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GENERALIZED BOUNDARY TRIPLES, II. SOME APPLICATIONS OF GENERALIZED BOUNDARY TRIPLES AND FORM DOMAIN INVARIANT NEVANLINNA FUNCTIONS

VOLODYMYR DERKACH, SEPPO HASSI, AND MARK MALAMUD

Devoted to the memory of our dear friend and colleague Hagen Neidhardt

Abstract. The paper is a continuation of Part I and contains several further results on generalized boundary triples, the corresponding Weyl functions, and applications of this technique to ordinary and partial differential operators. We establish a connection between Post's theory of boundary pairs of closed nonnegative forms on the one hand and the theory of generalized boundary triples of nonnegative symmetric operators on the other hand. Applications to the Laplacian operator on bounded domains with smooth, Lipschitz, and even rough boundary, as well as to mixed boundary value problem for the Laplacian are given. Other applications concern with the momentum, Schrödinger, and Dirac operators with local point interactions. These operators demonstrate natural occurrence of ES-generalized boundary triples involving essentially selfadjoint reference operators A_0.

1. Introduction

This paper is a continuation of the author's work [26] on generalized boundary triples and Weyl functions of symmetric operators. In particular, it is shown how various specific classes of generalized boundary triples appearing in Part I actually occur in the study of ordinary and partial differential operators. Both Part I and Part II are posed in Arxiv as a single paper [25]. We will freely use notations and terminology of Part I, but for the convenience of the reader we will recall here the most cited definitions and statements from Part I.

Let $\mathfrak{H}$ be a separable Hilbert space, let A be a not necessarily densely defined closed symmetric operator in $\mathfrak{H}$ with equal deficiency indices $n_+(A) = n_-(A) \le \infty$. The adjoint A^* of the operator A is a linear relation, see [17] and also [26] for the terminology.

Definition 1.1 ([30])**.** Let A_* be a linear relation in $\mathfrak{H}$ such that $A \subset A_* \subset \overline{A_*} = A^*$. Then the collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and $\Gamma = \{\Gamma_0, \Gamma_1\}$ is a single-valued linear mapping from A_* into $\mathcal{H}^2$, is said to be an *S-generalized boundary triple* for A^*, if:

(1) The following abstract Green's identity holds for all $\widehat{f} = \binom{f}{f'}, \widehat{g} = \binom{g}{g'} \in A_*$

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, \Gamma_0 \widehat{g})_{\mathcal{H}} - (\Gamma_0 \widehat{f}, \Gamma_1 \widehat{g})_{\mathcal{H}}; \tag{1.1}$$

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(2) $\Gamma := \operatorname{col}(\Gamma_0, \Gamma_1)$ is maximal in the sense that if $\widehat{g} = \binom{g}{g'} \in \mathfrak{H}^2$,$\widehat{k} = \binom{k}{k'} \in \mathcal{H}^2$ satisfies

$$(f', g) - (f, g') = (\Gamma_1 \widehat{f}, k)_{\mathcal{H}} - (\Gamma_0 \widehat{f}, k')_{\mathcal{H}}$$

for every $\widehat{f} = \binom{f}{f'} \in A_*$, then $\widehat{g} \in A_*$ and $\Gamma \widehat{g} = \widehat{k}$.

(3) $A_0 := \ker \Gamma_0$ is a selfadjoint extension of A.

A triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with $\mathcal{H}$, Γ as above, is called:

(1) *isometric boundary triple* for A^*, if (1) holds, see [26];
(2) *unitary boundary triple* for A^*, if (1), (2) hold, see [28];
(3) *essentially unitary boundary triple* for A^*, if the closure of $\Gamma : A_* \to \mathcal{H}^2$ is single-valued and it satisfies (1), (2) on the domain $\widetilde{A}_* = \operatorname{dom} \overline{\Gamma}$, see [25, 47];
(4) *ordinary boundary triple* for A^*, if $A_* = A^*$, (1) holds, and $\Gamma : A^* \to \mathcal{H}^2$ is surjective, see [50, 38, 33];
(5) *B-generalized boundary triple* for A^*, if (1), (3) hold and $\Gamma_0 : A_* \to \mathcal{H}$ is surjective, see [33];
(6) *ES-generalized boundary triple* for A^*, if (1), (2) hold and A_0 is essentially selfadjoint, see [26];
(7) *quasi-boundary triple* for A^*, if (1), (3) hold and the range of $\Gamma : A_* \to \mathcal{H}^2$ is dense in $\mathcal{H}^2$, see [12].

Definition 1.2 ([31, 32])**.** The abstract *Weyl function* and the *γ-field* of A, corresponding to a unitary boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ are defined by

$$M(\lambda)\Gamma_0 \widehat{f}_\lambda = \Gamma_1 \widehat{f}_\lambda, \quad \gamma(\lambda)\Gamma_0 \widehat{f}_\lambda = f_\lambda, \quad \lambda \in \rho(A_0),$$

where $\widehat{f}_\lambda \in \widehat{\mathfrak{N}}_\lambda(A_*) := \left\{ \binom{f_\lambda}{\lambda f_\lambda} : f_\lambda \in \mathfrak{H} \right\} \cap A_*$.

Recall that $\mathcal{R}[\mathcal{H}]$ (resp. $\mathcal{R}(\mathcal{H})$) denotes the Nevanlinna class of all operator valued holomorphic functions on $\mathbb{C}_+$ with values in the set of bounded dissipative (resp. maximal dissipative, not necessarily bounded) linear operators in $\mathcal{H}$. In addition, a Nevanlinna function $M(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which admits a holomorphic continuation to the negative real line $(-\infty, 0)$ (in the resolvent sense) and whose values $M(x)$ are nonnegative (nonpositive) selfadjoint operators for all $x < 0$ is called a *Stieltjes function* (an *inverse Stieltjes function*, respectively).

The set of Weyl functions corresponding to unitary boundary triples coincides with the set

$$\mathcal{R}^s(\mathcal{H}) := \{F(\cdot) \in \mathcal{R}(\mathcal{H}) : \operatorname{Im}(F(i)h, h) = 0 \Longrightarrow h = 0, \quad h \in \operatorname{dom} F(i)\} \tag{1.2}$$

of strict Nevanlinna functions, see [28, Theorem 3.9]. Notice, that the set of Weyl functions corresponding to B-generalized boundary triples coincides with the class $\mathcal{R}^s[\mathcal{H}] := \mathcal{R}^s(\mathcal{H}) \cap \mathcal{R}[\mathcal{H}]$, see [33, Theorem 6.1]. Weyl functions $M(\cdot)$ corresponding to S-generalized boundary triples are characterized by the following domain invariance property, see [30, Theorem 7.39], [26, Theorem 1.12].

Theorem 1.3. Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triple for A^* and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and γ-field, respectively. Then the following statements are equivalent:

(i) $A_0 = \ker \Gamma_0$ is selfadjoint, i.e. Π is an S-generalized boundary triple;

(ii) $A_* = A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda$ and $A_* = A_0 \widehat{+} \widehat{\mathfrak{N}}_\mu$ for some (equivalently for all) $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$;
(iii) $M(\cdot)$ is domain invariant and $\operatorname{dom} M(\lambda) = \operatorname{dom} M(\mu) = \operatorname{ran} \Gamma_0$ for all $\lambda \in \mathbb{C}_+$ and all $\mu \in \mathbb{C}_-$;
(iv) $\gamma(\lambda)$ and $\gamma(\mu)$ are bounded and densely defined in $\mathcal{H}$ for some (equivalently for all) $\lambda \in \mathbb{C}_+$ and $\mu \in \mathbb{C}_-$;
(v) $\operatorname{Im} M(\lambda)$ is bounded and densely defined for some (equivalently for all) $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(vi) the Weyl function $M(\cdot)$ belongs to $\mathcal{R}^s(\mathcal{H})$ and it admits a representation

$$M(\lambda) = E + M_0(\lambda), \quad M_0(\cdot) \in \mathcal{R}[\mathcal{H}], \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{1.3}$$

where $E = E^*$ is a selfadjoint (in general unbounded) operator in $\mathcal{H}$.

To characterize the class of ES-generalized boundary triples in terms of the corresponding Weyl functions we associate with each $M(\cdot) \in \mathcal{R}(\mathcal{H})$ a family of nonnegative quadratic forms $\mathfrak{t}'_{M(\lambda)}$ in $\mathcal{H}$:

$$\mathfrak{t}'_{M(\lambda)}[u,v] := \frac{1}{\lambda - \bar{\lambda}} \left[(M(\lambda)u, v) - (u, M(\lambda)v) \right], \quad u, v \in \operatorname{dom}(M(\lambda)), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.4}$$

According to [26, Theorem 1.14], Weyl functions $M(\cdot)$ corresponding to ES-generalized boundary triples are characterized by the following *form domain invariance* property: $\mathfrak{t}'_{M(\lambda)}$ is closable for each $\lambda \in \mathbb{C}_\pm$ and the domain of its closure $\mathfrak{t}_{M(\lambda)} := \overline{\mathfrak{t}'_{M(\lambda)}}$ [1], called *the form domain of the Weyl function* $M(\cdot)$, does not depend on $\lambda \in \mathbb{C}_\pm$. In this case $\operatorname{dom} M(\lambda)$ may, or may not, depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$, while for S-generalized boundary triples the equality $\operatorname{dom} M(\lambda) = \operatorname{ran} \Gamma_0$ follows from the decomposition $A_* = A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda$ in Theorem 1.3 (ii) and hence the Weyl function is always domain invariant and then also form domain invariant by [26, Proposition 5.30], see also [24] for general invariance results on operator-valued Nevanlinna functions. For an ES-generalized, but not S-generalized, boundary triple A_0 is only essentially selfadjoint and then one has just strict inclusions $A_* \subsetneqq A_0 \widehat{+} \widehat{\mathfrak{N}}_\lambda$. Therefore, the equality $\operatorname{dom} M(\lambda) = \operatorname{ran} \Gamma_0$ is violated and a strict inclusion $\operatorname{dom} M(\lambda) \subsetneqq \operatorname{ran} \Gamma_0$ holds; cf. discussions following [26, Theorems 1.13, 1.14]. Another characteristic difference between S-generalized and ES-generalized boundary triples appears in the γ-field: by Theorem 1.3 (iv) for S-generalized boundary triple $\gamma(\lambda)$ is a bounded operator for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, while for ES-generalized boundary triple $\gamma(\lambda)$ is, in general, an unbounded operator, which is closable for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Furthermore, as shown in [26, Theorem 5.24] for ES-generalized boundary triples the form domain of the Weyl function $M(\cdot)$ is directly connected to the closures of $\gamma(\lambda)$ and $\Gamma_0 : A_* \to \mathcal{H}$ by the following characteristic identities:

$$\operatorname{dom} \mathfrak{t}_{M(\lambda)} = \operatorname{dom} \overline{\gamma(\lambda)} = \operatorname{ran} \overline{\Gamma}_0.$$

These facts will be demonstrated in concrete boundary value problems: for Laplace operators on smooth domains in Theorem 3.1, where the form domain of the Weyl function associated with the Kreĭn - von Neumann Laplacian is described explicitly; see (3.13). Similarly it is shown that ES-generalized boundary triples occur naturally when describing mathematical models for various physical phenomena involving Schrödinger, Dirac, and momentum operators with local point interactions; cf. [3, 35]. In particular, in

[1]This notation is much better suited for applications than the notation in [26, (1.14)], where $\mathfrak{t}_{M(\lambda)}$ was used for the nonclosed form (1.4).

Proposition 4.8 such an ES-generalized boundary triple occurs in the connection with momentum operators. It is shown therein that the domain and the form domain of the associated Weyl function $M(\cdot)$ admit the following explicit descriptions:

$$\operatorname{dom} M(\lambda) = l^2\big(\mathbb{N}; \{d_n^{-2}\}\big) = \operatorname{dom} \mathfrak{t}'_{M(\lambda)} \subsetneqq \operatorname{dom} \mathfrak{t}_{M(\lambda)} = l^2\big(\mathbb{N}; \{d_n^{-1}\}\big) = \operatorname{ran} \Gamma_0, \quad \lambda \in \mathbb{C}_\pm;$$

see (4.13), (4.14). Here $X = \{x_n\}_1^\infty \subset \mathbb{R}_+$ is a strictly increasing sequence of point interactions satisfying two conditions $\lim_{n\to\infty} x_n = \infty$ and $d_* := \inf_{n\in\mathbb{N}} d_n = 0$; here $d_n = x_n - x_{n-1} > 0$. Similar explicit descriptions for the domain and form domain of the function $M(\lambda)$ are also presented for local point interactions involving Schrödinger operators in Theorem 4.10, see formulas (4.19) – (4.21), as well as in the case of Dirac operators in Proposition 4.19; see (4.48), (4.49).

Here is a short description of the contents of Part II. Section 2 contains a couple of further useful results which are of preparatory nature for applications of unitary and, in particular, ES-generalized boundary triples. Namely, it is shown how certain simple transforms of B-generalized boundary triples generate ES-generalized boundary triples; see Theorems 2.1, 2.2. On the other hand, by applying some proper renormalization procedures for ES-generalized boundary triples one can produce more regularly behaving boundary mappings; cf. Theorem 2.6. The key to find appropriate kind of transforms and renormalization procedures is based on the behaviour of the corresponding Weyl functions under such transforms, and hence these constructions are basically motivated by the analytic properties of the associated Weyl functions. The connection of various classes of boundary pairs for nonnegative forms as defined in Post [62] to various subclasses of generalized boundary triples is established in Theorem 2.16. For instance, we show that the so-called elliptically regular boundary pair as introduced in [62] generates an S-generalized boundary triple with a nonnegative operator A and vice versa.

Section 3 is devoted to applications of the general results in the PDE setting by treating Laplace operators in smooth bounded domain in Theorem 3.1 and for Lipschitz domains in Proposition 3.7. Mixed boundary value problems for Laplacian are also considered and again an ES-generalized boundary triple occurs in the connection of so-called Zaremba Laplacian; see Theorem 3.5. Laplacian on rough domains is shown to lead to a multivalued boundary mapping Γ and its multivalued transposed mapping $\Gamma^\top$ (called here unitary boundary pairs) where the corresponding Weyl function can even be multivalued; see Theorem 3.12.

In Section 4 spectral problems for momentum, Schrödinger and Dirac operators with local point interactions are treated from the point of view of boundary triples technique. The new subclasses of generalized boundary triplets from Part I and the corresponding analytic properties of associated Weyl functions allow to complete the results of [58], [52], [53], and [22]. In particular, it is shown, see Proposition 4.8, Theorem 4.10, and Proposition 4.19, that in each of these three cases the Weyl function is domain invariant and form domain invariant and we describe explicitly all of these domains. In these applications to local point interactions the underlying abstract results become demonstrated in a concrete way and the obtained results simultaneously allow, for instance, a straightforward verification of the specific properties of the corresponding Weyl functions associated with the different types of generalized boundary triples occurring therein.

We devote this paper to our dear friend and excellent mathematician Hagen Neidhardt who passed away in March, 2019. One of us collaborated with Hagen a lot in applications of boundary triples technique to the spectral and scattering theory. It is a great loss for us as well as for the whole spectral theory community.

2. Some classes of ES-generalized boundary triples

2.1. **Transforms of B-generalized boundary triples.** Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be an isometric (or unitary) boundary triple for A^* with domain A_*. Then $\Pi^\top = \{\mathcal{H}, \Gamma_0^\top, \Gamma_1^\top\} := \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ defines the so-called *transposed boundary triple* for A^*. It is well known that in the particular case of an ordinary boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*, also the transposed boundary triple $\Pi^\top$ is an ordinary boundary triple for A^*. Moreover, if W is any bounded $J_\mathcal{H}$-unitary operator, then the composition

$$\begin{pmatrix} \Gamma_0^W \\ \Gamma_1^W \end{pmatrix} = W \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} \quad \left(J_\mathcal{H} = \begin{pmatrix} 0 & -iI_\mathcal{H} \\ iI_\mathcal{H} & 0 \end{pmatrix} \right)$$

generates also an ordinary boundary triple $\{\mathcal{H}, \Gamma_0^W, \Gamma_1^W\}$ for A^* and, conversely, all ordinary boundary triples of A^* are connected via some $J_\mathcal{H}$-unitary operator W to each other in this way; cf. [33, 27, 29], see also [11, 34, 48].

The situation changes essentially when $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is not an ordinary boundary triple for A^*. In this section we treat the simplest case of a B-generalized boundary triple and show that a simple $J_\mathcal{H}$-unitary transform can produce a boundary triple for A^* which is not B-generalized and not even S-generalized. More precisely, the next result shows how any B-generalized boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^*, which is not an ordinary boundary triple, can be transformed to an ES-generalized boundary triple, whose γ-field becomes unbounded.

Theorem 2.1. Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a B-generalized boundary triple for A^* with $A_* = \operatorname{dom}\Gamma \subset A^*$, $A_* \neq A^*$, let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and γ-field, and let $A_0 = \ker \Gamma_0$. Then:

(i) for every fixed $\nu \in \mathbb{C} \setminus \mathbb{R}$ the transform

$$\begin{pmatrix} \Gamma_0^\nu \\ \Gamma_1^\nu \end{pmatrix} = \begin{pmatrix} -\operatorname{Re} M(\nu) & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}$$

defines a unitary boundary triple for A^* whose Weyl function and γ-field are given by

$$M_\nu(\lambda) = -(M(\lambda) - \operatorname{Re} M(\nu))^{-1}, \quad \gamma_\nu(\lambda) = \gamma(\lambda)(M(\lambda) - \operatorname{Re} M(\nu))^{-1}, \tag{2.1}$$

and, moreover, $M_\nu(\lambda)$ and $\gamma_\nu(\lambda)$ are unbounded operators for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(ii) $\{\mathcal{H}, \Gamma_0^\nu, \Gamma_1^\nu\}$ is an ES-generalized boundary triple for A^* with $\operatorname{dom}\Gamma^\nu = A_*$ and, hence, $M_\nu(\lambda)$ is form domain invariant and $\gamma_\nu(\lambda)$ is closable for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) the Weyl function $M_\nu(\cdot)$ (equivalently the γ-field $\gamma_\nu(\cdot)$) is domain invariant on $\mathbb{C} \setminus \mathbb{R}$ if and only if

$$\mathfrak{N}_\mu(A_*) \subset \operatorname{ran}(A_{0,\nu} - \lambda) \quad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \quad \text{where } A_{0,\nu} = \ker \Gamma_0^\nu.$$

Proof. (i) & (ii) Since $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a B-generalized boundary triple for A^*, we have $M \in \mathcal{R}^s[\mathcal{H}]$, see [28, Proposition 5.7], i.e., M is a strict Nevanlinna function whose values $M(\lambda)$ are bounded operators on $\mathcal{H}$ with $\ker \operatorname{Im} M(\lambda) = 0$ for every $\lambda \in \rho(A_0)$. In particular, the real part $\operatorname{Re} M(\nu)$ is a bounded operator when $\lambda \in \rho(A_0)$. Therefore, Γ^ν is a standard $J_\mathcal{H}$-unitary transform of Γ. According to [29, Proposition 3.11] this implies that Γ^ν is a unitary boundary triple (a boundary relation in the terminology of [29]) with

$\operatorname{dom}\Gamma^\nu = \operatorname{dom}\Gamma$ whose Weyl function and γ-field are given by (2.1). The assumption $A_* \neq A^*$ is equivalent to $\operatorname{ran}\Gamma \neq \mathcal{H}^2$ and therefore $0 \notin \rho(\operatorname{Im} M(\lambda))$, $\lambda \in \rho(A_0)$; see [28, Section 2]. It follows from (2.1) that

$$M_\nu(\nu) = i(\operatorname{Im} M(\nu))^{-1} = -M_\nu(\nu)^* \tag{2.2}$$

and then (1.1) shows that for all $h, k \in \operatorname{dom} M_\nu(\nu) = \operatorname{dom}\gamma_\nu(\nu)$,

$$(\nu - \bar\nu)(\gamma_\nu(\nu)h, \gamma_\nu(\nu)k)_{\mathfrak{H}} = (M_\nu(\nu)h, k)_{\mathcal{H}} - (h, M_\nu(\nu)k)_{\mathcal{H}} = 2i((\operatorname{Im} M(\nu))^{-1}h, k)_{\mathcal{H}}.$$

Hence, $M_\nu(\nu)$ and $\gamma_\nu(\nu)$ are unbounded operators at the point $\nu \in \mathbb{C}\setminus\mathbb{R}$. In this case $M_\nu(\lambda)$ is an unbounded operator for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$; see [28, Proposition 4.18].

Next consider the γ-field $\gamma_\nu(\cdot)$. Since $M(\lambda) - \operatorname{Re} M(\nu)$ is bounded, it follows from (2.1) that

$$\gamma_\nu(\lambda)^* = (M(\bar\lambda) - \operatorname{Re} M(\nu))^{-1}\gamma(\lambda)^*, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

This combined with (2.2) shows that

$$\gamma_\nu(\nu)^* = i(\operatorname{Im} M(\nu))^{-1}\gamma(\nu)^*, \quad \gamma_\nu(\bar\nu)^* = -i(\operatorname{Im} M(\nu))^{-1}\gamma(\bar\nu)^*. \tag{2.3}$$

Since

$$\gamma(\nu)^*\gamma(\nu) = \gamma(\bar\nu)^*\gamma(\bar\nu) = (\operatorname{Im}\nu)^{-1}\operatorname{Im} M(\nu),$$

it follows from (2.3) that

$$\operatorname{ran}\gamma(\nu) \oplus \ker\gamma(\nu)^* \subset \operatorname{dom}\gamma_\nu(\nu)^*, \quad \operatorname{ran}\gamma(\bar\nu) \oplus \ker\gamma(\bar\nu)^* \subset \operatorname{dom}\gamma_\nu(\bar\nu)^*.$$

Hence, $\gamma_\nu(\nu)^*$ and $\gamma_\nu(\bar\nu)^*$ are densely defined operators, which means that $\gamma_\nu(\nu)$ and $\gamma_\nu(\bar\nu)$ are closable operators. According to [26, Theorem 1.14] $A_{0,\nu} = \ker\Gamma_0^\nu$ is essentially selfadjoint and the assertions in (ii) hold. The fact that $\gamma_\nu(\lambda)$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, is an unbounded operator is seen e.g. from [26, eq: (5.36)]. Thus, (i) is proven.

(iii) This assertion is obtained directly from [26, Proposition 3.11]. □

Theorem 2.1 will now be specialized to a situation that appears often in system theory and in PDE setting where typically the underlying minimal symmetric operator A is nonnegative; the simplest situation occurs when the lower bound is positive. The first part of the next result follows the general formulation given in [30, Proposition 7.41] which was motivated by the papers of V. Ryzhov; see [63] and the references therein.

Theorem 2.2. Let A_0 be a selfadjoint relation in $\mathfrak{H}$ with $\ker A_0 = \{0\}$, let E be selfadjoint operator in $\mathcal{H}$, and let the operator $G : \mathcal{H} \to \mathfrak{H}$ be bounded and everywhere defined with $\ker G = \{0\}$. Moreover, let

$$A_* = \{\{A_0^{-1}f' + G\varphi, f'\} : \ f' \in \operatorname{ran} A_0,\ \varphi \in \operatorname{dom} E\} \tag{2.4}$$

and define the operators $\Gamma_0, \Gamma_1 : A_* \to \mathcal{H}$ by

$$\Gamma_0\widehat{f} = \varphi, \quad \Gamma_1\widehat{f} = G^*f' + E\varphi; \quad \widehat{f} = \{A_0^{-1}f' + G\varphi, f'\} \in A_*. \tag{2.5}$$

Then:

(i) $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an S-generalized boundary triple for $A^* = \overline{A_*}$ with $\ker\Gamma_0 = A_0$. For $\lambda \in \rho(A_0)$ and $\varphi \in \operatorname{dom} E$ the corresponding γ-field and the Weyl function are given by

$$\gamma(\lambda)\varphi = (I - \lambda A_0^{-1})^{-1}G\varphi, \quad M(\lambda)\varphi = E\varphi + \lambda G^*(I - \lambda A_0^{-1})^{-1}G\varphi;$$

(ii) Π is a B-generalized boundary triple for A^* if and only if E is bounded;

(iii) Π is an ordinary boundary triple for A^* if and only if E is bounded and $G^*(\operatorname{ran} A_0) = \mathcal{H}$, in particular, then $\operatorname{ran} G$ must be closed;

(iv) the transform $\{\Gamma_1 - E\Gamma_0, -\Gamma_0\}$ defines an essentially unitary boundary triple for A^* whose unitary closure $\widetilde{\Pi} = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ is given by

$$\begin{pmatrix} \widetilde{\Gamma}_0 \\ \widetilde{\Gamma}_1 \end{pmatrix} \widehat{f} = \begin{pmatrix} G^* f' \\ -\varphi \end{pmatrix}, \quad \widehat{f} \in \operatorname{dom} \widetilde{\Gamma} = \{\{A_0^{-1} f' + G\varphi, f'\} : \, f' \in \operatorname{ran} A_0, \, \varphi \in \mathcal{H}\}, \tag{2.6}$$

and whose Weyl function and γ-field are given by

$$\widetilde{M}(\lambda) = -(M_0(\lambda))^{-1}, \quad \widetilde{\gamma}(\lambda) = \overline{\gamma(\lambda)}(M_0(\lambda))^{-1}, \text{ where} \tag{2.7}$$
$$M_0(\lambda) = \overline{(M(\lambda) - E)} = \lambda G^*(I - \lambda A_0^{-1})^{-1} G, \quad \overline{\gamma(\lambda)} = (I - \lambda A_0^{-1})^{-1} G,$$

and, moreover, the transposed boundary triple $\widetilde{\Pi}^\top$ is B-generalized with the Weyl function $M_0(\cdot)$;

(v) if $0 \in \rho(A_0)$ then $\widetilde{\Pi}$ is an ES-generalized boundary triple for A^* and it is S-generalized if and only if $\operatorname{ran} G$ is closed, or equivalently, $\operatorname{dom} \widetilde{\Gamma}$ in (2.6) is closed, i.e., if and only if Π is an ordinary boundary triple;

(vi) the Weyl function $\widetilde{M}$ (equivalently the γ-field $\widetilde{\gamma}(\cdot)$) is domain invariant on $\mathbb{C} \setminus \mathbb{R}$ if and only if

$$\operatorname{ran} P_G(I - \mu A_0^{-1})^{-1} G = \operatorname{ran} P_G(I - \lambda A_0^{-1})^{-1} G \quad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}, \tag{2.8}$$

where P_G stands for the orthogonal projection onto $\overline{\operatorname{ran}}\, G$.

Proof. (i) It was proved in [30, Prop. 7.41] that Π is a unitary boundary triple for $A^* = \overline{A_*}$ and for (i) it suffices to note that $\ker \Gamma_0 = A_0$ is selfadjoint by assumption. Hence, Π is an S-generalized boundary triple.

(ii) & (iii) The formula for Γ_0 shows that $\operatorname{ran} \Gamma_0 = \mathcal{H}$ precisely when $\operatorname{dom} E = \mathcal{H}$ or equivalently, E is bounded. Since

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} \widehat{f} = \begin{pmatrix} I & 0 \\ E & G^* \end{pmatrix} \begin{pmatrix} \varphi \\ f' \end{pmatrix} = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & G^* \end{pmatrix} \begin{pmatrix} \varphi \\ f' \end{pmatrix} \tag{2.9}$$

and in the last product the triangular operator is bounded with bounded inverse when E is bounded, we conclude that $\operatorname{ran} \Gamma = \mathcal{H} \times \mathcal{H}$ if and only if $\operatorname{dom} E = \mathcal{H}$ and the diagonal operator in (2.9) is surjective, i.e., $G^*(\operatorname{ran} A_0) = \mathcal{H}$; in this case $\operatorname{ran} G^* = \mathcal{H}$ and $\operatorname{ran} G$ is closed.

(iv) It is clear from (2.5) that the transform $\{\Gamma_1 - E\Gamma_0, -\Gamma_0\}$ has the same domain A_* as Γ. Moreover, using (2.5) it is straightforward to check that the closure $\{\widetilde{\Gamma}_0, \widetilde{\Gamma}_1\} = \operatorname{clos} \{\Gamma_1 - E\Gamma_0, -\Gamma_0\}$ is given by (2.6). In fact, the transposed boundary triple $\{\widetilde{\Gamma}_1, -\widetilde{\Gamma}_0\}$ is S-generalized and of the same form as Γ in (2.5) when $E = 0$, i.e., in view of (ii) it is even B-generalized. Applying (i) to this transposed boundary triple one also concludes that the Weyl function and γ-field of the boundary triple $\{\widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ are given by (2.7).

(v) It follows from (2.6) that $\widetilde{A}_0 = \ker \widetilde{\Gamma}_0$ is given by

$$\widetilde{A}_0 = \{\{A_0^{-1} f' + G\varphi, f'\} : \, f' \in \operatorname{ran} A_0, \, G^* f' = 0, \, \varphi \in \mathcal{H}\}. \tag{2.10}$$

Using graph expressions one can write $\widetilde{A}_0 = A_0 \cap (\mathcal{H} \times \ker G^*) \widehat{+} (\operatorname{ran} G \times \{0\})$ and now using the properties of adjoints it is seen that

$$\widetilde{A}_0^* = \operatorname{clos} (A_0 \widehat{+} \overline{\operatorname{ran}}\, G \times \{0\}) \cap (\mathcal{H} \times \ker G^*).$$

Observe that $A_0 \cap (\overline{\operatorname{ran}}\, G \times \{0\}) = 0$, since $\ker A_0 = \{0\}$. If $0 \in \rho(A_0)$ then $A_0 \widehat{+} \overline{\operatorname{ran}}\, G \times \{0\}$ is a closed subspace of $\mathfrak{H}^2$ and this implies that $\widetilde{A}_0^* = \overline{\widetilde{A}_0}$. Hence, $\widetilde{A}_0$ is essentially selfadjoint. Since $0 \in \rho(A_0)$, it is clear from (2.10) that $\operatorname{ran} G$ is closed if and only if $\widetilde{A}_0 = \ker \widetilde{\Gamma}_0$ is closed, or equivalently, $\operatorname{dom} \widetilde{\Gamma}$ in (2.6) is closed.

