Below we will also need the following statement which follows a straightforward adaptation of the arguments of [Wal75, Theorem 4.7].

Lemma 4.8. *Let μ be a Borel probability measure on X. Let $(\beta_i)_{i\in I}$ be a net of finite partitions of X such that β_i is finer than β_j, whenever $i \geq j$. If α is a finite partition and $\mathcal{A}$ is the σ-algebra generated by $\bigcup_{i\in I} \beta_i$, then there holds $\lim_{i\in I} H^*_{\mu,p}(\alpha|\beta_i) = H^*_{\mu,p}(\alpha|\mathcal{A})$.*

On upper semicontinuity

Recall that we equip $\mathcal{M}(X)$ with the weak-* topology. It is well-known that the non-relative version $H_{(\cdot)}(\alpha)$ is continuous in all points $\mu \in \mathcal{M}(X)$ as soon as X is metrizable, whenever the finite partition α has *almost no boundary* with respect to μ, i.e whenever $\mu(\partial\alpha) = 0$ with $\partial\alpha = \bigcup_{A\in\alpha} \partial A$. The proof of the following lemma is inspired by [HYZ06, Lemma 3.4], where α is assumed to have empty boundary, i.e. that $\partial\alpha = \emptyset$ and X is assumed to be metrizable. We next present how to avoid these assumptions.

Lemma 4.9. *Let α and β be finite partitions of X and assume μ to be a Borel probability measure on X such that α and β have almost no boundary with respect to μ. Then the map $H^*_{(\cdot),p}(\alpha|\beta)\colon \mathcal{M}(X) \to [0,\infty)$ is upper semi-continuous in μ.*

Proof. Let $\epsilon > 0$ and recall that we denote the push forward of μ along p as $p_*\mu$. We furthermore denote by I the set of all finite partitions of Y that have almost no boundary with respect to $p_*\mu$. Now for any open subset $U \subseteq X$ we obtain from the regularity of μ that there is a compact subset $K_n \subseteq U$ such that $p_*\mu(U \setminus K_n) < 1/n$. Furthermore, by Lemma 2.6 there is a compact neighbourhood M_n of K_n such that M_n has almost no boundary with respect to $p_*\mu$ and such that $M_n \subseteq U$. Thus, clearly $\alpha_n := \{M_n, X \setminus M_n\} \in I$ and we obtain that $M := \bigcup_{n\in\mathbb{N}} M_n$ satisfies $p_*\mu(U\Delta M) = p_*\mu(U \setminus M) = 0$. We denote by $\bigvee_{\gamma\in I}\gamma$ the sigma algebra generated by $\bigcup_{\gamma\in I}\gamma$ and obtain that $\bigvee_{\gamma\in I}\gamma$ is up to $p_*\mu$ identical with $\mathcal{B}_Y$. We order I with the order $\preceq$ and set $\beta_\gamma := \beta \vee p^{-1}(\gamma)$ for $\gamma \in I$ to obtain a net $(\beta_\gamma)_{\gamma\in I}$ of finite partitions of X with almost no boundary with respect to μ. This net clearly satisfies $\beta_\gamma \preceq \beta_{\gamma'}$, whenever $\gamma \preceq \gamma'$. Furthermore, we obtain that up to μ that the σ-algebra $\bigvee_{\gamma\in I}\beta_\gamma$ generated by $\bigcup_{\gamma\in I}\beta_\gamma$ satisfies $\bigvee_{\gamma\in I}\beta_\gamma = \beta \vee p^{-1}\left(\bigvee_{\gamma\in I}\gamma\right) = \beta \vee p^{-1}(\mathcal{B}_Y)$ and thus in particular that $H^*_\mu(\alpha|\bigvee_{\gamma\in I}\beta_\gamma) = H^*_\mu(\alpha|\beta \vee p^{-1}(\mathcal{B}_Y)) = H^*_{\mu,p}(\alpha|\beta)$. By Lemma 4.8 we thus obtain that there is $\gamma \in I$ such that $H_\mu(\alpha|\beta_\gamma) \leq H_{\mu,p}(\alpha|\beta) + \epsilon$. As α and β_γ have almost no boundary with respect to μ one easily obtains the continuity of $H_{(\cdot)}(\alpha|\beta_\gamma)$ in μ and there is an open neighbourhood V of μ such that $H_\nu(\alpha|\beta_\gamma) \leq H_\mu(\alpha|\beta_\gamma) + \epsilon$ is satisfied for all $\nu \in V$. As clearly β_γ is a partition of $(\beta \vee p^{-1}(\mathcal{B}_Y))$-measurable sets we compute $H_{\nu,p}(\alpha|\beta) = H_\nu(\alpha|\beta \vee p^{-1}(\mathcal{B}_Y)) \leq H_\nu(\alpha|\beta_\gamma) \leq H_\mu(\alpha|\beta_\gamma) + \epsilon \leq H_{\mu,p}(\alpha|\beta) + 2\epsilon$ for $\nu \in V$. □

Properties of entropy at a certain scale

We finish this section with some statements about static relative topological and measure theoretical entropy at a certain scale, which are a direct consequence of Lemma 4.3, Proposition 4.4 and Proposition 4.5.

Proposition 4.10. *Let μ be a Borel probability measure. For $\eta, \kappa \in \mathbb{U}_X$ there holds*

(i) $H_{\mu,p}(\eta) \leq H_p(\eta) < \infty$.

(ii) $H_p(\eta) \leq H(\eta)$ *and* $H_{\mu,p}(\eta) \leq H_\mu(\eta)$.

(iii) $H_p(\eta \cap \kappa) \leq H_p(\eta) + H_p(\kappa)$ *and* $H_{\mu,p}(\eta \cap \kappa) \leq H_{\mu,p}(\eta) + H_{\mu,p}(\kappa)$.

(iv) $H_p(\eta) \leq H_p(\kappa)$ *and* $H_{\mu,p}(\eta) \leq H_{\mu,p}(\kappa)$, *whenever* $\kappa \subseteq \eta$.

Remark 4.11. *For $\eta \in \mathbb{U}_X$ there holds $\sup_{\mu\in\mathcal{M}(X)} H_{\mu,p}(\eta) < \infty$. Indeed, note that the map $(0,1] \ni x \mapsto -x\log(x)$ attains a maximum at $x = 1/e$ and that $-(1/e)\log(1/e) = 1/e \leq 1$. Thus considering any finite partition α at scale η we obtain for every Borel probability measure μ on X that $H_{\mu,p}(\eta) \leq H_\mu(\eta) \leq H^*_\mu(\alpha) \leq |\alpha|$.*

4.2 Relative entropy

In this section we consider an action π of a unimodular amenable group G on a compact Hausdorff space X. We furthermore consider a factor ϕ of π via a factor map $p\colon X \to Y$, i.e. $\pi \overset{p}{\to} \phi$.

4.2.1 Bowen entourages

For a precompact subset $A \subseteq G$ and $\eta \in \mathbb{U}_X$ we define the *Bowen entourage* as

$$\eta_A := \bigcap_{g \in A} \left\{(x,y) \in X^2;\ (g.x, g.y) \in \eta\right\}$$

and abbreviate $\eta_g := \eta_{\{g\}}$ for $g \in G$. This notion seems to origin from [Oll85], where it is studied in the context of discrete amenable groups. Clearly, whenever G is discrete, then η_A is a neighbourhood of the diagonal Δ_X in X^2 as it is a finite intersection of neighbourhoods of the diagonal. We thus obtain $\eta_A \in \mathbb{U}_X$, whenever G is discrete. Using the continuity of π and the compactness of X and $\overline{A}$ we will show in Lemma 4.12 below that this also holds true for non-discrete groups, which justifies the name "Bowen entourage". As we have not encountered the statement in the literature we include a full proof for the convenience of the reader. In order to omit brackets we will use the convention, that the operation of taking a Bowen entourage is stronger binding than the product of entourages.

Lemma 4.12. *For every $\eta \in \mathbb{U}_X$ and every precompact subset $A \subseteq G$ we have $\eta_A \in \mathbb{U}_X$.*

Proof. Note that $\eta_{\overline{A}} \subseteq \eta_A$. We can thus assume without lost of generality that A is closed and hence compact. Note that $\pi\colon A \times X \to X$ is uniformly continuous as a continuous mapping between compact Hausdorff spaces. Thus,

$$\{((a,x),(a',x')) \in (A \times X)^2;\ (a.x, a'.x') \in \eta\}$$

is a neighbourhood of the diagonal $\Delta_{A \times X}$ in $A \times X$. By the definition of the product topology we obtain the existence of $\kappa \in \mathbb{U}_A$ and $\rho \in \mathbb{U}_X$ such that for $(a,a') \in \kappa$ and $(x,x') \in \rho$ there holds $(a.x, a'.x') \in \eta$. In particular, for $(x,x') \in \rho$ there holds $(a.x, a.x') \in \eta$ for all $a \in A$ and we obtain $(x,x') \in \eta_A$. This shows $\eta_A \supseteq \rho \in \mathbb{U}_X$ and we conclude $\eta_A \in \mathbb{U}_X$. □

The following lemma summarizes some basic properties of the Bowen entourage that we will use frequently below. In particular, it justifies to write η_{AB} for $\eta_{(AB)} = (\eta_A)_B$. The proofs are straightforward, but as we do not know of any reference we included them for the convenience of the reader.

Lemma 4.13. *For $\eta, \kappa \in \mathbb{U}_X$ and precompact subsets $A, B \subseteq G$ there holds*

(i) $\eta_A \subseteq \eta_B$, *whenever* $B \subseteq A$,

(ii) $\eta_A \subseteq \kappa_A$, *whenever* $\eta \subseteq \kappa$,

(iii) $\eta_{(AB)} = (\eta_A)_B$,

(iv) $\eta_{A\cup B} = \eta_A \cap \eta_B$ *and*

(v) $\eta_A \kappa_A \subseteq (\eta\kappa)_A$.

Proof. The statements of (i) and (ii) follow directly from the definition. For $x, y \in X$ there holds $((ab).x, (ab).y) = (a.(b.x), a.(b.y)) \in \eta$ for all $a \in A$ and $b \in B$, from which we obtain that there holds $(x, y) \in \eta_{(AB)}$ if and only if $(x, y) \in (\eta_A)_B$. This shows (iii). Furthermore, $(x, y) \in \eta_A \cap \eta_B$ is equivalent to $(g.x, g.y) \in \eta$ for all $g \in A \cup B$, i.e. $(x, y) \in \eta_{A\cup B}$ and (iv) follows. To show (v) let $(x, z) \in \eta_A \kappa_A$. Then there exists $y \in X$ with $(x, y) \in \eta_A$ and $(y, z) \in \kappa_A$. Thus, for all $a \in A$ there holds $(a.x, a.y) \in \eta$ and $(a.y, a.z) \in \kappa$ and we obtain $(a.x, a.z) \in \eta\kappa$ for all $a \in A$, i.e. $(x, z) \in (\eta\kappa)_A$. □

The structure of the set of all Bowen entourages with respect to a fixed compact set can be described as follows.

Proposition 4.14. *For any non-empty and compact subset K of G the set $\{\eta_K; \eta \in \mathbb{U}_X\}$ is a base of the uniformity $\mathbb{U}_X$.*

Proof. Note first that Lemma 4.12 implies that $\mathbb{B}_K := \{\eta_K; \eta \in \mathbb{U}_X\} \subseteq \mathbb{U}_X$. We need to show that any $\eta \in \mathbb{U}_X$ contains an element of $\mathbb{B}_K$. To do this we consider $g \in K^{-1}$. As $e_G \in gK$ we obtain from Lemma 4.13 that there holds $\eta_{gK} \subseteq \eta$ and it remains to show $\eta_{gK} \in \mathbb{B}_K$. This however follows as Lemma 4.13 and $\eta_g \in \mathbb{U}_X$ allow to conclude $\eta_{gK} = (\eta_g)_K \in \mathbb{B}_K$. □

Remark 4.15. *The definition of the Bowen entourage is inspired by the definition of the Bowen metric d_A. This consept origins from R. Bowens visionary article [Bow71] and the related definition of topological entropy became in particular important in the elegant proof of the variational principle by M. Misiurewicz [Mis76]. To introduce this concept assume that X is metrizable and equipped with a metric d that generates the topology of X. For $A \subseteq G$ compact we define*

$$d_A(x, y) := \max_{a\in A} d(a.x, a.y)$$

for $x, y \in X$. This notion is well-defined as $A \ni g \mapsto d(\pi^g(x), \pi^g(y))$ is a continuous function on a compact space for all $x, y \in X$. It is straightforward to show, that d_A is a metric. The connection to the Bowen entourage can be formalized by $[d_A < \epsilon] = [d < \epsilon]_A$ and $[d_A \leq \epsilon] = [d \leq \epsilon]_A$ for all compact $A \subseteq G$ and $\epsilon > 0$. Note that Lemma 4.12 can *be seen as the natural generalization of the fact that all Bowen metrics with respect to π induce the same topology, as it easily follows from this lemma that they induce the same uniformity and the topology of a compact Hausdorff space can be reconstructed from the uniformity.*

We omit the straightforward proof of the following invariance properties of H^*_p, $H^*_{\mu,p}$, H_p and $H_{\mu,p}$. For the stataments in (i) and (ii) in the non-relative case see [Wal82, BS02]. The statement of (iii) follows from (i) and (ii).

Lemma 4.16. *Let μ be an invariant Borel probability measure. Then for any finite open cover $\mathcal{U}$, any finite partition α, any $\eta \in \mathbb{U}_X$ and any $g \in G$ there holds*

(i) $H^*_p(\mathcal{U}_g) = H^*_p(\mathcal{U})$.

(ii) $H^*_{\mu,p}(\alpha_g|\beta_g) = H^*_{\mu,p}(\alpha|\beta)$ *and in particular* $H^*_{\mu,p}(\alpha_g) = H^*_{\mu,p}(\alpha)$.

(iii) $H_p(\eta_g) = H_p(\eta)$ *and* $H_{\mu,p}(\eta_g) = H_{\mu,p}(\eta)$.

4.2.2 Relative topological and measure theoretical entropy

We can now define entropy for actions of non-discrete unimodular amenable groups. We will present below that this definition is equivalent to the classical one in the discrete case. Let μ be an invariant Borel probability measure on X and $\eta \in \mathbb{U}_X$. From Lemma 4.13(i) and Proposition 4.10(iv) we obtain that $\mathcal{K}(G) \ni A \mapsto H_p(\eta_A)$ and $\mathcal{K}(G) \ni A \mapsto H_{\mu,p}(\eta_A)$ are monotone. Furthermore, from Lemma 4.16(iii) it follows that these maps are right invariant and Lemma 4.13(iv) and Proposition 4.10(iii) imply the subadditivity. As G satisfies the Ornstein-Weiss Lemma the following limits exist independently from the choice of a Van Hove net $(A_i)_{i\in I}$ in G. We call

$$\mathrm{E}\left(\eta\middle|\pi \xrightarrow{p} \phi\right) := \lim_{i\in I} \frac{H_p(\eta_{A_i})}{\mu(A_i)}$$

the *relative topological entropy of p at scale η* and

$$\mathrm{E}_\mu\left(\eta\middle|\pi \xrightarrow{p} \phi\right) := \lim_{i\in I} \frac{H_{\mu,p}(\eta_{A_i})}{\mu(A_i)}$$

the *relative measure theoretical entropy of p at scale η*. Furthermore, we define

$$\mathrm{E}\left(\pi \xrightarrow{p} \phi\right) := \sup_{\eta\in\mathbb{U}_X} \mathrm{E}\left(\eta\middle|\pi \xrightarrow{p} \phi\right)$$

the *relative topological entropy of p* and

$$\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right) := \sup_{\eta\in\mathbb{U}_X} \mathrm{E}_\mu\left(\eta\middle|\pi \xrightarrow{p} \phi\right)$$

the *relative measure theoretical entropy of* p. If ϕ is an action on a single point, we write $\mathrm{E}(\pi) := \mathrm{E}\left(\pi \xrightarrow{p} \phi\right)$ and $\mathrm{E}_\mu(\pi) := \mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right)$ for the *topological entropy of* π and the *measure theoretical entropy of* π respectively and define similarly $\mathrm{E}(\eta|\pi)$ and $\mathrm{E}_\mu(\eta|\pi)$.

Remark 4.17. *(i) From Proposition 4.10(iv) and Lemma 4.13(ii) we obtain that there holds* $\mathrm{E}\left(\eta|\pi \xrightarrow{p} \phi\right) \geq \mathrm{E}\left(\kappa|\pi \xrightarrow{p} \phi\right)$ *and* $\mathrm{E}_\mu\left(\eta|\pi \xrightarrow{p} \phi\right) \geq \mathrm{E}_\mu\left(\kappa|\pi \xrightarrow{p} \phi\right)$, *whenever* $\eta \subseteq \kappa$. *Thus, there holds*

$$\mathrm{E}\left(\pi \xrightarrow{p} \phi\right) = \sup_{\eta \in \mathbb{B}_X} \mathrm{E}\left(\eta\middle|\pi \xrightarrow{p} \phi\right)$$

and a similar statement about measure theoretical entropy for any base $\mathbb{B}_X$ *of* $\mathbb{U}_X$.

(ii) If (X, d) is a compact metric space, we can choose the base $\{[d < \epsilon]; \epsilon > 0\}$. Whenever G is a discrete amenable group this yields the usual notion of topological entropy and measure theoretical entropy as discussed for example in [Wal75, BS02] for actions of $\mathbb{Z}$ and in [Oll85] for discrete amenable groups.

(iii) Recall that we denote $\langle \mathcal{U} \rangle := \bigcup_{u \in \mathcal{U}} U^2$ for any open cover $\mathcal{U}$ of X. Then the set of all $\langle \mathcal{U} \rangle$, where $\mathcal{U}$ is a finite open cover of X, is a base of the uniformity $\mathbb{U}_X$. Using this base one obtains the approach of [TZ91].

(iv) From Proposition 4.10(i) we obtain Goodwyn's half of the variational principle [Goo71]. For every $\eta \in \mathbb{U}_X$ and every invariant Borel probability measure μ on X there holds $\mathrm{E}_\mu(\eta|\pi \xrightarrow{p} \phi) \leq \mathrm{E}(\eta|\pi \xrightarrow{p} \phi)$ and in particular $\mathrm{E}_\mu(\pi \xrightarrow{p} \phi) \leq \mathrm{E}(\pi \xrightarrow{p} \phi)$.

(v) Consider the continuous rotation $\mathbb{R} \times \mathbb{T} \to \mathbb{T}$ with $(g, x) \mapsto g + x \mod 1$ and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. Then $A \mapsto H([d < \delta]_A) = H([d_A < \delta]) = H([d < \delta])$ is constant on $\mathcal{K}(\mathbb{R}) \setminus \{\emptyset\}$. Hence,

$$\sup_{A \in \mathcal{K}(\mathbb{R})} \frac{\log(H([d < \delta]_A))}{\theta(A)}$$

is not bounded. In order to define topological entropy we thus cannot add the assumption that f/θ is bounded to the assumptions on f in the Ornstein-Weiss Lemma if we want to apply this technique in order to define entropy.

(vi) Clearly, one would expect that the topological entropy of an action of a compact group is 0. Nevertheless, this is not satisfied for the given definition. Indeed, note that whenever G is a compact group, then $(G)_{n \in \mathbb{N}}$ is a Van Hove net in G and normalizing $\theta(G) = 1$ we obtain that

$$\mathrm{E}\left(\pi \xrightarrow{p} \phi\right) = \sup_{\eta \in \mathbb{U}_X} H_p(\eta_G) = \sup_{\eta \in \mathbb{U}_X} H_p(\eta).$$

In particular, whenever there is $y \in Y$ *such that* $p^{-1}(y)$ *consists of more than one element, then* $\mathrm{E}\left(\pi \overset{p}{\to} \phi\right) > 0$ *and in particular, whenever* X *consists of more than one element, then* $\mathrm{E}(\pi) > 0$.

(vii) *From Proposition 4.6 it is straightforward to conclude that whenever* $\pi \overset{p}{\to} \phi \overset{q}{\to} \psi$ *is a chain of factor maps and* μ *is an invariant Borel probability measure on the phase space of* π*, then there holds* $\mathrm{E}\left(\pi \overset{q\circ p}{\to} \psi\right) \geq \mathrm{E}\left(\pi \overset{p}{\to} \phi\right)$ *and* $\mathrm{E}_\mu\left(\pi \overset{q\circ p}{\to} \psi\right) \geq \mathrm{E}_\mu\left(\pi \overset{p}{\to} \phi\right)$.

It is natural to ask, whether entropy can also be defined using Følner or ergodic nets. These concepts are equivalent whenever G is discrete but already pairwise non-equivalent in $\mathbb{R}^d$. We will see in Remark 4.20 below that the concept of ergodic nets is not suitable in order to define entropy but present next, that one can indeed use Følner nets.

Proposition 4.18. *For every Følner net* $(A_i)_{i\in I}$ *there holds*

$$\mathrm{E}\left(\pi \overset{p}{\to} \phi\right) = \sup_{\eta\in\mathbb{U}_X} \liminf_{i\in I} \frac{H_p(\eta_{A_i})}{\theta(A_i)} = \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\eta_{A_i})}{\theta(A_i)}$$

and the statement remains valid, whenever we consider E_μ *and* $H_{\mu,p}$ *for an invariant Borel probability measure* μ.

Proof. For a compact neighbourhood M of 0 we obtain $(MA_i)_{i\in I}$ to be a Van Hove net. As $\eta_M \in \mathbb{U}_X$ for all $\eta \in \mathbb{U}_X$ and $\lim_{i\in I} \theta(MA_i)/\theta(A_i) = 1$ we compute

$$\begin{aligned}
\mathrm{E}\left(\pi \overset{p}{\to} \phi\right) &= \sup_{\eta\in\mathbb{U}_X} \lim_{i\in I} \frac{H_p(\eta_{MA_i})}{\theta(MA_i)} = \sup_{\eta\in\mathbb{U}_X} \lim_{i\in I} \frac{H_p((\eta_M)_{A_i})}{\theta(A_i)} \\
&\leq \sup_{\epsilon\in\mathbb{U}_X} \liminf_{i\in I} \frac{H_p(\epsilon_{A_i})}{\theta(A_i)} \leq \sup_{\epsilon\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\epsilon_{A_i})}{\theta(A_i)} \\
&\leq \sup_{\epsilon\in\mathbb{U}_X} \lim_{i\in I} \frac{H_p(\epsilon_{MA_i})}{\theta(MA_i)} = \mathrm{E}\left(\pi \overset{p}{\to} \phi\right).
\end{aligned}$$

The proof of the statement about measure theoretical entropy is similar. □

Remark 4.19. *It remains open, whether one can define relative entropy at a certain scale via Følner nets and whether the limit superior is a limit in the above formulas.*

Remark 4.20. *The formulas for topological and measure theoretical entropy as presented for Følner nets do not hold for ergodic nets. We will discuss this next for the case of topological entropy, but note that the arguments can be drawn similarly for measure theoretical entropy. Consider any action* π *of* $\mathbb{R}$ *with* $\mathrm{E}(\pi) \in (0,\infty)$ *as for example presented in Example 6.32 below. In Theorem 5.5 we will see that* $F_n := \{1, \cdots, n\}$

allows to compute

$$\mathrm{E}(\pi) = \sup_{\eta \in \mathbb{U}_X} \liminf_{n\to\infty} \frac{H_p(\eta_{F_n})}{n}.$$

As $\mathrm{E}(\pi) > 0$ *there is* $\eta \in \mathbb{U}_X$ *such that* $\liminf_{n\to\infty} H(\eta_{F_n})/n =: c > 0$*. In particular, we thus observe*

$$\liminf_{n\to\infty} H\left(\eta_{F_{n^2}}\right)/n^2 \geq c > 0.$$

From Proposition 2.22 we obtain that $(A_n)_{n\in\mathbb{N}}$ *with* $A_n := [0,n] \cup F_{n^2}$ *is an ergodic net and conclude from* $A_n \supseteq F_{n^2}$ *that*

$$\sup_{\eta\in\mathbb{U}_X} \liminf_{n\to\infty} \frac{H(\eta_{A_n})}{\theta(A_n)} \geq \liminf_{n\to\infty} \frac{H(\eta_{A_n})}{\theta(A_n)} \geq \liminf_{n\to\infty} \frac{H\left(\eta_{F_{n^2}}\right)}{n} = \infty > \mathrm{E}(\pi).$$

4.3 Some approaches to entropy

Topological and measure theoretical entropy are well-studied concepts. In this section we present that generalizing the classical theory of actions of $\mathbb{Z}$ to more general groups can be done into different non-equivalent directions. We will furthermore see that the well-known approach of [Bow71] via spanning and separating sets can be used in order to obtain a different approach to our notion of topological entropy.

4.3.1 Entropy along thin Følner nets

In [ST18] a variation of Følner nets is presented that is not equivalent to our definition but which we want to discuss next as this notion allows a non-equivalent approach to entropy. In order to separate this concept from our concept of Følner nets, we will refer to it as "thin Følner nets". To define it let us consider finite subsets E, F of a unimodular group G and an open neighbourhood V of e_G. We define the *V-matching number of E and F* as the maximal cardinality of a subset M of E such that there exists an injection $b\colon M \to F$ such that $\phi(e) \in Ve$ holds for any $e \in M$. We denote $\mathfrak{m}_V(E,F)$ for this number. We say that a net $(F_i)_{i\in I}$ of non-empty and finite subsets of G is a *thin Følner net*, whenever for any $g \in G$ and any open neighbourhood V of e_G there holds

$$\lim_{i\in I} \frac{\mathfrak{m}_V(F_i, gF_i)}{|F_i|} = 1.$$

In [ST18, Remark 4.6] it is presented that a unimodular group is amenable if and only there exists a thin Følner net in G. Furthermore, as one can consider $V = \{e_G\}$ in a discrete group the concepts of thin Følner nets and of Følner nets agree for discrete groups.

The advantage of thin Følner nets in comparison with Følner or Van Hove nets is that they consist of finite sets, which allows to use the concept of the common refinement

$\mathcal{U}_F$ and α_F for open covers $\mathcal{U}$ and finite partitions α respectively. It is natural to ask, whether taking the supremum over all finite open covers $\mathcal{U}$ in

$$\sup_{\mathcal{U}} \limsup_{i\in I} \frac{H^*_\mu(\alpha_{F_i})}{|F_i|} \tag{4.1}$$

yields the topological entropy $\mathrm{E}(\pi)$ for all thin Følner nets $(F_i)_{i\in I}$. Similarly one can ask, whether taking the supremum over all finite partitions α in

$$\sup_{\alpha} \limsup_{i\in I} \frac{H^*_\mu(\alpha_{F_i})}{|F_i|} \tag{4.2}$$

yields the measure theoretical entropy $\mathrm{E}_\mu(\pi)$ for an invariant Borel probability measure μ. We will see in this subsection that this is not the case already for actions of $\mathbb{R}$.

Remark 4.21. *Whenever G is a discrete amenable group, then we have already seen above that the concepts of thin Følner nets and Følner nets agree. From Remark 4.29 in combination with Proposition 4.28 and Corollary 5.17 below we will see that (4.1) and (4.2) indeed yield the topological and the measure theoretical entropy for actions of discrete groups. We will furthermore see that the Ornstein-Weiss Lemma can be applied for actions of discrete amenable groups to yield that the limit superior is a limit and that the notions are independent of the choice of a Følner net. It remains open, whether (4.1) and (4.2) are independent of the choice of a thin Følner net for all unimodular amenable groups and whether the limit superior is always a limit.*

In a communication Friedrich Martin Schneider suggested to consider the approach in [Oll85] and explained how to follow ideas of [Oll85, Proposition 3.1.9] to show that one can generalize another generalization of Fekete's lemma (Theorem 3.2) for thin Følner nets. To be a bit more precise it seems that one can show that for maps $f\colon \mathcal{F}(G) \to [0,\infty)$, that are strongly subadditve, right invariant and furthermore continuous with respect to a suitable topology on $\mathcal{F}(G)$, the limit $\lim_{i\in I} f(F_i)/|F_i|$ exists, is finite and independent of the choice of a thin Følner net $(F_i)_{i\in I}$. Such a theorem would be applicable to (4.2) (but not to (4.1) [DFR16]) and could be used to yield the existence of the limit as well as the independence from a thin Følner net in (4.2).

Following the ideas of [Oll85, Section 5.2] it also seems that one can show a version of the variational principle for the entropy along thin Følner nets, which could yield that (4.1) is also independent of the choice of a thin Følner net. Unfortunately carrying out these ideas would leave it open, whether the limit superior in (4.1) is a limit. As we will see next that this approach is not suitable in order to study the patch counting entropy, we did not include further details or investigations into this direction into this thesis.

We will discuss next that our approach to entropy and the approach along thin Følner nets are non-equivalent. Let us start our discussion with an example of a thin Følner net.

Example 4.22. *Let* $I := \mathbb{Z} \times \mathbb{Z}$ *be ordered component wise and consider* $F_{(n,m)} := [0, 2^n] \cap (2^{-m}\mathbb{Z})$*. Then* $(F_i)_{i \in I}$ *is a thin Følner net in* $\mathbb{R}$*.*

Proof. Consider $\epsilon > 0$, $g \in \mathbb{R}$ and set $V := B_\epsilon(0)$. For $\delta \in (0,1)$ we obtain for sufficiently large $m, n \in \mathbb{Z}$ that there holds $(2^{-m}\mathbb{Z} + g) \cap V \neq \emptyset$ and furthermore $\theta([\epsilon, 2^n - \epsilon] \cap [\epsilon + g, 2^n - \epsilon + g])/2^{n+1} \geq \delta$. Now consider

$$E := [\epsilon, 2^n - \epsilon] \cap [\epsilon + g, 2^n - \epsilon + g] \cap (2^{-m}\mathbb{Z}) \subseteq F_{n,m}.$$

We choose $h \in (2^{-m}\mathbb{Z} + g) \cap V$. Then for any $e \in E$ there holds

$$e + h \in [\epsilon + g, 2^n - \epsilon + g] + V \subseteq [g, 2^n + g].$$

Furthermore, we obtain $e + h \in 2^{-m}\mathbb{Z} + 2^{-m}\mathbb{Z} + g = 2^{-m}\mathbb{Z} + g$ and conclude that $e + h \in F_{(n,m)} + g$. As $b \colon E \to F_{(n,m)} + g$ that maps $e \mapsto e + h$ is an injection we obtain

$$\frac{\mathfrak{m}_V(F_{(n,m)}, F_{(n,m)} + g)}{|F_{(n,m)}|} \geq \frac{|E|}{2^{n+1} \cdot 2^m} \geq \frac{\theta([\epsilon, 2^n - \epsilon] \cap [\epsilon + g, g + 2^n - \epsilon]) \cdot 2^m}{2^{n+1} \cdot 2^m} \geq \delta.$$

This shows that there holds $\lim_{(n,m) \in I} \mathfrak{m}_V(F_{(n,m)}, F_{(n,m)} + g)/|F_{(n,m)}| = 1$. □

Let us denote $\mathcal{F} := (F_i)_{i \in I}$ for the thin Følner net from Example 4.22 and let π be an action on X. We denote the *topological entropy along* $\mathcal{F}$ by

$$\widetilde{\mathrm{E}}^{(\mathcal{F})}(\pi) := \sup_{\mathcal{U}} \limsup_{i \in I} \frac{H_\mu(\mathcal{U}_{F_i})}{|F_i|},$$

where the supremum is taken over all finite open covers $\mathcal{U}$ of X. Let us furthermore denote for all invariant Borel probability measures μ on X the *measure theoretic entropy along* $\mathcal{F}$ by

$$\widetilde{\mathrm{E}}_\mu^{(\mathcal{F})}(\pi) := \sup_{\alpha} \limsup_{i \in I} \frac{H_\mu(\alpha_{F_i})}{|F_i|},$$

where the supremum is taken over all finite partitions α of X.

Let us now consider any action π of $\mathbb{R}$ such that the topological entropy of π is finite but non-zero. From the variational principle (Theorem 5.33 below) we then obtain that there is also an invariant Borel probability measure μ on X such that the respective measure theoretical entropy is finite but non-zero. For an example of such an action see Example 6.32 below. We thus obtain from the following proposition that the approach along thin Følner nets to entropy and our approach to entropy are non-equivalent. In particular, this yields that the approach along thin Følner nets is not compatible with the patch counting entropy considered in the study of aperiodic order [Lag99, LP03, BLR07, HR15] as we will see in Chapter 6.

Proposition 4.23. *Let π be an action of $\mathbb{R}$ and denote for $r \in (0,\infty)$ by $\pi^{r(\cdot)}$ the action of $\mathbb{R}$ that is defined by $\mathbb{R} \times X \ni (g,x) \mapsto (r \cdot g, x)$. Then there holds*

(i) $\mathrm{E}\left(\pi^{r(\cdot)}\right) = r \cdot \mathrm{E}(\pi)$ *for any $r \in (0,\infty)$ and*

(ii) $\widetilde{\mathrm{E}}^{(\mathcal{F})}\left(\pi^{2(\cdot)}\right) = \widetilde{\mathrm{E}}^{(\mathcal{F})}(\pi)$.

For any invariant Borel probability measure μ on the phase space of π there holds

(iii) $\mathrm{E}_\mu\left(\pi^{r(\cdot)}\right) = r \cdot \mathrm{E}_\mu(\pi)$ *for any $r \in (0,\infty)$ and*

(iv) $\widetilde{\mathrm{E}}_\mu^{(\mathcal{F})}\left(\pi^{2(\cdot)}\right) = \widetilde{\mathrm{E}}_\mu^{(\mathcal{F})}(\pi)$.

Proof. We only show the statements (i) and (ii) as the statements (iii) and (iv) can be achieved with similar arguments. For the proof of (i) we need to keep track of the action we use to compute the Bowen entourage and thus for $\eta \in \mathbb{U}_X$ and $A \subseteq G$ compact we denote $\eta_A^{(s)} := \eta_A$ whenever we compute the Bowen entourage with respect to the action $\pi^{s(\cdot)}$ for $s \in \mathbb{R}$. In particular, we obtain $\eta_A^{(1)}$ to be the Bowen entourage with respect to π and furthermore that $\eta_{[0,n]}^{(r)} = \eta_{[0,r\cdot n]}^{(1)}$. As $([0,n])_{n\in\mathbb{N}}$ and $([0,r\cdot n])_{n\in\mathbb{N}}$ are Van Hove sequences in $\mathbb{R}$ we thus compute

$$\mathrm{E}\left(\eta|\pi^{r(\cdot)}\right) = \lim_{n\to\infty} \frac{H\left(\eta_{[0,n]}^{(r)}\right)}{\theta([0,n])} = r \cdot \lim_{n\to\infty} \frac{H\left(\eta_{[0,r\cdot n]}^{(1)}\right)}{\theta([0,r\cdot n])} = r \cdot \mathrm{E}\left(\eta|\pi\right)$$

and taking the supremum over all $\eta \in \mathbb{U}_X$ yields (i). To show (ii) note first that for $(n,m) \in I$ there holds

$$2 \cdot F_{(n,m)} = 2 \cdot \left([0,2^n] \cap (2^{-m}\mathbb{Z})\right) = \left([0,2^{n+1}] \cap (2^{-(m-1)}\mathbb{Z})\right) = F_{(n+1,m-1)}$$

and in particular $|F_{(n,m)}| = |2 \cdot F_{(n,m)}| = |F_{(n+1,m-1)}|$. Similar as above we need to add which action is considered in order to compute $\mathcal{U}_F$ for a finite open cover $\mathcal{U}$ and a finite set $F \subseteq G$. We will write $\mathcal{U}_F^{(s)}$, whenever we consider the action $\pi^{s(\cdot)}$ for some $s \in \mathbb{R}$. For a finite open cover $\mathcal{U}$ we compute

$$\limsup_{(n,m)\in\mathbb{Z}^2} \frac{H_\mu\left(\mathcal{U}^{(2)}_{F_{(n,m)}}\right)}{|F_{(n,m)}|} = \limsup_{(n,m)\in\mathbb{Z}^2} \frac{H_\mu\left(\mathcal{U}^{(1)}_{F_{(n+1,m-1)}}\right)}{|F_{(n+1,m-1)}|} = \limsup_{(n,m)\in\mathbb{Z}^2} \frac{H_\mu\left(\mathcal{U}^{(1)}_{F_{(n,m)}}\right)}{|F_{(n,m)}|}$$

and taking the supremum over all finite open covers $\mathcal{U}$ yields (ii). □

4.3.2 Topological generator entropy

In [Sch15] another non-equivalent approach to topological entropy of actions of certain not necessarily discrete groups was discussed. This approach also generalizes the original approach [AKM65] and defines entropy for certain compactly generated but not necessary amenable topological groups. It is inspired from the study of finitely generated (semi)groups, which is done for example in [Ave72, GLW88, Bis92, Bis04, BU06, MW11, WM15]. To present this approach we drop the assumption on G to be unimodular amenable and assume that G is a compactly generated topological group. Consider furthermore an action π of such a group on a compact Hausdorff space X. Let $A \subseteq G$ be a compact subsets and consider families $\mathcal{U}$ and $\mathcal{V}$ of subsets of X. Recall that we define $\mathcal{U}_g := \{(\pi^g)^{-1}(U);\, U \in \mathcal{U}\}$. We say that $\mathcal{V}$ *A-refines* $\mathcal{U}$ and write $\mathcal{U} \preceq_A \mathcal{V}$, if for any $g \in A$ there holds $\mathcal{U}_g \preceq \mathcal{V}$. It is presented in [Sch15, Lemma 3.2] that whenever A is a compact subset of G and $\mathcal{U}$ is a finite open cover of X, then there exists a finite open cover $\mathcal{V}$ of X that A-refines $\mathcal{U}$. We define $N_X(\mathcal{U}, A)$ as the minimum cardinality of an open cover of X that A-refines $\mathcal{U}$. Consider any compact subset $S \subseteq G$ that generates G, i.e. that satisfies $\bigcup_{n\in\mathbb{N}} S^n$ with $S^{n+1} := SS^n$ and $S^1 := S$. We define the *topological generator entropy of $\mathcal{U}$ with respect to* π (and S) as

$$\mathrm{E}_{\mathrm{gen}}(\mathcal{U}, \pi, S) := \limsup_{n\to\infty} \frac{\log(N_X(\mathcal{U}, S^n))}{n}.$$

The *topological generator entropy of* π (and S) is $\mathrm{E}_{\mathrm{gen}}(\pi, S) := \sup_{\mathcal{U}} E_{\mathrm{gen}}(\mathcal{U}, \pi, S)$, where the supremum is taken over all open covers $\mathcal{U}$ of X. Unfortunately this approach is not independent of the choice of a generating set S, but it can be shown that whenever the topological generator entropy is 0 with respect to any generating set, then it is 0 with respect to any other generating set [Sch15]. Thus, whenever G itself is compact, then there holds $G^n = G$ and we obtain the topological generator topology to be 0. This is not satisfied for topological entropy as presented in Remark 4.17 and gives a first hint that topological entropy and topological generator entropy are not equivalent concepts. In fact we will see in Subsection 4.3.3 below, that there holds

$$\mathrm{E}\left(\pi \overset{p}{\to} \phi\right) = \sup_{\mathcal{U}} \lim_{i\in I} \frac{\log(N_X(\mathcal{U}, A_i))}{\theta(A_i)}, \tag{4.3}$$

where the supremum is taken over all open covers $\mathcal{U}$ of X and $(A_i)_{i\in I}$ is a Van Hove net. We can thus consider the generator $S := [-1/2, 1/2]^d$ in $\mathbb{R}^d$ and define $A_n := S^n = [-n/2, n/2]^d$ to obtain $\mathrm{E}\left(\pi \overset{p}{\to} \phi\right) = \sup_{\mathcal{U}} \lim_{n\to\infty} \log(N_X(\mathcal{U}, S^n))/n^d$. Thus, these approaches are equivalent if and only if $d = 1$ and a similar argument shows that a similar statement holds for $\mathbb{Z}^d$. As hinted in (4.3) one can use some of the ideas of [Sch15] to obtain a notion of entropy that is equivalent to ours. We will discuss this approach next.

Remark 4.24. *As the topological generator entropy of an action* π *of* $\mathbb{R}$ *equals the topological entropy of* π *we obtain from the considerations in Subsection 4.3.1 that the topological generator entropy and the topological entropy along thin Følner nets are non-equivalent concepts. Thus, the concepts of toplogical entropy, topological generator entropy, and of the topological entropy along thin Følner nets are three pairwise non-equivalent concepts already for actions of* $\mathbb{R}^d$ *with* $d \neq 1$.

4.3.3 Relative topological entropy via open covers

We return to our standard assumption that G is a unimodular amenable group and present next how the ideas from [Sch15] can be used to give an approach to our notion of relative topological entropy that generalizes [AKM65]. Recall that for families $\mathcal{U}$ and $\mathcal{V}$ of subsets of X and a compact subset $A \subseteq G$ we say that $\mathcal{V}$ A-refines $\mathcal{U}$, whenever $\mathcal{U}_g \preceq \mathcal{V}$ holds for all $g \in A$. As mentioned above it is shown in [Sch15, Lemma 3.2] that whenever $\mathcal{U}$ is an open cover of X and A is compact there is an open cover of X that A-refines $\mathcal{U}$. Thus, in particular for any subset $M \subseteq X$ there is a finite family of open subsets of X that A-refines $\mathcal{U}$ and covers M. Extending the definition of [Sch15] to a relative setting we define $N_M(\mathcal{U}, A)$ as the minimal cardinality of such a family. We set

$$N_p(\mathcal{U}, A) := \sup_{y \in Y} N_{p^{-1}(y)}(\mathcal{U}, A)$$

and $H_p^*(\mathcal{U}, A) := \log N_p(\mathcal{U}, A)$. It is straightforward to show that $\mathcal{K}(G) \ni A \mapsto H_p^*(\mathcal{U}, A)$ is a monotone, right invariant and subadditive mapping. We can thus use the Ornstein-Weiss Lemma and any Van Hove net to define the *relative topological entropy of* p *and* $\mathcal{U}$ as

$$\mathrm{E}^*\left(\mathcal{U}|\pi \xrightarrow{p} \phi\right) := \lim_{i \in I} \frac{H_p^*(\mathcal{U}, A_i)}{\theta(A_i)}.$$

We furthermore define the *topological entropy of* $\mathcal{U}$ as the relative topological entropy with respect to the one point factor and denote $\mathrm{E}(\mathcal{U}|\pi)$. We will now show that the relative topological entropy of p is the supremum over the relative topological entropies of p of all open covers $\mathcal{U}$ of X.

Before we do this recall that we define $\langle \mathcal{U} \rangle := \bigcup_{U \in \mathcal{U}} U^2$ and thus obtain $\langle \mathcal{U} \rangle \in \mathbb{U}_X$ whenever $\mathcal{U}$ is an open cover of X. It is thus natural to ask, whether the defined concepts agree on a more fundamental level, i.e. whether there holds $\mathrm{E}^*(\mathcal{U}|\pi) = \mathrm{E}(\langle \mathcal{U} \rangle|\pi)$ or even $H^*(\mathcal{U}, A) = H(\langle \mathcal{U} \rangle_A)$ for all compact subsets $A \subseteq G$. The next example shows that this is not the case.

Example 4.25. *Let* $X = \{1,2,3\}^{\mathbb{Z}}$, *where we equip* $\{1,2,3\}$ *with the discrete topology and* X *with the the product topology. Considering the shift map* $\pi(a, (x_n)_{n \in \mathbb{N}}) := (x_{a+n})_{n \in \mathbb{N}}$ *we obtain an action on* X. *For* $i \in \{1,2,3\}$ *let* U_i *be the set of all sequences* $(x_n)_{n \in \mathbb{Z}}$ *such that* $x_0 \neq i$ *and* $\mathcal{U} := \{U_1, U_2, U_3\}$. *Then for any two sequences* $x, y \in X$ *there is* $i \in \{1,2,3\}$ *such that* $x, y \in U_i$ *and we obtain* $(x, y) \in \langle \mathcal{U} \rangle$. *This shows*

$\langle\mathcal{U}\rangle = X^2$ and in particular that $\langle\mathcal{U}\rangle_A = X^2$ for all $A \subseteq \mathbb{Z}$ compact. As $\{X\}$ is an open partition at scale X^2 we thus obtain $H(\langle\mathcal{U}\rangle_A) = \log 1 = 0$ for all $A \subseteq \mathbb{Z}$ compact and we obtain $\mathrm{E}(\langle\mathcal{U}\rangle|\pi) = 0$. Furthermore, for $A_i := \{-i, \cdots, i\}$ consider any open cover $\mathcal{V}$ of X that A_i-refines $\mathcal{U}$. Then for any $V \in \mathcal{V}$ there exists a sequence $(M_n)_{n=-i}^{i}$ of subsets of $\{1,2,3\}$ of cardinality 2, such that any sequence $(x_n)_{n\in\mathbb{Z}} \in V$ satisfies that $x_n \in M_n$ for $n \in A_i$. As one needs more than 2^{2i+1} of such V to cover X we obtain $|\mathcal{V}| \geq 2^{2i+1}$. Considering $\{U_1, U_2\}_{A_i}$ we obtain that in fact there are open covers of X of cardinality 2^{2i+1} that A_i-refine $\mathcal{U}$. Thus, there holds $H^*(\mathcal{U}, A_i) = 2^{2i+1}$. Taking the cardinality as a Haar measure on $\mathbb{Z}$ we thus obtain $\mathrm{E}^*(\mathcal{U}|\pi) = \log(2)$ and in particular $\mathrm{E}^*(\mathcal{U}|\pi) \neq \mathrm{E}(\langle\mathcal{U}\rangle|\pi)$.

In Example 4.25 we we have seen that there can be open covers $\mathcal{U}$ of X such that $\mathrm{E}^*(\mathcal{U}|\pi) > \mathrm{E}(\langle\mathcal{U}\rangle|\pi)$ and the question remains whether the inequality $\mathrm{E}^*(\mathcal{U}|\pi) \geq \mathrm{E}(\langle\mathcal{U}\rangle|\pi)$ is a general phenomenon. In fact as $\mathcal{U}$ is at scale $\langle\mathcal{U}\rangle$ we obtain that the inequality is always satisfied from the following.

Lemma 4.26. *Let $\mathcal{U}$ be an open cover of X at scale $\eta \in \mathbb{U}_X$. Then there holds $H_p^*(\mathcal{U}, A) \geq H_p(\eta_A)$ for any compact subset $A \subseteq G$. In particular, there holds*

$$\mathrm{E}^*(\mathcal{U}|\pi \xrightarrow{p} \phi) \geq \mathrm{E}(\eta|\pi \xrightarrow{p} \phi).$$

Proof. Let $y \in Y$ and choose a family $\mathcal{V}$ of open subsets of X that A-refines $\mathcal{U}$ and covers $p^{-1}(y)$ of minimal cardinality $N_{p^{-1}(y)}(\mathcal{U}, A)$. As $\mathcal{U}$ is at scale η and $\mathcal{U} \preceq_A \mathcal{V}$ we obtain that $\mathcal{V}$ is at scale η_A. Recall that we defined in Remark 4.2 that $\mathrm{cov}_{p^{-1}(y)}(\eta_A)$ is the minimal cardinality of a family of open and η_A-small sets that cover $p^{-1}(y)$. Thus, there holds $N_{p^{-1}(y)}(\mathcal{U}, A) = |\mathcal{V}| \geq \mathrm{cov}_{p^{-1}(y)}(\eta_A)$ and Remark 4.2 yields

$$H_p^*(\mathcal{U}, A) = \log\left(\sup_{y\in Y} N_{p^{-1}(y)}(\mathcal{U}, A)\right) \geq \log\left(\sup_{y\in Y} \mathrm{cov}_{p^{-1}(y)}(\eta_A)\right) = H_p(\eta_A).$$

□

Considering Lebesgue entourages we obtain the reverse inequality.

Lemma 4.27. *If η is a Lebesgue entourage of an open cover $\mathcal{U}$ of X, then there holds $H_p^*(\mathcal{U}, A) \leq H_p(\eta_A)$ for any compact subset $A \subseteq G$. Thus, in particular there holds*

$$\mathrm{E}^*(\mathcal{U}|\pi \xrightarrow{p} \phi) \leq \mathrm{E}(\eta|\pi \xrightarrow{p} \phi).$$

Proof. By Remark 4.2 there holds $H_p(\eta_A) = \log(\sup_{y\in Y} \mathrm{cov}_{p^{-1}(y)}(\eta_A))$. Let now $y \in Y$. To show $\mathrm{cov}_{p^{-1}(y)}(\eta_A) \leq N_{p^{-1}(y)}(\mathcal{U}, A)$ consider a family of open and η_A-small sets that cover $p^{-1}(y)$ of cardinality $\mathrm{cov}_{p^{-1}(y)}(\eta_A)$. For $g \in A$ observe that η_g is a Lebesgue entourage of $\mathcal{U}_g$. Thus, for non-empty $V \in \mathcal{V}$ there is $v \in V$ and we obtain the existence of $U \in \mathcal{U}_g$ such that $V = V^2[v] \subseteq \eta_g[v] \subseteq U$. Thus, $\mathcal{U}_g \preceq \mathcal{V}$ for all $g \in A$ and we have shown that $\mathcal{V}$ is a family of open subsets of X that A-refines $\mathcal{U}$ and covers $p^{-1}(y)$. We thus obtain $N_{p^{-1}(y)}(\mathcal{U}, A) \leq |\mathcal{V}| = \mathrm{cov}_{p^{-1}(y)}(\eta_A)$ for all $y \in Y$ and conclude from Remark 4.2 that there holds

$$H_p(\eta_A) = \log\left(\sup_{y\in Y} \mathrm{cov}_{p^{-1}(y)}(\eta_A)\right) \geq \log\left(N_p(\mathcal{U}, A)\right) = H_p^*(\mathcal{U}, A).$$

□

Proposition 4.28. *There holds*

$$\mathrm{E}\left(\pi \xrightarrow{p} \phi\right) = \sup_{\mathcal{U}} \mathrm{E}^*\left(\mathcal{U}|\pi \xrightarrow{p} \phi\right),$$

where the supremum is taken over all open covers $\mathcal{U}$ of X. Furthermore, for any Følner net $(A_i)_{i\in I}$ there holds

$$\mathrm{E}\left(\pi \xrightarrow{p} \phi\right) = \sup_{\mathcal{U}} \liminf_{i\in I} \frac{H_p^*(\mathcal{U}, A_i)}{\theta(A_i)} = \sup_{\mathcal{U}} \limsup_{i\in I} \frac{H_p^*(\mathcal{U}, A_i)}{\theta(A_i)}.$$

Proof. The first statement follows directly from the Lemmas 4.26, 2.5 and 4.27. Furthermore, these lemmas imply

$$\sup_{\mathcal{U}} \liminf_{i\in I} \frac{H_p^*(\mathcal{U}, A_i)}{\theta(A_i)} = \sup_{\eta\in\mathbb{U}_X} \liminf_{i\in I} \frac{H_p(\eta_{A_i})}{\theta(A_i)}$$

for any Følner net $(A_i)_{i\in I}$. Thus, Proposition 4.18 yields the statement about the limit inferior. Similarly one shows the statement about the limit superior. □

Remark 4.29. *If G is assumed to be discrete then there holds $H_p^*(\mathcal{U}_F) = H_p^*(\mathcal{U}, F)$ for any compact, i.e. finite subset $F \subseteq G$. Thus, the approach in this subsection restricts to the classical approach for discrete amenable groups considered first in [AKM65] and used for example in [Oll85, HYZ11, Yan15, YZ16], where we are far from giving a full list of the important references.*

4.3.4 Relative topological entropy via spanning and separating sets

It is well-known that one can also define topological entropy of actions of discrete groups on compact metric spaces in terms of separated and of spanning sets [Bow71, Yan15]. In [Hoo74] this approach is generalized to $\mathbb{Z}$-actions of compact Hausdorff spaces. We give a brief recap as this approach is important in the context of aperiodic order [BLR07, JLO16, FGJO18] and as it can be studied using the Ornstein-Weiss Lemma implicitly.

For $\eta \in \mathbb{U}_X$ a subset $S \subseteq X$ is called *η-separated*, if for every $s \in S$ there is no further element in S that is η-close to s. Furthermore, we say that $S \subseteq X$ is *η-spanning* for

$M \subseteq X$, if for all $m \in M$ there is $s \in S$ such that s is $(\eta \cup \eta^{-1})$-close to m.

Remark 4.30. *A subset S of a metric space (X, d) is $[d < \epsilon]$-separated, if any two distinct points in S are at least ϵ apart, i.e. $d(x, y) \geq \epsilon$ for all $x, y \in S$ with $x \neq y$. Furthermore, S is $[d < \epsilon]$-spanning for $M \subseteq X$, if and only if for every $m \in M$ there is $s \in S$ such that $d(s, m) < \epsilon$.*

Recall from Remark 4.2 that we denote by $\mathrm{cov}_M(\eta)$ the minimal cardinality of a finite family of open and η-small sets that covers M. With similar arguments as used in metric spaces we obtain the following lemma. See [Wal82, BS02] for reference.

Lemma 4.31. *For $\eta \in \mathbb{U}_X$ and $M \subseteq X$ the cardinality of every η-separated subset $S \subseteq M$ is bounded from above by $\mathrm{cov}_M(\eta) < \infty$. In particular, there are finite η-separated subsets of M of maximal cardinality. Every η-separated subset $S \subseteq M$ of maximal cardinality is η-spanning for M. In particular, there are finite subsets of M that are η-spanning for M.*

For $\eta \in \mathbb{U}_X$ and $M \subseteq X$ we define $\mathrm{sep}_M(\eta)$ as the maximal cardinality of a subset of M that is η-separated and $\mathrm{spa}_M(\eta)$ as the minimal cardinality of a subset of M that is η-spanning for M. Furthermore, we define $\mathrm{sep}_{p,M}(\eta) := \sup_{y \in Y} \mathrm{sep}_{M \cap p^{-1}(y)}(\eta)$ and abbreviate $\mathrm{sep}_p(\eta) := \mathrm{sep}_{p,X}(\eta)$. Similarly one defines $\mathrm{spa}_{p,M}(\eta)$ and $\mathrm{spa}_p(\eta)$.

Unfortunately the Ornstein-Weiss Lemma cannot be applied directly to these notions. One thus relates these notions to H_p to show that they can be used to define entropy independently from the choice of a Følner net. We will furthermore see that it suffices to consider only subsets of a dense subset of X. This observation will become useful in Chapter 6.

Lemma 4.32. *Let $\eta \in \mathbb{U}_X$ and $D \subseteq G$ be such that $D \cap p^{-1}(y)$ is dense in $p^{-1}(y)$ for each $y \in Y$. Then there exists an entourage $\epsilon \in \mathbb{U}_X$ with $\epsilon \subseteq \eta$ such that for every compact $A \subseteq G$ there holds $H_p(\eta_A) \leq \log \mathrm{spa}_{p,D}(\epsilon_A) \leq \log \mathrm{sep}_{p,D}(\epsilon_A) \leq H_p(\epsilon_A)$.*

Proof. Recall from Remark 4.2 that there holds

$$H_p(\epsilon_A) = \log \left(\sup_{y \in Y} \mathrm{cov}_{p^{-1}(y)}(\epsilon_A) \right)$$

for any $\epsilon \in \mathbb{U}_X$ and any compact subset $A \subseteq G$. Thus, the second and the third inequality follow from Lemma 4.31. In order to show the first one let $\epsilon \in \mathbb{U}_X$ be symmetric and such that $\epsilon\epsilon\epsilon\epsilon \subseteq \eta$. For $A \subseteq G$ compact we calculate $\epsilon_A\epsilon_A\epsilon_A\epsilon_A \subseteq (\epsilon\epsilon\epsilon\epsilon)_A \subseteq \eta_A$. Let now $\kappa \in \mathbb{U}_X$ be open and symmetric such that $\kappa \subseteq \epsilon_A$. Let $y \in Y$ and $S \subseteq p^{-1}(y) \cap D$ be ϵ_A-spanning for $p^{-1}(y) \cap D$ and of minimal cardinality. Then $\{\kappa\epsilon_A[s];\ s \in S\}$ is an open cover of $p^{-1}(y)$. A straightforward argument shows this cover to be at scale $\kappa\epsilon_A\epsilon_A\kappa$ and we obtain $\mathrm{cov}_{p^{-1}(y)}(\kappa\epsilon_A\epsilon_A\kappa) \leq |S| = \mathrm{spa}_{p^{-1}(y)\cap D}(\epsilon_A)$. Thus,

$$\begin{aligned}\mathrm{cov}_{p^{-1}(y)}(\eta_A) &\leq \mathrm{cov}_{p^{-1}(y)}(\epsilon_A\epsilon_A\epsilon_A\epsilon_A)\\ &\leq \mathrm{cov}_{p^{-1}(y)}(\kappa\epsilon_A\epsilon_A\kappa)\\ &\leq \mathrm{spa}_{p^{-1}(y)\cap D}(\epsilon_A)\\ &\leq \mathrm{spa}_{p,D}(\epsilon_A).\end{aligned}$$

Taking the supremum over all $y \in Y$ we obtain from Remark 4.2 that

$$H_p(\eta_A) = \log\left(\sup_{y\in Y} \mathrm{cov}_{p^{-1}(y)}(\eta_A)\right) \leq \log \mathrm{spa}_{p,D}(\epsilon_A).$$

□

Theorem 4.33. *If D is a subset of X such that $D \cap p^{-1}(y)$ is dense in $p^{-1}(y)$ for all $y \in Y$, then there holds*

$$\mathrm{E}(\pi \xrightarrow{p} \phi) = \sup_{\eta\in\mathbb{B}_X} \liminf_{i\in I} \frac{\log(\mathrm{spa}_{p,D}(\eta_{A_i}))}{\theta(A_i)} = \sup_{\eta\in\mathbb{B}_X} \limsup_{i\in I} \frac{\log(\mathrm{spa}_{p,D}(\eta_{A_i}))}{\theta(A_i)}$$

for any Følner net $(A_i)_{i\in I}$ and any base $\mathbb{B}_X$ of $\mathbb{U}_X$. A similar statement holds for $\mathrm{sep}_{p,D}$*. In particular, if D is dense in X we obtain*

$$\mathrm{E}(\pi) = \sup_{\eta\in\mathbb{B}_X} \limsup_{i\in I} \frac{\log(\mathrm{spa}_D(\eta_{A_i}))}{\theta(A_i)} = \sup_{\eta\in\mathbb{B}_X} \limsup_{i\in I} \frac{\log(\mathrm{sep}_D(\eta_{A_i}))}{\theta(A_i)}.$$

Proof. As $\eta \mapsto \limsup_{i\in I} \log(\mathrm{spa}_{p,D}(\eta_{A_i}))/\mu(A_i)$ and the other similar terms are decreasing, it suffices to show the statement for $\mathbb{B}_X = \mathbb{U}_X$. By Lemma 4.32 it is immediate that for any $\eta \in \mathbb{U}_X$ there holds

$$\begin{aligned}\mathrm{E}(\eta|\phi \xrightarrow{p} \psi) &\leq \sup_{\epsilon\in\mathbb{U}_X} \liminf_{i\in I} \frac{\log(\mathrm{spa}_{p,D}(\epsilon_{A_i}))}{\theta(A_i)}\\ &\leq \sup_{\epsilon\in\mathbb{U}_X} \limsup_{i\in I} \frac{\log(\mathrm{spa}_{p,D}(\epsilon_{A_i}))}{\theta(A_i)} \leq \mathrm{E}(\phi \xrightarrow{p} \psi).\end{aligned}$$

Taking the supremum over η yields the first equality. Similar arguments show the statements about $\mathrm{sep}_{p,D}$. □

Theorem 4.33 in particular allows to easily show that the entropy of a factor is always smaller than the entropy of the extension.

Proposition 4.34. *For actions π, ϕ and ψ of G such that ϕ is a factor of π via p and ψ is a factor of ϕ via q, i.e. $\pi \xrightarrow{p} \phi \xrightarrow{p} \psi$, there holds $\mathrm{E}\left(\pi \xrightarrow{q\circ p} \psi\right) \geq \mathrm{E}\left(\phi \xrightarrow{q} \psi\right)$ and in particular $\mathrm{E}(\pi) \geq \mathrm{E}(\phi)$.*

Proof. Let us denote X, Y and Z for the phase spaces of π, ϕ and ψ respectively. For $\eta \in \mathbb{U}_Y$ let us furthermore abbreviate $\hat{\eta} := \{(x, x') \in X^2;\ (p(x), p(x')) \in \eta\}$ and observe $\hat{\eta} \in \mathbb{U}_X$. Whenever we consider a subset $M \subseteq Y$ and an $\hat{\eta}$-spanning set S for $p^{-1}(M)$, then one easily shows $p(S)$ to be η-spanning for M and we deduce $\operatorname{spa}_{p^{-1}(M)}(\hat{\eta}) \geq \operatorname{spa}_M(\eta)$. In particular, for $z \in Z$ we obtain $\operatorname{spa}_{(q\circ p)^{-1}(z)}(\hat{\eta}) \geq \operatorname{spa}_{q^{-1}(z)}(\eta)$. This shows $\operatorname{spa}_{p\circ q}(\hat{\eta}) \geq \operatorname{spa}_q(\eta)$ for any $\eta \in \mathbb{U}_Y$.