(vi) Using for $\widetilde{A}_0$ the formula (2.10) and the equalities $\mathfrak{N}_\mu(\operatorname{dom} \widetilde{\Gamma}) = \operatorname{ran} \widetilde{\gamma}(\mu) = \overline{\operatorname{ran} \gamma(\mu)}$ the domain invariance condition in [26, Proposition 3.11] can be rewritten as follows: for every $h \in \mathcal{H}$ there exist $h_0 \in \mathcal{H}$ and $f' \in \operatorname{ran} A_0 \cap \ker G^*$ such that

$$(I - \mu A_0^{-1})^{-1} G h = (I - \lambda A_0^{-1}) f' + G h_0$$

or, equivalently,

$$(I - \lambda A_0^{-1})^{-1} (I - \mu A_0^{-1})^{-1} G h = f' + (I - \lambda A_0^{-1})^{-1} G h_0, \quad \mu, \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Applying resolvent identity to the product term it is seen that the previous condition is equivalent to

$$(I - \mu A_0^{-1})^{-1} G h = f_1' + (I - \lambda A_0^{-1})^{-1} G h_1, \tag{2.11}$$

for some $h_1 \in \mathcal{H}$ and $f_1' \in \operatorname{ran} A_0 \cap \ker G^*$. This condition is equivalent to the inclusion

$$\operatorname{ran} P_G (I - \mu A_0^{-1})^{-1} G \subset \operatorname{ran} P_G (I - \lambda A_0^{-1})^{-1} G.$$

Since $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$ are arbitrary, this last condition coincides with the condition (2.8). $\square$

Remark 2.3. (i) The boundary triples Γ and $\widetilde{\Gamma}$ are completely determined by A_0 (= $\ker \Gamma_0 = \ker \widetilde{\Gamma}_1$) and the operators G and $E = E^*$. If, in particular, $0 \in \rho(A_0)$, then the Weyl function $\widetilde{M}(\cdot)$ in (2.7) is form domain invariant (see [26, Theorem 1.14]) and the γ-field $\gamma(\cdot)$ and the Weyl function $M(\cdot)$ as well as $\widetilde{M}(\cdot)$ (in the resolvent sense) admit holomorphic continuations to the origin $\lambda = 0$ with

$$\gamma(0) = G, \quad M(0) = E.$$

If, in addition, E is bounded and G has closed range, then $\widetilde{\Gamma} = \{\Gamma_1 - E\Gamma_0, -\Gamma_0\}$ is an ordinary boundary triple and the condition (2.8) is satisfied. Indeed, in this case $\operatorname{dom} M(\lambda) = \operatorname{ran} M_0(\lambda) = \mathcal{H}$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

(ii) If E is bounded, no closure is needed in part (iv), i.e., $\widetilde{\Gamma} = \{\Gamma_1 - E\Gamma_0, -\Gamma_0\}$. In this case, Γ is a B-generalized boundary triple and Theorem 2.2 can be seen as an extension of Theorem 2.1 to a point on the real line. Here the results are formulated for $\nu = 0$. They can easily be reformulated also for $\nu \in \mathbb{R}$. In addition, for $\nu = \infty$ the results in Theorem 2.2 can be translated to analogous results for range perturbations (instead of domain perturbations as in Theorem 2.2); for general background see [30, Section 7.5]. For $\nu = \infty$ the operator E appears as the limit value $M(\infty)$, while A_0 and A_* should be replaced by their inverses; see (2.15).

(iii) The criterion (2.8) for domain invariance of $\widetilde{M}$ can be derived also directly using $\operatorname{dom} \widetilde{M}(\lambda) = \operatorname{ran} M_0(\lambda)$ and the explicit formula for $M_0(\lambda)$ given in part (iv) of Theorem 2.2; see also the equivalent condition in (2.11).

When $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is not an ordinary boundary triple for A^*, the condition (2.8) fails to hold in general. In particular, if $\operatorname{ran} A_0 \cap \ker G^* = \{0\}$ (if e.g. $\ker G^* = \{0\}$), then the condition (2.8) is equivalent to

$$\operatorname{ran} (I - \mu A_0^{-1})^{-1} G = \operatorname{ran} (I - \lambda A_0^{-1})^{-1} G \quad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{2.12}$$

Multiplying this identity from the left by $\frac{\mu}{\lambda}(I-\lambda A_0^{-1})$ it is seen that (2.12) implies

$$\operatorname{ran}(I-\mu A_0^{-1})^{-1}G \subset \operatorname{ran} G \quad \text{for all } \mu \in \mathbb{C}\setminus\mathbb{R}. \tag{2.13}$$

Similarly it can be seen that (2.13) implies (2.12). Thus, if $\operatorname{ran} A_0 \cap \ker G^* = \{0\}$ then $\widetilde{M}$ is domain invariant if and only if the operator range $\operatorname{ran} G$ is invariant under the resolvent $(I-\mu A_0^{-1})^{-1}$ for all $\mu \in \mathbb{C}\setminus\mathbb{R}$.

Corollary 2.4. Assume that A_0, E and G are as in Theorem 2.2, such that $\operatorname{ran} A_0 \cap \ker G^* = \{0\}$ and $\operatorname{mul} A_0 = \{0\}$. If $A = (A_*)^*$ is densely defined, then the function $\widetilde{M}(\cdot)$ defined in (2.7) is not domain invariant.

Proof. Since $\operatorname{ran} A_0 \cap \ker G^* = \{0\}$, $\widetilde{M}(\cdot)$ is domain invariant if and only if (2.13) holds. In other words, for every $\varphi \in \mathcal{H}$ there exists $h \in \mathcal{H}$ such that $(I-\mu A_0^{-1})^{-1}G\varphi = Gh$, or, equivalently,

$$(I+\mu(A_0-\mu)^{-1})G\varphi = Gh \quad \Leftrightarrow \quad \mu(A_0-\mu)^{-1}G\varphi = G(h-\varphi). \tag{2.14}$$

If A is densely defined, then $A^* \supset A_*$ is an operator. Since $\ker A_0 = \{0\}$ one concludes from (2.4) that A_* is an operator if and only if $\operatorname{dom} A_0 \cap \operatorname{ran} G = \{0\}$. This condition applied to (2.14) implies that $\varphi = 0$ and $h - \varphi = 0$, since $\ker G = \{0\}$. This proves the claim. $\square$

If A_0 in Theorem 2.2 is nonnegative, one can specify further the type of the Weyl function as follows.

Corollary 2.5. Assume that in Theorem 2.2 $A_0 = A_0^* \geq 0$ and $E = E^* \leq 0$. Then the Weyl functions

$$M(\lambda)\varphi = E\varphi + \lambda G^*(I-\lambda A_0^{-1})^{-1}G\varphi, \quad M_0(\lambda) = \lambda G^*(I-\lambda A_0^{-1})^{-1}G, \quad \lambda \in \rho(A_0),$$

are domain invariant inverse Stieltjes functions, while $\widetilde{M}(\cdot) = -M_0(\cdot)^{-1}$ in (2.7) is a Stieltjes function.

Proof. Since A_0 is nonnegative and selfadjoint with $\ker A_0 = \{0\}$ and $E = E^* \leq 0$, the Weyl function $M(\lambda) = E + \lambda G^*(I-\lambda A_0^{-1})^{-1}G$ admits a holomorphic extrapolation to the negative real line and, moreover,

$$M(x) = E + xG^*(I-xA_0^{-1})^{-1}G = M(x)^* \leq 0 \quad \text{for all } x < 0.$$

Consequently $M(\cdot)$ and $M_0(\cdot)$ are inverse Stieltjes functions. Moreover, $\ker M(x) = \{0\}$ and $\ker M_0(x) = \{0\}$, since $\ker G = \{0\}$. In view of

$$(\widetilde{M}(\lambda)-\mu I)^{-1} = -(I+\mu M_0(\lambda))^{-1}M_0(\lambda), \quad \mu \in \mathbb{C}\setminus\mathbb{R},$$

also the function $\widetilde{M}(\cdot) = -M_0(\cdot)^{-1}$ admits a holomorphic extrapolation to the negative real line with nonnegative selfadjoint values therein, i.e., it is a Stieltjes function. $\square$

Let us also mention that analogously the function $\widehat{M}(\lambda) = -M(1/\lambda) = G^*(A_0^{-1} - \lambda)^{-1}G - E$ admits a holomorphic continuation to the negative real line and

$$\widehat{M}(x) = G^*(A_0^{-1}-x)^{-1}G - E = M(x)^* \geq 0 \quad \text{for all } x < 0 \tag{2.15}$$

with $\ker \widehat{M}(x) = \{0\}$. Hence, $\widehat{M}(\cdot)$ is a Stieltjes function and the transposed function $\widehat{M}^\top(\cdot) = -\widehat{M}(\cdot)^{-1}$ is an inverse Stieltjes function. Observe, that $\widehat{M}(\cdot)$ is the Weyl function of the boundary triple $\{\mathcal{H}, \widehat{\Gamma}_0, \widehat{\Gamma}_1\}$ given by

$$\widehat{\Gamma}_0 \widehat{f} = \varphi, \quad \widehat{\Gamma}_1 \widehat{f} = -G^* f' - E\varphi; \quad \widehat{f} = \{f', A_0^{-1} f' + G\varphi\} \in T^{-1}$$

with $\ker \widehat{\Gamma}_0 = A_0^{-1}$ and $\ker \widehat{\Gamma}_1 = A_1^{-1}$.

We now assume that $0 \in \rho(A_0)$ and make explicit the renormalization procedure as described in [26, Theorem 5.32] in the case of the ES-generalized boundary triple $\widetilde{\Pi}$ in Theorem 2.2 (v). This also yields a representation for the form domain invariant Weyl function $\widetilde{M}(\cdot)$ in (2.7); cf. [26, Prop. 5.34]. To state the result decompose the bounded inverse A_0^{-1} according to $\mathfrak{H} = \overline{\operatorname{ran}}\, G \oplus (\operatorname{ran} G)^\perp$ as $A_0^{-1} = (A_{ij}^-)_{i,j=1}^2$. This generates the following expression for an associated Schur complement of the resolvent $(A_0^{-1} - 1/\lambda)^{-1}$,

$$S_0(\lambda) = A_{11}^- - 1/\lambda I - (A_{21}^-)^*(A_{22}^- - 1/\lambda I)^{-1} A_{21}^-, \quad \lambda \in \rho(A_0). \tag{2.16}$$

Theorem 2.6. Let the notations and assumptions be as in Theorem 2.2 and, moreover, let $0 \in \rho(A_0)$ and assume that $\operatorname{ran} G$ is not closed, so that the ES-generalized boundary triple $\widetilde{\Pi}$ is not an S-generalized (or ordinary) boundary triple; see Theorem 2.2 (v). Then:

(i) the closure of the γ-field $\widetilde{\gamma}$ satisfies $\operatorname{dom} \overline{\widetilde{\gamma}(\lambda)} = \operatorname{ran} G^*$, $\lambda \in \rho(A_0)$;

(ii) the renormalized boundary triple $\Pi_r = \{\overline{\operatorname{ran}}\, G, \Gamma_{0,r}, \Gamma_{1,r}\}$, being constructed as in [26, Theorem 5.32], is an ordinary boundary triple for $A^* = A_0 \widehat{+} (\overline{\operatorname{ran}}\, G \times \{0\})$ and is determined by

$$\begin{pmatrix} \Gamma_{0,r} \\ \Gamma_{1,r} \end{pmatrix} \{A_0^{-1} f' + h, f'\} = \begin{pmatrix} P_G f' \\ -h \end{pmatrix}, \quad f' \in \mathfrak{H}, \quad h \in \overline{\operatorname{ran}}\, G, \tag{2.17}$$

where P_G denotes the orthogonal projection onto $\overline{\operatorname{ran}}\, G$;

(iii) the Weyl function $M_r(\cdot)$ of Π_r coincides with the Schur complement in (2.16),

$$M_r(\lambda) = S_0(\lambda), \quad \lambda \in \rho(A_0)$$

and the form domain invariant Weyl function $\widetilde{M}(\cdot)$ in (2.7) has the form

$$\widetilde{M}(\lambda) = G^{-1} S_0(\lambda) G^{-(*)}, \quad \lambda \in \rho(A_0), \tag{2.18}$$

where $G^{(*)}$ is the adjoint when G is treated as an operator from $\mathcal{H}$ into $\overline{\operatorname{ran}}\, G$.

Proof. (i) By (v) Theorem 2.2 $\widetilde{\gamma}(\lambda) = \overline{\gamma(\lambda)}(M_0(\lambda))^{-1}$. Using the expressions for $M_0(\lambda)$ in (2.7) and $S_0(\lambda)$ in (2.16) one obtains

$$\widetilde{M}(\lambda) = G^{-1} S_0(\lambda) G^{-(*)}, \quad \widetilde{\gamma}(\lambda) = -(I - \lambda A_0^{-1})^{-1} I_{\operatorname{ran} G} S_0(\lambda) G^{-(*)}, \tag{2.19}$$

where $G^{-(*)}$ stands for the inverse of G^*, when G^* is treated as an injective mapping from $\overline{\operatorname{ran}}\, G$ to $\mathcal{H}$. Since $(I - \lambda A_0^{-1})^{-1}$, $I_{\operatorname{ran} G}$, and $S_0(\lambda)$ are bounded with bounded inverse for $\lambda \in \rho(A_0)$, we conclude that the form domain of $\widetilde{M}(\lambda)$ is equal to $\operatorname{ran} G^*$ and that the closure of the γ-field is given by

$$\overline{\widetilde{\gamma}(\lambda)} = \frac{1}{\lambda} (A_0^{-1} - \frac{1}{\lambda} I)^{-1} P_G S_0(\lambda) G^{-(*)} = \frac{1}{\lambda} \begin{pmatrix} I \\ -(A_{22}^- - \frac{1}{\lambda} I)^{-1} A_{21}^- \end{pmatrix} G^{-(*)},$$

$\lambda \in \rho(A_0)$. Here the last identity uses the standard block formula for the inverse $(A_0^{-1} - 1/\lambda)^{-1}$.

(ii) The assumption $0 \in \rho(A_0)$ implies that the closure of $\operatorname{dom}\widetilde{\Gamma}$ is $A^* = A_0 \widehat{+} (\overline{\operatorname{ran}}\, G \times \{0\})$. In view of (i) one can use $G^* : \overline{\operatorname{ran}}\, G \to \mathcal{H}$ as the renormalizing operator in [26, Theorem 5.32]. Now in view of expression for $\widetilde{\Gamma}$ in (2.6) this renormalization gives the formula

$$\begin{pmatrix} \Gamma_{0,r} \\ \Gamma_{1,r} \end{pmatrix} \widehat{f} = \begin{pmatrix} P_G f' \\ -G\varphi \end{pmatrix}, \quad \widehat{f} \in \{\{A_0^{-1} f' + G\varphi, f'\} : f' \in \mathfrak{H}, \varphi \in \mathcal{H}\}. \tag{2.20}$$

The final expression for the renormalized boundary triple Π_r is obtained by taking closure in (2.20); this leads to (2.17), since $0 \in \rho(A_0)$. Now clearly $\operatorname{dom}\Gamma_r = A^*$ and $\operatorname{ran}\Gamma_r = \overline{\operatorname{ran}}\, G \times \overline{\operatorname{ran}}\, G$, i.e., Π_r is an ordinary boundary triple for A^*.

(iii) This follows from (2.19). For the equality $M_r(\lambda) = S_0(\lambda)$ take the closure of $G\widetilde{M}(\lambda)G^* \restriction \overline{\operatorname{ran}}\, G$. □

According to Theorem 2.6 $A_{0,r} = \ker \Gamma_{0,r}$ is selfadjoint. Clearly, $A_{0,r}$ coincides with the closure of $\widetilde{A}_0 = \ker \widetilde{\Gamma}_0$ in Theorem 2.2; see (2.10). If, in particular, A_0 is strictly positive, then $A_{0,r} = \ker \Gamma_{0,r}$ is the *Kreĭn-von Neumann extension* A_K of A and we have the following identities

$$\overline{\ker \widetilde{\Gamma}_0} = \overline{\widetilde{A}_0} = A_{0,r} = \ker \Gamma_{0,r} = A \widehat{+} (\overline{\operatorname{ran}}\, G \times \{0\}) = A_K, \tag{2.21}$$

where A is the range restriction of A_0: $A = \{\{A_0^{-1}f', f'\} : f' \in \mathfrak{H},\, G^* f' = 0\}$. Observe, that A is densely defined if and only if A^* is an operator, i.e.,

$$\overline{\operatorname{dom}}\, A = \mathfrak{H} \quad \Leftrightarrow \quad \overline{\operatorname{ran}}\, G \cap \operatorname{dom} A_0 = \{0\}.$$

By (2.18) $\widetilde{M}(\cdot)$ is domain invariant if and only if the dense set $S_0(\lambda)^{-1}(\operatorname{ran} G)$ does not depend on λ; in the particular case $\ker G^* = \{0\}$ this also leads to Corollary 2.4.

In Theorem 2.2 we regularized the S-generalized triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ via the transform $\{\Gamma_0, \Gamma_1 - E\Gamma_0\}$ before transposing the mappings and closing up. In fact, the closure of this regularized triple $\operatorname{clos}\{\Gamma_0, \Gamma_1 - E\Gamma_0\}$ is of the same form as Γ in (2.5) with $E = 0$ and it is B-generalized; see Theorem 2.2 (iv).

The next example shows what happens for the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ in Theorem 2.2 if it is transposed without the indicated regularization of the mapping Γ_1.

Example 2.7. Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be as defined in (2.5). Then $\ker \Gamma_0 = A_0$ is the unperturbed relation and

$$A_1 = \{\{A_0^{-1}f' + G\varphi, f'\} : G^*f' + E\varphi = 0,\, f' \in \operatorname{ran} A_0,\, \varphi \in \operatorname{dom} E\},$$

$$A = \{\{A_0^{-1}f', f'\} : f' \in \ker G^*\} = \{\{f, f'\} \in A_0 : f' \in \ker G^*\}.$$

In particular, if A_0 is an operator then A is a range restriction of A_0 to $\ker G^*$ with $n_\pm(A) = \dim(\operatorname{ran} G)$.

Now, assume that $\ker E = \{0\}$ and $\operatorname{ran} G^* \cap \operatorname{ran} E = \{0\}$. Then the identity $G^*f' + E\varphi = 0$ implies that $G^*f' = E\varphi = 0$ and, consequently, $\varphi = 0$ and this means that $A_1 = A$. This means that A_1 is not essentially selfadjoint and thus the transposed boundary triple $\{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is not ES-generalized. The corresponding Weyl function is given by

$$M^\top(\lambda) = -(E + \lambda G^*(I - \lambda A_0^{-1})^{-1}G)^{-1}$$

and according to [26, Theorem 1.14] it cannot be form domain invariant.

If, in addition, $\ker G^* = \{0\}$, then

$$\operatorname{dom} M^\top(\lambda) \cap \operatorname{dom} M^\top(\mu) = \{0\}, \quad \text{for all } \lambda \neq \mu, \quad \lambda, \mu \in \rho(A_0).$$

To see this assume that

$$g = (E + \lambda G^*(I - \lambda A_0^{-1})^{-1} G) f_1 = (E + \mu G^*(I - \mu A_0^{-1})^{-1} G) f_2$$

holds for some $g, f_1, f_2 \in \mathcal{H}$. Then

$$E(f_2 - f_1) = G^*[\lambda(I - \lambda A_0^{-1})^{-1} G f_1 - \mu(I - \mu A_0^{-1})^{-1} G f_2] \tag{2.22}$$

and the assumptions $\operatorname{ran} G^* \cap \operatorname{ran} E = \{0\}$ and $\ker E = \{0\}$ imply $f_1 = f_2$. Now $\ker G^* = \{0\}$ and an application of the resolvent identity on the righthand side of (2.22) yields $g = 0$.

If, in particular, A_0 is a nonnegative selfadjoint operator with $\ker A_0 = \{0\}$ and $E = E^* \leq 0$, then the function $M(\cdot)$ is an inverse Stieltjes function and the transposed function $M^\top(\cdot) = -M(\cdot)^{-1}$ is a Stieltjes function, which need not be form domain invariant; cf. Corollary 2.5. Analogously the function

$$-M(1/\lambda) = G^*(A_0^{-1} - \lambda)^{-1} G - E$$

is a Stieltjes function while $M(1/\lambda)^{-1}$ is an inverse Stieltjes function, which need not be form domain invariant.

Finally, it should be mentioned that later, in Section 3, it is shown how the standard Dirichlet and Neumann trace operators on smooth, as well as on Lipschitz, domains can be included in the abstract boundary triple framework constructed in Theorem 2.2; hence the previous results can be made explicit in PDE setting.

2.2. **Graph continuity of boundary mappings.** It is known that for a boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ (as well as for a boundary pair $\{\mathcal{H}, \Gamma\}$, see [26, Definition 3.1]) to be an ordinary boundary triple it is necessary and sufficient that both boundary mappings Γ_0 and Γ_1 are continuous on A^* (with the graph norm on $\operatorname{dom} A^*$ in case A is densely defined). In general the mappings Γ_0 and Γ_1 both can be unbounded when $\dim \mathcal{H} = \infty$. In this section we establish analytic criteria for Γ_0 or Γ_1 to be continuous with the aid of the associated Weyl function. Recall that the kernels $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ are always symmetric and it is possible that $A_0 = A$ or $A_1 = A$; see e.g. Example 2.7.

The next result characterizes boundedness of the mapping Γ_1 for an ES-generalized boundary triple.

Proposition 2.8. For a unitary boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with $A_* = \operatorname{dom} \Gamma$ the following conditions are equivalent:

(i) $A_0 = \ker \Gamma_0$ is essentially selfadjoint and Γ_1 is a bounded operator (w.r.t. the graph norm) on A_*;

(ii) A_0 is selfadjoint and the restriction $\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*)$ is a bounded operator for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) the form associated with $\operatorname{Im}(-M^{-1}(\lambda))$ has a positive lower bound for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

If one of the conditions is satisfied, then the triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is B-generalized.

Proof. (i) $\Rightarrow$ (ii) If Γ_1 is bounded, the also $\Gamma_1 \upharpoonright A_0$ and $\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*)$ are bounded. Now by [26, Corollary 5.6] A_0 is closed and, therefore, $A_0 = A_0^*$.

(ii) $\Rightarrow$ (iii) Observe that $(\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*))^{-1} = \widehat{\gamma}^\top(\lambda)$ is the γ-field of the transposed boundary triple $\Pi^\top = \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$. Hence the condition that $\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*)$ is bounded means that $(\gamma^\top(\lambda))^*\gamma^\top(\lambda)$ has a positive lower bound or, equivalently, that the form corresponding to $\mathrm{Im}\,(-M^{-1}(\lambda))$ has a positive lower bound (cf. [26, eq:(3.6) and Definition 5.21]).

(iii) $\Rightarrow$ (i) As shown in the previous implication, the assumption concerning $\mathrm{Im}\,(-M^{-1}(\lambda))$ means that the restriction $\Gamma_1 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*)$ is bounded. On the other hand, if the form corresponding to $\mathrm{Im}\,(-M^{-1}(\lambda))$ has a positive lower bound, say $c > 0$, then

$$\|M^{-1}(\lambda)f\|_{\mathcal{H}}\|f\|_{\mathcal{H}} \geq |(M^{-1}(\lambda)f, f)| \geq \mathrm{Im}\,(-(M^{-1}(\lambda)f, f)) \geq c\|f\|^2_{\mathcal{H}}.$$

Consequently, $\|M(\lambda)\| \leq c^{-1}$, i.e., $M(\cdot)$ is a bounded Nevanlinna function. Now by Theorem 1.3 A_0 is selfadjoint and hence according to [26, Corollary 5.6] the restriction $\Gamma_1 \upharpoonright A_0$ is bounded. Moreover, by selfadjointness of A_0, one has the decomposition $A_* = A_0 \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(A_*)$. Since the angle between A_0 and $\widehat{\mathfrak{N}}_\lambda(A_*)$ is positive, one concludes that Γ_1 is bounded on A_*. This completes the proof of the implication.

Finally, if one of the equivalent conditions (i)–(iii) holds then, as shown above, $M(\cdot)$ is a bounded Nevanlinna function. This is a necessary and sufficient condition for the boundary triple Π to be B-generalized. □

By passing to the transposed boundary triple gives the following analog of Proposition 2.8.

Proposition 2.9. For a unitary boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with $A_* = \mathrm{dom}\,\Gamma$ the following conditions are equivalent:

(i) $A_1 = \ker \Gamma_1$ is essentially selfadjoint and Γ_0 is a bounded operator (w.r.t. the graph norm) on A_*;

(ii) A_1 is selfadjoint and the restriction $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*)$ is a bounded operator for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(iii) the form associated with $\mathrm{Im}\, M(\lambda)$ has a positive lower bound for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

If one of these conditions is satisfied, then the transposed boundary triple $\Pi^\top = \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is B-generalized.

Remark 2.10. (i) For infinite direct sums of ordinary boundary triples the extensions $A_j = \ker \Gamma_j$, $j = 1, 2$, are automatically essentially selfadjoint; see [52, Theorem 3.2]. If, in addition, Γ_1 is bounded, then $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a B-generalized boundary triple for A^* by Proposition 2.8; this implication was proved in another way in [52, Proposition 3.6]; see also Corollary 4.6 below.

(ii) Note that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a B-generalized boundary triple if and only if the composition $\Gamma_1\widehat{\gamma}(\lambda)$ $(= M(\lambda))$ is bounded for some (equivalently for all) $\lambda \in \mathbb{C} \setminus \mathbb{R}$. In particular, if $\Gamma_1\widehat{\gamma}(\lambda)$ is bounded, then also the γ-field $\gamma(\lambda)$ itself is bounded (see [26, eq:(3.6)]), $A_0 = A_0^*$ (by Theorem 1.3) and the restriction $\Gamma_1 \upharpoonright A_0$ is also bounded (by [26, Corollary 5.6]). However, in this case Γ_1 need not be bounded. Therefore, the conditions in Proposition 2.8 are sufficient, but not necessary, for Π to be a B-generalized boundary triple. An example is any B-generalized boundary triple Π, which is not an ordinary boundary triple, such that also the transposed boundary triple $\Pi^\top$ is B-generalized, since then $\Pi^\top$ cannot be an ordinary boundary triple. Then the condition in (iii) of Proposition 2.8 is not satisfied. For an explicit example of such a B-generalized boundary triple, see local point interactions of Dirac operators treated in Proposition 4.17. Also the S-generalized boundary triple for the Laplace operator associated with the Dirichlet-to-Neumann map in Theorem 3.1 (i) does not satisfy the properties in Proposition 2.9, but the transposed boundary triple is B-generalized. On the other hand, the ES-generalized boundary triple associated with the Kreĭn - von Neumann Laplacian in Theorem 3.1 (ii) satisfies the conditions in Proposition 2.9 and the transposed boundary triple therein is B-generalized.

The boundedness of the component mappings Γ_0 and Γ_1 can be used to derive the following new characterization of ordinary boundary triples.

Proposition 2.11. For a unitary boundary triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with $A_* = \operatorname{dom} \Gamma$ the following conditions are equivalent:

(i) Γ_0 is bounded and $\operatorname{ran} \Gamma_0 = \mathcal{H}$;

(ii) Γ_1 is bounded and $\operatorname{ran} \Gamma_1 = \mathcal{H}$;

(iii) $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple.

Proof. (i) $\Rightarrow$ (iii) By [26, Corollary 5.15] $\operatorname{ran} \Gamma_0 = \mathcal{H}$ implies that $A_0 = A_0^*$ and Π is a B-generalized boundary triple. In particular, the Weyl function $M(\cdot)$ of Π belongs to the class $R^s[\mathcal{H}]$ of bounded strict Nevanlinna functions. On the other hand, $(\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*))^{-1} = \widehat{\gamma}(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Now $\Gamma_0 \upharpoonright \widehat{\mathfrak{N}}_\lambda(A_*)$ is bounded and this means that $\gamma(\lambda)^*\gamma(\lambda)$ has a positive positive lower bound or, equivalently, that $0 \in \rho(\operatorname{Im}(M(\lambda))$. Hence, $M(\cdot) \in R^u[\mathcal{H}]$ and Π is an ordinary boundary triple; see [28, Proposition 2.18].

(ii) $\Rightarrow$ (iii) Apply the previous implication to the transposed boundary triple.