Now recall that p is a factor map which allows to show that for any compact subset $A \subseteq G$ there holds $\widehat{\eta_A} = (\hat{\eta})_A$. Considering any Van Hove net $(A_i)_{i\in I}$ in G we thus obtain

$$\begin{aligned}
\mathrm{E}\left(\pi \overset{q\circ p}{\to} \psi\right) &\geq \limsup_{i\in I} \frac{\log \operatorname{spa}_{q\circ p}(\hat{\eta}_{A_i})}{\theta(A_i)} \\
&\geq \limsup_{i\in I} \frac{\log \operatorname{spa}_q(\eta_{A_i})}{\theta(A_i)}
\end{aligned}$$

and taking the supremum over all $\eta \in \mathbb{U}_Y$ yields

$$\mathrm{E}\left(\pi \overset{q\circ p}{\to} \psi\right) \geq \sup_{\eta\in\mathbb{U}_Y} \limsup_{i\in I} \frac{\log \operatorname{spa}_q(\eta_{A_i})}{\theta(A_i)} = \mathrm{E}\left(\phi \overset{q}{\to} \psi\right).$$

Considering the one point flow for ψ we also obtain $\mathrm{E}(\pi) \geq \mathrm{E}(\phi)$. □

4.4 Properties of the relative entropy map

Consider a factor ϕ of an action π via factor map p. In this section we will collect properties of the *relative entropy map*

$$\mathrm{E}_{(\cdot)}\left(\pi \overset{p}{\to} \phi\right) : \mathcal{M}_G(X) \to [0, \infty] \colon \mu \mapsto \mathrm{E}_\mu\left(\pi \overset{p}{\to} \phi\right)$$

and of the *entropy map* $\mathrm{E}_{(\cdot)}(\pi)$ for later use.

4.4.1 Affinity

Recall that $\mathcal{M}_G(X)$ is always a convex set. It is well-known for actions of discrete amenable groups that the entropy map is always affine [HYZ11]. This statement is a straightforward generalization from [LW77, Wal82], where it is shown in the context of actions of $\mathbb{Z}$. Following a straightforward generalization of the corresponding arguments we next show that the (relative) entropy map is affine in our context. We will need this statement for later purposes and include the proof for the convenience of the reader.

Lemma 4.35. *Let $\lambda \in [0,1]$ and let μ and ν be Borel probability measures on X. For every entourage $\eta \in \mathbb{U}_X$ there holds*

$$0 \leq H_{\lambda\mu+(1-\lambda)\nu,p}(\eta) - \lambda H_{\mu,p}(\eta) - (1-\lambda)H_{\nu,p}(\eta) \leq 1.$$

Proof. For a finite partition α it is standard to show that

$$0 \leq H^*_{\lambda\mu+(1-\lambda)\nu,p}(\alpha) - \lambda H^*_{\mu,p}(\alpha) - (1-\lambda)H^*_{\nu,p}(\alpha) \leq -\lambda\log(\lambda) - (1-\lambda)\log(1-\lambda).$$

See [Wal82, Theorem 8.1] or [HYZ06, Lemma 3.3] for reference. As $-\lambda\log(\lambda) \leq 1/e \leq 1/2$ for $\lambda \in [0,1]$, we obtain that for every finite partition α of X there holds

$$0 \leq H^*_{\lambda\mu+(1-\lambda)\nu,p}(\alpha) - \lambda H^*_{\mu,p}(\alpha) - (1-\lambda)H^*_{\nu,p}(\alpha) \leq 1.$$

As the statement is trivial for $\lambda \in \{0,1\}$ we assume without lost of generality that $\lambda \in (0,1)$. Let now $\epsilon > 0$. Then by definition there are finite partitions α_1, α_2 and α_3 such that $H_{(\lambda\mu+(1-\lambda)\nu),p}(\eta) \geq H^*_{(\lambda\mu+(1-\lambda)\nu),p}(\alpha_1) - \epsilon$, $H_{\mu,p}(\eta) \geq H^*_{\mu,p}(\alpha_2) - \epsilon$ and $H_{\nu,p}(\eta) \geq H^*_{\nu,p}(\alpha_3) - \epsilon$. In particular, considering $\alpha := \alpha_1 \vee \alpha_2 \vee \alpha_3$ we obtain

$$\begin{aligned}
-\epsilon &\leq H^*_{\lambda\mu+(1-\lambda)\nu,p}(\alpha) - \epsilon - \lambda H^*_{\mu,p}(\alpha) - (1-\lambda)H^*_{\nu,p}(\alpha)\\
&\leq H_{\lambda\mu+(1-\lambda)\nu,p}(\eta) - \lambda H_{\mu,p}(\eta) - (1-\lambda)H_{\nu,p}(\eta)\\
&\leq H^*_{\lambda\mu+(1-\lambda)\nu,p}(\alpha) - \lambda H^*_{\mu,p}(\alpha) - (1-\lambda)H^*_{\nu,p}(\alpha) + \epsilon\\
&\leq 1 + \epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary we obtain the statement. □

With the arguments from [Wal82, Theorem 8.1] we obtain the following.

Theorem 4.36. *Whenever G is non-compact, then the relative entropy map $\mathrm{E}_{(\cdot)}\left(\pi \xrightarrow{p} \phi\right)$ and the entropy map $\mathrm{E}_{(\cdot)}(\pi)$ are affine. Furthermore, for every entourage $\eta \in \mathbb{U}_X$ the maps $\mathrm{E}_{(\cdot)}\left(\eta|\pi \xrightarrow{p} \phi\right)$ and $\mathrm{E}_{(\cdot)}(\eta|\pi)$ are affine.*

Proof. As the entropy map is the relative entropy map with respect to an action on a one point space it is sufficient to show the statements about the relative entropy map. Let $(A_i)_{i\in I}$ be a Van Hove net in G. From Lemma 4.35 we obtain that

$$\begin{aligned}
\lim_{i\in I}\frac{H_{(\lambda\mu+(1-\lambda)\nu),p}(\eta_{A_i})}{\theta(A_i)} &\leq \lambda\lim_{i\in I}\frac{H_{\mu,p}(\eta_{A_i})}{\theta(A_i)} + (1-\lambda)\lim_{i\in I}\frac{H_{\nu,p}(\eta_{A_i})}{\theta(A_i)} + \lim_{i\in I}\frac{1}{\theta(A_i)}\\
&= \lambda\lim_{i\in I}\frac{H_{\mu,p}(\eta_{A_i})}{\theta(A_i)} + (1-\lambda)\lim_{i\in I}\frac{H_{\nu,p}(\eta_{A_i})}{\theta(A_i)}\\
&\leq \lim_{i\in I}\frac{H_{(\lambda\mu+(1-\lambda)\nu),p}(\eta_{A_i})}{\theta(A_i)}.
\end{aligned}$$

Thus, we obtain

$$\mathrm{E}_{\lambda\mu+(1-\lambda)\nu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right) = \lambda\,\mathrm{E}_{\mu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right) + (1-\lambda)\,\mathrm{E}_{\nu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right).$$

Taking the supremum over all $\eta \in \mathbb{U}_X$ we obtain

$$\mathrm{E}_{\lambda\mu+(1-\lambda)\nu}\left(\pi \overset{p}{\to} \phi\right) \leq \lambda\,\mathrm{E}_{\mu}\left(\pi \overset{p}{\to} \phi\right) + (1-\lambda)\,\mathrm{E}_{\nu}\left(\pi \overset{p}{\to} \phi\right).$$

To show the remaining inequality let $\epsilon > 0$ and choose $\eta^\mu, \eta^\nu \in \mathbb{U}_X$ such that

$$\mathrm{E}_{\mu}\left(\eta^\mu\middle|\pi \overset{p}{\to} \phi\right) \geq \begin{cases} \mathrm{E}_{\mu}\left(\pi \overset{p}{\to} \phi\right) - \epsilon, & \text{whenever } \mathrm{E}_{\mu}\left(\pi \overset{p}{\to} \phi\right) < \infty \\ 1/\epsilon, & \text{whenever } \mathrm{E}_{\mu}\left(\pi \overset{p}{\to} \phi\right) = \infty. \end{cases}$$

and a similar statement concerning ν. Then $\eta := \eta^\mu \cap \eta^\nu \in \mathbb{U}_X$. If $\mathrm{E}_\mu(\pi \overset{p}{\to} \phi)$ and $\mathrm{E}_\nu(\pi \overset{p}{\to} \phi)$ are finite we compute

$$\begin{aligned}\mathrm{E}_{\lambda\mu+(1-\lambda)\nu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right) &= \lambda\,\mathrm{E}_{\mu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right) + (1-\lambda)\,\mathrm{E}_{\nu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right) \\ &\geq \lambda\,\mathrm{E}_{\mu}\left(\pi \overset{p}{\to} \phi\right) + (1-\lambda)\,\mathrm{E}_{\nu}\left(\pi \overset{p}{\to} \phi\right) - \epsilon.\end{aligned}$$

As $\epsilon > 0$ was arbitrary we obtain the statement in this case. Note that the statement is trivial for $\lambda \in \{0,1\}$. We thus assume w.l.o.g. that $\lambda \in (0,1)$. If $\mathrm{E}_\mu(\pi \overset{p}{\to} \phi) = \infty$ or $\mathrm{E}_\nu(\pi \overset{p}{\to} \phi) = \infty$, then

$$\begin{aligned}\mathrm{E}_{\lambda\mu+(1-\lambda)\nu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right) &= \lambda\,\mathrm{E}_{\mu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right) + (1-\lambda)\,\mathrm{E}_{\nu}\left(\eta\middle|\pi \overset{p}{\to} \phi\right) \\ &\geq \min\{\lambda, (1-\lambda)\}/\epsilon.\end{aligned}$$

As $\epsilon > 0$ was arbitrary we obtain $\mathrm{E}_{\lambda\mu+(1-\lambda)\nu}(\pi \overset{p}{\to} \phi) = \infty$ and the statement follows. □

4.4.2 Restriction to invariant subsets

Whenever $M \subseteq X$ is a closed and invariant subset, then $\pi|_{G\times M}$ is an action of G on M and we can identify an invariant Borel probability measure on X with support in M with its restriction to $\mathcal{B}_M$. This allows to identify $\mathcal{M}_G(M)$ as a subset of $\mathcal{M}_G(X)$. The relationship between the entropy maps of π and the restriction $\pi|_{G\times X'}$ can be summarized as follows. As we did not encounter this probably well-known statement in the literature we include a full and detailed proof.

Theorem 4.37. *Let $M \subseteq X$ be a closed and invariant subset. Then also $p(M)$ is closed and invariant and $\mathcal{M}_G(M)$ is a closed face of $\mathcal{M}_G(X)$. Furthermore, the respective relative entropy map $\mathrm{E}_{(\cdot)}\left(\pi|_{G\times M} \overset{p|_M}{\to} \phi|_{G\times p(M)}\right)$ is the restriction of $\mathrm{E}_{(\cdot)}(\pi \overset{p}{\to} \phi)$ to $\mathcal{M}_G(M)$ and in particular $\mathrm{E}_{(\cdot)}(\pi|_{G\times M})$ is the restriction of $\mathrm{E}_{(\cdot)}(\pi)$ to $\mathcal{M}_G(M)$.*

Before we present a proof we state the following corollary.

Corollary 4.38. *Let $M \subseteq X$ be a closed and invariant subset. Whenever the entropy map of π is upper semi-continuous, so is the entropy map of the restriction $\pi|_{G\times M}$.*

We begin the proof of the theorem with the following lemma.

Lemma 4.39. *Let $M \subseteq X$ be a closed subset. Then for all Borel measurable subsets A of X and all Borel measures μ with support in M the conditional expectation $\mathbb{E}_\mu\left(\chi_{A\cap M}\middle|(p|_M)^{-1}(\mathcal{B}_{p(M)})\right)$ is the restriction of $\mathbb{E}_\mu(\chi_A|p^{-1}(\mathcal{B}_Y))$ to $L^1(M,\mu)$.*

Proof. Denote by f the restriction of $\mathbb{E}_\mu(\chi_A|p^{-1}(\mathcal{B}_Y))$ to $L^1(M,\mu)$. We first show that f is measurable with respect to $(p|_M)^{-1}(\mathcal{B}_{p(M)})$. To do this consider a Borel measurable subset S of $\mathbb{R}$. As $(\mathbb{E}_\mu(\chi_A|p^{-1}(\mathcal{B}_Y)))^{-1}(S)$ is $p^{-1}(\mathcal{B}_Y)$-measurable there exists $N \in \mathcal{B}_Y$ such that $p^{-1}(N) = (\mathbb{E}_\mu(\chi_A|p^{-1}(\mathcal{B}_Y)))^{-1}(S)$. Thus

$$p^{-1}(N\cap p(M))\cap M \subseteq p^{-1}(N)\cap M \subseteq p^{-1}(N)\cap p^{-1}(p(M)) = p^{-1}(N\cap p(M))$$

implies

$$p^{-1}(N)\cap M = p^{-1}(N\cap p(M))\cap M = (p|_M)^{-1}(N\cap p(M)) \in (p|_M)^{-1}(\mathcal{B}_{p(M)})$$

and we obtain

$$f^{-1}(S) = \left(\left(\mathbb{E}_\mu(\chi_A|p^{-1}(\mathcal{B}_Y))\right)^{-1}(S)\right)\cap M = p^{-1}(N)\cap M \in (p|_M)^{-1}(\mathcal{B}_{p(M)}).$$

This shows that f is indeed $(p|_M)^{-1}(\mathcal{B}_{p(M)})$-measurable. As the support of μ is contained in M we compute

$$\begin{aligned}\int_M f d\mu &= \int_M \mathbb{E}_\mu(\chi_A|p^{-1}(\mathcal{B}_Y))d\mu = \int_X \mathbb{E}_\mu(\chi_A|p^{-1}(\mathcal{B}_Y))d\mu \\ &= \int_X \chi_A d\mu = \int_M \chi_{A\cap M} d\mu = \int_M \mathbb{E}_\mu\left(\chi_{A\cap M}\middle|(p|_M)^{-1}(\mathcal{B}_{p(M)})\right)d\mu.\end{aligned}$$

This shows that f is the conditional expectation $\mathbb{E}_\mu\left(\chi_{A\cap M}\middle|(p|_M)^{-1}(\mathcal{B}_{p(M)})\right)$. $\square$

We will furthermore need the next lemma.

Lemma 4.40. *Let $M \subseteq X$ be a closed subset. Then for all Borel measures μ with support in M and all $\eta \in \mathbb{U}_X$ there holds $H_{\mu,p}(\eta) = H_{\mu,p|_M}(\eta\cap(M)^2)$.*

Proof. We first show that $H_{\mu,p}(\eta) \leq H_{\mu,p|_M}(\eta\cap M^2)$. To do so, let α be a finite partition of M at scale $\eta\cap M^2$. Choose any finite partition γ of X at scale η and denote $\beta := \{C\setminus M; C\in\gamma\}$. Then $\alpha\cup\beta$ is a finite partition of X at scale η and we use

Lemma 4.39 to compute

$$\begin{aligned}
H_{\mu,p}(\eta) &\leq H^*_{\mu,p}(\alpha \cup \beta) \\
&= -\sum_{A\in\alpha} \int_X \mathbb{E}_\mu(\chi_A | p^{-1}(\mathcal{B}_Y)) \log(\mathbb{E}_\mu(\chi_A|p^{-1}(\mathcal{B}_Y)))d\mu \\
&\quad - \sum_{B\in\beta} \int_X \mathbb{E}_\mu(\chi_B | p^{-1}(\mathcal{B}_Y)) \log(\mathbb{E}_\mu(\chi_B|p^{-1}(\mathcal{B}_Y)))d\mu \\
&= -\sum_{A\in\alpha} \int_M \mathbb{E}_\mu \left(\chi_{A\cap M} \middle| (p|_M)^{-1}(\mathcal{B}_{p(M)})\right) \log\left(\mathbb{E}_\mu\left(\chi_{A\cap M}\middle|(p|_M)^{-1}(\mathcal{B}_{p(M)})\right)\right) d\mu \\
&\quad - \sum_{B\in\beta} \int_M \mathbb{E}_\mu \left(\chi_{B\cap M} \middle| (p|_M)^{-1}(\mathcal{B}_{p(M)})\right) \log\left(\mathbb{E}_\mu\left(\chi_{B\cap M}\middle|(p|_M)^{-1}(\mathcal{B}_{p(M)})\right)\right) d\mu \\
&= H^*_{\mu,p|_M}(\alpha) + 0,
\end{aligned}$$

where we have used the convention $0 \cdot \log 0 = 0$. Thus, taking the infimum over all considered α we obtain $H_{\mu,p}(\eta) \leq H_{\mu,p|_M}(\eta \cap M^2)$. To show the remaining inequality let α be a partition of X at scale η and consider $\beta := \{A \cap M;\ A \in \alpha\}$. A similar computation as above shows $H_{\mu,p|_M}(\eta \cap M^2) \leq H^*_{\mu,p|_M}(\beta) = H^*_{\mu,p}(\alpha)$ and taking the infimum over all considered α we obtain the remaining inequality $H_{\mu,p|_M}(\eta \cap M^2) \leq H_{\mu,p}(\eta)$. □

Proposition 4.41 is the key in order to prove Theorem 4.37 and can be seen as a scale wise version of the theorem.

Proposition 4.41. *Let $M \subseteq X$ be a closed and invariant subset. Then also $p(M)$ is closed and invariant. For every $\eta \in \mathbb{U}_X$ the relative entropy map*

$$\mathrm{E}_{(\cdot)}\left(\eta \cap M^2 \middle| \pi|_{G\times M} \overset{p|_M}{\to} \phi|_{G\times p(M)}\right)$$

is the restriction of $\mathrm{E}_{(\cdot)}(\eta|\pi \overset{p}{\to} \phi)$ to $\mathcal{M}_G(M)$. In particular, $\mathrm{E}_{(\cdot)}\left(\eta \cap M^2 \middle| \pi|_{G\times M}\right)$ is the restriction of $\mathrm{E}_{(\cdot)}(\eta|\pi)$ to $\mathcal{M}_G(M)$.

Proof. The set $p(M)$ is compact as the image of a compact set and hence also closed. It is furthermore straightforward to show that $p(M)$ is invariant. Let $\mu \in \mathcal{M}_G(M)$ and $\eta \in \mathbb{U}_X$. Then the invariance of M implies that for compact subsets $A \subseteq G$ there holds

$(\eta \cap M^2)_A = \eta_A \cap M^2$. Hence,

$$\begin{aligned}
\mathrm{E}_\mu\left(\eta \middle| \pi \overset{p}{\to} \phi\right) &= \lim_{i\in I} \frac{H_{\mu,p}(\eta_{A_i})}{\theta(A_i)} \\
&= \lim_{i\in I} \frac{H_{\mu,p|_M}(\eta_{A_i} \cap M^2)}{\theta(A_i)} \\
&= \lim_{i\in I} \frac{H_{\mu,p|_M}((\eta \cap M^2)_{A_i})}{\theta(A_i)} \\
&= \mathrm{E}_\mu\left(\eta \cap M^2 \middle| \pi|_{G\times M} \overset{p|_M}{\to} \phi|_{G\times p(M)}\right).
\end{aligned}$$

□

Proof of Theorem 4.37. Whenever μ and ν are invariant Borel probability measures on X and whenever there is $\lambda \in (0,1)$ such that $\lambda\mu + (1-\lambda)\nu$ has a support contained in M, then also μ and ν have support contained in M which shows that $\mathcal{M}_G(M)$ is a face of $\mathcal{M}_G(X)$. As $\mathcal{M}_G(M)$ is compact this face is closed. One easily shows $\mathbb{U}_M = \{\eta \cap M^2;\, \eta \in \mathbb{U}_X\}$. Thus, the restriction statement follows from Proposition 4.41. □

5 On relative entropy via discrete restriction

It is well-known that calculating entropy of an action π of $\mathbb{R}$ one can instead calculate the entropy of the restricted action $\pi|_{\mathbb{Z}}$ of $\mathbb{Z}$. This simplification allows to observe various properties of topological and measure theoretical entropy of actions of $\mathbb{Z}^d$ also for actions of $\mathbb{R}^d$. Motivated by the fact that not all unimodular amenable groups contain uniform lattices we have discussed in Chapter 4 approaches that are available in the general context. Nevertheless, we have also seen in the introduction of Chapter 4 that some approaches depend on the finiteness of Van Hove sets. As these approaches are important in the proofs of cornerstones of entropy theory, like the variational principle, we will thus present in this chapter that instead of considering uniform lattices, i.e. Delone subgroups, one can also consider certain other discrete substructures like Delone sets ω. We will show in Section 5.1 that replacing a Van Hove net $(A_i)_{i\in I}$ by the net of finite intersections $(A_i \cap \omega)_{i\in I}$ allows to reconsider well-known approaches like the original approaches [Kol58, Sin59, AKM65]. In this chapter we continue our discussion from the introduction. We discuss to what extend statements like Bowens formula, the Rokhlin-Abramov Theorem, the variational principle and sufficient conditions for the upper semi-continuity of the entropy map can be obtained by restricting to a uniform lattice. In particular, we will see how well-known results and techniques for discrete amenable acting groups can be used in this context. In this chapter we will consider a unimodular amenable group G and actions π and ϕ of G on compact Hausdorff spaces X and Y respectively such that ϕ is a factor of π via a factor map $p\colon X \to Y$.

5.1 Restriction to Delone sets

We start our considerations with the following lemma.

Lemma 5.1. *Let ω be a closed subset of G that is K-dense with respect to some compact subset K of G. Whenever $(A_i)_{i\in I}$ is a Van Hove net in G and whenever we denote $F_i := \omega \cap A_i$ for all $i \in I$, then $(KF_i)_{i\in I}$ is Van Hove and satisfies* $\lim_{i\in I} \theta(KF_i)/\theta(A_i) = 1$ *and* $\lim_{i\in I} \theta(KF_i \Delta A_i)/\theta(A_i) = 0$.

Remark 5.2. *We will use the statement of Lemma 5.1 in the following for locally finite ω in order to obtain finite F_i and in particular for Delone sets. Nevertheless, note that we do not assume ω to be discrete in the formulation of this lemma.*

Proof. Clearly, K is non-empty. We thus obtain from Proposition 2.26 the existence of a Van Hove net $(B_i)_{i\in I}$ that satisfies $K^{-1}B_i \subseteq A_i$ and $B_i^c \subseteq KA_i^c$ for all $i \in I$ and furthermore $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$. For $b \in B_i$ there are $k \in K$ and $v \in \omega$ such that $b = kv$ and we obtain $v = k^{-1}b \in K^{-1}B_i \subseteq A_i$, hence $v \in F_i$. Thus, $b \in KF_i$ and we have shown $B_i \subseteq KF_i$. Furthermore, Proposition 2.25 implies $(KA_i)_{i\in I}$ to be a Van Hove net and $\lim_{i\in I} \theta(KA_i)/\theta(A_i) = 1$ and we compute

$$1 \leftarrow \frac{\theta(B_i)}{\theta(A_i)} \leq \frac{\theta(KF_i)}{\theta(A_i)} \leq \frac{\theta(KA_i)}{\theta(A_i)} \to 1,$$

which shows $\lim_{i\in I} \theta(KF_i)/\theta(A_i) = 1$ and in particular that $\theta(KF_i) > 0$ for sufficiently large i. Above we have already seen that there holds $B_i \subseteq KF_i$ and we thus obtain $(KF_i)^c \subseteq B_i^c \subseteq KA_i^c$ for all $i \in I$. For a compact subset $M \subseteq G$ we compute

$$\partial_M(KF_i) = MKF_i \cap M\overline{(KF_i)^c} \subseteq MKA_i \cap MK\overline{A_i^c} = \partial_{MK}A_i.$$

Thus, there holds

$$0 \leq \limsup_{i\in I} \frac{\theta(\partial_M(KF_i))}{\theta(KF_i)} \leq \lim_{i\in I} \frac{\theta(\partial_{MK}A_i)}{\theta(A_i)} = 0$$

and we obtain that $(KF_i)_{i\in I}$ is Van Hove. It remains to show $\lim_{i\in I} \theta(KF_i\Delta A_i)/\theta(A_i) = 0$. To see this let $k \in K$. As $\mathcal{P}(X)$ is an abelian group under Δ, with the identity as inverse map and neutral element $\emptyset$, we compute

$$\begin{aligned} KF_i\Delta A_i &= KF_i\Delta\emptyset\Delta A_i \\ &= (KF_i\Delta kA_i)\Delta(kA_i\Delta A_i) \\ &\subseteq (KF_i\Delta kA_i) \cup (kA_i\Delta A_i) \\ &\subseteq (KA_i \setminus kA_i) \cup (kA_i \setminus KF_i) \cup (kA_i\Delta A_i). \end{aligned}$$

Now recall that $B_i \subseteq KF_i$ and $K^{-1}B_i \subseteq A_i$ for all $i \in I$. Thus, there holds $B_i \subseteq kA_i$ and we compute

$$0 \leq \frac{\theta(kA_i \setminus KF_i)}{\theta(A_i)} \leq \frac{\theta(kA_i \setminus B_i)}{\theta(A_i)} = \frac{\theta(kA_i)}{\theta(A_i)} - \frac{\theta(B_i)}{\theta(A_i)} \to 1 - 1 = 0.$$

As $(A_i)_{i\in I}$ is Van Hove it is in particular Følner and we obtain

$$0 \leq \frac{\theta(KF_i\Delta A_i)}{\theta(A_i)} \leq \frac{\theta(KA_i \setminus A_i)}{\theta(A_i)} + \frac{\theta(kA_i \setminus KF_i)}{\theta(A_i)} + \frac{\theta(kA_i\Delta A_i)}{\theta(A_i)} \to 0.$$

□

A Delone set ω does not necessarily have a uniform density. To see this consider for example the set $(-\mathbb{N}_0) \cup (2\mathbb{N}_0)$ and the Van Hove nets $([-n,0])_{n\in\mathbb{N}}$ and $([0,n])_{n\in\mathbb{N}}$. Nevertheless, the following lemma allows to control $|\omega \cap A_i|/\theta(A_i)$ for any Van Hove net $(A_i)_{i\in I}$.

Lemma 5.3. *Let ω be a Delone set in G, V be an open neighbourhood of 0 and $K \subseteq G$ be a compact subset such that ω is V-discrete and K-dense. Consider a Van Hove net $(A_i)_{i\in I}$ and abbreviate $F_i := \omega \cap A_i$.*

(i) Whenever E is a finite subset of G such that $K \subseteq EV$, then there holds

$$0 < \frac{1}{|E|\theta(V)} \leq \liminf_{i\in I} \frac{|F_i|}{\theta(A_i)}.$$

(ii) Whenever V is precompact, then there holds

$$\limsup_{i\in I} \frac{|F_i|}{\theta(A_i)} \leq \frac{1}{\theta(V)} < \infty.$$

Remark 5.4. *Note that as K is compact and V is open there always exist finite subsets E such that $K \subseteq EV$. Furthermore, as G is locally compact there always exist precompact open neighbourhoods V of 0 such that ω is V-discrete. Thus, we obtain that for any Delone set ω there are constants $a, b \in (0,\infty)$ such that for any Van Hove net there holds*

$$a \leq \liminf_{i\in I} \frac{|\omega \cap A_i|}{\theta(A_i)} \leq \limsup_{i\in I} \frac{|\omega \cap A_i|}{\theta(A_i)} \leq b.$$

Proof of Lemma 5.3. As $F_i = A_i \cap \omega$ is V-discrete we compute

$$\theta(KF_i) \leq \theta(EVF_i) \leq \theta\left(\bigcup_{g\in E} gVF_i\right) \leq |E|\theta(VF_i) = |E|\theta(V)|F_i|.$$

Now recall from Lemma 5.1 that $\lim_{i\in I} \theta(KF_i)/\theta(A_i) = 1$, which allows to observe (i). Furthermore, we obtain from the V-discreteness of V that there holds

$$|F_i|\theta(V) = \theta(VF_i) \leq \theta\left(\overline{V}A_i\right).$$

For (ii) we assume that V is precompact and obtain that $\overline{V}$ is compact. Thus, there holds $\lim_{i\in I} \theta(\overline{V}A_i)/\theta(A_i) = 1$ and we observe (ii). □

5.1.1 Topological entropy

Via scaled open covers

Theorem 5.5. *Let ω be a closed and relatively dense subset of G and let $(A_i)_{i\in I}$ be a Van Hove net in G. Set $F_i := A_i \cap \omega$ for any $i \in I$. Then there holds*

$$\mathrm{E}(\pi \xrightarrow{p} \phi) = \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\eta_{F_i})}{\theta(A_i)}.$$

This statement remains valid whenever we consider a limit inferior instead of a limit superior.

Remark 5.6. *Note that we do not assume ω to be discrete and that the statement also holds whenever F_i is not finite. This is somehow natural, as this approach to topological entropy does not depend on the finiteness of the Van Hove net.*

Remark 5.7. *One cannot expect Theorem 5.5 to holds whenever one considers Følner nets instead of Van Hove nets. Indeed, considering the Følner net $(A_n)_{n\in\mathbb{N}}$ with $A_n := [0,n] \setminus (\mathbb{Z} + B_{2^{-n}}(0))$ in $\mathbb{R}$ and the uniform lattice $\mathbb{Z}$ we obtain $F_n := A_n \cap \mathbb{Z} = \emptyset$ and thus the above formula always yields 0. As there are clearly $\mathbb{R}$-actions with non-zero topological entropy we observe that Følner nets are not the right concept for the above theorem.*

Proof of Theorem 5.5. Let K be a compact subset of G such that ω is K-dense and such that $e_G \in K$. Then for $\eta \in \mathbb{U}_X$ and $i \in I$ there holds $\eta_{F_i} \supseteq \eta_{KF_i} = (\eta_K)_{F_i}$ and $\eta_K \in \mathbb{U}_X$. We thus see that there holds

$$\begin{aligned}\limsup_{i\in I} \frac{H_p(\eta_{F_i})}{\theta(A_i)} &\leq \limsup_{i\in I} \frac{H_p((\eta_K)_{F_i})}{\theta(A_i)} \\ &\leq \sup_{\epsilon\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\epsilon_{F_i})}{\theta(A_i)}.\end{aligned}$$

Taking the supremum over $\eta \in \mathbb{U}_X$ we therefore obtain

$$\sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\eta_{F_i})}{\theta(A_i)} = \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\eta_{KF_i})}{\theta(A_i)}. \tag{5.1}$$

Recall from Lemma 5.1 that $(KF_i)_{i\in I}$ is a Van Hove net which furthermore satisfies $\lim_{i\in I} \theta(KF_i)/\theta(A_i) = 1$. Thus, for any $\eta \in \mathbb{U}_X$ we obtain from (5.1) that

$$\mathrm{E}\left(\pi \xrightarrow{p} \phi\right) = \sup_{\eta\in\mathbb{U}_X} \lim_{i\in I} \frac{H_p(\eta_{KF_i})}{\theta(KF_i)} = \sup_{\eta\in\mathbb{U}_X} \lim_{i\in I} \frac{H_p(\eta_{KF_i})}{\theta(A_i)} = \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\eta_{F_i})}{\theta(A_i)}.$$

A similar argument shows the statement for the limit inferior. □

Via refined open partitions

Theorem 5.8. *Let ω be a relatively dense and locally finite subset of G and let $(A_i)_{i\in I}$ be a Van Hove net in G. For $i \in I$ set $F_i := A_i \cap \omega$. Then there holds*

$$\mathrm{E}(\pi \xrightarrow{p} \phi) = \sup_{\mathcal{U}} \limsup_{i\in I} \frac{H_p^*(\mathcal{U}_{F_i})}{\theta(A_i)},$$

where the supremum is taken over all open covers $\mathcal{U}$ of X. The formula remains valid, whenever we consider a limit inferior.

Remark 5.9. *The approach via refining open covers depends on the finiteness of the Van Hove net and we thus need to assume ω to be locally finite. Note however that we can also use the notion $H_p^*(\mathcal{U}, F_i)$, introduced in Subsection 4.3.3, in order to obtain the statement for general closed and relatively dense ω.*

Remark 5.10. *Note that every Delone set is relatively dense and locally finite. An example of a relatively dense and locally finite set that is not Delone is $\mathbb{Z}\cup\{1/n;\, n \in \mathbb{N}\}$.*

Proof. We will use the notation and ideas from Subsection 4.3.3 to obtain this statement. As ω is assumed to be locally finite we obtain all F_i to be finite sets and we have already observed in Remark 4.29 that there holds $H_p^*(\mathcal{U}, F_i) = H_p^*(\mathcal{U}_{F_i})$ for all $i \in I$. For an open cover $\mathcal{U}$ of X we can thus consider a Lebesgue entourage η of $\mathcal{U}$ and obtain from Lemma 4.27 that there holds $H_p^*(\mathcal{U}_{F_i}) \leq H_p(\eta_{F_i})$. We thus obtain from Theorem 5.5 that there holds

$$\limsup_{i\in I} \frac{H_p^*(\mathcal{U}_{F_i})}{\theta(A_i)} \leq \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\eta_{F_i})}{\theta(A_i)} = \mathrm{E}(\pi \xrightarrow{p} \phi).$$

Taking the supremum over all open covers $\mathcal{U}$ reveals

$$\sup_{\mathcal{U}} \limsup_{i\in I} \frac{H_p^*(\mathcal{U}_{F_i})}{\theta(A_i)} \leq \mathrm{E}(\pi \xrightarrow{p} \phi).$$

Furthermore, whenever we consider $\eta \in \mathbb{U}_X$ and an open cover $\mathcal{U}$ of X at scale η, then $\mathcal{U}_{F_i}$ is at scale η_{F_i} for any $i \in I$ and we obtain from the definition of $H_p(\eta_{F_i})$ that $H_p^*(\mathcal{U}_{F_i}) \geq H_p(\eta_{F_i})$. Thus, we obtain from Theorem 5.5 that there holds

$$\sup_{\mathcal{U}} \limsup_{i\in I} \frac{H_p^*(\mathcal{U}_{F_i})}{|F_i|} \geq \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\eta_{F_i})}{|F_i|} = \mathrm{E}(\pi \xrightarrow{p} \phi).$$

□

Via spanning and separating sets

In Subsection 4.3.4 we have already seent that ideas of [Bow71] can be used in the general setting. We next present that a similar statement as Theorem 5.5 can also be achieved along this approach.

Theorem 5.11. *Let ω be a closed and relatively dense subset of G and let $(A_i)_{i\in I}$ be a Van Hove net in G. For $i \in I$ set $F_i := A_i \cap \omega$. Then there holds*

$$\mathrm{E}(\pi \xrightarrow{p} \phi) = \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{\log(\mathrm{sep}_p(\eta_{F_i}))}{\theta(A_i)}$$
$$= \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{\log(\mathrm{spa}_p(\eta_{F_i}))}{\theta(A_i)}.$$

Remark 5.12. *Also this approach does not depend on the finiteness of the Van Hove sets and we obtain the statement without any discreteness assumptions on ω.*

Remark 5.13. *The given formulas remain valid, whenever we consider a limit inferior instead of a limit superior. Furthermore, the formulas are also valid whenever we consider $\log(\mathrm{sep}_{p,D}(\cdot))$ and $\log(\mathrm{spa}_{p,D}(\cdot))$ for a subset $D \subseteq X$ such that $D\cap p^{-1}(y)$ is dense in $p^{-1}(y)$ for all $y \in Y$. Thus, in particular for a dense subset $D \subseteq X$ we obtain similar formulas for $\log(\mathrm{sep}_D(\cdot))$ and $\log(\mathrm{spa}_D(\cdot))$.*

Proof of Theorem 5.11. From Remark 4.2 and Lemma 4.32 we obtain that for $\eta \in \mathbb{U}_X$ there exists $\epsilon \in \mathbb{U}_X$ such that for all $i \in I$ there holds $H_p(\eta_{F_i}) \leq \log(\mathrm{spa}_p(\epsilon_{F_i})) \leq \log(\mathrm{sep}_p(\epsilon_{F_i})) \leq H_p(\epsilon_{F_i})$. We thus obtain from Theorem 5.5 that there holds

$$\begin{aligned}\limsup_{i\in I} \frac{H_p(\eta_{F_i})}{\theta(A_i)} &\leq \sup_{\epsilon\in\mathbb{U}_X} \limsup_{i\in I} \frac{\log(\mathrm{spa}_{p,D}(\epsilon_{F_i}))}{\theta(A_i)} \\ &\leq \sup_{\epsilon\in\mathbb{U}_X} \limsup_{i\in I} \frac{\log(\mathrm{sep}_{p,D}(\epsilon_{F_i}))}{\theta(A_i)} \\ &\leq \sup_{\epsilon\in\mathbb{U}_X} \lim_{i\in I} \frac{H_p(\epsilon_{F_i})}{\theta(A_i)} \\ &= \mathrm{E}(\pi \xrightarrow{p} \phi).\end{aligned}$$

Taking the supremum over η thus yields the statement. □

5.1.2 Measure theoretical entropy

Via scaled finite partitions

Theorem 5.14. *Let ω be a closed and relatively dense subset of G and let $(A_i)_{i\in I}$ be a Van Hove net in G. Set $F_i := A_i \cap \omega$ for any $i \in I$. Then for any invariant Borel probability measure μ on X there holds*

$$\mathrm{E}_\mu(\pi \overset{p}{\to} \phi) = \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_{\mu,p}(\eta_{F_i})}{\theta(A_i)}.$$

These statements remain valid if we consider a limit inferior instead of a limit superior.

Proof. The statement can be shown with similar arguments as presented in the proof of Theorem 5.5. □

Via refined finite partitions

As discussed above we have seen that the measure theoretical entropy is classically defined by considering certain refinements of finite partitions. This approach depends on the finiteness of the Van Hove sets for discrete amenable groups and it is natural to ask, whether it can be used also in the non-discrete context. We will see next that this is possible whenever we replace, similarly as for topological entropy, the Van Hove sets by the intersection of Van Hove sets with certain discrete substructures of G. To be more precise recall that we consider relatively dense and locally finite subsets of G in the case of the topological entropy. This is not possible for the measure theoretical entropy as we will see in the next example. Nevertheless, we will show below that one can consider the more restrictive class of Delone sets in order to give a formula for measure theoretical entropy.

Example 5.15. *Let us reconsider the action discussed in Remark 4.17, i.e. $\pi\colon \mathbb{R}\times\mathbb{T} \to \mathbb{T}$ that maps $\pi(g,x) \mapsto g + x \mod 1$. Let us denote the Lebesgue measure on $\mathbb{T}$ by λ and the Lebesgue measure on $\mathbb{R}$ by θ. Consider now the partition $\alpha = \{[0,1/2),[1/2,1)\}$ of $\mathbb{T}$ and the relatively dense and locally finite set*

$$\omega := \mathbb{Z} \cup \bigcup_{n\in\mathbb{N}} \left([n,n+1] \cap 2^{-n}\mathbb{Z}\right).$$

We furthermore consider the Van Hove sequence $([0,n])_{n\in\mathbb{N}}$ and denote as usual $F_n := [0,n] \cap \omega$. Then $\alpha_{F_n} = \alpha_{[0,1]\cap 2^{-n}\mathbb{Z}}$ consists of 2^n intervals of Lebesgue measure 2^{-n} and we obtain $H^*_\lambda(\alpha_{F_n}) = -2^n \cdot 2^{-n}\log(2^{-n}) = n\log(2)$. Thus,

$$\limsup_{n\to\infty} \frac{H_\lambda(\alpha_{F_n})}{\theta([0,n])} = \log(2).$$

Nevertheless, one easily shows that $\mathrm{E}_\lambda(\pi) = 0$*. Thus, one cannot expect to compute measure theoretical entropy via the restriction to such general* ω*.*

Considering Delone sets we next show the following formula.

Theorem 5.16. *Let* μ *be an invariant Borel probability measure on* X*. Let* ω *be a Delone set in* G*. Let furthermore* A_i *be a Van Hove net and define* $F_i := A_i \cap \omega$ *for any* $i \in I$*. Then there holds*

$$\mathrm{E}_\mu(\pi \xrightarrow{p} \phi) = \sup_\alpha \limsup_{i \in I} \frac{H^*_{\mu,p}(\alpha_{F_i})}{\theta(A_i)},$$

where the supremum is taken over all finite partitions α *of* X*. The formula remains valid if a limit inferior is considered.*

Before we present a proof note that whenever G is discrete, then G is a Delone set in G and furthermore any Følner net in G is Van Hove. As furthermore $\mathcal{F}(G) = \mathcal{K}(G) \ni F \mapsto H^*_{\mu,p}(\alpha_F)$ is monotone, right invariant and subadditive for any finite partition α, which can be seen from Proposition 4.5, Lemma 4.13 and Lemma 4.16, we can apply the Ornstein-Weiss Lemma to obtain that the following limit exists and its independence from the choice of a Følner net. We denote the *relative measure theoretical entropy of* p *and* α by

$$E^*_\mu\left(\alpha|\pi \xrightarrow{p} \phi\right) := \lim_{i \in I} \frac{H_{\mu,p}(\alpha_{F_i})}{|F_i|}.$$

We furthermore denote the *measure theoretical entropy of* α as $E^*_\mu(\alpha|\pi) := E^*_\mu\left(\alpha|\pi \xrightarrow{p} \phi\right)$, whenever Y is a one point flow and obtain as a corollary of Theorem 5.16 that our definition of measure theoretical entropy is equivalent to the probably most common definition in the literatureas for example discussed in [Kol58, Sin59, Oll85].

Corollary 5.17. *Let* μ *be an invariant Borel probability measure on* X*. Whenever* G *is a discrete amenable group, then there hold* $\mathrm{E}_\mu(\pi \xrightarrow{p} \phi) = \sup_\alpha \mathrm{E}^*_\mu\left(\alpha|\pi \xrightarrow{p} \phi\right)$*, and* $\mathrm{E}_\mu(\pi) = \sup_\alpha \mathrm{E}^*_\mu(\alpha|\pi)$*, where the suprema are taken over all finite partitions* α *of* X*.*

We will need the following lemma in order to give a proof of Theorem 5.16. The techniques that are used to show this lemma can be found at several places in the literature, for example in the proof of the variational principle in [Wal82, Theorem 8.6], or in [HYZ11, Theorem 3.5.].

Lemma 5.18. *Let* μ *be a Borel probability measure on* X*. For every* $\epsilon > 0$ *and every finite partition* α *of* X *there exists* $\eta \in \mathbb{U}_X$ *such that* α *is at scale* η *and such that* $H^*_{\mu,p}(\alpha|\beta) < \epsilon$ *holds for every finite partition* β *of* X *at scale* η*.*

Remark 5.19. *Note that in the proof of [HYZ11, Theorem 3.5.] it is shown that for every* $\epsilon > 0$ *and every finite partition* α *there exists an open cover* $\mathcal{U}$ *of* X *such that for* *any finite partition β that is finer than $\mathcal{U}$ there holds $H^*_\mu(\alpha|\beta) < \epsilon$. If we now consider a Lebesgue entourage η with respect to $\mathcal{U}$, then every open cover β at scale η is finer than $\mathcal{U}$ and thus satisfies the claimed properties. Nevertheless, to keep the proof self-contained and to present a proof that avoids the use of Lebesgue entourages, we include a full proof for the convenience of the reader.*

Proof of Lemma 5.18. Let r be the number of elements of α and write $\alpha = \{A_1, \cdots, A_r\}$. By Lemma 4.7 there is $\delta > 0$ such that if $\beta_j := \{B_1^j, \cdots, B_r^j\}$, $j = 1, 2$ are two partitions with $\sum_{i=1}^r \mu(B_i^1 \Delta B_i^2) < \delta$, then $H^*_{\mu,p}(\beta_1|\beta_2) < \epsilon$. As μ is regular there are compact subsets $D_i \subseteq A_i$ with $\mu(A_i \setminus D_i) < \delta/(2r^2)$. Set $D_0 := X \setminus \bigcup_{i=1}^r D_i$ and $U_i := D_0 \cup D_i$. Furthermore, set $\mathcal{U} := \{U_1, \cdots, U_k\}$ and $\eta := \langle \mathcal{U} \rangle = \bigcup_{U \in \mathcal{U}} U^2$. As $\mathcal{U}$ is an open cover of X we obtain $\eta \in \mathbb{U}_X$. Clearly, $\mathcal{U} \preceq \alpha$ and hence α is at scale η. Note furthermore that $\mu(D_0) < \delta/(2r)$.

Let now $\beta \in \mathcal{P}_X$ be at scale η. We show first, that there actually holds $\mathcal{U} \preceq \beta$ and consider $B \in \beta$. If $B \subseteq D_0$, then $B \subseteq U_i$ for all $i \in \{1, \cdots, r\}$ and otherwise there exists $d \in B \setminus D_0$. By the construction of D_0 this d is then contained in some D_i with $i \in \{1, \cdots, r\}$. As $d \notin D_0$ and as all the D_i are disjoint, we obtain that $d \notin U_j$ for $j \neq i$. Thus, for $b \in B$ we obtain from $(b, d) \in B^2 \subseteq \eta = \bigcup_{i=1}^r U_i^2$ that there also holds $b \in U_i$ and hence $B \subseteq U_i$. This shows that there holds indeed $\mathcal{U} \preceq \beta$.

From this we conclude, that there is a finite partition γ in r sets with $\mathcal{U} \preceq \gamma \preceq \beta$. We denote $\gamma = \{C_1, \cdots, C_r\}$ with $C_i \subseteq U_i = D_0 \cup D_i$. Thus, we obtain $D_i = X \setminus \bigcup_{j \neq i} U_j \subseteq C_i \subseteq U_i$ and $D_i \subseteq A_i$ implies

$$C_i \Delta A_i \subseteq (U_i \setminus D_i) \cup (A_i \setminus D_i) = D_0 \cup (A_i \setminus D_i).$$

Thus, $\mu(C_i \Delta A_i) \leq \mu(D_0) + \mu(A_i \setminus D_i) \leq \frac{\delta}{2r} + \frac{\delta}{2r^2} \leq \frac{\delta}{r}$. Hence,

$$\sum_{i=1}^r \mu(C_i \Delta A_i) < \delta.$$

We conclude by the choice of δ that $H^*_{\mu,p}(\alpha|\beta) \leq H^*_{\mu,p}(\alpha|\gamma) < \epsilon$. □

Proof of Theorem 5.16. If $\eta \in \mathbb{U}_X$ and if α is a finite partition at scale η, then α_{F_i} is at scale η_{F_i} for every $i \in I$. Thus, there holds $H_{\mu,p}(\eta_{F_i}) \leq H^*_{\mu,p}(\alpha_{F_i})$ and we obtain from Theorem 5.14 that

$$E_\mu(\pi \xrightarrow{p} \phi) = \sup_{\eta \in \mathbb{U}_X} \limsup_{i \in I} \frac{H_{\mu,p}(\eta_{F_i})}{\theta(A_i)} \leq \sup_{\alpha} \limsup_{i \in I} \frac{H^*_{\mu,p}(\alpha_{F_i})}{\theta(A_i)},$$

where the last supremum is taken over all finite partitions α of X. To show the reverse inequality let $\epsilon > 0$ and α be a finite partition of X. From Lemma 5.18 we obtain that there exists an entourage $\eta \in \mathbb{U}_X$ such that for any partition γ at scale η there holds $H^*_{\mu,p}(\alpha|\gamma) < \epsilon$. For $i \in I$, $g \in F_i$ and a finite partition β at scale η_{F_i} we obtain that $\beta_{g^{-1}}$

is at scale η, which implies $H^*_{\mu,p}(\alpha|\beta_{g^{-1}}) < \epsilon$. We thus compute

$$\begin{aligned} H^*_{\mu,p}(\alpha_{F_i}) &\leq H^*_{\mu,p}(\beta) + H^*_{\mu,p}(\alpha_{F_i}|\beta) \\ &\leq H^*_{\mu,p}(\beta) + \sum_{g\in F_i} H^*_{\mu,p}(\alpha_g|\beta) \\ &= H^*_{\mu,p}(\beta) + \sum_{g\in F_i} H^*_{\mu,p}(\alpha|\beta_{g^{-1}}) \\ &\leq H^*_{\mu,p}(\beta) + |F_i|\epsilon. \end{aligned}$$

Taking the infimum over all considered β, we obtain $H^*_{\mu,p}(\alpha_{F_i}) \leq H_{\mu,p}(\eta_{F_i}) + |F_i|\epsilon$ for all $i \in I$. Consider now a precompact and open neighbourhood V of e_G such that ω is V-discrete. From Lemma 5.3 and Theorem 5.14 we see

$$\begin{aligned} \limsup_{i\in I} \frac{H^*_{\mu,p}(\alpha_{F_i})}{\theta(A_i)} &\leq \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_{\mu,p}(\eta_{F_i})}{\theta(A_i)} + \epsilon \limsup_{i\in I} \frac{|F_i|}{\theta(A_i)} \\ &\leq E_\mu(\pi \xrightarrow{p} \phi) + \frac{\epsilon}{\theta(V)}. \end{aligned}$$

As $\epsilon > 0$ was chosen arbitrary we obtain the statement by taking the supremum over all finite partitions α. A similar argument shows that one can also consider a limit inferior instead of the limit superior. □

5.2 Restriction to uniform lattices

Recall that a Delone set ω is said to possess a uniform density, whenever for all Van Hove nets $(A_i)_{i\in I}$ the limit $\lim_{i\in I} |A_i \cap \omega|/\theta(A_i)$ exists and does not depend on the Van Hove net. Recall furthermore that this limit is denoted by $\mathrm{dens}(\omega)$. All regular model sets possess a uniform density [Moo02]. Thus, the following corollary of the Theorems 5.5, 5.8, 5.11, 5.14 and 5.16 can be applied in this context.

Corollary 5.20. *Let ω be a Delone set that possesses a uniform density* $\mathrm{dens}(\omega)$ *and consider a Van Hove net $(A_i)_{i\in I}$. Set $F_i := \omega \cap A_i$. Then there holds*

$$\mathrm{E}(\pi \xrightarrow{p} \phi) = \mathrm{dens}(\omega) \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_p(\eta_{F_i})}{|F_i|} = \mathrm{dens}(\omega) \sup_{\mathcal{U}} \limsup_{i\in I} \frac{H^*_p(\mathcal{U}_{F_i})}{|F_i|},$$

where the second supremum is taken over all finite open covers $\mathcal{U}$ of X. Similar formulas are valid, whenever we consider $\log(\mathrm{sep}_p(\cdot))$ *or* $\log(\mathrm{spa}_p(\cdot))$. *Furthermore, for any invariant Borel probability measure μ on X there holds*

$$\mathrm{E}_\mu(\pi \xrightarrow{p} \phi) = \mathrm{dens}(\omega) \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{H_{\mu,p}(\eta_{F_i})}{|F_i|} = \mathrm{dens}(\omega) \sup_{\alpha} \limsup_{i\in I} \frac{H^*_{\mu,p}(\alpha_{F_i})}{|F_i|},$$

where the second supremum is taken over all finite partitions α of X.

Now recall from Lemma 3.6 that uniform lattices are Delone sets that possess a uniform density. In Lemma 3.6 it was furthermore shown that in this case F_i is a Van Hove sequence in the uniform lattice. We thus obtain from the definition of topological and measure theoretical entropy and Corollary 5.20 the following.

Theorem 5.21. *If Λ is a uniform lattice in G, then there holds*

$$\mathrm{E}(\pi \overset{p}{\to} \phi) = \mathrm{dens}(\Lambda)\, \mathrm{E}\left(\pi|_{\Lambda\times X} \overset{p}{\to} \phi|_{\Lambda\times Y}\right).$$

Furthermore, any G-invariant Borel probability measure μ on X is also Λ-invariant and we obtain

$$\mathrm{E}_\mu(\pi \overset{p}{\to} \phi) = \mathrm{dens}(\Lambda)\, \mathrm{E}_\mu\left(\pi|_{\Lambda\times X} \overset{p}{\to} \phi|_{\Lambda\times Y}\right).$$

Remark 5.22. *Whenever $g\colon X \to X$ is a homeomorphism on a compact Hausdorff space we denote by $\mathrm{E}(g)$ the topological entropy of the action $\phi\colon \mathbb{Z}\times X \to X$ with $\phi(n,x) = g^n(x)$. As $\{0,\cdots,n-1\}$ is a fundamental domain for the uniform lattice $n\mathbb{Z}$ in $\mathbb{Z}$ for any $n \in \mathbb{N}$ we thus obtain from Theorem 5.21 the well-known statement that for every homeomorphism $f\colon X\to X$ there holds $n\,\mathrm{E}(f) = \mathrm{E}(f^n)$. Similarly one obtains the respective measure theoretical formula as a special case.*

Remark 5.23. *Entropy theory can also be studied in the context of actions of countable discrete sofic groups as demonstrated for example in [Bow10, KL11, Bow12, KL13], where we are far from giving a full list of the important references. It is well-known that the notion of measure theoretical and topological entropy from [Oll85] (for actions of countable amenable groups) is equivalent to the sofic measure theoretical and topological entropy respectively [Bow12, KL13]. From Remark 4.29 and Corollary 5.17 it thus follows that these notions are also equivalent to our notions whenever we consider actions of countable discrete amenable groups.*

In [Sin16] entropy theory for actions of (not necessarily discrete) locally compact sofic groups is studied. Theorem 5.21 allows to see that our notion of topological entropy is equivalent to the notion of topological entropy in [Sin16], whenever we consider an action π of a unimodular amenable group that contains a (countable) uniform lattice. Indeed, let us denote by $\mathrm{E}^\Sigma(\pi)$ for the sofic topological entropy[1] *of π. In [Sin16, Theorem 5.3], in analogy to Theorem 5.21, it is presented that whenever Λ is a uniform lattice in the acting group, then there holds $\mathrm{E}^\Sigma(\pi) = \mathrm{dens}(\Lambda)\,\mathrm{E}^\Sigma(\pi|_{\Lambda\times X})$. We thus obtain from Theorem 5.21 that there holds*

$$\mathrm{E}^\Sigma(\pi) = \mathrm{dens}(\Lambda)\,\mathrm{E}^\Sigma(\pi|_{\Lambda\times X}) = \mathrm{dens}(\Lambda)\,\mathrm{E}(\pi|_{\Lambda\times X}) = \mathrm{E}(\pi).$$

In [Sin16] also a notion of sofic measure theoretical entropy *$\mathrm{E}^\Sigma_\mu(\pi)$ is presented. It seems open, whether a similar statement as in Theorem 5.21 also holds for sofic measure*

[1] We will not present the technical definition of this notion or the notion of sofic measure theoretical entropy and refer to [Sin16] for the definition.

theoretical entropy. Nevertheless, it is shown in [Sin16, Theorem 6.2.1] that there holds $\mathrm{E}_\mu^\Sigma(\pi) \geq \mathrm{dens}(\Lambda)\, \mathrm{E}_\mu^\Sigma(\pi|_{\Lambda\times X})$*, whenever* Λ *is a uniform lattice. We thus only obtain*

$$\mathrm{E}_\mu^\Sigma(\pi) \geq \mathrm{dens}(\Lambda)\, \mathrm{E}_\mu^\Sigma(\pi|_{\Lambda\times X}) = \mathrm{dens}(\Lambda)\, \mathrm{E}_\mu(\pi|_{\Lambda\times X}) = \mathrm{E}_\mu(\pi).$$

and it remains open, whether there holds $\mathrm{E}_\mu^\Sigma(\pi) = \mathrm{E}(\pi)$ *for all actions of unimodular amenable groups that contain countable uniform lattices. This is of particular interest, as it is shown in [Sin16, Theorem 4.2.1] that the variational principle holds for sofic entropy, i.e. that there holds* $\mathrm{E}^\Sigma(\pi) = \sup_{\mu\in\mathcal{M}_G(X)} \mathrm{E}_\mu^\Sigma(\pi)$*. It furthermore remains open, whether the notions of sofic entropy of [Sin16] are equal to our respective notions for actions of general unimodular amenable groups.*

Remark 5.24. *With a similar argument as in Remark 5.23 one also shows that the notion of measure theoretical entropy of [Fel80] is equivalent to our notion of measure theoretical entropy. Note that the notion of [Fel80] is also used in [Oll85].*

Clearly the question arises, whether a similar statement as in Theorem 5.21 can be drawn also for entropy at a certain scale. This is the case, whenever we incorporate a modification of the "scale" into our formula.

Theorem 5.25. *If* Λ *is a uniform lattice in* G *and* K *is a compact subset such that* Λ *is* K*-dense, then for any* $\eta \in \mathbb{U}_X$ *there holds*

$$\mathrm{E}(\eta|\pi \overset{p}{\to} \phi) = \mathrm{dens}(\Lambda)\, \mathrm{E}\left(\eta_K \middle| \pi\big|_{\Lambda\times X} \overset{p}{\to} \phi\big|_{\Lambda\times Y}\right)$$

and furthermore for any G*-invariant Borel probability measure* μ *on* X *there holds*

$$\mathrm{E}_\mu(\eta|\pi \overset{p}{\to} \phi) = \mathrm{dens}(\Lambda)\, \mathrm{E}_\mu\left(\eta_K \middle| \pi\big|_{\Lambda\times X} \overset{p}{\to} \phi\big|_{\Lambda\times Y}\right).$$

Proof. Let $(A_i)_{i\in I}$ be a Van Hove net and consider $F_i := \Lambda \cap A_i$. From Lemma 3.6 and Lemma 5.1 we know that $(F_i)_{i\in I}$ and $(KF_i)_{i\in I}$ are Van Hove in Λ and G respectively and that there holds

$$\lim_{i\in I} \frac{|F_i|}{\theta(KF_i)} = \lim_{i\in I} \frac{|F_i|}{\theta(A_i)} \lim_{j\in I} \frac{\theta(A_j)}{\theta(KF_j)} = \mathrm{dens}(\Lambda)\cdot 1 = \mathrm{dens}(\Lambda).$$

We can thus use the invariance of the topological entropy at a certain scale from the

Van Hove net to compute

$$\begin{aligned}
\mathrm{E}_\mu(\eta|\pi \xrightarrow{p} \phi) &= \lim_{i\in I} \frac{H_p(\eta_{KF_i})}{\theta(KF_i)} \\
&= \mathrm{dens}(\Lambda) \lim_{i\in I} \frac{H_p\left((\eta_K)_{F_i}\right)}{|F_i|} \\
&= \mathrm{dens}(\Lambda)\, \mathrm{E}_\mu \left(\eta_K \middle| \pi\big|_{\Lambda\times X} \xrightarrow{p} \phi\big|_{\Lambda\times Y}\right).
\end{aligned}$$

A similar argument shows the formula for the measure theoretical entropy. □

Whenever G is a discrete amenable group all uniform lattices in G are F-dense with respect to a finite set F. We can thus show the following using the notion of relative entropy of open covers and finite partitions, introduced on page 82 and on page 100.

Theorem 5.26. *Let Λ be a uniform lattice in a discrete amenable group G and F be a finite subset such that Λ is F-dense. For any open partition $\mathcal{U}$ of X there holds*

$$\mathrm{E}^*(\mathcal{U}|\pi \xrightarrow{p} \phi) = \mathrm{dens}(\Lambda)\, \mathrm{E}^* \left(\mathcal{U}_F \middle| \pi\big|_{\Lambda\times X} \xrightarrow{p} \phi\big|_{\Lambda\times Y}\right)$$

and furthermore for any G-invariant Borel probability measure μ on X and any finite partition α there holds

$$\mathrm{E}^*_\mu(\alpha|\pi \xrightarrow{p} \phi) = \mathrm{dens}(\Lambda)\, \mathrm{E}^*_\mu \left(\alpha_F \middle| \pi\big|_{\Lambda\times X} \xrightarrow{p} \phi\big|_{\Lambda\times Y}\right).$$

Proof. Using the independence of the definition of relative topological entropy of an open cover from the choice of a Van Hove net we obtain the first statement from Lemma 3.6 and Lemma 5.1 with a similar argument as presented in the proof of Theorem 5.26. The statement for measure theoretical entropy follows analogously. □

5.3 Extrapolation of properties of entropy from uniform lattices

In Theorem 5.21 we have seen that one can compute entropy also by restricting to a uniform lattice that sits in the acting group. This allows an extrapolation technique for statements, i.e. to use results from the discrete theory and to restate them in the general context, whenever there exist uniform lattices in the acting group. The existence of a uniform lattice is in particular satisfied for the additive group $\mathbb{R}^d$ and the hope is that one easily generalizes the whole theory with this technique and in particular the variational principle as claimed in [TZ91]. In this section we will see that technique can indeed be performed for some theorems like for example the Rokhlin-Abramov

Theorem. Furthermore, we will see that the extrapolation technique causes problems with other statements, which unfortunately include the variational principle and the Kolmogorov-Sinai generator theorem. We will see below that some, but not all parts of these theorems can be drawn as a corollary. In fact it seems open, whether these statements hold in their full strength for all unimodular amenable groups and even under the assumption of the existence of a uniform lattice.

5.3.1 The Rokhlin-Abramov Theorem

For actions of $\mathbb{Z}$ this result was obtained by V. A. Rokhlin and J. G. Sinai in 1962 [WZ92]. In [WZ92, Theorem 4.4] the authors then extended the result to the context of countable discrete amenable groups and in this context the statement reappears in [GTW00, Dan01, DZ15, Yan15].

Theorem 5.27 (Rokhlin-Abramov Theorem - discrete version)**.** *Assume that G is a countable and discrete amenable group. Let ψ be a factor of ϕ via a factor map $q\colon Y \to Z$ and recall that we assume that ϕ is a factor of π via the factor map p, i.e. $\pi \overset{p}{\to} \phi \overset{q}{\to} \psi$. Let μ be an invariant Borel probability measure on X. Then there holds*

$$\mathrm{E}_\mu\left(\pi \overset{q\circ p}{\to} \psi\right) = \mathrm{E}_\mu\left(\pi \overset{p}{\to} \phi\right) + \mathrm{E}_{p_*\mu}\left(\phi \overset{q}{\to} \psi\right).$$

This statement seems to be open for actions of unimodular amenable groups. Whenever G is a unimodular amenable group and Λ is a uniform lattice in G, then any G-invariant Borel probability measure μ is also Λ-invariant. Thus, Theorem 5.21 and Theorem 5.27 imply

$$\begin{aligned}
\mathrm{E}_\mu\left(\pi \overset{q\circ p}{\to} \psi\right) &= \mathrm{dens}(\Lambda)\,\mathrm{E}_\mu\left(\pi\big|_{\Lambda\times X} \overset{q\circ p}{\to} \psi\big|_{\Lambda\times Z}\right)\\
&= \mathrm{dens}(\Lambda)\,\mathrm{E}_\mu\left(\pi\big|_{\Lambda\times X} \overset{p}{\to} \phi\big|_{\Lambda\times Y}\right) + \mathrm{dens}(\Lambda)\,\mathrm{E}_{p_*\mu}\left(\phi\big|_{\Lambda\times Y} \overset{q}{\to} \psi\big|_{\Lambda\times Z}\right)\\
&= \mathrm{E}_\mu\left(\pi \overset{p}{\to} \phi\right) + \mathrm{E}_{p_*\mu}\left(\phi \overset{q}{\to} \psi\right).
\end{aligned}$$

We obtain the following version of the Rokhlin-Abramov Theorem from the discussed extrapolation technique.

Corollary 5.28 (Rokhlin-Abramov Theorem - extrapolated version)**.** *Assume that G is a unimodular amenable group that contains a countable uniform lattice. Let ψ be a factor of ϕ via a factor map $q\colon Y \to Z$ and recall that we assume that ϕ is a factor of π via the factor map p, i.e. $\pi \overset{p}{\to} \phi \overset{q}{\to} \psi$. Let μ be an invariant Borel probability measure on X. Then there holds*

$$\mathrm{E}_\mu(\pi \overset{q\circ p}{\to} \psi) = \mathrm{E}_\mu(\pi \overset{p}{\to} \phi) + \mathrm{E}_{p_*\mu}(\phi \overset{q}{\to} \psi).$$

Remark 5.29. *It remains open, whether the Rokhlin-Abramov Theorem holds for all unimodular amenable groups and in particular for actions of the additive group $\mathbb{Q}_p$ of p-adic numbers.*

5.3.2 The variational principle

The variational principle is the main link between the topological and the measure theoretical entropy. In the non-relative context it was shown in [Goo69] by L. W. Goodwyn that topological entropy bounds measure theoretical entropy. In [Din71, Goo71] it was shown by E. I. Dinabourg and T. N. Goodman that considering all invariant Borel probability measures one obtains the topological entropy as the supremum over the respective measure theoretical entropies. A short and elegant proof of the variational principle is due to M. Misiurewicz and can be found in [Mis76]. Note that this proof already considers conditional entropy. In the context of actions of discrete amenable groups the statement seems to appear first in [STZ80, OP82] but we also need to mention the important work of [Goo72, Den72, Den74, Rue73, Wal75, Els77, OP79] that lead to this result. Clearly the question should be raised, whether this result holds for actions of all unimodular amenable groups. We are not able to answer this question completely. In [TZ91] it was claimed that the extrapolation technique can be used to obtain the statement at least for all unimodular amenable groups that contain uniform lattices. We will present below that this technique seems to allow to extrapolate only the result of L. W. Goodwyn, i.e. that topological entropy bounds measure theoretical entropy. As we are not aware how to easily extrapolate the full statement of the variational principle and as we need it in the context of aperiodic order we will present in Chapter 7 that the variational principle holds for actions of σ-compact LCA groups. We begin our discussion with the following citation of the discrete version of the variational principle. For a reference of the variational principle for relative entropy see [Yan15, Theorem 5.1], where a proof is given for compact metric spaces and note that the arguments easily generalize to compact Hausdorff spaces.

Theorem 5.30 (Variational Principle - discrete version)**.** *Whenever G is a countable discrete amenable group, then there holds*

$$\mathrm{E}\left(\pi \xrightarrow{p} \phi\right) = \sup_{\mu \in \mathcal{M}_G(X)} \mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right).$$

Unfortunately this statement cannot be extrapolated in its full strength. The reason behind this is that whenever we consider a unimodular amenable group G and assume that there exists a countable uniform lattice Λ in G, then any G-invariant measure is Λ-invariant, but the converse is not necessarily true. Thus, restricting to a uniform lattice and applying the variational principle for countable discrete amenable groups we simply obtain that

$$\begin{aligned}
\mathrm{E}(\pi \xrightarrow{p} \phi) &= \mathrm{dens}(\Lambda)\, \mathrm{E}\left(\pi\big|_{\Lambda\times X} \xrightarrow{p} \phi\big|_{\Lambda\times Y}\right) \\
&= \mathrm{dens}(\Lambda) \sup_{\mu\in\mathcal{M}_\Lambda(X)} \mathrm{E}_\mu\left(\pi\big|_{\Lambda\times X} \xrightarrow{p} \phi\big|_{\Lambda\times Y}\right) \\
&\geq \mathrm{dens}(\Lambda) \sup_{\mu\in\mathcal{M}_G(X)} \mathrm{E}_\mu\left(\pi\big|_{\Lambda\times X} \xrightarrow{p} \phi\big|_{\Lambda\times Y}\right) \\
&= \mathrm{dens}(\Lambda) \sup_{\mu\in\mathcal{M}_G(X)} \mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right) \\
&= \sup_{\mu\in\mathcal{M}_G(X)} \mathrm{E}_\mu(\pi \xrightarrow{p} \phi),
\end{aligned}$$

but it is not immediately clear why this inequality is an equality. Furthermore, care has to be taken also as $\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right)$ is not defined for $\mu \in \mathcal{M}_\Lambda(X) \setminus \mathcal{M}_G(X)$. With the extrapolation technique we obtain the following.

Corollary 5.31 (Goodwyn's theorem - extrapolated version)**.** *Whenever G is a unimodular amenable group and Λ is a uniform lattice in G, then there holds*

$$\mathrm{E}(\pi \xrightarrow{p} \phi) \geq \sup_{\mu\in\mathcal{M}_G(X)} \mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right).$$

Remark 5.32. *From Remark 4.17 we actually obtain that the statement of this corollary is also valid without the assumption of the existence of a uniform lattice. Thus, the extrapolation technique seems to give no new insights into the relation of topological and measure theoretical entropy.*

In Theorem 7.43 we will furthermore present a full proof of Goodwyn's theorem in the context of the topological pressure for arbitrary unimodular amenable groups. Corollary 5.31 is a special case of this statement.

In Chapter 7 we show the variational principle for the topological pressure in the context of actions of σ-compact LCA groups. As the following version of the variational principle is a special case of the statement of Theorem 7.49 we will see a proof for the following below.

Theorem 5.33 (Variational principle - LCA version)**.** *Whenever G is a σ-compact LCA group, then*

$$\mathrm{E}(\pi \xrightarrow{p} \phi) = \sup_{\mu\in\mathcal{M}_G(X)} \mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right).$$

Remark 5.34. *Note that Theorem 5.33 is also satisfied for actions of the p-adic numbers $\mathbb{Q}_p$, an LCA group that contains no uniform lattice.*

5.3.3 The Kolmogorov-Sinai generator theorem

The Kolmogorov-Sinai generator theorem is an important tool that simplifies the calculation of measure theoretical entropy for various examples. Recall that whenever G is a discrete amenable group it is common to use the formula

$$\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right) = \sup_\alpha \mathrm{E}_\mu^*\left(\alpha|\pi \xrightarrow{p} \phi\right),$$

from Corollary 5.17 as the definition of measure theoretic entropy. Recall furthermore that in this formula the supremum is taken over all finite open covers α of X. Naturally the question arises, whether this supremum can be a maximum and one might ask for sufficient conditions on α to attain this maximum. The matter was solved already by A. N. Kolmogorov and J. G. Sinai and they state in their pioneering works (for $\mathbb{Z}$-actions) that a sufficient condition is that α is generating [Kol58, Sin59]. To define this notion let $M \subseteq G$. A finite partition α of X is called *generating along* M, if $\bigcup_F \alpha_F$ generates the topology of X, where the union is taken over all finite subsets $F \subseteq M$. If α is generating along G, we simply say that α is *generating*. The Kolmogorov-Sinai theorem allows to reduce the computational afford in computing the entropy with respect to a certain measure. For a proof of the statement in the context of discrete amenable groups see [Oll85, Theorem 4.3.14] but note that the statement can also be seen as a special case of Theorem 5.37 below for which we present a full proof.

Theorem 5.35 (Kolmogorov-Sinai generator theorem - discrete version)**.** *Assume that G is a discrete amenable group and let α be a finite generating partition of X. Then there holds*

$$\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right) = \mathrm{E}_\mu^*\left(\alpha|\pi \xrightarrow{p} \phi\right).$$

If one wishes to generalize the Kolmogorov-Sinai generator theorem beyond discrete groups one first encounters the problem that the definition of $\mathrm{E}_\mu^*\left(\alpha|\pi \xrightarrow{p} \phi\right)$ depends heavily on the fact that the Følner nets in discrete groups consist of finite sets. Nevertheless, this theorem serves in order to simplify the computation of measure theoretical entropy. It is thus natural to ask, whether for an action of a unimodular amenable group G, a uniform lattice Λ in G and a suitable finite partition α of X one can calculate the (relative) measure theoretical entropy by simply computing $\mathrm{E}_\mu^*\left(\alpha\middle|\pi|_{\Lambda\times X} \xrightarrow{p} \phi|_{\Lambda\times Y}\right)$ scaled by the uniform density of the uniform lattice. Imposing the assumption on α that it is already generating along the uniform lattice Λ, we obtain from the extrapolation technique, i.e. from Theorem 5.21 and the discrete version of the statement, the following.

Corollary 5.36 (Kolmogorov-Sinai generator theorem - extrapolated version)**.** *Whenever G is a unimodular amenable group and α is a finite partition that is generating along a uniform lattice Λ, then there holds*

$$\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right) = \mathrm{dens}(\Lambda)\, \mathrm{E}_\mu^*\left(\alpha\middle|\pi\Big|_{\Lambda\times X} \xrightarrow{p} \phi\Big|_{\Lambda\times X}\right).$$

A generator theorem along a net of uniform lattices

We consider next the weaker assumption that the partition is generating along a certain dense subgroup and obtain the following formula to compute the topological entropy. In Proposition 5.44 below we will see that such partitions always exist for Delone dynamical systems of (not necessarily FLC) Delone sets in $\mathbb{R}^d$.

Theorem 5.37. *Let $(\Lambda_j)_{j\in J}$ be a net of uniform lattices in G such that $\Lambda_j \subseteq \Lambda_{j'}$ whenever $j \leq j'$ and such that $H := \bigcup_{j\in J} \Lambda_j$ is dense in G. If α is a finite partition that is generating along H, then for any $\mu \in \mathcal{M}_G(X)$ the net*

$$\left(\mathrm{dens}(\Lambda_j)\, \mathrm{E}^*_\mu \left(\alpha \middle| \pi|_{\Lambda_j\times X} \overset{p}{\to} \phi|_{\Lambda_j\times X}\right)\right)_{j\in J}$$

is monotone increasing in j and converges to $\mathrm{E}_\mu\left(\pi \overset{p}{\to} \phi\right)$.

Remark 5.38. *One can for example consider the sequence of uniform lattices $(2^{-n}\mathbb{Z}^d)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$. The statement can furthermore be applied considering the sequence $(H_3(2^{-n}\mathbb{Z}))_{n\in\mathbb{N}}$ in the Heisenberg group $H_3(\mathbb{R})$. For details on the notions concerning the Heisenberg group see Example 2.44. Whenever G is discrete, then we can consider the trivial sequence $(G)_{n\in\mathbb{N}}$ and obtain the classical Kolmogorov-Sinai generator theorem as stated in Theorem 5.35.*

Remark 5.39. *We will see in Remark 5.54 below that the convergence is not necessarily uniform in μ.*

In order to show this theorem we present the following generalization of [Wal82, Theorem 4.12(iv)]. We include the slightly adapted proof for the convenience of the reader.

Proposition 5.40. *If G is discrete and μ is an invariant Borel probability measure on X, then for all finite partitions α and β of X there holds*

$$\mathrm{E}^*_\mu \left(\alpha \middle| \pi \overset{p}{\to} \phi\right) \leq \mathrm{E}^*_\mu \left(\beta \middle| \pi \overset{p}{\to} \phi\right) + H^*_{\mu,p}(\alpha|\beta).$$

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be a Van-Hove net in G and note that $H^*_{\mu,p}(\alpha_{F_n}) \leq H^*_{\mu,p}(\alpha_{F_n} \vee \beta_{F_n}) \leq H^*_{\mu,p}(\beta_{F_n}) + H^*_{\mu,p}(\alpha_{F_n}|\beta_{F_n})$. Furthermore, there holds

$$H^*_{\mu,p}(\alpha_{F_n}|\beta_{F_n}) \leq \sum_{f\in F_n} H^*_{\mu,p}(\alpha_f|\beta_{F_n}) \leq \sum_{f\in F_n} H^*_{\mu,p}(\alpha_f|\beta_f) = |F_n| H^*_{\mu,p}(\alpha|\beta).$$

We thus have $H_{\mu,p}(\alpha_{F_n})/|F_n| \leq H_{\mu,p}(\beta_{F_n})/|F_n| + H^*_{\mu,p}(\alpha|\beta)$ and taking the limit $n \to \infty$ we obtain the statement. □

We will furthermore need the following lemma.

Lemma 5.41. *Let* Λ *and* Λ' *be uniform lattices in* G *with uniform densities* $\operatorname{dens}(\Lambda)$ *and* $\operatorname{dens}(\Lambda')$ *respectively. Whenever there holds* $\Lambda \subseteq \Lambda'$*, then* Λ *is a uniform lattice in* Λ' *with respective uniform density* $\operatorname{dens}(\Lambda)/\operatorname{dens}(\Lambda')$*.*

Proof. To show that Λ is a uniform lattice in Λ' note first that Λ is a discrete subgroup of Λ'. Let us consider a compact subset K of G such that Λ is K-dense. As Λ' is discrete we obtain $F := K \cap \Lambda'$ to be finite. Furthermore, for $l' \in \Lambda' \subseteq G = K\Lambda$ there are $k \in K$ and $l \in \Lambda$ such that $l' = kl$ and we obtain $k = l'l^{-1} \in \Lambda'\Lambda = \Lambda'$. Hence, $k \in F$ and we have shown $\Lambda' \subseteq F\Lambda \subseteq \Lambda'$. As F is finite this shows Λ to be cocompact in Λ'. Thus, Λ is a uniform lattice in Λ'. From Lemma 3.6 we obtain that Λ has a uniform density with respect to Λ'. To compute this value consider a Van Hove net $(A_i)_{i\in I}$ in G and denote $F_i' := A_i \cap \Lambda'$ and $F_i := A_i \cap \Lambda = F_i' \cap \Lambda$. From Lemma 3.6 we obtain that $(F_i')_{i\in I}$ and $(F_i)_{i\in I}$ are Van Hove nets in Λ' and Λ respectively and furthermore that the density of Λ with respect to Λ' can be calculated as

$$\lim_{i\in I} \frac{|F_i|}{|F_i'|} = \lim_{i\in I} \frac{|F_i|}{\theta(A_i)} \frac{\theta(A_i)}{|F_i'|} = \lim_{i\in I} \frac{|F_i|}{\theta(A_i)} \lim_{j\in I} \frac{\theta(A_j)}{|F_j'|} = \frac{\operatorname{dens}(\Lambda)}{\operatorname{dens}(\Lambda')}.$$

□

Proof of Theorem 5.37. For $j \in J$ let us abbreviate $\pi_j := \pi|_{\Lambda_j \times X}$ and similarly $\phi_j := \phi|_{\Lambda_j \times Y}$. Consider $j, j' \in J$ such that $j \leq j'$. Then $\Lambda_j \subseteq \Lambda_{j'}$ and we obtain from Lemma 5.41 that Λ_j is a uniform lattice in $\Lambda_{j'}$ with respective uniform density $\operatorname{dens}(\Lambda_j)/\operatorname{dens}(\Lambda_{j'})$. Considering any respective (finite) fundamental domain $C \subseteq \Lambda'$ Theorem 5.26 yields

$$\begin{aligned}\operatorname{dens}(\Lambda_j)\, \mathrm{E}_\mu^*\left(\alpha \middle| \pi_j \xrightarrow{p} \phi_j\right) &\leq \operatorname{dens}(\Lambda_j)\, \mathrm{E}_\mu^*\left(\alpha_C \middle| \pi_j \xrightarrow{p} \phi_j\right) \\ &= \operatorname{dens}(\Lambda_{j'})\, \mathrm{E}_\mu^*\left(\alpha \middle| \pi_j \xrightarrow{p} \phi_j\right).\end{aligned}$$

This shows that the considered net is indeed monotone increasing and it remains to show that the supremum over this net is $\mathrm{E}_\mu(\pi \xrightarrow{p} \phi)$. To show this we fix $l \in J$ and consider $\epsilon > 0$. From Corollary 5.17 we obtain that there exists a finite partition β of X such that $\mathrm{E}(\beta|\pi_l \xrightarrow{p} \phi_l) + \epsilon \geq \mathrm{E}(\pi_l \xrightarrow{p} \phi_l)$. We denote by I the set of all finite subsets of M and order I by inclusion. As we assume that α generates along H we apply Lemma 4.8 to the net $(\alpha_F)_{F\in I}$ to obtain that there exists a finite set $F \in I$ such that $H_{\mu,p}^*(\beta|\alpha_F) \leq \epsilon + H_{\mu,p}^*(\beta|\mathcal{B}_X) = \epsilon$. As F is finite there exists $j \geq l$ such that $F \subseteq \Lambda_j$. Considering any Van Hove net $(E_\iota)_{\iota\in\tilde{I}}$ in Λ_j we thus obtain

$$\mathrm{E}_\mu^*\left(\alpha_F|\pi_j \xrightarrow{p} \phi_j\right) = \lim_{\iota\in\tilde{I}} \frac{H_{\mu,p}^*(\alpha_{FE_\iota})}{|E_\iota|} = \lim_{\iota\in\tilde{I}} \frac{H_{\mu,p}^*(\alpha_{FE_\iota})}{|FE_\iota|} = \mathrm{E}_\mu^*\left(\alpha|\pi_j \xrightarrow{p} \phi_j\right). \tag{5.2}$$

From Proposition 5.40 we obtain the following

$$\begin{aligned}
\mathrm{E}_\mu^*\left(\pi \xrightarrow{p} \phi\right) &= \mathrm{dens}(\Lambda_l)\,\mathrm{E}_\mu^*\left(\phi_l \xrightarrow{p} \phi_l\right) \\
&\leq \mathrm{dens}(\Lambda_l)\left(\mathrm{E}_\mu^*\left(\beta|\phi_l \xrightarrow{p} \phi_l\right) + \epsilon\right) \\
&\leq \mathrm{dens}(\Lambda_l)\left(\mathrm{E}_\mu^*\left(\alpha_F|\pi_l \xrightarrow{p} \phi_l\right) + H_{\mu,p}^*(\beta|\alpha_F) + \epsilon\right) \\
&\leq \mathrm{dens}(\Lambda_l)\,\mathrm{E}_\mu^*\left(\alpha_F|\pi_l \xrightarrow{p} \phi_l\right) + \mathrm{dens}(\Lambda_l)2\epsilon.
\end{aligned}$$

From the already shown monotonicity and (5.2) we then compute

$$\begin{aligned}
\mathrm{E}_\mu^*\left(\pi \xrightarrow{p} \phi\right) &\leq \mathrm{dens}(\Lambda_j)\,\mathrm{E}_\mu^*\left(\alpha_F|\pi_j \xrightarrow{p} \phi_j\right) + \mathrm{dens}(\Lambda_l)2\epsilon \\
&= \mathrm{dens}(\Lambda_j)\,\mathrm{E}_\mu^*\left(\alpha|\pi_j \xrightarrow{p} \phi_j\right) + \mathrm{dens}(\Lambda_l)2\epsilon \\
&\leq \sup_{j\in J}\mathrm{dens}(\Lambda_j)\,\mathrm{E}_\mu^*\left(\alpha|\pi_j \xrightarrow{p} \phi_j\right) + \mathrm{dens}(\Lambda_l)2\epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary and independent of l we thus observe

$$\mathrm{E}_\mu^*\left(\pi \xrightarrow{p} \phi\right) \leq \sup_{j\in J}\mathrm{dens}(\Lambda_j)\,\mathrm{E}_\mu^*\left(\alpha|\pi_j \xrightarrow{p} \phi_j\right).$$

From Corollary 5.17 and Theorem 5.21 we furthermore see for any $j \in J$ that there holds

$$\begin{aligned}
\mathrm{dens}(\Lambda_j)\,\mathrm{E}_\mu^*\left(\alpha|\pi_j \xrightarrow{p} \phi_j\right) &\leq \mathrm{dens}(\Lambda_j)\,\mathrm{E}_\mu^*\left(\pi_j \xrightarrow{p} \phi_j\right) \\
&= \mathrm{E}_\mu^*\left(\pi \xrightarrow{p} \phi\right)
\end{aligned}$$

and we have proven the statement. □

Naturally one asks whether the existence of a uniform lattice is sufficient to guarantee the existence of a monotone sequence of uniform lattice with dense union. We will next see that this is not the case even for compactly generated LCA groups.

Example 5.42. *Consider the additive group of p-adic integers $\mathbb{Z}_p$ for some prime p. As argued in Example 2.42 $\mathbb{Q}_p$ has no non-trivial discrete subgroup and we obtain that also $\mathbb{Z}_p$ contains no non-trivial discrete subgroup. Thus, there is no net of uniform lattices such that the union over this family is dense in $\mathbb{Z}_p$. Nevertheless, as $\mathbb{Z}_p$ is compact we obtain that $\{0\}$ is a uniform lattice. Similarly one argues that the non-compact but compactly generated LCA group $\mathbb{R} \times \mathbb{Z}_p$ contains no net of uniform lattices with a dense union.*

Generating partitions for Delone actions of $\mathbb{R}^d$

Naturally the question arises which actions allow the construction of a generating partition. It is well-known that the full shift on a discrete group has a generating partition. See for example [Kel98, Example 3.2.20] for the case of $\mathbb{Z}^d$. The following can also be seen from a simple and similar argument.

Example 5.43. *Let G be a discrete LCA group and ω be a Delone set in G. Then it is straightforward to show that the partition of X_ω that consists of the sets $\{\xi \in X_\omega;\, 0 \in \xi\}$ and $\{\xi \in X_\omega;\, 0 \notin \xi\}$ is a generating partition for the Delone action π_ω.*

The matter somehow complicates whenever one considers non-discrete groups. We illustrate this considering the case of $\mathbb{R}^d$ and present next that Theorem 5.37 can be applied to the Delone actions of $\mathbb{R}^d$.

Proposition 5.44. *Let ω be a Delone set in $\mathbb{R}^d$. Then there exists a finite partition α of X_ω that is generating along any dense subgroup H of $\mathbb{R}^d$. In particular, for any open neighbourhood V of 0 such that ω is V-discrete and any $\delta > 0$ that satisfies $[-\delta,\delta]^d \subseteq V$ the partition α that consists of the sets $\{\xi \in X_\omega;\, \xi \cap [-\delta,\delta]^d \neq \emptyset\}$ and $\{\xi \in X_\omega;\, \xi \cap [-\delta,\delta]^d = \emptyset\}$ satisfies the claimed properties.*

Remark 5.45. *It remains open, whether one can also construct generating partitions for Delone actions of arbitrary LCA groups.*

For the proof of Proposition 5.44 we will need the following lemma. Recall that we abbreviate $\langle\alpha\rangle = \bigcup_{A\in\alpha} A^2$ for any partition α of a set X.

Lemma 5.46. *Let $\{\alpha_i\}_{i\in I}$ be a countable family of finite partitions of a compact Hausdorff space X and $\mathbb{B}$ be a base of the uniformity $\mathbb{U}_X$. If for any $\eta \in \mathbb{B}$ there exists $i \in I$ such that $\langle\alpha_i\rangle \subseteq \eta$, then the σ-algebra $\sigma(\bigcup_{i\in I}\alpha_i)$ generated by $\bigcup_{i\in I}\alpha_i$ is the Borel σ-algebra.*

Proof. As α_i consists of Borel measurable sets for any $i \in I$ we obtain that $\sigma(\bigcup_{i\in I}\alpha_i)$ is contained in the Borel σ-algebra. To show the reverse let O be an open set and consider for any $x \in X$ an $\eta_x \in \mathbb{B}$ such that $B_{\eta_x}(x) \subseteq O$. Our assumption furthermore allows to choose $i_x \in I$ such that $\langle\alpha_{i_x}\rangle \subseteq \eta_x$ and we obtain in particular that for any $x \in X$ there holds $x \in \langle\alpha_{i_x}\rangle[x] \subseteq \eta_x[x] = B_{\eta_x}(x) \subseteq O$. Thus, $O = \bigcup_{x\in O}\langle\alpha_{i_x}\rangle[x]$. Now recall that I is assumed to be countable and that all α_i are finite. Thus, $\bigcup_{i\in I}\alpha_i$ is countable. As $\alpha_{i_x}[x]$ is the set in the partition α_{i_x} that contains x we thus obtain $\{\langle\alpha_{i_x}\rangle[x];\, x \in O\} \subseteq \bigcup_{i\in I}\alpha_i$ to be countable and thus $O = \bigcup_{x\in O}\langle\alpha_{i_x}\rangle[x]$ is contained in $\sigma(\bigcup_{i\in I}\alpha_i)$. □

Proof of Proposition 5.44. Let V be an open neighbourhood of 0 such that ω is V-discrete and let $\delta > 0$ such that $C := [-\delta,\delta]^d \subseteq V$. Consider the partition α that consists of the sets $\{\xi \in X_\omega;\, \xi\cap[-\delta,\delta]^d \neq \emptyset\}$ and $\{\xi \in X_\omega;\, \xi\cap[-\delta,\delta]^d = \emptyset\}$. In order to show that this partition consists of measurable sets we will show that $A := \{\xi \in X_\omega;\, \xi \cap C \neq \emptyset\}$

is a closed subset of X_ω. To do this consider a net $(\xi_i)_{i\in I}$ in A that converges to some $\xi \in X_\omega$. For $i \in I$ we know that there holds $\xi_i \cap C \neq \emptyset$ and we can choose $g_i \in \xi_i \cap C$. As C is compact and as V is a neighbourhood of C there exists an open neighbourhood U of 0 such that $C + U \subseteq V$ and we assume without lost of generality that $U \subseteq V$. For an open neighbourhood W of 0 with $W \subseteq -U$ we then know that for large i there holds $(\xi_i, \xi) \in \epsilon(C, W)$ and in particular $g_i \in \xi_i \cap C \subseteq \xi + W$. Thus, there is $v_i \in W$ such that $g_i \in \xi + v_i$. We obtain $g_i - v_i \in \xi \cap (C - W) \subseteq \xi \cap (C + U) \subseteq \xi \cap V$ and as ξ is V-discrete there is a unique $g \in \xi$ with $g = g_i - v_i$ for all sufficiently large i. In particular, there holds $g_i - g \in W$ for all large i and we have shown that g_i converges to g. As C is closed we thus obtain $g \in \xi \cap C$ and we have shown $\xi \in A$. This shows that A is indeed closed and we have proven α to be measurable.

To show that α is generating let $\check{H}$ be a countable and dense subgroup of H and consider the set I of all finite subsets of $\check{H}$. Then I is countable and we can apply Lemma 5.46 to the local rubber base to obtain that it is sufficient to show that for all compact $K \subseteq G$ and all open neighbourhoods U of 0 there exists $F \in I$ such that $\langle \alpha_F \rangle \subseteq \epsilon(K, U)$.

Let W be an open neighbourhood of 0 such that $W - W \subseteq U$. Choose furthermore $E \subseteq (-\delta, \delta)^d \cap H$ finite such that $\bigcap_{e\in E}(C - e) = \bigcap_{e\in E}[-\delta, \delta]^d - e \subseteq W$. Note that $\bigcap_{e\in E}(C - e)$ is a neighbourhood of 0 and we thus obtain from the compactness of K that there is $M \subseteq \check{H}$ finite such that $K \subseteq (\bigcap_{e\in E}(C - e)) + M$. We then obtain $F := E - M \subseteq \check{H} - \check{H} = \check{H}$, i.e. $F \in I$.

Let $(\xi, \zeta) \in \langle \alpha_F \rangle$. We only show that $\xi \cap K \subseteq \zeta + U$ as the other inclusion can be shown similarly. Let $g \in \xi \cap K$. As $g \in K$ there is $m \in M$ such that $g \in \bigcap_{e\in E}(C - e + m)$. For $e \in E$ there holds $g + (e - m) \in C$, which implies $\pi^{e-m}(\xi) = \xi + (e - m) \in A$. This shows $\xi \in \bigcap_{e\in E} \pi^{-(e-m)}(A)$. Now recall that $F = E - M \supseteq E - m$ and thus α_F is a finer partition than α_{E-m}. From $(\xi, \zeta) \in \langle \alpha_F \rangle$ we thus obtain that ξ and ζ are contained in the same partition element $\bigcap_{e\in E} \pi^{-(e-m)}(A)$ of α_{E-m} and in particular $\zeta \in \bigcap_{e\in E} \pi^{-(e-m)}(A)$. Thus, for any $e \in E$ there holds $\zeta + (e - m) \in A$, i.e. $(\zeta + (e - m)) \cap C \neq \emptyset$. For each $e \in E$ choose $h_e \in \zeta \cap (C - e + m)$. Then for $e, e' \in E \subseteq (-\delta, \delta)^d \subseteq V$ there holds $h_e - h_{e'} \in V - V$. As ζ is V-discrete we obtain $h_e = h_{e'}$ and there is $h \in \zeta \cap \bigcap_{e\in E}((C - e + m)) \subseteq (\bigcap_{e\in E} C - e) + m \subseteq W + m$. Recall that also $g \in W + m$. Thus, $h - z \in W - W \subseteq U$ and hence $g \in h + U \subseteq \zeta + U$. This proves $\xi \cap C \subseteq \zeta + U$ and we have shown the statement. □

Remark 5.47. *Note that with similar arguments one also shows that for an open neighbourhood V of 0 there always exists a finite partition α of $\mathcal{D}_V(\mathbb{R}^d)$ that is generating along any dense subgroup H of $\mathbb{R}^d$ with respect to the shift on $\mathcal{D}_V(\mathbb{R}^d)$.*

5.3.4 On upper semi-continuity of the entropy map

We will see in Chapter 7 that the upper semi-continuity of the entropy map $\mu \mapsto \mathrm{E}_\mu(\pi)$ is an important property of an action that simplifies many aspects of the thermodynamic formalism. It is well-known that for discrete actions the Kolmogorov-Sinai generator theorem can be used to formulate sufficient conditions for the upper semi-continuity of the entropy map. To discuss how this condition can be extrapolated we present next the following well-known statement. We include the short proof for the convenience of the reader. For reference see [Wal82].

Proposition 5.48. *Let G be a discrete amenable group and μ be an invariant Borel probability measure on X. Then, whenever α is a finite partition that has almost no boundary with respect to μ the map $\mathcal{M}_G(X) \ni \nu \to \mathrm{E}_\nu (\alpha|\pi \to \phi)$ is upper semi-continuous in μ.*

Proof. As discussed above we know that $\mathcal{K}(G) \ni F \mapsto H^*_{\mu,p}(\alpha_F)$ is monotone, right invariant and subadditive. In fact it can be shown that this map is even strongly subadditive [Wal82]. Considering any Van Hove net $(F_i)_{i\in I}$ in G we thus obtain from Theorem 3.2 that there holds

$$\mathrm{E}_\mu (\alpha|\pi \to \phi) = \inf_{i\in I} \frac{H^*_{\mu,p}(\alpha_{F_i})}{|F_i|}.$$

As α has almost no boundary with respect to μ the same is valid for all refinements α_F with $F \subseteq G$ finite. Thus, we obtain from Lemma 4.9 that $\mathcal{M}_G(X) \ni \mu \mapsto H_{\mu,p}(\alpha_F)/|F|$ is upper semi-continuous. As the infimum of a family of functions which are upper semi-continuous in μ is upper semi-continuous μ the statement follows. □

From the discrete version of the Kolmogorov-Sinai generator theorem formulated in Theorem 5.35 we conclude the following.

Corollary 5.49. *Let G be a discrete amenable group and μ be an invariant Borel probability measure. Then whenever there exists a finite partition of X that is generating and that has almost no boundary with respect to μ, then the entropy map $\mathcal{M}_G(X) \ni \mu \mapsto \mathrm{E}_\mu (\pi \to \phi)$ is upper semi-continuous in μ.*

From Theorem 5.21 we thus obtain the following.

Corollary 5.50. *Let G be a unimodular amenable group and μ be an invariant Borel probability measure. Then whenever there exists a finite partition that is generating along a uniform lattice in G and that has almost no boundary with respect to μ, then the entropy map $\mathcal{M}_G(X) \ni \mu \mapsto \mathrm{E}_\mu (\pi \to \phi)$ is upper semi-continuous in μ.*

Remark 5.51. *Considering a non-discrete unimodular amenable group G and an invariant Borel probability measure it is natural to ask whether the existence of a finite partition that is generating (along G) and that has almost no boundary with respect to μ is sufficient to ensure that that the entropy map $\mathcal{M}_G(X) \ni \mu \mapsto \mathrm{E}_\mu (\pi \to \phi)$ is upper semi-continuous in μ. This is not the case already for $G = \mathbb{R}$. Indeed, we will discuss in Example 6.37 below an action on a compact Hausdorff space X with an entropy map that* *is not upper semi-continuous but such that for any invariant Borel probability measure μ on X there exists a finite partition that is generating along $\mathbb{R}$ and that has almost no boundary with respect to μ.*

To be more precise we will consider Delone actions of $\mathbb{R}$ to observe this situation. In fact we will see next that all Delone actions allow the construction of a finite partition that has almost no boundary with respect to a given invariant Borel probability measure, but which is generating along $\mathbb{R}$.

Proposition 5.52. *Let ω be a Delone set in $\mathbb{R}^d$ and consider a π_ω-invariant Borel probability measure μ on X_ω. Then there exists a finite partition α of X_ω with almost no boundary respect to μ such that α is generating along any dense subgroup of $\mathbb{R}^d$.*

Proof. Let $\overline{\delta} > 0$ such that $\left[-2\overline{\delta}, 2\overline{\delta}\right]^d \subseteq V$. For $\delta \in \left(0, \overline{\delta}\right)$ define $O_\delta := \{\xi \in X_\omega;\, \xi \cap (-\delta,\delta)^d \neq \emptyset\}$ and $A_\delta := \{\xi \in X_\omega;\, \xi \cap [-\delta,\delta]^d \neq \emptyset\}$. Now recall from Proposition 5.44 that for any $\delta \leq \overline{\delta}$ the partition $\{A_\delta, X_\omega \setminus A_\delta\}$ is generating along any dense subgroup of $\mathbb{R}^d$. It thus suffices to find δ such that A_δ has no boundary with respect to μ. We will do this by applying Froda's theorem.

We first show that O_δ is open for any $\delta \in (0, \overline{\delta})$. Let $\xi \in O_\delta$. As any $\xi \in X_\omega$ is $[-\delta,\delta]^d \subseteq V$-discrete and $\xi \cap (-\delta,\delta) \neq \emptyset$ there is $x \in \mathbb{R}^d$ such that $\{x\} = \xi \cap (-\delta,\delta)^d = \xi \cap [-\delta,\delta]^d$. Let $\epsilon > 0$ such that $B_\epsilon(x) \subseteq (-\delta,\delta)^d$ and set $\eta := \epsilon_\omega([-\delta,\delta]^d, B_\epsilon(0))$. Then for $\zeta \in \eta[\xi]$ there holds $\zeta + B_\epsilon(0) \supseteq \xi \cap [-\delta,\delta]^d = \{x\}$. Thus, there exists $b \in B_\epsilon(0)$ such that $x - b \in \zeta$. As $x - b \in B_\epsilon(x) \subseteq (-\delta,\delta)^d$ this yields $\zeta \in O_\delta$. This shows that $\eta[\zeta] \subseteq O_\delta$ and we obtain O_δ to be open.

Let us next show that A_δ is the topological closure $\overline{O_\delta}$ of O_δ for $\delta \in (0, \overline{\delta})$. We first show $A_\delta \supseteq \overline{O_\delta}$. Let $\xi \in \overline{O_\delta}$. Then there exists a net $(\xi_i)_{i\in I}$ in O_δ such that $\xi_i \to \xi$. For $i \in I$ we obtain as above the existence of $x_i \in \mathbb{R}^d$ such that $\{x_i\} = \xi_i \cap (-\delta,\delta)^d = \xi_i \cap [-\delta,\delta]^d$. As $(x_i)_{i\in I}$ is a net in the compact set $[-\delta,\delta]^d$ there exists a subnet that converges to some $x \in [-\delta,\delta]^d$. Restricting to the corresponding subnet of $(\xi_i)_{i\in I}$ we thus assume without lost of generality that already $x_i \to x$. As $\xi_i \to \xi$ and by the V-discreteness of ξ we know $\{x\} = \xi \cap (-\delta,\delta)^d = \xi \cap [-\delta,\delta]^d$ and in particular $\xi \in A_\delta$. To show the reverse inclusion let $\xi \in A_\delta$. As above there is $x \in \mathbb{R}^d$ such that $\{x\} = \xi \cap [-\delta,\delta]^d$. Let $(x_i)_{i\in I}$ be any net in $(-\delta,\delta)^d$ such that $x_i \to x$. Then $\xi_i := \xi - x + x_i \in O_\delta$ satisfies $\xi_i \to \xi$ and we obtain $\xi \in \overline{O_\delta}$.

We can now show that there is indeed $\delta \in (0, \overline{\delta})$ such that $\mu(\partial A_\delta) = 0$. Consider $f\colon (0, \overline{\delta}) \to [0,1]\colon \delta \mapsto \mu(A_\delta)$ and note that f is monotone. Thus, by Froda's theorem there is $\delta \in (0, \delta_0)$ such that f is continuous in δ and we obtain $\mu(A_\delta) = \sup_{r\in(0,\delta)} \mu(A_r)$. Note that $A_r \subseteq O_\delta$ for $r \in (0,\delta)$. Furthermore, for $r \in (0,\delta)$ we know that A_r is closed as the closure of O_δ and as X_ω is compact we obtain A_r to be compact. As O_δ is open and μ is regular we get

$$\mu(O_\delta) = \sup_{A\subseteq O_\delta \text{ compact}} \mu(A) \geq \sup_{r\in(0,\delta)} \mu(A_r) = \mu(A_\delta) \geq \mu(O_\delta).$$

Thus, $0 \leq \mu(\partial A_\delta) \leq \mu(A_\delta) - \mu(O_\delta) = 0$ and we have shown the statement. □

Remark 5.53. *As above one obtains with similar arguments the analogue statement for the shift on $\mathcal{D}_V(\mathbb{R}^d)$ for any open neighbourhood V of 0.*

Remark 5.54. *Whenever $(f_i)_{i\in I}$ is a sequence of maps that are upper semi-continuous in x that converges uniformly*[2] *to a function f, then it can be seen with an easy $\epsilon/3$-argument that f is also upper semi-continuous in x. From Proposition 5.48 we know that whenever α is a finite partition with almost no boundary with respect to μ, then*

$$\mathcal{M}_G(X) \ni \nu \mapsto \mathrm{E}^*_\nu\left(\alpha \middle| \pi\big|_{\Lambda\times X} \overset{p}{\to} \phi\big|_{\Lambda\times X}\right)$$

is upper semi-continuous in μ for all uniform lattices $\Lambda \subseteq G$. Hoping to prove the entropy map to be upper semi-continuous it is thus natural to ask, whether the convergence of the net

$$\left(\mathrm{dens}(\Lambda_j)\,\mathrm{E}^*_\mu\left(\alpha \middle| \pi|_{\Lambda_j\times X} \overset{p}{\to} \phi|_{\Lambda_j\times X}\right)\right)_{j\in J}$$

to $\mathrm{E}_\mu\left(\pi \overset{p}{\to} \phi\right)$ as shown in Theorem 5.37 for finite partitions α that generate along the dense subgroup $\bigcup_{j\in J}\Lambda_j$ is actually uniform in μ. Proposition 5.52 allows to see that this already fails for $G = \mathbb{R}$. Indeed in Example 6.37 we will see that there are Delone sets ω in $\mathbb{R}$ such that the entropy function of π_ω is not upper semi-continuous. Nevertheless, from Proposition 5.52 we obtain that for any invariant Borel probability measure μ there exists a finite partition that is generating along $\bigcup_{j\in J}\Lambda_j$ and which has furthermore no boundary with respect to μ.

5.3.5 Bowen's formula

Bowens's formula can be seen as the topological analogue of the Rokhlin-Abramov Theorem. In fact it is not hard to deduce it from the Rokhlin-Abramov Theorem via the variational principle. Unfortunately we do not know of a reference of Bowen's formula in the case of discrete amenable groups. For the convenience of the reader we thus include a proof, but note that some of the corresponding ideas can be found in the literature, such as for example the proof of [Yan15, Theorem 5.7].

For the proof we will need to consider the push forward $p_*\mu$ of a Borel probability measure μ along $p\colon X \to Y$. As p is assumed to be a factor map we obtain that $p_*\mu \in \mathcal{M}_G(Y)$, whenever $\mu \in \mathcal{M}_G(X)$. With a Krylov–Bogolyubov argument one can show that p_* is a surjection. We include the argument for the convenience of the reader.

Proposition 5.55. *The map $p_*\colon \mathcal{M}_G(X) \to \mathcal{M}_G(Y)$ is surjective whenever G is a discrete amenable group.*

[2] A sequence $(f_i)_{i\in I}$ of maps $f_i\colon K \to [0,\infty)$ is said to converge uniformly to $f\colon K \to [0,\infty)$ whenever for all $\epsilon > 0$ there exists $j \in I$ such that for all $i \geq j$ and for any $y \in K$ there holds $|f_i(y) - f(y)| < \epsilon$.

Remark 5.56. *With some more technical afford the statement can also be shown with similar methods for general unimodular amenable groups. As we do not need the statement in this generality we only present the proof for discrete groups.*

Remark 5.57. *Taking ϕ as the one point flow, we obtain the Krylov–Bogolyubov theorem, i.e. that $\mathcal{M}_G(X)$ is non-empty for discrete amenable groups.*

Proof of Proposition 5.55. Note first that the pull back $p^*\colon C(Y) \to X(Y)$ that maps $p^*(f) := f \circ p$ is an injective, linear and continuous operator and thus $C(Y)$ can be identified with a subspace of $C(X)$. As we identify the topological dual of $C(X)$ with $\mathcal{M}(X)$ the Hahn-Banach theorem allows to deduce that $p_*\colon \mathcal{M}(X) \to \mathcal{M}(Y)$ is surjective. Thus, for $\nu \in \mathcal{M}_G(Y) \subseteq \mathcal{M}(Y)$ there is $\mu \in \mathcal{M}(X)$ such that $p_*\mu = \nu$. Considering an ergodic net $(F_i)_{i\in I}$ in G we define

$$\nu_i := \frac{1}{|F_i|} \sum_{g\in F_i} (\pi_*^g \nu)$$

for any $i \in I$. Then clearly $(\nu_i)_{i\in I}$ is a net in the weak-* compact set $\mathcal{M}(Y)$ and thus there exists a weak-* limit point $\hat{\nu}$ and we can assume without lost of generality that $\nu_i \to \hat{\nu}$. Now for any $g \in G$ and any $f \in C(Y)$ there holds $|\pi_*^g\nu(f)| \leq \int_X |f| d\pi_*^g \nu \leq \|f\|_\infty$. Thus, for any $g' \in G$ we compute

$$\left|(\pi_*^{g'}\hat{\nu} - \hat{\nu})(f)\right| \leq \frac{1}{|F_i|} \sum_{(g'F_i)\Delta F_i} |\pi_*^g \nu(f)| d\theta(g) \leq \frac{|(g'F_i)\Delta F_i|}{|F_i|} \|f\|_\infty.$$

As we assume $(F_i)_{i\in I}$ to be ergodic we obtain from the arbitrary choice of f that $\pi_*^{g'}\hat{\nu} = \hat{\nu}$ for any $g' \in G$. This proves $\hat{\nu} \in \mathcal{M}_G(Y)$ and it remains to show that $p_*\mu = \hat{\nu}$. For this we use that p is a factor map and that μ is invariant and compute for $g \in G$ that

$$\phi_*^g \nu = \phi_*^g p_* \mu = (\phi^g \circ p)_* \mu = (p \circ \pi^g)_* \mu = p_* \pi_*^g \mu = p_* \mu.$$

As p_* is affine we thus obtain that for any $i \in I$ there holds $p_*\mu = \phi_*^g \nu_i$. As $\nu_i \to \hat{\nu}$ we obtain from the weak-* continuity of p_* that there also holds $p_*\mu = \hat{\nu}$. □

Theorem 5.58 (Bowen's formula - discrete version)**.** *Assume that G is a countable discrete amenable group. Let ψ be a factor of ϕ via a factor map $q\colon Y \to Z$ and recall that we assume that ϕ is a factor of π via the factor map p, i.e. $\pi \xrightarrow{p} \phi \xrightarrow{q} \psi$. There holds*

$$\max\{\mathrm{E}(\pi \xrightarrow{p} \phi), \mathrm{E}(\phi \xrightarrow{q} \psi)\} \leq \mathrm{E}(\pi \xrightarrow{q\circ p} \psi) \leq \mathrm{E}(\pi \xrightarrow{p} \phi) + \mathrm{E}(\phi \xrightarrow{q} \psi).$$

Whenever ϕ is uniquely ergodic, then there holds

$$\mathrm{E}(\pi \xrightarrow{q\circ p} \psi) = \mathrm{E}(\pi \xrightarrow{p} \phi) + \mathrm{E}(\phi \xrightarrow{q} \psi).$$

Proof. The first inequality follows from Remark 4.17 and Proposition 4.34. As $p_*\colon \mathcal{M}_G(X) \to \mathcal{M}_G(Y)$ is surjective we furthermore observe

$$\mathrm{E}\left(\phi \xrightarrow{q} \psi\right) = \sup_{\nu\in\mathcal{M}_G(Y)} \mathrm{E}_\nu\left(\phi \xrightarrow{q} \psi\right) = \sup_{\mu\in\mathcal{M}_G(X)} \mathrm{E}_{p_*\mu}\left(\phi \xrightarrow{q} \psi\right).$$

We thus obtain the statement from Theorem 5.27 and the following computation

$$\begin{aligned}
\mathrm{E}(\pi \xrightarrow{q\circ p} \psi) &= \sup_{\mu\in\mathcal{M}_G(X)} \mathrm{E}_\mu\left(\pi \xrightarrow{q\circ p} \phi\right)\\
&= \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right) + \mathrm{E}_{p_*\mu}\left(\phi \xrightarrow{q} \psi\right)\right)\\
&\leq \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right)\right) + \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_{p_*\mu}\left(\phi \xrightarrow{q} \psi\right)\right)\\
&= \mathrm{E}\left(\pi \xrightarrow{p} \phi\right) + \mathrm{E}\left(\phi \xrightarrow{q} \psi\right).
\end{aligned}$$

To show the second statement let us denote by ν the unique invariant Borel probability measure on Y. Then there holds $p_*\mu = \nu$ for any $\mu \in \mathcal{M}_G(X)$ and thus $E_{p_*\mu}\left(\phi \xrightarrow{p} \psi\right) = \mathrm{E}\left(\phi \xrightarrow{p} \psi\right)$ by the variational principle and we compute

$$\begin{aligned}
\mathrm{E}(\pi \xrightarrow{q\circ p} \psi) &= \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right) + \mathrm{E}_{p_*\mu}\left(\phi \xrightarrow{q} \psi\right)\right)\\
&= \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right)\right) + \mathrm{E}\left(\phi \xrightarrow{q} \psi\right)\\
&= \mathrm{E}\left(\pi \xrightarrow{p} \phi\right) + \mathrm{E}\left(\phi \xrightarrow{q} \psi\right).
\end{aligned}$$

□

The proof of the previous theorem shows that the combination of the variational principle and the Rokhlin-Abramov Theorem gives the Bowen formula. As we know of neither to hold for actions of general unimodular amenable groups we cannot deduce the general version of the Bowen formula so far. Nevertheless, it is straightforward to extrapolate the first statement of the theorem with Theorem 5.21.

Corollary 5.59 (Bowen's formula - extrapolated version)**.** *Assume that G is a unimodular amenable group that contains a countable uniform lattice. Let ψ be a factor of ϕ via a factor map q and recall that we assume that ϕ is a factor of π via the factor map p, i.e. $\pi \xrightarrow{p} \phi \xrightarrow{q} \psi$. There holds*

$$\max\{\mathrm{E}(\pi \xrightarrow{p} \phi), \mathrm{E}(\phi \xrightarrow{q} \psi)\} \leq \mathrm{E}(\pi \xrightarrow{q\circ p} \psi) \leq \mathrm{E}(\pi \xrightarrow{p} \phi) + \mathrm{E}(\phi \xrightarrow{q} \psi).$$

Remark 5.60. *Note that $\mathrm{E}(\pi \xrightarrow{p} \phi) \leq \mathrm{E}(\pi \xrightarrow{q\circ p} \psi)$ and $\mathrm{E}(\phi \xrightarrow{q} \psi) \leq \mathrm{E}(\pi \xrightarrow{q\circ p} \psi)$ also hold without the assumption of the existence of a uniform lattice as seen in Remark 4.17 and*

Proposition 4.34. Nevertheless, it remains open, whether we also have

$$\mathrm{E}(\pi \overset{q\circ p}{\to} \psi) \leq \mathrm{E}(\pi \overset{p}{\to} \phi) + \mathrm{E}(\phi \overset{q}{\to} \psi)$$

for all actions of unimodular amenable groups.

Remark 5.61. *Note that whenever ϕ is uniquely ergodic, then the restriction of ϕ to a uniform lattice is not necessarily uniquely ergodic. To see this consider for example the rotation $\phi\colon \mathbb{R} \times \mathbb{T} \to \mathbb{T}$ with $\phi(r, x) = x + r$ and note that ϕ is uniquely ergodic (already the restriction to $\alpha\mathbb{Z}$ is uniquely ergodic for α irrational) while the restriction to $\mathbb{Z}$ is not uniquely ergodic. Thus, one cannot use the extrapolation technique to show that there holds $\mathrm{E}(\pi \overset{q\circ p}{\to} \psi) = \mathrm{E}(\pi \overset{p}{\to} \phi) + \mathrm{E}(\phi \overset{q}{\to} \psi)$ whenever ϕ is uniquely ergodic. Nevertheless, whenever the variational principle holds in G, then one can use the extrapolated version of the Rokhlin-Abramov Theorem as stated in Corollary 5.28 to argue as in the proof of Theorem 5.58 to obtain the statement also for G. In particular, by Theorem 5.33 there holds $\mathrm{E}(\pi \overset{q\circ p}{\to} \psi) = \mathrm{E}(\pi \overset{p}{\to} \phi) + \mathrm{E}(\phi \overset{q}{\to} \psi)$ whenever ϕ is uniquely ergodic and G is a σ-compact LCA group.*

Letting ψ be the one point flow we obtain Bowen's formulation for unimodular amenable groups that contain uniform lattices.

Corollary 5.62 (Bowen's formula - classical version)**.** *Assume that G is a unimodular amenable group that contains a countable uniform lattice. Then there holds*

$$\mathrm{E}(\pi) \leq \mathrm{E}(\pi \overset{p}{\to} \phi) + \mathrm{E}(\phi).$$

We have already seen that whenever ϕ is uniquely ergodic (and whenever the variational principle holds in G), then we obtain an equality in Corollary 5.62. We next extrapolate some sufficient conditions that ensure this equality. Note first that the equality is satisfied whenever the relative topological entropy of p is 0, i.e. $\mathrm{E}(\pi \overset{p}{\to} \phi) = 0$. In the context of countable discrete amenable groups this topic is discussed in [Yan15]. To discuss the results from [Yan15] we will need the following notions. Recall that $p\colon X \to Y$ is assumed to be a factor map. We say that p is *countable to one*, whenever $p^{-1}(y)$ is countable for all $y \in Y$. We furthermore call two points $x, x' \in X$ *distal*, whenever there is $\eta \in \mathbb{U}_X$ such that $(g.x, g.x') \notin \eta$ for all $g \in G$. The factor map p is said to be *distal*, whenever for all $y \in Y$ two distinct points in $p^{-1}(y)$ are distal. From [Yan15, Theorem 5.7] and [Yan15, Corollary 6.7] we quote the following result.

Proposition 5.63. *If G is a (non-compact) countable discrete amenable group and X and Y are metrizable, then $\mathrm{E}(\pi \overset{p}{\to} \phi) = 0$ is satisfied, whenever p is countable to one or distal.*

Note that the property that p is a countable to one factor map is independent of the acting group. We thus obtain from the extrapolation technique the following corollary.

Corollary 5.64. *If G is a (non-compact) unimodular amenable group that contains a countable uniform lattice and whenever X and Y are metrizable, then* $\mathrm{E}(\pi) = \mathrm{E}(\phi)$ *and* $\mathrm{E}(\pi \xrightarrow{p} \phi) = 0$, *whenever p is countable to one.*

Distality of a factor map clearly depends on the acting group. Nevertheless, whenever Λ is a subgroup of G and $x, x' \in X$ are distal with respect to the action of G, then they are also distal with respect to the action of Λ. In particular, one obtains that whenever p is a distal factor map with respect to the action of G, then it is also a distal factor map with respect to the action of Λ. We can thus apply the extrapolation technique to obtain the following.

Corollary 5.65. *If G is a (non-compact) unimodular amenable group that contains a countable uniform lattice and whenever X and Y are metrizable, then* $\mathrm{E}(\pi) = \mathrm{E}(\phi)$ *and* $\mathrm{E}(\pi \xrightarrow{p} \phi) = 0$, *whenever p is distal.*

6 On entropy of Delone sets

In this chapter we will use the developed tools in order to answer the questions raised in the introduction of this thesis concerning the patch counting entropy. In order to do this we present in Section 6.1 a geometric approach to the topological entropy of a Delone set that avoids the construction of the Delone dynamical system and works for all LCA groups. In Section 6.2 this approach will be related to the patch counting entropy for FLC Delone sets as discussed in the introduction and it will be shown that whenever one computes the patch counting entropy in a compactly generated LCA group and along a certain type of Van Hove net, then the patch counting entropy equals the topological entropy. Note that this extends the result of [BLR07], where it is shown that the patch counting entropy equals the topological entropy for FLC Delone sets in $\mathbb{R}^d$, whenever computed along a Van Hove sequence of centred closed balls. We show furthermore that in this context the patch counting entropy can be computed as a limit. In Section 6.3 we will see that already for FLC Delone in $\mathbb{R}$ the patch counting entropy depends on the choice of a Van Hove sequence. In Section 6.4 we will present different Delone sets in the additive group of the 2-adic numbers $\mathbb{Q}_2$. Computing the patch counting entropy along the Van Hove sequence of closed centred balls we will see that the limit in the formula does not always exist and furthermore that the patch counting entropy is not always the topological entropy of the associated action. These examples heavily use the geometrical approach to topological entropy of Section 6.1 as this one allows to define FLC Delone sets in $\mathbb{Q}_2$ inductively while controlling the topological entropy. We finish this chapter with the computation of the topological entropy of the full shift. We show that this entropy is finite if and only if the considered LCA group is discrete.

Recall that $\mathrm{Pat}_\omega(A)$ is the set of all A-patches of a Delone set ω in an LCA group G, where A is a compact subset of G. Recall furthermore that we define the *patch counting entropy of* ω along a Van Hove net $\mathcal{A} = (A_i)_{i\in I}$ as

$$\limsup_{i\in I} \frac{\log|\mathrm{Pat}_\omega(A_i)|}{\theta(A_i)}.$$

We consider next different approaches to patch counting in order to see that the Ornstein-Weiss Lemma can indeed not be applied directly in order to study patch counting entropy.

Example 6.1. *Let* $\omega := \{5n;\, n\in\mathbb{Z}\}\cup\{5n+1;\, n\in\mathbb{Z}\}$*. Then* ω *is a FLC Delone set in* $\mathbb{R}$*. For* $A := [0,1]$ *we obtain* $\mathrm{Pat}_\omega(A) = \{\{0\},\{0,1\}\}$ *and* $\mathrm{Pat}_\omega(A+2) = \mathrm{Pat}_\omega([2,3]) = \{\emptyset\}$*. This shows that* $\mathcal{K}(\mathbb{R}) \ni A \mapsto |\mathrm{Pat}_\omega(A)|$ *is not right invariant. Note that one can* *interpret ω as a FLC Delone set in $\mathbb{Z}$ and consider $F = \{0,1\}$ and $F + 2 = \{2,3\}$ in order to see that $\mathcal{K}(\mathbb{Z}) \ni F \mapsto |\operatorname{Pat}_\omega(F)|$ is also not right invariant.*

In order to overcome the lack of right invariance one could consider the following. Let us denote by $\operatorname{NPat}_\omega(A)$ the non-centred A-patches of ω, i.e. all sets of the form $(\omega - g) \cap A$ for $g \in G$. It is then straightforward to show that $\mathcal{K}(\mathbb{R}) \ni A \mapsto |\operatorname{NPat}_\omega(A)|$ is monotone right invariant and subadditive. Unfortunately the considered cardinality $|\operatorname{NPat}_\omega(A)|$ is infinite whenever A contains an infinite open set and in particular for all sufficiently invariant subsets of a non-discrete group such as $\mathbb{R}$. In particular, we would always obtain $\limsup_{i\in I} \log|\operatorname{NPat}_\omega(A_i)|/\theta(A_i) = \infty$, whenever we consider a FLC Delone set in a non-discrete LCA group G and a Van Hove net $(A_i)_{i\in I}$ in G. One can try to resolve this problem by considering the non-centred patches only up to translation. We denote by $|\operatorname{NPat}_\omega^\sim(A)|$ the number of elements of $\operatorname{NPat}_\omega(A)$ up to translation. Then $A \mapsto \log|\operatorname{NPat}_\omega^\sim(A)|$ is still monotone and right invariant and one could hope that it is also subadditive. The next example shows that this is not the case for FLC Delone sets in $\mathbb{R}$ or $\mathbb{Z}$.

Example 6.2. *Let $a_k := 5 + (k \mod 5)$ for $k \in \mathbb{N}$ and $\omega := \{\sum_{k=1}^n a_k;\ k \in \mathbb{N}\} \cup \{0\} \cup \{-\sum_{k=1}^n a_k;\ k \in \mathbb{N}\}$. Then ω is a subset of $\mathbb{R}$ and easily seen to be an FLC Delone set in $\mathbb{R}$. Consider now $A := [0,4]$. As two points in ω are at least 5 apart we obtain that $\operatorname{NPat}_\omega(A) = \{\emptyset\} \cup \{\{x\};\ x \in A\}$, i.e. all non-centred A-patches of ω are either empty or consist of one point. In particular, we obtain that $|\operatorname{NPat}_\omega^\sim(A)| = 2$. Similarly one obtains $|\operatorname{NPat}_\omega^\sim(B)| = 2$ for $B := [5,9]$. Note that all distances $5, 6, 7, 8$ and 9 can be realized between points in ω. Thus, considering $A \cup B = [0,4] \cup [5,9]$ we note that $\operatorname{NPat}_\omega(A \cup B)$ contains the sets $\{0, x\}$ with $x \in \{5,6,7,8,9\}$ and thus even up to translation at least 5 elements. We thus observe*

$$\begin{aligned}\log|\operatorname{NPat}_\omega^\sim(A)| + \log|\operatorname{NPat}_\omega^\sim(A)| &= \log(2) + \log(2) = \log(4)\\ &< \log(5) \leq \log|\operatorname{NPat}_\omega^\sim(A \cup B)|\end{aligned}$$

and thus $\mathcal{K}(\mathbb{R}) \ni A \mapsto |\operatorname{NPat}_\omega^\sim(A)|$ is not subadditive. Note that $\omega \subseteq \mathbb{Z}$ and that one can argue similarly considering $E = \{0,1,2,3,4\}$ and $F = \{5,6,7,8,9\}$ in order to obtain that $\mathcal{K}(\mathbb{Z}) \ni F \mapsto |\operatorname{NPat}_\omega^\sim(F)|$ is not subadditive.

Remark 6.3. *Clearly, one can also consider the number $|\operatorname{Pat}_\omega^\sim(A)|$ of centred A-patches up to translation. Note that the arguments from Example 6.2 also shows that the map $\mathcal{K}(\mathbb{R}) \ni A \mapsto |\operatorname{Pat}_\omega^\sim(A)|$ is not subadditive for the considered ω.*

In this chapter we assume G to be a non-compact LCA group.

6.1 Topological entropy via patch counting

We will start our considerations by presenting a geometrical approach to topological entropy which is inspired by the concept of patch counting entropy. This is motivated as we will see in Example 6.30 and in Theorem 6.47 that the patch counting entropy along certain Van Hove sequences and the topological entropy are not necessarily equal, even for FLC Delone sets.