(iii) $\Rightarrow$ (i), (ii) This is clear, since for ordinary boundary triple $\Gamma : A_* \to \mathcal{H}^2$ is bounded and surjective. $\square$

2.3. **Extrapolation of Weyl functions via a real regular point.** The main result here contains an analytic extrapolation principle for Weyl functions in the case when the underlying minimal operator A admits a regular type point on the real line $\mathbb{R}$. The proof relies on the so-called *main transform* of boundary relations (called here boundary pairs) introduced in [28]. The main transform makes a connection between subspaces of the Hilbert space $(\mathfrak{H} \oplus \mathcal{H})^2$ and linear relations from the Kreĭn space $(\mathfrak{H}^2, J_\mathfrak{H})$ to the Kreĭn space $(\mathcal{H}^2, J_\mathcal{H})$. It is a linear mapping $\mathcal{J}$ from $\mathfrak{H}^2 \times \mathcal{H}^2$ to $(\mathfrak{H} \oplus \mathcal{H})^2$ which establishes a one-to-one correspondence between all (closed) linear relations $\Gamma : \mathfrak{H}^2 \to \mathcal{H}^2$ and all (closed) linear relations $\widetilde{A}$ in $\widetilde{\mathfrak{H}} = \mathfrak{H} \oplus \mathcal{H}$ via

$$\Gamma \mapsto \widetilde{A} := \mathcal{J}(\Gamma) = \left\{ \left\{ \begin{pmatrix} f \\ h \end{pmatrix}, \begin{pmatrix} f' \\ -h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\}. \tag{2.23}$$

According to [28, Proposition 2.10] $\mathcal{J}$ establishes a one-to-one correspondence between the sets of contractive, isometric, and unitary relations Γ from $(\mathfrak{H}^2, J_\mathfrak{H})$ to $(\mathcal{H}^2, J_\mathcal{H})$ and the sets of dissipative, symmetric, and selfadjoint relations $\widetilde{A}$ in $\mathfrak{H} \oplus \mathcal{H}$, respectively. Recall that a boundary pair $\{\mathcal{H}, \Gamma\}$ is called *minimal*, if

$$\mathfrak{H} = \mathfrak{H}_{min} := \overline{\operatorname{span}}\, \{ \mathfrak{N}_\lambda(A_*) : \lambda \in \mathbb{C}_+ \cup \mathbb{C}_- \}.$$

The next result shows usefulness of the main transform for analytic extrapolation of Weyl functions $M(\cdot)$ from a single real point $x \in \mathbb{R}$ to the complex plane, when $x \in \hat{\rho}(A)$ is a regular type point of the minimal operator A. In the special case when the analytic extrapolation of $M(x)$ is a uniformly strict Nevanlinna function the extrapolation principle formulated for Weyl functions in the next theorem, yields a solution to the following general inverse problem: given a pair $\{\Gamma_0, \Gamma_1\}$ of boundary mappings from A^* to $\mathcal{H}$ determine the selfadjoint extension A_Θ of A (up to unitary equivalence) when the boundary condition $\Gamma_1 \widehat{f} = \Theta\Gamma_0 \widehat{f}$ is fixed by some operator Θ acting on $\mathcal{H}$. It is emphasized that for this result it suffices to know initially only the value of $M(x)$ at the single point $x \in \hat{\rho}(A)$. In this case the value $M(x)$ is defined in the same way as $M(\lambda)$ is defined for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (see [26, Definition 3.2]): $M(x) = \Gamma(\widehat{\mathfrak{N}}_x(A_*))$ or, more precisely,

$$M(x) := \left\{ \widehat{h} \in \mathcal{H}^2 : \{\widehat{f}_x, \widehat{h}\} \in \Gamma \ \text{ for some } \widehat{f}_x = \begin{pmatrix} f_x \\ x f_x \end{pmatrix} \in \mathfrak{H}^2, \ x \in \mathbb{R} \right\}. \tag{2.24}$$

Theorem 2.12. Let $\{\Gamma, \mathcal{H}\}$ be an isometric boundary pair for A^* with domain $A_* = \operatorname{dom}\Gamma$, $\operatorname{clos} A_* = A^*$, (i.e. Green's identity (1.1) holds for $\widehat{f}, \widehat{g} \in A_*$; cf. [26, Def. 3.1, eq:(3.1)]), let $\widetilde{A} = \mathcal{J}(\Gamma)$ be the main transform (2.23) of Γ. Assume that there exists a selfadjoint extension $H \subset A_* = \operatorname{dom}\Gamma$ of A with $x \in \rho(H) \cap \mathbb{R}$ and let the mapping $M(x)$ at this point x be defined by (2.24). Then the following assertions hold:

- *(i)* The following two conditions are equivalent:
  - (a) $M(x)$ is selfadjoint in $\mathcal{H}$ and $0 \in \rho(M(x) + xI)$;
  - (b) $x \in \rho(\widetilde{A}) \cap \mathbb{R}$.
- *(ii)* If the conditions (a), (b) in (i) hold then $\{\Gamma, \mathcal{H}\}$ is a unitary boundary pair for A^* and $M(x)$ admits an analytic extrapolation from the point x to the half-planes $\mathbb{C}_\pm$ as the Weyl family $M(\cdot)$ which necessarily belongs to the class $\widetilde{\mathcal{R}}(\mathcal{H})$ of Nevanlinna families, see Definition in [26, Section 2.1].
- *(iii)* If the boundary pair $\{\Gamma, \mathcal{H}\}$ is minimal then all the intermediate extensions A_Θ of A determined by $\Gamma(A_\Theta) = \Theta \Leftrightarrow A_\Theta = \Gamma^{-1}(\Theta)$ are, up to unitary equivalence, uniquely determined by $M(\cdot)$.

Proof. (i) To prove the equivalence of (a) and (b) consider the main transform $\widetilde{A}$ of Γ in (2.23). The range of $\widetilde{A} - x$ is given by

$$\operatorname{ran}(\widetilde{A} - xI) = \left\{ \begin{pmatrix} f' - xf \\ -h' - xh \end{pmatrix} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\}. \tag{2.25}$$

(a) $\Rightarrow$ (b) For $\widehat{f}_x \in \widehat{\mathfrak{N}}_x(A_*)$ one has $f'_x = x f_x$ and $\{h, h'\} \in M(x)$ by the definition in (2.24). Since $-x \in \rho(M(x))$ and $-h' - xh \in \operatorname{ran}(-M(x) - xI) = \mathcal{H}$ it follows from (2.25) that

$$\begin{pmatrix} 0 \\ \mathcal{H} \end{pmatrix} \subset \operatorname{ran}(\widetilde{A} - xI). \tag{2.26}$$

Since $H \subset \operatorname{dom}\Gamma$ and $x \in \rho(H) \cap \mathbb{R}$ one has $\operatorname{ran}(H - x) = \mathfrak{H}$ which combined with (2.25) and (2.26) shows that $\operatorname{ran}(\widetilde{A} - xI) = \mathfrak{H} \oplus \mathcal{H}$. This implies that $\widetilde{A}$ is a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$, since $\widetilde{A}$ is symmetric by isometry of Γ; cf. [28, Proposition 2.10]. In particular, $x \in \rho(\widetilde{A}) \cap \mathbb{R}$.

(b) $\Rightarrow$ (a) If $x \in \rho(\widetilde{A}) \cap \mathbb{R}$ then $\operatorname{ran}(\widetilde{A} - xI) = \mathfrak{H} \oplus \mathcal{H}$ and, in particular, (2.26) is satisfied. In view of (2.23) and (2.24) this means that $\{f, f'\} \in \widehat{\mathfrak{N}}_x(A_*)$ and $\{h, h'\} \in M(x)$ and therefore $\operatorname{ran}(-M(x) - xI) = \mathcal{H}$. On the other hand, it follows from (2.24) and Green's identity (1.1) that $M(x)$ is symmetric, i.e., $(h', h) = (h, h')$ for all $\{h, h'\} \in M(x)$. Therefore, $M(x)$ is selfadjoint and $-x \in \rho(M(x))$.

(ii) The proof of (i) shows that if (a) or, equivalently, (b) holds then $\widetilde{A}$ is a selfadjoint relation in $\mathfrak{H} \oplus \mathcal{H}$. Thus the (inverse) main transform $\Gamma = J^{-1}(\widetilde{A})$ defines is a unitary boundary pair $\{\Gamma, \mathcal{H}\}$ for A^*. By the main realization result in [28, Theorem 3.9] one concludes that $M \in \widetilde{\mathcal{R}}(\mathcal{H})$.

(iii) To prove this assertion first recall that according to [28, Theorem 3.9] the Weyl function uniquely determines Γ, as well as $\widetilde{A}$, by the minimality of Γ. Uniqueness of Γ here means that if there exists another minimal boundary pair $\{\mathcal{H}, \widetilde{\Gamma}\}$ associated with the symmetric operator $\widehat{A} = \ker \Gamma$ in some Hilbert space $\widehat{\mathfrak{H}}$ having the same Weyl function $M(\cdot)$, then there exists a standard unitary operator $U : \mathfrak{H} \to \widehat{\mathfrak{H}}$ such that

$$\widehat{\Gamma} = \left\{ \left\{ \begin{pmatrix} Uf \\ Uf' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ f' \end{pmatrix}, \begin{pmatrix} h \\ h' \end{pmatrix} \right\} \in \Gamma \right\}. \tag{2.27}$$

Hence, if the extension A_Θ of A in the Hilbert space $\mathfrak{H}$ and the extension $\widehat{A}_\Theta$ of $\widehat{A}$ in the Hilbert space $\widehat{\mathfrak{H}}$ are associated with the same "boundary condition" Θ then (2.27) implies that

$$\widehat{A}_\Theta = \{\{Uf, Uf'\} : \{f, f'\} \in A_\Theta\} = U A_\Theta U^{-1}.$$

Thus A_Θ and $\widehat{A}_\Theta$ are unitarily equivalent via the same unitary operator U for every linear relation Θ in $\mathcal{H}$. □

Remark 2.13. The proof of item (i) in Theorem 2.12 shows that (b) $\Rightarrow$ (a) without the assumption on the existence of a selfadjoint extension $H \subset A_*$ with $x \in \rho(H)$.

As to item (iii) of Theorem 2.12 it should be mentioned that if the analytic extrapolation $M(\cdot)$ belongs to the class $\mathcal{R}^u[\mathcal{H}]$, then each selfadjoint extension A_Θ ($\Theta = \Theta^*$) of A is uniquely (up to the unitary equivalence) defined by the Weyl function $M_\Theta(\cdot) \in \mathcal{R}[\mathcal{H}]$ as well as by the (non-orthogonal) spectral measure $\Sigma(t)$ from the integral representation of $M_\Theta(\cdot)$, see [20, 27, 29, 32, 33] for details.

Some further developments concerning uniqueness of boundary triples and connections between $\sigma(A_\Theta)$ and the spectral functions $\Sigma(t)$ can be found in [42].

Theorem 2.12 offers also a useful analytic tool to check whether an isometric boundary triple (or boundary pair) is actually unitary or, equivalently, if the Weyl function of some isometric boundary triple is in fact from the class $\mathcal{R}(\mathcal{H})$ of Nevanlinna functions. We use this result to construct a unitary boundary pair for Laplacians defined on rough domains in Section 3.4 and to associate unitary boundary triples with boundary pairs of nonnegative forms in the next subsection.

2.4. **Boundary pairs of nonnegative operators and boundary triples.** The notion of boundary pairs involves initially only one boundary map associated with a closed nonnegative form $\mathfrak{h}$ or a pair of nonnegative selfadjoint operators. The purpose in this section is to show that, after introducing a second boundary map Γ_1 (via the first Green's identity), the boundary pair $(\mathcal{H}, \widetilde{\Gamma}_0)$ generates a unitary boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$. Furthermore, various special cases of boundary pairs are connected to specific classes of unitary boundary triples. In applications to PDE's $\mathfrak{h}$ is often the Neumann form and in abstract setting the form $\mathfrak{h}_K$ associated to the Kreĭn extension A_K, which is the smallest nonnegative selfadjoint extension of A. The notion of a boundary pair can be seen to arise from the works of Kreĭn, Birman, and Višik and has been treated in later papers by G. Grubb (PDE setting) and Yu. M. Arlinskii (abstract setting).

A (basic) positive boundary pair $\{\mathcal{H}, \widetilde{\Gamma}_0\}$ involving the form domain of the Kreĭn extension was introduced in [8]. This notions leads to positive boundary triples $\{\mathcal{H}, \widetilde{\Gamma}_0, \Gamma_1\}$, where $\ker \widetilde{\Gamma}_0 = A_F$ and $A_K = \ker \Gamma_1$ are the Friedrichs and the Kreĭn extension of a nonnegative operator A; see [50, 7]. Boundary pairs which lead to B-generalized boundary triples appear in [9]. A more general class of boundary pairs $(\mathcal{H}, \widetilde{\Gamma}_0)$ has been studied recently by O. Post [62]; who relaxed the surjectivity condition on $\widetilde{\Gamma}_0$ and replaced it by the weaker requirement that $\operatorname{ran} \widetilde{\Gamma}_0$ is dense in $\mathcal{H}$. We recall the definition more explicitly here (using present notations):

Definition 2.14 ([62]). Let $\mathfrak{h}$ be a closed nonnegative form on a Hilbert space $\mathfrak{H}$ and let $\widetilde{\Gamma}_0$ be a bounded linear map from $\mathfrak{H}^1 := (\operatorname{dom} \mathfrak{h}, \|\cdot\|_1)$, where $\|f\|_1^2 = \mathfrak{h}(f) + \|f\|^2$, into another Hilbert space $\mathcal{H}$. Then $(\mathcal{H}, \widetilde{\Gamma}_0)$ is said to be a *boundary pair associated with the form* $\mathfrak{h}$, if:

(a) $(\mathfrak{H}^{1,D} :=) \ker \widetilde{\Gamma}_0$ is dense in $\mathfrak{H}$;

(b) $(\mathcal{H}^{1/2} :=) \operatorname{ran} \widetilde{\Gamma}_0$ is dense in $\mathcal{H}$.

A pair $(\mathcal{H}, \widetilde{\Gamma}_0)$ is said to be *bounded* if $\operatorname{ran} \widetilde{\Gamma}_0 = \mathcal{H}$, otherwise it is said to be unbounded.

Since $\widetilde{\Gamma}_0$ is bounded its kernel defines a closed restriction of the form $\mathfrak{h}$, which we denote here by $\mathfrak{h}_0(f) = \mathfrak{h}(f)$, $f \in \ker \widetilde{\Gamma}_0$. By assumption (a) the forms $\mathfrak{h}_0$ and $\mathfrak{h}$ are densely defined in $\mathfrak{H}$ and we denote by H_0 and H the nonnegative selfadjoint operators associated with the closed forms $\mathfrak{h}_0$ and $\mathfrak{h}$, respectively. Next we associate a symmetric operator and its adjoint with the boundary pair $(\mathcal{H}, \widetilde{\Gamma}_0)$ via

$$A := H_0 \cap H, \quad A^* = \operatorname{clos}\left(H_0 \mathbin{\widehat{+}} H\right).$$

In general, A need not be densely defined, in which case A^* is multivalued; in what follows we assume that A is densely defined. By definition H_0 and H are disjoint selfadjoint extensions of A. Recall that $\operatorname{dom} A^* = \operatorname{dom} H_0 \dot{+} \ker(A^* - \lambda I)$, $\lambda \in \rho(H_0)$, and there is similar decomposition with H. Since $\mathfrak{h}_0 \subset \mathfrak{h}$, one has $H \le H_0$ or, equivalently, $(H+a)^{-1} \ge (H_0 + a)^{-1}$ for all $a > 0$. Then one can write,

$$\operatorname{dom} H^{1/2} = \operatorname{dom} H_0^{1/2} + \operatorname{ran}\left((H+a)^{-1} - (H_0+a)^{-1}\right)^{1/2}, \tag{2.28}$$

and since clearly $\operatorname{ran}((H+a)^{-1} - (H_0+a)^{-1}) \subset \ker(A^* + a)$, one obtains

$$\operatorname{dom} \mathfrak{h} = \operatorname{dom} \mathfrak{h}_0 + (\mathfrak{N}_{-a} \cap \operatorname{dom} \mathfrak{h}), \quad a > 0; \tag{2.29}$$

This sum is not in general direct, since $\mathfrak{N}_{-a} \cap \operatorname{dom} \mathfrak{h}_0$ is nontrivial, whenever $H_0 \neq A_F$. Formulas (2.29) and (2.28) go back to the classical papers by Kreĭn [54] and Birman [19], respectively. Boundary triples approach to (2.28) as well as its further development including the case of operators with zero lower bound can be found in [56] (see also [43], [64, Theorem 14.24], and [34, Theorem 8.78]). A simple different proof of (2.28) was also given in [44, Lemma 2.2].

The sum in (2.29) can be made direct with a restriction on $\mathfrak{N}_\lambda$. As shown in [62, Propositions 2.9] the set of so-called weak solutions with a fixed $\lambda \in \mathbb{C}$ defined by

$$\mathfrak{N}^1_\lambda := \left\{ f \in \mathfrak{H}^1 : \mathfrak{h}(f,g) - \lambda(f,g)_{\mathfrak{H}} = 0, \, \forall\, g \in \operatorname{dom} \mathfrak{h}_0 \right\} \tag{2.30}$$

leads to the following direct sum decomposition for every $\lambda \in \rho(H_0)$:

$$\operatorname{dom} \mathfrak{h} = \operatorname{dom} \mathfrak{h}_0 \dot{+} \mathfrak{N}^1_\lambda. \tag{2.31}$$

Here $\mathfrak{N}^1_\lambda$ ($\subset \mathfrak{N}_\lambda \cap \operatorname{dom} \mathfrak{h}$) is closed in $\mathfrak{H}^1$, $\mathfrak{N}^1_\lambda$ is dense in $\ker(A^* - \lambda)$, and $\mathfrak{N}^1_\lambda \cap \operatorname{dom} \mathfrak{h}_0 = \{0\}$. The restriction $\widetilde{\Gamma}_0 \upharpoonright \mathfrak{N}^1_\lambda$ is a bounded operator from $\mathfrak{N}^1_\lambda$ into $\mathcal{H}$ and the decomposition (2.31) implies that it is injective and its range is equal to $\operatorname{ran} \widetilde{\Gamma}_0$. The inverse operator

$$S(\lambda) := (\widetilde{\Gamma}_0 \upharpoonright \mathfrak{N}^1_\lambda)^{-1} : \mathcal{H}^{1/2} \to \mathfrak{N}^1_\lambda \tag{2.32}$$

is closed as an operator from $\mathcal{H}$ to $\mathfrak{H}^1$ with domain $\mathcal{H}^{1/2} = \operatorname{ran} \widetilde{\Gamma}_0$.

Definition 2.15 ([62])**.** The boundary pair $(\mathcal{H}, \widetilde{\Gamma}_0)$ associated with the form $\mathfrak{h}$ is said to be *elliptically regular*, if the operator $S := S(-1)$ is bounded as an operator from $\mathcal{H}$ to $\mathfrak{H}$, i.e. $\|Sh\|_{\mathfrak{H}} \le C\|h\|_{\mathcal{H}}$ for all $h \in \mathcal{H}^{1/2}$ and some $C \ge 0$. Moreover, the boundary pair $(\mathcal{H}, \widetilde{\Gamma}_0)$ is said to be *(uniformly) positive*, if there is a constant $c > 0$, such that $\|Sh\|_{\mathfrak{H}} \ge c\|h\|_{\mathcal{H}}$ for all $h \in \mathcal{H}^{1/2}$.

Let $\lambda = -1$ and define the form $\mathfrak{l}[h,k]$ on $\mathcal{H}$ by

$$\mathfrak{l}[h,k] = (Sh, Sk)_{\mathfrak{H}^1}, \quad h,k \in \mathcal{H}^{1/2}.$$

The form $\mathfrak{l}$ is closed in $\mathcal{H}$, since $S : \mathcal{H} \to \mathfrak{H}^1$ is a closed operator. By the first representation theorem, see [49], there is a unique selfadjoint operator Λ in $\mathcal{H}$ characterized by the equality

$$\mathfrak{l}[h,k] = (\Lambda h, k)_{\mathcal{H}}, \qquad h \in \operatorname{dom} \Lambda, \quad k \in \operatorname{dom} \mathfrak{l} = \mathcal{H}^{1/2}. \tag{2.33}$$

It is clear that $\Lambda = S^*S$, where $S^* : \mathfrak{H}^1 \to \mathcal{H}$ is the usual Hilbert space adjoint. The operator Λ is called the *Dirichlet-to-Neumann operator* at the point $\lambda = -1$ associated with the boundary pair $(\mathcal{H}, \widetilde{\Gamma}_0)$. The *(strong) Dirichlet-to-Neumann operator* at a point $\lambda \in \rho(H_0)$ is defined as follows ([62, Section 2.4]):

$$\operatorname{dom} \Lambda(\lambda) := \left\{ \varphi \in \mathcal{H}^{1/2} : \exists \psi \in \mathcal{H} \text{ s.t. } (\mathfrak{h} - \lambda)(S(\lambda)\varphi, S\eta) = (\psi, \eta)_{\mathcal{H}}, \, \forall \eta \in \mathcal{H}^{1/2} \right\} \tag{2.34}$$

and then $\Lambda(\lambda)\varphi := \psi$. The operator $\Lambda(\lambda)$ is closed in $\mathcal{H}$ and it has bounded inverse $\Lambda(\lambda)^{-1} \in \mathcal{B}(\mathcal{H})$ for all $\lambda \in \rho(H_0)$; see [62, Proposition 2.17].

Next consider the restriction of A^* to the form domain of $\mathfrak{h}$

$$\mathfrak{H}^1_0 := \left\{ f \in \mathfrak{H}^1 \cap \operatorname{dom} A^* : \mathfrak{h}(f,g) = (A^*f, g)_{\mathfrak{H}}, \, \forall\, g \in \operatorname{dom} \mathfrak{h}_0 \right\} \tag{2.35}$$

and equip it with the norm defined by $\|f\|^2_{\mathfrak{H}^1_0} = \mathfrak{h}(f) + \|f\|^2 + \|A^*f\|^2$, which makes $\mathfrak{H}^1_0$ a Hilbert space. Now using the rigged Hilbert space $\mathcal{H}^{1/2} \subset \mathcal{H} \subset \mathcal{H}^{-1/2}$ introduce a bounded operator $\check{\Gamma}_1 : \mathfrak{H}^1_0 \to \mathcal{H}^{-1/2}$ such that

$$(\check{\Gamma}_1 f, \widetilde{\Gamma}_0 g)_{-1/2,1/2} = (A^*f, g)_{\mathfrak{H}} - \mathfrak{h}(f,g) \tag{2.36}$$

holds for all $f \in \mathfrak{H}^1_0$, $g \in \mathfrak{H}^1$; this map is well defined by the formulas (2.35), (2.36). Finally, introduce the restriction A_* of A^* by

$$\operatorname{dom} A_* := \left\{ g \in \mathfrak{H}^1_0 : \check{\Gamma}_1 g \in \mathcal{H} \right\}$$

and denote $\Gamma_0 = \widetilde{\Gamma}_0 \upharpoonright \operatorname{dom} A_*$, $\Gamma_1 = \check{\Gamma}_1 \upharpoonright \operatorname{dom} A_*$. By definition (the first Green's identity)

$$\mathfrak{h}(f,g) = (A^* f, g)_{\mathfrak{H}} - (\Gamma_1 f, \widetilde{\Gamma}_0 g)_{\mathcal{H}} \tag{2.37}$$

holds for all $f \in \operatorname{dom} A_*$ and $g \in \mathfrak{H}^1$. In what follows the triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ with the domain $\operatorname{dom} A_* = \operatorname{dom} \Gamma_0 \cap \operatorname{dom} \Gamma_1$ is called a boundary triple generated by the boundary pair $(\mathcal{H}, \widetilde{\Gamma}_0)$. The next result characterizes the central properties of the boundary pair $(\mathcal{H}, \widetilde{\Gamma}_0)$ by means of the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$. In particular, it shows that the notion of boundary pair in Definition 2.14 can be included in the framework of unitary boundary triples whose Weyl functions are Nevanlinna functions from the class $\mathcal{R}^s(\mathcal{H})$.

Theorem 2.16. Let $(\mathcal{H}, \widetilde{\Gamma}_0)$ be a boundary pair for the closed nonnegative form $\mathfrak{h}$ in $\mathfrak{H}$, let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the corresponding triple as defined above and let $S(\cdot)$ and $\Lambda(\cdot)$ be defined by (2.32) and (2.33), respectvely. Then:

(i) $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unitary boundary triple for A^*;

(ii) $A_0 := A^* \upharpoonright \ker \Gamma_0$ is a symmetric restriction of H_0, while $A_1 := A^* \upharpoonright \ker \Gamma_1$ is selfadjoint and equal to H;

(iii) the γ-field and the Weyl function $M(\cdot)$ of the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ are given by
$$\gamma(\lambda) = S(\lambda) \upharpoonright \operatorname{dom} \Lambda(\lambda), \quad M(\lambda) = -\Lambda(\lambda), \quad \lambda \in \rho(H_0);$$

(iv) the transposed triple $\{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is a B-generalized boundary triple for A^*;

(v) $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is ES-generalized, i.e., $\operatorname{clos} A_0 = H_0$ if and only if $S(\lambda)$ is closable when treated as an operator from $\mathcal{H} \to \mathfrak{H}$ for some (equivalently for all) $\lambda \in \rho(H_0)$;

(vi) $(\widetilde{\Gamma}_0, \mathcal{H})$ is elliptically regular if and only if $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an S-generalized boundary triple;

(vii) $(\widetilde{\Gamma}_0, \mathcal{H})$ is uniformly positive if and only if $\Gamma_0 : A_* \to \mathcal{H}$ is a bounded operator (w.r.t. the graph norm on A_*) or, equivalently, the form $\mathfrak{t}_{M(\lambda)}$ has a positive lower bound for some (equivalently for every) $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(viii) $(\widetilde{\Gamma}_0, \mathcal{H})$ is bounded if and only if $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a B-generalized boundary triple;

(ix) $(\widetilde{\Gamma}_0, \mathcal{H})$ is bounded and uniformly positive if and only if $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple.

Proof. (i) First observe that the first Green's identity (2.37) applied to $h[f,g]$ and $\overline{h[g,f]}$ with $f, g \in \operatorname{dom} A_*$ leads to the second Green's identity (1.1) by symmetry of the form $\mathfrak{h}$. The second Green's identity (1.1) implies that the restrictions $A_0 = A^* \upharpoonright \ker \Gamma_0$ and $A_1 = A^* \upharpoonright \ker \Gamma_1$ are symmetric operators extending A.

Next we prove that the (graph) closure of A_* is A^* and that $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unitary boundary triple for A^*. It is clear from (2.35) that the set of weak solutions $\mathfrak{N}^1_\lambda$ belongs to $\mathfrak{H}^1_0$. Since H_0 is the selfadjoint operator associated with the form $\mathfrak{h}_0$ by the first representation theorem of Kato and $\mathfrak{h}_0 \subset \mathfrak{h}$, one concludes from (2.35) that $\operatorname{dom} H_0 \subset \mathfrak{H}^1_0$. Similarly H is the selfadjoint operator associated with the form $\mathfrak{h}$ and, hence also $\operatorname{dom} H \subset \mathfrak{H}^1_0$. Now applying (2.36) with $f \in \operatorname{dom} H$, $g \in \mathfrak{H}^1$ and taking into account that $\operatorname{ran} \Gamma_0$ is dense in $\mathcal{H}$ by assumption (b) in Definition 2.14 one concludes that $\check{\Gamma}_1 f = 0$. Hence, $\operatorname{dom} H \subset \operatorname{dom} A_*$ and $\Gamma_1(\operatorname{dom} H) = \{0\}$. Thus, $H \subset A_1$ and since A_1 is symmetric this implies that $A_1 = H$ is selfadjoint. Now consider the operator $\Lambda = S^* S$. Since $\operatorname{dom} \Lambda$ is a core for the form $\mathfrak{l}$ it is also a core for the operator S. This implies that $S(\operatorname{dom} \Lambda)$ is dense in $\mathfrak{N}^1_{-1}$ w.r.t. the topology in $\mathfrak{H}^1$, since S has bounded inverse. We claim that

$S(\operatorname{dom}\Lambda) \subset \operatorname{dom} A_*$. To see this consider the form

$$\mathfrak{h}(f,g) - (A^* f, g)_{\mathfrak{H}}, \qquad f \in \mathfrak{H}_0^1, \quad g \in \mathfrak{H}^1. \tag{2.38}$$

Notice that $\mathfrak{N}_\lambda^1 \subset \mathfrak{H}_0^1$, see (2.30), (2.35), and that the decomposition (2.31) for $\lambda = -1$ is orthogonal in $\mathfrak{H}^1$. Hence, one can write $g = g_0 + g_1 \in \operatorname{dom} \mathfrak{h}_0 \oplus_1 \mathfrak{N}_{-1}^1$, $g \in \mathfrak{H}^1 = \operatorname{dom} \mathfrak{h}$. Now for $h \in \operatorname{dom}\Lambda$ one has $Sh \in \mathfrak{N}_{-1}^1$ and for all $g = g_0 \in \operatorname{dom} \mathfrak{h}_0 = \ker \Gamma_0$,

$$\mathfrak{h}(Sh, g_0) - (A^* Sh, g_0)_{\mathfrak{H}} = \mathfrak{h}(Sh, g_0) + (Sh, g)_{\mathfrak{H}} = (Sh, g_0)_{\mathfrak{H}^1} = 0.$$

On the other hand, when $g = g_1 \in \mathfrak{N}_{-1}^1$, then $k = \Gamma_0 g_1 \in \mathcal{H}^{1/2}$ satisfies $g_1 = Sk$. This leads to

$$\begin{aligned}\mathfrak{h}(Sh, g_1) - (A^* Sh, Sk)'_{\mathfrak{H}} &= \mathfrak{h}(Sh, Sk) + (Sh, Sk)_{\mathfrak{H}} = (Sh, Sk)_{\mathfrak{H}^1} \\ &= (\Lambda h, k)_{\mathfrak{H}} = (\Lambda h, \Gamma_0 g_1)_{\mathfrak{H}}.\end{aligned}$$

One concludes that for $f = Sh$, $h \in \operatorname{dom}\Lambda$, and all $g \in \mathfrak{H}^1$ the form (2.38) can be rewritten as follows

$$\mathfrak{h}(Sh, g) - (A^* Sh, g)_{\mathfrak{H}} = (\Lambda h, \Gamma_0 g)_{\mathfrak{H}}.$$

Comparing this with (2.36) leads to $\check{\Gamma}_1 Sh = \Gamma_1 Sh = -\Lambda h \in \mathcal{H}$, which proves the claim $S(\operatorname{dom}\Lambda) \subset \operatorname{dom} A_*$.