For a closed subset $\omega \subseteq G$, a compact subset $A \subseteq G$ and an open neighbourhood V of 0 we say that $F \subseteq \omega$ is an *A-patch representation* at *scale* V for ω, if for any $g \in \omega$ there is $f \in F$ s.t.

$$\omega - f \overset{A,V}{\approx} \omega - g.$$

Note that $F \subseteq \omega$ is an A-patch representation if and only if $\{\omega - g;\, g \in F\}$ is $\epsilon(A, V)$-spanning for $\{\omega - g; g \in \omega\}$. Considering the compact Hausdorff space $\mathcal{A}(G)$ Lemma 4.31 thus yields that there exists a finite A-patch representation at scale V for ω and we denote $\mathrm{pat}_\omega(A, V)$ for the minimal cardinality of an A-patch representation at scale V for ω. Clearly, $\{\omega - g;\, g \in \omega\}$ is a much simpler object than X_ω. In this section we will show that the topological entropy of π_ω can be simplified and computed using pat_ω.

Theorem 6.4. *For every Delone set ω and every Van Hove net $(A_i)_{i\in I}$ in a non-compact LCA group G there holds*

$$\mathrm{E}(\pi_\omega) = \sup_V \liminf_{i\in I} \frac{\log(\mathrm{pat}_\omega(A_i, V))}{\theta(A_i)} = \sup_V \limsup_{i\in I} \frac{\log(\mathrm{pat}_\omega(A_i, V))}{\theta(A_i)},$$

where the suprema are taken over all open neighbourhoods V of 0*.*

We start our investigations concerning the proof of the theorem with the following formula, which simplifies the computation of Bowen entourages of members of the local rubber base and thus gives a link between the dynamics of the full shift on G and a static notion like the local rubber base.

Lemma 6.5. *For compact subsets A and K of G and any open neighbourhood V of* 0 *there holds*

$$\epsilon(K, V)_A = \epsilon(K - A, V).$$

In particular, there holds $\epsilon_\omega(K, V)_A = \epsilon_\omega(K - A, V)$ for any Delone set ω in G.

Proof. Let us denote the full shift on G by π. To show $\epsilon(K - A, V) \subseteq \epsilon(K, V)_A$, let $(\xi, \zeta) \in \epsilon(K - A, V)$. For $g \in A$ we obtain $\xi \cap (K - g) \subseteq \xi \cap (K - A) \subseteq \zeta + V$, hence

$$\pi^g(\xi) \cap K = (\xi + g) \cap K \subseteq \zeta + g + V = \pi^g(\zeta) + V.$$

Similarly one shows $\pi^g(\zeta) \cap K \subseteq \pi^g(\xi) + V$. This proves $(\pi^g(\xi), \pi^g(\zeta)) \in \epsilon(K, V)$, i.e. $(\xi, \zeta) \in \epsilon(K, V)_g$ for every $g \in A$. We thus obtain

$$\epsilon(K - A, V) \subseteq \bigcap_{g\in A} \epsilon(K, V)_g = \epsilon(K, V)_A.$$

We next show $\epsilon(K,V)_A \subseteq \epsilon(K-A,V)$. For $(\xi,\zeta) \in \epsilon(K,V)_A$ there holds $(\pi^g(\xi),\pi^g(\zeta)) \in \epsilon(K,V)$ for every $g \in A$, hence

$$(\xi+g)\cap K = \pi^g(\xi)\cap K \subseteq \pi^g(\zeta)+V = \zeta+g+V.$$

We obtain $\xi \cap (K-g) \subseteq \zeta + V$ for all $g \in A$ and compute

$$\xi \cap (K-A) = \xi \cap \left(\bigcup_{g\in A}(K-g)\right) = \bigcup_{g\in A}(\xi\cap(K-g)) \subseteq \zeta+V.$$

As one shows similarly that $\zeta \cap (K-A) \subseteq \xi + V$, we conclude $(\xi,\zeta) \in \epsilon(K-A,V)$.

For a Delone set ω in G we compute

$$\begin{aligned}\epsilon_\omega(K,V)_A &= \bigcap_{g\in A}\epsilon_\omega(K,V)_g = \bigcap_{g\in A}\epsilon(K,V)_g \cap X_\omega^2 = \epsilon(K,V)_A \cap X_\omega^2 \\ &= \epsilon(K-A,V)\cap X_\omega^2 = \epsilon_\omega(K-A,V)\end{aligned}$$

and the second statement follows. □

6.1.1 Topological entropy and non-centred patch counting

In order to show that the topological entropy of a Delone dynamical system can be calculated using pat_ω we introduce the following intermediate concept between A-patch representations and spanning sets in the corresponding Delone dynamical system. Let $A \subseteq G$ be a compact subset and V be an open neighbourhood of 0. For a closed subset $\omega \subseteq G$ we say that $F \subseteq G$ is a *non-centred A-patch representation* at *scale* V for ω, if for any $g \in G$ there is $f \in F$ s.t.

$$\omega - g \overset{A,V}{\approx} \omega - f.$$

Note that this is equivalent to $\{\omega - f;\ f \in F\}$ being $\epsilon(A,V)$-spanning for $D_\omega = \{\omega + g;\ g \in G\}$. Thus, by Lemma 4.32 we obtain that there is a finite non-centred A-patch representation at scale V for ω. We define $\mathrm{npat}_\omega(A,V)$ as the minimal cardinality of a non-centred A-patch representation at scale V for ω, i.e. $\mathrm{npat}_\omega(A,V) := \mathrm{spa}_{D_\omega}(\epsilon(A,V))$ with respect to the full shift on G. In particular, we obtain that whenever ω is a Delone set, then there holds $\mathrm{npat}_\omega(A,V) := \mathrm{spa}_{D_\omega}(\epsilon_\omega(A,V))$. Using Lemma 6.5 and the considerations about entropy and dense subsets from Subsection 4.3.4 we next show the following formulas for the topological entropy of a Delone dynamical system.

Proposition 6.6. *For every Van Hove net $(A_i)_{i\in I}$ there holds*

$$\mathrm{E}(\pi_\omega) = \sup_{V\in\mathcal{N}(G)} \liminf_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i,V))}{\theta(A_i)} = \sup_{V\in\mathcal{N}(G)} \limsup_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i,V))}{\theta(A_i)}.$$

Proof. Let V be an open neighbourhood of 0 and set $\epsilon := \epsilon_\omega(\{0\}, V)$. Then Lemma 6.5 yields

$$\mathrm{npat}_\omega(A_i, V) = \mathrm{spa}_{D_\omega}(\epsilon_\omega(A_i, V)) = \mathrm{spa}_{D_\omega}(\epsilon_{(-A_i)})$$

and we obtain from Proposition 4.33 that

$$\mathrm{E}(\pi_\omega) \geq \limsup_{i\in I} \frac{\log(\mathrm{spa}_{D_\omega}(\epsilon_{(-A_i)}))}{\theta(A_i)} = \limsup_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i, V))}{\theta(A_i)}.$$

Taking the supremum over all open neighbourhoods V of 0 we thus obtain

$$\mathrm{E}(\pi_\omega) \geq \sup_{V\in\mathcal{N}(G)} \limsup_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i, V))}{\theta(A_i)}.$$

Let now $\eta \in \mathbb{U}_X$ and choose $\epsilon \in \mathbb{U}_X$ as in Lemma 4.32, i.e. such that $H(\eta_A) \leq \log\Big(\mathrm{spa}_{D_\omega}(\epsilon_A)\Big)$ is satisfied for all compact subsets $A \subseteq G$. There are $K \subseteq G$ compact and an open neighbourhood V of 0 such that $\epsilon_\omega(K, V) \subseteq \epsilon$. As also $(-A_i)_{i\in I}$ is a Van Hove net in G we obtain from Proposition 2.26 the existence of a Van Hove net $(B_i)_{i\in I}$ that satisfies $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$ and $B_i + (-K) \subseteq -A_i$ for all $i \in I$. Lemma 6.5 thus allows to see $e^{H(\eta_{B_i})} \leq \mathrm{spa}_{D_\omega}(\epsilon_\omega(K, V)_{B_i}) = \mathrm{npat}_\omega(K - B_i, V) \leq \mathrm{npat}_\omega(A_i, V)$. Hence,

$$\mathrm{E}(\eta|\pi_\omega) = \lim_{i\in I} \frac{H(\eta_{B_i})}{\theta(B_i)} \leq \liminf_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i, V))}{\theta(A_i)}$$

and taking the supremum over all open neighbourhoods V of 0 yields

$$\mathrm{E}(\eta|\pi_\omega) \leq \sup_{V\in\mathcal{N}(G)} \liminf_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i, V))}{\theta(A_i)}.$$

Taking the supremum over all $\eta \in \mathbb{U}_X$ we obtain

$$\mathrm{E}(\pi_\omega) \leq \sup_{V\in\mathcal{N}(G)} \liminf_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i, V))}{\theta(A_i)} \leq \sup_{V\in\mathcal{N}(G)} \limsup_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i, V))}{\theta(A_i)}$$

and the statement follows. □

6.1.2 Centred and non-centred patch counting

We now establish the connection between non-centred and centred A-patch representations. To do this we will need the following version of the triangle inequality for the local rubber base.

Lemma 6.7. *For all compact subsets K of G and all precompact open neighbourhoods V of 0 there holds $\epsilon\left(K-\overline{V},V\right)\epsilon\left(K-\overline{V},V\right)\subseteq\epsilon\left(K,V+V\right)$. In particular, for a Delone set ω in G there holds $\epsilon_\omega\left(K-\overline{V},V\right)\epsilon_\omega\left(K-\overline{V},V\right)\subseteq\epsilon_\omega\left(K,V+V\right)$.*

Proof. Let $(\xi_1,\xi_2)\in\epsilon\left(K-\overline{V},V\right)\epsilon\left(K-\overline{V},V\right)$. Then there exists $\zeta\in\mathcal{A}(G)$ such that $(\xi_1,\zeta),(\zeta,\xi_2)\in\epsilon\left(K-\overline{V},V\right)$ and in particular there holds

$$\xi_1\cap K=\xi_1\cap K\cap K\subseteq\xi_1\cap\left(K-\overline{V}\right)\cap K\subseteq(\zeta+V)\cap K$$

As $(\zeta,\xi_2)\in\epsilon\left(K-\overline{V},V\right)$ any $v\in V$ satisfies $\zeta\cap(K-v)\subseteq\zeta\cap\left(K-\overline{V}\right)\subseteq\xi_2+V$ and we compute

$$\xi_1\cap K\subseteq(\zeta+V)\cap K=\bigcup_{v\in V}(\zeta+v)\cap K\subseteq\bigcup_{v\in V}(\xi_2+V+v)=\xi_2+V+V.$$

Similarly one shows $\xi_2\cap K\subseteq\xi_1+V+V$ and we conclude $(\xi_1,\xi_2)\in\epsilon\left(K,V+V\right)$. This shows the first statement. The second statement easily follows from the first one as $\epsilon_\omega\left(K-\overline{V},V\right)\epsilon_\omega\left(K-\overline{V},V\right)\subseteq X_\omega^2\cap\left(\epsilon\left(K-\overline{V},V\right)\epsilon\left(K-\overline{V},V\right)\right)$. □

Note that a non-centred A-patch representation at scale V is not necessarily contained in ω and thus not necessarily an A-patch representation at scale V. Nevertheless, we have the following.

Lemma 6.8. *Let ω be a Delone set. Let $A\subseteq G$ be a compact subset of G and consider a precompact and open neighbourhood V of 0. Then for every non-centred $\left(A-\overline{V}\right)$-patch representation F at scale V for ω there exists an A-patch representation E at scale $V+V$ for ω such that $|E|\leq|F|$.*

Proof. For $f\in F$ let $[f]$ be the set of all $g\in\omega$ such that $\omega-g$ is $\epsilon\left(A-\overline{V},V\right)$-close to $\omega-f$. For $f\in F$ choose $g_f\in[f]$, whenever $[f]\neq\emptyset$. Otherwise choose $g_f\in\omega$ arbitrary. Set $E:=\{g_f;\ f\in F\}$. As F is a non-centred $\left(A-\overline{V}\right)$-patch representation for any $g\in\omega$ there is $f\in F$ such that $\omega-g$ and $\omega-f$ are $\epsilon\left(A-\overline{V},V\right)$-close. Such an f in particular satisfies $g\in[f]$, i.e. $[f]\neq\emptyset$ and we thus know $g_f\in[f]$. Then $\omega-g$ is $\epsilon\left(A-\overline{V},V\right)$-close to $\omega-f$ and $\omega-f$ is $\epsilon\left(A-\overline{V},V\right)$-close to $\omega-g_f$. Thus, Lemma 6.7 allows to conclude that $\omega-g$ and $\omega-g_f$ are $\epsilon\left(A,V+V\right)$-close, i.e.

$$\omega-g\overset{A,V+V}{\approx}\omega-g_f$$

and we obtain E to be an A-patch representation at scale $V+V$. □

The following Lemma shows that one can also control the minimal cardinality of certain non-centred patch representations by certain centred patch representations.

Lemma 6.9. *Let ω be a Delone set. Consider a compact subset K of G such that ω is K-dense. Consider furthermore an open neighbourhood V of 0. Then there exists a finite set $E \subseteq K$ such that $F + E$ is a non-centred A-patch representation at scale $V + V$ for ω, whenever F is an $(A + K)$-patch representation at scale V for ω and A is a compact subset of G.*

Proof. As K is compact and as $V \cap (-V)$ is an open neighbourhood of 0, there is a finite set $E \subseteq K$ such that $K \subseteq E + (V \cap (-V))$. Let now A be a compact subset of G and F be an $(A + K)$-patch representation at scale V. To show that $F + E$ is a non-centred A-patch representation at scale $V + V$ let $g \in G$. Then as ω is K-dense we obtain the existence of $k \in K$ and $u \in \omega$ such that $g = k + u$. Furthermore, from the choice of E we obtain that there are $e \in E$ and $v \in V \cap (-V)$ such that $k = e + v$. As F is an $(A + K)$-patch representation at scale V there exists $f \in F$ such that $\omega - f$ and $\omega - u$ are $\epsilon_\omega(A + K, V)$-close. We will now show that $\omega - g$ and $\omega - (e + f)$ are $\epsilon(A, V + V)$-close. As $e + f \in E + F$ this will allow to conclude that $E + F$ is an A-patch representation at scale $V + V$.

As $\omega - f$ and $\omega - u$ are $\epsilon(A + K, V)$-close and as $e \in E \subseteq K$ we observe

$$(\omega - f) \cap (A + e) \subseteq (\omega - f) \cap (A + K) \subseteq (\omega - u) + V.$$

From $e + u = k - v + g - k = g - v$ we thus compute

$$\begin{aligned}(\omega - (f + e)) \cap A &\subseteq (\omega - u) + V - e \\ &= \omega - g + v + V \\ &\subseteq (\omega - g) + (V + V).\end{aligned}$$

As $e + v = k \in K$ and as $\omega - u$ and $\omega - f$ are $\epsilon(A + K, V)$-close we furthermore know that there holds

$$(\omega - u) \cap (A + e + v) \subseteq (\omega - u) \cap (A + K) \subseteq (\omega - f) + V.$$

From $g = k + u = e + v + u$ and $v \in -V$ we thus obtain

$$\begin{aligned}(\omega - g) \cap A &= ((\omega - u) \cap (A + e + v)) - e - v \\ &\subseteq (\omega - f) + V - e - v \\ &\subseteq (\omega - (f + e)) + (V + V)\end{aligned}$$

and we conclude that $\omega - g$ and $\omega - (e + f)$ are indeed $\epsilon_\omega(A, V + V)$-close. □

The previous lemmas allow to relate pat_ω and npat_ω in the following way.

Proposition 6.10. *Let ω be a Delone set in G and consider a compact subset K of G such that ω is K-dense and furthermore a precompact and open neighbourhood V of 0. Then there exists a constant $N \in \mathbb{N}$ such that for all compact subsets $A \subseteq G$ there holds $\mathrm{pat}_\omega(A, V+V) \leq \mathrm{npat}_\omega\left(A - \overline{V}, V\right)$ and $\mathrm{npat}_\omega(A, V+V) \leq N \cdot \mathrm{pat}_\omega(A+K, V)$.*

Proof. From Lemma 6.8 we obtain that the minimal cardinality of an A-patch representation at scale $V+V$ is smaller than the minimal cardinality of a non-centred $\left(A - \overline{V}\right)$-patch representation at scale V, i.e. the first inequality. Furthermore, we choose a finite set $E \subseteq K$ as in Lemma 6.9. Setting $N := |E|$ we then obtain that whenever F is an $(A+K)$-patch representation at scale V of minimal cardinality, then $F+E$ is an A-patch representation at scale $V+V$. As $|F+E| \leq N|F| = N\,\mathrm{pat}_\omega(A+K, V)$ we thus obtain the second statement. □

We have now collected all tools that we will need for the proof of Theorem 6.4.

Proof of Theorem 6.4. First recall that we assume G to be non-compact and thus there holds $\lim_{i\in I} 1/\theta(A_i) = 0$ for the Van Hove net $(A_i)_{i\in I}$. Let $K \subseteq G$ be compact such that ω is K-dense and choose a Van Hove net $(B_i)_{i\in I}$ such that $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$ and such that $B_i + K \subseteq A_i$ for all $i \in I$. Let $V \subseteq G$ be an open neighbourhood of 0. Then there exists a precompact and open neighbourhood W of 0 such that $W + W \subseteq V$. By Proposition 6.10 there is $N \in \mathbb{N}$ such that for all $i \in I$ there holds

$$\mathrm{npat}_\omega(B_i, V) \leq \mathrm{npat}(B_i, W+W) \leq N\,\mathrm{pat}_\omega(B_i + K, W) \leq N\,\mathrm{pat}_\omega(A_i, W)$$

and we obtain

$$\begin{aligned}
\liminf_{i\in I} \frac{\log(\mathrm{npat}_\omega(B_i, V))}{\theta(B_i)} &\leq \liminf_{i\in I} \frac{\log(N) + \log(\mathrm{pat}_\omega(A_i, W))}{\theta(A_i)} \\
&= \liminf_{i\in I} \frac{\log(\mathrm{pat}_\omega(A_i, W))}{\theta(A_i)} \\
&\leq \sup_{U\in\mathcal{N}(G)} \liminf_{i\in I} \frac{\log(\mathrm{pat}_\omega(A_i, U))}{\theta(A_i)}.
\end{aligned}$$

Taking the supremum over all open neighbourhoods V of 0 we thus obtain from Proposition 6.6 that there holds

$$\mathrm{E}(\pi_\omega) = \sup_{V\in\mathcal{N}(G)} \liminf_{i\in I} \frac{\log(\mathrm{npat}_\omega(B_i, V))}{\theta(B_i)} \leq \sup_{U\in\mathcal{N}(G)} \liminf_{i\in I} \frac{\log(\mathrm{pat}_\omega(A_i, U))}{\theta(A_i)}.$$

To show the reverse inequality let V be an open neighbourhood of 0. Then V contains a precompact open neighbourhood W of 0 that satisfies $W + W \subseteq V$ and we can consider a Van Hove net $(B_i)_{i\in I}$ such that $B_i - \overline{W} \subseteq A_i$. Thus, Proposition 6.10 yields

$$\mathrm{pat}_\omega(B_i, V) \leq \mathrm{pat}(B_i, W + W) \leq \mathrm{npat}_\omega\left(B_i - \overline{W}, W\right) \leq \mathrm{npat}_\omega(A_i, W).$$

Another application of Proposition 6.6 thus allows to see that there holds

$$\begin{aligned}\mathrm{E}(\pi_\omega) &\geq \limsup_{i\in I} \frac{\log(\mathrm{npat}_\omega(A_i, W))}{\theta(A_i)} \\ &\geq \limsup_{i\in I} \frac{\log(\mathrm{pat}_\omega(B_i, V))}{\theta(A_i)} \\ &= \limsup_{i\in I} \frac{\log(\mathrm{pat}_\omega(B_i, V))}{\theta(B_i)}.\end{aligned}$$

Taking the supremum over all open neighbourhoods V of 0 we obtain the statement. □

6.2 Patch counting for FLC Delone sets

In this section we study the classical definition of patch counting entropy presented in the introduction and at the beginning of this chapter. To give a representation analogue of the classical notion of patch counting we define the following. For closed subsets ω and compact subsets $A \subseteq G$ we call a subset $F \subseteq \omega$ an *exact A-patch representation*, if for all $g \in \omega$ there is $f \in F$ such that $(\omega - g) \cap A = (\omega - f) \cap A$. Then the minimal cardinality of an exact A-patch representation is $|\mathrm{Pat}_\omega(A)|$. Furthermore, for every open neighbourhood V of 0 we obtain that every exact A-patch representation is an A-patch representation at scale V. Thus, for every compact $A \subseteq G$ there holds $\mathrm{pat}_\omega(A, V) \leq |\mathrm{Pat}_\omega(A)|$ and we obtain from Theorem 6.4 the following general relation between the patch counting entropy of a Delone set ω and the topological entropy of the corresponding Delone dynamical system π_ω.

Proposition 6.11. *For every Van Hove net $(A_i)_{i\in I}$ there holds*

$$\mathrm{E}(\pi_\omega) \leq \liminf_{i\in I} \frac{\log|\mathrm{Pat}_\omega(A_i)|}{\theta(A_i)} \leq \limsup_{i\in I} \frac{\log|\mathrm{Pat}_\omega(A_i)|}{\theta(A_i)}.$$

The equality in Proposition 6.11 cannot be achieved for all Van Hove nets as we will present below. In order to give a condition on the Van Hove net that ensures equality we will need the following notion. Let $C \subseteq G$ be a compact subset. We say that $A \subseteq G$ is *C-connected to* 0, if for all $a \in A$ there are $a_1, \cdots, a_n \in A$ that satisfy $a_n = a$ such that defining $a_0 := 0$ yields $a_i - a_{i-1} \in C$ for every $i \in \{1, \cdots, n\}$. Furthermore, we say that a net of compact sets $(A_i)_{i\in I}$ is *C-connected to* 0, if A_i is C-connected to 0 for all $i \in I$. A net is called *compactly connected to* 0, if it is C-connected to 0 for some compact set $C \subseteq G$.

Example 6.12. *Let us consider a Van Hove net $(A_i)_{i\in I}$ in $\mathbb{R}^d$. $(A_i)_{i\in I}$ is compactly connected to* 0*, if and only if there exists $R \geq 0$ such that for all $i \in I$ the set $\overline{B}_R(0)+A_i$ is connected and contains* 0*. In particular any Van Hove net $(A_i)_{i\in I}$ in $\mathbb{R}^d$ that consists of connected sets that contain* 0 *is compactly connected to* 0*.*

Indeed, if we assume that $(A_i)_{i\in I}$ in $\mathbb{R}^d$ is compactly connected to 0 *there exists a compact set $K \subseteq \mathbb{R}^d$ such that A_i is K-connected to* 0 *for all $i \in I$. Let $R \geq 0$ such that $K \subseteq \overline{B}_R(0)$. Let $i \in I$. Clearly $\overline{B}_R(0) + A_i$ contains* 0*. Considering $\mathfrak{F} := \{\overline{B}_R(a); a \in A_i\}$ we obtain a family of connected sets with the property that for $B, B' \in \mathfrak{F}$ there is always a finite sequence $(B_j)_{j=1}^k$ such that $B_1 = B$, $B_k = B'$ and $B_j \cap B_{j+1} \neq \emptyset$ for all $j = 1, \cdots, k-1$. With a standard argument*[1] *we obtain that $\overline{B}_R(0) + A_i = \bigcup_{B\in\mathfrak{F}} B$ is connected.*

To also show the reverse direction let $R \geq 0$ such that $\overline{B}_R(0) + A_i$ is connected and contains 0 *for all $i \in I$. To show that each A_i is $\overline{B}_R(0)$-connected to* 0 *let $i \in I$ and notice that A_i is compact. Thus there is a finite set $F \subseteq A_i$ such that $\overline{B}_R(0) + F \supseteq A_i \cup \{0\}$. As A_i is connected we obtain that $\overline{B}_R(0) + F = A_i \cup (\overline{B}_R(0) + F) = A_i \cup \left(\bigcup_{x\in F} \overline{B}_R(x)\right)$ is connected. From this one easily observes A_i to be $\overline{B}_R(0)$-connected to* 0*.*

Example 6.13. *The sequences $(\{1, \cdots, n\})_{n\in\mathbb{N}}$ and $(\{-n, \cdots, n\})_{n\in\mathbb{N}}$ are Van Hove sequences in $\mathbb{Z}$ that are compactly connected to* 0*.*

Example 6.14. *All compactly generated LCA groups contain a Van Hove sequence that is compactly connected to* 0*. Indeed, a compactly generated LCA group is isomorphic (as a topological group) to $\mathbb{R}^a \times \mathbb{Z}^b \times H$ for $a, b \in \mathbb{N}_0$ and a compact group H. As the latter contains the Van Hove sequence $([0,n]^a \times \{0, \cdots, n\}^b \times H)_{n\in\mathbb{N}}$ that is clearly $\left([0,1]^a \times \{0,1\}^b \times H\right)$-connected to* 0*, we obtain the statement.*

Example 6.15. *Let p be a prime. The additive group of the p-adic numbers $\mathbb{Q}_p$ is a metrizable, σ-compact LCA group that contains no Van Hove net that is compactly connected to* 0*. We omit the proof of this claim as the statement can be deduced from Proposition 6.23 below.*

Example 6.16. *Let p be a prime. The additive group $G := \mathbb{R} \times \mathbb{Q}_p$ is a metrizable, σ-compact LCA group but contains no Van Hove net that is compactly connected to* 0*. We will also obtain this statement from Proposition 6.23 below. Note that G even contains a uniform lattice as we have seen in Example 3.9 above.*

We will use the local matching base $\mathbb{B}_{lm}(\omega)$ of $\mathbb{U}_{X_\omega}$ in order to establish the equality in Proposition 6.11 for Van Hove nets that are compactly connected to 0. Unfortunately the formula in Lemma 6.5, which gives the tool to calculate the Bowen entourages of members of the local rubber base $\mathbb{B}_{lr}(\omega)$ seems not to hold for members of the local matching base $\mathbb{B}_{lm}(\omega)$. Nevertheless, it is straightforward to show the following.

[1] See [Kel55, Problem 1.R].

Lemma 6.17. *For all compact $K \subseteq G$ and every open neighbourhood V of 0, and $g \in G$ there holds $\eta_\omega(K,V)_g = \eta_\omega(K-g,V)$.*

We can now proof the following key lemma.

Lemma 6.18. *Let ω be a FLC Delone set. Let C be a compact subset of G such that ω is C-dense. Then there is $\eta \in \mathbb{U}_{X_\omega}$ such that for all compact $A \subseteq G$ that are C-connected to 0 and contain 0 there holds*

$$|\operatorname{Pat}_\omega(A)| \leq \operatorname{sep}_{D_\omega}(\eta_{(-A)}).$$

Proof. Let V be a precompact and open neighbourhood of 0 such that ω is V-discrete. We set $K := C + C + \overline{V} - C - C - \overline{V}$ and $\eta := \eta_\omega(K,V)$. To show the statement it is sufficient to show that every exact A-patch representation $F \subseteq \omega$ of minimal cardinality $|\operatorname{Pat}_\omega(A)|$ satisfies that $\{\omega - g;\, g \in F\}$ is $\eta_{(-A)}$-separated. To argue by contraposition assume $F \subseteq \omega$ to be an exact A-patch representation $F \subseteq \omega$ for which $\{\omega - g;\, g \in F\}$ is not $\eta_{(-A)}$-separated. Thus, there are $x, y \in F$ such that $(\omega - x, \omega - y) \in \eta_{(-A)}$. We will argue below that $(\omega - x) \cap A = (\omega - y) \cap A$ and thus obtain that F is not a minimal exact A-patch representation.

It remains to show that $(\omega - x) \cap A = (\omega - y) \cap A$. Assume $a \in (\omega - x) \cap A$. As A is C-connected to 0 and contains 0 there are $a_0, \cdots, a_n \in A$ such that $a_0 = 0, a_n = a$ and $a_{i+1} - a_i \in C$ for all $i = 0, \cdots, n-1$. Set $x_0 := 0$ and $x_n := a$. As $\omega - x$ is C-dense there are $x_i \in \omega - x$ such that $a_i - x_i \in C$ for $i = 1, \cdots, n-1$. Note that $x_0 = 0 \in (\omega - y) \cap A$. We will show $a = x_n \in (\omega - y)$ by induction and thus obtain that $(\omega - x) \cap A \subseteq (\omega - y) \cap A$.

Assume now $x_i \in \omega - y$ for some $i = 0, \cdots, n-1$. As $a_i \in A$ we obtain that $(\omega - x, \omega - y) \in \eta_{(-A)} \subseteq \eta_{(-a_i)} = \eta_\omega(K + a_i, V)$. Thus, there are $u, v \in V$ such that $(\omega - x + u) \cap (K + a_i) = (\omega - y + v) \cap (K + a_i)$. From $a_i - x_i \in C$ we obtain furthermore $x_i + u = (x_i - a_i) + a_i + u \in -C + a_i + \overline{V} \subseteq K + a_i$. Now recall that we have chosen x_i such that $x_i \in \omega - x$, which allows to observe

$$x_i + u \in (\omega - x + u) \cap (K + a_i) = (\omega - y + v) \cap (K + a_i) \subseteq \omega - y + v.$$

We have thus shown that $x_i + (u - v) \in \omega - y$. Recall from the hypothesis of our induction that there furthermore holds $x_i \in \omega - y$. As $\omega - y$ is V-discrete and $u, v \in V$ we thus obtain $x_i = x_i + u - v$, i.e. $u = v$. In particular, we obtain $(\omega - x + u) \cap (K + a_i) = (\omega - y + u) \cap (K + a_i)$ and observe

$$(\omega - x) \cap (K + a_i + u) = (\omega - y) \cap (K + a_i + u).$$

Now abbreviate $M := C + C - C$. We then obtain from

$$M + x_i = M + (x_i - a_i) - u + (a_i + u) \subseteq M - C - \overline{V} + (a_i + u) \subseteq K + (a_i + u)$$

that there holds $(\omega - x) \cap (M + x_i) = (\omega - y) \cap (M + x_i)$. Then

$$x_{i+1} - x_i = (x_{i+1} - a_{i+1}) + (a_{i+1} - a_i) + (a_i - x_i) \in -C + C + C = M$$

implies $x_{i+1} \in M + x_i$. As by construction $x_{i+1} \in \omega - x$ we thus observe

$$x_{i+1} \in (\omega - x) \cap (M + x_i) = (\omega - y) \cap (M + x_i) \subseteq \omega - y,$$

which finishes the step of our induction. From this induction we obtain that $a = x_n \in \omega - y$ for all $a \in (\omega - x) \cap A$ and have thus shown that $(\omega - x) \cap A \subseteq (\omega - y) \cap A$. As one obtains the reversed inclusion similarly we thus know that $(\omega - x) \cap A = (\omega - y) \cap A$ is valid, which yields that F can indeed not be of minimal cardinality. □

We can now show that the topological entropy equals the patch counting entropy along Van Hove nets that are compactly connected to 0 and contain 0. To also include Van Hove nets that are compactly connected to 0 but do not necessarily contain 0, such as $(\{1, \cdots, n\})_{n\in\mathbb{N}}$ in $\mathbb{Z}$ we will need the following lemma.

Lemma 6.19. *Let $(A_i)_{i\in I}$ be a Van Hove net and set $B_i := A_i \cup \{0\}$ for all $i \in I$. Then $(B_i)_{i\in I}$ is a Van Hove net and there holds*

$$\limsup_{i\in I} \frac{\log |\operatorname{Pat}_\omega(A_i)|}{\theta(A_i)} = \limsup_{i\in I} \frac{\log |\operatorname{Pat}_\omega(B_i)|}{\theta(B_i)}.$$

Proof. Let $K \subseteq G$ be compact. From

$$\begin{aligned}\partial_K B_i &= (K + (A_i \cup \{0\})) \cap \overline{K + (A_i \cup \{0\})^c} \\ &\subseteq ((K + A_i) \cup K) \cap (K + \overline{A_i^c}) \subseteq (\partial_K A_i) \cup K\end{aligned}$$

we obtain that $0 \leq \theta(\partial_K B_i)/\theta(B_i) \leq (\theta(\partial_K A_i) + \theta(K))/\theta(A_i) \to_{i\in I} 0$. Thus, $(B_i)_{i\in I}$ is a Van Hove net. Furthermore, $1 \leq \theta(B_i)/\theta(A_i) \leq (\theta(A_i) + \theta(\{0\}))/\theta(A_i) \to 1 + 0$ implies $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$. As $\operatorname{Pat}_\omega(B_i) \subseteq \operatorname{Pat}_\omega(A_i) \cup \{P \cup \{0\};\, P \in \operatorname{Pat}_\omega(A_i)\}$ for all $i \in I$ we obtain $|\operatorname{Pat}_\omega(A_i)| \leq |\operatorname{Pat}_\omega(B_i)| \leq 2|\operatorname{Pat}_\omega(A_i)|$ and a straightforward argument yields the statement. □

Theorem 6.20. *Let G be a non-compact LCA group and ω be a FLC Delone set in G. For all compact subsets $C \subseteq G$ there exists an entourage $\eta \in \mathbb{U}_X$ such that for every Van Hove net $(A_i)_{i\in I}$ that is C-connected to 0 the following limit exists and there holds*

$$\mathrm{E}(\pi_\omega) = \mathrm{E}(\eta|\pi_\omega) = \lim_{i\in I} \frac{\log |\operatorname{Pat}_\omega(A_i)|}{\theta(A_i)}.$$

Remark 6.21. *To give a precise formula for η we need to consider a precompact open neighbourhood V of 0 such that ω is V-discrete and a compact subset $K \subseteq G$ such that*

ω is K-dense. Then a carefull look into the following proof and the proof of Lemma 6.18 allows to conclude that we can consider

$$\eta = \eta_\omega\left((K \cup C) + (K \cup C) + \overline{V} - (K \cup C) - (K \cup C) - \overline{V}, V\right).$$

Proof of Theorem 6.20. Let C be a compact subset of G. Let furthermore $K \subseteq G$ be a compact subset such that ω is K-dense. Then ω is $K \cup C$-dense and Lemma 6.18 yields the existence of $\eta \in \mathbb{U}_X$ such that for all compact subsets $B \subseteq G$ that are $(K \cup C)$-connected to 0 and furthermore contain 0 there holds $|\operatorname{Pat}_\omega(B)| \leq \operatorname{sep}_{D_\omega}(\eta_{(-B)})$. Now recall from Lemma 4.32 that we have $\operatorname{sep}_{D_\omega}(\eta_{(-B)}) \leq e^{H(\eta_{(-B)})}$ for any compact subset $B \subseteq G$. As any Van Hove net $(A_i)_{i\in I}$ that is C-connected to 0 is in particular $(K \cup C)$-connected to 0 we denote $B_i := A_i \cup \{0\}$ and obtain from Lemma 6.19 that there holds

$$\begin{aligned}
\limsup_{i\in I} \frac{\log|\operatorname{Pat}_\omega(A_i)|}{\theta(A_i)} &= \limsup_{i\in I} \frac{\log|\operatorname{Pat}_\omega(B_i)|}{\theta(B_i)} \\
&\leq \limsup_{i\in I} \frac{\log(\operatorname{sep}_{D_\omega}(\eta_{(-B_i)}))}{\theta(B_i)} \\
&\leq \lim_{i\in I} \frac{H(\eta_{(-B_i)})}{\theta(B_i)} \\
&= \lim_{i\in I} \frac{H(\eta_{(-B_i)})}{\theta(-B_i)} \\
&= \mathrm{E}(\eta|\pi_\omega) \leq \mathrm{E}(\pi_\omega).
\end{aligned}$$

We thus conclude from Proposition 6.11 that

$$\mathrm{E}(\pi_\omega) \leq \liminf_{i\in I} \frac{\log|\operatorname{Pat}_\omega(A_i)|}{\theta(A_i)} \leq \limsup_{i\in I} \frac{\log|\operatorname{Pat}_\omega(A_i)|}{\theta(A_i)} \leq \mathrm{E}(\eta|\pi_\omega) \leq \mathrm{E}(\pi_\omega)$$

and the statement follows. □

Corollary 6.22. *Let G be a non-compact LCA group and ω be a FLC Delone set in G. For every Van Hove net $(A_i)_{i\in I}$ that is compactly connected to 0 there holds*

$$\mathrm{E}(\pi_\omega) = \lim_{i\in I} \frac{\log|\operatorname{Pat}_\omega(A_i)|}{\theta(A_i)}.$$

Naturally the question arises which LCA groups contain Van Hove nets that are compactly connected to 0. In Example 6.14 above we have already seen that all compactly generated LCA groups do. We next show that we have already found all examples.

Proposition 6.23. *An LCA group G is compactly generated if and only if it contains a Van Hove net that is compactly connected to 0.*

Proof. By Example 6.14 it is sufficient to show that the existence of a Van Hove net $(A_i)_{i\in I}$ which is K-connected to 0 for some compact subset $K \subseteq G$ implies G to be generated by K, i.e. $G = \bigcup_{n\in\mathbb{N}} K_n$, where we abbreviate $K_1 := K$ and $K_{n+1} := K_n + K$. We assume without lost of generality that K is symmetric and contains 0. To show that G is generated by K let us consider $g \in G$. As $(A_i)_{i\in I}$ is in particular ergodic we know that there exists $i \in I$ such that $\theta(A_i) > 0$ and such that $\theta((A_i + g) \setminus A_i)/\theta(A_i) < 1$. This in particular implies that A_i and $A_i + g$ intersect and there is $h \in A_i \cap (A_i + g)$. Now recall that A_i is K-connected to 0 and thus there are $a_1, \cdots, a_n \in A_i$ with $a_n = h$ and such that $a_0 := 0$ gives $a_j - a_{j-1} \in K$ for all $1 \leq j \leq n$. Furthermore, also $h - g \in A_i$ and we find $b_1, \cdots, b_m \in A_i$ with $b_m = h - g$ and such that $b_0 := 0$ gives $b_l - b_{l-1} \in K$ for all $1 \leq l \leq m$. We define $a_j := b_{n+m-j} + g$ for all $n \leq j \leq n+m$. Note that $a_n = b_{n+m-n} + g = h - g + g = h$ fits our earlier choice of a_n and that we have $a_{n+m} = b_0 + g = g$. Furthermore, there holds $a_j - a_{j-1} = b_{n+m-j} + g - b_{n+m-j+1} - g = -(b_{n+m-j+1} - b_{n+m-j}) \in -K = K$ for all $n \leq j \leq m+n$. Summarizing we have found elements $a_0, \cdots, a_{n+m}$ in G with $a_0 = 0$, $a_{n+m} = g$ and such that $a_j - a_{j-1} \in K$ for all $1 \leq j \leq n+m$. Clearly, $a_0 \in K = K_1$ and we can proceed inductively to obtain that $a_j \in K + a_{j-1} \subseteq K + K_{(j-1)+1} = K_{j+1}$ for any $1 \leq j \leq n+m$. We thus observe $g = a_{n+m} \in K_{n+m+1}$. As $g \in G$ was arbitrary we have shown G to be compactly generated. □

Remark 6.24. *Note that the proof of Proposition 6.23 only uses that the considered net is ergodic and thus also shows that an LCA group is compactly generated if and only if it contains an ergodic net that is compactly connected to* 0*. The same argument also gives that a similar statement holds about Følner nets.*

From Proposition 6.23 and Theorem 6.20 we obtain the following corollary.

Corollary 6.25. *Let G be a non-compact but compactly generated LCA group. Then for any FLC Delone set ω there exists $\eta \in \mathbb{U}_{X_\omega}$ such that* $\mathrm{E}(\pi_\omega) = \mathrm{E}(\eta|\pi_\omega)$.

Remark 6.26. *It remains open, whether a similar statement is valid, whenever G is not compactly generated, i.e. whenever it contains no Van Hove net that is compactly connected to* 0.

As another consequence of Theorem 6.20 we obtain the following.

Corollary 6.27. *Let G be a non-compact but compactly generated LCA group. Let furthermore ω be a FLC Delone set G. Then there exists an open neighbourhood V of* 0 *such that for all Van Hove nets $(A_i)_{i\in I}$ and all open neighbourhoods W of* 0 *that are contained in V the following limit exists and satisfies*

$$\mathrm{E}(\pi_\omega) = \lim_{i\in I} \frac{\log(\mathrm{pat}_\omega(A_i, W))}{\theta(A_i)}.$$

Remark 6.28. *Note that the Van Hove nets considered in Corollary 6.27 are not assumed to be compactly connected to* 0*. Again the statement remains open for LCA groups that are not compactly generated.*

Proof of Corollary 6.27. By Corollary 6.25 there exists $\eta \in \mathbb{U}_X$ such that there holds $\mathrm{E}(\pi_\omega) = \mathrm{E}(\eta|\pi_\omega)$. By Lemma 4.32 there is $\epsilon \in \mathbb{U}_X$ that satisfies $H(\eta_A) \leq \log\left(\mathrm{spa}_{D_\omega}(\epsilon_A)\right)$ for all compact subsets $A \subseteq G$. Let $K' \subseteq G$ be a compact subset and V' be an open neighbourhood of 0 such that $\epsilon_\omega(K', V') \subseteq \epsilon$. Let $K \subseteq G$ be compact such that ω is K-dense and V a precompact, symmetric and open neighbourhood of 0 such that $V + V \subseteq V'$.

Now consider a Van Hove net $(A_i)_{i\in I}$. Then by Proposition 2.26 there exists a Van Hove net $(B_i)_{i\in I}$ such that $B_i + K' + K \subseteq A_i$ for all $i \in I$ and such that $\lim_{i\in I} \theta(B_i)/\theta(A_i) = 1$. From Proposition 6.10 we obtain the existence of a $N \in \mathbb{N}$ such that for all $i \in I$ there holds

$$\begin{aligned} e^{H(\eta_{(-B_i)})} &\leq \mathrm{spa}_{D_\omega}(\epsilon_\omega(K', V')_{(-B_i)}) = \mathrm{spa}_{D_\omega}(\epsilon_\omega(B_i + K', V')) = \mathrm{npat}_\omega(B_i + K', V') \\ &\leq \mathrm{npat}_\omega(B_i + K', V + V) \leq N\, \mathrm{pat}_\omega(B_i + K' + K, V) \leq N\, \mathrm{pat}_\omega(A_i, V). \end{aligned}$$

Thus, for any open neighbourhood W of 0 that is contained in V we obtain $H(\eta_{(-B_i)}) \leq \log(N) + \log(\mathrm{pat}_\omega(A_i, V)) \leq \log(N) + \log(\mathrm{pat}_\omega(A_i, W))$. As we assume that G is non-compact we know that $\log(N)/\theta(A_i) \to 0$. Furthermore, there holds $\lim_{i\in I} \theta(A_i)/\theta(-B_i) = \lim_{i\in I} \theta(A_i)/\theta(B_i) = 1$. Thus, our choice of η and Theorem 6.4 yield

$$\begin{aligned} \mathrm{E}(\pi_\omega) &= \mathrm{E}(\eta|\pi_\omega) \\ &= \lim_{i\in I} \frac{H(\eta_{(-B_i)})}{\theta(-B_i)} \\ &\leq \liminf_{i\in I} \frac{\log(N) + \log(\mathrm{pat}_\omega(A_i, W))}{\theta(A_i)} \\ &= \liminf_{i\in I} \frac{\log(\mathrm{pat}_\omega(A_i, W))}{\theta(A_i)} \\ &\leq \limsup_{i\in I} \frac{\log(\mathrm{pat}_\omega(A_i, W))}{\theta(A_i)} \leq \mathrm{E}(\pi_\omega). \end{aligned}$$

□

Remark 6.29. *Clearly, it would be also interesting to have a formula for* V *in terms of parameters of* ω *in Corollary 6.27. Unfortunately we do not know how to give such a formula. The problem seems to be, that for compact* K *and an open neighbourhood* V *of* 0 *one would need formulas for a compact subset* $C(K, V)$ *and an open neighbourhood* $W(K, V)$ *of* 0 *which satisfy* $\epsilon_\omega(C(K, V), W(K, V)) \subseteq \eta_\omega(K, V)$*. Nevertheless, we are not aware of such formulas. The fact that the local matching base yields the local rubber topology for FLC Delone sets is achieved everywhere, where we encountered it, by an abstract topological argument [BL04].*

6.3 Patch counting in $\mathbb{R}$

We will next demonstrate that the classical formula of the patch counting entropy yields different values for different choices of Van Hove sequences already for FLC Delone sets in $\mathbb{R}$.

Example 6.30. *Consider the FLC Delone set $\omega := (-\mathbb{N}_0) \cup \alpha\mathbb{N}_0 \subseteq \mathbb{R}$ for $\alpha \in [0,1]$ irrational. Then for $\kappa \in [0,\infty]$ there exists a Van Hove sequence $(A_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} = \kappa.$$

Proof. Let us first consider the case $\kappa < \infty$ and define $A_n := [0,n] + e^{\kappa n}$ for any $n \in \mathbb{N}$. We will now show that $F_n := \mathbb{Z} \cap [-(n+1) - e^{\kappa n}, 0]$ is an exact A_n-patch representation for ω. To do this we consider $g \in \omega \setminus F_n$. If $g > 0$, then $(\omega - g) \cap A_n = (\alpha\mathbb{Z}) \cap A_n = (\omega - 0) \cap A_n$ and we can represent g by $0 \in F_n$. If $g \leq 0$, then $g \in \mathbb{Z}$ and we obtain from $g \notin F_n$ that there holds $g < -(n+1) - e^{\kappa n}$. Thus, we obtain $g, (\min F_n) \leq -n - e^{\kappa n}$ and observe $(\omega - g) \cap A_n = \mathbb{Z} \cap A_n = (\omega - \min F_n) \cap A_n$. This shows that F_n is indeed an exact A_n-patch representation for ω and we obtain $|\operatorname{Pat}_\omega(A_n)| \leq |F_n| \leq (n+1) + e^{\kappa n}$. Now as $\kappa \geq 0$ for sufficiently large n there holds $1 \leq e^{\kappa n}$ and we get $\log|\operatorname{Pat}_\omega(A_n)| \leq \log((n+2)e^{\kappa n}) = \log(n+2) + \kappa n$. Thus, there holds

$$\limsup_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} \leq \limsup_{n\to\infty} \left(\frac{\log(n+2)}{n} + \frac{\kappa n}{n}\right) = \kappa$$

and the statement follows, whenever $\kappa = 0$. Otherwise let us consider $E_n := \mathbb{Z} \cap (-e^{\kappa n}, 0] \cap \mathbb{Z}$. Then for $g \in E_n$ there holds $|g| < e^{\kappa n} = \min A_n$. Thus, the elements of $(\omega - g) \cap A_n$ are of the form $|g| + k\alpha$ for $k \in \mathbb{N}$. Furthermore, as $\alpha \leq 1$ there is at least one such number contained in $(\omega - g) \cap A_n$. Thus, whenever we consider distinct $g, g' \in E_n$ we obtain from α being irrational, that the corresponding patches $(\omega - g) \cap A_n$ and $(\omega - g') \cap A_n$ do not agree. This yields that $|\operatorname{Pat}_\omega(A_n)| \geq |E_n| \geq e^{\kappa n} - 1$. Now as we assume that $\kappa > 0$ we obtain that for large n there holds $2 \leq e^{\kappa n}$ and in particular that $\log|\operatorname{Pat}_\omega(A_n)| \geq \log(e^{\kappa n} - (1/2)e^{\kappa n}) = \log(1/2) + \kappa n$. This allows to compute

$$\liminf_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} \geq \liminf_{n\to\infty} \frac{\log(1/2) + \kappa n}{n} = \kappa$$

and we obtain the claimed statement for all $\kappa < \infty$. Similarly one shows the result for $\kappa = \infty$ using $A_n := [0,n] + e^{(n^2)}$. □

With the observations from Example 6.30 we also obtain that the limit superior in the formula for the patch counting entropy is not always a limit.

Example 6.31. *Consider again the finite local complexity Delone set* $\omega := (-\mathbb{N}_0) \cup \alpha\mathbb{N}_0 \subseteq \mathbb{R}$ *for* $\alpha \in [0,1]$ *irrational. Then there exists a Van Hove sequence* $(A_n)_{n\in\mathbb{N}}$ *such that* $\log|\operatorname{Pat}_\omega(A_n)|/\theta(A_n)$ *does not converge.*

Proof. For $n \in \mathbb{N}$ we define $A_n := [0,n] + e^{(1+(-1)^n)n}$. We then obtain from Example 6.30 and $A_{2n+1} = [0,n] + e^{0\cdot(2n+1)}$ that there holds

$$\liminf_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} \leq \lim_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_{2n+1})|}{\theta(A_{2n+1})} = 0.$$

Similarly we obtain from $A_{2n} = [0,n] + e^{2\cdot(2n)}$ that

$$\limsup_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} \geq \lim_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_{2n})|}{\theta(A_{2n})} = 2$$

and the statement follows. □

Recall from Proposition 4.18 that one can also use Følner nets instead of Van Hove nets in order to define topological entropy. It is thus natural to ask, whether one can use Følner nets that are compactly connected to 0 in order to compute the patch counting entropy. The following example shows that this is not possible.

Example 6.32. *Consider the Delone set of finite local complexity*

$$\omega := \{n \in \mathbb{N};\, \xi_n = 1\} \cup (\mathbb{Z} + 1/2),$$

where $(\xi_n)_{n\in\mathbb{N}}$ *is a sequence containing all finite words in* $\{0,1\}$*, i.e. for all finite sequences* $(x_j)_{j=1}^n$ *there exists* $i \in \mathbb{N}$ *such that* $\xi_{i+j} = x_j$ *for* $j = 1, \cdots, n$*. Then* $\mathrm{E}_{pc}(\omega) = \log(2)$ *and for all* $\kappa \in [0, \log(2)]$ *there is a Følner sequence* $(A_n)_{n\in\mathbb{N}}$*, which is compactly connected to* 0*, such that*

$$\limsup_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} = \kappa.$$

Remark 6.33. *Note that for any FLC Delone set* ω *and for any Følner net that is compactly connected to* 0 *there holds*

$$\limsup_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} \leq \mathrm{E}(\pi_\omega).$$

Indeed, this can be seen by considering any compact neighbourhood K *of* 0*. Then* $(KA_i)_{i\in I}$ *is a Van Hove net that is compactly connected to* 0*. As this Van Hove net furthermore satisfies* $\lim_{i\in I}\theta(KA_i)/\theta(A_i) = 1$ *we obtain from Corollary 6.22 that*

$$\limsup_{i\in I} \frac{\log|\operatorname{Pat}_\omega(A_i)|}{\theta(A_i)} \leq \limsup_{i\in I} \frac{\log|\operatorname{Pat}_\omega(KA_i)|}{\theta(A_i)} = \lim_{i\in I} \frac{\log|\operatorname{Pat}_\omega(KA_i)|}{\theta(KA_i)} = \mathrm{E}(\pi_\omega).$$

Remark 6.34. *Note that with a similar construction as in Example 6.31 above we obtain from Example 6.32 that there are Følner sequences* $(A_n)_{n\in\mathbb{N}}$ *that are compactly connected to* 0 *such that the limit superior in the patch counting entropy formula is not a limit, i.e. such that* $\log|\operatorname{Pat}_\omega(A_n)|/\theta(A_n)$ *does not converge.*

Proof of the claims of Example 6.32. We abbreviate $\rho := \kappa/\log(2) \in [0,1]$ and set

$$A_n := [0,\rho n] \cup \left([0,n] \setminus \left(\frac{1}{2}\mathbb{Z} + B_{(n+2)^{-1}}(0)\right)\right).$$

We first show that $(A_n)_{n\in\mathbb{N}}$ is a Følner sequence and thus consider a compact and non-empty subset $K \subseteq \mathbb{R}$. Then there is $k \in \mathbb{N}$ such that $K \subseteq [-k,k]$ and we compute

$$\begin{aligned}\theta(KA_n \setminus A_n) &\leq \theta\left([-k,n+k] \setminus \left([0,n] \setminus \left(\frac{1}{2}\mathbb{Z} + B_{(n+2)^{-1}}(0)\right)\right)\right)\\ &\leq \theta\left([-k,0] \cup [n,n+k] \cup \left([0,n] \cap \left(\frac{1}{2}\mathbb{Z} + B_{(n+2)^{-1}}(0)\right)\right)\right)\\ &\leq 2\frac{n}{n+2} + 2k.\end{aligned}$$

Thus, it follows that $\lim_{n\to\infty} \theta(KA_n \setminus A_n)/\theta(A_n) = 0$ and we have shown $(A_n)_{n\in\mathbb{N}}$ to be a Følner net.

Note next that ω is contained in $(1/2)\mathbb{Z}$, which implies $\operatorname{Pat}_\omega(A_n) = \operatorname{Pat}_\omega([0,\rho n])$ for $n \in \mathbb{N}$. This allows to observe

$$\begin{aligned}\operatorname{Pat}_\omega(A_n) = &\left\{W \cup \left([0,\rho n] \cap \left(\frac{1}{2} + \mathbb{Z}\right)\right);\, W \subseteq [0,\rho n] \cap \mathbb{Z}\right\}\\ &\cup \left\{W \cup ([0,\rho n] \cap \mathbb{Z});\, W \subseteq [0,\rho n] \cap \left(\mathbb{Z} + \frac{1}{2}\right)\right\}.\end{aligned}$$

We thus obtain that $2^{\rho n} \leq |\operatorname{Pat}_\omega(A_n)| \leq 2^{\rho n+2}$ and a straightforward argument shows $\limsup_{n\to\infty} \log|\operatorname{Pat}_\omega(A_n)|/\theta(A_n) = \kappa$.

Now for $\kappa = \log(2)$ we obtain $A_n = [0,\rho n]$ for any $n \in \mathbb{N}$ and thus $(A_n)_{n\in\mathbb{N}}$ is a Van Hove net that is compactly connected to 0. We thus see from Corollary 6.22 that $\mathrm{E}(\pi_\omega) = \limsup_{n\to\infty} \log|\operatorname{Pat}_\omega(A_n)|/\theta(A_n) = \log(2)$. □

Now recall that we have seen that the patch counting at a certain scale via $\operatorname{pat}_\omega$ is a concept which is closely related to the topological entropy of the corresponding Delone dynamical system and that the corresponding formula can be used for all Van Hove nets. Furthermore, the topological entropy is a concept that can be computed via Følner nets as we have seen in Proposition 4.18. It is thus natural to ask, whether the formula of Theorem 6.4 also holds for all Følner nets $(A_i)_{i\in I}$, i.e. whether there holds

$$\mathrm{E}(\pi_\omega) = \sup_V \liminf_{i\in I} \frac{\log(\operatorname{pat}_\omega(A_i,V))}{\theta(A_i)} = \sup_V \limsup_{i\in I} \frac{\log(\operatorname{pat}_\omega(A_i,V))}{\theta(A_i)},$$

where the suprema are taken over all open neighbourhoods V of 0. In the next example we present that this is not the case.

Example 6.35. *Consider the FLC Delone set $\omega \subseteq \mathbb{R}$ as in Example 6.32. Then for all $\kappa \in [0, \log(2)]$ there is a Følner sequence $(A_n)_{n\in\mathbb{N}}$, which is compactly connected to 0 such that there holds*

$$\sup_V \limsup_{n\to\infty} \frac{\log(\mathrm{pat}_\omega(A_n, V))}{\theta(A_n)} = \kappa,$$

where the supremum is taken over all open neighbourhoods V of 0.

Proof. Clearly, it suffices to consider open neighbourhoods V of 0 that are contained in $B_{1/2}(0)$. Let V be such a neighbourhood and consider the Følner sequence $(A_n)_{n\in\mathbb{N}}$ as in the proof of the claims of Example 6.32. Now recall that ω is contained in $1/2\mathbb{Z}$ and consider $g, g' \in \omega$ such that $\omega - g$ and $\omega - g'$ are $\epsilon(A_n, V)$-close. Then for $x \in (\omega - g) \cap A_n$ we know that x is also contained in $1/2\mathbb{Z}$ and observe furthermore

$$x \in (\omega - g) \cap A_n \subseteq \omega - g' + V \subseteq \omega - g' + B_{1/2}(0).$$

Thus, there is $b \in B_{1/2}(0)$ such that $x \in \omega - g' + b$. We have $b \in -\omega + g' + x \subseteq 1/2\mathbb{Z}$ and we obtain $b = 0$. Thus, there holds $x \in \omega - g'$ and we have shown $(\omega - g) \cap A_n \subseteq (\omega - g') \cap A_n$. A similar argument gives the reversed inclusion and we have shown that for $g, g' \in \omega$ the A_n-patches $(\omega - g) \cap A_n$ and $(\omega - g') \cap A_n$ are actually equal, whenever $\omega - g$ and $\omega - g'$ are $\epsilon(A_n, V)$-close. Thus, whenever F is an A_n-patch representation at scale V, then it is actually an exact A_n-patch representation. As any exact A_n-patch representation is an A_n-patch representation at scale V we thus obtain $\mathrm{pat}_\omega(A_n, V) = |\operatorname{Pat}_\omega(A_n)|$. Now our choice of $(A_n)_{n\in\mathbb{N}}$ yields that

$$\limsup_{n\to\infty} \frac{\log(\mathrm{pat}_\omega(A_n, V))}{\theta(A_n)} = \limsup_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} = \kappa.$$

Taking the supremum over all considered V thus yields the statement. □

We have already seen, that one cannot use Følner nets, which are compactly connected to 0 in the patch counting formula. Nevertheless, we have seen in Remark 6.33 that this formula always yields a value which is bounded by the topological entropy of the corresponding Delone dynamical system. Thus, naturally the question arises, whether this can also be achieved for the more general class of ergodic nets that are compactly connected to 0. We next show that this is not the case.

Example 6.36. *Consider $\omega \subseteq \mathbb{R}$ as in Example 6.32. Then for all $\kappa \in [0, \infty]$ there is an ergodic sequence $(A_n)_{n\in\mathbb{N}}$, which is compactly connected to 0, such that*

$$\limsup_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} = \kappa.$$

Proof. As every Følner sequence is ergodic it remains to consider $\kappa \in [\log(2), \infty]$. We first consider the case $\kappa < \infty$, set $\rho := \kappa/\log(2)$ and define

$$A_n := \Big([0,n] \setminus \Big(\frac{1}{2}\mathbb{Z} + B_{(n+2)^{-1}}(0)\Big)\Big) \cup \Big([0,\rho n] \cap \mathbb{Z}\Big).$$

Then with a similar argument as presented in Example 6.32 we show that the sequence $\Big([0,n] \setminus \Big(\frac{1}{2}\mathbb{Z} + B_{(n+2)^{-1}}(0)\Big)\Big)_{n\in\mathbb{N}}$ is a Følner sequence in $\mathbb{R}$. We thus obtain that $(A_n)_{n\in\mathbb{N}}$ is ergodic from Proposition 2.22. This net is clearly $[0,1]$-connected.

As ω is contained in $1/2\mathbb{Z}$ we obtain $|\operatorname{Pat}_\omega(A_n)| = |\operatorname{Pat}_\omega([0,\rho n] \cap \mathbb{Z})|$. We thus observe $\operatorname{Pat}_\omega(A_n) = \{W;\, W \subseteq [0,\rho n] \cap \mathbb{Z}\}$ and in particular $2^{\rho n} \leq |\operatorname{Pat}_\omega(A_n)| \leq 2^{\rho n+1}$. Thus, a straightforward argument shows $\limsup_{n\to\infty} \log|\operatorname{Pat}_\omega(A_n)|/\theta(A_n) = \kappa$. Whenever $\kappa = \infty$ we use

$$A_n := \Big([0,n] \setminus \Big(\frac{1}{2}\mathbb{Z} + B_{(n+2)^{-1}}(0)\Big)\Big) \cup \Big([0,n^2] \cap \mathbb{Z}\Big)$$

and a similar argument as above yields the statement. □

The following example will serve to answer the questions raised around the upper semi-continuity and the Kolmogorov-Sinai generator theorem in Section 5.3.

Example 6.37. *There exists a Delone set ω in $\mathbb{R}$ such that $\mathrm{E}(\pi_\omega) = \infty$ and such that the entropy map $\mu \mapsto \mathrm{E}_\mu(\pi_\omega)$ is not upper semi-continuous.*

Remark 6.38. *Recall from Proposition 5.52 that for any invariant Borel probability measure μ on X_ω there exists a finite partition with almost no boundary with respect to μ that is generating along any dense subgroup of $\mathbb{R}$. Nevertheless, for any uniform lattice Λ in $\mathbb{R}$ there exists an invariant Borel probability measure μ on X_ω such that there is no finite partition that is generating along Λ and which has no boundary with respect to μ. Indeed, whenever for each $\mu \in \mathcal{M}_G(X)$ there would exist such a finite partition, then Corollary 5.50 would imply the entropy map to be upper semi-continuous, a contradiction.*

Proof of the claims of Example 6.37. Let us first see that it is sufficient to construct a Delone set ω such that π_ω has infinite topological entropy. Indeed, any upper semi-continuous function attains its maximum on a compact set. Thus, if the entropy map would be upper semi-continuous the variational principle as stated in Theorem 5.33 would imply $\mathrm{E}(\pi_\omega)$ to be finite.