Since $S(\operatorname{dom}\Lambda)$ is dense in $\mathfrak{N}_{-1}^1$ and $\operatorname{dom} H \subset \operatorname{dom} A_*$, the closure of A_* is equal to the closure of $H + \widehat{\mathfrak{N}}_{-1}^1$, which coincides with A^*. Hence, the domain of $\{\Gamma_0, \Gamma_1\}$ is dense in $\operatorname{dom} A^*$ w.r.t. the graph topology. As was shown above $\Gamma_1 Sh = -\Lambda h$ for all $h \in \operatorname{dom}\Lambda$ and, in addition, $\Gamma_0 Sh = h$. Since $S(\operatorname{dom}\Lambda) \subset \mathfrak{N}_{-1}(A_*)$ this implies that for the regular point $\lambda = -1 \in \rho(H)$ one has $-\Lambda \subset M(-1)$.

Here equality $M(-1) = -\Lambda$ prevails, since $M(-1)$ is necessarily symmetric by Green's identity (1.1). Clearly, $M(-1) - I = -\Lambda - I \le -I$ and thus $0 \in \rho(M(-1) - I)$. Therefore, we can apply Theorem 2.12 to conclude that $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unitary boundary triple for A^* with dense domain A_*.

(ii) The equality $A_1 = H$ was already proved in item (i). Next we prove the inclusion $A_0 \subset H_0$. The first Green's identity (2.37) shows that

$$\mathfrak{h}(f,g) = (A^* f, g)_{\mathfrak{H}}, \quad \text{for all } f \in \operatorname{dom} A_*, \quad g \in \ker \widetilde{\Gamma}_0 = \operatorname{dom} \mathfrak{h}_0. \tag{2.39}$$

If, in particular, $f \in \operatorname{dom} A_0$ i.e. $\Gamma_0 f = 0$, then $f \in \operatorname{dom} \mathfrak{h}_0$ and (2.39) can be rewritten as

$$\mathfrak{h}_0(f,g) = (A_0 f, g)_{\mathfrak{H}}, \quad \text{for all } g \in \operatorname{dom} \mathfrak{h}_0.$$

Now by the first representation theorem (see [49]) one has $f \in \operatorname{dom} H_0$ and $A_0 f = H_0 f$. Thus, $A_0 \subset H_0$.

(iii) It was shown in part (i) that $\operatorname{ran} S(\lambda) = \mathfrak{N}_\lambda^1 \subset \mathfrak{H}_0^1$ for each $\lambda \in \rho(H_0)$. Now assume in addition that $h \in \operatorname{dom}\Lambda(\lambda)$ and let $g \in \mathfrak{H}^1$. Then the definition of $\Lambda(\lambda)$ shows that

$$\mathfrak{h}(S(\lambda)h, g) - (A^* S(\lambda)h, g)_{\mathfrak{H}} = (\mathfrak{h} - \lambda I)[S(\lambda)h, g] = (\Lambda(\lambda)h, \Gamma_0 g)_{\mathfrak{H}}.$$

Comparing this with (2.36) gives $\check{\Gamma}_1 S(\lambda)h = \Gamma_1 S(\lambda)h = -\Lambda(\lambda)h \in \mathcal{H}$. Hence $S(\operatorname{dom}\Lambda(\lambda)) \subset \operatorname{dom} A_*$ and, moreover, one has $M(\lambda)h = -\Lambda(\lambda)h$. Therefore, $-\Lambda(\lambda) \subset M(\lambda)$, $\lambda \in \rho(H_0)$. Equivalently, $\Lambda(\lambda)^{-1} \subset -M(\lambda)^{-1}$ and since $M(\cdot)$ is the Weyl function of a single-valued unitary boundary triple, $M(\cdot) \in \mathcal{R}^s(\mathcal{H})$, in particular, $\ker M(\lambda) = \{0\}$; see (1.2). On the other hand, $\Lambda(\lambda)^{-1} \in \mathcal{B}(\mathcal{H})$ and, hence, the equality $\Lambda(\lambda)^{-1} = -M(\lambda)^{-1}$ follows. The equality $\gamma(\lambda) = S(\lambda) \upharpoonright \operatorname{dom} M(\lambda)$ is clear. The formulas for $\gamma(\lambda)$ and $M(\lambda)$ are proven.

(iv) Since $\Lambda(\cdot)^{-1} \in \mathcal{B}(\mathcal{H})$ and $-M(\lambda)^{-1} = \Lambda(\cdot)^{-1}$ by part (iii) the transposed boundary triple is B-generalized; see [33, Theorem 6.1].

(v) By definition $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is ES-generalized if and only if A_0 is essentially selfadjoint, which in view of (ii) means that $\operatorname{clos} A_0 = H_0$. On the other hand, by [26, Theorem 1.14] $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is ES-generalized if and only if $\gamma(\lambda)$ is closable for some (equivalently for all) $\lambda, \overline{\lambda} \in \rho(H_0)$.

Since $\gamma(\lambda) \subset S(\lambda)$, it is clear that if $S(\lambda)$ is closable then also $\gamma(\lambda)$ is closable. On the other hand, it follows from [62, Theorems 2.11, Proposition 2.17] that $\operatorname{dom} \Lambda(\lambda)$ is dense w.r.t. the $\mathcal{H}^{1/2}$-topology on $\mathcal{H}^{1/2}$ and that

$$\overline{S(\lambda)\upharpoonright \operatorname{dom} \Lambda(\lambda)}^{\mathcal{H}^{1/2}\to\mathfrak{H}^1} = S(\lambda),$$

since $S(\lambda) : \mathcal{H}^{1/2} \to \mathfrak{N}^1_\lambda$ is a topological isomorphism. Since the topologies on $\mathcal{H}^{1/2}$ and $\mathfrak{H}^1$ are stronger than the topologies on $\mathcal{H}$ and $\mathfrak{H}$ it follows that if $\gamma(z) : \mathcal{H} \to \mathfrak{H}$ is closable, then also $S(z) : \mathcal{H} \to \mathfrak{H}$ is closable and

$$\overline{\gamma(\lambda)}^{\mathcal{H}\to\mathfrak{H}} = \overline{S(\lambda)}^{\mathcal{H}\to\mathfrak{H}}.$$

(vi) When $(\widetilde{\Gamma}_0, \mathcal{H})$ is elliptically regular, then $S : \mathfrak{H}^1 \to \mathcal{H}$ is a bounded operator. Then equivalently the γ-field $\gamma(\lambda)$ is bounded for all $\lambda \in \rho(H_0)$, cf. [62, Theorem 2.11], and the statement is obtained from Theorem 1.3.

(vii) If $(\widetilde{\Gamma}_0, \mathcal{H})$ is (uniformly) positive then $S(\lambda)$, $\lambda \in \rho(H_0)$ is bounded from below; cf. [62, Theorem 2.11]. The connection between the γ-field and the Weyl function (see [26, eq:(3.10)]) combined with (1.4) gives

(2.40)
$$(\gamma(\lambda)u, \gamma(\lambda)v) = \frac{(M(\lambda)u, v) - (u, M(\lambda)v)}{\lambda - \bar{\lambda}} = \mathfrak{t}_{M(\lambda)}[u, v], \quad u, v \in \operatorname{dom} M(\lambda),\ \lambda \in \mathbb{C} \setminus \mathbb{R},$$

and this implies that the form $\mathfrak{t}_{M(\lambda)}$ has a positive lower bound. Now the statement follows from Proposition 2.9, because $A_1 = H$ is selfadjoint by part (iii).

(viii) If $(\widetilde{\Gamma}_0, \mathcal{H})$ is bounded, i.e., $\operatorname{ran} \widetilde{\Gamma}_0 = \mathcal{H}^{1/2} = \mathcal{H}$, then $S : \mathcal{H} \to \mathfrak{H}^1$ is closed (as an inverse of a bounded operator $\widetilde{\Gamma}_0 \upharpoonright \mathfrak{N}^1_{-1}$), everywhere defined, and bounded by the closed graph theorem. In particular, $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a B-generalized boundary triple. On the other hand, we conclude that the form $(\mathfrak{h}+1)(Sh, Sk)$ is closed and defined everywhere on $\mathcal{H}$. Now it follows from (2.34) that $\operatorname{dom} \Lambda(-1) = \mathcal{H}$. This implies that $M(\cdot) \in \mathcal{R}^s[\mathcal{H}]$; see e.g. (1.3) in Theorem 1.3. Therefore, $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a B-generalized boundary triple; see [33, Theorem 6.1].

The converse statement is clear, since $\operatorname{ran} \Gamma_0 = \mathcal{H}$ implies that also $\operatorname{ran} \widetilde{\Gamma}_0 = \mathcal{H}$.

(ix) This follows directly e.g. from Proposition 2.11. Alternatively, by (vii) and (viii) the conditions mean that $M(\cdot) \in \mathcal{R}^u[\mathcal{H}]$, which characterizes ordinary boundary triples (cf. [26, Theorem 1.4]). $\square$

Remark 2.17. (a) Characterizations (viii) and (ix) have been announced (without proofs) in [62, Theorem 1.8]. Moreover, elliptic regularity has been characterized in [62, Theorem 1.8] using equivalence to quasi boundary triples. However, as indicated the conditions defining a quasi boundary triple are not sufficient to guarantee that the corresponding Weyl function belongs to the class of Nevanlinna functions. In this sense the characterization of elliptic regularity presented in (vi) is more precise and complete. As to (vii) a characterization of positive boundary pairs via uniform positivity of the form valued function $z \to -\mathfrak{l}_z$ appears in [62, Theorem 3.13], while the other characterization that $\Gamma_0 : A_* \to \mathcal{H}$ is a bounded operator, as well as the statements (i) – (v) in Theorem 2.16 are obviously new.

(b) Since H_0 and H are nonnegative selfadjoint operators, the Weyl functions $M(\cdot)$ and $-M(\cdot)^{-1}$ admit analytic continuations (in the resolvent sense) to the negative real line. In fact, $M(\cdot)$ belongs to the class of operator valued (in general unbounded) *inverse Stieltjes functions*, while $-M(\cdot)^{-1}$ belongs to the class of operator valued *Stieltjes functions*. These facts follow from the following formula:

$$(M(x)h, h) = -(\mathfrak{h} - x)[(H_0 + 1)(H_0 - x)^{-1}h, h] \leq 0, \quad h \in \operatorname{dom} M(x), \quad x < 0.$$

3. Applications to Laplace operators

In this section the applicability of the abstract theory developed in the preceding sections is demonstrated for the analysis of some classes of differential operators. First we consider the most standard case of elliptic PDE by treating Laplacians in smooth bounded domains; in this case many of the abstract results take a rather explicit form.

3.1. **The Kreĭn - von Neumann Laplacian.** Let Ω be a bounded domain in $\mathbb{R}^d$ $(d \geq 2)$ with a smooth boundary $\partial\Omega$. Consider the differential expression $\ell := -\Delta$, where Δ is a Laplacian operator in Ω and denote by $A := A_{\min}$ and $A_{\max}$ the minimal and the maximal differential operators generated in $H^0(\Omega) := L^2(\Omega)$ by the differential expression ℓ.

It is well known (see, for instance, [18], [40, Chapter 9], [55]) that

$$\operatorname{dom} A_{\min} = H_0^2(\Omega) := \{f \in H^2(\Omega) : \gamma_D f = \gamma_N f = 0\}.$$

Here γ_D and γ_N are the Dirichlet and Neumann trace operators originally defined for any $f \in H^2(\Omega)$ by

$$\gamma_D : f \mapsto f|_{\partial\Omega}, \quad \gamma_N : f \mapsto \left.\frac{\partial f}{\partial n}\right|_{\partial\Omega},$$

and n denotes the outward unit normal to the boundary $\partial\Omega$. Then the mapping

$$(3.1) \qquad \begin{pmatrix} \gamma_D \\ \gamma_N \end{pmatrix} : f \in H^2(\Omega) \mapsto \begin{pmatrix} \gamma_D f \\ \gamma_N f \end{pmatrix} \in \begin{pmatrix} H^{3/2}(\partial\Omega) \\ H^{1/2}(\partial\Omega) \end{pmatrix} \quad \text{is bounded and onto}$$

(see [55, Thm 1.8.3], [2], [40]). It is known (see, for instance, [18]) that $A_{\max} = A_{\min}^* \, (= A^*)$.

Clearly, $\operatorname{dom} A_{\max} \supset H^2(\Omega)$. However, $\operatorname{dom} A_{\max} \neq H^2(\Omega)$ and a description of $\operatorname{dom} A^*$ was given via trace mappings by Lions and Magenes [55] (see also [40, Chapter 9]) who have shown that the mappings γ_D and γ_N admit extensions

$$(3.2) \qquad \gamma_D : \operatorname{dom} A_{\max} \to H^{-1/2}(\partial\Omega), \quad \gamma_N : \operatorname{dom} A_{\max} \to H^{-3/2}(\partial\Omega)$$

to $\operatorname{dom} A_{\max}$ equipped with the graph norm and these mappings are surjective and continuous.

Denote by $H^s_\Delta(\Omega)$ the following space

$$(3.3) \qquad H^s_\Delta(\Omega) := H^s(\Omega) \cap \operatorname{dom} A_{\max} = \left\{f \in H^s(\Omega) : \Delta f \in L^2(\Omega)\right\}, \quad 0 \leq s \leq 2,$$

where Δf is understood in the sense of distributions. We equip the space with the graph norm $\|f\|_{H^s_\Delta(\Omega)} = (\|f\|^2_{H^s} + \|A_{\max} f\|^2_{L^2(\Omega)})^{1/2}$ of $-\Delta$ on $H^s(\Omega)$. In particular, $H^0_\Delta(\Omega)$ is a domain of maximal operator $A_{\max}$.

According to the Lions-Magenes result ([55, Theorem 2.7.3] (see also [40, Chapter 9])) the restrictions of the trace mappings γ_D and γ_N in (3.2) to $H^s_\Delta(\Omega)$,

$$\gamma^s_D : H^s_\Delta(\Omega) \to H^{s-1/2}(\partial\Omega), \quad \gamma^s_N : H^s_\Delta(\Omega) \to H^{s-3/2}(\partial\Omega), \quad 0 \le s \le 2, \tag{3.4}$$

are continuous and surjective. It is emphasized that the values $s = 1/2$ and $s = 3/2$ are not excluded here in contrast to the case of the trace mappings $\gamma^s_D : H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ and $\gamma^s_N : H^s(\Omega) \to H^{s-3/2}(\partial\Omega)$ that are continuous if and only if $s > 1/2$ and $s > 3/2$, respectively, (see ([55, Theorems 1.9.4, 1.9.5] and [2]). In the latter case both mappings in (3.4) are surjective and the mapping $\gamma^s_D \times \gamma^s_N : H^s(\Omega) \to H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega)$ is also surjective for $s > 3/2$.

When treating the traces γ^s_D and γ^s_N as mappings into $L^2(\partial\Omega)$ a natural choice for the index is $s = 3/2$; see Remark 3.3 below. Namely, we introduce a pre-maximal operator A_* by setting

$$A_* := A_{\max}\restriction \operatorname{dom} A_*, \quad \operatorname{dom} A_* = H^{3/2}_\Delta(\Omega) = H^{3/2}(\Omega) \cap \operatorname{dom} A_{\max}. \tag{3.5}$$

It is well known (see e.g. [55], [2], [40, Chapter 9]) that two classical realizations of the expression ℓ, the Dirichlet Laplacian $-\Delta_D$ and the Neumann Laplacian $-\Delta_N$, given by ℓ on the domains

$$\operatorname{dom} \Delta_D = \{f \in H^2(\Omega) : \gamma_D f = 0\} = H^{2,0}(\Omega) := H^2(\Omega) \cap H^1_0(\Omega), \tag{3.6}$$

$$\operatorname{dom} \Delta_N = \{f \in H^2(\Omega) : \gamma_N f = 0\}, \tag{3.7}$$

respectively, are selfadjoint. Moreover, the Dirichlet Laplacian $-\Delta_D$ is invertible in $L^2(\Omega)$ with a discrete spectrum $\sigma_p(-\Delta_D)$. Define a solution operator $\mathcal{P}(z) : L^2(\partial\Omega) \to H^{1/2}(\Omega)$ for $z \in \mathbb{C} \setminus \sigma_p(-\Delta_D)$. Let $\varphi \in L^2(\partial\Omega)$ and let $f_z \in \operatorname{dom} A_{\max}$ be the unique solution of the Dirichlet problem

$$-\Delta f_z - z f_z = 0, \quad \gamma_D f_z = \varphi. \tag{3.8}$$

Then the operator $\mathcal{P}(z) : \varphi \mapsto f_z$ is continuous as an operator from $L^2(\partial\Omega)$ onto $H^{1/2}(\Omega)$ and it maps $H^1(\partial\Omega)$ onto $H^{3/2}(\Omega)$; see [39]. Hence the Poincaré-Steklov operator $\Lambda(z)$ defined by

$$\Lambda(z)\varphi := \gamma_N \mathcal{P}(z)\varphi, \tag{3.9}$$

maps $H^1(\partial\Omega)$ into $H^0(\partial\Omega)$ with continuous extension from $H^{-1/2}(\partial\Omega)$ into $H^{-3/2}(\partial\Omega)$. Moreover, the Dirichlet-to-Neumann map $\Lambda := \Lambda(0)$ treated as an operator in $L^2(\partial\Omega)$ is selfadjoint on the domain $\operatorname{dom} \Lambda = H^1(\partial\Omega)$; see [57] and also [59].

It was shown in [65] and [39] that the regularized trace operators $\widetilde{\Gamma}_{0,\Omega} f = (\gamma_N - \Lambda(0)\gamma_D) f$, $\widetilde{\Gamma}_{1,\Omega} f = \gamma_D f$, $f \in \operatorname{dom} A_{\max}$, are well defined on $\operatorname{dom} A_{\max}$ and meet the following regularity properties:

$$\widetilde{\Gamma}_{0,\Omega} = \gamma_N - \Lambda(0)\gamma_D : H^0_\Delta(\Omega) \to H^{1/2}(\partial\Omega), \quad \widetilde{\Gamma}_{1,\Omega} = \gamma_D : H^0_\Delta(\Omega) \to H^{-1/2}(\partial\Omega), \tag{3.10}$$

and here both mappings are continuous and surjective. These properties have allowed one to extend the Green formula to dom $A_{\max}$ with pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. The main ingredient of the proof in Višik [65] and Grubb [39] is the following decomposition $\operatorname{dom} A_{\max} = \operatorname{dom} \Delta_D \dotplus \ker A^*$ $(0 \in \rho(-\Delta_D))$ and the properties (3.2), (3.3), and (3.6) (see [39]).

With these preliminaries we are ready to describe complete analogs of the abstract results in Theorem 2.2 for Laplacians on smooth bounded domains. Let $\widetilde{A}_*$ be a restriction of $A_{\max}$ to the domain

$$\widetilde{A}_* = A_{\max}\upharpoonright \operatorname{dom}\widetilde{A}_*, \quad \operatorname{dom}\widetilde{A}_* = H^{1/2}_{\Delta}(\Omega) = \{f \in \operatorname{dom} A_{\max} : \gamma_D f \in L^2(\partial\Omega)\}. \tag{3.11}$$

Theorem 3.1. Let the operators γ_N, γ_D, $\Lambda(z)$, A_* and $\widetilde{A}_*$ be as above. Then:

(i) the triple $\Pi = \{L^2(\partial\Omega), \gamma_D\upharpoonright \operatorname{dom} A_*, -\gamma_N\upharpoonright \operatorname{dom} A_*\}$ is an S-generalized boundary triple for A^*, and the transposed triple Π is B-generalized. The corresponding Weyl function $M(\cdot)$ coincides with $-\Lambda(\cdot)$ and, in particular, $M(\cdot)$ is domain invariant;

(ii) the triple $\widetilde{\Pi} = \{L^2(\partial\Omega), \widetilde{\Gamma}_{0,\Omega}\upharpoonright \operatorname{dom}\widetilde{A}_*, \widetilde{\Gamma}_{1,\Omega}\upharpoonright \operatorname{dom}\widetilde{A}_*\}$ defined by (3.10) is an ES-generalized boundary triple for A^*. The transposed triple $\widetilde{\Pi}^\top$ is B-generalized. Moreover, the extension $\widetilde{A}_0 := \widetilde{A}_*\upharpoonright \ker\widetilde{\Gamma}_{0,\Omega}$ is essentially selfadjoint and its closure coincides with the Kreĭn - von Neumann extension A_K of $A_{\min}$;

(iii) The Weyl function $\widetilde{M}(\cdot)$ and (the closure of) the γ-field corresponding to the ES-generalized boundary triple $\widetilde{\Pi}$ are given by

$$\widetilde{M}(z) = (\overline{\Lambda(z) - \Lambda(0)})^{-1}, \quad \overline{\widetilde{\gamma}(z)} = (\widetilde{\Gamma}_{0,\Omega}\upharpoonright \mathfrak{N}_z(A_{\max}))^{-1}, \qquad z \in \mathbb{C}\setminus\mathbb{R}. \tag{3.12}$$

where the closures are taken in the L^2-space sense. In particular, $\widetilde{M}(\cdot)$ is form domain invariant, $\overline{\widetilde{\gamma}(z)}$ is unbounded domain invariant and, furthermore,

$$\operatorname{dom}\widetilde{M}(z) \subsetneqq \operatorname{dom}\mathfrak{t}_{\widetilde{M}(z)} = \operatorname{dom}\overline{\widetilde{\gamma}(z)} = H^{1/2}(\partial\Omega), \quad \operatorname{ran}\overline{\widetilde{\gamma}(z)} = \mathfrak{N}_z(A_{\max}), \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{3.13}$$

Proof. (i) It is first shown that the transposed triple $\Pi^\top = \{L^2(\partial\Omega), \gamma_N\upharpoonright \operatorname{dom} A_*, \gamma_D\upharpoonright \operatorname{dom} A_*\}$ is a B-generalized boundary triple for A^*. Indeed, the Green's identity is obviously satisfied. Furthermore, it follows from (3.5) and (3.4) with $s = 3/2$ that $\operatorname{ran}(\gamma_N\upharpoonright \operatorname{dom} A_*) = \operatorname{ran}(\gamma_N^{3/2}) = L^2(\partial\Omega)$, i.e. the mapping $\gamma_N\upharpoonright \operatorname{dom} A_* = \gamma_N^{3/2}$ is surjective. On the other hand, combining definition (3.5) with (3.7) shows that

$$\ker(\gamma_N\upharpoonright \operatorname{dom} A_*) \supseteq \{f \in H^2(\Omega) : \gamma_N f = 0\} = \operatorname{dom}\Delta_N.$$

Since $A_0 := A_*\upharpoonright \ker(\gamma_N\upharpoonright \operatorname{dom} A_*)$ is a symmetric extension of A, and $\Delta_N = \Delta_N^*$, one gets $A_0 = \Delta_N$.

Clearly, the triple Π is unitary because it is transposed to the B-generalized triple $\Pi^\top$. Moreover, as above from (3.6) one concludes that $\ker(\gamma_D\upharpoonright \operatorname{dom} A_*) = \operatorname{dom}\Delta_D$. Since $-\Delta_D = -\Delta_D^*$, the triple Π is an S-generalized boundary triple for A^*. By definition, the corresponding Weyl function coincides with $-\Lambda(z)$.

(ii) Again it is first shown that the corresponding transposed triple $\widetilde{\Pi}^\top$ is B-generalized. Green's identity is clearly satisfied. Hence, it suffices to show that $\widetilde{\Gamma}_{1,\Omega}\upharpoonright \operatorname{dom}\widetilde{A}_*$ maps onto $L^2(\partial\Omega)$ and that $\ker\widetilde{\Gamma}_{1,\Omega}$ defines a domain of a selfadjoint extension. To this end observe that $H^{2,0}(\Omega) \subset H^{1/2}(\Omega)$ and hence the decomposition $\operatorname{dom} A_{\max} = \operatorname{dom}\Delta_D \dotplus \mathfrak{N}_z$, $z \in \rho(\Delta_D)$ together with (3.6), (3.11) implies that

$$\operatorname{dom}\widetilde{A}_* = H^{2,0}(\Omega) \dotplus \mathfrak{N}_z \cap H^{1/2}(\Omega), \qquad z \in \rho(\Delta_D). \tag{3.14}$$

It follows from this decomposition of $\operatorname{dom}\widetilde{A}_*$ (with $z = 0 \in \rho(\Delta_D)$) and (3.4), (3.10) that

$$\operatorname{ran}(\widetilde{\Gamma}_{1,\Omega}\upharpoonright \operatorname{dom}\widetilde{A}_*) = \gamma_D(\mathfrak{N}_0) \cap \gamma_D(H^{1/2}(\Omega)) = H^{-1/2}(\partial\Omega) \cap H^0(\partial\Omega) = H^0(\partial\Omega).$$

On the other hand, $\ker(\widetilde{\Gamma}_{1,\Omega}\upharpoonright \operatorname{dom}\widetilde{A}_*) = H^{2,0}(\Omega) = \operatorname{dom}\Delta_D$ and, since $\Delta_D = \Delta_D^*$, the transposed triple $\widetilde{\Pi}^\top$ is B-generalized. To complete the proof of (ii) consider the Kreĭn extension A_K; see [54]. One has $\operatorname{dom} A_K = \ker(\widetilde{\Gamma}_{0,\Omega}\upharpoonright \operatorname{dom} A_{\max}) = \operatorname{dom} A \dotplus \mathfrak{N}_0$ and hence

$$\operatorname{dom}\widetilde{A}_0 = \ker(\widetilde{\Gamma}_{0,\Omega}\upharpoonright \operatorname{dom}\widetilde{A}_*) = \operatorname{dom} A \dotplus \mathfrak{N}_0 \cap H^{1/2}(\Omega).$$

Since $\mathfrak{N}_0 \cap H^{1/2}(\Omega)$ is dense in $\mathfrak{N}_0$, $\operatorname{dom}\widetilde{A}_0$ is a core for the operator $A_K = A_K^*$. Thus, $\widetilde{A}_0 = A_K\upharpoonright \operatorname{dom}\widetilde{A}_0$ is essentially selfadjoint and the triple $\widetilde{\Pi}$ is an ES-generalized triple.

(iii) To prove the first formula in (3.12). note that it follows from (3.9) that the mapping $\Lambda(z) - \Lambda(0)$ takes $H^1(\partial\Omega)$ into $H^0(\partial\Omega)$. Hence, from the definition of the Weyl function $-\widetilde{M}(z)^{-1}$, one gets that it is an extension of $-\Lambda(z) + \Lambda(0)$, i.e. $\Lambda(z) - \Lambda(0) \subset \widetilde{M}(z)^{-1}$. Since by (ii) the transposed boundary triple $\widetilde{\Pi}^\top$ is B-generalized, the operator $-\widetilde{M}(z)^{-1}$ is bounded for each $z \in \rho(\Delta_D)$. This implies the required formula in (3.12).

The γ-field $\widetilde{\gamma}(z)$ corresponding to $\widetilde{\Pi}$ is given by $\widetilde{\gamma}(z) = (\widetilde{\Gamma}_{0,\Omega}\upharpoonright \mathfrak{N}_z(\widetilde{A}_*))^{-1}$. Hence, $\widetilde{\gamma}(z)^{-1} = \widetilde{\Gamma}_{0,\Omega}\upharpoonright \mathfrak{N}_z(\widetilde{A}_*)$. Combining (3.10) with the decomposition (3.14) implies that $\widetilde{\Gamma}_{0,\Omega}$ maps $\mathfrak{N}_z(A_{\max})$ continuously onto $H^{1/2}(\partial\Omega)$. Hence the mapping $\widetilde{\Gamma}_{0,\Omega} : \mathfrak{N}_z(A_{\max}) \to L^2(\partial\Omega)$, as well as its restriction to $\mathfrak{N}_z(\widetilde{A}_*)$, is also continuous. Since $\mathfrak{N}_z(\widetilde{A}_*) = \mathfrak{N}_z(A^*) \cap H^{1/2}(\Omega)$ is dense in $\mathfrak{N}_z(A_{\max})$, the L^2-closure of $\widetilde{\gamma}(z)^{-1}$ coincides with the mapping $\widetilde{\Gamma}_{0,\Omega} : \mathfrak{N}_z(A_{\max}) \to L^2(\partial\Omega)$, $z \in \mathbb{C}\setminus\mathbb{R}$. This proves the second formula in (3.12). In particular, $\operatorname{dom}\overline{\widetilde{\gamma}(z)} = H^{1/2}(\partial\Omega)$ and $\operatorname{ran}\overline{\widetilde{\gamma}(z)} = \mathfrak{N}_z(A_{\max})$, while the identity $\operatorname{dom}\mathfrak{t}_{\widetilde{M}(\lambda)} = \operatorname{dom}\overline{\widetilde{\gamma}(z)}$, $z \in \mathbb{C}\setminus\mathbb{R}$, follows from (2.40).

Finally, to see the strictness of the inclusion $\operatorname{dom}\widetilde{M}(\lambda) \subsetneqq \operatorname{ran}\widetilde{\Gamma}_{0,\Omega}$ observe that for all $z \in \mathbb{C}\setminus\mathbb{R}$ the mapping $\widetilde{\Gamma}_{0,\Omega} : \mathfrak{N}_z(A_{\max}) \to H^{1/2}(\partial\Omega)$ is bijective (because $A_K = A_K^*$). Since $\mathfrak{N}_z(\widetilde{A}_*)$ is a proper dense subset of $\mathfrak{N}_z(A_{\max})$, it follows that $\operatorname{dom}\widetilde{M}(z) = \widetilde{\Gamma}_{0,\Omega}(\mathfrak{N}_z(A_*))$ is a proper dense subset of $H^{1/2}(\partial\Omega)$. $\square$

Remark 3.2. (i) Another proof for Theorem 3.1 can be extracted from the abstract result in Theorem 2.2 (for details see [25]). Indeed, take $A_0 = -\Delta_D$ and fix the mappings $G := \mathcal{P}(0)$ and $E := -\Lambda(0)$; see Remark 2.3. By definition $\gamma_D G\varphi = \varphi$ and $\gamma_D A_0^{-1} f = 0$ for all $\varphi \in H^0(\partial\Omega)$ and $f \in L^2(\Omega)$. Moreover, a direct calculation (see e.g. [63] with smooth functions f) leads to

$$G^* f = -\gamma_N A_0^{-1} f, \quad f \in L^2(\Omega),$$

cf. [26, eq: (3.11), (4.1)]. It follows that the abstract boundary mappings Γ_0 and Γ_1 in (2.5) of the boundary triple Π in Theorem 2.2 (i) coincide with the trace operators $\gamma_D\upharpoonright \operatorname{dom} A_*$ and $-\gamma_N\upharpoonright \operatorname{dom} A_*$, respectively. Similarly, the boundary mappings $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ constructed in Theorem 2.2 (iv) coincide with the regularized boundary mappings in Theorem 3.1 (ii). Notice that $\operatorname{ran} G = H^{1/2}(\Omega)$ is not closed in $H^0(\Omega)$ and hence also by Theorem 2.2 (v) the triple $\{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ cannot be S-generalized. Indeed, by Theorem 3.1 (iii) the property (iii) in Theorem 1.3 is violated since the strict inclusion $\operatorname{dom}\widetilde{M}(\lambda) \subsetneqq \operatorname{ran}\widetilde{\Gamma}_{0,\Omega}$ holds; cf. [26, Section 1.5]. In fact, a more explicit characterization of the domain of the Weyl function $\widetilde{M}(\lambda)$ will be given later on elsewhere.

The above proof shows that the boundary triple $\widetilde{\Pi}$ in Theorem 3.1 satisfies the condition (ii) in Proposition 2.9 and hence the fact that the transposed triple $\widetilde{\Pi}^\top$ is B-generalized can be deduced also from Proposition 2.9.

(ii) By Theorem 3.1 (i) the Weyl function $M(\cdot) = -\Lambda(\cdot)$ of Π in Theorem 3.1 (i) is domain invariant with $\operatorname{dom} M(z) = H^1(\partial\Omega)$; see (3.9). In fact, (cf. [25, Prop. 7.6]) $M(\cdot)$ belongs to the class of inverse Stieltjes functions of unbounded operators and is associated with the Friedrichs extension of the minimal operator $A_{\min}$. The inverse $-M(\cdot)^{-1}$ belongs to the class of Stieltjes functions of compact operators, because the embedding $H^1(\partial\Omega) \to L^2(\partial\Omega)$ is compact. The form domain invariant Weyl function $\widetilde{M}(\cdot)$ of $\widetilde{\Pi}$ in Theorem 3.1 (ii) belongs to the class of Stieltjes functions of unbounded operators, while the inverse $-\widetilde{M}(\cdot)^{-1}(\cdot)$ belongs to the class of inverse Stieltjes functions of bounded operators.

(iii) Another B-generalized boundary triple for A^* was constructed and used in [15] to compute the scattering matrix. In that case a B-generalized boundary triple has been selected to satisfy $\operatorname{dom} A_* = H^1_\Delta(\Omega)$.

(iv) We expect the relation $\operatorname{dom} \widetilde{M}(\cdot) = H^1(\partial\Omega)$ in (3.13), in which case $\widetilde{M}(\cdot)$ is domain invariant, while the triplet $\widetilde{\Pi}$ is ES-generalized. We postpone a discussion of this fact to another place.

Remark 3.3. (i) Using the above mentioned properties of the traces γ_D^s and γ_N^s, it is easily seen that for the values $3/2 \le s \le 2$ the boundary triple

$$\Pi_s = \{L^2(\partial\Omega), \gamma_D^s \upharpoonright \operatorname{dom} A_*, -\gamma_N^s \upharpoonright \operatorname{dom} A_*\}$$

as well as the transposed boundary triple $\Pi_s^\top$ are quasi boundary triples (compare [13, Theorem 6.11]) and, in particular, AB-generalized boundary triples. Indeed, since Green's identity holds for $s = 3/2$ (by Theorem 3.1), it holds also for $s \in [3/2, 2]$, moreover, $\operatorname{dom} \Delta_D = \ker \gamma_D^s$, $\operatorname{dom} \Delta_N = \ker \gamma_N^s$, and surjectivity of $\gamma_D^s \times \gamma_N^s : H^s(\Omega) \to H^{s-1/2}(\partial\Omega) \times H^{s-3/2}(\partial\Omega)$ shows that the range of $\gamma_D^s \times \gamma_N^s$ is dense in $(L^2(\partial\Omega))^2$.

(ii) More precisely, for every $s \in [3/2, 2]$ the quasi boundary triples in (i) are, in fact, essentially unitary. The choice $s = 3/2$ in Theorem 3.1 is motivated by the following statement: for every $s \in [3/2, 2]$ the closure of the graph of $(\gamma_D^s \times \gamma_N^s)$ in $(H^0_\Delta(\Omega)) \times (L^2(\partial\Omega))^2$ coincides with the graph of $(\gamma_D^{3/2} \times \gamma_N^{3/2})$ in $L^2(\Omega) \times (L^2(\partial\Omega))^2$. By Theorem 3.1 this closure is an S-generalized boundary triple for $A_{\max}$ and, hence, it is unitary.

(iii) It follows from (ii) that the Weyl function $M^s(\cdot)$ of the quasi boundary triple Π^s for any $s \in (3/2, 2)$ is not closed in $L^2(\partial\Omega)$, hence it is not a Nevanlinna function. However, the closure of $M^s(\cdot)$ in $L^2(\partial\Omega)$ is just the Weyl function $M^{3/2}(\cdot) = M(\cdot)$ of $\Pi_{3/2} = \Pi$ in Theorem 3.1 (i).

Remark 3.4. (i) General theory of, not necessarily local, boundary value problems for elliptic operators in bounded domains with smooth boundary was built in the pioneering works by Višik [65] and Grubb [39]. In terms of boundary triples Grubb's results were adapted and further developed in Malamud [57] (see also [1, 14, 21, 36, 40] for some further developments and applications).

(ii) The description of the Kreĭn - von Neumann Laplacian (see Theorem 3.1 (ii)) in terms of boundary conditions for domains with smooth boundary is immediate by combining Krein's description of A_K [54] with trace theory by Lions and Magenes (see [55]) and goes back to the works [65] and [60, Section 12.3] (see also [57]). For Lipschitz domains a similar description of the Kreĭn - von Neumann Laplacian in terms of extended trace operators was recently given in [10]; see also Section 3.3 below for another construction.

(iii) Finally, it is mentioned that the abstract renormalization result in [26, Theorem 5.32], when specialized to the case of the ES-generalized boundary triple in Theorem 3.1 (ii), leads to an ordinary boundary triple for $A_{\max}$; for further discussion see [25, Cor. 7.7] and comments therein.

3.2. **Mixed boundary value problem for Laplacian.** Let Ω be a bounded open set in $\mathbb{R}^d$ ($d \geq 2$) with a smooth boundary $\partial\Omega$. Let Σ_+ be a compact smooth submanifold of $\partial\Omega$, Σ_+° be the interior of Σ_+ and let $\Sigma_- := \partial\Omega \setminus \Sigma_+^\circ$, so that $\Sigma = \Sigma_+ \cup \Sigma_-$. Let $-\Delta_Z$ be the Zaremba Laplacian, i.e. the restriction of the maximal operator $A_{\max}$ to the set of functions, which satisfy Dirichlet boundary condition on Σ_- and Neumann boundary condition on Σ_+.

Let $H^1_{\Sigma_+}(\Omega) = \{u \in H^1(\Omega) : \operatorname{supp} \gamma_D u \subset \Sigma_+\}$. It is known (see for instance Grubb [41]), that the operator $-\Delta_Z$ is associated with the nonnegative closed quadratic form

$$\mathfrak{a}_{\Sigma_+}(u,v) = \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx, \quad \operatorname{dom} \mathfrak{a}_{\Sigma_+} = H^1_{\Sigma_+}(\Omega),$$

hence it is selfadjoint in $H^0(\Omega)$. Clearly, its spectrum $\sigma(-\Delta_Z)$ is discrete.

Here we construct an ES-generalized boundary triple, associated with the Zaremba Laplacian.

Let $A_{\min}$ and A_* be the minimal and pre-maximal operators, respectively, associated with $-\Delta$, $\operatorname{dom}(A_*) = H^{3/2}_\Delta(\Omega) = H^{3/2}(\Omega) \cap \operatorname{dom} A_{\max}$ (see Section 3.1). Let $A_{*,-}$ be a realization of $-\Delta$ given by

$$\operatorname{dom} A_{*,-} = \{f \in H^{3/2}_\Delta(\Omega) : (\gamma_N f)|_{\Sigma_+} = 0\}, \tag{3.15}$$

and let $A_- := (A_{*,-})^*$. Then A_- is an intermediate extension of $A := A_{\min}$ in the sense of [29], i.e.

$$A_{\min} \subset A_- \subset A_{*,-} \subset (A_-)^* \subset A_{\max},$$

More precisely we have the following result.

Theorem 3.5. Let the operator $A_{*,-}$ be defined by (3.15) and let $A_- = (A_{*,-})^*$. Then:

(i) A_- is a symmetric realization of the Laplacian $-\Delta$ on the domain

$$\operatorname{dom} A_- = \{f \in H^2(\Omega) : \gamma_N f = (\gamma_D f)|_{\Sigma_-} = 0\} \subset \operatorname{dom} \Delta_N; \tag{3.16}$$

(ii) the triple $\Pi^- = (L^2(\Sigma_-), P_{L^2(\Sigma_-)}\gamma_N, P_{L^2(\Sigma_-)}\gamma_D)$ is a B-generalized boundary triple for $(A_-)^*$;

(iii) the Weyl function corresponding to the boundary triple Π^- equals to

$$M_-(z) = P_{L^2(\Sigma_-)}\Lambda(z)^{-1} \upharpoonright L^2(\Sigma_-),$$

where $\Lambda(z)^{-1}$ is the Neumann-to-Dirichlet map;

(iv) the triple $(\Pi^-)^\top = (L^2(\Sigma_-), P_{L^2(\Sigma_-)}\gamma_D, -P_{L^2(\Sigma_-)}\gamma_N)$ is an ES-generalized boundary triple for $(A_-)^*$.

Proof. (i) Since $A_N \subset A_{*,-}$ it follows that $A_- \subset A_N$ and hence $\operatorname{dom}(A_-) \subset H^2(\Omega)$ and $\gamma_N f = 0$ for $f \in \operatorname{dom}(A_-)$. Since $\gamma_N \operatorname{dom}(A_*) = L^2(\partial\Omega)$ for every $\varphi \in L^2(\Sigma_-)$ there exists $f \in \operatorname{dom}(A_*)$ such that

$$(\gamma_N f)(x) = \begin{cases} \varphi(x), & x \in \Sigma_-; \\ 0, & x \in \partial\Omega \setminus \Sigma_-. \end{cases}$$

Then for every $g \in \operatorname{dom}(A_-)$ one obtains from

$$0 = (-\Delta f, g)_{L^2(\Omega)} - (f, -\Delta g)_{L^2(\Omega)} = -(\gamma_N f, \gamma_D g)_{L^2(\Sigma_-)} = -(\varphi, \gamma_D g)_{L^2(\Sigma_-)}.$$

Hence $P_{L^2(\Sigma_-)}\gamma_D g = 0$. This proves the inclusion

$$\operatorname{dom} A_- = \{f \in H^2(\Omega) : \gamma_N f = (\gamma_D f)|_{\Sigma_-} = 0\} \subset \operatorname{dom} \Delta_N.$$

The converse inclusion is immediate from (3.15).

(ii) As is proved in Theorem 3.1(i) the triple $\Pi = (L^2(\partial\Omega), \gamma_N, \gamma_D)$ is a B-generalized boundary triple for A^*. Since S_- is an intermediate extension of A, also $\Pi^- = (L^2(\Sigma_-), P_{L^2(\Sigma_-)}\gamma_N, P_{L^2(\Sigma_-)}\gamma_D)$ is a B-generalized boundary triple for $(S_-)^*$; see [29, Proposition 4.1]. Notice that $\operatorname{ran}(P_{L^2(\Sigma_-)}\gamma_N) = L^2(\Sigma_-)$, and the operator $A_{0,-}$ defined as the restriction of $-\Delta$ to the domain

$$\begin{aligned}\operatorname{dom} A_{0,-} &= \ker \Gamma_0^+ = \{f \in \operatorname{dom}(A_{*,-}) : P_{L^2(\Sigma_-)}\gamma_N f = 0\} \\ &= \{f \in H^{3/2}_\Delta(\Omega) : \gamma_N f = 0\} = \{f \in H^2(\Omega) : \gamma_N f = 0\} = \operatorname{dom}(-\Delta_N).\end{aligned}$$

is selfadjoint, since it coincides with the Neumann Laplacian.

(iii) This statement is implied by the fact that the Weyl function of the operator A, corresponding to the boundary triple $\Pi = (L^2(\partial\Omega), \gamma_N, \gamma_D)$, coincides with $\Lambda(z)^{-1}$; see [29, Proposition 4.1].

(iv) Consider the operator $A_{1,-}$ defined as the restriction of $-\Delta$ to the domain

$$\operatorname{dom} A_{1,-} = \{f \in \operatorname{dom}(S_{*,-}) : P_{L^2(\Sigma_-)}\gamma_D f = 0\} = \{f \in H^{3/2}_\Delta(\Omega) : (\gamma_N f)|_{\Sigma_+} = (\gamma_D f)|_{\Sigma_-} = 0\}.$$

Observe, that $\operatorname{dom}(-\Delta_Z) \subset H^{3/2-\varepsilon}(\Omega)$ for each $\varepsilon > 0$ while $\operatorname{dom}(-\Delta_Z) \not\subset H^{3/2}(\Omega)$ (see [41]). Therefore, the operator $A_{1,-}$ is a proper symmetric restriction of Zaremba Laplacian $-\Delta_Z$, hence $A_{1,-}$ is not selfadjoint.

To prove the statement (iv) it suffices to show that the operator $A_{1,-}$ is essentially selfadjoint. Assuming the contrary one finds $\lambda_0 = \bar{\lambda}_0 \notin \sigma_p(-\Delta_Z)$ and a vector $g \in L^2(\Omega)$ such that $g \perp \operatorname{ran}(A_{1,-} - \lambda_0)$, i.e.

$$(g, (-\Delta - \lambda_0)f)_{L^2(\Omega)} = 0 \quad \text{for all} \quad f \in \operatorname{dom} A_{1,-}. \tag{3.17}$$

This relation with $f \in \operatorname{dom} A$ implies $g \in \operatorname{dom}(A_{\max})$ and $(-\Delta - \lambda_0)g = 0$. Letting $f \in \operatorname{dom} A_-$ and applying the Green formula one obtains from (3.16) and (3.17) that

$$\begin{aligned}0 &= (g, -\Delta f)_{L^2(\Omega)} - (\lambda_0 g, f)_{L^2(\Omega)} \\ &= (g, -\Delta f)_{L^2(\Omega)} - (-\Delta g, f)_{L^2(\Omega)} \\ &= \langle \gamma_N g, \gamma_D f\rangle_{-3/2,3/2} - \langle \gamma_D g, \gamma_N f\rangle_{-1/2,1/2} = \big\langle (\gamma_N g)|_{\Sigma_+}, (\gamma_D f)|_{\Sigma_+}\big\rangle_{-3/2,3/2},\end{aligned}$$

Here $\langle\cdot,\cdot\rangle_{-s,s}$ denotes duality between $H^{-s}(\partial\Omega)$ and $H^s(\partial\Omega)$, $s \in \mathbb{R}$. It follows from (3.1) that $\gamma_D(\operatorname{dom} \Delta_N) = H^{3/2}(\partial\Omega)$. Hence $\gamma_D(\operatorname{dom} A_-) = H^{3/2}(\Sigma_+)$ and the latter implies

$$(\gamma_N g)|_{\Sigma_+} = 0. \tag{3.18}$$

Similarly, it follows from (3.1) that $\gamma_N(\operatorname{dom} \Delta_D) = H^{1/2}(\partial\Omega)$. For a subset $\mathcal{L}$ of $\operatorname{dom} A_{1,-}$

$$\mathcal{L} = \{f \in H^2(\Omega) : (\gamma_N f)|_{\Sigma_+} = \gamma_D f = 0\}$$

one obtains

$$\gamma_N \mathcal{L} = H^{1/2}(\Sigma_-). \tag{3.19}$$

Let now $f \in \mathcal{L}$. Then using the Green formula the equality (3.17) can be rewritten as

$$0 = (g, -\Delta f)_{L^2(\Omega)} - (\lambda_0 g, f)_{L^2(\Omega)} = -\langle \gamma_D g, \gamma_N f \rangle_{-1/2,1/2} \tag{3.20}$$

and (3.19), (3.20) lead to

$$(\gamma_D g)|_{\Sigma_-} = 0. \tag{3.21}$$

Since $g \in \operatorname{dom}(A_{\max})$, relations (3.18) and (3.21) mean that $g \in \operatorname{dom}(-\Delta_Z)$. Thus $g \in \ker(-\Delta_Z - \lambda_0) = \{0\}$, hence $g = 0$. This completes the proof. □

Remark 3.6. As follows from [26, Theorem 5.24] the statement (iii) in Theorem 3.5 is equivalent to the fact that the γ-field $\gamma(\lambda)$ admits a single-valued closure for all $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ with constant domain and the M-function $-M_-(z)^{-1}$ is form domain invariant. As was mentioned in the proof, $A_{1,-}$ is essentially selfadjoint, while is not selfadjoint. By [26, Theorem 1.12, Theorem 5.17], this implies that the operators $\gamma(\lambda)$ are not bounded; this fact was apparently first mentioned in [62, Theorem 6.23]. In particular, the corresponding boundary triple $(\Pi^-)^\top$ is neither S-generalized, nor an AB-generalized or a quasi boundary triple in the sense of [12].

3.3. **Laplacians on Lipschitz domains.** Here the smoothness properties on Ω are relaxed; it is assumed that Ω is a bounded Lipschitz domain. In this case the Dirichlet and Neumann traces

$$\gamma_D : H^s_\Delta(\Omega) \to H^{s-1/2}(\partial\Omega), \quad \gamma_N : H^s_\Delta(\Omega) \to H^{s-3/2}(\partial\Omega),$$

are still continuous operators for all $1/2 \le s \le 3/2$ and, in addition, both are surjective when $s = 1/2$ and $s = 3/2$; see Gesztesy and Mitrea [36, Lemmas 3.1, 3.2]. In this case the results, which are analogous to those in Section 3.1, will be derived directly from the abstract setting treated in Section 2.1.

The following analog of Theorem 3.1 is obtained from Theorem 2.2 using the 3/2 regularity of the selfadjoint extensions $-\Delta_D$ and $-\Delta_N$; cf. [45, 46, 36]. Since $0 \in \rho(-\Delta_D)$, one can decompose

$$\operatorname{dom} A_{\max} = \operatorname{dom} \Delta_D \dot{+} \ker A_{\max}.$$

Proposition 3.7. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded Lipschitz domain. Let the operators γ_N, γ_D, $\mathcal{P}(z)$, $\Lambda(z)$ and A_* be defined by (3.2), (3.8), (3.9), and (3.5). Then:

(i) $\{L^2(\partial\Omega), \gamma_D \upharpoonright \operatorname{dom} A_*, -\gamma_N \upharpoonright \operatorname{dom} A_*\}$ is an S-generalized boundary triple for A^* with domain $\operatorname{dom} A_* = H^{3/2}_\Delta(\Omega)$, the transposed boundary triple is B-generalized, moreover, the corresponding γ-field $\gamma(\cdot)$ is bounded and coincides with $\mathcal{P}(z)$ and the Weyl function $M(\cdot)$ coincides with $-\Lambda(\cdot)$;

(ii) $\{L^2(\partial\Omega), \widetilde{\Gamma}_{0,\Omega}, \widetilde{\Gamma}_{1,\Omega}\}$, where

$$\begin{pmatrix} \widetilde{\Gamma}_{0,\Omega} \\ \widetilde{\Gamma}_{1,\Omega} \end{pmatrix} (f + \overline{\gamma(0)} h) = \begin{pmatrix} -\gamma(0)^* \Delta_D f \\ -h \end{pmatrix}, \quad f \in \operatorname{dom} \Delta_D, \quad h \in L^2(\partial\Omega)\}, \tag{3.22}$$

defines an ES-generalized boundary triple for $A_{\max}$ with dense domain

$$\operatorname{dom} A_* = \operatorname{dom} \Delta_D + \operatorname{ran} \overline{\gamma(0)} \subset \operatorname{dom} A^*,$$

the transposed boundary triple is B-generalized, and the corresponding Weyl function is the $L^2(\partial\Omega)$-closure

$$\widetilde{M}(z) = \operatorname{clos}(\Lambda(z) - \Lambda(0))^{-1};$$

(iii) the extension $\widetilde{A}_0 := A_{\max}\upharpoonright \ker \widetilde{\Gamma}_{0,\Omega}$ is essentially selfadjoint and its closure coincides with the Kreĭn - von Neumann extension of the operator $A_{\min}$.

Proof. (i) Green's identity holds: this can be obtained for instance from the formula (3.21) in [36] (cf. proof of Theorem 3.12 below). Moreover, according to [45, 46, 36], see also [16],

$$\Delta_D = \Delta\upharpoonright \{y \in H^{3/2}_\Delta(\Omega) : \gamma_D y = 0\} \quad \text{and} \quad \Delta_N = \Delta\upharpoonright \{y \in H^{3/2}_\Delta(\Omega) : \gamma_N y = 0\},$$

are selfadjoint operators in $L^2(\partial\Omega)$ and, in addition, $0 \in \rho(-\Delta_D)$. Hence,

$$\operatorname{dom} M(\cdot) = \operatorname{ran}\gamma_D = H^1(\partial\Omega), \quad \operatorname{ran} M(\cdot) = \operatorname{ran}\gamma_N = H^0(\partial\Omega).$$

Thus, $\{L^2(\partial\Omega), \gamma_D\upharpoonright \operatorname{dom} A_*, -\gamma_N\upharpoonright \operatorname{dom} A_*\}$ is an AB-generalized boundary triple. Moreover, according to [36, Theorem 5.7] the corresponding Weyl function $M(\cdot)$ is a bounded operator from $H^1(\partial\Omega)$ to $L^2(\partial\Omega)$. Since $M(z)$, $z \in \rho(-\Delta_D)$, is surjective, the inverse $M(z)^{-1}$, $z \in \rho(-\Delta_D) \cap \rho(-\Delta_N)$, is bounded from $L^2(\partial\Omega)$ to $H^1(\partial\Omega)$; in particular, $-M(z)^{-1}$ is bounded in $L^2(\partial\Omega)$. From [26, Corollary 4.7] one concludes that the AB-generalized boundary triple $\{L^2(\partial\Omega), \gamma_D\upharpoonright \operatorname{dom} A_*, -\gamma_N\upharpoonright \operatorname{dom} A_*\}$ is unitary, i.e., it is S-generalized. The assertion concerning the γ-field is obtained from Theorem 1.3. The transposed boundary triple is B-generalized, since $\gamma_N : H^{3/2}_\Delta(\Omega) \to H^0(\partial\Omega)$ is surjective, or since the corresponding Weyl function $-M(z)^{-1}$ is bounded.

(ii) This result is obtained directly from Theorem 2.2 with $A_0 = -\Delta_D$, $G := \overline{\gamma(0)}$ which is bounded by item (i) and $E := -\Lambda(0)$ which is selfadjoint, since $0 \in \rho(-\Delta_D)$.

(iii) This follows from Theorem 2.2 and Theorem 2.6; see (2.21). □

Next the renormalization in Theorem 2.6 is applied to the ES-generalized boundary triple in Proposition 3.7.

Proposition 3.8. With the notations and assumptions as in Proposition 3.7, let $\{L^2(\partial\Omega), \Gamma_{0,\Omega}, \Gamma_{1,\Omega}\}$ be the ES-generalized boundary triple with the Weyl function $\widetilde{M}(\cdot)$ and let P_0 be the orthogonal projection onto $\mathfrak{N}_0 := \ker A_{\max}$. Then:

(i) the Weyl function $\widetilde{M}(z) = \operatorname{clos}(\Lambda(z) - \Lambda(0))^{-1}$ is form domain invariant,

$$\operatorname{dom} \mathfrak{t}_{\widetilde{M}(z)} = \operatorname{ran}\gamma(0)^*, \quad z \in \rho(-\Delta_D);$$

(ii) the renormalized boundary triple $\{\mathfrak{N}_0, \Gamma_{0,r}, \Gamma_{1,r}\}$, where

$$\begin{pmatrix} \Gamma_{0,r} \\ \Gamma_{1,r} \end{pmatrix} (f + h) = \begin{pmatrix} -P_0\Delta_D f \\ -h \end{pmatrix}, \quad f \in \operatorname{dom}\Delta_D, \quad h \in \mathfrak{N}_0,$$

is an ordinary boundary triple for $A_{\max}$;

(iii) the corresponding Weyl function is given by

$$M_r(\lambda) = A^-_{11} - 1/\lambda - (A^-_{21})^*(A^-_{22} - 1/\lambda)^{-1}A^-_{21}, \quad \lambda \in \rho(-\Delta_D).$$

where $-\Delta_D^{-1} = (A^-_{ij})^2_{i,j=1}$ is decomposed according to $\mathfrak{H} = \mathfrak{N}_0 \oplus (\mathfrak{N}_0)^\perp$.

Proof. The result is obtained by applying Theorem 2.6 to Proposition 3.7 with the choices $A_0 = -\Delta_D$ and $G := \overline{\gamma(0)}$. □

As a consequence one has the following result:

Corollary 3.9. The inverse of the regularized Dirichlet-to-Neumann map $\operatorname{clos}(\Lambda(z)-\Lambda(0))$ has the form

$$\widetilde{M}(z)=\operatorname{clos}(\Lambda(z)-\Lambda(0))^{-1}=\gamma(0)^{(-1)}M_r(z)\gamma(0)^{-(*)}$$

and, consequently, the Dirichlet-to-Neumann map has the representation

$$\Lambda(z)=\Lambda(0)+\gamma(0)^*M_r(z)^{-1}\gamma(0),\quad z\in\rho(-\Delta_D).$$

Notice that here by definition $M_r(0)^{-1}=(\infty^{-1}=)\,0$.

Comparing Proposition 3.7 (ii) with Proposition 3.8 (i) we get the following equality

$$\operatorname{ran}\Gamma_{0,\Omega}=\operatorname{dom}\mathfrak{t}_{\widetilde{M}(z)}=\operatorname{ran}\gamma(0)^*,\quad z\in\rho(-\Delta_D).$$

Furthermore, it is clear from (3.22) that

$$\operatorname{ran}\Gamma_{0,\Omega}\times\Gamma_{1,\Omega}=\operatorname{ran}\gamma(0)^*\times L^2(\partial\Omega).$$

In particular, one can renormalize the regularized boundary mappings $\Gamma_{0,\Omega}=\gamma_N-\Lambda(0)\gamma_D$, $\Gamma_{1,\Omega}=\gamma_D$ also by any bounded operator G acting in the boundary space $L^2(\partial\Omega)$ such that $\operatorname{ran}G=\operatorname{ran}\gamma(0)^*$ and $\ker G=\{0\}$. This leads to an isomorphic copy of the results in Proposition 3.8. In this case the parametrization of all intermediate extensions of $A_{\min}$ can be expressed via boundary conditions involving $G^{-1}(\gamma_N-\Lambda(0)\gamma_D)$ and $G^*\gamma_D$; cf. Remark 3.4 (iii).

3.4. **Laplacian on rough domains.** Let Ω be a bounded domain in $\mathbb{R}^d$ $(d\geq 2)$ whose boundary $\partial\Omega$ is equipped with a finite $(d-1)$-dimensional Hausdorff measure σ, $\sigma(\partial\Omega)<\infty$. To construct an analog for the boundary triple appearing in Theorem 3.1 (i) in nonsmooth domains Ω we make use of some results established in [23] and [4, 5, 6]. Following Arendt and ter Elst [4, Definition 3.1] we first recall the notion of a trace $\varphi\in L^2(\sigma)$ for a class of functions $u\in H^1(\Omega)$.