It remains to construct a Delone set ω in $\mathbb{R}$ that satisfies $\mathrm{E}(\pi_\omega) = \infty$. To do this consider the countable set $D := \bigcup_{M\in\mathbb{N}} ((0,1] \cap (M^{-1}\mathbb{Z}))$. Thus, the set of all finite sequences $s = (x_i)_{i=1}^n$ in D is countable and there exists a sequence $(s^{(k)})_{k\in\mathbb{N}}$ of finite sequences in D such that any finite sequence in D appears as $s^{(k)}$ for some $k \in \mathbb{N}$. Let now $(a_m)_{m\in\mathbb{N}}$ be the sequence in D that is constructed by first following the sequence $s^{(1)}$, then $s^{(2)}$ and so on. To be precise denote by n_k the length of the finite sequence

$s^{(k)} = (s_i^{(k)})_{i=1}^{n_k}$ and set $N_0 := 0$ and $N_k := \sum_{i=1}^{k} n_i$. Then for any $m \in \mathbb{N}$ there is a unique $k \in \mathbb{N}$ such that $N_{k-1} < m \leq N_k$ and we define $a_m := s^{(k)}_{m-N_{k-1}}$. We furthermore set $g_0 := 0$ and $g_m := \sum_{i=1}^{m}(1+a_i)$ for $m \in \mathbb{N}$ and define

$$\omega := (-\mathbb{N}) \cup \{g_m;\, m \in \mathbb{N}_0\}\,.$$

As $a_m \in D \subseteq [0,1]$ one easily observes that ω is $(-1/2, 1/2)$-discrete and $[0,2]$-dense. Thus, ω is a Delone set.

Let us denote $A_n := [-2n, 0]$, $V_M := (-1/M, 1/M)$ and $\epsilon^{(M)} := \epsilon_\omega(\{0\}, V_M)$ for $n, M \in \mathbb{N}$. Then by Lemma 6.5 there holds

$$\epsilon^{(M)}_{A_n} = \epsilon_\omega(\{0\}, V_M)_{-[0,2n]} = \epsilon_\omega([0,2n], V_M).$$

To show that there holds $\mathrm{sep}_{X_\omega}\left(\epsilon^{(M)}_{A_n}\right) \geq M^n$ consider all finite sequences in $(0,1] \cap (M^{-1}\mathbb{Z})$ of length n and note that there are M^n such sequences as $(0,1] \cap (M^{-1} \cap \mathbb{Z})$ has cardinality M. By the construction of $(a_m)_{m\in\mathbb{N}}$ there exist $m_1, \cdots, m_{(M^n)}$ in $\mathbb{N}_0$ such that $(a_{m_j+k})_{k=1}^n$ with $j = 1, \cdots, M^n$ are exactly the finite sequences in $(0,1] \cap (M^{-1}\mathbb{Z})$ of length n. Now consider distinct $j, j' \in \{1, \cdots, M^n\}$. As $(a_{m_j+k})_{k=1}^n \neq (a_{m_{j'}+k})_{k=1}^n$ there is a minimal $k \in \{1, \cdots, n\}$ such that $a_{m_j+k} \neq a_{m_{j'}+k}$ and we assume without lost of generality that $a_{m_j+k} < a_{m_{j'}+k}$. As a_{m_j+k} and $a_{m_{j'}+k}$ are contained in $(0,1] \cap (M^{-1}\mathbb{Z})$ we thus obtain in particular that $a_{m_j+k} + 1/M \leq a_{m_{j'}+k}$. Let us consider $h := \sum_{i=1}^{k}(1 + a_{m_j+i})$. There holds $0 \leq h \leq \sum_{i=1}^{k} 2 \leq 2k \leq 2n$. Furthermore, we compute

$$h = \sum_{i=1}^{m_j+k}(1+a_i) - \sum_{i=1}^{m_j}(1+a_i) = g_{m_j+k} - g_{m_j} \in \omega - g_{m_j}$$

and obtain $h \in (\omega - g_{m_j}) \cap [0, 2n]$. As

$$g_{(m_{j'}+k-1)} - g_{m_{j'}} + \frac{1}{M} \leq \left(\sum_{i=1}^{k-1}(1 + a_{m_{j'}+i})\right) + 1 = \left(\sum_{i=1}^{k-1}(1 + a_{m_j+i})\right) + 1 \leq h$$

and

$$\begin{aligned} h &= \sum_{i=1}^{k}(1 + a_{m_j+i}) \\ &\leq \left(\sum_{i=1}^{k-1}(1 + a_{m_j+i})\right) + (1 + a_{m_{j'}+k}) - \frac{1}{M} \\ &= \sum_{i=1}^{k}(1 + a_{m_{j'}+i}) - \frac{1}{M} = g_{(m_{j'}+k)} - g_{m_{j'}} - \frac{1}{M}, \end{aligned}$$

we observe that $h \notin \omega - g_{m_{j'}} + V_M$. This shows that $(\omega - g_{m_j}) \cap [0, 2n] \not\subseteq \omega - g_{m_{j'}} + V_M$

and we have shown that for distinct $j, j' \in \{1, \cdots, M^n\}$ the sets $\omega - g_{m_j}$ and $\omega - g_{m_{j'}}$ are not $\epsilon_{A_n}^{(M)} = \epsilon_{X_\omega}([0,2n], V_M)$-close. In particular, we have shown that $\mathrm{sep}_{X_\omega}\left(\epsilon_{A_n}^{(M)}\right) \geq M^n$.

As $(A_n)_{n\in\mathbb{N}}$ is a Van Hove net in $\mathbb{R}$ we then obtain from Theorem 4.33 that for any $M \in \mathbb{N}$ there holds

$$\mathrm{E}(\pi_\omega) \geq \limsup_{n\to\infty} \frac{\log\left(\mathrm{sep_{X_\omega}}\left(\epsilon_{\mathrm{A_n}}^{(\mathrm{M})}\right)\right)}{\theta(A_n)} \geq \limsup_{n\to\infty} \frac{n\log(M)}{2n} = \frac{\log(M)}{2}.$$

This shows $\mathrm{E}(\pi_\omega) = \infty$. □

Remark 6.39. *It is shown in [Lag99, Theorem 2.3] that the patch counting entropy of a FLC Delone set in $\mathbb{R}^d$ is always finite. We thus obtain from Proposition 6.11 that the topological entropy of the Delone dynamical system of a FLC Delone set is also always finite. It remains open, whether FLC Delone sets have an upper semi-continuous entropy map with respect to the corresponding Delone system. This is in particular of interest as some of the statements of Chapter 7 like the converse variational principle or statements about the existence of equilibrium states depend on this property.*

6.4 Patch counting in $\mathbb{Q}_2$

Recall from Example 6.15 that $\mathbb{Q}_2$ contains no Van Hove net that is compactly connected to 0. Thus, in particular the Van Hove sequence of closed centred balls $\left(\overline{B}_n(0)\right)_{n\in\mathbb{N}}$ is not uniformly compactly connected to 0 in $\mathbb{Q}_2$ and we cannot apply Corollary 6.22 to obtain that the sequence $\log|\mathrm{Pat}_\omega(\overline{B}_n(0))|/\theta(\overline{B}_n(0))$ converges to $\mathrm{E}(\pi_\omega)$. Nevertheless, $\mathbb{Q}_2$ is actually a complete field whose topology comes from an absolute value. One could thus hope that these strong properties, which are similar to the ones of $\mathbb{R}$, are sufficient to show that the formula for the patch counting entropy along the sequence of centred closed balls $(\overline{B}_n(0))_{n\in\mathbb{N}}$ yields the topological entropy or that $\log|\mathrm{Pat}_\omega(\overline{B}_n(0))|/\theta(\overline{B}_n(0))$ converges as n tends to ∞. In this section we will construct examples of Delone sets such that these properties are not satisfied. In particular, we will see in Theorem 6.47 that there are examples of FLC Delone sets in $\mathbb{Q}_2$ for which $\log|\mathrm{Pat}_\omega(\overline{B}_n(0))|/\theta(\overline{B}_n(0))$ converges to $\log(2)$, but whose Delone dynamical system has 0 topological entropy. Furthermore, we will see in Theorem 6.51 that there exist FLC Delone sets ω in $\mathbb{Q}_2$ such that $\log|\mathrm{Pat}_\omega(\overline{B}_n(0))|/\theta(\overline{B}_n(0))$ does not converge. Note that we restrict to 2-adic numbers for simplicity and that similar examples can constructed also within $\mathbb{Q}_p$ for any prime number p. It remains open, whether one can construct model sets in $\mathbb{Q}_2$ with the mentioned properties.

6.4.1 Construction tools

The construction of the Delone sets below will be done by first constructing inductively a monotone family $(\omega_n)_{n\in\mathbb{N}_0}$ of finite subsets of $\mathbb{Q}_2$ with appropriate properties such that the union $\bigcup_n \omega_n$ is a FLC Delone set with the required properties. In particular, in the induction step we will need to expand from ω_n to ω_{n+1}. In this subsection we summarize the technical tools to do this. We will construct the ω_n to consist of rational numbers. This will allow us to consider the ω_n also as subsets of $\mathbb{R}$ and in particular to use the metric structure of $\mathbb{R}$. In order to get not confused with the closed balls in $\mathbb{Q}_2$, we say *real ball*, whenever we mean a closed ball in $\mathbb{R}$. We furthermore denote $\overline{R}_r(g)$ for the real ball of radius r and centre $g \in \mathbb{R}$. Nevertheless, we will most of the times consider closed balls in $\mathbb{Q}_2$ for which we shortly say *ball* and use the usual notation $\overline{B}_r(g)$ in this section. Recall that the metric of $\mathbb{Q}_2$ only takes values in $\{2^z;\, z \in \mathbb{Z}\}$. It will be thus convenient to use the notation $A_n := \overline{B}_{2^n}(0)$ for $n \in \mathbb{N}_0$. We furthermore abbreviate $a_n := 2^{2^n}$ and $V_n := \overline{B}_{2^{-n}}(0)$ for any $n \in \mathbb{N}_0$.

Basic tools

Let us start with the following observation which links balls and centred real balls.

Lemma 6.40. *Whenever $B \subseteq \mathbb{Q}_2$ is a (not necessarily centred) ball and $R \subseteq \mathbb{R}$ is a centred real ball and both have a radius which is greater or equal to 1, then they intersect.*

Proof. Choose any $b \in B$ such that $\overline{B}_1(b) \subseteq B$. Now recall that b can be written as $\sum_{i=-n}^{\infty} b_i 2^i$ for some $n \in \mathbb{N}_0$ and a sequence $(b_i)_{i=-n}^{\infty}$ in $\{0, 1\}$. Note that the convergence of the series is given with respect to the topology of $\mathbb{Q}_2$. We now define $\tilde{b} := \sum_{i=-n}^{-1} b_i 2^i$. Then there holds

$$\left|\tilde{b} - b\right|_2 = \left|\sum_{i=0}^{\infty} b_i 2^i\right|_2 \leq 1$$

and in particular we obtain that $\tilde{b} \in \overline{B}_1(b) \subseteq B$. Clearly, $\tilde{b} = \sum_{i=-n}^{-1} b_i 2^i = \sum_{i=1}^{n} b_{-i} 2^{-i}$ is contained in $\mathbb{Q}$ and thus also in $\mathbb{R}$. Furthermore, interpreting $\tilde{b}$ and the following series as elements of $\mathbb{R}$ we obtain $0 \leq_{\mathbb{R}} \tilde{b} \leq_{\mathbb{R}} \sum_{i=1}^{\infty} 2^{-i} = 1$, which implies $\tilde{b} \in \overline{R}_1(0) \subseteq R$ and we have shown $\tilde{b} \in R \cap B$. $\square$

We will furthermore frequently use the following observations.

Lemma 6.41. *Let $m \in \mathbb{N}$ and $M \subseteq \mathbb{Q}_2$ and consider $F := M \cap A_{2^m}$. Then for any $n \leq 2^m$ the following statements are valid.*

(i) For any $g \in A_{2^m}$ there holds

$$(F - g) \cap A_n = (M - g) \cap A_n.$$

(ii) For any $g \in A_{2^m}$ and any open neighbourhood V of 0 there holds

$$F - g \overset{A_n, V}{\approx} M - g.$$

(iii) Every A_n-patch of F is an A_n-patch of M, i.e.

$$\mathrm{Pat}_F(A_n) \subseteq \mathrm{Pat}_M(A_n).$$

Proof. For $g \in A_{2^m}$ we use that A_{2^m} is a subgroup to obtain that $A_{2^m} + g = A_{2^m}$. We thus compute

$$(F - g) \cap A_{2^m} = (F \cap A_{2^m}) - g = (M \cap A_{2^m}) - g = (M - g) \cap A_{2^m}.$$

As for $n \leq 2^m$ there holds $A_n \subseteq A_{2^m}$ we thus obtain $(F - g) \cap A_n = (M - g) \cap A_n$, which proves (i). To show (ii) consider any $g \in A_{2^m}$ and any open neighbourhood V of 0 and observe that (i) implies $(F - g) \cap A_n = (M - g) \cap A_n \subseteq (M - g) + V$. As one observes similarly $(M - g) \cap A_n \subseteq (F - g) + V$ we have shown (ii). To see (iii) note that the A_n-patches of F are of the form $(F - g) \cap A_n$ for $g \in F \subseteq A_{2^m}$. We then obtain from (i) that $(F - g) \cap A_n = (M - g) \cap A_n$ and thus also conclude (iii). □

In order to handle A_{2^n}-patch representations at scale V_n the following observation will become useful.

Lemma 6.42. *For all $n \in \mathbb{N}$ the set $\epsilon(A_{2^n}, V_n) \subseteq \mathcal{A}(\mathbb{Q}_2)^2$ is an equivalence relation.*

Proof. Clearly, $\epsilon(A_{2^n}, V_n)$ is a reflexive and symmetric relation. To show that it is also transitive consider $(\xi_1, \xi_2) \in \epsilon(A_{2^n}, V_n)$ and consider furthermore ξ_3 such that $(\xi_2, \xi_3) \in \epsilon(A_{2^n}, V_n)$. For $v \in V_n$ there holds $v \in V_n \subseteq A_{2^n}$ and as A_{2^n} is a subgroup we obtain $A_{2^n} - v = A_{2^n}$. Thus, there holds $\xi_2 \cap (A_{2^n} - v) = \xi_2 \cap A_{2^n} \subseteq \xi_3 + V_n$. As V_n is also a subgroup we therefore compute

$$\begin{aligned}\xi_1 \cap A_{2^n} \subseteq (\xi_2 + V) \cap A_{2^n} &= \bigcup_{v \in V_n} ((\xi_2 + v) \cap A_{2^n}) = \bigcup_{v \in V_n} (\xi_2 \cap (A_{2^n} - v)) + v \\ &\subseteq \bigcup_{v \in V_n} \xi_3 + V_n + v = \xi_3 + V_n.\end{aligned}$$

Similarly one shows $\xi_3 \cap A_{2^n} \subseteq \xi_1 + V_n$ and we have shown $(\xi_1, \xi_3) \in \epsilon(A_{2^n}, V_n)$. □

The expanding lemmas

We will next present three lemmas, which are the corner stones of the constructions in this section. Lemma 6.43 gives the tool to expand from ω_n to ω_{n+1} without improving the number of a certain type of patches.

Lemma 6.43. *Let $n \in \mathbb{N}_0$ and let $F \subseteq \mathbb{Q}_2$ be a finite subset. Assume that F is contained in $A_{2^n} \cap \mathbb{Q}$ and that F contains exactly one element from each ball of radius 1 that is contained in A_{2^n}. Then there exists a finite subset E of $\mathbb{Q}_2$ such that*

(a) $F = E \cap A_{2^n}$.

(b) E is contained in $A_{2^{n+1}} \cap \mathbb{Q}$ and contains exactly one element from each ball of radius 1 that is contained in $A_{2^{n+1}}$.

(c) F is an exact A_{2^n}-patch representation for E.

Proof. Let us denote by $\hat{\mathcal{B}}$ the set of all balls of radius a_n that are contained in $A_{2^{n+1}}$ and that do not intersect A_{2^n}. As $\mathbb{Q}$ is dense in $\mathbb{Q}_2$ we can choose a rational number $g_{\hat{B}} \in \hat{B}$ for each $\hat{B} \in \hat{\mathcal{B}}$. We set

$$E := F \cup \bigcup_{\hat{B} \in \hat{\mathcal{B}}} (F + g_{\hat{B}}).$$

Now observe that $F + g_{\hat{B}} \subseteq A_{2^n} + g_{\hat{B}} = \hat{B}$ is disjoint from A_{2^n} for each $\hat{B} \in \hat{\mathcal{B}}$. This implies $F = E \cap A_{2^n}$ and we have shown (a). Now note that $A_{2^{n+1}} \cap \mathbb{Q}$ is a subgroup of $\mathbb{Q}_2$ that contains F and all $g_{\hat{B}}$ for $\hat{B} \in \hat{\mathcal{B}}$. We thus obtain that E is contained in $A_{2^{n+1}} \cap \mathbb{Q}$. To show the rest of (b) let B be a ball of radius 1 contained in $A_{2^{n+1}}$. If $B \subseteq A_{2^n}$, then the statement follows from (a) and the respective assumption on F. We thus assume without lost of generality that $B \subseteq A_{2^{n+1}} \setminus A_{2^n}$ and obtain that there exists $\hat{B} \in \hat{\mathcal{B}}$ such that $B \subseteq \hat{B}$. As $g \mapsto g + g_{\hat{B}}$ is a bijective isometry it maps the balls of radius 1 contained in A_{2^n} bijectively to the balls of radius 1 contained in $A_{2^n} + g_{\hat{B}} = \hat{B}$. Using that $\hat{\mathcal{B}} \cup \{F\}$ is a partition of $A_{2^{n+1}}$ it is furthermore straightforward to see that there holds $\hat{B} \cap E = F + g_{\hat{B}}$. This allows to use the respective assumption on F to observe that E contains exactly one element of B and we have shown (b).

To show (c) let $g \in E \setminus F$. Then there is $\hat{B} \in \hat{\mathcal{B}}$ such that $g \in \hat{B}$. We then obtain that $g \in E \cap \hat{B} = F + g_{\hat{B}}$ and thus $h := g - g_{\hat{B}} \in F$. As $F \subseteq A_{2^n}$ and as A_{2^n} is a subgroup we thus obtain $A_{2^n} + g = A_{2^n} + h + g_{\hat{B}} = A_{2^n} + g_{\hat{B}} = \hat{B}$ and compute

$$(E - g) \cap A_{2^n} = \left(E \cap \hat{B}\right) - g = (F + g_{\hat{B}}) - g = F - h = (F - h) \cap A_{2^n}.$$

This shows that F is indeed an exact A_{2^n}-patch representation for E. □

The next lemma will allow us to introduce complexity into the FLC Delone sets.

Lemma 6.44. *Let $n \in \mathbb{N}_0$ and let $F \subseteq \mathbb{Q}_2$ be a finite subset. Assume that F is contained in $A_{2^n} \cap \mathbb{Q}$ and that F contains exactly one element from each ball of radius 1 that is contained in A_{2^n}. Then there exists a finite subset E of $\mathbb{Q}_2$ such that*

(a) $F = E \cap A_{2^n}$.

(b) *E is contained in $A_{2^{n+1}} \cap \mathbb{Q}$ and contains exactly one element from each ball of radius 1 that is contained in $A_{2^{n+1}}$.*

(c) *F is an exact A_n-patch representation for E.*

(d) *F is an A_{2^n}-patch representation for E at scale V_n.*

(e) $a_{n+1}/2^{n+2} \leq |\operatorname{Pat}_E(A_{n+1})|$.

Proof. Again we denote the set of all balls of radius a_n that are contained in $A_{2^{n+1}}$ and that do not intersect A_{2^n} by $\hat{\mathcal{B}}$. The set of all balls of radius 2^{n+1} that are contained in $A_{2^{n+1}}$ and which do not intersect A_{2^n} will be denoted by $\mathcal{B}$. As $\mathbb{Q}$ is dense in $\mathbb{Q}_2$ we can choose a rational number $g_{\hat{B}} \in \hat{B}$ for each $\hat{B} \in \hat{\mathcal{B}}$.

Let us next consider $B \in \mathcal{B}$. Then there is $\hat{B} \in \hat{\mathcal{B}}$ such that $B \subseteq \hat{B}$ and we compute $B - g_{\hat{B}} \subseteq \hat{B} - g_{\hat{B}} = \overline{B}_{a_n}(0) = A_{2^n}$. Considering a ball of radius 1 contained in $B - g_{\hat{B}}$ we thus obtain from our assumption on F that $F \cap (B - g_{\hat{B}})$ is non-empty and we can choose $h_B \in (F + g_{\hat{B}}) \cap B$. Then $\overline{B}_{2^n}(h_B)$ is one of the two balls of radius 2^n contained in B. With a similar argument as above one shows that also the other ball intersects $F + g_{\hat{B}}$, chooses h'_B from this intersection, and identifies the second ball of radius 2^n contained in B as $\overline{B}_{2^n}(h'_B)$.

As $F \subseteq \mathbb{Q}$ is finite we know that there exists $r \in \mathbb{N}$ with $r \geq n$ such that F is contained in the centred real ball $\overline{R}_{2^r}(0)$. We denote $W := \{5 \cdot 2^r \cdot j;\ j \in \{1, \cdots, a_{n+1}\}\}$. Then any $v \in W$ satisfies $|v|_2 \leq 2^{-r} \leq 2^{-n}$ and we obtain $W \subseteq V_n (\subseteq \overline{B}_1(0))$. Furthermore, $\mathcal{B}$ contains less then $2^{2^{n+1}}/2^{n+1} = a_{n+1}/2^{n+1}$ elements and thus the cardinality of $\mathcal{B}$ is bounded by the cardinality a_{n+1} of W. We thus obtain the existence of an injective mapping $B \mapsto v_B$ from $\mathcal{B}$ to W. Having collected these notions we can now define

$$E := F \cup \bigcup_{\hat{B} \in \hat{\mathcal{B}}} \bigcup_{B \in \mathcal{B};\, B \subseteq \hat{B}} \left[(F + g_{\hat{B}}) \cap \overline{B}_{2^n}(h_B)\right] \cup \left[(F + g_{\hat{B}} + v_B) \cap \overline{B}_{2^n}(h'_B)\right].$$

As $\left\{\overline{B}_{2^n}(h_B);\ B \in \mathcal{B}\right\} \cup \left\{\overline{B}_{2^n}(h'_B);\ B \in \mathcal{B}\right\}$ is a partition of $A_{2^{n+1}} \setminus A_{2^n}$ we obtain (a). As above it is furthermore straightforward to show that E is contained in $A_{2^{n+1}} \cap \mathbb{Q}$. To show that E contains exactly one element from each ball of radius 1 that is contained in $A_{2^{n+1}}$ let $\check{B}$ be such a ball. If $\check{B}$ is contained in A_{2^n}, then we use the respective property of F and (a) to obtain that $\check{B} \cap E = \check{B} \cap F$ contains exactly one element. If $\check{B}$ is not contained in A_{2^n}, then there exist $B \in \mathcal{B}$ and $\hat{B} \in \hat{\mathcal{B}}$ such that $\check{B} \subseteq B \subseteq \hat{B}$. Recall that B consists of the balls $\overline{B}_{2^n}(h_B)$ and $\overline{B}_{2^n}(h'_B)$. If $\check{B} \subseteq \overline{B}_{2^n}(h_B)$ we obtain $\check{B} \cap E = \check{B} \cap (F + g_{\hat{B}})$. As $\check{B} - g_{\hat{B}}$ is a ball of radius 1 contained in A_{2^n} we thus have that $\check{B} \cap E$ contains exactly one element from our assumptions on F. Similarly one considers the case $\check{B} \subseteq \overline{B}_{2^n}(h'_B)$ and obtains $\check{B} \cap E = \check{B} \cap (F + g_{\hat{B}} + v_B)$. In this case we use that $\check{B} - g_{\hat{B}} - v_B$ is a ball of radius 1 contained in A_{2^n} and conclude (b).

To show (c) we need to show that F is an exact A_n-patch representation for E. To do this consider $g \in E \setminus F$. Then there exist $B \in \mathcal{B}$ and $\hat{B} \in \hat{\mathcal{B}}$ such that $g \in B \subseteq \hat{B}$. If $g \in \overline{B}_{2^n}(h_B)$, then we obtain $g \in \overline{B}_{2^n}(h_B) \cap E = \overline{B}_{2^n}(h_B) \cap (F + g_{\hat{B}})$ and in particular $h := g - g_{\hat{B}} \in F$. As $\overline{B}_{2^n}(h_B) = \overline{B}_{2^n}(g) = A_n + g$ we compute

$$\begin{aligned}((E-g) \cap A_n) + g &= E \cap \overline{B}_{2^n}(h_B)\\ &= (F + g_{\hat{B}}) \cap \overline{B}_{2^n}(h_B)\\ &= ((F + g_{\hat{B}} - g) \cap A_n) + g\\ &= ((F - h) \cap A_n) + g.\end{aligned}$$

Now recall from (a) and our assumptions on F that $F \cap A_{2^n} = F = E \cap A_{2^n}$. As A_{2^n} is furthermore a subgroup we observe that $A_{2^n} = A_{2^n} + h = A_{2^n} - h$, which allows to compute

$$(F-h) \cap A_{2^n} = (F \cap A_{2^n}) - h = (E \cap A_{2^n}) - h = (E-h) \cap A_{2^n}. \tag{6.1}$$

We thus combine our observations to compute

$$\begin{aligned}(E-g) \cap A_n = (F-h) \cap A_n &= (F-h) \cap A_{2^n} \cap A_n\\ &= (E-h) \cap A_{2^n} \cap A_n = (E-h) \cap A_n.\end{aligned}$$

Similarly one considers the case $g \in \overline{B}_{2^n}(h'_B)$, shows that $h' := g - g_{\hat{B}} - v_B$ is contained in F, and obtains that there holds $(E-g) \cap A_n = (E-h') \cap A_n$. This shows that F is indeed an exact A_n-patch representation for E, i.e. (c).

To show (d) we need to show that F is an A_{2^n}-patch representation for E at scale V_n. Let $g \in E \setminus F$ and consider $B \in \mathcal{B}$ and $\hat{B} \in \hat{\mathcal{B}}$ such that $g \in B \subseteq \hat{B}$. Then whenever $g \in \overline{B}_{2^n}(h_B)$, we can argue as above to obtain that $h := g - g_{\hat{B}}$ is contained in F and an argument as in (6.1) yields that $(F-h) \cap A_{2^n} = (E-h) \cap A_{2^n}$. As $g \in \hat{B}$ we furthermore obtain that

$$A_{2^n} = \hat{B} - g = \bigcup_{C \in \mathcal{B}; C \subseteq \hat{B}} \left[\overline{B}_{2^n}(h_C - g) \cup \overline{B}_{2^n}(h'_C - g)\right].$$

Using $v_C \in W \subseteq V_n$ for all $C \in \mathcal{B}$ this allows to compute

$$\begin{aligned}
&(E-h)\cap A_{2^n}\\
&\quad= (F-h)\cap A_{2^n}\\
&\quad= (F-(g-g_{\hat{B}}))\cap\left(\bigcup_{C\in\mathcal{B};C\subseteq\hat{B}}\left[\overline{B}_{2^n}(h_C-g)\cup\overline{B}_{2^n}(h'_C-g)\right]\right)\\
&\quad= \left(\bigcup_{C\in\mathcal{B};C\subseteq\hat{B}}\left[(F+g_{\hat{B}})\cap\overline{B}_{2^n}(h_C)\right]\cup\left[(F+g_{\hat{B}})\cap\overline{B}_{2^n}(h'_C)\right]\right)-g\\
&\quad\overset{A_{2^n},V_n}{\approx} \left(\bigcup_{C\in\mathcal{B};C\subseteq\hat{B}}\left[(F+g_{\hat{B}})\cap\overline{B}_{2^n}(h_C)\right]\cup\left[(F+g_{\hat{B}}+v_C)\cap\overline{B}_{2^n}(h'_C)\right]\right)-g\\
&\quad= \left(E\cap\hat{B}\right)-g\\
&\quad= (E-g)\cap A_{2^n}.
\end{aligned}$$

We have thus shown that $E-g$ and $E-h$ are $\epsilon(A_{2^n}, V_n)$-close. Similarly we argue whenever $g \in \overline{B}_{2^n}(h'_B)$. In this case we consider $h' := g - g_{\hat{B}} - v_B$ and obtain $h' \in F$ and furthermore that $E-g$ and $E-h'$ are $\epsilon(A_{2^n}, V_n)$-close. This shows that F is indeed an A_{2^n}-patch representation at scale V_n for E and we have shown (d).

To show (e) we need to show $a_{n+1}/2^{n+2} \le |\operatorname{Pat}_E(A_{n+1})|$. To do this recall that $\mathcal{B}$ is the set of all balls of radius 2^{n+1} that are contained in the ball $A_{2^{n+1}} = \overline{B}_{a_{n+1}}(0)$ but which do not intersect the ball $A_{2^n} = \overline{B}_{a_n}(0)$. Thus, $\mathcal{B}$ contains exactly $a_{n+1}/2^{n+1} - a_n/2^{n+1}$ elements. Now $a_n \ge a_0 = 2$ implies $2a_n - 2 \ge a_n$ and we obtain

$$a_{n+1} = a_n^2 \le a_n(2a_n-2) = 2(a_n^2 - a_n) = 2(a_{n+1}-a_n).$$

Thus, $\mathcal{B}$ contains at least $a_{n+1}/2^{n+2}$ elements. Note furthermore that for any $B \in \mathcal{B}$ and $\hat{B} \in \hat{\mathcal{B}}$ with $B \subseteq \hat{B}$ there holds $h_B \in (F+g_{\hat{B}}) \cap \overline{B}_{2^n}(h_B) \subseteq E$ and thus the sets $(E-h_B)\cap A_{n+1}$ with $B \in \mathcal{B}$ are A_{n+1}-patches of E. In order to show

$$a_{n+1}/2^{n+2} \le |\mathcal{B}| \le |\operatorname{Pat}_E(A_{n+1})|,$$

i.e. (e) it is thus sufficient to show that for distinct balls B and B' in $\mathcal{B}$ we have $(E-h_B)\cap A_{n+1} \ne (E-h_{B'})\cap A_{n+1}$.

To show this we first consider $B \in \mathcal{B}$ and $\hat{B} \in \mathcal{B}$ such that $B \subseteq \hat{B}$. Now recall from our choice of r that $F \subseteq \overline{R}_{2^r}(0)$. Recall furthermore that we have chosen h_B to be contained in $(F+g_{\hat{B}})\cap B$. We thus obtain $A_{n+1}+h_B = \overline{B}_{2^{n+1}}(h_B) = B$ and that there holds

$$\begin{aligned}
(E - h_B) \cap A_{n+1} &= (E \cap (A_{n+1} + h_B)) - h_B \\
&= (E \cap B) - h_B \\
&= \Big[\Big((F + g_{\hat{B}}) \cap \overline{B}_{2^n}(h_B)\Big) \cup \Big((F + g_{\hat{B}} + v_B) \cap \overline{B}_{2^n}(h'_B)\Big)\Big] - h_B \\
&\subseteq (F + g_{\hat{B}} - h_B) \cup (F + g_{\hat{B}} + v_B - h_B) \\
&\subseteq (F + g_{\hat{B}} - (F + g_{\hat{B}})) \cup (F + g_{\hat{B}} + v_B - (F + g_{\hat{B}})) \\
&= (F - F) \cup (F + v_B - F) \\
&\subseteq \overline{R}_{2 \cdot 2^r}(0) \cup \overline{R}_{2 \cdot 2^r}(v_B).
\end{aligned}$$

We furthermore observe $h'_B + v_B \in h'_B + V_n = h'_B + B_{2^n}(0) \subseteq h'_B + \overline{B}_{2^n}(0) = \overline{B}_{2^n}(h'_B)$. As also $h'_B \in (F + g_{\hat{B}}) \cap B$ this implies $h'_B + v_B \in (F + g_{\hat{B}} + v_B) \cap \overline{B}_{2^n}(h'_B) \subseteq E$. From

$$h'_B - h_B + v_B \in B - h_B + v_B \subseteq A_{n+1} + \overline{B}_1(0) = A_{n+1}$$

we therefore obtain $h'_B - h_B + v_B \in (E - h_B) \cap A_{n+1}$. As

$$h'_B - h_B + v_B \in (F + g_{\hat{B}}) - (F + g_{\hat{B}}) + v_B = F - F + v_B \subseteq \overline{R}_{2 \cdot 2^r}(v_B),$$

we have thus shown that $((E - h_B) \cap A_{n+1}) \cap \overline{R}_{2 \cdot 2^r}(v_B)$ is not empty. Furthermore, for distinct $B, B' \in \mathcal{B}$ we know that v_B and $v_{B'}$ are distinct and contained in W and thus have at least $\mathbb{R}$-distance $5 \cdot 2^r$. Thus, whenever $B, B' \in \mathcal{B}$ are distinct, then the non-empty sets $((E - h_B) \cap A_{n+1}) \cap \overline{R}_{2 \cdot 2^r}(v_B)$ and $((E - h_{B'}) \cap A_{n+1}) \cap \overline{R}_{2 \cdot 2^r}(v_{B'})$ do not intersect, which shows in particular that $(E - h_B) \cap A_{n+1} \neq (E - h_{B'}) \cap A_{n+1}$. We have thus shown (e). □

We next present a third lemma of this sort, which will help us to construct a FLC Delone set in $\mathbb{Q}_2$ with an infinite patch counting entropy along $(\overline{B}_n(0))_{n \in \mathbb{N}}$. Note that we expand this time from balls of radius $2^{a_n} = 2^{2^{2^n}}$ to balls of radius $2^{a_{n+1}}$.

Lemma 6.45. *Let $n \in \mathbb{N}_0$ and let $F \subseteq \mathbb{Q}_2$ be a finite subset. Assume that F is contained in $A_{a_n} \cap \mathbb{Q}$ and that F contains exactly one element from each ball of radius 1 that is contained in A_{a_n}. Then there exists a finite subset E of $\mathbb{Q}_2$ such that*

(a) $F = E \cap A_{a_n}$.

(b) *E is contained in $A_{a_{n+1}}$ and contains exactly one element from each ball of radius 1 that is contained in $A_{a_{n+1}}$.*

(c) *F is an exact A_n-patch representation for E.*

(d) $2^{a_n} \leq |\operatorname{Pat}_E(A_{n+1})|$.

Proof. Let us this time denote the set of all balls of radius 2^{n+1} that are contained in $A_{a_{n+1}}$ but do not intersect A_{a_n} by $\mathcal{B}$. Note that $F \subseteq \mathbb{Q}$ is finite and hence bounded in $\mathbb{R}$. Thus, there exists $r \in \mathbb{N}$ such that F is contained in the real ball $\overline{R}_r(0)$. We define $W := \{5rk;\, k = 1, \cdots, 2^{a_{n+1}}\}$. Then W contains $2^{a_{n+1}} \geq (2^{a_{n+1}} - 2^{a_n})/2^{n+1} = |\mathcal{B}|$ elements and thus there exists an injection $B \mapsto v_B$ from $\mathcal{B}$ to W. We furthermore choose a rational number $g_B \in B$ for each $B \in \mathcal{B}$. Then $\check{B}^{(1)} := \overline{B}_{2^n}(g_B)$ is a ball of radius 2^n contained in B. We denote by $\check{B}^{(2)}$ the other ball of radius 2^n contained in the ball B. Let us define

$$E := F \cup \bigcup_{B \in \mathcal{B}} \left((F + g_B) \cap \check{B}^{(1)}\right) \cup \left((F + g_B + v_B) \cap \check{B}^{(2)}\right).$$

Similarly as above we then deduce (a) and (b).

To show (c) we argue similarly as above. We consider $g \in E \setminus F$, choose $B \in \mathcal{B}$ that contains g and distinguish the cases $g \in \check{B}^{(1)}$ and $g \in \check{B}^{(2)}$. In the first case one shows as above that $h := g - g_B \in F$ satisfied $(E - g) \cap A_n = (A - h) \cap A_n$. In the second case one shows that $h' := g - g_B - v_B \in F$ satisfies $(E - g) \cap A_n = (A - h') \cap A_n$ and concludes (c).

It remains to show (d). We use (b) to observe that $(F + g_B) \cap \check{B}^{(1)}$ is non-empty and choose $h_B \in (F + g_B) \cap \check{B}^{(1)}$. Similarly as above we next show that the A_{n+1}-patches $(E - h_B) \cap A_{n+1}$ and $(E - h_{B'}) \cap A_{n+1}$ are distinct for distinct $B, B' \in \mathcal{B}$. As above it is straightforward to deduce that

$$(E - h_B) \cap A_{n+1} \subseteq (F - F) \cup (F - F + v_B) \subseteq \overline{R}_{2\cdot r}(0) + \overline{R}_{2\cdot r}(v_B)$$

and that there are elements from $(E - h_B) \cap A_{n+1}$ contained in $\overline{R}_{2\cdot r}(v_B)$. By construction of W we know that for distinct B, B' the elements v_B and $v_{B'}$ have a real distance which is at least $5r$ and thus $\overline{R}_{2\cdot r}(v_B)$ and $\overline{R}_{2\cdot r}(v_{B'})$ cannot intersect. We thus obtain that the corresponding A_{n+1} are indeed distinct. Thus,

$$|\operatorname{Pat}_E(A_{n+1})| \geq |\mathcal{B}| \geq (2^{a_{n+1}} - 2^{a_n})/2^{n+1}.$$

Note now that $n + 2 \leq 2^{2^n} = a_n$ and thus $a_{n+1} = a_n^2 \geq 2a_n \geq a_n + n + 2$. From this we compute $2^{a_{n+1}} \geq 2^{a_n} 2^{n+2} \geq 2^{a_n}(2^{n+1} + 1)$ and hence

$$|\operatorname{Pat}_E(A_{n+1})| \geq (2^{a_{n+1}} - 2^{a_n})/2^{n+1} \geq 2^{a_n}.$$

We have thus shown (d). $\square$

6.4.2 About patch counting and topological entropy

In this subsection we construct an example of a Delone set ω in $\mathbb{Q}_2$ for which

$$\frac{\log\left|\mathrm{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta\left(\overline{B}_n(0)\right)}$$

converges to log(2) but for which the corresponding Delone dynamical system has 0 topological entropy. Recall that we abbreviate $a_n := 2^{2^n}$, $A_n := \overline{B}_{2^n}(0)$ and $V_n := \overline{B}_{2^{-n}}(0)$ for $n \in \mathbb{N}_0$.

Lemma 6.46. *There exists a family $(\omega_n)_{n\in\mathbb{N}_0}$ of finite subsets of $\mathbb{Q}_2$ such that for all $n \in \mathbb{N}_0$ there holds*

(i) $\omega_n = \omega_{n+1} \cap A_{2^n}$.

(ii) ω_n *is contained in* $A_{2^n} \cap \mathbb{Q}$ *and contains exactly one element from each ball of radius* 1 *that is contained in* A_{2^n}.

(iii) ω_n *is an exact* A_n*-patch representation for* ω_m*, whenever* $m \geq n$.

(iv) ω_n *is an* A_{2^n}*-patch representation for* ω_m *at scale* V_n*, whenever* $m \geq n$.

(v) There holds $a_n/2^{n+1} \leq |Pat_{\omega_n}(A_n)|$.

Proof. We start our induction by defining $\omega_0 := \{0, 1/2\}$. Then $A_{2^0} = A_1 = \overline{B}_2(0)$ contains two balls of radius 1. As $|0 - 1/2|_2 = |2^{-1}|_2 = 2$ we obtain that each such ball contains exactly one element of ω_0, i.e. (ii) for $n = 0$. Now whenever ω_n is chosen for some $n \in \mathbb{N}$ such that (ii) is satisfied we apply Lemma 6.44 in order to obtain ω_{n+1} with the following properties.

(a) $\omega_n = \omega_{n+1} \cap A_{2^n}$.

(b) ω_{n+1} is contained in $A_{2^{n+1}} \cap \mathbb{Q}$ and contains exactly one element from each ball of radius 1 that is contained in $A_{2^{n+1}}$.

(c) ω_n is an exact A_n-patch representation for ω_{n+1}.

(d) ω_n is an A_{2^n}-patch representation for ω_{n+1} at scale V_n.

(e) $a_{n+1}/2^{n+2} \leq |\,\mathrm{Pat}_{\omega_{n+1}}(A_{n+1})|$.

Then (i) and (ii) are trivially satisfied. Furthermore, ω_0 has trivially at least one A_0-patch and we obtain (v) from $a_0/2^{0+2} = 2^{2^0}/4 = 1/2$ and (e). It remains to show (iii) and (iv).

To show (iii) we fix $n \in \mathbb{N}_0$. We will perform an induction over $m \geq n$. Clearly, ω_n is an exact A_n-patch representation for itself and we obtain the statement for $m = n$.

Let us now assume that ω_n is an exact A_n-patch representation for ω_m for some $m \geq n$. To proceed inductively we consider $g \in \omega_{m+1}$. Now recall from (c) that ω_m is an exact A_m-patch representation for ω_{m+1} and thus there is $g' \in \omega_m$ such that $(\omega_{m+1} - g) \cap A_m = (\omega_{m+1} - g') \cap A_m$. From $m \geq n$ we thus obtain in particular that $(\omega_{m+1} - g) \cap A_n = (\omega_{m+1} - g') \cap A_n$. As $g' \in \omega_m$ we obtain from the induction hypothesis the existence of some $g'' \in \omega_n$ such that $(\omega_m - g') \cap A_n = (\omega_m - g'') \cap A_n$. Now recall from (i) that $\omega_{m+1} \cap A_{2^m} = \omega_m$. As $g', g'' \in A_{2^m}$ and as $n \leq 2^m$ we can apply Lemma 6.41 to obtain that

$$\begin{aligned}(\omega_{m+1} - g) \cap A_n &= (\omega_{m+1} - g') \cap A_n = (\omega_m - g') \cap A_n \\ &= (\omega_m - g'') \cap A_n = (\omega_{m+1} - g'') \cap A_n.\end{aligned}$$

As $g'' \in \omega_n$ this shows that ω_n is indeed an exact A_n-patch representation for ω_{m+1} and we have shown (iii).

To show (iv) we argue similarly as for (iii). Let us fix $n \in \mathbb{N}_0$ and note again that the statement is trivial for $m = n$. Let us now assume that ω_n is an A_{2^n}-patch representation for ω_m at scale V_n for some $m \geq n$. To proceed with an induction we consider $g \in \omega_{m+1}$. Then (d) yields the existence of $g' \in \omega_m$ such that $\omega_{m+1} - g$ and $\omega_{m+1} - g'$ are $\epsilon(A_{2^m}, V_m)$-close. As $A_{2^m} \supseteq A_{2^n}$ and as $V_m \subseteq V_n$ we obtain in particular that $\omega_{m+1} - g$ and $\omega_{m+1} - g'$ are $\epsilon(A_{2^n}, V_n)$-close. Now $g' \in \omega_m$ and our induction hypothesis give the existence of $g'' \in \omega_n$ such that $\omega_m - g'$ and $\omega_m - g''$ are $\epsilon(A_{2^n}, V_n)$-close. As $g' \in A_{2^m}$ and as $2^n \leq 2^m$ we can furthermore use Lemma 6.41 in order to see that $\omega_m - g'$ and $\omega_{m+1} - g'$ are $\epsilon(A_{2^n}, V_n)$-close. Similarly we observe that also $\omega_m - g''$ and $\omega_{m+1} - g''$ are $\epsilon(A_{2^n}, V_n)$-close and summarize

$$\omega_{m+1} - g \overset{A_{2^n}, V_n}{\approx} \omega_{m+1} - g' \overset{A_{2^n}, V_n}{\approx} \omega_m - g' \overset{A_{2^n}, V_n}{\approx} \omega_m - g'' \overset{A_{2^n}, V_n}{\approx} \omega_{m+1} - g''.$$

From Lemma 6.42 we know that $\epsilon(A_{2^n}, V_n)$ is an equivalence relation and obtain that $\omega_{m+1} - g$ and $\omega_{m+1} - g''$ are $\epsilon(A_{2^n}, V_n)$-close. As $g'' \in \omega_n$ we have proven that ω_n is indeed an A_{2^n}-patch representation for ω_{m+1} at scale V_n and conclude (iv). □

Theorem 6.47. *There exists a FLC Delone set ω in $\mathbb{Q}_2$ such that the topological entropy* $\mathrm{E}(\pi_\omega)$ *is* 0*, but for which the following limit exists and satisfies*

$$\lim_{n\to\infty} \frac{\log \left|\mathrm{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta\left(\overline{B}_n(0)\right)} = \log(2).$$

Remark 6.48. *Recall from Proposition 6.11 that for all Van Hove nets $\mathcal{A}$ and all Delone sets the patch counting entropy along $\mathcal{A}$ is always larger than the topological entropy, a behaviour that we also observe in the previous theorem.*

Remark 6.49. *In [BLR07] it is shown that for FLC Delone sets in $\mathbb{R}^d$ (with "uniform cluster frequencies") "pure point diffraction" implies* 0 *patch counting entropy (along*

$(\overline{B}_n(0))_{n\in\mathbb{N}}$). In order to do this the authors show first that the measure theoretical entropy of the unique ergodic measure of the Delone dynamical system is 0*. Then they apply the variational principle for actions of $\mathbb{R}^d$ cited from [TZ91] and conclude that also the topological entropy of the considered dynamical system is* 0*. Showing that the topological entropy and the patch counting entropy are equal in this context they then conclude the statement. Note that parts of the proof can be generalized via [FGL18], but it seems open, whether the variational principle holds for actions of $\mathbb{Q}_2$. Theorem 6.47 shows that an analogue of the proof of [BLR07] does not work for the metrizable and σ-compact LCA group $\mathbb{Q}_2$. It remains open, whether "pure point diffraction" implies* 0 *patch counting entropy (along $(\overline{B}_n(0))_{n\in\mathbb{N}}$) also in $\mathbb{Q}_2$ for all FLC Delone sets.*

Proof of Theorem 6.47. Let $(\omega_n)_{n\in\mathbb{N}}$ be a family of finite subsets of $\mathbb{Q}_2$ that satisfies the properties (i),$\cdots$,(v) of Lemma 6.46. We define $\omega := \bigcup_{n\in\mathbb{N}} \omega_n$ and obtain from (i) and (ii) that ω contains exactly one element from each ball of radius 1. Thus, ω is 1-discrete and 1-dense and in particular a Delone set. Furthermore, for $n \in \mathbb{N}$ and $g \in \omega$ there is $m \geq n$ such that $g \in \omega_m$ and we obtain from (iii) that there exists $g' \in \omega_n$ such that $(\omega_m - g) \cap A_n = (\omega_m - g') \cap A_n$. As $g, g' \in A_{2^m}$ and as (i), (ii) and the definition of ω yield $\omega_m = \omega \cap A_{2^m}$ we obtain from Lemma 6.41 that

$$(\omega - g) \cap A_n = (\omega_m - g) \cap A_n = (\omega_m - g') \cap A_n = (\omega - g') \cap A_n.$$

As $g \in \omega_n$ we obtain that ω_n is an exact A_n-patch representation for ω for any $n \in \mathbb{N}$. Thus, (ii) implies

$$|\operatorname{Pat}_\omega(A_n)| \leq |\omega_n| = 2^{2^n}.$$

Clearly, for compact $K \subseteq \mathbb{Q}_2$ there is $n \in \mathbb{N}$ such that $K \subseteq A_n$ and thus $|\operatorname{Pat}_\omega(K)| \leq 2^{2^n}$ and we have proven ω to be FLC. Furthermore, this observation allows to compute

$$\limsup_{n\to\infty} \frac{\log |\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} \leq \limsup_{n\to\infty} \frac{\log\left(2^{2^n}\right)}{2^n} = \log(2). \tag{6.2}$$

Now recall that for any $n \in \mathbb{N}$ there holds $\omega_n = \omega \cap A_{2^n}$. As clearly $n \leq 2^n$ Lemma 6.41 implies that $\operatorname{Pat}_{\omega_n}(A_n) \subseteq \operatorname{Pat}_\omega(A_n)$ and we observe from (v) that

$$2^{2^n-(n+1)} = a_n/2^{n+1} \leq |\operatorname{Pat}_\omega(A_n)|.$$

This allows to compute

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log |\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} &\geq \liminf_{n\to\infty} \frac{\log(2^{2^n-(n+1)})}{2^n} \\ &= \liminf_{n\to\infty} \left(\frac{2^n}{2^n} - \frac{n+1}{2^n}\right)\log(2) \\ &= \log(2). \end{aligned}$$

As the metric of $\mathbb{Q}_2$ only takes values in $\{2^n;\, n \in \mathbb{Z}\}$ we conclude from (6.2) that there holds

$$\lim_{n\to\infty} \frac{\log \left|\mathrm{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta\left(\overline{B}_n(0)\right)} = \lim_{n\to\infty} \frac{\log |\mathrm{Pat}_\omega(A_n)|}{\theta(A_n)} = \log(2).$$

Consider now an open neighbourhood V of 0. Then there exists $N \in \mathbb{N}$ that satisfies $V_N \subseteq V$ and we consider $n \geq N$. To show that ω_n is an A_{2^n}-patch representation for ω at scale V let us consider $g \in \omega$. Clearly, there exists $m \geq n$ such that $g \in \omega_m$ and (iv) allows to observe the existence of $g' \in \omega_n$ such that $\omega_m - g$ and $\omega_m - g'$ are $\epsilon(A_{2^n}, V_n)$-close. We use once more that $\omega_m = \omega \cap A_{2^m}$, that $g \in A_{2^m}$ and furthermore $2^n \leq 2^m$ to apply Lemma 6.41 in order to obtain that $\omega_m - g$ and $\omega - g$ are $\epsilon(A_{2^n}, V)$-close. Similarly we see that $\omega_m - g'$ and $\omega - g'$ are $\epsilon(A_{2^n}, V)$-close. We thus summarize

$$\omega - g \overset{A_{2^n},V}{\approx} \omega_m - g \overset{A_{2^n},V}{\approx} \omega_m - g' \overset{A_{2^n},V}{\approx} \omega - g'.$$

Thus, by Lemma 6.42 the sets $\omega - g$ and $\omega - g'$ are $\epsilon(A_{2^n}, V)$-close. As $g' \in \omega_n$ we have shown that ω_n is indeed an A_{2^n}-patch representation for ω at scale V for any $n \geq N$. We thus obtain from (ii) that there holds $\mathrm{pat}_\omega(A_{2^n}, V) \leq |\omega_n| = 2^{2^n}$ for all $n \geq N$ and compute

$$0 \leq \limsup_{n\to\infty} \frac{\log\left(\mathrm{pat}_\omega(A_{2^n}, V)\right)}{\theta(A_{2^n})} \leq \limsup_{n\to\infty} \frac{\log\left(2^{2^n}\right)}{\theta\left(\overline{B}_{2^{2^n}}(0)\right)} = \log(2) \lim_{n\to\infty} \frac{2^n}{2^{2^n}} = 0.$$

As $(A_{2^n})_{n\in\mathbb{N}}$ is a Van Hove net in $\mathbb{Q}_2$ Theorem 6.4 allows to compute

$$\mathrm{E}(\pi_\omega) = \sup_V \limsup_{n\to\infty} \frac{\log\left(\mathrm{pat}_\omega(A_{2^n}, V)\right)}{\theta(A_{2^n})} = 0,$$

where the supremum is considered over all open neighbourhoods V of 0. □

6.4.3 About the limit in the patch counting formula

We will now use the developed tools to construct an example of a FLC Delone set for which

$$\frac{\log \left|\mathrm{Pat}_\omega(\overline{B}_n(0))\right|}{\theta\left(\overline{B}_n(0)\right)}$$

does not converge. Again we abbreviate $a_n := 2^{2^n}$, $A_n := \overline{B}_{2^n}(0)$ and $V_n := B_{2^{-n}}(0)$ for $n \in \mathbb{N}_0$.

Lemma 6.50. *There exists a family $(\omega_n)_{n\in\mathbb{N}_0}$ of finite subsets of $\mathbb{Q}_2$ and a strictly increasing sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{N}_0$ such that for all $n \in \mathbb{N}_0$ there holds* *(i)* $\omega_n = \omega_{n+1} \cap A_{2^n}$.

(ii) ω_n *is contained in* $A_{2^n} \cap \mathbb{Q}$ *and contains exactly one element from each ball of radius* 1 *that is contained in* A_{2^n}.

(iii) ω_{b_n} *is an exact* $A_{2^{b_n}}$*-patch representation for* ω_m*, whenever* $m \geq b_n$.

(iv) There holds $a_{b_n}/2^{b_n+1} \leq |Pat_{\omega_{b_n}}(A_{b_n})|$.

Proof. We define $b_0 := 0$ and inductively $b_{n+1} := 2^{b_n} + 1$ for any $n \in \mathbb{N}_0$ and obtain a strictly increasing sequence $(b_n)_{n\in\mathbb{N}}$. We define $\omega_0 := \{0, 1/2\}$ and obtain (ii) for $n = 0$ as above.

Now whenever ω_n is chosen to satisfy (ii) for $n \in \mathbb{N}_0$ we distinguish two cases. Whenever n is of the form $b_k - 1$ for some $k \in \mathbb{N}_0$, then we use Lemma 6.44 to obtain a finite set ω_{n+1} that satisfies[2]

(a) $\omega_n = \omega_{n+1} \cap A_{2^n}$.

(b) ω_{n+1} is contained in $A_{2^{n+1}} \cap \mathbb{Q}$ and contains exactly one element from each ball of radius 1 that is contained in $A_{2^{n+1}}$.

(c) ω_n is an exact A_n-patch representation for ω_{n+1}.

(e) $a_{n+1}/2^{n+2} \leq |\operatorname{Pat}_{\omega_{n+1}}(A_{n+1})|$.

Whenever n is not of the form $b_k - 1$ for some $k \in \mathbb{N}_0$, we use Lemma 6.43 to obtain a finite set ω_{n+1} that satisfies (a), (b) and furthermore

(c)' ω_n is an exact A_{2^n}-patch representation for ω_{n+1}.

Then (i) and (ii) are trivially satisfied. Furthermore, from $b_0 = 0$ we obtain $a_{b_0}/2^{b_0+1} = 2^{2^0}/2^{0+1} = 1$ and as ω_0 has at least one A_0-patch we obtain $a_{b_0}/2^{b_0+1} \leq |Pat_{\omega_{b_0}}(A_{b_0})|$. Furthermore, for $k \geq 1$ we know that $b_k \geq 3$ and thus $b_k - 1 \in \mathbb{N}$. Thus, $\omega_{b_k} = \omega_{(b_k-1)+1}$ was chosen via Lemma 6.44 and in particular satisfies

$$a_{b_k}/2^{b_k+1} = a_{(b_k-1)+1}/2^{(b_k-1)+2} \leq |\operatorname{Pat}_{\omega_{(b_k-1)+1}}(A_{(b_k-1)+1})| = |\operatorname{Pat}_{\omega_{b_k}}(A_{b_k})|.$$

This shows (iv) and it remains to show (iii).

Before we show (iii) let us fix $k \in \mathbb{N}_0$ and first show that ω_m is an exact $A_{2^{(b_k)}}$-patch representation for ω_{m+1} for all $m \geq b_k$. Whenever m is not of the form $b_j - 1$ for some $j \in \mathbb{N}_0$, then (c)' implies that ω_m is an exact A_{2^m}-patch representation for ω_{m+1}. As we assume $m \geq b_k$ we obtain $A_{2^{b_k}} \subseteq A_{2^m}$ and we have shown the statement in this case. We next consider the case where $m = b_j - 1$ for some $j \in \mathbb{N}_0$. We then obtain $b_k \leq m = b_j - 1$ and in particular $b_k < b_j$. As $(b_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence we thus obtain

[2] Note that we did not recite (d). This property will not be used here.

$k < j$ and in particular $k+1 \leq j$. An application of the monotonicity of $(b_n)_{n\in\mathbb{N}}$ as well as the definition of this sequence thus yields that there holds $2^{b_k}+1 = b_{k+1} \leq b_j = m+1$ and we observe $2^{b_k} \leq m$. In particular, we obtain $A_{2^{(b_k)}} \subseteq A_m$. As $m = b_j - 1$ for some $j \in \mathbb{N}_0$ we have constructed ω_{m+1} from ω_m via Lemma 6.44. In particular, we recall from (c) that ω_m is an exact A_m-patch representation for ω_{m+1}. We thus obtain from $A_{2^{(b_k)}} \subseteq A_m$ that ω_m is an exact $A_{2^{(b_k)}}$-patch representation for ω_{m+1}. We have thus shown that for any $m \geq b_k$ the set ω_m is an exact $A_{2^{(b_k)}}$-patch representation for ω_{m+1}.

To show (iii) we will fix $k \in \mathbb{N}_0$ and perform an induction over all $m \geq b_k$. Trivially ω_{b_k} is an exact $A_{2^{(b_k)}}$-patch representation for itself. To proceed inductively let us assume that ω_{b_k} is an exact $A_{2^{(b_k)}}$-patch representation for ω_m for some $m \geq b_k$ and consider $g \in \omega_{m+1}$. We have shown above that ω_m is an exact $A_{2^{(b_k)}}$-patch representation for ω_{m+1} and thus there is $g' \in \omega_m$ such that $(\omega_{m+1}-g)\cap A_{2^{(b_k)}} = (\omega_{m+1}-g')\cap A_{2^{(b_k)}}$. Furthermore, from our induction hypothesis we obtain that $g' \in \omega_m$ implies that there is $g'' \in \omega_{b_k}$ such that $(\omega_m - g') \cap A_{2^{(b_k)}} = (\omega_m - g'') \cap A_{2^{(b_k)}}$. Now recall from (i) that $\omega_m = \omega_{m+1} \cap A_{2^m}$. As $2^m \geq 2^{b_k}$ and $g', g'' \in \omega_m \subseteq A_{2^m}$ we thus obtain from Lemma 6.41 that there holds $(\omega_m - g') \cap A_{2^{(b_k)}} = (\omega_{m+1} - g') \cap A_{2^{(b_k)}}$ and $(\omega_m - g'') \cap A_{2^{(b_k)}} = (\omega_{m+1} - g'') \cap A_{2^{(b_k)}}$. We can now combine our observations to compute

$$
\begin{aligned}
(\omega_{m+1} - g) \cap A_{2^{(b_k)}} &= (\omega_{m+1} - g') \cap A_{2^{(b_k)}} = (\omega_m - g') \cap A_{2^{(b_k)}} \\
&= (\omega_m - g'') \cap A_{2^{(b_k)}} = (\omega_{m+1} - g'') \cap A_{2^{(b_k)}}.
\end{aligned}
$$

As $g'' \in \omega_{b_k}$ we have shown that ω_{b_k} is an exact $A_{2^{(b_k)}}$-patch representation for ω_{m+1} and conclude (iii). □

Theorem 6.51. *There exists a FLC Delone set ω in $\mathbb{Q}_2$ such that*

$$
\frac{\log \left|\mathrm{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta\left(\overline{B}_n(0)\right)}
$$

does not converge as n tends to ∞.

Proof. Let $(\omega_n)_{n\in\mathbb{N}}$ be a family of finite subsets of $\mathbb{Q}_2$ that satisfies the properties (i),$\cdots$,(iv) of Lemma 6.50. Again we define $\omega := \bigcup_{n\in\mathbb{N}} \omega_n$ and obtain as above that ω is a Delone set.

To show that for any $n \in \mathbb{N}$ the set ω_{b_n} is an exact $A_{2^{(b_n)}}$-patch representation for ω we consider $g \in \omega$. Then there exists $m \geq b_n$ such that $g \in \omega_m$ and (iii) gives the existence of an $g' \in \omega_{b_n}$ such that $(\omega_m - g) \cap A_{2^{(b_n)}} = (\omega_m - g') \cap A_{2^{(b_n)}}$. Now recall from (i), (ii) and our definition of ω that $\omega \cap A_{2^m} = \omega_m$. As there holds furthermore $2^{(b_n)} \leq 2^m$ and $g, g' \in \omega_m \subseteq A_{2^m}$ we thus obtain from Lemma 6.41 that there holds

$$
(\omega - g) \cap A_{2^{(b_n)}} = (\omega_m - g) \cap A_{2^{(b_n)}} = (\omega_m - g') \cap A_{2^{(b_n)}} = (\omega - g') \cap A_{2^{(b_n)}}.
$$

As $g' \in \omega_{b_n}$ we have thus shown that ω_{b_n} is an exact $A_{2^{(b_n)}}$-patch representation for ω

for all $n \in \mathbb{N}$ and obtain in particular from (ii) that

$$|\operatorname{Pat}_\omega(A_{2^{(b_n)}})| \leq |\omega_{b_n}| = 2^{2^{b_n}}. \tag{6.3}$$

Whenever K is a compact subset of $\mathbb{Q}_2$ we can use that $(b_n)_{n\in\mathbb{N}}$ is strictly increasing to obtain that $\bigcup_{n\in\mathbb{N}} A_{2^{b_n}}$. Thus, in particular there exists $n \in \mathbb{N}$ such that $K \subseteq A_{2^{b_n}}$. Hence, $|\operatorname{Pat}_\omega(K)| \leq |\operatorname{Pat}_\omega(A_{2^{(b_n)}})| \leq 2^{2^{b_n}}$ and we obtain ω to be FLC. Furthermore, (6.3) allows to compute

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\log\left|\operatorname{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta\left(\overline{B}_n(0)\right)} &\leq \liminf_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_{2^{(b_n)}})|}{\theta(A_{2^{(b_n)}})} \\
&\leq \liminf_{n\to\infty} \frac{\log\left(2^{2^{b_n}}\right)}{2^{2^{b_n}}} \\
&= 0.
\end{aligned}$$

Let us now consider $n \in \mathbb{N}$. Then (i) and our definition of ω yield $\omega_{b_n} = \omega \cap A_{2^{b_n}}$. As $b_n \leq 2^{b_n}$ Lemma 6.41 allows to observe $\operatorname{Pat}_{\omega_{b_n}}(A_{b_n}) \subseteq \operatorname{Pat}_\omega(A_{b_n})$ and we obtain from (iv) that

$$2^{2^{b_n}-(b_n+1)} = a_{b_n}/2^{b_n+1} \leq |\operatorname{Pat}_{\omega_{b_n}}(A_{b_n})| \leq |\operatorname{Pat}_\omega(A_{b_n})|.$$

As any strictly increasing sequence in $\mathbb{N}_0$ tends to ∞ we compute

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(\overline{B}_n(0))|}{\theta(\overline{B}_n(0))} &\geq \limsup_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_{b_n})|}{\theta(A_{b_n})} \\
&\geq \limsup_{n\to\infty} \frac{\log\left(2^{2^{b_n}-(b_n+1)}\right)}{2^{b_n}} \\
&= \log(2) \lim_{n\to\infty} \frac{\left(2^{b_n} - (b_n+1)\right)}{2^{b_n}} \\
&= \log(2).
\end{aligned}$$

□

Remark 6.52. *Note that it it follows from Proposition 6.11 that for FLC Delone sets in $\mathbb{Q}_2$ there holds*

$$\mathrm{E}(\pi_\omega) \leq \liminf_{n\to\infty} \frac{\log\left|\operatorname{Pat}_\omega(\overline{B}_n(0))\right|}{\theta(\overline{B}_n(0))}.$$

Thus, also the FLC Delone set ω constructed in the proof of Theorem 6.51 satisfies

$\mathrm{E}(\pi_\omega) < \mathrm{E}_{pc}(\omega)$. *Nevertheless, note that Theorem 6.47 shows in addition that also*

$$\liminf_{n\to\infty} \frac{\log\left|\mathrm{Pat}_\omega(\overline{B}_n(0))\right|}{\theta(\overline{B}_n(0))}$$

can be strictly larger than $\mathrm{E}(\pi_\omega)$ *for FLC Delone sets in* $\mathbb{Q}_2$.

6.4.4 About the finiteness of patch counting entropy

It is well-known that the patch counting entropy along $(\overline{B}_n(0))_{n\in\mathbb{N}}$ of a FLC Delone set in $\mathbb{R}^d$ is always finite [Lag99, Theorem 2.3]. Furthermore, it is known that any model set in $\mathbb{Q}_2$ has finite patch counting entropy (along $(\overline{B}_n(0))_{n\in\mathbb{N}}$) [HR15, Theorem 4.5]. It is thus surprising that this statement does not hold for general FLC Delone sets in $\mathbb{Q}_2$. We will next construct a FLC Delone set in $\mathbb{Q}_2$ with infinite patch counting entropy along $(\overline{B}_n(0))_{n\in\mathbb{N}}$. Again we abbreviate $a_n := 2^{2^n}$ and $A_n := \overline{B}_{2^n}(0)$ for $n \in \mathbb{N}_0$.

Lemma 6.53. *There exists a family* $(\omega_n)_{n\in\mathbb{N}_0}$ *of finite subsets of* $\mathbb{Q}_2$ *such that for all* $n \in \mathbb{N}_0$ *there holds*

(i) $\omega_n = \omega_{n+1} \cap A_{a_n}$.

(ii) ω_n *is contained in* $A_{a_n} \cap \mathbb{Q}$ *and contains exactly one element from each ball of radius* 1 *that is contained in* A_{a_n}.

(iii) ω_n *is an exact* A_n*-patch representation for* ω_m *for any* $m \geq n$.

(iv) $2^{a_n} \leq |\,\mathrm{Pat}_{\omega_{n+1}}(A_{n+1})|$.

Proof. Let $\omega_0 := \{0, 1/2, 1/4, 3/4\}$. As $A_{a_0} = \overline{B}_{2^2}(0)$ we obtain that ω_0 indeed contains exactly one element from each ball of radius 1 contained in A_{a_0}. As ω_0 is contained in $\mathbb{Q}$ it thus satisfies (ii). To proceed inductively assume that ω_n is chosen such that (ii) is satisfied and choose ω_{n+1} according to Lemma 6.45, i.e. such that (i), (ii) and (iv) are fulfilled and such that

(c) ω_n is an exact A_n-patch representation for ω_{n+1}.

Then (iii) can be shown similarly as above by fixing n and performing an induction over $m \geq n$ while using (c) and (i) in the induction step. □

Theorem 6.54. *There exists a FLC Delone set* ω *in* $\mathbb{Q}_2$ *such that the following limit exists and such that*

$$\lim_{n\to\infty} \frac{\log\left|\mathrm{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta\left(\overline{B}_n(0)\right)} = \infty.$$

Proof. Similarly as above we show that the union ω over the family $(\omega_n)_{n\in\mathbb{N}}$ constructed in Lemma 6.53 is a Delone set. An induction as above also allows to deduce from (iii) and Lemma 6.41 that ω is FLC. From (i), (iv) and Lemma 6.41 we obtain as above $2^{a_n} \leq |\operatorname{Pat}_\omega(A_{n+1})|$ and compute

$$\begin{aligned}\liminf_{n\to\infty} \frac{\log|\operatorname{Pat}_\omega(A_n)|}{\theta(A_n)} &\geq \liminf_{n\to\infty} \frac{\log(2^{a_{n-1}})}{2^n}\\ &= \log(2)\lim_{n\to\infty}\frac{a_{n-1}}{2^n} = \infty.\end{aligned}$$

As the metric of $\mathbb{Q}_2$ only takes values in $\{2^n;\, n\in\mathbb{Z}\}$ we thus obtain

$$\liminf_{n\to\infty} \frac{\log\left|\operatorname{Pat}_\omega\left(\overline{B}_n(0)\right)\right|}{\theta(\overline{B}_n(0))} = \infty.$$

□

6.5 About the topological entropy of the full shift

In this subsection we will briefly consider the full shift on an LCA group. We will see that the topological entropy of this action is finite if and only if the group is discrete. It is well-known that it takes the value log(2) in the discrete case. Nevertheless, for the convenience of the reader, we include the short proof of this statement. Note that one does not need the assumption of commutativity for the arguments of this section. Nevertheless, we only present the statements for abelian groups as we want to stick to our additive notion.

Example 6.55. *The topological entropy of the full shift on a discrete LCA group is* $\log(2)$.

Proof. Note first that $\mathbb{B} := \{\epsilon(K,\{0\});\, K\in\mathcal{K}(G)\}$ is a base for $\mathbb{U}_{\mathcal{A}(G)}$. Now consider $K, F\in\mathcal{K}(G)$ and note that $F+K$ is finite. Then $\mathfrak{S} = \{E\subseteq F+K\}$ consists of finite and closed sets and is a maximal $\epsilon(F+K,\{0\})$-separated set. From Lemma 6.5 we obtain that

$$\operatorname{sep}_{\mathcal{A}(G)}(\epsilon(K,\{0\})_{(-F)}) = \operatorname{sep}_{\mathcal{A}(G)}(\epsilon(K+F,\{0\})) = |\mathfrak{S}| = 2^{|F+K|}.$$

As $\lim_{i\in I}|F_i+K|/|F_i| = 1$ for any Van Hove net $(F_i)_{i\in I}$ we obtain

$$\limsup_{i\in I}\frac{\log(\operatorname{sep}_{\mathcal{A}(G)}(\epsilon(K,\{0\})_{(-F_i)}))}{|F_i|} = \log(2)\limsup_{i\in I}\frac{|F_i+K|}{|F_i|} = \log(2).$$

We thus obtain from Theorem 4.33 that

$$\mathrm{E}(\pi) = \sup_{\epsilon\in\mathbb{B}} \limsup_{i\in I} \frac{\log(\mathrm{sep}_{\mathcal{A}(G)}(\epsilon_{(-F_i)}))}{|F_i|} = \log(2).$$

□

It is less well-known how the topological entropy of the full shift on a non-discrete LCA group behaves.

Example 6.56. *The topological entropy of the full shift on a non-discrete LCA group is infinite.*

Remark 6.57. *This in particular implies that the entropy map of the full shift as considered in Section 4.4 is never upper semi-continuous whenever G is a non-discrete LCA group.*

Proof of the claims of Example 6.56. Let ω be a Delone set in G and consider a neighbourhood V of 0 such that ω is V-discrete. Now consider a finite set $E \subseteq V$ such that $0 \notin E$. From the finiteness of E we know that there exists an open neighbourhood U of 0 such that $E \cup \{0\}$ is U-discrete and we assume without lost of generality that $U \subseteq V$. It is straightforward to show that $E + \omega$ and ω are disjoint and that $(E + \omega) \cup \omega$ is U-discrete. For a compact subset $A \subseteq G$ let us now consider

$$\mathfrak{S} := \{F \cup \omega;\ F \subseteq E + (A \cap \omega)\}.$$

To show that $\mathfrak{S}$ is $\epsilon(E,U)_{(-A)}$-separated in $\mathcal{A}(G)$ let us consider $F, F' \subseteq E + (A \cap \omega)$ such that $F \cup \omega$ and $F' \cup \omega$ are $\epsilon(E,U)_{(-A)}$-close. We need to show that $F \cup \omega = F' \cup \omega$ and consider $g \in F \cup \omega$. Let us assume without lost of generality that $g \in F$. Then $g \in F \subseteq E + (A \cap \omega)$ and we can choose $e \in E$ and $a \in A \cap \omega$ such that $g = e + a$. As $\pi^{-a}(F \cup \omega)$ and $\pi^{-a}(F' \cup \omega)$ are $\epsilon(E,U)$-close there holds

$$e = g - a \in ((F \cup \omega) - a) \cap E \subseteq (F' \cup \omega) - a + U.$$

We observe $g \in (F' \cup \omega) + U$ and there is $u \in U$ such that $g - u \in F' \cup \omega \subseteq (E + \omega) \cup \omega$. We thus obtain from $g \in F \cup \omega \subseteq (E + \omega) \cup \omega$ and from $(E + \omega) \cup \omega$ being U-discrete that there holds $g = g - u \in F' \cup \omega$. We have thus shown $F \cup \omega \subseteq F' \cup \omega$ and similarly one shows the other inclusion and obtains $\mathfrak{S}$ to be $\epsilon(E,U)_{(-A)}$-separated. Now recall that $E + (A \cap \omega)$ and ω are disjoint. Thus, different $F \subseteq E + (A \cap \omega)$ yield different $F \cup \omega$. This shows that $\mathrm{sep}_{\mathcal{A}(G)}(\epsilon(E,U)_{-A}) \geq |\mathcal{S}| = 2^{|E|} 2^{|A \cap \omega|}$ for any compact subset $A \subseteq G$.

Considering a Van Hove net $(A_i)_{i\in I}$ in G we know $\lim_{i\in I} \theta(A_i)/\theta(-A_i) = 1$ and obtain from Lemma 5.3 that there exists a constant $c > 0$ only dependent on ω, such that

$$\mathrm{E}(\pi) \geq \liminf_{i\in I} \frac{\log \mathrm{sep}_{\mathcal{A}(G)}(\epsilon(E,U)_{-A_i})}{\theta(A_i)} \geq |E| \log(2) \liminf_{i\in I} \frac{|A_i \cap \omega|}{\theta(A_i)} \geq |E| \log(2) c.$$

Now recall that we assume G to be non-discrete. Thus, there are finite sets $E \subseteq V$ of arbitrary finite cardinality and we obtain $\mathrm{E}(\pi) = \infty$. $\square$

Remark 6.58. *Note that the previous proof can be simplified by considering*

$$\mathfrak{S}' := \{F;\, F \subseteq E + (A \cap \omega)\}$$

instead of $\mathfrak{S}$. *Nevertheless, we chose to consider* $\mathfrak{S}$ *as it consists of Delone sets. The proof thus shows that already the Delone sets in* $\mathcal{A}(G)$ *force the topological entropy to be infinite.*

7 On topological pressure

The concept of topological pressure is the natural generalization of topological entropy and sheds new light on statements like the variational principle as illustrated in [Wal75, Wal82, Oll85]. In this chapter we first show that the Ornstein-Weiss Lemma can be used in order to define the topological pressure for actions of unimodular amenable groups. In Section 7.1 we discuss the generalizations of several approaches and their equivalence to topological pressure. Clearly, one can discuss similarly as in Section 4.3 further non-equivalent approaches but to avoid unnecessary repetition we did not include such a discussion into this thesis. In Section 7.2 we follow the ideas of [Wal75, Wal82] and show with some minor changes on well-known arguments that one can easily generalize some properties of the topological pressure from the context of actions of discrete amenable groups as for example studied in [Oll85]. In Section 7.3 we show that similar results as in Chapter 5 can also be observed for the topological pressure whenever one modifies also the potential. In Section 7.4 we then give a proof for Goodwyn's half of the variational principle for actions of unimodular amenable groups. We also present a proof of the variational principle for actions of σ-compact LCA groups. In order to give these proofs we will use the results of Chapter 5. The Sections 7.5 and 7.6 are used to demonstrate that several well-known ideas about equilibrium states and the converse variational principle as presented in [Wal82] also work in the context of aperiodic order.

7.1 Topological pressure

In this section we define the topological pressure of a function $f \in C(X)$ for actions of a unimodular amenable group G following ideas from [Wal75, STZ80, Wal82, Oll85, Kel98, Buf11]. In particular, we will see that one can use the Ornstein-Weiss Lemma to achieve that the topological pressure can be defined by averaging over a Van Hove net and furthermore that this notion is independent of the choice of a Van Hove net. This independence is well know in the context of actions of countable discrete amenable groups but we have encountered no reference which shows the statement using the Ornstein-Weiss Lemma. Most of the proofs we found in the literature use the variational principle for the topological pressure to achieve this independence but note that there also exists a direct proof in the context that uses tiling methods [Buf11]. In this section consider an action π of a unimodular amenable group G on a compact Hausdorff space X.

7.1.1 Via Bowen entourages

To define the topological pressure consider $f \in C(X)$, which is called a *potential* in this context. For an open cover $\mathcal{U}$ of X we denote

$$\mathrm{P}_f^*(\mathcal{U}) := \log\left(\sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f(x)}\right).$$

For $\eta \in \mathbb{U}_X$ we define the *static topological pressure of f at scale η* as

$$\mathrm{P}_f(\eta) := \inf_{\mathcal{U}} \mathrm{P}_f^*(\mathcal{U}),$$

where the infimum is taken over all finite open covers $\mathcal{U}$ of X, which are at scale η.

Remark 7.1. *There holds $\mathrm{P}_f^*(\mathcal{U}) \leq \mathrm{P}_{f'}^*(\mathcal{U})$ and $\mathrm{P}_f(\eta) \leq \mathrm{P}_{f'}(\eta)$ for potentials $f, f' \in C(X)$ that satisfy $f \leq f'$.*

For $f \in C(X)$, $x \in X$ and a precompact and measurable subset A of G the map $A \ni g \mapsto f(g.x)$ is continuous and we define[1] $(f_A)(x) := \left(\int_A f\right)(x) := \int_A f(g.x)d\theta(g)$, where θ denotes the restricted Haar measure of G to A. We furthermore define $\left(\sum_F f\right)(x) := \sum_{g\in F} f(g.x)$ for finite sets $F \subseteq G$.

Lemma 7.2. *Let $f \in C(X)$ and A be a precompact subset of G. Then $f_A\colon X \to \mathbb{R}$ is continuous and satisfies $\|f_A\|_\infty \leq \theta(A)\|f\|_\infty$. Furthermore, for any G-invariant Borel probability measure there holds $\mu(f_A) = \theta(A)\mu(f)$.*

Proof. We obtain from the uniform continuity of the map $\overline{A} \times X \to \mathbb{R}$ that sends $(g, x) \mapsto f(g.x)$, that for any $\epsilon > 0$ there are $\eta \in \mathbb{U}_{\overline{A}}$ and $\delta \in \mathbb{U}_X$ such that for $(g, h) \in \eta$ and $(x, y) \in \delta$ there holds $|f(g.x) - f(h.y)| \leq \epsilon/(\theta(A) + 1)$. Thus, in particular for $(x, y) \in \delta$ we obtain

$$|f_A(x) - f_A(y)| \leq \int_A |f(g.x) - f(g.y)|d\theta(g) \leq \epsilon.$$

This shows that $f_A \in C(X)$. Furthermore, we compute

$$\begin{aligned}\|f_A\|_\infty &= \sup_{x\in X}\left|\int_A f(g.x)d\theta(g)\right| \leq \sup_{x\in X}\int_A |f(g.x)|\, d\theta(g)\\ &\leq \int_A \|f\|_\infty\, d\theta(g) = \theta(A)\,\|f\|_\infty\,.\end{aligned}$$

[1] See [Rud91, Theorem 3.27] for an approach to this notion via vector-valued integration.

For $\mu \in \mathcal{M}_G(X)$ we observe

$$\mu(f_A) = \int_X \int_A f(g.x) d\theta(g) d\mu(x) = \int_A \int_X f(g.x) d\mu(x) d\theta(g)$$
$$= \int_A \mu(f) d\theta(g) = \theta(A)\mu(f).$$

□

We will now apply the Ornstein-Weiss Lemma to define the topological pressure.

Theorem 7.3. *Let $f \in C(X)$. Then the following limit exists, is finite and does not depend on the choice of a Van Hove net $(A_i)_{i \in I}$. We define the* topological pressure of f at scale η *as*

$$\mathrm{p}_f(\eta|\pi) := \lim_{i \in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{A_i})}{\theta(A_i)}.$$

We define the topological pressure *of a potential $f \in C(X)$ with respect to π as*

$$\mathrm{p}_f(\pi) := \sup_{\eta \in \mathbb{U}_X} \mathrm{p}_f(\eta|\pi).$$

Before we can show this we will need to show the following standard statements. We include a proof for the convenience of the reader.

Lemma 7.4. *Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of X, A be a compact subset of G and $f, f' \in C(X)$ be potentials.*

(i) There holds $\mathrm{P}^*_{f_{Ag}}(\mathcal{U}_g) = \mathrm{P}^*_{f_A}(\mathcal{U})$ for all $g \in G$.

(ii) There holds $\mathrm{P}^*_{f+f'}(\mathcal{U} \vee \mathcal{V}) \le \mathrm{P}^*_f(\mathcal{U}) + \mathrm{P}^*_{f'}(\mathcal{V})$.

(iii) There holds $\mathrm{P}^*_{(f+c)_A}(\mathcal{U}) = \mathrm{P}^*_{f_A}(\mathcal{U}) + c\theta(A)$ for all $c \in \mathbb{R}$.

Proof. In order to show (i) we consider $g \in G$ and compute

$$f_{Ag}(x) = \int_{Ag} f(h.x) d\theta(h) = \int_A f((ag).x) d\theta(a) = f_A(g.x).$$

Thus, there holds

$$\mathrm{P}^*_{f_{Ag}}(\mathcal{U}_g) = \sum_{V \in \mathcal{U}_g} \sup_{x \in V} e^{f_{Ag}(x)} = \sum_{U \in \mathcal{U}} \sup_{x \in (\pi^g)^{-1}(U)} e^{f_A(g.x)}$$
$$= \sum_{U \in \mathcal{U}} \sup_{y \in U} e^{f_A\left(g.(g^{-1}.y)\right)} = \sum_{U \in \mathcal{U}} \sup_{y \in U} e^{f_A(y)} = \mathrm{P}^*_{f_A}(\mathcal{U}).$$

To show (ii) we compute

$$
\begin{aligned}
e^{\mathrm{P}^*_{f+f'}(\mathcal{U}\vee\mathcal{V})} &= \sum_{W\in\mathcal{U}\vee\mathcal{V}} \sup_{x\in W} e^{(f+f')(x)} = \sum_{U\in\mathcal{U},V\in\mathcal{V}:U\cap V\neq\emptyset} \sup_{x\in U\cap V} e^{f(x)+f'(x)} \\
&\leq \sum_{U\in\mathcal{U},V\in\mathcal{V}:U\cap V\neq\emptyset} \sup_{x\in U,y\in V} e^{f(x)+f'(y)} \leq \sum_{U\in\mathcal{U},V\in\mathcal{V}} \sup_{x\in U,y\in V} e^{f(x)+f'(y)} \\
&= \sum_{U\in\mathcal{U},V\in\mathcal{V}} \left(\sup_{x\in U} e^{f(x)}\right)\left(\sup_{y\in V} e^{f'(y)}\right) = \left(\sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f(x)}\right)\left(\sum_{V\in\mathcal{V}} \sup_{y\in V} e^{f'(y)}\right) \\
&= e^{\mathrm{P}^*_f(\mathcal{U})} \cdot e^{\mathrm{P}^*_{f'}(\mathcal{V})} = e^{\mathrm{P}^*_f(\mathcal{U})+\mathrm{P}^*_{f'}(\mathcal{V})}.
\end{aligned}
$$

To see (iii) let $c \in \mathbb{R}$ and note that there holds $(f+c)_A = f_A + c\theta(A)$. Thus,

$$
\begin{aligned}
e^{\mathrm{P}^*_{(f+c)_A}(\mathcal{U})} &= \sum_{U\in\mathcal{U}} \sup_{x\in U} e^{(f+c)_A(x)} = \sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f_A(x)+c\theta(A)} \\
&= \left(\sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f_A(x)}\right) e^{c\theta(A)} = e^{\mathrm{P}^*_{f_A}(\mathcal{U})+c\theta(A)}.
\end{aligned}
$$

□

Proof of Theorem 7.3. We will first show the statement under the additional assumption that the potential f is positive, i.e. $f \geq 0$. Let us consider the map

$$
\mathcal{K}(G) \ni A \mapsto \mathrm{P}_{f_A}(\eta_A) \tag{7.1}
$$

and note that by the Ornstein-Weiss Lemma it suffices to show that this map is monotone, right invariant and subadditive. To show the monotonicity consider compact subsets A and B of G that satisfy $A \subseteq B$. As f is positive we obtain that $f_A \leq f_B$. Furthermore, any open cover at scale η_B is at scale η_A and considering in the following all open cover $\mathcal{U}$ and $\mathcal{V}$ at scale η_A and η_B respectively we compute

$$
\mathrm{P}_{f_A}(\eta_A) = \inf_{\mathcal{U}} P^*_{f_A}(\mathcal{U}) \leq \inf_{\mathcal{V}} P^*_{f_A}(\mathcal{V}) \leq \inf_{\mathcal{V}} P^*_{f_B}(\mathcal{V}) = \mathrm{P}_{f_B}(\eta_B),
$$

which shows the claimed monotonicity. To show the right invariance let $g \in G$. Then an open cover $\mathcal{U}$ is at scale η, if and only if $\mathcal{U}_g$ is at scale η_g and considering the infima over all finite open cover $\mathcal{U}$ and $\mathcal{V}$ at scale η and η_g respectively we obtain from Lemma 7.4 that

$$
\mathrm{P}_{f_{Ag}}(\eta_{Ag}) = \inf_{\mathcal{V}} \mathrm{P}^*_{f_{Ag}}(\mathcal{V}) = \inf_{\mathcal{U}} \mathrm{P}^*_{f_{Ag}}(\mathcal{U}_g) = \inf_{\mathcal{U}} \mathrm{P}^*_{f_A}(\mathcal{U}) = \mathrm{P}_{f_A}(\eta_A).
$$

To show that (7.1) is subadditive let A and B be subsets of G and observe that the positivity of f implies that there holds $f_{A\cup B} \leq f_A + f_B$. Now let $\mathcal{U}$ and $\mathcal{V}$ be finite open covers of X at scale η_A and η_B respectively and note that $\mathcal{U} \vee \mathcal{V}$ is at scale $\eta_{A\cup B}$.

Thus, we consider an infimum over all open cover $\mathcal{W}$ of X at scale $\eta_{A\cup B}$ to obtain from Lemma 7.4 that there holds

$$\mathrm{P}_{f_{A\cup B}}(\eta_{A\cup B}) = \inf_{\mathcal{W}} \mathrm{P}^*_{f_{A\cup B}}(\mathcal{W}) \leq \mathrm{P}^*_{f_{A\cup B}}(\mathcal{U}\vee\mathcal{V}) \leq \mathrm{P}^*_{f_A+f_B}(\mathcal{U}\vee\mathcal{V}) \leq \mathrm{P}^*_{f_A}(\mathcal{U}) + \mathrm{P}^*_{f_B}(\mathcal{V}).$$

Taking the infima over all considered $\mathcal{U}$ and $\mathcal{V}$ respectively, we obtain

$$\mathrm{P}_{f_{A\cup B}}(\eta_{A\cup B}) \leq \mathrm{P}_{f_A}(\eta_A) + \mathrm{P}_{f_B}(\eta_B),$$

i.e. the subadditivity of (7.1). This proofs the claimed statement under the additional assumption that $f \geq 0$.

To obtain the statement for all $f \in C(X)$ note that for general $f \in C(X)$ there exists $c \in \mathbb{R}$ such that $f+c$ is positive. Thus, the already proven statement yields the existence and finiteness of the limit $\lim_{i\in I} \mathrm{P}_{(f+c)_{A_i}}(\eta_{A_i})/\theta(A_i)$ and furthermore its independence from the choice of a Van Hove net. Considering infima over all open cover $\mathcal{U}$ at scale η_{A_i} in the following we obtain from Lemma 7.4 that there holds

$$P_{(f+c)_{A_i}}(\eta_{A_i}) = \inf_{\mathcal{U}} P^*_{(f+c)_{A_i}}(\mathcal{U}) = \inf_{\mathcal{U}} P^*_{f_{A_i}}(\mathcal{U}) + c\theta(A_i) = \mathrm{P}_{f_{A_i}}(\eta_{A_i}) + c\theta(A_i).$$

Thus,

$$\frac{\mathrm{P}_{f_{A_i}}(\eta_{A_i})}{\theta(A_i)} = \frac{\mathrm{P}_{(f+c)_{A_i}}(\eta_{A_i})}{\theta(A_i)} - c \tag{7.2}$$

converges to a finite limit independent of the choice of a Van Hove net. □

Remark 7.5. *(i) Note that* $\mathrm{p}_f(\kappa|\pi) \leq \mathrm{p}_f(\eta|\pi)$ *for* $\kappa, \eta \in \mathbb{U}_X$ *with* $\kappa \supseteq \eta$. *We thus obtain for any base* $\mathbb{B}_X$ *of* $\mathbb{U}_X$ *that* $\mathrm{p}_f(\pi) = \sup_{\eta\in\mathbb{B}_X} \mathrm{p}_f(\eta|\pi)$.

(ii) From (7.2) above one easily obtains that for $f \in C(X)$, $c \in \mathbb{R}$ *and* $\eta \in \mathbb{U}_X$ *there holds* $\mathrm{p}_{f+c}(\eta|\pi) = \mathrm{p}_f(\eta|\pi) + c$. *It follows that* $\mathrm{p}_{f+c}(\pi) = \mathrm{p}_f(\pi) + c$.

7.1.2 Via open covers

In Subsection 4.3.3 we have seen that topological entropy can also be defined using open covers that refine other open covers. As this approach is the natural generalization of the approach of [AKM65] to topological entropy, it is natural to ask, whether the corresponding ideas can also be used to define topological pressure. We will present next that this is the case. Recall that for open covers $\mathcal{U}$ and $\mathcal{V}$ of X and a compact subset $A \subseteq G$ we say that $\mathcal{V}$ A-refines $\mathcal{U}$, whenever $\mathcal{V}$ is finer than $\mathcal{U}_g$ for all $g \in A$, and that for all open covers $\mathcal{U}$ of X and all compact subsets $A \subseteq G$ there exists a finite open cover that A-refines $\mathcal{U}$. Let $f \in C(X)$ be a potential and $\mathcal{U}$ be an open cover of X. We define $\mathrm{P}^*_f(\mathcal{U}, A) := \inf_{\mathcal{V}} \mathrm{P}^*_f(\mathcal{V})$, for compact subsets $A \subseteq G$, where the infimum is taken over all finite open covers $\mathcal{V}$ that cover X and that A-refine $\mathcal{U}$.

Proposition 7.6. *Let $f \in C(X)$, $\mathcal{U}$ an open cover of X and $(A_i)_{i\in I}$ be a Van Hove net in G. Then the following limit exists, is finite and does not depend on the choice of the Van Hove net. We define the* topological pressure of f with respect to $\mathcal{U}$ *as*

$$\mathrm{p}^*_f(\mathcal{U}|\pi) := \lim_{i\in I} \frac{\mathrm{P}^*_{f_{A_i}}(\mathcal{U}, A_i)}{\theta(A_i)}.$$

Proof. Considering the following infima over all open covers $\mathcal{V}$ of X that A_i-refine $\mathcal{U}$, we obtain from Lemma 7.4 that for any $i \in I$ and any $c \in \mathbb{R}$ there holds

$$\mathrm{P}^*_{(f+c)_{A_i}}(\mathcal{U}, A_i) = \inf_{\mathcal{V}} \mathrm{P}^*_{(f+c)_{A_i}}(\mathcal{V}) = \inf_{\mathcal{V}} \mathrm{P}^*_{f_{A_i}}(\mathcal{V}) + c\theta(A_i) = \mathrm{P}^*_{f_{A_i}}(\mathcal{U}, A_i) + c\theta(A_i)$$

and thus as above it suffices to show the statement for positive $f \in C(X)$. Again this will be done by an application of the Ornstein-Weiss Lemma. We consider the map

$$\mathcal{K}(G) \ni A \mapsto \mathrm{P}^*_{f_A}(\mathcal{U}, A) \tag{7.3}$$

and it remains to show that this map is monotone, right invariant and subadditive. For compact subsets A and B of G with $A \subseteq B$ the positivity of f implies $f_A \leq f_B$. Furthermore, a finite open cover $\mathcal{V}$ that B-refines $\mathcal{U}$, also A-refines $\mathcal{U}$. Thus, considering the infima over all finite open cover $\mathcal{V}$ and $\mathcal{W}$ that A-refine and B-refine $\mathcal{U}$ respectively we obtain

$$\mathrm{P}^*_{f_A}(\mathcal{U}, A) = \inf_{\mathcal{V}} \mathrm{P}^*_{f_A}(\mathcal{V}) \leq \inf_{\mathcal{W}} \mathrm{P}^*_{f_A}(\mathcal{W}) \leq \inf_{\mathcal{W}} \mathrm{P}^*_{f_B}(\mathcal{W}) = \mathrm{P}^*_{f_B}(\mathcal{U}, B)$$

and (7.3) is monotone. Furthermore, for $g \in G$ and a compact subset $A \subseteq G$ we obtain that an open cover $\mathcal{V}$ A-refines $\mathcal{U}$ if and only if $\mathcal{V}_g$ Ag-refines $\mathcal{U}$. Considering the infima over all finite open covers $\mathcal{V}$ and $\mathcal{W}$ that A-refine and Ag-refine $\mathcal{U}$ respectively we obtain from Lemma 7.4 that

$$\mathrm{P}^*_{f_{Ag}}(\mathcal{U}, Ag) = \inf_{\mathcal{W}} \mathrm{P}^*_{f_{Ag}}(\mathcal{W}) = \inf_{\mathcal{V}} \mathrm{P}^*_{f_{Ag}}(\mathcal{V}_g) = \inf_{\mathcal{V}} \mathrm{P}^*_{f_A}(\mathcal{V}) = \mathrm{P}^*_{f_A}(\mathcal{U}, A).$$

To show that (7.1) is subadditive let A and B be subsets of G and observe that the positivity of f implies that there holds $f_{A\cup B} \leq f_A + f_B$. Now let $\mathcal{V}$ and $\mathcal{W}$ be finite open covers of X that A-refine and B-refine $\mathcal{U}$ respectively and note that $\mathcal{V} \vee \mathcal{W}$ $(A \cup B)$-refines $\mathcal{U}$. Thus, we consider the infimum over all open cover $\tilde{\mathcal{U}}$ that $(A \cup B)$-refine $\mathcal{U}$ to obtain from Lemma 7.4 that there holds

$$\mathrm{P}^*_{f_{A\cup B}}(\mathcal{U}, A\cup B) = \inf_{\tilde{\mathcal{U}}} \mathrm{P}^*_{f_{A\cup B}}(\tilde{\mathcal{U}}) \leq \mathrm{P}^*_{f_{A\cup B}}(\mathcal{V}\vee\mathcal{W}) \leq \mathrm{P}^*_{f_A+f_B}(\mathcal{V}\vee\mathcal{W}) \leq \mathrm{P}^*_{f_A}(\mathcal{V})+\mathrm{P}^*_{f_B}(\mathcal{W}).$$

Taking the infimum over all considered $\mathcal{V}$ and $\mathcal{W}$ respectively, we obtain

$$\mathrm{P}^*_{f_{A\cup B}}(\mathcal{U}, A\cup B) \leq \mathrm{P}^*_{f_A}(\mathcal{U}, A) + \mathrm{P}^*_{f_B}(\mathcal{U}, B),$$

i.e. the subadditivity of (7.1). □

The following lemma allows to relate the presented notion to topological pressure at a certain scale.

Lemma 7.7. *Let $f \in C(X)$, $\mathcal{U}$ be an open cover of X and $\eta \in \mathbb{U}_X$.*

(i) If $\mathcal{U}$ is at scale η, then there holds $\mathrm{P}_f(\eta_A) \leq \mathrm{P}^*_f(\mathcal{U}, A)$ for any compact subset $A \subseteq G$ and we obtain

$$\mathrm{p}_f(\eta|\pi) \leq \mathrm{p}^*_f(\mathcal{U}|\pi).$$

(ii) If η is a Lebesgue entourage of $\mathcal{U}$, then there holds $\mathrm{P}^*_f(\mathcal{U}, A) \leq \mathrm{P}_f(\eta_A)$ for any compact subset $A \subseteq G$ and thus there holds

$$\mathrm{p}^*_f(\mathcal{U}|\pi) \leq \mathrm{p}_f(\eta|\pi).$$

Proof. Clearly, it suffices to show the static statements about $\mathrm{P}_f(\eta_A)$ and $\mathrm{P}^*_f(\mathcal{U}, A)$ for compact subsets A of G. Whenever $\mathcal{U}$ is at scale η and $\mathcal{V}$ is an open cover of X that A-refines $\mathcal{U}$, then $\mathcal{V}$ is at scale η_A. One thus obtains the first statement. To show the second statement consider an open cover $\mathcal{V}$ at scale η_A. Then $\mathcal{V}$ is at scale $\eta_A \subseteq \eta_g$ for all $g \in A$. Furthermore, η_g is a Lebesgue entourage of $\mathcal{U}_g$ for every $g \in A$ and we obtain that for $V \in \mathcal{V}$ and $x \in V$ there exists $U \in \mathcal{U}_g$ such that $V \subseteq \eta_g[x] \subseteq U$, i.e. that $\mathcal{V}$ is at scale $\mathcal{U}_g$. This shows that $\mathcal{V}$ A-refines $\mathcal{U}$ and we obtain $\mathrm{P}^*_f(\mathcal{U}, A) \leq \mathrm{P}^*_f(\mathcal{V})$. Taking the infimum over all open covers $\mathcal{V}$ at scale η_A we thus obtain the second statement. □

As for every entourage η there exists a finite open cover of X at scale η, and as for every open cover of X there exists a Lebesgue entourage we obtain the following as a direct consequence of Lemma 7.7.

Theorem 7.8. *For every potential $f \in C(X)$ there holds*

$$\mathrm{p}_f(\pi) = \sup_{\mathcal{U}} \mathrm{p}^*_f(\mathcal{U}|\pi),$$

where the supremum is taken over all finite open covers $\mathcal{U}$ of X.

Remark 7.9. *Whenever G is a discrete amenable group and $f \in C(X)$ is a potential, then for any Følner net $(F_i)_{i\in I}$ and any open cover $\mathcal{U}$ of X the following limit exists and satisfies $\mathrm{p}^*_f(\mathcal{U}|\pi) = \lim_{i\in I} \mathrm{P}^*_{f_{F_i}}(\mathcal{U}_{F_i})/|F_i|$. Thus, Theorem 7.8 yields*

$$\mathrm{p}_f(\pi) = \sup_{\mathcal{U}} \lim_{i\in I} \frac{\mathrm{P}^*_{f_{F_i}}(\mathcal{U}_{F_i})}{|F_i|},$$

where we consider again a supremum over all open covers $\mathcal{U}$ of X. To see this observe that, whenever G is discrete amenable, then G is a relatively dense subset of itself. Furthermore, any Følner net $(F_i)_{i\in I}$ is Van Hove and consists of finite sets and we can simplify $\mathrm{P}^*_{f_{A_i}}(\mathcal{U}, F_i) = \mathrm{P}^*_{f_{F_i}}(\mathcal{U}_{F_i})$. We thus obtain the claimed formula from Proposition 7.3.

Note that this in particular shows that our notion of topological pressure is equivalent to the standard notion [Wal82, Oll85, Buf11] in the context of actions of discrete amenable groups.

7.1.3 Via separated sets

In Subsection 4.3.4 we have seen that topological entropy can also be defined via separated subsets of X. We will need this approach to show the variational principle of the topological pressure for actions in the context of aperiodic order. Many of the arguments are analogous to the ones given in the standard literature [Rue73, Wal75, OP82, Wal82, Oll85, Kel98], but as we need some statements hidden in the proofs and for convenience of the reader, full proofs are presented. For a potential $f \in C(X)$ and $\eta \in \mathbb{U}_X$ we define

$$\mathrm{Q}_f(\eta) := \sup_E \log\left(\sum_{x\in E} e^{f(x)}\right),$$

where the supremum is taken over all η-separated finite subsets $E \subseteq X$. We furthermore define

$$\mathrm{q}_f^{\mathcal{A}}(\eta|\pi) := \limsup_{i\in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta_{A_i})}{\theta(A_i)}$$

for a Van Hove net $\mathcal{A} = (A_i)_{i\in I}$.

Remark 7.10. *Here we have to keep track of Van-Hove nets, as there seems to be no possibility to use the Ornstein-Weiss Lemma to show independence.*

Remark 7.11. *For all $f \in C(X)$ and $\eta \in \mathbb{U}_X$ there holds $\mathrm{Q}_f(\eta) \leq \mathrm{P}_f(\eta)$. Thus, for all $f \in C(X)$, all $\eta \in \mathbb{U}_X$ and all Van Hove nets $\mathcal{A}$ in G there holds*

$$\mathrm{q}_f^{\mathcal{A}}(\eta|\pi) \leq \mathrm{p}_f(\eta|\pi).$$

Indeed, whenever E is an η-separated subset of X and $\mathcal{V}$ is an open cover at scale η, then each $V \in \mathcal{V}$ cannot contain more than one element of E. Thus, we obtain $\log\left(\sum_{x\in E} e^{f(x)}\right) \leq \log\left(\sum_{V\in\mathcal{V}} \sup_{x\in V} e^{f(x)}\right) = \mathrm{P}^*_f(\mathcal{V})$ and taking the the infimum over all considered $\mathcal{V}$ and the supremum over all considered E we conclude the first statement. The second statement is a direct consequence.

Theorem 7.12. *Let $\mathcal{A}$ be a Van Hove net in G and $f \in C(X)$. Then there holds*

$$\mathrm{p}_f(\pi) = \sup_{\eta\in\mathbb{U}_X} \mathrm{q}_f^{\mathcal{A}}(\eta|\pi).$$

Before we present a proof we will need to show some technical lemma. The arguments of the following lemma are inspired from [Oll85, Theorem 5.2.2] and seem to go back to [Rue73] and [Wal75].

Lemma 7.13. *Let $\mathcal{A}$ be a Van Hove net in G and $f \in C(X)$. Let $\epsilon > 0$ and $\delta \in \mathbb{U}_X$ and consider a symmetric $\eta \in \mathbb{U}_X$ that satisfies $\eta\eta\eta\eta \subseteq \delta$. If for all $(x, y) \in \delta$ there holds $|f(x) - f(y)| < \epsilon$, then we obtain*

$$\mathrm{p}_f(\delta|\pi) \leq \mathrm{q}_f^{\mathcal{A}}(\eta|\pi) + \epsilon.$$

Proof. Let $A \subseteq G$ be compact and consider a maximal η_A-separated subset $E \subseteq X$. As E is maximal one obtains that E is also η_A-spanning. Fixing an open and symmetric $\kappa \in \mathbb{U}_X$ with $\kappa \subseteq \eta_A$ we obtain that $\mathcal{U} := \{\kappa\eta_A[x]; x \in E\}$ is an open cover of X. Furthermore, $\mathcal{U}$ is at scale $\eta_A\eta_A\eta_A\eta_A \subseteq (\eta\eta\eta\eta)_A \subseteq \delta_A$. Thus, for $x \in E$ and $y \in \kappa\eta_A[x]$ we obtain $(x, y) \in \delta_A$, i.e. $(g.x, g.y) \in \delta$ and in particular $|f(g.x) - f(g.y)| < \epsilon$ for all $g \in A$. We thus compute

$$|f_A(x) - f_A(y)| \leq \int_A |f(g.x) - f(g.y)d\theta(g)| \leq \epsilon\theta(A).$$

Hence,

$$\begin{aligned}
\mathrm{P}_{f_A}(\delta_A) &\leq \log\left(\sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f_A(x)}\right) \\
&\leq \log\left(\sum_{x\in E} e^{f_A(x)+\epsilon\theta(A)}\right) \\
&= \log\left(\sum_{x\in E} e^{f_A(x)}\right) + \epsilon\theta(A) \\
&\leq \mathrm{Q}_{f_A}(\eta_A) + \epsilon\theta(A).
\end{aligned}$$

Dividing by $\theta(A)$ and considering the limit superior along $\mathcal{A}$ yields the statement. □

Proof of Theorem 7.12. By Remark 7.11 it remains to show $\mathrm{p}_f(\pi) \leq \sup_{\eta\in\mathbb{U}_X} \mathrm{q}_f^{\mathcal{A}}(\eta|\pi)$. Let $\epsilon > 0$. As X is compact we obtain that f is uniformly continuous and thus there is $\kappa \in \mathbb{U}_X$ such that $(x, y) \in \kappa$ implies $|f(x) - f(y)| < \epsilon$. Let $\delta \in \mathbb{U}_X$ such that $\delta \subseteq \kappa$. Thus, from Lemma 7.13 we obtain that for any symmetric $\eta \in \mathbb{U}_X$ that satisfies $\eta\eta\eta\eta \subseteq \delta$ there holds

$$\mathrm{p}_f(\delta|\pi) \leq \mathrm{q}_f^{\mathcal{A}}(\eta|\pi) + \epsilon \leq \sup_{\eta\in\mathbb{U}_X} \mathrm{q}_f^{\mathcal{A}}(\eta|\pi) + \epsilon.$$

Thus, considering the base $\mathbb{B}_\kappa = \{\delta \in \mathbb{U}_X;\ \delta \subseteq \kappa\}$ of $\mathbb{U}_X$ we obtain

$$\mathrm{p}_f(\pi) = \sup_{\delta \in \mathbb{B}_\kappa} \mathrm{p}_f(\delta|\pi) \leq \sup_{\eta \in \mathbb{U}_X} \mathrm{q}_f^{\mathcal{A}}(\eta|\pi) + \epsilon.$$

As $\epsilon > 0$ was arbitrary we conclude the claimed inequality. □

7.2 Properties of the topological pressure

In this section we present properties of the *topological pressure map*

$$\mathrm{p}_{(\cdot)}(\pi)\colon C(X) \to \mathbb{R} \cup \{\infty\},$$

which we will use in the following. These properties are shown in [Wal75] in the context of actions of $\mathbb{Z}$ and in this section we follow closely [Wal75, Wal82]. See also [Oll85, Kel98] for the context of actions of $\mathbb{Z}^d$ or countable discrete amenable groups respectively. For the convenience of the reader we present the proofs, which are inspired from [Wal82] and use similar methods. Nevertheless, note that these proofs use a different approach to topological pressure which also allows to also investigate the *topological pressure map at scale* $\eta \in \mathbb{U}_X$

$$\mathrm{p}_{(\cdot)}(\eta|\pi)\colon C(X) \to \mathbb{R}.$$

7.2.1 Basic properties

Proposition 7.14. *Let π be an action of a unimodular amenable group G on a compact Hausdorff space X.*

(i) The topological pressure of the potential constant 0 *is the topological entropy, i.e. there holds* $\mathrm{E}(\pi) = \mathrm{p}_0(\pi)$*. Furthermore, for* $\eta \in \mathbb{U}_X$ *there holds* $\mathrm{E}(\eta|\pi) = \mathrm{p}_0(\eta|\pi)$*.*

(ii) The topological pressure map is monotone, i.e. there holds $\mathrm{p}_f(\pi) \leq \mathrm{p}_{f'}(\pi)$ *for all* $f, f' \in C(X)$ *that satisfy* $f \leq f'$*. For* $\eta \in \mathbb{U}_X$ *there furthermore holds* $\mathrm{p}_f(\eta|\pi) \leq \mathrm{p}_{f'}(\eta|\pi)$*, whenever* $f \leq f'$*.*

(iii) For any $f \in C(X)$ *there holds*

$$\mathrm{E}(\pi) + \inf_{x \in X} f(x) \leq \mathrm{p}_f(\pi) \leq \mathrm{E}(\pi) + \sup_{x \in X} f(x).$$

For $\eta \in \mathbb{U}_X$ *there holds* $\mathrm{E}(\eta|\pi) + \inf_{x \in X} f(x) \leq \mathrm{p}_f(\eta|\pi) \leq \mathrm{E}(\eta|\pi) + \sup_{x \in X} f(x)$*.*

Proof. Note that $\mathrm{P}_0^*(\mathcal{U}) = \log\left(\sum_{U \in \mathcal{U}} \sup_{x \in U} e^0\right) = \log\left(|\mathcal{U}|\right)$ for all finite open cover $\mathcal{U}$ of X. Thus, whenever $\mathcal{U}$ is an open cover of X at scale $\eta \in \mathbb{U}_X$ of minimal cardinality, then there holds $\mathrm{P}_0(\eta) = \log(|\mathcal{U}|) = H(\eta)$ and (i) follows. The statement of (ii) is a consequence of Remark 7.1. From (i), (ii) and Remark 7.5 we obtain

$$\mathrm{p}_f(\pi) \leq \mathrm{p}_{(\sup_{x\in X} f(x))}(\pi) = \mathrm{p}_0(\pi) + \sup_{x\in X} f(x) = \mathrm{E}(\pi) + \sup_{x\in X} f(x)$$

and similarly $\mathrm{E}(\pi) + \inf_{x\in X} f(x) \leq \mathrm{p}_f(\pi)$. A similar argument shows the statement with respect to a certain scale. □

Later we will also need the following statement, which can also be found in [Wal82, Theorem 9.7] in the context of actions of $\mathbb{Z}$. Another proof in the context of actions of discrete abelian groups can be found in [Oll85, Corollary 5.2.6].

Proposition 7.15. *For $f, f' \in C(X)$ and $g \in G$ there holds $\mathrm{p}_f(\pi) = \mathrm{p}_{f+f'\circ\pi^g - f'}(\pi)$.*

Proof. Let $f, f' \in C(X)$ and $g \in G$. Then for compact subsets $A \subseteq G$ there holds

$$(f + f' \circ \pi^g - f')_A = f_A + f'_{(Ag)\Delta A} \leq f_A + \theta((Ag)\Delta A) \left\|f'\right\|_\infty$$

and for any $\eta \in \mathbb{U}_X$ we get from Remark 7.5 that

$$\mathrm{P}_{(f+f'\circ\pi^g - f')_A}(\eta_A) \leq \mathrm{P}_{f_{A_i}+\theta((Ag)\Delta A)\|f'\|_\infty}(\eta_A) = \mathrm{P}_{f_{A_i}}(\eta_A) + \theta((Ag)\Delta A) \left\|f'\right\|_\infty .$$

As any Van Hove net $(A_i)_{i\in I}$ is ergodic we thus compute

$$\begin{aligned}
\mathrm{p}_{f+f'\circ\pi^g - f'}(\eta|\pi) &= \lim_{i\in I} \frac{\mathrm{P}_{(f+f'\circ\pi^g - f')_{A_i}}(\eta_{A_i})}{\theta(A_i)} \\
&\leq \limsup_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{A_i}) + \theta((A_i g)\Delta A_i) \left\|f'\right\|_\infty}{\theta(A_i)} \\
&= \lim_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{A_i})}{\theta(A_i)} + 0 = \mathrm{p}_f(\eta|\pi).
\end{aligned}$$

Similarly one shows $\mathrm{p}_f(\eta|\pi) \leq \mathrm{p}_{f+f'\circ\pi^g - f'}(\eta|\pi)$ and obtains the statement by considering the supremum over all $\eta \in \mathbb{U}_X$. □

7.2.2 Continuity

It is presented in [Wal82, Theorem 9.7] that the topological pressure map of an action of $\mathbb{Z}$ is continuous. This statement seems to go back to [Rue73, Wal75]. We present next that this is also the case for actions of any unimodular amenable group.

Theorem 7.16. *Whenever* $\mathrm{E}(\pi) = \infty$*, then the topological pressure map is constantly* ∞*. Whenever* $\mathrm{E}(\pi) < \infty$*, then the topological pressure map only takes values in* $\mathbb{R}$ *and satisfies*

$$\left|\mathrm{p}_f(\pi) - \mathrm{p}_{f'}(\pi)\right| \leq \|f - f'\|_\infty$$

for any $f, f' \in C(X)$*. Thus, the topological pressure map is continuous for any topology on* $\mathbb{R} \cup \{\infty\}$ *that extends the topology of* $\mathbb{R}$*.*

In order to show this and for later purposes we will need the following lemma.

Lemma 7.17. *For* $f, f' \in C(X)$*, an open cover* $\mathcal{U}$ *of* X *and* $\eta \in \mathbb{U}_X$ *there holds* $\left|\mathrm{P}^*_f(\mathcal{U}) - \mathrm{P}^*_{f'}(\mathcal{U})\right| \leq \|f - f'\|_\infty$ *and* $|\mathrm{P}_f(\eta) - \mathrm{P}_{f'}(\eta)| \leq \|f - f'\|_\infty$*.*

Remark 7.18. *In the following proof we will use the following simple inequalities, which are also used in [Wal82]. For two families* $(a_i)_{i\in I}$ *and* $(b_i)_{i\in I}$ *in* $(0,\infty)$ *there holds* $\sup_{i\in I} a_i / \sup_{j\in I} b_j \leq \sup_{i\in I} a_i/b_i$ *and* $\sum_{i\in I} a_i / \sum_{j\in I} b_j \leq \sup_{i\in I} a_i/b_i$*. Indeed, for* $i \in I$ *there holds* $a_i/b_i \geq a_i/\sup_{j\in I} b_j$ *and we obtain the first statement by considering the supremum over all* $i \in I$*. Furthermore,* $\sum_{i\in I} a_i \leq \sum_{i\in I}\left(b_i \sup_{l\in I}(a_l/b_l)\right) = \left(\sum_{i\in I} b_i\right)\left(\sup_{l\in I}(a_l/b_l)\right)$ *yields the second statement.*

Proof. We compute

$$\begin{aligned}
\mathrm{P}^*_f(\mathcal{U}) - \mathrm{P}^*_{f'}(\mathcal{U}) &= \log\left(\frac{\sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f(x)}}{\sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f'(x)}}\right) \\
&\leq \log\left(\sup_{U\in\mathcal{U}} \frac{\sup_{x\in U} e^{f(x)}}{\sup_{x\in U} e^{f'(x)}}\right) \\
&\leq \log\left(\sup_{U\in\mathcal{U}} \sup_{x\in U} \frac{e^{f(x)}}{e^{f'(x)}}\right) \\
&= \sup_{x\in X}\left(f(x) - f'(x)\right) \\
&\leq \|f - f'\|_\infty .
\end{aligned}$$

Similarly one shows $\mathrm{P}^*_{f'}(\mathcal{U}) - \mathrm{P}^*_f(\mathcal{U}) \leq \|f - f'\|_\infty$ and the first statement follows. To see the second statement consider $\epsilon > 0$ and choose an open cover $\mathcal{U}$ of X such that $\mathrm{P}^*_{f'}(\mathcal{U}) \leq \mathrm{P}_{f'}(\eta) + \epsilon$. One obtains from the first statement that

$$\mathrm{P}_f(\eta) - \mathrm{P}_{f'}(\eta) \leq \mathrm{P}^*_f(\mathcal{U}) - \mathrm{P}^*_{f'}(\mathcal{U}) + \epsilon \leq \|f - f'\|_\infty + \epsilon.$$

As $\epsilon > 0$ was arbitrary we observe $\mathrm{P}_f(\eta) - \mathrm{P}_{f'}(\eta) \leq \|f - f'\|_\infty$ and a similar argument shows $\mathrm{P}_{f'}(\eta) - \mathrm{P}_f(\eta) \leq \|f - f'\|_\infty$. Thus, the second statement follows. □

We can now deduce the following continuity properties of the topological pressure at a certain scale or with respect to an open cover.

Proposition 7.19. *There holds* $\left|\mathrm{p}_f(\eta|\pi) - \mathrm{p}_{f'}(\eta|\pi)\right| \leq \|f - f'\|_\infty$ *for any* $\eta \in \mathbb{U}_X$ *and any* $f, f' \in C(X)$.

Remark 7.20. *Recall from Theorem 7.3 that the map* $C(X) \ni f \mapsto \mathrm{p}_f(\eta|\pi)$ *takes values in* $\mathbb{R}$. *The proposition shows that this map is (Libschitz) continuous for any* $\eta \in \mathbb{U}_X$.

Proof of Proposition 7.19. Considering a Van Hove net $(A_i)_{i\in I}$ Lemma 7.17 yields for any $i \in I$ that

$$\left|\mathrm{P}_{f_{A_i}}(\eta_{A_i}) - \mathrm{P}_{f'_{A_i}}(\eta_{A_i})\right| \leq \left\|f_{A_i} - f'_{A_i}\right\|_\infty \leq \theta(A_i)\,\|f - f'\|_\infty .$$

Recall from Theorem 7.3 that $\mathrm{p}_f(\eta|\pi)$and $\mathrm{p}_{f'}(\eta|\pi)$ are finite, which allows to compute

$$\left|\mathrm{p}_f(\eta|\pi) - \mathrm{p}_{f'}(\eta|\pi)\right| \leq \limsup_{i\in I} \frac{\left|\mathrm{P}_{f_{A_i}}(\eta_{A_i}) - \mathrm{P}_{f'_{A_i}}(\eta_{A_i})\right|}{\theta(A_i)} \leq \|f - f'\|_\infty .$$

□

Proof of Theorem 7.16. From Proposition 7.14 one easily obtains that the topological pressure map is finite valued if and only if the topological entropy is finite. In this case Proposition 7.19 yields that for $f, f' \in C(X)$ there holds

$$\left|\mathrm{p}_f(\pi) - \mathrm{p}_{f'}(\pi)\right| \leq \|f - f'\|_\infty$$

and the statement follows. □

Remark 7.21. *With similar arguments as presented above one can also show that for any open cover* $\mathcal{U}$ *of* X *and* $f, f' \in C(X)$ *there holds* $\left|\mathrm{p}^*_f(\mathcal{U}|\pi) - \mathrm{p}^*_{f'}(\mathcal{U}|\pi)\right| \leq \|f - f'\|_\infty$.

7.2.3 Subadditivity

In [Wal75] it is presented that the topological pressure map is affine for actions of $\mathbb{Z}$. The corresponding proof can be easily adopted in order to obtain the statement for all discrete amenable groups. With some minor changes but following closely ideas of [Wal75, Wal82] we next present that the topological pressure map is subadditive for all unimodular amenable groups.

Theorem 7.22. *For* $f, f' \in C(X)$ *there holds* $\mathrm{p}_{f+f'}(\pi) \leq \mathrm{p}_f(\pi) + \mathrm{p}_{f'}(\pi)$.

Remark 7.23. *Note that by Theorem 7.16 the statement is only interesting in the case where the topological entropy is finite, as otherwise the topological pressure map is constantly* ∞.

Again we will use similar methods as in [Wal82] applied to our approach to obtain the statement. We first show that the subadditivity holds at every scale.

Proposition 7.24. *There holds* $\mathrm{p}_{f+f'}(\eta\cap\eta'|\pi) \leq \mathrm{p}_f(\eta|\pi) + \mathrm{p}_{f'}(\eta'|\pi)$ *for all* $\eta, \eta' \in \mathbb{U}_X$ *and all* $f, f' \in C(X)$.

Remark 7.25. *For* $\eta = \eta'$ *one obtains* $\mathrm{p}_{f+f'}(\eta|\pi) \leq \mathrm{p}_f(\eta|\pi) + \mathrm{p}_{f'}(\eta|\pi)$.

Proof. Consider a Van Hove net $(A_i)_{i\in I}$ and let $i \in I$. Whenever $\mathcal{U}$ and $\mathcal{V}$ are open covers at scale η_{A_i} and η'_{A_i} respectively, then $\mathcal{U} \vee \mathcal{V}$ is at scale $\eta_{A_i} \cap \eta'_{A_i} = (\eta \cap \eta')_{A_i}$ and Lemma 7.4 yields

$$\mathrm{P}_{f+f'}((\eta\cap\eta')_{A_i}) \leq \mathrm{P}^*_{f+f'}(\mathcal{U}\vee\mathcal{V}) \leq \mathrm{P}^*_f(\mathcal{U}) + \mathrm{P}^*_{f'}(\mathcal{V}).$$

Taking the infima over all considered $\mathcal{U}$ and $\mathcal{V}$ respectively we obtain $\mathrm{P}_{f+f'}((\eta\cap\eta')_{A_i}) \leq \mathrm{P}_f(\eta_{A_i}) + \mathrm{P}_{f'}(\eta'_{A_i})$. We thus compute

$$\mathrm{p}_{f+f'}(\eta\cap\eta'|\pi) = \lim_{i\in I} \frac{\mathrm{P}_{f+f'}((\eta\cap\eta')_{A_i})}{\theta(A_i)} \leq \lim_{i\in I} \frac{\mathrm{P}_f(\eta_{A_i})}{\theta(A_i)} + \lim_{i\in I} \frac{\mathrm{P}_{f'}(\eta'_{A_i})}{\theta(A_i)} = \mathrm{p}_f(\eta|\pi) + \mathrm{p}_{f'}(\eta'|\pi)$$

and we obtain the statement of the proposition. □

Proof of Theorem 7.22. For $\eta \in \mathbb{U}_X$ and $f, f' \in C(X)$ we obtain from Proposition 7.24 that there holds

$$\mathrm{p}_{f+f'}(\eta|\pi) \leq \mathrm{p}_f(\eta|\pi) + \mathrm{p}_{f'}(\eta|\pi) \leq \mathrm{p}_{f'}(\pi) + \mathrm{p}_f(\pi).$$

Thus, taking the supremum over all $\eta \in \mathbb{U}_X$ yields the statement. □

Remark 7.26. *With similar arguments as presented in Proposition 7.24 one can also show that for all open covers* $\mathcal{U}$ *and* $\mathcal{U}'$ *of* X *there holds* $\mathrm{p}^*_{f+f'}(\mathcal{U}\vee\mathcal{U}'|\pi) \leq \mathrm{p}^*_f(\mathcal{U}|\pi) + \mathrm{p}^*_{f'}(\mathcal{U}'|\pi)$.

7.2.4 Convexity

Also the convexity of the topological pressure map of an action of $\mathbb{Z}$ is shown in [Wal82] and we present a full proof which uses classical arguments adjusted to our context. Recall from Theorem 7.16 that the topological pressure map is constantly ∞, whenever the topological entropy is not finite. It is thus natural to assume that the topological entropy is finite in the following.

Theorem 7.27. *The topological pressure map is convex whenever* $\mathrm{E}(\pi)$ *is finite, i.e. for* $f, f' \in C(X)$ *and* $\lambda \in [0,1]$ *there holds* $\mathrm{p}_{\lambda f+(1-\lambda)f'}(\pi) \leq \lambda\,\mathrm{p}_f(\pi) + (1-\lambda)\,\mathrm{p}_{f'}(\pi)$.

Again we will first present that the statement holds at all scales. To do this we use the following lemma.

Lemma 7.28. *There holds* $\mathrm{P}_{\lambda f+(1-\lambda)f'}(\eta) \leq \lambda\,\mathrm{P}_f(\eta) + (1-\lambda)\,\mathrm{P}_{f'}(\eta)$ *for* $\eta \in \mathbb{U}_X$*, potentials* $f, f' \in C(X)$ *and* $\lambda \in [0,1]$.