Definition 3.10. A function $\varphi\in L^2(d\sigma)$ is said to be a trace of $u\in H^1(\Omega)$, if there is a sequence $u_n\in H^1(\Omega)\cap C(\overline{\Omega})$, such that

$$\lim_{n\to\infty}u_n=u\quad(\text{in } H^1(\Omega))\qquad\text{and}\qquad\lim_{n\to\infty}u_n|_{\partial\Omega}=\varphi\quad(\text{in } L^2(\sigma)).$$

Denote by $H^1_\sigma(\Omega)$ the set of elements of $H^1(\Omega)$ for which there exists a trace. In general, the trace is not uniquely defined. It is possible that $u\mid\Omega=0$ while its trace $\gamma_D u=u\mid\partial\Omega$ in $L^2(\sigma)$ is nontrivial; for an example see e.g. [4, Example 4.4]. Define the linear relation γ_D by

$$\gamma_D:=\{\{u,\varphi\}:\,u\in H^1_\sigma(\Omega),\,\varphi\in L^2(\sigma),\,\varphi\text{ is a trace of }u\}.$$

Then γ_D can be considered as a mapping from $H^1(\Omega)$ to $L^2(\sigma)$, which is linear but in general multivalued on the domain $H^1_\sigma(\Omega)$ and it has dense range in $L^2(\sigma)$; cf. [4]. If u and φ are as in Definition 3.10 we shall write

$$\varphi\in\gamma_D u.$$

The space $H^1_\sigma(\Omega)$ coincides with the closure of $H^1(\Omega)\cap C(\overline{\Omega})$ in the norm

$$\|u\|^2_{1,\sigma}=\|u\|^2_{H^1(\Omega)}+\int_{\partial\Omega}|u|^2\,d\sigma. \tag{3.23}$$

Following [4] denote by $\widetilde{H}^1(\Omega)$ the closure of $H^1(\Omega)\cap C(\overline{\Omega})$ in $H^1(\Omega)$. In view of (3.23) $H^1_\sigma(\Omega)$ is a subset of $\widetilde{H}^1(\Omega)$. Without additional conditions on Ω the space $\widetilde{H}^1(\Omega)$ need not be dense in $H^1(\Omega)$. Some sufficient conditions, like Ω being starshaped or having a continuous boundary, can be found e.g. in [61, Section 1.1.6]. Consequently, $H^1_\sigma(\Omega)$ is not necessarily a dense subset of $H^1(\Omega)$.

For associating an appropriate boundary triple in this setting, we impose the following additional assumption.

Assumption 3.11. $H^1_\sigma(\Omega) = \widetilde{H}^1(\Omega)$.

A list of conditions equivalent to Assumption 3.11 is given in [4, Theorem 6.1]. Notice that the space $H^1_{\mathcal{H}}(\Omega)$ appearing in [4, Section 5] has a norm which is equivalent to norm of $H^1_\sigma(\Omega)$ defined in (3.23) due to the following special case of Maz'ya inequality: there exists a constant $c_M > 0$ such that

$$\int_\Omega |u|^2\,dx \le c_M \left(\int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} |u|^2\,d\sigma \right) \tag{3.24}$$

holds for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$; see [61, Section 3.6], [4, eq. (5)]. The inequality (3.24) is a generalization of Friedrichs inequality to the case of rough domains.

In [4, Definition 3.2] the (weak) normal derivative is defined implicitly via Green's (first) formula as follows: a function $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ is said to have a weak normal derivative in $L^2(\sigma)$ if there exists $\psi \in L^2(\sigma)$ such that

$$\int_\Omega (\Delta u)\overline{v}\,dx + \int_\Omega \nabla u \cdot \overline{\nabla v}\,dx = \int_{\partial\Omega} \psi\overline{v}\,d\sigma \tag{3.25}$$

holds for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$, where Δu denotes the Laplacian understood in distributional sense. Since the functions $v \upharpoonright \partial\Omega$, $v \in H^1(\Omega) \cap C(\overline{\Omega})$, form a dense set in $L^2(\sigma)$, the function $\psi \in L^2(\sigma)$ is uniquely determined by u and the mapping $u \to \psi$ is denoted by γ_N:

$$\gamma_N u := \psi, \quad u \in \operatorname{dom} \gamma_N \subset H^1(\Omega) \cap \operatorname{dom} A_{\max}.$$

Assume that for some $\varphi, \psi \in L^2(\sigma)$, $u \in H^1(\Omega)$, and $x \le 0$ one has

$$(-\Delta - xI)u = 0, \quad \varphi \in \gamma_D u, \quad \psi = \gamma_N u, \quad x \le 0. \tag{3.26}$$

The operator $\Lambda(x)$ which maps φ to ψ is called the *Dirichlet-to-Neumann map*. A slight modification of the proof of [4, Theorem 3.3] shows, that $\Lambda(x)$ is a nonnegative selfadjoint operator on $L^2(\sigma)$ which is uniquely determined by the three properties listed in (3.26).

Now consider the differential expression $-\Delta$, where $\Delta = \nabla \cdot \nabla$ is the (distributional) Laplacian operator in Ω. Recall (see [6, Example 3.1]) that for an open set Ω (without any regularity on the boundary) the Dirichlet Laplacian $-\Delta_D$ is defined as the selfadjoint operator associated with the closed (Dirichlet) form

$$\tau_D(f,g) = \int_\Omega \nabla f \cdot \overline{\nabla g}\,dx, \quad \operatorname{dom} \tau_D = H^1_0(\Omega).$$

Similarly the Neumann Laplacian $-\Delta_N$ is defined as the selfadjoint operator associated with the closed form (see [6, Example 3.2])

$$\tau_N(f,g) = \int_\Omega \nabla f \cdot \overline{\nabla g}\,dx, \quad \operatorname{dom} \tau_N = \widetilde{H}^1(\Omega). \tag{3.27}$$

Theorem 3.12. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, whose boundary $\partial\Omega$ is equipped with a finite $(d-1)$-dimensional Hausdorff measure σ, let Assumption 3.11 be in force, and let the linear relation Γ be defined by

$$\Gamma = \left\{ \left\{ f, \begin{pmatrix} \varphi \\ -\psi \end{pmatrix} \right\} : \begin{array}{ccc} f \in \widetilde{H}^1(\Omega) \cap \operatorname{dom} \gamma_N, & \varphi, \psi \in L^2(\sigma), & \Delta f \in L^2(\Omega) \\ \varphi \in \gamma_D f, & \psi = \gamma_N f & \end{array} \right\}. \tag{3.28}$$

Then:

(i) the pair $\{L^2(\sigma), \Gamma\}$ is a positive unitary boundary pair for $-\Delta$ on $A_* := \operatorname{dom} \Gamma$;
(ii) for every $x < 0$ the Weyl function $M(x)$ of the pair $\{L^2(\sigma), \Gamma\}$ coincides (up to the sign) with the Dirichlet-to-Neumann map $\Lambda(x)$:

$$M(x) = -\Lambda(x), \quad x < 0, \tag{3.29}$$

in particular, the function $M(\cdot)$ is an inverse Stieltjes function whose values $M(z)$, $z \in \mathbb{C} \setminus [0, \infty)$, are (unbounded) operators with $\ker M(z) = \operatorname{mul} \Gamma_0$;
(iii) the operator $A_1 := -\Delta \upharpoonright \ker \Gamma_1$ coincides with the Neumann Laplacian $-\Delta_N$;
(iv) the transposed pair $\{L^2(\sigma), \Gamma^\top\}$ is S-generalized and the corresponding Weyl function $-M(\cdot)^{-1}$ is a multivalued domain invariant Stieltjes function.

Proof. (i)–(iii) If $f \in \operatorname{dom} \Gamma$, then the (first) Green's identity (3.25) holds with $u = f$ and $v \in H^1(\Omega) \cap C(\overline{\Omega})$. Then in view of (3.23) this identity can be extended to hold for all $v \in H^1_\sigma(\Omega)$. Thus, in particular, it holds for all $g := v \in \operatorname{dom} \Gamma$:

$$\int_\Omega (\Delta f)\overline{g}\, dx + \int_\Omega \nabla f \cdot \overline{\nabla g}\, dx = \int_{\partial\Omega} \psi \overline{\widetilde{\varphi}}\, d\sigma, \quad \psi = \gamma_N f,\ \widetilde{\varphi} \in \gamma_D g. \tag{3.30}$$

Similarly, one gets from (3.25) with $u = g \in \operatorname{dom} \Gamma$ and $v = f \in \operatorname{dom} \Gamma$:

$$\int_\Omega (\Delta g)\overline{f}\, dx + \int_\Omega \nabla g \cdot \overline{\nabla f}\, dx = \int_{\partial\Omega} \widetilde{\psi} \overline{\varphi}\, d\sigma, \quad \widetilde{\psi} = \gamma_N g,\ \varphi \in \gamma_D f.$$

Taking conjugates in the last identity and subtracting the identity (3.30) from that leads to Green's (second) formula in [26, eq: (3.1)] for $-\Delta$ with $f, g \in A_* = \operatorname{dom} \Gamma$. Thus, $\{L^2(\sigma), \Gamma\}$ is an isometric boundary pair.

To prove that $\{L^2(\sigma), \Gamma\}$ is a unitary boundary pair, we proceed by proving (ii) and (iii). With $x < 0$ it follows from (3.28) that $\varphi \in \operatorname{dom} M(x)$ and $M(x)\varphi = -\psi$ precisely when there exists $u \in \widetilde{H}^1(\Omega) \cap \operatorname{dom} \gamma_N$, such that

$$-\Delta u - xu = 0, \quad \varphi \in \gamma_D u, \quad \psi = \gamma_N u.$$

In view of (3.26) this means that the operator $-M(x)$ coincides with the Dirichlet-to-Neumann map $\Lambda(x)$, which is a nonnegative selfadjoint operator in $L^2(\sigma)$. This proves (3.29). The definition of Γ shows that $\operatorname{mul} \Gamma = \operatorname{mul} \Gamma_0 \times \{0\}$ and hence by [26, Lemma 3.6] $\ker M(z) = \operatorname{mul} \Gamma_0$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The assertion that $M(\cdot)$ is an inverse Stieltjes function is a consequence of $M(x) \leq 0$, $x < 0$, (the nonnegativity of the main transform $\widetilde{A}$, which is shown below, implies that $M(x)$ is also holomorphic at $x < 0$). This proves (ii).

By definition every $f \in \operatorname{dom}(-\Delta_N)$ belongs to $\widetilde{H}^1(\Omega)$. On the other hand, by Assumption 3.11 $\widetilde{H}^1(\Omega) = H^1_\sigma(\Omega) = \operatorname{dom} \gamma_D$ and hence, in particular, for every $f \in \operatorname{dom}(-\Delta_N)$ there exists a Dirichlet trace $\varphi \in \gamma_D f$. Next it is shown that for every

$f \in \operatorname{dom}(-\Delta_N)$ also the Neumann trace $\gamma_D u$ exists. Indeed, by definition the Neumann Laplacian $-\Delta_N$ is the selfadjoint operator associated with the closed form (3.27). Hence, (3.27) implies that for all $f \in \operatorname{dom}(-\Delta_N)$ and $g \in \widetilde{H}^1(\Omega)$,

$$\int_\Omega \nabla f \cdot \overline{\nabla g}\, dx = \int_\Omega (-\Delta f)\overline{g}\, dx.$$

Comparing this identity with the definition of γ_N it is seen that the equality (3.25) is satisfied with the choice $\psi = 0$. Therefore, $f \in \operatorname{dom}\gamma_N$ and $\gamma_N f = 0$. This implies that $\operatorname{dom}(-\Delta_N) \subset \operatorname{dom} A_*$ and, moreover, that $\operatorname{dom}(-\Delta_N) \subset \ker\gamma_N = \operatorname{dom} A_1$. Since $-\Delta_N$ is a selfadjoint operator in $L^2(\Omega)$ and A_1 is symmetric (see [26, Section 3.3], the equality $A_1 = -\Delta_N$ follows. This proves the assertion (iii).

Next we complete the proof of (i) by showing that $\{L^2(\sigma), \Gamma\}$ is a positive unitary boundary pair, i.e., that the main transform $\widetilde{A}$ of Γ given by

$$\widetilde{A} := \left\{ \left\{ \begin{pmatrix} f \\ \varphi \end{pmatrix}, \begin{pmatrix} -\Delta f \\ \psi \end{pmatrix} \right\} : \left\{ \begin{pmatrix} f \\ -\Delta f \end{pmatrix}, \begin{pmatrix} \varphi \\ -\psi \end{pmatrix} \right\} \in \Gamma \right\}$$

is a nonnegative selfadjoint relation in $L^2(\Omega) \times L^2(\sigma)$; see (2.23). Nonnegativity of $\widetilde{A}$ follows immediately from (3.30). On the other hand, by item (ii) the Weyl function satisfies $-M(x) = \Lambda(x) \geq 0$, $x < 0$, and hence it is a nonpositive selfadjoint operator with $-x \in \rho(M(x))$. Since $\operatorname{dom}(-\Delta_N) \subset \operatorname{dom} A_*$ and $-\Delta_N \geq 0$ is selfadjoint it follows from Theorem 2.12 that $x \in \rho(\widetilde{A})$ and hence $\widetilde{A} = \widetilde{A}^* \geq 0$, which proves the claim.

(iv) Since $A_1 = -\Delta_N$ is selfadjoint, the transposed pair $\{L^2(\sigma), \Gamma^\top\}$ is S-generalized; see [26, Definition 5.11]. Moreover, the value of the corresponding Weyl function $-M(x)^{-1} \geq 0$ is a nonnegative selfadjoint relation in $L^2(\sigma)$ for every $x < 0$. This implies that $-M(\cdot)^{-1}$ is a (multivalued) Stieltjes family (see Definition in [26, Section 2.1]). It is domain invariant by [26, Theorem 5.17]. □

In this general setting, the multivalued part of Γ can be nontrivial, since the trace γ_D need not be uniquely determined. For unitary boundary pairs the multivalued part is described in [28, Lemma 4.1] and for isometric boundary pairs in [26, Lemma 3.6]. In the present setting a more explicit description of the multivalued part can be given with the aid of a result of Daners in [23]; see also [6] for an other proof of Daners result via capacity arguments.

Corollary 3.13. There exists a Borel set $B_0 \subset \partial\Omega$, such that

$$\operatorname{mul}\gamma_D = L^2(B_0), \quad \operatorname{mul}\Gamma = \operatorname{mul}\gamma_D \times \{0\}.$$

and, in particular, $\operatorname{mul}\gamma_D = \ker M(\lambda)$, $\lambda \in \mathbb{C} \setminus [0, \infty)$.

Hence, Γ is single-valued if and only if $L^2(B_0) = \{0\}$, i.e., $\sigma(B_0) = 0$. The set B_0 is unique up to σ-equivalence $\sigma(B_0 \Delta \widetilde{B}_0) = 0$. Since $\operatorname{mul}\gamma_D \neq 0$ corresponds to $\sigma(B_0) > 0$, B_0 can be considered to represent an irregular part of the boundary.

Remark 3.14. In this general setting we do not know if the operator $A_0 := -\Delta \upharpoonright \ker\Gamma_0$ coincides with the Dirichlet Laplacian $-\Delta_D$. In other words, we do not know if the Neumann trace $\gamma_N u$ exists for every $u \in \operatorname{dom}(-\Delta_D)$.

4. Differential operators with local point interactions

4.1. **Abstract results on direct sums of boundary triples.** A general class of unitary boundary triples, which are more general than generalized boundary triples is obtained by considering an infinite orthogonal sum of ordinary boundary triples. Here we mainly follow the considerations in [52]; see also the references given therein.

Let S_n be a densely defined symmetric operator with equal defect numbers $n_+(S_n) = n_+(S_n)$ in the Hilbert space $\mathfrak{H}_n$, $n \in \mathbb{N}$. Consider the operator $A = \bigoplus_{n=1}^{\infty} S_n$ in the Hilbert space

$$\mathfrak{H} := \bigoplus_{n=1}^{\infty} \mathfrak{H}_n = \Big\{\bigoplus_{n=1}^{\infty} f_n : f_n \in \mathfrak{H}_n, \ \sum_{n=1}^{\infty} \|f_n\|^2 < \infty\Big\}.$$

Then A is symmetric with equal defect numbers and its adjoint A^* is given by

$$A^* = \bigoplus_{n=1}^{\infty} S_n^*, \quad \operatorname{dom} A^* = \Big\{\bigoplus_{n=1}^{\infty} f_n \in \mathfrak{H} : f_n \in \operatorname{dom} S_n^*, \ \sum_{n=1}^{\infty} \|S_n^* f_n\|^2 < \infty\Big\}. \tag{4.1}$$

Now let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be an ordinary boundary triple for S_n^*, $n \in \mathbb{N}$. Let $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$,
$\Gamma^{(n)} := \mathrm{col}\{\Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ and let the mapping Γ_0' and Γ_1' be defined by
(4.2)

$$\Gamma_j{}' := \bigoplus_{n=1}^{\infty} \Gamma_j^{(n)}, \quad \operatorname{dom} \Gamma_j' = \Big\{\bigoplus_{n=1}^{\infty} f_n \in \operatorname{dom} A^* : \sum_{n\in\mathbb{N}} \|\Gamma_j^{(n)} f_n\|^2_{\mathcal{H}_n} < \infty\Big\}, \quad j \in \{0,1\}.$$

We also put

$$\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} := \begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} \upharpoonright \operatorname{dom}\Gamma, \quad \text{where} \quad \operatorname{dom}\Gamma = \operatorname{dom}\Gamma_1' \cap \operatorname{dom}\Gamma_0'. \tag{4.3}$$

Then $\Gamma_j' = \overline{\Gamma_j}$, $j = 0, 1$. Denote by $\mathfrak{H}_+$ the domain $\operatorname{dom} A^*$ equipped with the graph norm of A^*. Clearly, $\operatorname{dom}\Gamma$ is dense in $\mathfrak{H}_+$. Define the operators $S_{n,j} := S_n^* \upharpoonright \ker \Gamma_j^{(n)}$ and $A_j' := \bigoplus_{n=1}^{\infty} S_{n,j}$, $j \in \{0,1\}$. Then A_0' and A_1' are selfadjoint extensions of A, which are are disjoint but not necessarily transversal. Finally, denote

$$A_* := A^* \upharpoonright \operatorname{dom}\Gamma \quad \text{and} \quad A_j := A_* \upharpoonright \ker \Gamma_j, \quad j \in \{0,1\}. \tag{4.4}$$

Clearly, $\overline{A_j} = A_j'$, hence A_j is essentially selfadjoint, $j \in \{0,1\}$.

The following result is contained in [52] (and stated here in the terminology of the present paper).

Theorem 4.1 ([52])**.** *Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be an ordinary boundary triple for S_n^*, let also $S_{n,j} = S_n^* \upharpoonright \ker \Gamma_j^{(n)}$, $j \in \{0,1\}$, and let $M_n(\cdot)$, $n \in \mathbb{N}$, be the corresponding Weyl function. Moreover, let the operators A^*, Γ_j' and Γ_j, $j \in \{0,1\}$, be given by (4.1), (4.2) and (4.3). Then:*

- *(i)* $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a unitary boundary triple for A^*;
- *(ii)* the Weyl function is the orthogonal sum $M(z) = \bigoplus_{n=1}^{\infty} M_n(z)$;
- *(iii)* the mapping $\Gamma_j : \mathfrak{H}_+ \to \mathcal{H}$ is closable and $\overline{\Gamma_j} = \Gamma_j'$, $j \in \{0,1\}$;
- *(iv)* The operator A_j given by (4.4) is essentially selfadjoint and $\overline{A_j} = \bigoplus_{n=1}^{\infty} S_{n,j} = A_j'$, $j \in \{0,1\}$.

The triple $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ in Theorem 4.1 is called the direct sum of Π_n and is denoted by $\Pi = \oplus_{n=1}^{\infty}\Pi_n$. The following result characterizes selfadjointness of A_j, $j \in \{0,1\}$, and completes Theorem 3.2 from [52].

Proposition 4.2. Let the assumptions be as in Theorem 4.1 and let $A_j = \ker \Gamma_j$, $j \in \{0,1\}$. Then

$$A_j = \bigoplus_{n=1}^{\infty} S_{n,j} \quad \Longleftrightarrow \quad \Gamma_{j'} \upharpoonright A_j \quad \text{is bounded} \quad (j' = 1 - j \in \{0,1\}). \tag{4.5}$$

In particular, A_0 satisfies (4.5) (i.e. $A_0 = A_0^*$) if and only if the corresponding Weyl function $M(\cdot)$ and the γ-field $\gamma(\cdot)$ satisfy one of the equivalent conditions in Theorem 1.3.

Similarly, A_1 satisfies (4.5) if and only if the Weyl function $-M^{-1}(\cdot)$ and γ-field $\gamma(\cdot)M^{-1}(\cdot)$ corresponding to the transposed boundary triple $\Pi^\top = \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ satisfy one of the equivalent conditions in Theorem 1.3.

Proof. Indeed, by [26, Proposition 5.5 (i)] $\Gamma_1 H(\lambda) = \gamma(\bar{\lambda})^*$ and hence $\Gamma_1 H(\lambda)$ is closed. Since A_0 is essentially selfadjoint, the equivalence $A_0 = A_0^* \Longleftrightarrow \Gamma_1 \upharpoonright A_0$ is bounded, is obtained from [26, Lemma 5.3 (iii), (v)]. All the other equivalent conditions for $A_0 = A_0^*$ hold by Theorem 1.3.

The criterion (4.5) and the other equivalent statements for $A_1 = A_1^*$ are obtained by passing to the transposed boundary triple $\{\mathcal{H}, \Gamma_1, -\Gamma_0\}$. □

Remark 4.3. The criterion (4.5) implies the sufficient conditions for A_0 and A_1 to be selfadjoint as established in [52, Theorem 3.2]. Namely, if Γ_1 or Γ_0 is bounded, then also the restriction $\Gamma_1 \upharpoonright A_0$ or $\Gamma_0 \upharpoonright A_1$, respectively, is bounded. Moreover, if A_0 and A_1 are transversal, i.e. $\operatorname{dom} A_0 + \operatorname{dom} A_1 = \operatorname{dom} A^*$, then clearly $\Gamma_{j'} \upharpoonright A_j$ is bounded $\Leftrightarrow$ $\Gamma_{j'}$ is bounded, since $\ker \Gamma_j = \operatorname{dom} A_j$ $(j' = 1 - j \in \{0,1\})$.

A criterion for a direct sum of ordinary boundary triples to form also an ordinary boundary triple can be formulated in terms of the corresponding Weyl functions (see [58, 52, 22]).

Theorem 4.4. Let $\Pi_n = \{\mathcal{H}_n, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be a boundary triple for S_n^*, let $M_n(\cdot)$ be the corresponding Weyl function, $n \in \mathbb{N}$, and let $A^* = \oplus_{n=1}^{\infty} S_n^*$.

- *(i)* The direct sum $\Pi = \oplus_{n=1}^{\infty}\Pi_n$ is an ordinary boundary triple for the operator A^* if and only if
$$C_1 = \sup_n \|M_n(i)\|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_2 = \sup_n \|(\operatorname{Im} M_n(i))^{-1}\|_{\mathcal{H}_n} < \infty.$$
- *(ii)* The direct sum $\Pi = \oplus_{n=1}^{\infty}\Pi_n$ is a B-generalized boundary triple for the operator A^* if and only if $C_1 < \infty$.
- *(iii)* If, in addition, the operators $\{S_{n,0}\}_{n\in\mathbb{N}}$ have a common gap $(a-\varepsilon, a+\varepsilon)$, then the direct sum $\Pi = \bigoplus_{n=1}^{\infty}\Pi_n$ is a B-generalized boundary triple for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if
$$C_3 := \sup_{n\in\mathbb{N}} \|M_n(a)\|_{\mathcal{H}_n} < \infty \quad \text{and} \quad C_4 := \sup_{n\in\mathbb{N}} \|M_n'(a)\|_{\mathcal{H}_n} < \infty, \tag{4.6}$$
where $M_n'(a) := (dM_n(z)/dz)|_{z=a}$.
- *(iv)* The direct sum $\Pi = \bigoplus_{n=1}^{\infty}\Pi_n$ is an ordinary boundary triple for A^* if and only if in addition to (4.6) the following condition is fulfilled $C_5 := \sup_{n\in\mathbb{N}} \|\big(M_n'(a)\big)^{-1}\|_{\mathcal{H}_n} < \infty$.

The next result contains analogous characterization for S-generalized boundary triples.

Proposition 4.5. Assume the conditions of Theorem 4.1. Then the direct sum $\Pi = \bigoplus_{n=1}^{\infty} \Pi_n$ forms an S-generalized boundary triple for $A^* = \bigoplus_{n=1}^{\infty} S_n^*$ if and only if

$$\sup_n \|\mathrm{Im}\, M_n(i)\|_{\mathcal{H}_n} < \infty. \tag{4.7}$$

Similarly, if the operators $(S_{n,0})$ have a common gap $(a-\varepsilon, a+\varepsilon)$, then Π forms an S-generalized boundary triple for A^* if and only if $C_4 < \infty$ where C_4 is given by (4.6).

Proof. The condition (4.7) means that $\mathrm{Im}\, M(z)$ is bounded for some (equivalently for every) $z \in \mathbb{C}_\pm$. By Theorem 1.3, this amounts to saying that Π is an S-generalized boundary triple for A^*.

Similarly, in case of a common spectral gap $(a-\varepsilon, a+\varepsilon)$ the condition (4.7) is equivalent to the condition $C_4 < \infty$ in (4.6) as can be seen by the same argument that was used in [26, Remark 5.25]. □

The next result is immediate by combining Proposition 2.8 in (4.6) with Proposition 2.9.

Corollary 4.6. Assume the assumptions of Theorem 4.1. Then the following statements are equivalent:

(i) $\Gamma_0 : A_* \to \mathcal{H}$ is bounded;
(ii) $C_2 = \sup_n \|(\mathrm{Im}\, M_n(i))^{-1}\|_{\mathcal{H}_n} < \infty$.

In this case the transposed boundary triple $\Pi^\top = \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ is B-generalized.

Similarly, the following conditions are equivalent:

(i)′ $\Gamma_1 : A_* \to \mathcal{H}$ is bounded;
(ii)′ $C_2^\top := \sup_n \|(\mathrm{Im}\,(M_n^{-1}(i)))^{-1}\|_{\mathcal{H}_n} < \infty$.

In this case the triple Π is a B-generalized boundary triple.

Proof. By Theorem 4.1 (see [52, Theorem 3.2]) Π is a unitary boundary triple such that $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ are essentially selfadjoint. Now the first part of the statement follows easily from Proposition 2.9, while the second part is implied by Proposition 2.8. □

4.2. **Momentum operators with local point interactions.** Let $X = \{x_n\}_1^\infty$ be a strictly increasing sequence of positive numbers satisfying $\lim_{n\to\infty} x_n = \infty$ and let

$$d_n := x_n - x_{n-1} > 0, \quad (x_0 := 0), \quad 0 \le d_* := \inf_{n\in\mathbb{N}} d_n, \quad d^* := \sup_{n\in\mathbb{N}} d_n \le \infty. \tag{4.8}$$

Define a symmetric differential operator D_n in $\mathcal{H}_n := L^2([x_{n-1}, x_n])$ by

$$D_n = -i\frac{d}{dx}, \qquad \mathrm{dom}\, D_n = W_0^{1,2}([x_{n-1}, x_n]), \qquad n \in \mathbb{N}.$$

In quantum mechanics this operator in 1-D case appears in the form $-i\hbar\frac{d}{dx}$, where $\hbar = h/2\pi$ is the reduced Planck constant and whose eigenvalues are measuring the momentum of a particle.

The adjoint of the operator D_n is given by $D_n^* = -i\frac{d}{dx}$ with $\mathrm{dom}\, D_n^* = W^{1,2}([x_{n-1}, x_n])$, $n \in \mathbb{N}$. Following [58] associate with D_n^* a boundary triple $\Pi_n = \{\mathbb{C}, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ by setting

$$\Gamma_0^{(n)} f_n := i\frac{f_n(x_n - 0) - f_n(x_{n-1} + 0)}{\sqrt{2}}, \quad \Gamma_1^{(n)} f_n := \frac{f_n(x_n - 0) + f_n(x_{n-1} + 0)}{\sqrt{2}}. \tag{4.9}$$

The Weyl function $M_n(\cdot)$ corresponding to the triple Π_n is given by

$$M_n(z) = -i\frac{e^{izx_n} + e^{izx_{n-1}}}{e^{izx_n} - e^{izx_{n-1}}} = -\cot(2^{-1}zd_n), \qquad z \in \mathbb{C}_\pm. \tag{4.10}$$

Let $D_X := \bigoplus_1^\infty D_n$. Then $D_X^* = \bigoplus_1^\infty D_n^*$ and $\operatorname{dom} D_X^* = W^{1,2}(\mathbb{R}_+\backslash X) = \bigoplus_{n=1}^\infty W^{1,2}([x_{n-1}, x_n])$.

Next we describe the main properties of a boundary triple $\Pi := \bigoplus_{n=1}^\infty \Pi_n$ assuming that $d_* = 0$ partially treated in [58]. To this end we first recall a complete trace characterization of the space $W^{1,2}(\mathbb{R}_+ \setminus X)$ (see [22, Proposition 3.5]). Due to the embedding theorem, the trace mappings

$$\pi_\pm : W^{1,2}(\mathbb{R}_+ \setminus X) \to l^2(\mathbb{N}), \quad \pi_+(f) = \{f(x_{n-1}+)\}_1^\infty, \quad \pi_-(f) = \{f(x_n-)\}_1^\infty,$$

are well defined for functions with compact supports, i.e., $f \in \bigoplus_1^N W^{1,2}[x_{n-1}, x_n]$, $N \in \mathbb{N}$. We assume $\pi_\pm$ to be defined on its maximal domain

$$\operatorname{dom}(\pi_\pm) := \{f \in W^{1,2}(\mathbb{R}_+ \setminus X) : \ \pi_\pm f \in l^2(\mathbb{N})\}.$$

Clearly, $\operatorname{dom}(\pi_\pm)$ is dense in $W^{1,2}(\mathbb{R}_+\backslash X)$ although, in general, $\operatorname{dom}(\pi_\pm) \neq W^{1,2}(\mathbb{R}_+\backslash X)$.

Lemma 4.7 ([22])**.** Let $X = \{x_n\}_{n=1}^\infty$ be as above with $x_0 = 0$ and $X \subset \overline{\mathbb{R}}_+$. Then:

(i) For any pair of sequences $a^\pm = \{a_n^\pm\}_1^\infty$ satisfying

$$a^\pm = \{a_n^\pm\}_1^\infty \in l^2(\mathbb{N}; \{d_n\}) \qquad \text{and} \qquad \{a_n^+ - a_n^-\}_1^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\}), \tag{4.11}$$

there exists a (non-unique) function $f \in W^{1,2}(\mathbb{R}_+ \setminus X)$ such that $\pi_\pm(f) = a^\pm$. Moreover, the mapping $\pi_+ - \pi_- : W^{1,2}(\mathbb{R}_+ \setminus X) \to l^2(\mathbb{N}; \{d_n^{-1}\})$ is surjective and contractive, i.e.

$$\sum_{n\in\mathbb{N}} d_n^{-1}|f(x_n-) - f(x_{n-1}+)|^2 \leq \|f\|^2_{W^{1,2}(\mathbb{R}_+\setminus X)}, \quad f \in W^{1,2}(\mathbb{R}_+ \setminus X).$$

(ii) Assume in addition, that $d^* < \infty$. Then the mapping $\pi_\pm$ can be extended to a bounded surjective mapping from $W^{1,2}(\mathbb{R}_+ \setminus X)$ onto $l^2(\mathbb{N}; \{d_n\})$. Moreover, the following estimate holds

$$\sum_{n\in\mathbb{N}} d_n \left(|f(x_{n-1}+)|^2 + |f(x_n-)|^2\right) \leq C_1\|f\|^2_{W^{1,2}(\mathbb{R}_+\setminus X)}, \tag{4.12}$$

for any $f \in W^{1,2}(\mathbb{R}_+ \setminus X)$ where $C_1 := 4\max\{(d^*)^2, 1\}$. Besides, the traces $a^\pm := \pi_\pm(f)$ of each $f \in W^{1,2}(\mathbb{R}_+ \setminus X)$ satisfy conditions (4.11). Moreover, the assumption $d^* < \infty$ is necessary for the inequality (4.12) to hold with some $C_1 > 0$.

Now we are ready to state and prove the main result of this subsection.

Proposition 4.8. Let X be as above, let $d_* = 0$ and $d^* < \infty$, let $\Pi^{(n)} = \{\mathbb{C}, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be the boundary triple for the operator D_n^* defined by (4.9). Let $\Pi = \bigoplus_{n=1}^\infty \Pi^{(n)} =: \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\mathcal{H} = l^2(\mathbb{N})$, $D_X := \oplus_{n=1}^\infty D_n$, $D_{X,*} = D_X^* |\operatorname{dom}\Gamma$ and Γ_j', Γ_j, $j \in \{0, 1\}$ are given by (4.2) and (4.3). Then:

(i) The mapping $\Gamma_0' \times \Gamma_1'$ can be extended to the mapping

$$\Gamma_0'' \times \Gamma_1'' : W^{1,2}(\mathbb{R}_+ \setminus X) \ \mapsto l^2\big(\mathbb{N}; \{d_n^{-1}\}\big) \times l^2\big(\mathbb{N}; \{d_n\}\big),$$

which is well defined and surjective. Besides, $\ker(\Gamma_0'' \times \Gamma_1'') = W_0^{1,2}(\mathbb{R}_+ \setminus X)$.

(ii) The mapping

$$\Gamma_0 \times \Gamma_1 : \operatorname{dom} D_{X,*} = \operatorname{dom} \Gamma \mapsto l^2\big(\mathbb{N}; \{d_n^{-1}\}\big) \times l^2\big(\mathbb{N}\big)\big(\subset l^2\big(\mathbb{N}\big) \otimes \mathbb{C}^2\big),$$

is well defined and surjective. Moreover, Γ_0 boundedly maps $\operatorname{dom} D_{X,*}$ in $l^2(\mathbb{N})$.

(iii) The Weyl function $M(\cdot)$ is domain invariant and its domain is given by

$$\operatorname{dom} M(z) = l^2\big(\mathbb{N}; \{d_n^{-2}\}\big)\big(\subsetneq \operatorname{ran} \Gamma_0 = \Gamma_0(\operatorname{dom} A_*) = l^2\big(\mathbb{N}; \{d_n^{-1}\}\big)\big), \quad z \in \mathbb{C}_\pm. \tag{4.13}$$

(iv) The domain of the form $\mathfrak{t}_{M(z)}$ associated with the imaginary part $\operatorname{Im} M(z)$ is given by

$$\operatorname{dom} \mathfrak{t}_{M(z)} = \left\{\{a_n\}_{n=1}^\infty \in l^2(\mathbb{N}) : \{a_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\})\right\}, \quad z \in \mathbb{C}_\pm. \tag{4.14}$$

(v) The triple Π is an ES-generalized boundary triple for D_X^* and $A_0 \neq A_0^*$. Moreover, the imaginary part $\operatorname{Im} M(\cdot)$ of the Weyl function $M(\cdot)$ takes values in $\mathcal{C}(\mathcal{H}) \setminus \mathcal{B}(\mathcal{H})$.

(vi) The transposed triple $\Pi^\top$ is B-generalized (but not an ordinary) boundary triple for the operator D_X^*. In particular, the Weyl function $-M(\cdot)^{-1}$ takes values in $\mathcal{B}(\mathcal{H})$, and $A_1 = A_1^*$.

Proof. (i) The proof is immediate from Lemma 4.7(1).

(ii) Since $d^* < \infty$, the space $l^2(\mathbb{N})$ is (continuously) embedded in $l^2\big(\mathbb{N}; \{d_n\}\big)$. Therefore the surjectivity is immediate from (*i*). By Lemma 4.7 (i), the mapping $\Gamma_0 : \operatorname{dom} D_{X,*} \mapsto l^2\big(\mathbb{N}; \{d_n\}^{-1}\big)$ is bounded. To prove the boundedness of $\Gamma_0 :\ \operatorname{dom} D_{X,*} = \operatorname{dom} \Gamma \mapsto l^2(\mathbb{N})$ it remains to note that the embedding $l^2\big(\mathbb{N}; \{d_n\}^{-1}\big) \hookrightarrow l^2(\mathbb{N})$ is continuous since $d^* < \infty$.

(iii) In accordance with (4.10) $M_n(z) = -\cot(2^{-1}d_n z)$. Therefore the description of $\operatorname{dom} M(\cdot)$ follows from the obvious relation

$$\cot(2^{-1}zd_n) \sim 2z^{-1}d_n^{-1} \qquad \text{as} \qquad d_n \to 0, \qquad z \in \mathbb{C}_\pm. \tag{4.15}$$

(iv) Notice that $\{a_n\}_{n=1}^\infty \in \operatorname{dom} \mathfrak{t}_{M(z)}$ if and only if $\sum_{n=1}^\infty (\operatorname{Im} M_n(z)a_n, a_n) < \infty$. It follows from (4.10) and (4.15) that $\operatorname{Im} M_n(x+iy) \sim \frac{2y}{x^2+y^2} d_n^{-1}$ as $n \to \infty$. Therefore,

$$\sum_{n=1}^\infty (\operatorname{Im} M_n(z)a_n, a_n) < \infty \Longleftrightarrow \sum_{n=1}^\infty |a_n|^2 d_n^{-1} < \infty.$$

(v) Being a direct sum of ordinary boundary triples, the triple $\Pi = \bigoplus_{n=1}^\infty \Pi^{(n)}$ is an ES-generalized boundary triple in accordance with Theorem 4.1(iv). The relation $A_0 \neq A_0^*$ is implied by item (iii) since the inclusion $\operatorname{dom} M(z) \subsetneq \operatorname{ran} \Gamma_0$ is strict.

Furthermore, relation (4.10) implies $M_n(i) = i \operatorname{cth}(2^{-1}d_n)$. It follows that $\operatorname{Im} M_n(i) = \operatorname{cth}(2^{-1}d_n)$, $n \in \mathbb{N}$. Hence the values of imaginary part $\operatorname{Im} M(\cdot)$ are unbounded, $\operatorname{Im} M(\cdot) \in \mathcal{C}(\mathcal{H}) \setminus \mathcal{B}(\mathcal{H})$. Due to Theorem 1.3 this last property gives another proof for the fact that the triple Π is not S-generalized.

(vi) It follows from (4.10) that $-M_n^{-1}(z) = \tan(2^{-1}d_n z)$. Therefore the Weyl function $-M^{-1}(\cdot) = \oplus_1^\infty(-M_n^{-1}(\cdot)) \in R^s[\mathcal{H}]$. By [26, Theorem 1.7] the transposed triple $\Pi^\top$ is B-generalized. $\square$

Remark 4.9. (i) Note that statements (iii)–(vi) remain valid for $d^* = \infty$.

(ii) Assuming that $d^* < \infty$ it is shown in [58] that the triple $\Pi = \oplus_{n\in\mathbb{N}}\Pi_n$ is an ordinary boundary triple for D_X^* if and only if $d_* > 0$. This result remains true also in the case $d^* = \infty$.

(iii) Let $G = \mathrm{diag}\,\{(\widetilde{d}_1)^{1/2}, \dots, (\widetilde{d}_n)^{1/2}, \dots\}$ be the diagonal operator defined on $\mathcal{H} = l^2(\mathbb{N})$, with $\widetilde{d}_n = \min\{1, d_n\}$, $n \in \mathbb{N}$. In accordance with Theorem 4.8 (iv),

$$\mathrm{ran}\, G = \mathrm{dom}\, G^{-1} = \mathrm{dom}\, \mathfrak{t}_{M(i)}.$$

Hence the renormalization in [26, Theorem 5.32] is determined via the formulas $\widetilde{\Gamma}_0 = G^{-1}\Gamma_0$, $\widetilde{\Gamma}_1 = G\Gamma_0$ and the corresponding Weyl function is given by

$$M_G(z) = G^* M(z) G = -\bigoplus_{n=1}^{\infty} \widetilde{d}_n \cot(2^{-1} z d_n).$$

Since $\widetilde{d}_n \mathrm{Im}\, M_n(i) \to 2$ as $d_n \to 0$, we conclude that (the closure of) $M_G(\cdot)$ is a bounded uniformly strict Nevanlinna function, $M_G(\cdot) \in R^u[\mathcal{H}]$. Thus, the renormalization procedure in this case leads to an ordinary boundary triple for D_X^*. In the case $d^* < \infty$ this renormalization procedure was firstly applied in [58] to construct the above mentioned ordinary boundary triple for D_X^*; see Examples 3.2, 3.8 and Theorem 3.6 in [58].

4.3. **Schrödinger operators with local point interactions.** Let $X = \{x_n\}_1^\infty$ be a strictly increasing sequence of positive numbers satisfying $\lim_{n\to\infty} x_n = \infty$ and let d_n, d_*, and d^* be defined by (4.8). Let also H_n be a minimal operator associated with expression $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in $L^2[x_{n-1}, x_n]$. Clearly, H_n is a closed symmetric, $n_\pm(\mathrm{H}_n) = 2$, and its domain is $\mathrm{dom}\,(\mathrm{H}_n) = W_0^{2,2}[x_{n-1}, x_n]$.

It is easily seen that a boundary triple $\Pi_n = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ for H_n^* can be chosen as

$$\Gamma_0^{(n)} f := \begin{pmatrix} f'(x_{n-1}+) \\ f'(x_n-) \end{pmatrix}, \quad \Gamma_1^{(n)} f := \begin{pmatrix} -f(x_{n-1}+) \\ f(x_n-) \end{pmatrix}, \qquad f \in W_2^2[x_{n-1}, x_n]. \tag{4.16}$$

The corresponding Weyl function M_n is given by

$$M_n(z) = \frac{-1}{\sqrt{z}} \begin{pmatrix} \cot(\sqrt{z} d_n) & -\frac{1}{\sin(\sqrt{z} d_n)} \\ -\frac{1}{\sin(\sqrt{z} d_n)} & \cot(\sqrt{z} d_n) \end{pmatrix}. \tag{4.17}$$

Consider in $L^2(\mathbb{R}_+)$ the direct sum of symmetric operators H_n, $\mathrm{H} := \mathrm{H}_{\min} = \oplus_{n=1}^\infty \mathrm{H}_n$, $\mathrm{dom}\,(\mathrm{H}_{\min}) = W_0^{2,2}(\mathbb{R}_+ \setminus X) = \bigoplus_{n=1}^\infty W_0^{2,2}[x_{n-1}, x_n]$.

We note that $\mathrm{dom}\,\Gamma$ is dense in $\mathrm{dom}\,(\mathrm{H}^*)$ equipped with the graph norm while in general it is narrower than $\mathrm{dom}\,(\mathrm{H}^*)$. As was shown in [51], the triple $\Pi = \oplus_{n\in\mathbb{N}} \Pi_n := \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for the operator $\mathrm{H}_{\max} := \mathrm{H}_{\min}^*$ whenever

$$0 \;<\; d_* = \inf_{n\in\mathbb{N}} d_n \quad\le\quad d^* = \sup_{n\in\mathbb{N}} d_n \;<\; +\infty.$$

The converse statement is also true (see [52]): the condition $d_* > 0$ is necessary for the direct sum $\Pi = \oplus_{n\in\mathbb{N}} \Pi_n$ to form a boundary triple for $\mathrm{H}_{\max} := \mathrm{H}_{\min}^*$.

Such type of boundary triples have naturally arisen in investigation of spectral properties of the Hamiltonian $\mathrm{H}_{X,\alpha}$ associated in $L^2(\mathbb{R}_+)$ with a formal differential expression

$$\ell_{X,\alpha} := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{x_n \in X} \alpha_n \delta(x - x_n), \qquad \alpha = (\alpha_n)_{n=0}^\infty \subset \mathbb{R}, \tag{4.18}$$

when treating $\mathrm{H}_{X,\alpha}$ as an extension of $\mathrm{H}_{\min}$ (see [51, 52], and Remark 4.15 below).

Theorem 4.10. Let Π_n, $n \in \mathbb{N}$, be the boundary triple given by (4.16), let $M_n(\cdot)$ be the corresponding Weyl function, $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^2$, let $\Pi := \bigoplus_{n=1}^\infty \Pi_n = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the direct sum of triples Π_n given by (4.2) and (4.3) and let $H_* := H^*|_{\operatorname{dom}\Gamma}$. Assume also that $d_* = 0$ and $d^* \le \infty$. Then the following statements hold:

(i) The triple Π is an ES-generalized boundary triple for $H^*_{\min}$ such that $A_0 \ne A_0^*$.

(ii) The Weyl function $M(\cdot)$ is domain invariant and with $z \in \mathbb{C}_\pm$ one has

$$\operatorname{dom} M(z) = \left\{ \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n - b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-2}\}) \right\}. \tag{4.19}$$

(iii) Let in addition $d^* < \infty$. Then the range of Γ_0 is given by

$$\operatorname{ran}\Gamma_0 = \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n - b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\}) \right\}, \tag{4.20}$$

in particular, the proper inclusions $\operatorname{dom} M(\pm i) \subsetneqq \operatorname{ran}\Gamma_0$ hold.

(iv) The domain of the form $\mathfrak{t}_{M(z)}$ generated by the imaginary part $\operatorname{Im} M(z)$, $z \in \mathbb{C}_\pm$, is given by

$$\operatorname{dom} \mathfrak{t}_{M(z)} = \left\{ \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n - b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\}) \right\}. \tag{4.21}$$

In particular, if $d^* < \infty$, then $\operatorname{dom} \mathfrak{t}_{M(z)} = \operatorname{ran}\Gamma_0$.

(v) The transposed triple $\Pi^\top$ is an S-generalized boundary triple for $H^*_{\min}$, i.e. $A_1 = A_1^*$. However, it is not a B-generalized boundary triple for $H^*_{\min}$.

(vi) The Weyl function $M^\top(\cdot) = -M(\cdot)^{-1}$ corresponding to the transposed boundary triple $\Pi^\top$ is domain invariant and its domain for $z \in \mathbb{C}_\pm$ is given by

$$\operatorname{dom} M^\top(z) = \left\{ \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n + b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-2}\}) \right\}.$$

(vii) The domain of the form $\mathfrak{t}_{M^\top(z)}$ generated by the imaginary part $\operatorname{Im} M^\top(z)$, $z \in \mathbb{C}_\pm$, is given by

$$\operatorname{dom} \mathfrak{t}_{M^\top(z)} = \left\{ \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2 : \{a_n + b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\}) \right\}.$$

Proof. (i) By Theorem 4.1(iv), the triple Π is an ES-generalized boundary triple for $\mathrm{H}^*_{\min}$. Fix $z \in \mathbb{C} \setminus \mathbb{R}$. It follows from (4.17) that

$$\lim_{d_n \to 0} d_n M_n(z) = \frac{-1}{z} K, \qquad \lim_{d_n \to 0} d_n \operatorname{Im} M_n(i) = K, \quad \text{where} \quad K = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{4.22}$$

Since $d_* = 0$, the last relation yields $\sup_n \|\operatorname{Im} M_n(i)\| = \infty$. Therefore, Proposition 4.5 implies $A_0 \ne A_0^*$.

(ii) By Theorem 4.1(ii), the Weyl function of $\Pi = \bigoplus \Pi_n$ is $M(\cdot) = \bigoplus_{n=1}^\infty M_n(\cdot)$, where $M_n(\cdot)$ is given by (4.17). By definition, $\{h_n\}_{n=1}^\infty \in \operatorname{dom} M(z)$ if and only if

$$\sum_{n=1}^\infty \|M_n(z) h_n\|^2 < \infty; \quad \{h_n\}_{n=1}^\infty = \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2. \tag{4.23}$$

It follows from (4.17) that $\|M_n(z)\|$ as a function of d_n is bounded on the intervals $[\delta, \infty)$, $\delta > 0$.

Combining this fact with the first relation in (4.22) and noting that $d_* = 0$ and $\frac{\sin(\sqrt{z}d_n)}{\sqrt{z}d_n} \sim 1$ as $d_n \to 0$, one concludes that the convergence of the series in (4.23) is equivalent to

$$\sum_{n=1}^{\infty} \frac{|a_n - b_n|^2}{d_n^2} < \infty,$$

i.e. to the inclusion $\{a_n - b_n\}_{n=1}^{\infty} \in l^2(\mathbb{N}; \{d_n^{-2}\})$.

(iii) The proof is postponed after Lemma 4.12.

(iv) The proof is similar to that of the item (ii). First notice that $\{h_n\}_{n=1}^{\infty} \in \operatorname{dom} \mathfrak{t}_{M(z)}$ if and only if

$$\sum_{n=1}^{\infty} (\operatorname{Im} M_n(z)h_n, h_n) < \infty, \quad \{h_n\}_{n=1}^{\infty} = \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right\}_{n=1}^{\infty} \in l^2(\mathbb{N}) \otimes \mathbb{C}^2. \tag{4.24}$$

Note that $\operatorname{Im} M_n(z)$ as a function of d_n is bounded on the intervals $[\delta, \infty)$, $\delta > 0$. It follows from (4.17) that

$$\lim_{d_n \to 0} \left(M_n(z) + \frac{1}{d_n z} K \right) = 0, \quad \text{where} \quad K = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

Hence the convergence of the series in (4.24) is equivalent to

$$\sum_{n=1}^{\infty} \operatorname{Im} \left(\frac{(Kh_n, h_n)}{zd_n} \right) = \frac{y}{x^2 + y^2} \sum_{n=1}^{\infty} \frac{|a_n - b_n|^2}{d_n} < \infty, \quad z = x + iy \in \mathbb{C}_+.$$

This proves the statement.

(v) The Weyl function $M^\top(\cdot)$ corresponding to the transposed boundary triple $\Pi^\top$ is $M^\top(\cdot) = \oplus_1^\infty M_n^\top(\cdot)$, where $M_n^\top(\cdot) = -M_n^{-1}(\cdot)$ is given by

$$M_n^\top(z) = -\sqrt{z} \begin{pmatrix} \cot(\sqrt{z}d_n) & \frac{1}{\sin(\sqrt{z}d_n)} \\ \frac{1}{\sin(\sqrt{z}d_n)} & \cot(\sqrt{z}d_n) \end{pmatrix}.$$

It follows that

$$\lim_{d_n \to \infty} M_n^\top(z) = \pm i\sqrt{z} I_2, \qquad \lim_{d_n \to 0} d_n M_n^\top(z) = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad z \in \mathbb{C}_\pm. \tag{4.25}$$

Since $d_* = 0$, the last relation shows that the Weyl function $M^\top(\cdot)$ takes unbounded values.

On the other hand, using the Laurent series expansions for $\cot z$ and $(\sin z)^{-1}$ at 0 gives

$$\lim_{d_n \to 0} d_n^{-1} \operatorname{Im} M_n^\top(z) = (\operatorname{Im} z) \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}, \quad z \in \mathbb{C}_\pm.$$

Hence, $\operatorname{Im} M_n^\top(z)$ is uniformly bounded as a function of $d_n \in (0, \infty)$ for every $z \in \mathbb{C} \setminus \mathbb{R}$. Therefore Proposition 4.5 ensures that the transposed boundary triple $\Pi^\top$ is S-generalized. At the same time $\Pi^\top$ is not B-generalized, since $M^\top(\cdot)$ takes values in $\mathcal{C}(\mathcal{H}) \setminus \mathcal{B}(\mathcal{H})$.

(vi) The proof is similar to that of the statement (ii). One should only use relations (4.25) instead of (4.22).

(vii) The proof is similar to that of (iv). $\square$

Remark 4.11. Here we show that the triple $\Pi = \oplus_{n\in\mathbb{N}}\Pi_n := \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is an ordinary boundary triple for the operator $\mathrm{H}^*_{\min}$ if and only if $d_* > 0$. This statement extends the corresponding results from [51, 52], to the case $d^* = \infty$.

Consider the behavior of $M_n(z)$ as $d_n \to \infty$. The functions $\cot(\sqrt{z}d)$ and $(\sin(\sqrt{z}d))^{-1}$ depend continuously on $d \in (0,\infty)$ and $\lim_{d\to\infty}\cot(\sqrt{z}d) = -i$ and $\lim_{d\to\infty}(\sin(\sqrt{z}d))^{-1} = 0$. Therefore for any fixed $z \in \mathbb{C}\setminus\mathbb{R}$ the matrix function $M_n(z)$ in (4.17) is continuous and bounded in $d_n \in [\delta,\infty)$ for every $\delta > 0$.

Furthermore, clearly $\lim_{d_n\to\infty} \operatorname{Im} M_n(z) = \pm I_2$ for $z \in \mathbb{C}_\pm$ and this implies that for every fixed $z \in \mathbb{C}_+$ there exists $c_\delta(z) > 0$ such that

$$\operatorname{Im} M_n(z) \ge c_\delta(z) I_2, \quad d_n \in [\delta, d^*], \quad \delta > 0.$$

Thus, by Theorem 4.4 (i) Π is an ordinary boundary triple for the operator $H^*_{\min}$, whenever $d_* > 0$ and, in particular, $A_0 = A_0^*$ and $A_1 = A_1^*$ are transversal extensions of $H_{\min}$ in this case.

It remains to prove the assertion (iii) of Theorem 4.10. It is more involved and to this end we describe traces of functions $f \in W^{2,2}(\mathbb{R}_+\setminus X)$ as well as traces of their first derivatives and prove an analog of Lemma 4.7.

Lemma 4.12. Let $X = \{x_n\}_{n=1}^\infty$ be as above and let $0 \le d_* \le d^* < \infty$. Then the mapping $\Gamma_0'' : W^{2,2}(\mathbb{R}_+\setminus X)\otimes\mathbb{C}^2 \to l^2(\mathbb{N};\{d_n^3\})$ defined by

$$\Gamma_0'' : f \to \left\{ \begin{pmatrix} f'(x_{n-1}+) \\ f'(x_n-) \end{pmatrix} \right\}_{n\in\mathbb{N}}$$

is well defined and bounded and its range $\operatorname{ran}\Gamma_0''$ is given by

$$\Gamma_0''\big(W^{2,2}(\mathbb{R}_+\setminus X)\big) = \left\{ \begin{pmatrix} a_n \\ b_n \end{pmatrix}_{n\in\mathbb{N}} \in l^2(\mathbb{N};\{d_n^3\})\otimes\mathbb{C}^2 : \{a_n - b_n\}_{n\in\mathbb{N}} \in l^2(\mathbb{N};\{d_n^{-1}\}) \right\}. \tag{4.26}$$

Proof. Denote temporarily the right-hand side of (4.26) by $\mathcal{R}(\Gamma_0'')$. First we prove the inclusion $\operatorname{ran}\Gamma_0'' = \Gamma_0''(W^{2,2}(\mathbb{R}_+\setminus X)) \subset \mathcal{R}(\Gamma_0'')$. Let $f \in W^{2,2}(\mathbb{R}_+\setminus X)$. This inclusion implies $f \in W^{2,2}[x_{n-1}, x_n]$ for each $n \in \mathbb{N}$ and, it is easy to check that

$$d_n|f(x)|^2 \le 2\left(\|f\|^2_{L^2(\Delta_n)} + d_n^2\|f'\|^2_{L^2(\Delta_n)}\right), \qquad x \in \Delta_n := [x_{n-1}, x_n], \quad n \in \mathbb{N}. \tag{4.27}$$

Moreover, by the Sobolev embedding theorem (cf. e.g. [2], [49, p. 192]) there exist constants $c_0, c_1 > 0$ not depending on f and $n \in \mathbb{N}$ such that

$$\|f'\|^2_{L^2(\Delta_n)} \le c_1 d_n^2\|f''\|^2_{L^2(\Delta_n)} + c_0 d_n^{-2}\|f\|^2_{L^2(\Delta_n)}, \qquad x \in \Delta_n, \quad n \in \mathbb{N}. \tag{4.28}$$

By applying (4.27) to f' and combining the result with (4.28) shows that

$$d_n^2|f'(x)|^2 \le C_1 d_n^3\|f''\|^2_{L^2(\Delta_n)} + C_0 d_n^{-1}\|f\|^2_{L^2(\Delta_n)}, \qquad x \in \Delta_n, \quad n \in \mathbb{N},$$

where C_0 and C_1 do not depend on f and $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} \sum_n d_n^3\big(|f'(x_n-)|^2 + |f'(x_{n-1}+)|^2\big) &\le 2C_1\sum_n d_n^4\|f''\|^2_{L^2(\Delta_n)} + 2C_0\sum_n \|f\|^2_{L^2(\Delta_n)} \\ &\le 2C_1(d^*)^4\|f''\|^2_{L^2(\mathbb{R}_+)} + 2C_0\|f\|^2_{L^2(\mathbb{R}_+)} \le C_3\|f\|^2_{W^{2,2}(\mathbb{R}_+\setminus X)}, \end{aligned} \tag{4.29}$$

where $C_3 = 2\max\{C_0, C_1(d^*)^4\}$. Hence, the mapping Γ_0'' is bounded.

Furthermore, since $f \in W^{2,2}[x_{n-1}, x_n]$, $n \in \mathbb{N}$, and $f'' \in L^2(\mathbb{R}_+)$, one gets

(4.30)
$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{|f'(x_n-) - f'(x_{n-1}+)|^2}{d_n} &= \sum_{n\in\mathbb{N}} \frac{1}{d_n} \left| \int_{x_{n-1}}^{x_n} f''(x)\,dx \right|^2 \\ &\le \sum_{n\in\mathbb{N}} \int_{x_{n-1}}^{x_n} |f''(x)|^2\,dx = \int_{\mathbb{R}_+} |f''(x)|^2\,dx \le \|f\|^2_{W^{2,2}(\mathbb{R}_+\setminus X)}.\end{aligned}$$

Combining (4.29) with (4.30) yields $\operatorname{ran}\Gamma_0'' = \Gamma_0''(W^{2,2}(\mathbb{R}_+ \setminus X)) \subset \mathcal{R}(\Gamma_0'')$.

To prove the reverse inclusion choose any vector $\{\binom{a_n}{b_n}\}_{n\in\mathbb{N}}$ from $\mathcal{R}(\Gamma_0'')$. Setting

$$g_n(x) = a_n(x - x_{n-1}) + 2^{-1}d_n^{-1}(x - x_{n-1})^2(b_n - a_n), \quad x \in [x_{n-1}, x_n] \tag{4.31}$$

and $g := \oplus_1^\infty g_n$ one easily checks that

$$\|g_n\|^2_{L^2(\Delta_n)} \le d_n^3 \left[\frac{2}{3}|a_n|^2 + \frac{1}{10}|b_n - a_n|^2\right] \le d_n^3 \left(|a_n|^2 + |b_n|^2\right),$$

hence $g = \oplus_1^\infty g_n \in L^2(\mathbb{R}_+)$. Moreover, the condition $\{a_n - b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\})$ yields the inclusion $g'' \in L^2(\mathbb{R}_+)$. Thus $g \in W^{2,2}(\mathbb{R}_+ \setminus X)$. To complete the proof it remains to note that

$$g_n'(x_{n-1}+) = a_n, \quad g_n'(x_n-) = a_n + (b_n - a_n) = b_n, \tag{4.32}$$

i.e. $\Gamma_0'' g = \left\{\binom{a_n}{b_n}\right\}_{n\in\mathbb{N}}$. □

Remark 4.13. Notice that the relation (4.26) cannot be extracted from Proposition 4.7(i) applied to the derivative f', since the embedding $W^{2,2}(\mathbb{R}_+ \setminus X) \to W^{1,2}(\mathbb{R}_+ \setminus X)$ holds if and only if $d_* > 0$ (see [53]).

We are now ready to prove the assertion (iii) in Theorem 4.10, i.e. to prove relation (4.20).

Proof of item (iii) in Theorem 4.10. Let the righthand side of (4.20) be denoted temporarily by $\mathcal{R}_0(\Gamma_0)$. The inclusion $\operatorname{ran}(\Gamma_0) = \Gamma_0(\operatorname{dom} \mathrm{H}_*) \subset \mathcal{R}_0(\Gamma_0)$ is immediate from Lemma 4.12.