Proof. Let $\epsilon > 0$. Let $\mathcal{U}$ be an open cover at scale η such that $\mathrm{P}^*_f(\mathcal{U}) \leq \mathrm{P}_f(\eta) + \epsilon$ and $\mathrm{P}^*_{f'}(\mathcal{U}) \leq \mathrm{P}_{f'}(\eta) + \epsilon$. Using Hölders inequality we compute

$$\begin{aligned} e^{\mathrm{P}^*_{\lambda f+(1-\lambda)f'}(\eta)} &= \sum_{U\in\mathcal{U}} \sup_{x\in U} e^{\lambda f(x)+(1-\lambda)f'(x)} \\ &\leq \sum_{U\in\mathcal{U}} \left(\sup_{x\in U} e^{\lambda f(x)}\right)\left(\sup_{y\in U} e^{(1-\lambda)f'(y)}\right) \\ &\leq \left(\sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f(x)}\right)^{\lambda}\left(\sum_{U\in\mathcal{U}} \sup_{y\in U} e^{f'(y)}\right)^{(1-\lambda)} \\ &= e^{\lambda\,\mathrm{P}^*_f(\mathcal{U})} e^{(1-\lambda)\,\mathrm{P}^*_{f'}(\mathcal{U})} \\ &= e^{\lambda\,\mathrm{P}^*_f(\mathcal{U})+(1-\lambda)\,\mathrm{P}^*_{f'}(\mathcal{U})}. \end{aligned}$$

Thus, there holds

$$\begin{aligned} \mathrm{P}_{\lambda f+(1-\lambda)f'}(\eta) &\leq \mathrm{P}^*_{\lambda f+(1-\lambda)f'}(\mathcal{U}) \\ &\leq \lambda\,\mathrm{P}^*_f(\mathcal{U}) + (1-\lambda)\,\mathrm{P}^*_{f'}(\mathcal{U}) \\ &\leq \lambda\,\mathrm{P}_f(\eta) + (1-\lambda)\,\mathrm{P}_{f'}(\eta) + \epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary the statement follows. □

Proposition 7.29. *For $\eta \in \mathbb{U}_X$, potentials $f, f' \in C(X)$ and $\lambda \in [0,1]$ there holds $\mathrm{p}_{\lambda f+(1-\lambda)f'}(\eta|\pi) \leq \lambda\,\mathrm{p}_f(\eta|\pi) + (1-\lambda)\,\mathrm{p}_{f'}(\eta|\pi)$.*

Proof. Let $(A_i)_{i\in I}$ be any Van Hove net in G. Then there holds $(\lambda f + (1-\lambda)f')_{A_i} = \lambda f_{A_i} + (1-\lambda)f'_{A_i}$. Thus, Lemma 7.28 allows to compute

$$\begin{aligned} \mathrm{p}_{\lambda f+(1-\lambda)f'}(\eta|\pi) &= \lim_{i\in I} \frac{\mathrm{P}_{(\lambda f+(1-\lambda)f')_{A_i}}(\eta_{A_i})}{\theta(A_i)} \\ &\leq \lim_{i\in I} \frac{\lambda\,\mathrm{P}_{f_{A_i}}(\eta_{A_i})}{\theta(A_i)} + \lim_{i\in I} \frac{(1-\lambda)\,\mathrm{P}_{f'_{A_i}}(\eta_{A_i})}{\theta(A_i)} \\ &= \lambda\,\mathrm{p}_f(\eta|\pi) + (1-\lambda)\,\mathrm{p}_{f'}(\eta|\pi). \end{aligned}$$

□

Proof of Theorem 7.27. For $\eta \in \mathbb{U}_X$ we obtain from Proposition 7.29 that there holds

$$\mathrm{p}_{\lambda f+(1-\lambda)f'}(\eta|\pi) \leq \lambda\,\mathrm{p}_f(\eta|\pi) + (1-\lambda)\,\mathrm{p}_{f'}(\eta|\pi) \leq \lambda\,\mathrm{p}_f(\pi) + (1-\lambda)\,\mathrm{p}_{f'}(\pi)$$

and taking the supremum over all $\eta \in \mathbb{U}_X$ yields the claimed statement. □

Remark 7.30. *A similar argument shows that for an open cover $\mathcal{U}$, potentials $f, f' \in C(X)$ and $\lambda \in [0,1]$ there holds $\mathrm{p}^*_{\lambda f+(1-\lambda)f'}(\mathcal{U}|\pi) \leq \lambda\,\mathrm{p}^*_f(\mathcal{U}|\pi) + (1-\lambda)\,\mathrm{p}^*_{f'}(\mathcal{U}|\pi)$.*

7.3 Topological pressure via discrete restriction

In Chapter 5 we have seen that measure theoretical and topological entropy can be calculated also along the finite intersections of Van Hove sets with Delone sets. In this section we will present similar statements in the context of topological pressure. These results will allow us to obtain further approaches to topological pressure and in particular allow us to show Goodwyn's half of the variational principle for actions of general unimodular amenable groups in Subsection 7.4.1.

7.3.1 Via scaled open covers

Generalizing Theorem 5.5 we next proof the following.

Theorem 7.31. *Let $\omega \subseteq G$ be a relatively dense subset and let $(A_i)_{i\in I}$ be a Van Hove net. Set $F_i := A_i \cap \omega$. Then for all $f \in C(X)$ there holds*

$$\mathrm{p}_f(\pi) = \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{\log \mathrm{P}_{f_{A_i}}(\eta_{F_i})}{\theta(A_i)}$$

and the formula remains valid, whenever the limit superior is replaced by a limit inferior.

For the proof of this statement we will need the following lemma.

Lemma 7.32. *Let $f \in C(X)$ and $\eta \in \mathbb{U}_X$. Then for any Van Hove nets $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$, which satisfy $\lim_{i\in I} \theta(A_i \Delta B_i)/\theta(B_i) = 0$ the topological pressure of f at scale η can be computed as*

$$\mathrm{p}_f(\eta|\pi) = \lim_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{B_i})}{\theta(B_i)}.$$

Proof. For $i \in I$ with $\theta(B_i) > 0$ we get from Lemma 7.17 that

$$\begin{aligned}\left|\frac{\mathrm{P}_{f_{B_i}}(\eta_{B_i})}{\theta(B_i)} - \frac{\mathrm{P}_{f_{A_i}}(\eta_{B_i})}{\theta(B_i)}\right| &\leq \frac{\|f_{B_i} - f_{A_i}\|_\infty}{\theta(B_i)} \\ &\leq \frac{\sup_{x\in X} \int_{A_i\Delta B_i} |f(g.x)| d\theta(g)}{\theta(B_i)} \\ &\leq \frac{\int_{A_i\Delta B_i} \|f\|_\infty \, d\theta(g)}{\theta(B_i)} \\ &= \frac{\theta(A_i\Delta B_i)\,\|f\|_\infty}{\theta(B_i)} \to 0.\end{aligned}$$

As $\mathrm{P}_{f_{B_i}}(\eta_{B_i})/\theta(B_i)$ converges to $\mathrm{p}_f(\eta|\pi)$ as presented in Theorem 7.3 we obtain the statement. □

Proof of Theorem 7.31. Let K be a compact subset of G such that ω is K-dense and such that $e_G \in K$. From Lemma 5.1 we know that $(KF_i)_{i\in I}$ is a Van Hove net that satisfies $\lim_{i\in I} \theta(KF_i)/\theta(A_i) = 1$ and $\lim_{i\in I} \theta(KF_i \Delta A_i)/\theta(A_i) = 0$. We thus obtain from Lemma 7.32 that for $\eta \in \mathbb{U}_X$ there holds

$$\mathrm{p}_f(\eta|\pi) = \lim_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{KF_i})}{\theta(KF_i)} = \lim_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{KF_i})}{\theta(A_i)}.$$

Note that $e_G \in K$ implies $\epsilon_{F_i} \supseteq \epsilon_{KF_i}$ for all $\epsilon \in \mathbb{U}_X$ and all $i \in I$. As there holds furthermore $\eta_K \in \mathbb{U}_X$, we see

$$\mathrm{p}_f(\eta|\pi) \leq \sup_{\epsilon\in\mathbb{U}_X} \limsup_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\epsilon_{F_i})}{\theta(A_i)} \leq \sup_{\epsilon\in\mathbb{U}_X} \lim_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\epsilon_{KF_i})}{\theta(A_i)} = \mathrm{p}_f(\pi).$$

Thus, taking the supremum over all $\eta \in \mathbb{U}_X$ yields the statement. □

7.3.2 Via open covers

Also the following generalization of Theorem 5.8 is valid. We will use this generalization in order to show Goodwyn's half of the variational principle.

Theorem 7.33. *Let ω be a relatively dense and locally finite subset of G and $(A_i)_{i\in I}$ be a Van Hove net in G. Abbreviate $F_i := \omega \cap A_i$. Then for all $f \in C(X)$ there holds*

$$\mathrm{p}_f(\pi) = \sup_{\mathcal{U}} \limsup_{i\in I} \frac{\mathrm{P}^*_{f_{A_i}}(\mathcal{U}_{F_i})}{\theta(A_i)},$$

where the supremum is taken over all open covers $\mathcal{U}$ of X. The formula remains valid, whenever the limit superior is replaced by a limit inferior.

Proof. Note that F_i is finite for all $i \in I$ as we assume ω to be locally finite. Thus, whenever an open cover $\mathcal{U}$ is at scale $\eta \in \mathbb{U}_X$, then $\mathcal{U}_{F_i}$ is at scale η_{F_i} and we obtain $\mathrm{P}_{f_{A_i}}(\eta_{F_i}) \leq \mathrm{P}^*_{f_{A_i}}(\mathcal{U}_{F_i})$. Thus, taking the supremum over all open covers $\mathcal{U}$ we get that for all $\eta \in \mathbb{U}_X$ there holds

$$\limsup_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{F_i})}{\theta(A_i)} \leq \sup_{\mathcal{U}} \limsup_{i\in I} \frac{\mathrm{P}^*_{f_{A_i}}(\mathcal{U}_{F_i})}{\theta(A_i)}.$$

We can now take the supremum over all $\eta \in \mathbb{U}_X$ and apply Theorem 7.31 to see $\mathrm{p}_f(\pi) \leq \sup_{\mathcal{U}} \limsup_{i\in I} \mathrm{P}^*_{f_{A_i}}(\mathcal{U}_{F_i})/\theta(A_i)$.

To show the reverse inequality let $\mathcal{U}$ be an open cover of X and consider a Lebesgue entourage η of $\mathcal{U}$. For such η we know from Lemma 7.7 that there holds $\mathrm{P}^*_{f_{A_i}}(\mathcal{U}_{F_i}) =$

$\mathrm{P}^*_{f_{A_i}}(\mathcal{U}, F_i) \leq \mathrm{P}_{f_{A_i}}(\eta_{F_i})$. We thus obtain from Theorem 7.31 that

$$\limsup_{i\in I} \frac{\mathrm{P}^*_{f_{A_i}}(\mathcal{U}_{F_i})}{\theta(A_i)} \leq \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{F_i})}{\theta(A_i)} = \mathrm{p}_f(\pi)$$

and taking the supremum over all open covers $\mathcal{U}$ yields the statement. □

7.3.3 Via separated sets

Theorem 7.34. *Let ω be a relatively dense subset of G and $(A_i)_{i\in I}$ be a Van Hove net in G. Abbreviate $F_i := \omega \cap A_i$. Then for all $f \in C(X)$ there holds*

$$\mathrm{p}_f(\pi) = \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta_{F_i})}{\theta(A_i)}.$$

The formula remains valid, whenever the limit superior is replaced by a limit inferior.

Proof. From Remark 7.11 and Theorem 7.31 we obtain that for all $\eta \in \mathbb{U}_X$ there holds

$$\limsup_{i\in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta_{F_i})}{\theta(A_i)} \leq \limsup_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{F_i})}{\theta(A_i)} \leq \sup_{\eta'\in\mathbb{U}_X} \limsup_{i\in I} \frac{\mathrm{P}_{f_{A_i}}(\eta'_{F_i})}{\theta(A_i)} = \mathrm{p}_f(\pi)$$

and taking the supremum over all $\eta \in \mathbb{U}_X$ we obtain

$$\sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta_{F_i})}{\theta(A_i)} \leq \mathrm{p}_f(\pi).$$

To show the reverse inequality let $\epsilon > 0$. Let K be a compact subset of G such that ω is K dense and recall from Lemma 5.1 that $(KF_i)_{i\in I}$ is a Van Hove net in G that satisfies $\lim_{i\in I} \theta(KF_i)/\theta(A_i) = 1$ and $\lim_{i\in I} \theta(KF_i \Delta A_i)/\theta(A_i) = 0$. As X is compact we obtain that f is uniformly continuous and thus there is $\delta \in \mathbb{U}_X$ such that $(x, y) \in \delta$ implies $|f(x) - f(y)| \leq \epsilon$. We consider the base $\mathbb{B}_\delta := \{\eta \in \mathbb{U}_X;\ \eta \subseteq \delta\}$ of $\mathbb{U}_X$.

Then for $\eta \in \mathbb{B}_\delta$ there exists $\kappa \in \mathbb{U}_X$ symmetric such that $\kappa\kappa\kappa\kappa \subseteq \eta$. For $i \in I$ we consider an κ_{KF_i}-separated subset E of X of maximal cardinality. Such E are also κ_{KF_i}-spanning. We can thus consider an open and symmetric $\rho \in \mathbb{U}_X$ that satisfies $\rho \subseteq \kappa_{KF_i}$ to obtain an open cover $\mathcal{U} := \{\rho\kappa_{KF_i}[x];\ x \in E\}$ at scale $\rho\kappa_{KF_i}\kappa_{KF_i}\rho \subseteq (\kappa\kappa\kappa\kappa)_{KF_i} \subseteq \eta_{KF_i}$. In particular, for $x \in E$ and $y \in \rho\kappa_{KF_i}[x]$ there holds $(x, y) \in \delta_{KF_i}$ and thus

$(g.x, g.y) \in \delta$ for all $g \in KF_i$. From $A_i \subseteq KF_i \cup (KF_i \Delta A_i)$ we thus obtain

$$\begin{aligned}
|f_{A_i}(x) - f_{A_i}(y)| &\leq \int_{A_i} |f(g.x) - f(g.y)| d\theta(g) \\
&\leq \int_{KF_i} |f(g.x) - f(g.y)| d\theta(g) + \int_{KF_i \Delta A_i} |f(g.x) - f(g.y)| d\theta(g) \\
&\leq \int_{KF_i} \epsilon d\theta(g) + \int_{KF_i \Delta A_i} 2 \|f\|_\infty d\theta(g) \\
&= \epsilon\theta(KF_i) + 2 \|f\|_\infty \theta(KF_i \Delta A_i).
\end{aligned}$$

Since $\mathcal{U}$ is at scale η_{KF_i} we compute

$$\begin{aligned}
\mathrm{P}_{f_{A_i}}(\eta_{KF_i}) &\leq \mathrm{P}_{f_{A_i}}(\mathcal{U}) \\
&= \log\left(\sum_{U \in \mathcal{U}} \sup_{y \in U} e^{f_{A_i}(y)}\right) \\
&\leq \log\left(\sum_{x \in E} e^{f_{A_i}(x) + \epsilon\theta(KF_i) + 2\|f\|_\infty \theta(KF_i \Delta A_i)}\right) \\
&= \log\left(e^{\epsilon\theta(KF_i) + 2\|f\|_\infty \theta(KF_i \Delta A_i)} \sum_{x \in E} e^{f_{A_i}(x)}\right) \\
&= \log\left(\sum_{x \in E} e^{f_{A_i}(x)}\right) + \epsilon\theta(KF_i) + 2\|f\|_\infty \theta(KF_i \Delta A_i) \\
&\leq \mathrm{Q}_{f_{A_i}}(\kappa_{KF_i}) + \epsilon\theta(KF_i) + 2\|f\|_\infty \theta(KF_i \Delta A_i).
\end{aligned}$$

As $\kappa_K \in \mathbb{U}_X$ we thus obtain

$$\begin{aligned}
\limsup_{i \in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{KF_i})}{\theta(A_i)} &\leq \limsup_{i \in I} \frac{\mathrm{Q}_{f_{A_i}}(\kappa_{KF_i})}{\theta(A_i)} + \epsilon \lim_{i \in I} \frac{\theta(KF_i)}{\theta(A_i)} + 2\|f\|_\infty \lim_{i \in I} \frac{\theta(KF_i \Delta A_i)}{\theta(A_i)} \\
&= \limsup_{i \in I} \frac{\mathrm{Q}_{f_{A_i}}(\kappa_{KF_i})}{\theta(A_i)} + \epsilon \cdot 1 + 2\|f\|_\infty \cdot 0 \\
&\leq \sup_{\eta' \in \mathbb{U}_X} \limsup_{i \in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta'_{F_i})}{\theta(A_i)} + \epsilon.
\end{aligned}$$

As $(A_i)_{i \in I}$ and $(KF_i)_{i \in I}$ are Van Hove nets in G that satisfy $\lim_{i \in I} \theta(KF_i)/\theta(A_i) = 1$ and $\lim_{i \in I} \theta(KF_i \Delta A_i)/\theta(A_i) = 0$ we thus obtain from Lemma 7.32 that

$$\mathrm{p}_f(\eta|\pi) = \lim_{i \in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{KF_i})}{\theta(KF_i)} = \lim_{i \in I} \frac{\mathrm{P}_{f_{A_i}}(\eta_{KF_i})}{\theta(A_i)} \leq \sup_{\eta' \in \mathbb{U}_X} \limsup_{i \in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta'_{F_i})}{\theta(A_i)} + \epsilon.$$

As $\eta \in \mathbb{B}_\delta$ was arbitrary we consider the supremum over all η and obtain

$$\mathrm{p}_f(\pi) = \sup_{\eta\in\mathbb{B}_\delta} \mathrm{p}_f(\eta|\pi) \leq \sup_{\eta'\in\mathbb{U}_X} \limsup_{i\in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta'_{F_i})}{\theta(A_i)} + \epsilon.$$

As $\epsilon > 0$ was arbitrary we have shown the remaining inequality

$$\mathrm{p}_f(\pi) \leq \sup_{\eta\in\mathbb{U}_X} \limsup_{i\in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta_{F_i})}{\theta(A_i)}.$$

□

7.3.4 Discrete restriction to uniform lattices

If ω is a uniform lattice the formulas simplify as follows. Note that we have shown in Proposition 2.45 that for any uniform lattice there always exists a regular and precompact fundamental domain.

Theorem 7.35. *Let $f \in C(X)$, Λ be a uniform lattice in G and K be the closure of a regular and precompact fundamental domain of Λ. Then for all $\eta \in \mathbb{U}_X$ there holds*

$$\mathrm{p}_f(\eta|\pi) = \mathrm{dens}(\Lambda)\, \mathrm{p}_{f_K}\left(\eta_K \middle| \pi|_{\Lambda\times X}\right).$$

Furthermore, there holds

$$\mathrm{p}_f(\pi) = \mathrm{dens}(\Lambda)\, \mathrm{p}_{f_K}(\pi|_{\Lambda\times X}).$$

Remark 7.36. *The second formula is shown in [Wal75, Theorem 2.2] in the context of actions of $\mathbb{Z}$.*

Proof. Let $(A_i)_{i\in I}$ be a Van Hove net in G and denote $F_i := A_i \cap \Lambda$. Let C be a regular and precompact fundamental domain such that $\overline{C} = K$. As $\theta(\partial C) = 0$ we compute

$$\begin{aligned}\theta(C)|F_i| = \theta(CF_i) \leq \theta(KF_i) &\leq \sum_{g\in F_i} \theta(Kg) = \theta(K)|F_i| \\ &\leq (\theta(C) + \theta(\partial C))|F_i| = \theta(C)|F_i|\end{aligned}$$

and obtain $\theta(KF_i) = \theta(C)|F_i| = \theta(CF_i)$. As $CF_i \subseteq KF_i$ this implies $\theta(KF_i \setminus CF_i) = 0$. Furthermore, as C is regular we know $\theta(K \setminus C) = 0$ and compute for $x \in X$ that

$$\begin{aligned}f_{KF_i}(x) &= \int_{KF_i} f(g.x)d\theta(g) = \int_{CF_i} f(g.x)d\theta(g) = \sum_{g\in F_i} \int_{Cg} f(h.x)d\theta(h) \\ &= \sum_{g\in F_i} \int_C f((hg).x)d\theta(h) = \sum_{g\in F_i} \int_K f((hg).x)d\theta(h) = \sum_{g\in F_i} f_K(g.x).\end{aligned}$$

This shows $f_{KF_i} = \sum_{F_i} f_K$ for all $i \in I$. From Lemma 3.6 we furthermore obtain that $(F_i)_{i\in I}$ is a Van Hove net in Λ and that $\operatorname{dens}(\Lambda) = \theta(C)^{-1}$. As C is assumed to be precompact we know that K is compact and thus as Λ is K-dense we get from Lemma 5.1 that $(KF_i)_{i\in I}$ is a Van Hove net. As the definition of the topological pressure at scale η does not depend on the choice of a Van Hove net we thus compute

$$\mathrm{p}_f(\eta|\pi) = \lim_{i\in I} \frac{\mathrm{P}_{f_{KF_i}}(\eta_{KF_i})}{\theta(KF_i)} = \frac{1}{\theta(C)} \lim_{i\in I} \frac{\mathrm{P}_{\left(\sum_{F_i} f_K\right)}\left((\eta_K)_{F_i}\right)}{|F_i|} = \operatorname{dens}(\Lambda)\, \mathrm{p}_{f_K}\left(\eta_K \middle| \pi|_{\Lambda\times X}\right),$$

which shows the first formula. Now recall from Proposition 4.14 that $\mathbb{B} := \{\eta_K;\ \eta \in \mathbb{U}_X\}$ is a base of $\mathbb{U}_X$. Considering Remark 7.5 we thus compute

$$\begin{aligned}
\mathrm{p}_f(\pi) &= \sup_{\eta\in\mathbb{U}_X} \mathrm{p}_f(\eta|\pi) \\
&= \operatorname{dens}(\omega) \sup_{\eta\in\mathbb{U}_X} \mathrm{p}_{f_K}\left(\eta_K \middle| \pi|_{\Lambda\times X}\right) \\
&= \operatorname{dens}(\omega) \sup_{\epsilon\in\mathbb{B}} \mathrm{p}_{f_K}\left(\epsilon \middle| \pi|_{\Lambda\times X}\right) \\
&= \operatorname{dens}(\omega)\, \mathrm{p}_{f_K}\left(\pi|_{\Lambda\times X}\right).
\end{aligned}$$

□

7.4 The variational principle

A first proof of the variational principle of the topological pressure goes back to the pioneering works [Rue73, Wal75]. [Mis76] gives a short and elegant proof of the variational principle for actions of $\mathbb{Z}^d$ which has influenced heavily other works, for example [DGS76, LW77, OP82]. The variational principle of the topological pressure of actions of countable amenable groups can be found in [STZ80, OP82, Tem84, Oll85, Kel98] and finer statements related to this variational principle are discussed in [Buf11, Zha18, HLZ19]. In [Chu13] the topic is considered for actions of countable discrete sofic groups.

The variational principle is one of the cornerstones of the thermodynamic formalism and well-known for actions of discrete amenable groups. For reference see [Oll85, Theorem 5.2.7]. In this setting it states as follows.

Theorem 7.37 (Variational principle for pressure - discrete version)**.** *Whenever G is a discrete amenable group and $f \in C(X)$ is a potential there holds*

$$\mathrm{P}_f(\pi) = \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_\mu(\pi) + \mu(f)\right).$$

Note that applying Proposition 7.14 with $f = 0$ this formula reduces to the variational principle of (non-relative) topological entropy. Similarly as in the context of topological entropy one can now use the structure of a uniform lattice in a unimodular amenable group via Theorem 7.35 in order to obtain the following.

Theorem 7.38. *(Goodwyn's theorem - extrapolated version) Whenever G is a unimodular amenable group and Λ is a uniform lattice in G, then there holds*

$$\mathrm{P}_f(\pi) \geq \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_\mu(\pi) + \mu(f)\right).$$

Proof. Let K be the closure of a precompact and regular fundamental domain of Λ. As K is the closure of a regular fundamental domain Lemma 3.6 implies that there holds $\mathrm{dens}(\Lambda) = \theta(K)^{-1}$. Thus, Lemma 7.2 yields $\mu(f_K) = \theta(K)\mu(f)$. As every G-invariant measure is Λ-invariant, Theorem 7.35 and Theorem 7.37 allow to conclude

$$\begin{aligned}
\mathrm{P}_f(\pi) &= \mathrm{dens}(\Lambda)\, \mathrm{P}_{f_K}\left(\pi|_{\Lambda\times X}\right)\\
&= \mathrm{dens}(\Lambda) \sup_{\mu\in\mathcal{M}_\Lambda(X)} \left(\mathrm{E}_\mu(\pi|_{\Lambda\times X}) + \mu(f_K)\right)\\
&\geq \mathrm{dens}(\Lambda) \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_\mu(\pi|_{\Lambda\times X}) + \theta(K)\mu(f)\right)\\
&= \sup_{\mu\in\mathcal{M}_G(X)} \left(\mathrm{E}_\mu(\pi) + \mu(f)\right).
\end{aligned}$$

□

Recall that the entropy map is not necessarily upper semi-continuous. We thus cannot simply use integration techniques in order to construct G-invariant measures from Λ-invariant measures in order to obtain G-invariant measures of sufficiently large entropy. It is thus not clear how to use the extrapolation technique in order to show the variational principle.

Another problem occurs as there are groups, like $\mathbb{Q}_p$, which are important in the study of aperiodic order, but which contain no uniform lattice. Clearly, the extrapolation technique cannot be used in order to obtain even Goodwyn's half of the variational principle for such groups. We thus present next that Goodwyn's theorem holds without the assumption of the existence of a uniform lattice. We will furthermore present that the variational principle holds for actions of σ-compact LCA groups.

To prove Goodwyn's theorem for general actions of unimodular amenable groups we will follow techniques from [Oll85]. These techniques depend heavily on the finiteness of Følner nets and in order to recycle them we will replace the uniform lattice with a Delone set, which exists in all locally compact groups. We will then carefully adjust the methods to the new setting. Care has to be taken as the lack of the group structure of the discrete subset causes difficulties. This investigation will be done in Subsection 7.4.1 below.

The proof of the variational principle for actions of σ-compact LCA groups does also use this discretization technique. In addition to the techniques to overcome the lack of a group structure on the Delone set, it also requires a careful analysis of the interplay of static entropy and integration of measure-valued functions. The arguments are carried out in Subsection 7.4.2 below.

7.4.1 Goodwyn's theorem for actions of unimodular amenable groups

We will need the following notions for the proof of Godwyn's theorem. Note that these notions are studied in more detail in [Oll85]. A finite partition α of X is said to be *adapted* to an open cover $\mathcal{U}$ of X, whenever there exists an injective mapping $\mathfrak{U}\colon \alpha \to \mathcal{U}$ such that $A \subseteq \mathfrak{U}(A)$ for all $A \in \alpha$. For a Borel probability measure μ on X and an open cover $\mathcal{U}$ on X we define the *overlap ratio* as

$$\mathrm{R}_\mu^*(\mathcal{U}) := \sup_{\alpha,\beta} H_\mu^*(\alpha|\beta),$$

where the supremum is taken over all finite open covers α and β of X, which are adapted to $\mathcal{U}$. The overlap ratio satisfies the following simple invariance property.

Lemma 7.39. *For an open cover $\mathcal{U}$ of X and a Borel probability measure μ on X there holds $\mathrm{R}_\mu^*(\mathcal{U}) = \mathrm{R}_\mu^*(\mathcal{U}_g)$ for any $g \in G$.*

Proof. If α and β are finite partitions of X adapted to $\mathcal{U}$ and $g \in G$, then α_g and β_g are adapted to $\mathcal{U}_g$ and we obtain $H_\mu^*(\alpha|\beta) = H_\mu^*(\alpha_g|\beta_g) \leq \mathrm{R}_\mu^*(\mathcal{U}_g)$. Taking the supremum over the considered α and β we see $\mathrm{R}_\mu^*(\mathcal{U}) \leq \mathrm{R}_\mu^*(\mathcal{U}_g)$. Similarly one shows the reverse inequality and obtains the statement. □

Slightly modifying arguments presented in [Oll85, 5.2.12] we obtain the following.

Lemma 7.40. *Let μ be a Borel probability measure on X. For any finite open cover α of X and any $\epsilon > 0$ there exists an open cover $\mathcal{U}$ such that α is adapted to $\mathcal{U}$ and such that $\mathrm{R}_\mu^*(\mathcal{U}) \leq \epsilon$.*

Proof. We assume without lost of generality that $\emptyset \notin \alpha$ and denote $r := |\alpha|$. From Lemma 4.7 we obtain the existence of $\delta > 0$ such that for any two finite partitions $\beta^{(1)} = \{B_1^{(1)}, \cdots, B_r^{(1)}\}$ and $\beta^{(2)} = \{B_1^{(2)}, \cdots, B_r^{(2)}\}$ that satisfy

$$\sum_{j=1}^{r} \mu\left(B_j^{(1)} \Delta B_j^{(2)}\right) < \delta$$

there holds $H_\mu^*(\beta^{(1)}|\beta^{(2)}) + H_\mu^*(\beta^{(2)}|\beta^{(1)}) < \epsilon$. As μ is regular we can choose for all $A \in \alpha$ an open neighbourhood O_A of A and a non-empty and compact subset $K_A \subseteq A$ such that

$$\mu(O_A \setminus K_A) = \mu(O_A \setminus A) + \mu(A \setminus K_A) < \frac{\delta}{3r}.$$

We define $U_A := O_A \setminus \left(\bigcup_{A'\in\alpha; A'\neq A} K_A\right)$ for $A \in \alpha$ and set $\mathcal{U} := \{U_A;\ A \in \alpha\}$. Now note that for each $A \in \alpha$ the set U_A is an open neighbourhood of A and furthermore that K_A is disjoint from $U_{A'}$ for all $A' \in \alpha$ with $A' \neq A$. In particular, we observe that $A \mapsto U_A$ is injective and obtain that α is adapted to $\mathcal{U}$. Our observation furthermore allows to see that $\mathcal{U}$ is an open cover without proper subcovers. Thus, for any finite partition β that is adapted to $\mathcal{U}$ satisfies $|\beta| = |\mathcal{U}| = |\alpha|$.

Let us now consider two finite partitions β and β' that are adapted to $\mathcal{U}$ and write $\beta = \{B_A;\ A \in \alpha\}$ and $\beta' = \{B'_A;\ A \in \alpha\}$ such that $B_A \subseteq U_A$ and such that $B'_A \subseteq U_A$. As K_A is disjoint from $U_{A'}$ for all $A' \in \alpha$ with $A' \neq A$ we obtain $K_A \subseteq B_A$ and $K_A \subseteq B'_A$ for all $A \in \alpha$. We compute

$$\sum_{A\in\alpha} \mu\left(B_A \Delta B'_A\right) \leq \sum_{A\in\alpha} \left(\mu\left(U_A \setminus K_A\right) + \mu\left(U_A \setminus K_A\right)\right) \leq 2r\frac{\delta}{3r} < \delta$$

From the choice of δ we thus obtain $H^*_\mu(\beta|\beta') + H^*_\mu(\beta'|\beta) < \epsilon$. Taking the supremum over all considered β and β' we conclude $\mathrm{R}^*_\mu(\mathcal{U}) \leq \epsilon$. □

The following combinatorial lemma and its proof can be found in [Wal82]. We include the short proof for the convenience of the reader.

Lemma 7.41. *Let $k \in \mathbb{N}$ and $a_1, \cdots, a_k, p_1, \cdots, p_k$ be given real numbers such that $p_i \geq 0$ and $\sum_{i=1}^k p_i = 1$. Then there holds*

$$\sum_{i=1}^k p_i(a_i - \log(p_i)) \leq \log\left(\sum_{i=1}^k e^{a_i}\right)$$

and equality holds, if and only if $p_i = e^{a_i}/(\sum_{i=1}^k e^{a_i})$.

Proof. It is straightforward to show with elementary methods of calculus that

$$\Phi(x) := \begin{cases} x\log(x) & , \ x > 0 \\ 0 & , \ x = 0 \end{cases}$$

is strictly convex, i.e. that $\Phi(\sum_{i=1}^k \lambda_i x_i) \leq \sum_{i=1}^k \lambda_i \Phi(x_i)$ holds for $k \in \mathbb{N}$, $x_i \in [0,\infty)$ and $\lambda_i \in [0,1]$ with $\sum_{i=1}^k \lambda_i = 1$. With the same methods one shows that equality holds, if and only if all the x_i corresponding to non-zero λ_i are equal. For details see [Wal82, Theorem 4.2]. Letting $M := \sum_{i=1}^k e^{a_i}$ and setting $\lambda_i := e^{a_i}/M$ and $x_i := p_i M/e^{a_i}$ we thus obtain

$$0 = \Phi(1) \leq \sum_{i=1}^k \frac{e^{a_i}}{M}\frac{p_i M}{e^{a_i}} \log\left(\frac{p_i M}{e^{a_i}}\right) = \sum_{i=1}^k p_i(\log(p_i) + \log M - a_i),$$

i.e. $\sum_{i=1}^k p_i(a_i - \log(p_i)) \leq \log M$. We furthermore obtain that equality holds, if and only if $(p_i M/e^{a_i})$ is independent of i, i.e. $p_i = e^{a_i}/M$. □

Remark 7.42. *Let $F \subseteq G$ be a finite set, β be a finite partition and $\mathcal{U}$ be a finite open cover such that β is adapted to $\mathcal{U}_F$. Then for any $g \in F$ there exists a finite partition γ that is adapted to $\mathcal{U}_g$ and satisfies $\gamma \preceq \beta$. Indeed, from our assumptions we obtain $\mathcal{U}_g \preceq \mathcal{U}_F \preceq \beta$. Thus, for each $B \in \beta$ there exists $U_B \in \mathcal{U}_g$ such that $B \subseteq U_B$. Considering $\gamma := \left\{\bigcup_{B\in\beta; U_B=U} B;\ U \in \mathcal{U}_g\right\}\setminus\{\emptyset\}$ we obtain a finite partition that is clearly adapted to $\mathcal{U}_g$ and such that β is finer than γ.*

The proof of Goodwyn's theorem presented in [Oll85] uses the approach to measure theoretical entropy via the (finite) refinement of finite partitions and thus depends on the finiteness of Van Hove nets. In Section 5.1.2 we have seen how this approach can be recycled also for general unimodular amenable groups by considering the intersections of Van Hove sets with Delone sets. We are now ready to prove Goodwyn's theorem for actions of general unimodular amenable groups. For this we use the Theorems 5.16 and 7.33 in combination with a modification[2] of the arguments from [Oll85, Section 5.2].

Theorem 7.43 (Goodwyn's theorem - general version)**.** *Let G be a unimodular amenable group and let π be an action of G on a compact Hausdorff space X. Then for all $f \in C(X)$ and all $\mu \in \mathcal{M}_G(X)$ there holds*

$$\mathrm{E}_\mu(\pi) + \mu(f) \leq \mathrm{p}_f(\pi).$$

Proof. From Remark 2.37 we get the existence of a Delone set ω in G and we choose a compact neighbourhood V of e_G such that ω is V-discrete. Let furthermore $(A_i)_{i\in I}$ be a Van Hove net in G and denote $F_i := \omega \cap A_i$.

Let $\epsilon > 0$ and consider a finite partition α of X. From Lemma 7.40 we obtain the existence of an open cover $\mathcal{U}$ of X such that α is adapted to $\mathcal{U}$ and such that $\mathrm{R}^*_\mu(\mathcal{U}) \leq \theta(V)\epsilon$. Let $i \in I$ and consider a finite partition β that is adapted to $\mathcal{U}_{F_i}$. Using Lemma 7.41 we compute

$$\begin{aligned} H^*_\mu(\beta) + \mu(f_{A_i}) &\leq \sum_{B\in\beta} \mu(B)\left(\sup_{x\in B} f_{A_i}(x) - \log(\mu(B))\right) \\ &\leq \log\left(\sum_{B\in\beta} e^{\sup_{x\in B} f_{A_i}(x)}\right) \\ &= \log \sum_{B\in\beta} \sup_{x\in B} e^{f_{A_i}(x)} \\ &\leq \log \sum_{U\in\mathcal{U}_{F_i}} \sup_{x\in U} e^{f_{A_i}(x)} \\ &= \mathrm{P}^*_{f_{A_i}}(\mathcal{U}_{F_i}). \end{aligned}$$

From Remark 7.42 we obtain that for any $g \in F_i$ there exists a finite partition $\gamma^{(g)}$ that is adapted to $\mathcal{U}_g$ and such that $\gamma^{(g)} \preceq \beta$. Clearly, also α_g is adapted to $\mathcal{U}_g$. Thus, the

[2] See Remark 7.44 below.

basic properties of the static entropy summarized in Proposition 4.5 and the invariance of the overlap ratio allow to estimate

$$H_\mu^*(\alpha_{F_i}|\beta) \leq \sum_{g\in F_i} H_\mu^*(\alpha_g|\beta) \leq \sum_{g\in F_i} H_\mu^*(\alpha_g|\gamma^{(g)}) \leq \sum_{g\in F_i} \mathrm{R}_\mu^*(\mathcal{U}_g) \leq |F_i|\,\mathrm{R}_\mu^*(\mathcal{U}) \leq |F_i|\theta(V)\epsilon.$$

Combining our observations we thus obtain

$$H_\mu^*(\alpha_{F_i}) + \mu(f_{A_i}) \leq H_\mu^*(\beta) + \mu(f_{A_i}) + H_\mu^*(\alpha_{F_i}|\beta) \leq \mathrm{P}_{f_{A_i}}^*(\mathcal{U}_{F_i}) + |F_i|\theta(V)\epsilon.$$

Recall from Lemma 7.2 that there holds $\mu(f_{A_i}) = \theta(A_i)\mu(f)$. Using Lemma 5.3 and Theorem 7.33 we thus conclude

$$\begin{aligned}\limsup_{i\in I} \frac{H_\mu^*(\alpha_{F_i})}{\theta(A_i)} + \mu(f) &\leq \limsup_{i\in I} \frac{\mathrm{P}_{f_{A_i}}^*(\mathcal{U}_{F_i})}{\theta(A_i)} + \limsup_{i\in I} \frac{|F_i|}{\theta(A_i)}\theta(V)\epsilon \\ &\leq \mathrm{p}_f(\pi) + \epsilon.\end{aligned}$$

Taking the supremum over all finite partitions α of X we obtain from Theorem 5.16 that

$$\mathrm{E}_\mu(\pi) + \mu(f) = \sup_\alpha \limsup_{i\in I} \frac{H_\mu^*(\alpha_{F_i})}{\theta(A_i)} + \mu(f) \leq \mathrm{p}_f(\pi) + \epsilon.$$

The statement follows as $\epsilon > 0$ was arbitrary. □

Remark 7.44. *In [OP82] and in [Oll85, 5.2.12] the proof of Goodwyn's half of the variational principle makes heavy use of subadditivity properties of the overlap ratio [Oll85, Proposition 5.2.11] and the following claim. For $\mu \in \mathcal{M}_G(X)$ and all finite open covers $\mathcal{U}$ it is claimed that whenever α is a finite partition that is adapted to $\mathcal{U}$ via an injective map $\mathfrak{U}\colon \alpha \to \mathcal{U}$ such that $\mu(\mathfrak{U}(A)\setminus A)$ is small, then α_F is adapted to $\mathcal{U}_F$ for any finite set $F \subseteq G$. Considering $X = \{1,2,3\}$ and the action of $\mathbb{Z}$ on X introduced by $\pi^1(1) = 3$, $\pi^1(2) = 2$ and $\pi^1(3) = 1$ we can consider the partition $\alpha = \{\{1,2\},\{3\}\}$ that is adapted to $\mathcal{U} = \{X, \{1,3\}\}$ via $\mathfrak{U}\colon \alpha \to \mathcal{U}$ that sends $\{1,2\} \mapsto X$ and $\{3\} \mapsto \{1,3\}$. Considering the Dirac measure δ_2 we obtain furthermore an invariant Borel probability measure that satisfies $\delta_2(\mathfrak{U}(A)\setminus A) = 0$ for all $A \in \alpha$. Nevertheless, there holds $\alpha_{\{0,1\}} = \{\{1\},\{2\},\{3\},\emptyset\}$ and $\mathcal{U}_{\{0,1\}} = \mathcal{U}$ and thus clearly $\alpha_{\{0,1\}}$ is not adapted to $\mathcal{U}_{\{0,1\}}$.*

In a correspondence with J. M. Ollagnier it was discussed how to repair the proof for the statement in the context of actions of discrete amenable groups. The new ideas in combination with the techniques from [OP82, Oll85] are worked out in details above and presented in combination with the Theorems 5.16 and 7.33 in order to give the statement also for all unimodular amenable groups.

7.4.2 The variational principle for actions of σ-compact LCA groups

In this subsection let G be a σ-compact LCA group and let π be an action of G on a compact Hausdorff space X. We will now present a full proof of the variational principle for π. Note that [Oll85] contains a proof of the variational principle which works also for uncountable discrete groups. Unfortunately we are not aware how to use the corresponding ideas in order to give a proof in the non-discrete setting. Instead we will follow the ideas of [Mis76] and use that all σ-compact LCA groups contain Van Hove sequences which consist of tiling sets. In order to construct Borel probability measures we will need to take averages. In our setting we need to define them by integration. Unlike in the discrete context, where one can use sums instead, care has to be taken as the map $\mathcal{M}(X) \ni \mu \mapsto H^*_\mu(\alpha)$ is not necessarily upper semi-continuous. We thus begin our investigations with some results on the interplay of integration theory and static entropy, which are trivial in the discrete context.

Considering a compact subset $A \subseteq G$ and a Borel probability measure σ on X we denote $\int_A \pi^g_* \sigma d\theta(g)$ for the mapping

$$C(X) \ni f \mapsto \int_A \pi^g_* \sigma(f) d\theta(g).$$

Lemma 7.45. *For $A \subseteq G$ compact with $\theta(A) > 0$ and $\sigma \in \mathcal{M}(X)$ we have*

$$\frac{1}{\theta(A)} \int_A \pi^g_* \sigma d\theta(g) \in \mathcal{M}(X).$$

Proof. Clearly $\mu := 1/\theta(A) \int_A \pi^g_* \sigma d\theta(g)$ is linear, positive and satisfies $\mu(x \mapsto 1) = 1$. For any $f \in C(X)$ we furthermore have $|\pi^g_* \sigma(f)| = |\sigma(f \circ \pi^g)| \leq \|f\|_\infty$. We thus obtain the continuity of μ from $|\mu(f)| \leq 1/\theta(A) \int_A |\pi^g_* \sigma(f)| d\theta(g) \leq \|f\|_\infty$. □

Remark 7.46. *Note that $\int_A \pi^g_* \sigma d\theta(g)$ can be interpreted via the theory of vector-valued integration. For reference see [Rud91, Chapter 3] and [EW11, Appendix B]. We chose to avoid the application of this abstract machinery in order to simplify the presentation.*

A Borel probability measure σ is called *finitely supported*, whenever there exists a finite set E and $p_x \in [0,1]$ with $x \in E$ such that $\sum_{x \in E} p_x = 1$ and $\sigma = \sum_{x \in E} p_x \delta_x$.

Lemma 7.47. *Let $A \subseteq G$ be a compact subset with $\theta(A) > 0$. Let $\sigma \in \mathcal{M}(X)$ be finitely supported and denote $\mu := 1/\theta(A) \int_A \pi^g_* \sigma d\theta(g)$.*

(i) For any $f \in C(X)$ there holds $\mu(f) = 1/\theta(A)\sigma(f_A)$.

(ii) Let α be a finite partition of X that has almost no boundary with respect to μ. The map $A \ni g \mapsto H^*_{\pi^g_* \sigma}(\alpha)$ is Lebesgue integrable and satisfies

$$\frac{1}{\theta(A)} \int_A H^*_{\pi^g_* \sigma}(\alpha) d\theta(g) \leq H^*_\mu(\alpha).$$

Proof. There is a finite set $E \subseteq X$ and $p_x \in [0,1]$ with $\sum_{x\in E} p_x = 1$ such that $\sigma = \sum_{x\in E} p_x \delta_x$. For $f \in C(X)$ we observe $\pi_*^g\sigma(f) = \sum_{x\in E} p_x f(g.x)$ for any $g \in A$. We obtain (i) from

$$\begin{aligned}\mu(f) &= \frac{1}{\theta(A)} \int_A \sum_{x\in E} p_x f(g.x) d\theta(g) = \frac{1}{\theta(A)} \sum_{x\in E} p_x \left(\int_A f(g.x) d\theta(g) \right) \\ &= \frac{1}{\theta(A)} \sum_{x\in E} p_x f_A = \frac{1}{\theta(A)} \sigma(f_A).\end{aligned}$$

To show (ii) note first that the map $G \ni g \mapsto H^*_{\pi_*^g \sigma_i}(\alpha_{F_j})$ is constant on the elements of the finite and measurable partition $\bigvee_{x\in E}\{\pi(\cdot,x)^{-1}(A);\, A \in \alpha\}$ of G. Restricted to the compact subset A this map is thus Lebesgue integrable.

In order to show the claimed inequality and rescaling the Haar measure if necessary, we assume without lost of generality that there holds $\theta(A) = 1$. Let $\mathfrak{R}$ be the set of all pairs $(\beta, \mathfrak{g})$ such that β is a finite (and measurable) partition of A and such that $\mathfrak{g} \colon \beta \to A$ satisfies $\mathfrak{g}(B) \in B$ for any $B \in \beta$. We order $\mathfrak{R}$ by setting $(\beta, \mathfrak{g}) \leq (\beta', \mathfrak{g}')$, whenever β' is finer than β and obtain $\mathfrak{R}$ to be a directed partially ordered set. We will now show that the net

$$\left(\sum_{B\in\beta} \theta(B) \pi_*^{\mathfrak{g}(B)} \sigma \right)_{(\beta,\mathfrak{g})\in\mathfrak{R}}$$

converges to μ with respect to the weak*-topology. We thus consider $f \in C(X)$ and let $\epsilon > 0$. Note that $A \ni g \mapsto \pi_*^g\sigma(f) = \sum_{x\in E} f(g.x) \in \mathbb{R}$ is continuous. Thus there exists a finite partition β_1 of A such that for any $B \in \beta_1$ and $g, g' \in B$ there holds

$$\left| \pi_*^g \sigma(f) - \pi_*^{g'} \sigma(f) \right| \leq \epsilon.$$

By the definition of the Lebesgue integral there is furthermore a finite partition β_2 of A that is finer than β_1 and satisfies

$$\left| \int_A \pi_*^g \sigma(f) d\theta(g) - \sum_{B\in\beta_2} \theta(B) \iota_B \right| \leq \epsilon,$$

where we abbreviate $\iota_B := \inf_{g\in B} \pi_*^g\sigma(f)$ for $B \in \beta_2$. Now let $(\beta, \mathfrak{g}) \in \mathfrak{R}$ such that β is finer than β_2. From our choice of β_1 we then obtain that $\iota_{\hat{B}} \leq \pi_*^{\mathfrak{g}(B)}\sigma(f) \leq \iota_{\hat{B}} + \epsilon$ for any $B \in \beta$ and $\hat{B} \in \beta_2$ with $B \subseteq \hat{B}$. We thus observe

$$\sum_{B\in\beta_2} \theta(B)\iota_B \leq \sum_{B\in\beta} \theta(B)(\pi_*^{\mathfrak{g}(B)}\sigma(f)) \leq \sum_{B\in\beta_2} \theta(B)(\iota_B + \epsilon) = \sum_{B\in\beta_2} \theta(B)\iota_B + \epsilon$$

and hence

$$\left|\sum_{B\in\beta_2}\theta(B)\iota_B-\sum_{B\in\beta}\theta(B)(\pi_*^{\mathfrak{g}(B)}\sigma(f))\right|\leq\epsilon$$

follows. From our choice of β_2 we thus obtain

$$\left|\int_A\pi_*^g\sigma(f)d\theta(g)-\sum_{B\in\beta}\theta(B)(\pi_*^{\mathfrak{g}(B)}\sigma(f))\right|\leq 2\epsilon.$$

As $\epsilon>0$ was arbitrary we have shown that for any $f\in C(X)$ we have

$$\left(\sum_{B\in\beta}\theta(B)\pi_*^{\mathfrak{g}(B)}\sigma\right)(f)=\sum_{B\in\beta}\theta(B)\left(\pi_*^{\mathfrak{g}(B)}\sigma(f)\right)\overset{(\beta,\mathfrak{g})\in\mathfrak{R}}{\to}\int_A\pi_*^g\sigma(f)d\theta(g)=\mu(f).$$

Thus we have indeed

$$\sum_{B\in\beta}\theta(B)\pi_*^{\mathfrak{g}(B)}\sigma\overset{(\beta,\mathfrak{g})\in\mathfrak{R}}{\to}\mu$$

with respect to the weak*-topology. Now recall that α has almost no boundary with respect to μ. We thus obtain $\mathcal{M}(X)\ni\nu\mapsto H_\nu^*(\alpha)$ to be continuous in μ and in particular that

$$H^*_{\sum_{B\in\beta}\theta(B)\pi_*^{\mathfrak{g}(B)}\sigma}(\alpha)\overset{(\beta,\mathfrak{g})\in\mathfrak{R}}{\to}H_\mu^*(\alpha).$$

For $(\beta,\mathfrak{g})\in\mathfrak{R}$ with β finer than the induced partition of $\bigvee_{x\in E}\pi(\cdot,x)^{-1}(\alpha)$ on A we observe from Lemma 4.35 that

$$\int_A H^*_{\pi_*^g\sigma}(\alpha)d\theta(g)=\sum_{B\in\beta}\theta(B)H^*_{\pi_*^{\mathfrak{g}(B)}\sigma}(\alpha)\leq H^*_{\sum_{B\in\beta}\theta(B)\pi_*^{\mathfrak{g}(B)}\sigma}(\alpha)$$

and the claimed inequality follows. □

Lemma 7.48. *Let $M\subseteq\mathcal{M}(X)$ be a countable subset and $\eta\in\mathbb{U}_X$. Then there exists a finite measurable partition α of X that has almost no boundary with respect to all $\mu\in M$ and that is at scale η.*

Proof. Denote $M:=\{\mu_n;\,n\in\mathbb{N}\}$ and $\mu:=\sum_{n\in\mathbb{N}}2^n\mu_n$. Then there holds $\mu\in\mathcal{M}(X)$ and it suffices to find a finite measurable partition α at scale η that has almost no boundary with respect to μ.

Let $\epsilon\in\mathbb{U}_X$ be an open and symmetric entourage that satisfies $\epsilon\epsilon\epsilon\epsilon\subseteq\eta$. As X is compact there exist $k\in\mathbb{N}$ and $x_1,\cdots,x_k\in X$ such that $\bigcup_{i=1}^k B_\epsilon(x_i)=X$. For $i=1,\cdots,k$ we obtain from Lemma 2.6 the existence of a measurable set B_i with $\mu(\partial B_i)=0$ and that satisfies $\overline{B_\epsilon(x_i)}\subseteq B_i\subseteq B_{\epsilon\epsilon}(x_i)$. A straight forward computations shows $B_i^2\subseteq\epsilon\epsilon\epsilon\epsilon\subseteq\eta$. Considering $A_1:=B_1$ and $A_i:=B_i\setminus A_{i-1}$ for $i=2,\cdots,k$ we obtain a finite partition $\alpha=\{A_1,\ldots,A_k\}$ at scale η that has no boundary with respect to μ. □

We are now prepared to prove the variational principle in the mentioned setting.

Theorem 7.49 (Variational principle for pressure - LCA version)**.** *Let π be an action of a σ-compact LCA group G. For any $f \in C(X)$ there holds*

$$\mathrm{p}_f(\pi) = \sup_{\mu \in \mathcal{M}_G(X)} \left(\mathrm{E}_\mu(\pi) + \mu(f)\right).$$

Proof. From Proposition 3.4 we know that any σ-compact LCA group contains a Van Hove sequence $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ of symmetric and tiling sets. Recall that we denote

$$\mathrm{q}_f^{\mathcal{A}}(\eta|\pi) = \limsup_{i \to \infty} \frac{\mathrm{Q}_{f_{A_i}}(\eta_{A_i})}{\theta(A_i)}.$$

From Theorem 7.12 we furthermore recall $\mathrm{p}_f(\pi) = \sup_{\eta \in \mathbb{U}_X} \mathrm{q}_f^{\mathcal{A}}(\eta|\pi)$. In order to show the theorem it thus suffices to show that for any $\eta \in \mathbb{U}_X$ and any $\epsilon > 0$ there exists $\mu \in \mathcal{M}_G(X)$ such that $\mathrm{q}_f^{\mathcal{A}}(\eta|\pi) - \epsilon \leq \mathrm{E}_\mu(\pi) + \mu(f)$.

Let $\eta \in \mathbb{U}_X$ and $\epsilon > 0$. For $i \in \mathbb{N}$ let us consider a finite (η_{A_i})-separated subset $E_i \subseteq X$ such that

$$\mathrm{Q}_{f_{A_i}}(\eta_{A_i}) - \epsilon \leq \log\left(\sum_{x \in E_i} e^{f_{A_i}(x)}\right).$$

For $i \in I$ we denote $Z_i := \sum_{y \in E_i} e^{f_{A_i}(y)}$. We define

$$\sigma_i := \sum_{x \in E_i} \frac{e^{f_{A_i}(x)}}{Z_i} \delta_x$$

and furthermore

$$\mu_i := \frac{1}{\theta(A_i)} \int_{A_i} \pi_*^g \sigma_i d\theta(g).$$

Restricting to a subsequence of $(A_i)_{i \in \mathbb{N}}$ if necessary we assume without lost of generality that $(\mu_i)_{i \in \mathbb{N}}$ converges in $\mathcal{M}(X)$ to some μ and that $\mathrm{q}_f^{\mathcal{A}}(\eta|\pi) = \lim_{i \to \infty} \mathrm{Q}_{f_{A_i}}(\eta_{A_i})/\theta(A_i)$. With a Krylov-Bogolyubov argument[3] we obtain that $\mu \in \mathcal{M}_G(X)$. In order to complete the proof it remains to show that μ indeed satisfies $\mathrm{q}_f^{\mathcal{A}}(\eta|\pi) - \epsilon \leq \mathrm{E}_\mu(\pi) + \mu(f)$.

Let ω be a Delone set in G that is K-dense with respect to some compact and symmetric subset $K \subseteq G$. Furthermore Lemma 7.48 allows to choose a finite partition α at scale η_K that has almost no boundary with respect to μ and to all μ_i with $i \in \mathbb{N}$. We abbreviate $F_i := (K + A_i) \cap \omega$, $A_i^{[2]} := A_i + A_i$ and $A_i^{[3]} := A_i + A_i + A_i$ for $i \in I$.

Consider now $i, j \in I$. As A_j is a tile there is $\omega_j \subseteq G$ such that $A_j + \omega_j = G$ and such that $\theta((A + v)\Delta(A + v')) = 0$ for all distinct $v, v' \in \omega$. We denote

[3] See [EW11, Theorem 8.10].

$\check{C} := \{g \in \omega_j;\ (g + A_j^{[2]}) \subseteq A_i\}$ and $\hat{C} := \{g \in \omega_j;\ (g + A_j^{[2]}) \cap A_i \neq \emptyset\}$ and observe

$$\begin{aligned}\hat{C} \setminus \check{C} &\subseteq \{g \in G;\ (g + A_j^{[2]}) \cap A_i \neq \emptyset, g + A_j^{[2]} \nsubseteq A_i\} \\ &= (A_j^{[2]} + A_i) \cap (A_j^{[2]} + A_i^c) \subseteq \partial_{A_j^{[2]}} A_i.\end{aligned}$$

We thus have $A_j + (\hat{C} \setminus \check{C}) \subseteq A_j \partial_{A_j^{[2]}} A_i \subseteq \partial_{A_j^{[3]}} A_i$ and obtain

$$A_j + \hat{C} \subseteq (A_j + \check{C}) \cup \partial_{A_j^{[3]}} A_i.$$

We next show that for all $a \in A_j$ the partition $\alpha_{F_j + \hat{C} + a}$ is at scale η_{A_i}. For $g \in A_i - a$ we obtain from $G = A_j + \omega_j$ the existence of $a' \in A_j$ and $v \in \omega_j$ such that $g = a' + v$. In particular there holds $v = g - a' \in A_i - a - a'$ and we observe $v + a + a' \in (v + A_j^{[2]}) \cap A_i$ and hence $v \in \hat{C}$. Thus there holds $g = a' + v \in A_j + \hat{C}$ and we observe $A_i \subseteq A_j + \hat{C} + a$. For $a' \in A_j$ and $G = K + \omega$ we furthermore know that there are $k \in K$ and $v \in \omega$ such that $a' = k + v$. From the symmetry of K we obtain $v = a - k \in \omega \cap (A_j + K) = F_j$ and hence $a' = k + v \in K + F_j$. We obtain

$$A_i \subseteq A_j + \hat{C} + a \subseteq K + F_j + \hat{C} + a.$$

Thus $\alpha_{F_j + \hat{C} + a}$ is indeed at scale

$$\eta_{A_i} \supseteq \eta_{K + F_j + \hat{C} + a} = (\eta_K)_{F_j + \hat{C} + a}.$$

As E_i was chosen to be η_{A_i}-separated we thus obtain that every element of the partition $\alpha_{F_j + \hat{C} + a}$ contains at most one element of E_i. This observation in combination with Lemma 7.41 allows to compute

$$\begin{aligned}\mathrm{Q}_{f_{A_i}}(\eta_{A_i}) - \epsilon &\leq \log\left(\sum_{x \in E_i} e^{f_{A_i}(x)}\right) \\ &= \sum_{x \in E_i} \frac{e^{f_{A_i}(x)}}{Z_i}\left(f_{A_i}(x) - \log\left(\frac{e^{f_{A_i}(x)}}{Z_i}\right)\right) \\ &= H^*_{\sigma_i}(\alpha_{F_j + \hat{C} + a}) + \sigma_i(f_{A_i}) \\ &\leq \sum_{g \in \hat{C}} H^*_{\pi_*^{(g+a)}\sigma_i}(\alpha_{F_j}) + \sigma_i(f_{A_i}).\end{aligned}$$

From a similar argument as in Remark 4.11 we observe

$$\left(\sup_{\nu \in \mathcal{M}(X)} H^*_\nu(\alpha_{F_j})\right) \leq |\alpha_{F_j}| < \infty.$$

Now recall that α_{F_j} has almost no boundary with respect to μ_i and that σ_i is finitely supported. From Lemma 7.47(ii) we thus know that $g \mapsto H^*_{\pi^g_*\sigma_i}(\alpha_{F_j})$ is Lebesgue integrable on any compact subset of G and that

$$\int_{A_i} H^*_{\pi^g_*\sigma_i}(\alpha_{F_j}) d\theta(g) \leq \theta(A_i) H^*_\mu(\alpha_{F_j}).$$

Integrating over all $a \in A_j$ with respect to θ we thus obtain

$$\begin{aligned}
\theta(A_j)(\mathrm{Q}_{f_{A_i}}(\eta_{A_i}) - \epsilon - \sigma_i(f_{A_i})) &\leq \int_{A_j} \sum_{g\in\hat{C}} H^*_{\pi^{(g+a)}_*\sigma_i}(\alpha_{F_j}) d\theta(a)\\
&= \int_{A_j+\hat{C}} H^*_{\pi^g_*\sigma_i}(\alpha_{F_j}) d\theta(g)\\
&\leq \int_{A_j+\check{C}} H^*_{\pi^g_*\sigma_i}(\alpha_{F_j}) d\theta(g) + \int_{\left(\partial_{A_j^{[3]}} A_i\right)} H^*_{\pi^g_*\sigma_i}(\alpha_{F_j}) d\theta(g)\\
&\leq \int_{A_i} H^*_{\pi^g_*\sigma_i}(\alpha_{F_j}) d\theta(g) + |\alpha_{F_j}| \theta\left(\partial_{A_j^{[3]}} A_i\right)\\
&\leq \theta(A_i) H^*_{\mu_i}(\alpha_{F_j}) + |\alpha_{F_j}| \theta\left(\partial_{A_j^{[3]}} A_i\right).
\end{aligned}$$

From Lemma 7.47 we know $\mu_i(f) = 1/\theta(A_i)\sigma_i(f_{A_i})$ and thus $\mu(f) = \lim_{i\in I} \sigma_i(f_{A_i})/\theta(A_i)$. Now recall that the partition α_{F_j} has no boundary with respect to μ. We thus observe that $\nu \mapsto H^*_\nu(\alpha_{F_j})$ is continuous in μ. As $(A_i)_{i\in I}$ is a Van Hove sequence we compute

$$\begin{aligned}
\mathrm{q}^{\mathcal{A}}_f(\eta|\pi) - \mu(f) &= \lim_{i\in I} \frac{\mathrm{Q}_{f_{A_i}}(\eta_{A_i}) - \sigma_i(f_{A_i})}{\theta(A_i)}\\
&\leq \frac{1}{\theta(A_j)} \left(\lim_{i\to\infty} H^*_\mu(\alpha_{F_j})\right) + \frac{1}{\theta(A_j)} \lim_{i\to\infty} \left(|\alpha_{F_j}| \frac{\theta\left(\partial_{A_j^{[3]}} A_i\right)}{\theta(A_i)} + \frac{\epsilon}{\theta(A_i)}\right)\\
&\leq \frac{H^*_\mu(\alpha_{F_j})}{\theta(A_j)} + \frac{\epsilon}{\theta(A_j)}.
\end{aligned}$$

In the last inequality we have used that there holds $\theta(A_i) \to \infty$ whenever G is not compact. In case G is compact we can rescale θ such that $\theta(A_i) \to \theta(G) = 1$, which yields the last inequality in both settings. From the same arguments we observe $\lim_{j\to\infty} \epsilon/\theta(A_j) \leq \epsilon$. As $(K + A_j)_{j\in\mathbb{N}}$ is a Van Hove sequence and $\theta(K + A_i)/\theta(A_i) \to 1$ we obtain from Theorem 5.16 that

$$\mathrm{q}^{\mathcal{A}}_f(\eta|\pi) - \mu(f) - \epsilon \leq \limsup_{j\to\infty} \frac{H^*_\mu(\alpha_{F_j})}{\theta(A_j)} \leq \sup_{\alpha'} \limsup_{j\to\infty} \frac{H^*_\mu(\alpha'_{F_j})}{\theta(K + A_j)} = \mathrm{E}_\mu(\pi),$$

where the supremum is taken over all finite partitions α' of X. □

Remark 7.50. *It remains open to give a proof of the variational principle for actions of general unimodular amenable groups. It seems possible that one can use the slightly improved version of the quasi-tiling technique, developed in Chapter 3, in order to provide a proof also in the general context. Nevertheless, note that the given proof is sufficient for the setting of aperiodic order and shows the statement also for groups like* $\mathbb{Q}_p$ *that contain no uniform lattice.*

7.5 The converse variational principle

The variational principle shows that the pressure map can be constructed from the knowledge of the entropy map (and its domain, the set of all invariant Borel probability measures). The following shows that this construction can also be reversed as long as $\mathrm{p}_0(\pi) = \mathrm{E}(\pi) < \infty$ and as long as the entropy map is upper semi-continuous. The ideas behind this converse variational principle are already contained in [Rue73] and [DGS76] in the context of actions of $\mathbb{Z}^d$. In [Oll85, Remark 5.3.8] the converse variational principle can be found for actions of discrete amenable groups. Following closely the proofs given in [Kel98, Theorem 4.2.9] and [Wal82, Theorem 9.11 and Theorem 9.12] we next investigate under which assumptions the converse variational principle holds in our context.

Recall from Theorem 7.16 that the topological pressure map is either constantly ∞ or real valued. Clearly, one cannot expect the topological pressure map to contain a lot of information in the first case. In the latter case however the information whether a measure is invariant or not is contained in the knowledge of the topological pressure map. To show this we follow ideas from [Wal82] and apply our version of Goodwyn's theorem (Theorem 7.43).

Theorem 7.51. *Let* π *be an action of a unimodular amenable group* G *on a compact Hausdorff space* X *such that the topological entropy of* π *is finite. Then a Borel probability measure* μ *on* X *is invariant if and only if there holds* $\mu(f) \leq \mathrm{p}_f(\pi)$ *for all* $f \in C(X)$*.*

Proof. To show the first statement note first that whenever μ is invariant we obtain from the Theorem 7.43 that

$$\mu(f) \leq \mathrm{E}_\mu(\pi) + \mu(f) \leq \mathrm{p}_f(\pi) \tag{7.4}$$

holds for all $f \in C(X)$. It thus remains to show that this condition is also sufficient and we assume that $\mu(f) \leq \mathrm{p}_f(\pi)$ for all $f \in C(X)$. To show that μ is invariant let $g \in G$. Then by Proposition 7.15 for $n \in \mathbb{Z}$ and $f \in C(X)$ there holds

$$n\mu(f \circ \pi^g - f) = \mu(n(f \circ \pi^g - f)) \leq \mathrm{p}_{n(f\circ\pi^g - f)}(\pi) = \mathrm{p}_0(\pi) = \mathrm{E}(\pi).$$

As the topological entropy $\mathrm{E}(\pi)$ is finite we obtain for positive n that

$$\mu(f \circ \pi^g) - \mu(f) = \mu(f \circ \pi^g - f) \leq \frac{1}{n} \mathrm{E}(\pi) \overset{n \to \infty}{\to} 0.$$

Similarly one obtains with $n \to -\infty$ that

$$\mu(f \circ \pi^g) - \mu(f) \geq \frac{1}{n} \mathrm{E}(\pi) \overset{n \to -\infty}{\to} 0$$

and we conclude that μ is invariant. □

To reverse the variational principle it is natural to assume that the action satisfies the variational principle. This assumption is automatically satisfied whenever G is a discrete amenable group or a σ-compact LCA group, but as we do not know whether the variational principle holds for all actions of unimodular amenable groups we need to add the variational principle to our assumptions. Following exactly the proof of [Wal82] we obtain the following converse variational principle. We include the proof for the convenience of the reader.

Theorem 7.52 (Converse variational principle)**.** *Let π be an action of a unimodular amenable group G that satisfies the variational principle*

$$\mathrm{p}_f(\pi) = \sup_{\mu \in \mathcal{M}_G(X)} (\mathrm{E}_\mu(\pi) + \mu(f))$$

for any potential $f \in C(X)$ and furthermore $\mathrm{E}(\pi) < \infty$. Then for any $\mu \in \mathcal{M}_G(X)$ the measure theoretical entropy can be calculated via the converse variational principle as

$$\mathrm{E}_\mu(\pi) = \inf_{f \in C(X)} \left(\mathrm{p}_f(\pi) - \mu(f) \right),$$

if and only if the entropy map is upper semi-continuous in μ.

Proof. Let us first show that the upper semi-continuity in μ is necessary for the converse variational principle. Let $\epsilon > 0$ and consider $f \in C(X)$ such that $\mathrm{p}_f(\pi) - \mu(f) \leq \mathrm{E}_\mu(\pi) + \epsilon$. Then for any net $(\mu_i)_{i \in I}$ in $\mathcal{M}_G(X)$ that converges to μ with respect to the weak-* topology we know in particular that $\mu_i(f) \to \mu(f)$ and use the variational principle to compute

$$\limsup_{i \in I} \mathrm{E}_{\mu_i}(\pi) \leq \limsup_{i \in I} \left(\mathrm{p}_f(\pi) - \mu_i(f) \right) = \mathrm{p}_f(\pi) - \mu(f) \leq \mathrm{E}_\mu(\pi) + \epsilon.$$

As $\epsilon > 0$ was arbitrary this shows that $\limsup_{\nu \to \mu} \mathrm{E}_\nu(\pi) \leq \mathrm{E}_\mu(\pi)$, i.e. the upper semi-continuity of the entropy map in μ.

To show that the upper semi-continuity in μ of the entropy map is also sufficient for the converse variational principle let $E > \mathrm{E}_\mu(\pi)$ and set

$$C := \{(\nu, t) \in \mathcal{M}_G(X) \times \mathbb{R};\ 0 \leq t \leq \mathrm{E}_\nu(\pi)\}.$$

Recall from Theorem 4.36 that the entropy map $\mu \mapsto \mathrm{E}_\mu(\pi)$ is affine. We thus observe that C is a convex set. Let us now consider $t \in \mathbb{R}$ with $(\mu, t) \in \overline{C}$. For $\epsilon > 0$ we obtain from the upper semi-continuity of the entropy map in μ that there exists an open neighbourhood U of μ such that any $\nu \in U$ satisfies $\mathrm{E}_\nu(\pi) \leq \mathrm{E}_\mu(\pi) + \epsilon$. Now $U \times (t - \epsilon, t + \epsilon)$ is an open neighbourhood of (μ, t) in $C(X)^* \times \mathbb{R}$ and thus intersects C. Considering any (ν, s) in this intersection we compute

$$t \leq s + \epsilon \leq \mathrm{E}_\nu(\pi) + \epsilon \leq \mathrm{E}_\mu(\pi) + 2\epsilon$$

and as ϵ was arbitrary we conclude that $t \leq \mathrm{E}_\mu(\pi)$ and in particular that there holds $(\mu, E) \notin \overline{C}$. Now recall that we identify $\mathcal{M}_G(X)$ as a subset of the topological dual $C(X)^*$ equipped with the weak-* topology via the Riesz–Markov–Kakutani representation theorem. Thus, $\overline{C}$ is a closed and convex subset of the locally convex linear topological space $C(X)^* \times \mathbb{R}$. By a standard theorem about separation of points from compact convex sets in locally convex topological spaces [DS88, V.2. Theorem 10], we obtain the existence of a continuous and linear functional $F\colon C(X)^* \times \mathbb{R} \to \mathbb{R}$ such that $F(\nu, t) < F(\mu, E)$ for all $(\nu, t) \in C$. As F is linear and continuous with respect to the weak-* topology, we obtain the existence of $f \in C(X)$ and $c \in \mathbb{R}$ such that there holds $F(\nu, t) = \nu(f) + tc$ for $(\nu, t) \in C(X)^* \times \mathbb{R}$. In particular, as $(\mu, 0) \in C$ there holds $\mu(f) + 0c < \mu(f) + Ec$ and as $E > \mathrm{E}_\mu(\pi) \geq 0$ we obtain $c > 0$. Furthermore, for $\nu \in \mathcal{M}_G(X)$ there holds $(\nu, \mathrm{E}_\nu(\pi)) \in C$ and thus $\nu(f) + \mathrm{E}_\nu(\pi)c < \mu(f) + Ec$. Dividing by $c > 0$ we obtain $\nu(f/c) + \mathrm{E}_\nu(\pi) < \mu(f/c) + E$ for all $\nu \in \mathcal{M}_G(X)$. Thus, taking the supremum over all such ν we obtain from the variational principle that $\mathrm{p}_{f/c}(\pi) \leq \mu(f/c) + E$. As $E > \mathrm{E}_\mu(\pi)$ was arbitrary we thus see

$$\inf_{g \in C(X)} \left(\mathrm{p}_g(\pi) - \mu(g)\right) \leq \mathrm{p}_{f/c}(\pi) - \mu\left(\frac{f}{c}\right) \leq E_\mu(\pi).$$

From the variational principle we furthermore obtain $\mathrm{E}_\mu(\pi) \leq \mathrm{p}_f(\pi) - \mu(f)$ for all $f \in C(X)$ and thus conclude the statement. □

Remark 7.53. *The assumption that* $\mathrm{E}(\pi) < \infty$ *is necessary in Theorem 7.52, even for Delone actions. Indeed, consider the Delone set* ω *constructed in Example 6.37 and recall that* $\mathrm{E}(\pi_\omega) = \infty$ *and that* $\omega \cap (-\infty, 0] = -\mathbb{N}_0$. *From the latter property we obtain in particular that* X_ω *contains the closed and invariant set* $X_\mathbb{Z} = \{\mathbb{Z} + g;\ g \in \mathbb{R}\}$. *Considering any invariant Borel probability measure* μ *on* $X_\mathbb{Z}$ *we obtain that* $0 \leq \mathrm{E}_\mu(\pi_\omega) = \mathrm{E}_\mu(\pi_\mathbb{Z}) \leq \mathrm{E}(\pi_\mathbb{Z}) = 0$. *Nevertheless, as* $\mathrm{E}(\pi_\omega) = \infty$ *we observe that the* *topological pressure map is constantly* ∞ *and thus*

$$\mathrm{E}_\mu(\pi_\omega) = 0 \neq \infty = \inf_{f \in C(X_\omega)} (\mathrm{p}_f(\pi_\omega) - \mu(f)).$$

As the variational principle is satisfied in the context of aperiodic order we next summarize the results concerning aperiodic order.

Corollary 7.54 (Converse variational principle - LCA version)**.** *Let* π *be an action of a* σ*-compact LCA group* G *of finite topological entropy. We have*

$$\mathcal{M}_G(X) = \{\mu \in \mathcal{M}(X);\, \forall f \in C(X)\colon \mu(f) \leq \mathrm{p}_f(\pi)\}.$$

For any $\mu \in \mathcal{M}_G(X)$ *we have*

$$\mathrm{E}_\mu(\pi) = \inf_{f \in C(X)} \left(\mathrm{p}_f(\pi) - \mu(f)\right),$$

if and only if the entropy map of π *is upper semi-continuous in* μ*.*

7.6 Equilibrium states

For a potential $f \in C(X)$ we call $\mu \in \mathcal{M}_G(X)$ an *equilibrium state* for f, whenever $\mathrm{E}_\mu(\pi) + \mu(f)$ is maximal. Note that whenever the variational principle is satisfied for π this maximum is $\mathrm{p}_f(\pi)$ but as we do not know whether the variational principle holds in full generality we define this notion independent of the variational principle. We denote the set of all equilibrium states for f by $\mathcal{M}_G^f(X)$. The study of this important concept is another cornerstone of the thermodynamic formalism as illustrated by [Wal82, Oll85, Kel98] and we next present that the statements from [Wal82] can easily be generalized to the non-discrete context.

7.6.1 The structure of equilibrium states

The basic structure statements about the set of the equilibrium states can be summarized as follows and the arguments of the statements (except (ii)) can be found in [Wal82]. We include the short proof for the convenience of the reader.

Proposition 7.55. *Let* π *be an action of a unimodular amenable and non-compact group* G *and consider a potential* $f \in C(X)$*. The set* $\mathcal{M}_G^f(X)$ *is*

(i) convex.

(ii) a face of $\mathcal{M}_G(X)$*, whenever the entropy map of* π *is bounded.*

(iii) a closed and non-empty face of $\mathcal{M}_G(X)$*, whenever the entropy map of* π *is upper semi-continuous.*

(iv) non-empty, whenever the entropy map of π *is unbounded.*

Remark 7.56. *Note that we define the notion of a face such that empty sets are also faces. This aspect differs slightly from the notion in [JL01]. For examples of actions of* $\mathbb{Z}$ *such that* $\mathcal{M}_G^f(X)$ *is empty see [Wal82, Section 8.3].*

Proof of Proposition 7.55. Let us abbreviate $M := \sup_{\mu\in\mathcal{M}_G(X)}(\mathrm{E}_\mu(\pi) + \mu(f))$. Recall from Theorem 4.36 that the entropy map is affine. Thus, also the map $\mu \mapsto \mathrm{E}_\mu(\pi)+\mu(f)$ is affine and for $\mu, \nu \in \mathcal{M}_G^f(X)$ and $\lambda \in [0, 1]$ we obtain

$$\begin{aligned}\mathrm{E}_{\lambda\mu+(1-\lambda)\nu}(\pi) + (\lambda\mu + (1-\lambda)\nu)(f) &= \lambda(\mathrm{E}_\mu(\pi) + \mu(f)) + (1-\lambda)(\mathrm{E}_\nu(\pi) + \nu(f))\\ &= \lambda M + (1-\lambda)M = M.\end{aligned}$$

This shows $\lambda\mu + (1-\lambda)\nu \in \mathcal{M}_G^f(X)$ and we have proven $\mathcal{M}_G^f(X)$ to be convex.

Recall that $\mathcal{M}_G(X)$ is compact with respect to the weak-* topology and that $\mu \mapsto \mu(f)$ is continuous with respect to the weak-* topology. Thus, $\mu \mapsto \mu(f)$ is bounded and whenever the entropy map is bounded we obtain $M < \infty$. To show that $\mathcal{M}_G^f(X)$ is a face consider $\mu, \nu \in \mathcal{M}_G(X)$ and furthermore $\lambda \in (0, 1)$ such that $\lambda\mu + (1-\lambda)\nu \in \mathcal{M}_G^f(X)$. Now if $\mu \notin \mathcal{M}_G^f(X)$, then there is $\epsilon > 0$ such that $\mathrm{E}_\mu(\pi) + \mu(f) \leq M - \epsilon$. Thus,

$$\begin{aligned}M &= \mathrm{E}_{\lambda\mu+(1-\lambda)\nu}(\pi) + (\lambda\mu + (1-\lambda)\nu)(f)\\ &= \lambda(\mathrm{E}_\mu(\pi) + \mu(f)) + (1-\lambda)(\mathrm{E}_\nu(\pi) + \nu(f))\\ &\leq \lambda(M-\epsilon) + (1-\lambda)M = M - \lambda\epsilon,\end{aligned}$$

a contradiction. Thus, $\mathcal{M}_G^f(X)$ is indeed a face of $\mathcal{M}_G(X)$, whenever $\mathrm{E}(\pi)$ is finite and we have shown (ii).

To show (iii) note first that any upper semi-continuous map on a compact set is bounded and thus (ii) implies that $\mathcal{M}_G^f(X)$ is a face of $\mathcal{M}_G(X)$. Furthermore, in this case also the map $\mathcal{M}_G(X) \ni \mu \mapsto \mathrm{E}_\mu(\pi) + \mu(f)$ is upper semi-continuous. As upper semi-continuous maps on compact sets attain there finite maximum on a closed and non-empty set we obtain $\mathcal{M}_G^f(X)$ to be closed and non-empty.

To show (iv) let we assume that the entropy map is unbounded. Then for $n \in \mathbb{N}$ there exists $\mu_n \in \mathcal{M}_G(X)$ such that $\mathrm{E}_{\mu_n}(\pi) \geq 2^n$. Let us consider $\mu := \sum_{n\in\mathbb{N}} 2^{-n}\mu_n$, which is easily seen to be an invariant Borel probability measure. From the affinity of the entropy map shown in Theorem 4.36 we obtain for any $N \in \mathbb{N}$ that

$$\mathrm{E}_\mu(\pi) + \mu(f) = \sum_{n=1}^{N} 2^{-n}\,\mathrm{E}_{\mu_n}(\pi) + \mathrm{E}_{\sum_{n=N+1}^{\infty} 2^{-n}\mu_n}(\pi) + \mu(f) \geq N + \mu(f),$$

which shows $\mathrm{E}_\mu(\pi) + \mu(f) = \infty$, i.e. $\mu \in \mathcal{M}_G^f(X)$. □

Remark 7.57. *Note that the assumption of boundedness of the entropy map in (ii) cannot be dropped. To see this consider again the Delone set ω from Example 6.37. As the variational principle is satisfied for $\mathbb{R}$-actions and as the topological entropy of π_ω is infinite we obtain that the entropy map is unbounded. Thus, Proposition 7.55 implies that there exists $\mu \in \mathcal{M}_G^0(X)$ which satisfies $\mathrm{E}_\mu(\pi) = \infty$. Now recall from Remark 7.53 that $X_\mathbb{Z} \subseteq X_\omega$, which implies the existence of $\nu \in \mathcal{M}_G(X)$ such that $\mathrm{E}_\nu(\pi) = 0$. Nevertheless, for any $\lambda \in (0,1)$ there holds $\mathrm{E}_{\lambda\mu+(1-\lambda)\nu}(\pi) = \lambda \cdot 0 + (1-\lambda)\cdot\infty = \infty$ and we deduce $\lambda\mu + (1-\lambda)\nu \in \mathcal{M}_G^0(X)$. As $\mu \notin \mathcal{M}_G^0(X)$ we obtain that $\mathcal{M}_G^0(X)$ is not a face of $\mathcal{M}_G(X)$.*

We have just seen that whenever the entropy map is bounded, then $\mathcal{M}_G^f(X)$ is a face of $\mathcal{M}_G(X)$. It is thus natural to consider the relation of faces of convex sets. For further details see [JL01]. We include the short proof for the convenience of the reader.

Lemma 7.58. *If $\mathcal{X}$ is a topological vector space and K is a convex subset of $\mathcal{X}$, then the following statements are valid.*

(i) Whenever F is a face of K and E is a face of F, then E is a face of K.

(ii) Whenever $F \subseteq K$ is a convex subset and whenever E is a face of K that is contained in F, then E is a face of F.

Proof. To show (i) let $x, y \in K$ and $\lambda \in (0,1)$ such that $\lambda x + (1-\lambda)y \in E$. As E is a face of F we obtain $\lambda x + (1-\lambda)y \in E \subseteq F$ and as F is a face of K we observe $x, y \in F$. Using that E is a face of F and $\lambda x + (1-\lambda)y \in E \subseteq F$ we obtain $x, y \in E$ and we have shown that E is a face of K.

To show (ii) let $x, y \in F$ and $\lambda \in (0,1)$ such that $\lambda x + (1-\lambda)y \in E$. As $x, y \in F \subseteq K$ we obtain from E being a face of K that $x, y \in E$, which shows E to be a face of F. □

As singleton faces are exactly the extreme points of a convex set we obtain the following corollary from Proposition 7.55.

Corollary 7.59. *Let π be an action of a non-compact unimodular amenable group G and consider a potential $f \in C(X)$. Whenever $\mathrm{E}(\pi)$ is finite, then the extreme points of $\mathcal{M}_G^f(X)$ are exactly the extreme points of $\mathcal{M}_G(X)$ that are contained in $\mathcal{M}_G^f(X)$.*

Remark 7.60. *Note that the extreme points of $\mathcal{M}_G(X)$ are exactly the* ergodic measures, *i.e. invariant Borel probability measures μ on X such that $\mu(A) \in \{0,1\}$, for all Borel sets A that satisfy $\pi^g(A) = A$ for all $g \in G$ [Phe01, Proposition 12.4]. Thus, the previous corollary can be reformulated as follows. The extreme points of $\mathcal{M}_G^f(X)$ are exactly the ergodic equilibrium states for f. For further details on the notion of ergodic measures see [Wal82, Phe01].*

7.6.2 On uniqueness of equilibrium states

The variational principle gives the tool to relate the entropy map with the topological pressure. Again we add the statement to our assumptions to present the theory in the context of unimodular amenable groups. As before this assumption can be dropped whenever we consider actions of σ-compact LCA groups. The following can be found in [Wal82, Theorem 9.14 and Theorem 9.15]. We include the proof for the convenience of the reader.

Theorem 7.61. *Let π be an action of a unimodular amenable group G and assume that π satisfies the variational principle*

$$\mathrm{p}_f(\pi) = \sup_{\mu \in \mathcal{M}_G(X)} (\mathrm{E}_\mu(\pi) + \mu(f))$$

for all potentials $f \in C(X)$ and that the entropy map of π is upper semi-continuous. Then a Borel measure μ on X is en equilibrium state for a potential $f \in C(X)$, if and only if for all $h \in C(X)$ there holds

$$\mu(h) \leq \mathrm{p}_{f+h}(\pi) - \mathrm{p}_f(\pi). \tag{7.5}$$

Proof. Let us first consider $\mu \in \mathcal{M}_G^f(X)$. Then by the assumed variational principle there holds $\mathrm{p}_f(\pi) = \mathrm{E}_\mu(\pi) + \mu(f)$. From the variational principle we furthermore obtain that for all $h \in C(X)$ there holds

$$\mathrm{p}_{f+h}(\pi) - \mathrm{p}_f(\pi) \geq \mathrm{E}_\mu(\pi) + \mu(f+h) - \mathrm{E}_\mu(\pi) - \mu(f) = \mu(h)$$

and we have shown (7.5).

Let us next consider a Borel measure μ that satisfies (7.5) for all $h \in C(X)$. From (7.5) and Remark 7.5 it follows with $h = 1$ that

$$\mu(X) = \mu(1) \geq \mathrm{p}_{f+1}(\pi) - \mathrm{p}_f(\pi) = \mathrm{p}_f(\pi) + 1 - \mathrm{p}_f(\pi) = 1.$$

With $h = -1$ one furthermore computes

$$\mu(X) = -\mu(-1) \leq -\mathrm{p}_{f-1}(\pi) + \mathrm{p}_f(\pi) = -\mathrm{p}_f(\pi) + 1 + \mathrm{p}_f(\pi) = 1.$$

and we obtain μ to be a probability measure. To show that μ is invariant let $h \in C(X)$ and $g \in G$. Then by Proposition 7.15 and (7.5) there holds

$$\pm(\mu(h \circ \pi^g) - \mu(h)) = \mu(\pm(h \circ \pi^g - h)) \leq \mathrm{p}_{f \pm (h \circ \pi^g - h)}(\pi) - \mathrm{p}_f(\pi) = 0,$$

hence $\mu(g \circ \pi^g) = \mu(g)$ and it remains to show that μ is an equilibrium state.

From (7.5) we obtain that for $h \in C(X)$ there holds

$$\mathrm{p}_h(\pi) - \mathrm{p}_f(\pi) = \mathrm{p}_{f+(h-f)}(\pi) - \mathrm{p}_f(\pi) \geq \mu(h-f) = \mu(h) - \mu(f).$$

As we assume that the entropy map is upper semi-continuous (and that the variational principle holds for π) we obtain from the converse variational principle, i.e. Theorem 7.52, that

$$\mathrm{E}_\mu(\pi) = \inf_{h \in C(X)} (\mathrm{p}_h(\pi) - \mu(h)) \geq \mathrm{p}_f(\pi) - \mu(f).$$

We thus obtain $\mathrm{p}_f(\pi) \geq \mathrm{E}_\mu(\pi) + \mu(f) \geq \mathrm{p}_f(\pi)$ from the variational principle, which proves that μ is an equilibrium state for f. □

Remark 7.62. *Recall that we identify $C(X)^*$ with the space of all finite signed Borel measures on X by the Riesz–Markov representation theorem and equip this space with the weak-* topology. In fact if a finite signed Borel measure μ on X satisfies (7.5) for all $g \in C(X)$, then μ is called a* tangent functional *(on the pressure map $f \mapsto \mathrm{p}_f(\pi)$) in the literature [DS88, Wal82, Kel98]. Such tangent functionals μ are always positive, i.e. finite Borel measures. Indeed, for positive $h \in C(X)$ and $\epsilon > 0$ we obtain from the basic properties of the pressure map that*

$$\begin{aligned}
\mu(h) + \epsilon\mu(1) &= \mu(h+\epsilon) \\
&= -\mu(-(h+\epsilon)) \\
&\geq -\mathrm{p}_{f-h+\epsilon}(\pi) + \mathrm{p}_f(\pi) \\
&\geq -\mathrm{p}_f(\pi) - \inf_{x \in X} h(x) + \epsilon + \mathrm{p}_f(\pi) \\
&= \inf_{x \in X} h(x) + \epsilon > 0
\end{aligned}$$

and as $\epsilon > 0$ was arbitrary we observe $\mu(h) \geq 0$, i.e. that μ is positive. Thus, the previous theorem can be rephrased as follows. $\mathcal{M}_G^f(X)$ is the set of all tangent functionals to the pressure map, whenever the entropy map is upper semi-continuous (and whenever the variational principle holds). See [DS88] for further details on the notion of a tangent functional.

Similarly to [Wal82, Corollary 9.15.1] we can draw the following corollary. From [DS88, V.9. Theorem 8] we know that a convex function on the separable Banach space $C(X)$ has a unique tangent functional at a dense set of points.

Corollary 7.63. *Let π be an action of a unimodular amenable group G such that the entropy map is upper semi-continuous (and that satisfies the variational principle). Then there exists a dense subset of $C(X)$ such that each member f of this set has a unique equilibrium state, i.e. such that $\mathcal{M}_G^f(X)$ consists of a single measure.*

We will finish our discussion by discussing which forms $\mathcal{M}_G^f(X)$ can take. From [Dow91] we obtain the following.

Example 7.64. *For any metrizable Choquet simplex K there exists an action of $\mathbb{Z}$ and a non-zero potential f on the corresponding phase space such that $\mathcal{M}_{\mathbb{Z}}^f(X)$ is affinely homeomorphic[4] to K. Indeed, in [Dow91] it is shown that for any metrizable Choquet simplex there exists a model set[5] in $\mathbb{Z}$, such that $\mathcal{M}_{\mathbb{Z}}(X)$ is affinely homeomorphic to K. As stated in [DS03] the constructed actions have 0 topological entropy. Thus, the variational principle implies the entropy map to be constant 0 and considering the potential f that is constantly c we obtain that*

$$\mathcal{M}_{\mathbb{Z}}(X_\omega) \ni \mu \mapsto \mathrm{E}_\mu(\pi_\omega) + \mu(f) = 0 + c$$

is constant. Thus, there holds $\mathcal{M}_{\mathbb{Z}}(X_\omega) = \mathcal{M}_{\mathbb{Z}}^f(X_\omega)$ and $\mathcal{M}_{\mathbb{Z}}^f(X_\omega)$ is also affinely homeomorphic to K.

Remark 7.65. *Naturally the question arises, whether for any unimodular amenable group G and any metrizable Choquet simplex K one can find an action π of G and a non-zero potential f on the respective phase space X, such that $\mathcal{M}_G^f(X)$ is affinely homeomorphic to K. From Theorem 7.43 we know that whenever an action of a unimodular amenable group has zero topological entropy, then the entropy map is constantly 0. With similar arguments as in Example 7.64 this question can thus be answered, whenever one shows that for any metrizable Choquet simplex K there exists an action π of G with $\mathrm{E}(\pi) = 0$ and such that $\mathcal{M}_G(X)$ is affinely homeomorphic to K. This question is partially answered in the literature but seems open in full generality.*

In [Cor06] it is shown that such a realization of metrizable Choquet simplices is possible whenever one considers $G = \mathbb{Z}^d$ or $G = \mathbb{R}^d$ and it is in particular shown that one obtains such examples considering actions associated with Delone sets in the respective groups. The matter was pushed further in [Dow08, CP14, FH18, CC19] where certain countable discrete amenable groups are studied. Unfortunately the matter remains open in particular for all LCA groups that contain no uniform lattice.

Example 7.66. *Note that with a result of [DS03] one can deduce that whenever K is a metrizable Choquet simplex and whenever F is a closed face of K, then there exists an action π of $\mathbb{Z}$ and a non-zero potential f on the respective phase space X such that $\mathcal{M}_{\mathbb{Z}}(X)$ is affinely homeomorphic to K and such that this affine homeomorphism restricted to $\mathcal{M}_{\mathbb{Z}}^f(X)$ is a homeomorphism onto F. Indeed, consider a metrizable Choquet simplex K and a closed face F of K. Then by [JL01, Corollary 3.13] there exists a non-negative, continuous and affine map $a\colon K \to [0,\infty)$ such that $F = \{x \in K;\, a(x) = 0\}$. As K is compact we obtain a to be bounded and consider the non-negative continuous and affine map $b\colon K \to [0,\infty)$ with $b(x) = a(x) - \max_{y\in K} a(y)$. Applying [DS03, Theorem 1] to (K, b) we then obtain an action π of $\mathbb{Z}$ on a compact Hausdorff space*

[4] Two convex sets K and K' are called *affinely homeomorphic*, whenever there exists an affine homeomorphism between K and K'.

[5] This model set can actually be chosen to be a "Toeplitz set". Note that all Toeplitz sets in $\mathbb{Z}$ are model sets. For the definition of Toeplitz sets and reference see [Dow05, BJL16].

X such that there exists a affine homeomorphism $\iota \colon K \to \mathcal{M}_{\mathbb{Z}}(X)$ such that the entropy map satisfies $\mathrm{E}_{(\cdot)}(\pi) \circ \iota = b$. Now recall from the definition of b that $F = \{x \in K;\, b(x) = \max_{y \in K} b(y)\}$. One thus easily obtains that $\iota(F) = \{\mu \in \mathcal{M}_{\mathbb{Z}}(X);\, \mathrm{E}_{\mu}(\pi) = \max_{\nu \in \mathcal{M}_{\mathbb{Z}}(\pi)} \mathrm{E}_{\nu}\}$. Considering any non-zero and constant potential $f = c$, we obtain $\mathrm{E}_{\mu}(\pi) + \mu(f) = \mathrm{E}_{\mu}(\pi) + c$ for any $\mu \in \mathcal{M}_{\mathbb{Z}}(X)$ and deduce $\iota(F) = \mathcal{M}_{\mathbb{Z}}^{f}(X)$.

Bibliography

[AKM65] R. L. Adler, A. G. Konheim, and M. H. McAndrew. Topological entropy. *Trans. Amer. Math. Soc.*, 114:309–319, 1965.

[Ave72] A Avez. Entropie des groupes de type fini. *C. R. Acad. Sci. Paris Sér. A-B*, 275:A1363–A1366, 1972.

[BG13] M. Baake and U. Grimm. *Aperiodic Order. Vol. 1: A Mathematical Invitation*, volume 149 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2013.

[BH15] M. Baake and C. Huck. Ergodic properties of visible lattice points. *Proc. Steklov Inst. Math.*, 288(1):165–188, 2015.

[BH18] M. Björklund and T. Hartnick. Approximate lattices. *Duke Math. J.*, 167(15):2903–2964, 2018.

[BHP18] M. Björklund, T. Hartnick, and F. Pogorzelski. Aperiodic order and spherical diffraction, I: auto-correlation of regular model sets. *Proc. Lond. Math. Soc. (3)*, 116(4):957–996, 2018.

[Bis92] D. Bisch. Entropy of groups and subfactors. *J. Funct. Anal.*, 103(1):190–208, 1992.

[Bis04] A. Bis. Entropies of a semigroup of maps. *Discrete Contin. Dyn. Syst.*, 11(2-3):639–648, 2004.

[BJL16] M. Baake, T. Jäger, and D. Lenz. Toeplitz flows and model sets. *Bull. Lond. Math. Soc.*, 48(4):691–698, 2016.

[BL04] M. Baake and D. Lenz. Dynamical systems on translation bounded measures: pure point dynamical and diffraction spectra. *Ergodic Theory Dynam. Systems*, 24(6):1867–1893, 2004.

[BLR07] M. Baake, D. Lenz, and C. Richard. Pure point diffraction implies zero entropy for Delone sets with uniform cluster frequencies. *Lett. Math. Phys.*, 82(1):61–77, 2007.

[Bow71] R. Bowen. Entropy for group endomorphisms and homogeneous spaces. *Trans. Amer. Math. Soc.*, 153:401–414, 1971.

[Bow10] L. Bowen. Measure conjugacy invariants for actions of countable sofic groups. *J. Amer. Math. Soc.*, 23(1):217–245, 2010.

[Bow12] L. Bowen. Sofic entropy and amenable groups. *Ergodic Theory Dynam. Systems*, 32(2):427–466, 2012.

[BS02] M. Brin and G. Stuck. *Introduction to dynamical systems.* Cambridge University Press, Cambridge, 2002.

[BU06] A. Biś and M. Urbański. Some remarks on topological entropy of a semigroup of continuous maps. *Cubo*, 8(2):63–71, 2006.

[Buf11] A. I. Bufetov. Pressure and equilibrium measures for the action of amenable groups on the configuration space. *Mat. Sb.*, 202(3):37–46, 2011.

[CC19] P. Cecchi and M. I. Cortez. Invariant measures for actions of congruent monotileable amenable groups. *Groups Geom. Dyn.*, 13(3):821–839, 2019.

[CdlH16] Y. Cornulier and P. de la Harpe. *Metric geometry of locally compact groups*, volume 25 of *EMS Tracts in Mathematics.* European Mathematical Society (EMS), Zürich, 2016.

[CEPT86] H. S. M. Coxeter, M. Emmer, R. Penrose, and M. L. Teuber. M. C. Escher: art and science. *North-Holland Publishing Co., Amsterdam*, pages xiv+402, 1986.

[Chu13] N. Chung. Topological pressure and the variational principle for actions of sofic groups. *Ergodic Theory Dynam. Systems*, 33(5):1363–1390, 2013.

[Cor06] M. I. Cortez. Realization of a Choquet simplex as the set of invariant probability measures of a tiling system. *Ergodic Theory Dynam. Systems*, 26(5):1417–1441, 2006.

[CP14] M. I. Cortez and S. Petite. Invariant measures and orbit equivalence for generalized Toeplitz subshifts. *Groups Geom. Dyn.*, 8(4):1007–1045, 2014.

[CSCK14] T. Ceccherini-Silberstein, M. Coornaert, and F. Krieger. An analogue of Fekete's lemma for subadditive functions on cancellative amenable semigroups. *J. Anal. Math.*, 124:59–81, 2014.

[Dan01] A. I. Danilenko. Entropy theory from the orbital point of view. *Monatsh. Math.*, 134(2):121–141, 2001.

[DE14] A. Deitmar and S. Echterhoff. *Principles of harmonic analysis.* Universitext. Springer, Cham, second edition, 2014.

[Den72] M. Denker. Une démonstration nouvelle du théorème de Goodwyn. *C. R. Acad. Sci. Paris Sér. A-B*, 275:A735–A738, 1972.

[Den74] M. Denker. Remarques sur la pression pour les transformations continues. *C. R. Acad. Sci. Paris Sér. A*, 279:967–970, 1974.

[DFR16] T. Downarowicz, B. Frej, and P. Romagnoli. Shearer's inequality and infimum rule for Shannon entropy and topological entropy. In *Dynamics and numbers*, volume 669 of *Contemp. Math.*, pages 63–75. Amer. Math. Soc., Providence, RI, 2016.

[DGS76] M. Denker, C. Grillenberger, and K. Sigmund. *Ergodic theory on compact spaces*. Lecture Notes in Mathematics, Vol. 527. Springer-Verlag, Berlin-New York, 1976.

[DHZ19] T. Downarowicz, D. Huczek, and G. Zhang. Tilings of amenable groups. *J. Reine Angew. Math.*, 747:277–298, 2019.

[Din71] E. I. Dinaburg. A connection between various entropy characterizations of dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:324–366, 1971.

[Dow91] T. Downarowicz. The Choquet simplex of invariant measures for minimal flows. *Israel J. Math.*, 74(2-3):241–256, 1991.

[Dow05] T. Downarowicz. Survey of odometers and Toeplitz flows. In *Algebraic and topological dynamics*, volume 385 of *Contemp. Math.*, pages 7–37. Amer. Math. Soc., Providence, RI, 2005.

[Dow08] T. Downarowicz. Faces of simplexes of invariant measures. *Israel J. Math.*, 165:189–210, 2008.

[Dow11] T. Downarowicz. *Entropy in dynamical systems*, volume 18 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2011.

[DS88] N. Dunford and J. T. Schwartz. *Linear operators. Part I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988.

[DS03] T. Downarowicz and J. Serafin. Possible entropy functions. *Israel J. Math.*, 135:221–250, 2003.

[DZ15] A. H. Dooley and G. Zhang. Local entropy theory of a random dynamical system. *Mem. Amer. Math. Soc.*, 233(1099):vi+106, 2015.

[EFHN15] T. Eisner, B. Farkas, M. Haase, and R. Nagel. *Operator theoretic aspects of ergodic theory*, volume 272 of *Graduate Texts in Mathematics*. Springer, Cham, 2015.

[Els77] S. A. Elsanousi. A variational principle for the pressure of a continuous Z^2-action on a compact metric space. *Amer. J. Math.*, 99(1):77–106, 1977.

[Esc75] M. C. Escher. The graphic work of mc escher: Introduced and explained by the artist (john e. brigham, trans.). *London & Sydney: Pan Books*, 1975.

[EW11] M. Einsiedler and T. Ward. *Ergodic theory with a view towards number theory*, volume 259 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[Fel80] J. Feldman. r-entropy, equipartition, and Ornstein's isomorphism theorem in $\mathbf{R}^n$. *Israel J. Math.*, 36(3-4):321–345, 1980.

[FGJO18] G. Fuhrmann, E. Glasner, T. Jäger, and C. Oertel. Irregular model sets and tame dynamics. *arXiv preprint arXiv:1811.06283*, 2018.

[FGL18] G. Fuhrmann, M. Gröger, and D. Lenz. The structure of mean equicontinuous group actions. *arXiv preprint arXiv:1812.10219*, 2018.

[FH18] B. Frej and D. Huczek. Faces of simplices of invariant measures for actions of amenable groups. *Monatsh. Math.*, 185(1):61–80, 2018.

[Føl55] E. Følner. On groups with full Banach mean value. *Math. Scand.*, 3:243–254, 1955.

[Fol99] G. B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[GK82] C. Grillenberger and U. Krengel. On the spatial constant of superadditive set functions in $\mathbf{R}^d$. In *Ergodic theory and related topics (Vitte, 1981)*, volume 12 of *Math. Res.*, pages 53–57. Akademie-Verlag, Berlin, 1982.

[GLW88] É. Ghys, R. Langevin, and P. Walczak. Entropie géométrique des feuilletages. *Acta Math.*, 160(1-2):105–142, 1988.

[Goo69] L. W. Goodwyn. Topological entropy bounds measure-theoretic entropy. *Proc. Amer. Math. Soc.*, 23:679–688, 1969.

[Goo71] T. N. T. Goodman. Relating topological entropy and measure entropy. *Bull. London Math. Soc.*, 3:176–180, 1971.

[Goo72] L. W. Goodwyn. Comparing topological entropy with measure-theoretic entropy. *Amer. J. Math.*, 94:366–388, 1972.

[Gou97] F. Q. Gouvêa. *p-adic numbers An Introduction*. Universitext. Springer-Verlag, Berlin, second edition, 1997.

[Gro99] M. Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps. I. *Math. Phys. Anal. Geom.*, 2(4):323–415, 1999.

[GTW00] E. Glasner, J. Thouvenot, and B. Weiss. Entropy theory without a past. *Ergodic Theory Dynam. Systems*, 20(5):1355–1370, 2000.

[Hau20a] T. Hauser. A note on entropy of delone sets. *to appear (accepted for publication in Math. Nachr.), arXiv: 1902.05441*, 2020.

[Hau20b] T. Hauser. On entropy of delone sets of p-adic numbers. *to appear, arXiv: 2011.12880*, 2020.

[Hau20c] T. Hauser. Relative topological entropy for actions of non-discrete groups on compact spaces in the context of cut and project schemes. *Journal of Dynamics and Differential Equations*, pages 1–22, 2020.

[HJ19] T. Hauser and T. Jäger. Monotonicity of maximal equicontinuous factors and an application to toral flows. *Proc. Amer. Math. Soc.*, 147(10):4539–4554, 2019.

[HLZ19] X. Huang, Y. Lian, and C. Zhu. A Billingsley-type theorem for the pressure of an action of an amenable group. *Discrete Contin. Dyn. Syst.*, 39(2):959–993, 2019.

[Hoo74] B. M. Hood. Topological entropy and uniform spaces. *J. London Math. Soc. (2)*, 8:633–641, 1974.

[HR79] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I*, volume 115 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, second edition, 1979.

[HR15] C. Huck and C. Richard. On pattern entropy of weak model sets. *Discrete Comput. Geom.*, 54(3):741–757, 2015.

[HYZ06] W. Huang, X. Ye, and G. Zhang. A local variational principle for conditional entropy. *Ergodic Theory Dynam. Systems*, 26(1):219–245, 2006.

[HYZ11] W. Huang, X. Ye, and G. Zhang. Local entropy theory for a countable discrete amenable group action. *J. Funct. Anal.*, 261(4):1028–1082, 2011.

[JL01] W. B. Johnson and J. Lindenstrauss. *Basic concepts in the geometry of Banach spaces*. North-Holland, Amsterdam, 2001.

[JLO16] T. Jäger, D. Lenz, and C. Oertel. Model sets with positive entropy in euclidean cut and project schemes. *arXiv preprint arXiv:1605.01167*, 2016.

[Kel55] J. L. Kelley. *General topology.* D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[Kel98] G. Keller. *Equilibrium states in ergodic theory*, volume 42 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 1998.

[KL11] D. Kerr and H. Li. Entropy and the variational principle for actions of sofic groups. *Invent. Math.*, 186(3):501–558, 2011.

[KL13] D. Kerr and H. Li. Soficity, amenability, and dynamical entropy. *Amer. J. Math.*, 135(3):721–761, 2013.

[Kol58] A. N. Kolmogorov. A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces. *Dokl. Akad. Nauk SSSR (N.S.)*, 119:861–864, 1958.

[Kri07] F. Krieger. Le lemme d'Ornstein-Weiss d'après Gromov. In *Dynamics, ergodic theory, and geometry*, volume 54 of *Math. Sci. Res. Inst. Publ.*, pages 99–111. Cambridge Univ. Press, Cambridge, 2007.

[Kri10] F. Krieger. The Ornstein–Weiss Lemma for discrete amenable groups. *Max Planck Institute for Mathematics Bonn, MPIM Preprint*, 48:2010, 2010.

[Lag99] J. C. Lagarias. Geometric models for quasicrystals I. Delone sets of finite type. *Discrete Comput. Geom.*, 21(2):161–191, 1999.

[LP03] J. C. Lagarias and P. A. B. Pleasants. Repetitive Delone sets and quasicrystals. *Ergodic Theory Dynam. Systems*, 23(3):831–867, 2003.

[LW77] F. Ledrappier and P. Walters. A relativised variational principle for continuous transformations. *J. London Math. Soc. (2)*, 16(3):568–576, 1977.

[LW00] E. Lindenstrauss and B. Weiss. Mean topological dimension. *Israel J. Math.*, 115:1–24, 2000.

[Mac20] S. Machado. Approximate lattices and Meyer sets in nilpotent Lie groups. *Discrete Anal.*, pages Paper No. 1, 18, 2020. arXiv:1810.10870.

[Mey72] Y. Meyer. *Algebraic numbers and harmonic analysis.* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1972. North-Holland Mathematical Library, Vol. 2.

[Mis76] M. Misiurewicz. A short proof of the variational principle for a Z_+^n action on a compact space. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 24(12):1069–1075, 1976.

[Moo02] R. V. Moody. Uniform distribution in model sets. *Canad. Math. Bull.*, 45(1):123–130, 2002.

[Mor15] D. W. Morris. *Introduction to arithmetic groups*. Deductive Press, [place of publication not identified], 2015.

[Mun00] J. R. Munkres. *Topology*. Prentice Hall, Inc., Upper Saddle River, NJ, 2000.

[MW11] D. Ma and M. Wu. Topological pressure and topological entropy of a semigroup of maps. *Discrete Contin. Dyn. Syst.*, 31(2):545–557, 2011.

[Oll85] J. M. Ollagnier. *Ergodic theory and statistical mechanics*, volume 1115 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985.

[OP79] J. M. Ollagnier and D. Pinchon. Groupes pavables et principe variationnel. *Z. Wahrsch. Verw. Gebiete*, 48(1):71–79, 1979.

[OP82] J. M. Ollagnier and D. Pinchon. The variational principle. *Studia Math.*, 72(2):151–159, 1982.

[OW80] D. S. Ornstein and B. Weiss. Ergodic theory of amenable group actions. I. The Rohlin lemma. *Bull. Amer. Math. Soc. (N.S.)*, 2(1):161–164, 1980.

[OW83] D. Ornstein and B. Weiss. The Shannon-McMillan-Breiman theorem for a class of amenable groups. *Israel J. Math.*, 44(1):53–60, 1983.

[OW87] D. S. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–141, 1987.

[Pen86] R. Penrose. Escher and the visual representation of mathematical ideas. In *M. C. Escher: art and science (Rome, 1985)*, pages 143–157. North-Holland, Amsterdam, 1986. With a note by M. Emmer.

[Phe01] R. R. Phelps. *Lectures on Choquet's theorem*, volume 1757 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 2001.

[Pie84] J. Pier. *Amenable locally compact groups*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. A Wiley-Interscience Publication.

[Pog13] F. Pogorzelski. Almost-additive ergodic theorems for amenable groups. *J. Funct. Anal.*, 265(8):1615–1666, 2013.

[PS16] F. Pogorzelski and F. Schwarzenberger. A Banach space-valued ergodic theorem for amenable groups and applications. *J. Anal. Math.*, 130:19–69, 2016.

[Rud76] W. Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. International Series in Pure and Applied Mathematics.

[Rud91] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[Rue73] D. Ruelle. Statistical mechanics on a compact set with Z^v action satisfying expansiveness and specification. *Trans. Amer. Math. Soc.*, 187:237–251, 1973.

[SBGC84] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn. Metallic phase with long-range orientational order and no translational symmetry. *Physical review letters*, 53(20):1951, 1984.

[Sch00] M. Schlottmann. Generalized model sets and dynamical systems. In *Directions in mathematical quasicrystals*, volume 13 of *CRM Monogr. Ser.*, pages 143–159. Amer. Math. Soc., Providence, RI, 2000.

[Sch15] F. M. Schneider. Topological entropy of continuous actions of compactly generated groups. *arXiv preprint arXiv:1502.03980*, 2015.

[Sin59] J. Sinaĭ. On the concept of entropy for a dynamic system. *Dokl. Akad. Nauk SSSR*, 124:768–771, 1959.

[Sin16] S. Singh. *Entropy theory for locally compact sofic groups*. Dissertation, University of Texas at Austin, 2016.

[ST18] F. M. Schneider and A. Thom. On Følner sets in topological groups. *Compos. Math.*, 154(7):1333–1361, 2018.

[STZ80] A. M. Stepin and A. T. Tagi-Zade. Variational characterization of topological pressure of the amenable groups of transformations. *Dokl. Akad. Nauk SSSR*, 254(3):545–549, 1980.

[Tem84] A. Tempelman. Specific characteristics and variational principle for homogeneous random fields. *Z. Wahrsch. Verw. Gebiete*, 65(3):341–365, 1984.

[Tem92] A. Tempelman. *Ergodic theorems for group actions*, volume 78 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1992. Informational and thermodynamical aspects, Translated and revised from the 1986 Russian original.

[TZ91] A. T. Tagi-Zade. A variational characterization of the topological entropy of continuous groups of transformations. The case of $\mathbf{R}^n$ actions. *Mat. Zametki*, 49(3):114–123, 160, 1991.

[Wal75] P. Walters. A variational principle for the pressure of continuous transformations. *Amer. J. Math.*, 97(4):937–971, 1975.

[Wal82] P. Walters. *An introduction to ergodic theory*, volume 79 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[Wei01] B. Weiss. Monotileable amenable groups. In *Topology, ergodic theory, real algebraic geometry*, volume 202 of *Amer. Math. Soc. Transl. Ser. 2*, pages 257–262. Amer. Math. Soc., Providence, RI, 2001.

[Wei03] B. Weiss. Actions of amenable groups. In *Topics in dynamics and ergodic theory*, volume 310 of *London Math. Soc. Lecture Note Ser.*, pages 226–262. Cambridge Univ. Press, Cambridge, 2003.

[WM15] Y. Wang and D. Ma. On the topological entropy of a semigroup of continuous maps. *J. Math. Anal. Appl.*, 427(2):1084–1100, 2015.

[WZ92] T. Ward and Q. Zhang. The Abramov-Rokhlin entropy addition formula for amenable group actions. *Monatsh. Math.*, 114(3-4):317–329, 1992.

[Yan15] K. Yan. Conditional entropy and fiber entropy for amenable group actions. *J. Differential Equations*, 259(7):3004–3031, 2015.

[YZ16] K. Yan and F. Zeng. Topological entropy, pseudo-orbits and uniform spaces. *Topology Appl.*, 210:168–182, 2016.

[Zha18] R. Zhang. Topological pressure of generic points for amenable group actions. *J. Dynam. Differential Equations*, 30(4):1583–1606, 2018.

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Symbol index

Sets of numbers

$\mathbb{N}$	set of natural numbers excluding 0
$\mathbb{N}_0$	set of natural numbers including 0
$\mathbb{Z}$	set of integer numbers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{T}$	torus
$\mathbb{Z}_p$	set of p-adic integers
$\mathbb{Q}_p$	set of p-adic numbers

Sets and functions

$\lvert A\rvert$	cardinality of a set A	11
A^c	complement of a set A (in another set)	11
χ_A	characteristic function of A	11
$A\Delta B$	symmetric difference	21
$f\|_M$	restriction of a function f to a subset M of the domain	11

Sets of subsets

$\mathcal{P}(X)$	set of all subsets of X (power set)	11
$\mathcal{F}(X)$	set of all finite subsets of X	11
$\mathcal{A}(X)$	set of all closed subsets of X	11

$\mathcal{K}(X)$	set of all compact subsets of X	11
$\mathcal{D}(X)$	set of all discrete subsets of X	11
$\mathcal{B}_X$	set of all Borel measurable subsets of X (Borel σ-algebra)	11
$\mathcal{D}_V(G)$	set of all V-discrete subsets of G	33
$\mathcal{D}_{K,V}(G)$	set of all V-discrete and K-dense subsets of G	33
$\mathcal{N}(G)$	set of all neighbourhoods of the unit element in G	16
$\mathrm{Pat}_\omega(A)$	set of A-patches of the Delone set ω	33

Topological spaces

$\overline{A}$	topological closure of A	11
$\mathrm{int}(A)$	topological interior of A	11
∂A	topological boundary of A	11
d	(usually) a (pseudo) metric	13
μ, ν	(usually) Borel probability measures (often invariant)	14
$p_*\mu$	push forward of the measure μ	14
$C(X)^*$	space of all linear bounded functionals on $C(X)$ equipped with the weak-* topology	11
$\mathcal{M}(X)$	set of Borel probability measures on X with weak-* topology	11
$\mathbb{E}_\mu(f\|\mathcal{A})$	conditional expectation of a real valued function f given a sub σ-algebra $\mathcal{A}$ of $\mathcal{B}_X$	68
$\mathcal{U}, \mathcal{V}$	(usually) finite open covers	11
α, β, γ	(usually) finite Borel measurable partitions	11
$\partial\alpha$	$\bigcup_{A\in\alpha} \partial A$ for a Borel measurable partition α	70
$\preceq$	"finer"-relation	67
$C(X)$	Banach space of all bounded and continuous real valued functions equipped with supremum norm and component wise order	11
$f \leq g$	componentwise order relation on $C(X)$	11

Compact Hausdorff spaces

X, Y, Z	(usually) compact Hausdorff spaces	12
$\mathbb{U}_X$	uniformity of a compact Hausforff space X	12
Δ_X	diagonal in $X \times X$	12
$\mathbb{B}, \mathbb{B}_X$	(usually) bases of uniformities	13
η, κ, θ	(usually) entourages	12
$\eta\kappa$	$\{(x,z) \in X^2;\ \exists y \in X : (x,y) \in \eta, (y,z) \in \kappa\}$ for $\eta, \kappa \in \mathbb{U}_X$	13
η^{-1}	$\{(y,x);\ (x,y) \in \eta\}$ for $\eta \in \mathbb{U}_X$	13
$\langle \mathcal{U} \rangle$	$\bigcup_{U \in \mathcal{U}} U^2$ for a family of sets $\mathcal{U}$	13
$\eta[x]$	$\{x' \in X;\ (x', x) \in \eta\}$ for $\eta \in \mathbb{U}_X$	13
$B_\eta(x)$	ball with radius $\eta \in \mathbb{U}_X$ and center $x \in X$	13
$B^d_\epsilon(x)$	open ball of radius $\epsilon > 0$ and centre $x \in X$	14
$\overline{B}^d_\epsilon(x)$	closed ball of radius $\epsilon > 0$ and centre $x \in X$	14
$[d < \epsilon]$	$\{(x,y) \in X^2;\ d(x,y) < \epsilon\}$ for a (pseudo) metric d	13
$[d \leq \epsilon]$	$\{(x,y) \in X^2;\ d(x,y) \leq \epsilon\}$ for a (pseudo) metric d	13

Topological groups

G, H	(usually) unimodular amenable groups	16
e, e_G, e_H	(usually) neutral elements in groups	16
0	(usually) the neutral element in an Abelian group	16
$\theta, \theta_G, \theta_H$	(usually) Haar measures	16
$A + B, AB, A^{-1}$	Minkowski operations	16
Λ	(usually) a uniform lattice	33
$\mathrm{dens}(\omega)$	uniform denisty of a Delone set ω	33
$\partial_K A$	K-boundary/ Van Hove boundary of a set A	20
$\alpha(A, K)$	$\theta(\partial_K A)/\theta(A)$	21
$\mathfrak{m}_V(E, F)$	V-matching number of E and F	77

Dynamical systems

π, ϕ, ψ	(usually) actions	38
π^g	$\pi(g, \cdot)$	38
p, q	(usually) factor mappings	38
$\pi \xrightarrow{p} \phi$	ϕ is factor of π via factor mapping p	38
$\mathcal{M}_G(X)$	set of all invariant Borel probability measures on X with weak-* topology	38

Delone dynamical systems

ω, ξ, ζ	(usually) Delone sets	33
D_ω	$\{\omega + g;\ g \in G\}$	39
X_ω	closure of D_ω with respect to local rubber topology of $\mathcal{A}(G)$	39
π_ω	Delone dynamical system of a Delone set ω	39
$\xi \overset{K,V}{\approx} \zeta,$	abbreviates $\xi \cap K \subseteq \zeta + V$ and $\zeta \cap K \subseteq \xi + V$	38
$\epsilon(K, V)$	set of all $(\xi, \zeta) \in \mathcal{A}(G)^2$ that satisfy $\xi \overset{K,V}{\approx} \zeta$	39
$\epsilon_X(K, V)$	set of all $(\xi, \zeta) \in X^2$ that satisfy $\xi \overset{K,V}{\approx} \zeta$ for $X \subseteq \mathcal{A}(G)$	39
$\epsilon_\omega(K, V)$	set of all $(\xi, \zeta) \in X_\omega^2$ that satisfy $\xi \overset{K,V}{\approx} \zeta$ for a Delone set ω	39
$\mathbb{B}_{lr}$	local rubber base	39
$\eta_\omega(K, V)$	$\{(\xi, \zeta) \in X_\omega^2;\ \exists x, z \in V \colon (\xi + x) \cap K = (\zeta + z) \cap K\}$	39
$\mathbb{B}_{lm}$	local matching base	39

Topological entropy

$\mathcal{U} \vee \mathcal{V}$	common refinement of open covers	67
$\mathcal{U}_g$	$\{(\pi^g)^{-1}(U);\ U \in \mathcal{U}\}$	66
$\mathcal{U}_F$	$\bigvee_{g \in F} \mathcal{U}_g$	66
η_A, η_g	Bowen entourage	72
$N_M(\mathcal{U})$	minimal cardinality of a subset of $\mathcal{U}$ that covers M	68

$N_M(\mathcal{U}, A)$	minimal cardinality of a finite family of open subsets of X that A-refines $\mathcal{U}$ and covers M	82
$N_p(\mathcal{U})$	$\sup_{y\in Y} N_{p^{-1}(y)}(\mathcal{U})$ for a factor map p and an open cover $\mathcal{U}$	68
$N_p(\mathcal{U}, A)$	$\sup_{y\in Y} N_{p^{-1}(y)}(\mathcal{U}, A)$ for a factor map p and an open cover $\mathcal{U}$	82
$\mathrm{cov}_M(\eta)$	minimal cardinality of a family of open and η-small subsets of X that cover M	68
$H_p^*(\mathcal{U})$	$\log N_p(\mathcal{U})$ for a factor map and an open cover $\mathcal{U}$	68
$H_p^*(\mathcal{U}, A)$	$\log N_p(\mathcal{U}, A)$ for a factor map and an open cover $\mathcal{U}$	82
$H(\eta)$	static topological entropy of X at scale η	68
$H_p(\eta)$	static topological entropy of p at scale η	68
$\mathrm{E}^*(\mathcal{U}\vert\pi)$	topological entropy of $\mathcal{U}$	82
$\mathrm{E}(\eta\vert\pi)$	topological entropy at scale η of an action π	75
$\mathrm{E}(\pi)$	topological entropy of an action π	75
$\mathrm{E}^*\left(\mathcal{U}\vert\pi \xrightarrow{p} \phi\right)$	relative topological entropy of p and $\mathcal{U}$	82
$\mathrm{E}\left(\eta\middle\vert\pi \xrightarrow{p} \phi\right)$	relative topological entropy at scale η of a factor map p	74
$\mathrm{E}\left(\pi \xrightarrow{p} \phi\right)$	relative topological entropy of a factor map p	74

Measure theoretical entropy

$\alpha \vee \beta$	common refinement of partitions	67
α_g	$\{(\pi^g)^{-1}(A);\ A \in \alpha\}$	66
α_F	$\bigvee_{g\in F} \alpha_g$	66
η_A, η_g	Bowen entourage	72
$H_\mu^*(\alpha)$	$-\sum_{A\in\alpha} \mu(A)\log(\mu(A))$	69
$H_\mu^*(\alpha\vert\mathcal{A})$	$-\sum_{A\in\alpha} \int_X \mathbb{E}_\mu(\chi_A\vert\mathcal{A})\log(\mathbb{E}_\mu(\chi_A\vert\mathcal{A}))d\mu$	69
$H_{\mu,p}^*(\alpha)$	$H_\mu^*(\alpha\vert p^{-1}(\mathcal{B}_Y))$	69
$H_{\mu,p}^*(\alpha\vert\mathcal{A})$	$H_\mu^*(\alpha\vert p^{-1}(\mathcal{B}_Y) \vee \mathcal{A})$	69
$H_\mu(\eta)$	static measure theoretical entropy of X at scale η	69
$H_{\mu,p}(\eta)$	static measure theoretical entropy of p at scale η	69
$\mathrm{E}_\mu^*(\alpha\vert\pi)$	measure theoretical entropy of a finite partition α	100
$\mathrm{E}_\mu(\eta\vert\pi)$	measure theoretical entropy at scale η of an action π	75
$\mathrm{E}_\mu(\pi)$	measure theoretical entropy of an action π	75
$\mathrm{E}_\mu^*\left(\alpha\middle\vert\pi \xrightarrow{p} \phi\right)$	relative measure theoretical entropy of a finite partition α	100

$\mathrm{E}_\mu\left(\eta \middle| \pi \xrightarrow{p} \phi\right)$	relative measure theoretical entropy at scale η of p	74
$\mathrm{E}_\mu\left(\pi \xrightarrow{p} \phi\right)$	relative measure theoretical entropy of a factor map p	75

Topological pressure

$\mathrm{P}_f(\eta)$	static topological pressure of f at scale η	166
f_A	$f_A(x) := \int_A f(g.x)d\theta(g)$	166
$\mathrm{p}_f(\eta\|\pi)$	topological pressure of f at scale η	167
$\mathrm{p}_f(\pi)$	topological pressure of f with respect to π	167
$\mathrm{P}^*_f(\mathcal{U})$	$\log\left(\sum_{U\in\mathcal{U}} \sup_{x\in U} e^{f(x)}\right)$	166
$\mathrm{p}^*_f(\mathcal{U}\|\pi)$	topological pressure of f with respect to $\mathcal{U}$	170
$\mathrm{P}^*_f(\mathcal{U}, A)$	-	169
$\mathrm{Q}_f(\eta)$	-	172
$\mathrm{q}^{\mathcal{A}}_f(\eta\|\pi)$	-	172
$\mathrm{R}^*_\mu(\mathcal{U})$	overlap ratio	187
$\mathcal{M}^f_G(X)$	set of equilibrium states of a potential f	200

Abbreviations

CPS	cut and project scheme	36
FLC	finite local complexity	33
LCA	locally compact Abelian	16
u.s.c.	upper semi-continuous	12

Ehrenwörtliche Erklärung

Hiermit erkläre ich,

- dass mir die Promotionsordnung der Fakultät bekannt ist,
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