To prove the reverse inclusion we choose any vector $\{\binom{a_n}{b_n}\}_{n\in\mathbb{N}} \in l^2(\mathbb{N}) \otimes \mathbb{C}^2$ that satisfies $\{a_n - b_n\}_{n=1}^\infty \in l^2(\mathbb{N}; \{d_n^{-1}\})$ and consider the functions g_n and $g = \oplus_1^\infty g_n$ as defined in (4.31). As shown in Lemma 4.12 $g \in W^{2,2}(\mathbb{R}_+ \setminus X)$ and g' satisfies the equalities (4.32). Besides,

$$g_n(x_{n-1}+) = 0 \quad \text{and} \quad g_n(x_n-) = a_n d_n + 2^{-1}d_n(b_n - a_n) = 2^{-1}(a_n + b_n)d_n \in l^2(\mathbb{N}).$$

Note that the latter inclusion holds since $d^* < \infty$. Summing up we get

$$\Gamma_0 g = \begin{pmatrix} g'(x_{n-1}+) \\ g'(x_n-) \end{pmatrix} = \left\{\begin{pmatrix} a_n \\ b_n \end{pmatrix}\right\}_{n=1}^\infty, \quad \Gamma_1 g = \left\{\begin{pmatrix} 0 \\ 2^{-1}(a_n + b_n)d_n \end{pmatrix}\right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2.$$

Thus, $g \in \operatorname{dom}\Gamma_0' \cap \operatorname{dom}\Gamma_1' = \operatorname{dom}\mathrm{H}_*$ and this completes the proof. □

One gets from Lemma 4.12 a description for the ranges of the closures of Γ_0 and Γ_1.

Corollary 4.14. Assume the conditions of Theorem 4.10 and let $d^* < \infty$. Then

$$\operatorname{ran}\overline{\Gamma_0} = \operatorname{ran}\Gamma_0 \quad \text{and} \quad \operatorname{ran}\overline{\Gamma_1} = l^2(\mathbb{N}) \otimes \mathbb{C}^2. \tag{4.33}$$

Proof. Recall that, by definition, $\operatorname{dom}\Gamma_0 = \operatorname{dom}\Gamma_1 = \operatorname{dom}\mathrm{H}_*$. Clearly, $\Gamma_0 = \Gamma_0'' \upharpoonright \operatorname{dom}\mathrm{H}_*$ and

$$\operatorname{ran}\Gamma_0 \subseteq \operatorname{ran}\overline{\Gamma}_0 \subseteq \operatorname{ran}\Gamma_0'' \cap \big(l^2(\mathbb{N}) \otimes \mathbb{C}^2\big). \tag{4.34}$$

On the other hand, it follows from Lemma 4.12 and Theorem 4.10 (iii) that

$$\operatorname{ran}\Gamma_0 = \operatorname{ran}\Gamma_0'' \cap \big(l^2(\mathbb{N}) \otimes \mathbb{C}^2\big).$$

Combining this relation with (4.34) and applying Theorem 4.10 (iii) yields the first relation in (4.33).

The second relation is proved similarly. □

Remark 4.15. (i) Recall that according to Theorem 1.3 the condition

$$\operatorname{ran}\Gamma_0 = \operatorname{dom}M(z), \quad z \in \mathbb{C}\setminus\mathbb{R}, \tag{4.35}$$

ensures selfadjointness of $A_0 = \ker\Gamma_0$. Theorem 4.10 (iii) gives an explicit example showing that condition (4.35) cannot be replaced by the weaker domain invariance condition

$$\operatorname{dom}M(z) = \operatorname{dom}M(i)\ (\subsetneqq \operatorname{ran}\Gamma_0), \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

In other words, domain invariance property does not imply the property of a boundary triple to be S-generalized (see also [26, Example 5.29]). Such Weyl functions cannot be written in the form (1.3) without a renormalization of the boundary triple as in [26, Theorem 5.32].

(ii) In the case $d_* = 0$ and $d^* < \infty$ an abstract regularization procedure from [58, 52] has first been applied in [52] to the direct sum $\Pi = \oplus_{n=1}^{\infty}\Pi_n = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ of triples (4.16) for H_n^* to obtain a (regularized) ordinary boundary triple $\Pi^r = \{\mathcal{H}, \Gamma_0^r, \Gamma_1^r\}$ satisfying $\ker\Gamma_0 = \ker\Gamma_0^r$. A special construction of a regularized triple Π^r in [52] has been motivated by the following circumstance: the boundary operator $B_{X,\alpha}$ corresponding to the Hamiltonian $\mathrm{H}_{X,\alpha}$ of the form (4.18), i.e. operator satisfying $\operatorname{dom}(\mathrm{H}_{X,\alpha}) = \ker(\Gamma_1^r - B_{X,\alpha}\Gamma_0^r)$, is a Jacobi matrix. It is shown in [52] that certain spectral properties of $\mathrm{H}_{X,\alpha}$ strictly correlate with that of $B_{X,\alpha}$.

Finally, it is mentioned that boundary triple models are motivated by and naturally appear in various physical problems as exactly solvable models that describe complicated physical phenomena; see e.g. [3, 14, 35, 53] for further details.

Next we apply the renormalization result in Theorem 2.6 to the ES-generalized boundary triple Π in Theorem 4.10. The transposed boundary triple $\Pi^\top$ can be renormalized by a suitable modification of [26, Theorem 4.4, Theorem 4.11], using a regular point (here $z = -1$) on the real line; cf. Theorem 2.2.

Proposition 4.16. Assume the conditions of Theorem 4.10 and let $\widetilde{d}_n = \min\{d_n, 1\}$. Then:

(i) The direct sum $\widetilde{\Pi} = \oplus_{n=1}^{\infty}\widetilde{\Pi}_n = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ of the triples $\widetilde{\Pi}_n = \{\mathbb{C}^2, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$, where $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^2$ and the mappings $\widetilde{\Gamma}_j^{(n)} : W_2^2[x_{n-1}, x_n] \to \mathbb{C}^2$, $n \in \mathbb{N}$, $j \in \{0, 1\}$ are given by

$$\widetilde{\Gamma}_0^{(n)} f := \begin{pmatrix} \widetilde{d}_n^{-1/2} f'(x_{n-1}+) \\ \widetilde{d}_n^{-1/2} f'(x_n-) \end{pmatrix}, \quad \widetilde{\Gamma}_1^{(n)} f := \begin{pmatrix} -\widetilde{d}_n^{1/2} f(x_{n-1}+) \\ \widetilde{d}_n^{1/2} f(x_n-) \end{pmatrix},$$

is a B-generalized boundary triple for H^*. Moreover, $\widetilde{\Pi}$ is an ordinary boundary triple if and only if $d_* > 0$.

(ii) The direct sum $\Pi^{(r)} = \oplus_{n=1}^{\infty}\Pi_n^{(r)} = \{\mathcal{H}, \Gamma_0^{(r)}, \Gamma_1^{(r)}\}$ of the triples $\Pi_n^r = \{\mathbb{C}^2, \Gamma_0^{(r,n)}, \Gamma_1^{(r,n)}\}$, where the mappings $\Gamma_j^{(r,n)} : W_2^2[x_{n-1}, x_n] \to \mathbb{C}^2$, $n \in \mathbb{N}$, $j \in \{0, 1\}$ are given by

$$\Gamma_0^{(r,n)} f = \begin{pmatrix} \widetilde{d}_n^{1/2} f(x_{n-1}+) \\ -\widetilde{d}_n^{1/2} f(x_n-) \end{pmatrix}, \quad \Gamma_1^{(r,n)} f = \begin{pmatrix} \widetilde{d}_n^{-1/2} f'(x_{n-1}+) + \widetilde{d}_n^{-3/2}(f(x_{n-1}+) - f(x_n-) \\ \widetilde{d}_n^{-1/2} f'(x_n-) + \widetilde{d}_n^{-3/2}(f(x_{n-1}+) - f(x_n-) \end{pmatrix},$$

is an ordinary boundary triple for $H_{\min}^*$.

The proof is similar to that of Theorem 4.10 and is omitted.

4.4. **Dirac operators with local point interactions.** Let D be a differential expression

$$D = -i\, c \frac{d}{dx} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3 = \begin{pmatrix} c^2/2 & -i\, c \frac{d}{dx} \\ -i\, c \frac{d}{dx} & -c^2/2 \end{pmatrix} \tag{4.36}$$

acting on $\mathbb{C}^2$-valued functions of a real variable. Here $c > 0$ denotes the velocity of light and $\sigma_1, \sigma_2, \sigma_3$ stand for the Pauli matrices in $\mathbb{C}^2$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Furthermore, let $X = \{x_n\}_1^{\infty}$ be a strictly increasing sequence of positive numbers satisfying $\lim_{n\to\infty} x_n = \infty$, let d_n, d_*, and d^* be defined by (4.8) and let D_n be the minimal operator generated in $L^2[x_{n-1}, x_n] \otimes \mathbb{C}^2$ by the differential expression (4.36)

$$D_n = D \upharpoonright \operatorname{dom}(D_n), \qquad \operatorname{dom}(D_n) = W_0^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2.$$

Recall that D_n is a symmetric operator with deficiency indices $n_{\pm}(D_n) = 2$ and its adjoint D_n^* is given by

$$D_n^* = D \upharpoonright \operatorname{dom}(D_n^*), \qquad \operatorname{dom}(D_n^*) = W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2.$$

Next following [22] we recall the construction of a boundary triple for D_n^* and compute the corresponding Weyl function. Namely, the boundary triple $\Pi^{(n)} = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$, where

$$\Gamma_0^{(n)} f := \Gamma_0^{(n)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1(x_{n-1}+) \\ i\, c\, f_2(x_n-) \end{pmatrix}, \quad \Gamma_1^{(n)} f := \Gamma_1^{(n)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} i\, c\, f_2(x_{n-1}+) \\ f_1(x_n-) \end{pmatrix}, \tag{4.37}$$

forms a boundary triple for D_n^*. Clearly, $D_{n,0} := D_n^* \upharpoonright \ker \Gamma_0^{(n)} = D_{n,0}^*$ and

$$\operatorname{dom}(D_{n,0}) = \{\{f_1, f_2\}^{\tau} \in W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2 : f_1(x_{n-1}+) = f_2(x_n-) = 0\}.$$

Moreover, the spectrum of the operator $D_{n,0}$ is discrete,

$$\sigma(D_{n,0}) = \sigma_d(D_{n,0}) = \left\{ \pm\sqrt{\frac{c^2\pi^2}{d_n^2}\left(j + \frac{1}{2}\right)^2 + \left(\frac{c^2}{2}\right)^2}, \quad j \in \mathbb{N} \right\}. \tag{4.38}$$

The defect subspace $\mathfrak{N}_z := \ker(D_n^* - z)$ is spanned by the vector functions

$$f_n^{\pm}(x, z) := \begin{pmatrix} e^{\pm i\, k(z)\, x} \\ \pm k_1(z) e^{\pm i\, k(z)\, x} \end{pmatrix}.$$

Moreover, the Weyl function $M_n(\cdot)$ corresponding to the triple $\Pi^{(n)}$ is (cf. [22])

$$M_n(z) = \frac{1}{\cos(d_n\, k(z))}\begin{pmatrix} c\, k_1(z)\, \sin(d_n\, k(z)) & 1 \\ 1 & (c\, k_1(z))^{-1}\sin(d_n\, k(z)) \end{pmatrix}, \tag{4.39}$$

where $z \in \rho(D_{n,0})$,

$$k(z) := c^{-1}\sqrt{z^2 - (c^2/2)^2}, \quad \text{and} \quad k_1(z) := \frac{c\, k(z)}{z + c^2/2} = \sqrt{\frac{z - c^2/2}{z + c^2/2}}, \quad z \in \mathbb{C}. \tag{4.40}$$

Next we construct a boundary triple for the operator $D_X^* := \bigoplus_{n=1}^{\infty} D_n^*$ in the general case $0 \le d_* < d^* \le \infty$. It appears that the result in the case $d^* = \infty$ remains analogous to what was obtained in [22] for the case $d^* < \infty$.

Define $D_X := \bigoplus_1^{\infty} D_n$,

$$\operatorname{dom}(D_X^*) = W^{1,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 = \bigoplus_1^{\infty} W^{1,2}[x_{n-1}, x_n] \otimes \mathbb{C}^2.$$

Next following [22] certain properties of the direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)}$ of the boundary triples $\Pi^{(n)}$ given by (4.37) are collected.

Proposition 4.17. Let X be as above, let $0 \le d_* < d^* \le \infty$, and let $\Pi^{(n)} = \{\mathbb{C}^2, \Gamma_0^{(n)}, \Gamma_1^{(n)}\}$ be the boundary triple for the operator D_n^* defined in (4.37). Let $\mathcal{H} = l^2(\mathbb{N}) \otimes \mathbb{C}^2$ and $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where the operators Γ_j, $j \in \{0, 1\}$ are given by (4.3), i.e.

$$\Gamma_0 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left\{\begin{pmatrix} f_1(x_{n-1}+) \\ i\, c\, f_2(x_n-) \end{pmatrix}\right\}_{n\in\mathbb{N}}, \quad \Gamma_1 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left\{\begin{pmatrix} i\, c\, f_2(x_{n-1}+) \\ f_1(x_n-) \end{pmatrix}\right\}_{n\in\mathbb{N}}, \tag{4.41}$$

where $f = \binom{f_1}{f_2} \in \operatorname{dom} D_{X,*} := \operatorname{dom}\Gamma$ and $D_{X,*} := D_X^* \upharpoonright \operatorname{dom} D_{X,*}$. Then:

- *(i)* The domain Γ is given by $\operatorname{dom} D_{X,*} := \operatorname{dom}\Gamma (= \operatorname{dom}\Gamma_0 = \operatorname{dom}\Gamma_1)$.
- *(ii)* The direct sum $\Pi := \bigoplus_{n=1}^{\infty} \Pi^{(n)}$ forms a B-generalized boundary triple for D_X^*.
- *(iii)* The transposed triple $\Pi^\top = \{\mathcal{H}, \Gamma_1, -\Gamma_0\}$ also forms a B-generalized boundary triple for D_X^*.
- *(iv)* The triple Π (equivalently the triple $\Pi^\top$) is an ordinary boundary triple for the operator $D_X^* = \bigoplus_{n=1}^{\infty} D_n^*$ if and only if $d_* > 0$ (with $d^* \le \infty$).

Proof. (i), (ii) The Weyl function of the boundary triple $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is the orthogonal sum $M = \oplus_1^{\infty} M_n$ of the Weyl functions defined by (4.39). It follows from (4.40) that $k(0) = i\, c/2$ and $k_1(0) = i$ and hence

$$M_n(0) = \frac{1}{\operatorname{ch}(d_n\, c/2)}\begin{pmatrix} -c\operatorname{sh}(d_n\, c/2) & 1 \\ 1 & c^{-1}\operatorname{sh}(d_n\, c/2) \end{pmatrix}. \tag{4.42}$$

Hence,

$$M_n(0) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{as} \quad d_n \to 0 \quad \text{and} \quad M_n(0) \to \begin{pmatrix} -c & 0 \\ 0 & c^{-1} \end{pmatrix} \quad \text{as} \quad d_n \to \infty. \tag{4.43}$$

It follows that the sequence $\{M_n(0)\}_{n\in\mathbb{N}}$ is bounded.

Furthermore, one gets from (4.40) that $k'(0) = 0$, $k_1'(0) = -i\, 2/c^2$, and

$$M_n'(0) = \begin{pmatrix} \frac{2}{c}\operatorname{th}(d_n\, c/2) & 0 \\ 0 & \frac{2}{c^3}\operatorname{th}(d_n\, c/2) \end{pmatrix} \ge 0, \qquad n \in \mathbb{N}. \tag{4.44}$$

This description implies that

$$M_n'(0) \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{as} \quad d_n \to 0 \quad \text{and} \quad M_n'(0) \to \begin{pmatrix} \frac{2}{c} & 0 \\ 0 & \frac{2}{c^3} \end{pmatrix} \quad \text{as} \quad d_n \to \infty. \tag{4.45}$$

Thus, the sequence $\{M_n'(0)\}_{n\in\mathbb{N}}$ is bounded too. Combining the formulas in (4.43) with (4.45) and applying Theorem 4.4 (iii) one concludes that $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a B-generalized boundary triple for D_X^*.

(iii) It follows from (4.42) that $\det(M_n(0)) = -1$, hence the sequence of inverses $\{M_n(0)^{-1}\}_{n\in\mathbb{N}}$ is bounded alongside the sequence $\{M_n(0)\}_{n\in\mathbb{N}}$. Combining this fact with boundedness of the sequence $\{M_n'(0)\}_{n\in\mathbb{N}}$ of the derivatives and using the identities

$$-(M_n^{-1})'(0) = M_n^{-1}(0)\, M_n'(0)\, M_n^{-1}(0), \quad n \in \mathbb{N},$$

we obtain that the sequence $\{(M_n^{-1})'(0)\}_{n\in\mathbb{N}}$ is bounded too. It remains to apply Theorem 4.4 (iii).

(iv) It follows from (4.44) that the sequence $\{M_n'(0)\}_{n\in\mathbb{N}}$ of the derivatives is uniformly positive if and only if $d_* > 0$. One completes the proof by combining Theorem 4.4 (iv) with the above proved items (ii), (iii). □

Remark 4.18. Note that if $d^* = \infty$ then in view of (4.38) $\pm\frac{c^2}{2} \in \sigma(D_0)$, while $(-\frac{c^2}{2}, \frac{c^2}{2}) \subset \rho(D_0)$. Therefore as distinguished from the considerations in [22] treating the case $d^* < \infty$, here we consider the behavior of the Weyl function at $z = 0 \in \rho(D_0)$.

We now apply a modification of Theorem 2.1 and Theorem 2.2 to produce an ES-generalized boundary triple for D_X^* from the B-generalized boundary triple $\Pi := \bigoplus_{n=1}^\infty \Pi^{(n)} = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$. In this modification we subtract from the Weyl function M_n the limit value $\lim_{d_n\to 0} M_n(0)$, instead of the value $M_n(0)$, to get a transform of boundary mappings in a simple form.

Proposition 4.19. *Let X be as above, let $0 \le d_* < d^* < \infty$, let $\widetilde{\Pi}^{(n)} = \{\mathbb{C}^2, \widetilde{\Gamma}_0^{(n)}, \widetilde{\Gamma}_1^{(n)}\}$ be the boundary triple for the operator D_n^*, $n \in \mathbb{N}$, defined by*

$$\widetilde{\Gamma}_0^{(n)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} i\, c\, (f_2(x_n-) - f_2(x_{n-1}+)) \\ f_1(x_{n-1}+) - f_1(x_n-) \end{pmatrix}, \quad \widetilde{\Gamma}_1^{(n)} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1(x_{n-1}+) \\ i\, c\, f_2(x_n-) \end{pmatrix},$$

let $\widetilde{\Gamma}_j' = \bigoplus_{n=1}^\infty \widetilde{\Gamma}_j^{(n)}$, $j \in \{0, 1\}$, and let $\widetilde{\Pi} = \bigoplus_{n=1}^\infty \widetilde{\Pi}_n = \{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\}$ be the boundary triple for D_X^*, where

$$\widetilde{\Gamma}_j := \widetilde{\Gamma}_j' \upharpoonright \operatorname{dom}(D_{X,*}), \quad \operatorname{dom}(D_{X,*}) := \operatorname{dom}\widetilde{\Gamma} := \operatorname{dom}\widetilde{\Gamma}_0' \cap \operatorname{dom}\widetilde{\Gamma}_1'.$$

Then:

(i) The mapping $\widetilde{\Gamma}_0 \times \widetilde{\Gamma}_1$ is naturally extended to the mapping $\widetilde{\Gamma}_0'' \times \widetilde{\Gamma}_1''$ defined by the same formulas on $W^{1,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2$. Moreover, the following mapping is well defined and surjective:

$$\widetilde{\Gamma}_0'' \times \widetilde{\Gamma}_1'' : W^{1,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 \to (l^2(\mathbb{N}; \{d_n^{-1}\}) \otimes \mathbb{C}^2) \times (l^2(\mathbb{N}; \{d_n\}) \otimes \mathbb{C}^2)$$

(ii) The mapping

$$\widetilde{\Gamma}_0 \times \widetilde{\Gamma}_1 : \operatorname{dom} D_{X,*} \to (l^2(\mathbb{N}; \{d_n^{-1}\}) \otimes \mathbb{C}^2) \times (l^2(\mathbb{N}) \otimes \mathbb{C}^2)(\subset l^2(\mathbb{N}) \otimes \mathbb{C}^4), \tag{4.46}$$

is well defined and surjective. Moreover, $\operatorname{dom} D_{X,*} = \operatorname{dom}\widetilde{\Gamma} = \operatorname{dom}\overline{\widetilde{\Gamma}_1}$, while $\operatorname{dom}\overline{\widetilde{\Gamma}_0} = \operatorname{dom} D_X^* = W^{1,2}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2$.

(iii) The Weyl function is of the form $\widetilde{M}(\cdot) = \bigoplus_{n=1}^{\infty} \widetilde{M}_n(\cdot)$, where

$$\widetilde{M}_n(z) = -2^{-1} \begin{pmatrix} \dfrac{\sin(d_n k(z))}{ck_1(z)(1-\cos(d_n k(z)))} & -1 \\ -1 & \dfrac{ck_1(z)\sin(d_n k(z))}{1-\cos(d_n k(z))} \end{pmatrix}. \tag{4.47}$$

This function is domain invariant and with $z \in \mathbb{C}_\pm$ one has

$$\operatorname{dom} \widetilde{M}(z) = l^2\big(\mathbb{N}; \{d_n^{-2}\}\big) \otimes \mathbb{C}^2 \subseteq \widetilde{\Gamma}_0(\operatorname{dom} D_{X,*}) = l^2\big(\mathbb{N}; \{d_n^{-1}\}\big) \otimes \mathbb{C}^2. \tag{4.48}$$

Here the strict inclusion $\operatorname{dom} M(z) \subsetneqq \widetilde{\Gamma}_0(\operatorname{dom} D_{X,*})$ holds if and only if $d_* = 0$.

(iv) The Weyl function $\widetilde{M}(\cdot)$ is also form domain invariant with

$$\operatorname{dom} \mathfrak{t}_{\widetilde{M}(z)} = l^2\big(\mathbb{N}; \{d_n^{-1}\}\big) \otimes \mathbb{C}^2 = \widetilde{\Gamma}_0(\operatorname{dom} D_{X,*}), \quad z \in \mathbb{C}_\pm. \tag{4.49}$$

(v) $\widetilde{\Pi} = \big\{\mathcal{H}, \widetilde{\Gamma}_0, \widetilde{\Gamma}_1\big\}$ is an ES-generalized boundary triple for D_X^*. Moreover, $\widetilde{\Pi}$ is an S-generalized boundary triple for D_X^* if and only if $d_* > 0$ and in this case $\widetilde{\Pi}$ is in fact and ordinary boundary triple for D_X^*.

(vi) The triple $\Pi^\top = \{\mathcal{H}, \widetilde{\Gamma}_0^\top, \widetilde{\Gamma}_1^\top\}$ for D_X^* is B-generalized. In particular, $A_1 = D_X^* \upharpoonright \ker \widetilde{\Gamma}_1$ is selfadjoint.

Proof. (i) The proof is immediate from Lemma 4.7.

(ii) Due to $d^* < \infty$ one has the following chain of continuous embeddings

$$l^2\big(\mathbb{N}; \{d_n^{-1}\}\big) \otimes \mathbb{C}^2 \subset l^2\big(\mathbb{N}\big) \otimes \mathbb{C}^2 \subset l^2\big(\mathbb{N}; \{d_n\}\big) \otimes \mathbb{C}^2. \tag{4.50}$$

Since $l^2\big(\mathbb{N}\big) \otimes \mathbb{C}^2$ is a part of $l^2\big(\mathbb{N}; \{d_n\}\big) \otimes \mathbb{C}^2$, the surjectivity of $\widetilde{\Gamma} = (\widetilde{\Gamma}_0'' \times \widetilde{\Gamma}_1'') \upharpoonright \operatorname{dom} D_{X,*}$ is immediate from (i). The inclusion in (4.46) as well as the relation $\operatorname{dom} \widetilde{\Gamma} = \operatorname{dom} \overline{\widetilde{\Gamma}_1}$ is implied by the first inclusion in (4.50).

(iii) The Weyl function of $\widetilde{\Pi}$ is the direct sum $\widetilde{M}(\cdot) = \bigoplus_{n=1}^{\infty} \widetilde{M}_n(\cdot)$, where

$$\widetilde{M}_n(z) := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - M_n(z) \right)^{-1}, \quad z \in \rho(M_n).$$

This immediately leads to formula (4.47) for $\widetilde{M}_n(z)$. Using (4.40), and the Taylor series expansions for $\sin(z)$ and $\cos(z)$ we easily derive

$$\widetilde{M}_n(z) + \frac{1}{d_n} \begin{pmatrix} (z - c^2/2)^{-1} & 0 \\ 0 & c^2(z + c^2/2)^{-1} \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{as } d_n \to 0, \ \ z \in \mathbb{C}_\pm. \tag{4.51}$$

This formula shows that $\widetilde{M}(z)$, as well as $\operatorname{Im} \widetilde{M}(z)$, is bounded if and only if $d_* > 0$, $z \in \mathbb{C}_\pm$. Moreover, it follows from (4.51) that $\{\binom{a_n}{b_n}\}_{n \in \mathbb{N}} \in \operatorname{dom} \widetilde{M}(z)$, $z \in \mathbb{C}_\pm$, precisely when

$$\sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{d_n^2} < \infty.$$

The inclusion (in fact the continuous embedding) in (4.48) follows from the estimate

$$\sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{d_n} \le d^* \sum_{n=1}^{\infty} \frac{|a_n|^2 + |b_n|^2}{d_n^2}.$$

The reverse inequality holds if and only if $d_* > 0$. Indeed, writing down the reverse inequality and inserting here $\{a_n\} = \{\delta_{jn}\}_{n\in\mathbb{N}}$ and $\{b_n\} = \{0\}_{n\in\mathbb{N}}$, one arrives at the inequalities $1 \le cd_j,\ j \in \mathbb{N}$, so that $d_* \ge 1/c > 0$.

(iv) By definition, $\{h_n\}_{n=1}^\infty \in \operatorname{dom} \mathfrak{t}_{\widetilde{M}(z)}$ if and only if

$$\sum_{n=1}^{\infty} \left(\operatorname{Im} \widetilde{M}_n(z)h_n, h_n\right) < \infty; \quad \{h_n\}_{n=1}^\infty = \left\{\begin{pmatrix} a_n \\ b_n \end{pmatrix}\right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2. \tag{4.52}$$

As a function of d_n the imaginary part $\operatorname{Im} \widetilde{M}_n(z)$ is bounded on the intervals $[\delta, \infty)$, $\delta > 0$, and hence it follows from (4.51) that the convergence of the series in (4.52) is equivalent to

$$\sum_{n=1}^{\infty} \frac{(\operatorname{Im} K(z)h_n, h_n)}{d_n} < \infty; \quad \{h_n\}_{n=1}^\infty = \left\{\begin{pmatrix} a_n \\ b_n \end{pmatrix}\right\}_{n=1}^\infty \in l^2(\mathbb{N}) \otimes \mathbb{C}^2,$$

where $K(z)$ denotes the diagonal matrix function in the left-hand side of (4.51). Clearly, $\operatorname{Im} K(z)$ is bounded with bounded inverse for each $z \in \mathbb{C}_\pm$ and this yields the stated description of $\operatorname{dom} \mathfrak{t}_{\widetilde{M}(z)}$.

(v) By Theorem 4.1(iv), the triple $\widetilde{\Pi}$ being a direct sum of ordinary boundary triples, is an ES-generalized boundary triple. On the other hand, by (iii) the strict inclusion $\operatorname{dom} M(z) \subsetneqq \widetilde{\Gamma}_0(\operatorname{dom} D_{X,*})$ is equivalent to $d_* = 0$. Therefore, Theorem 1.3 applies and ensures that in the latter case $\widetilde{\Pi}$ is not an S-generalized triple.

(vi) The Weyl function corresponding to the transposed boundary triple $\Pi^\top$ is $-\widetilde{M}(\cdot)^{-1} = \bigoplus_1^\infty(-\widetilde{M}_n(\cdot)^{-1})$. In particular, one gets from (4.51) (or from (4.39)) that

$$-\widetilde{M}_n(z)^{-1} = M_n(z) - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim d_n \begin{pmatrix} z - c^2/2 & 0 \\ 0 & c^{-2}(z + c^2/2) \end{pmatrix} \quad \text{as} \quad d_n \to 0.$$

This shows that $-\widetilde{M}(\cdot)^{-1} \in \mathcal{R}^s[\mathcal{H}]$. Thus $\Pi^\top$ is a B-generalized boundary triple; cf. [33, Chapter 5]. □

Remark 4.20. Apart from statements (ii) and the formula for $\widetilde{\Gamma}_0(\operatorname{dom} D_{X,*})$ in statement (iii) the results in Proposition 4.19 remain valid for $d^* = \infty$. Indeed, statement (i) is still immediate from Proposition 4.7(i) which holds in this case, too. All the other statements can easily be extracted from the fact that the limit value of the Weyl function $\widetilde{M}_n(z)$ as well as its inverse $\widetilde{M}_n(z)^{-1}$ remain bounded when $d_n \to \infty$.

Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence from $\mathbb{R}$. Gesztesy-Šeba realization of Dirac operator (see [37]) is defined by $D_{X,\alpha} = D|_{\operatorname{dom} D_{X,\alpha}}$, where

$$\operatorname{dom} D_{X,\alpha} = \left\{ \begin{matrix} f \in W_{comp}(\mathbb{R}_+ \setminus X) \otimes \mathbb{C}^2 : \ f_1 \in AC_{loc}(\mathbb{R}_+), f_2 \in AC_{loc}(\mathbb{R}_+ \setminus X) \\ f_2(a+) = 0,\ f_2(x_n+) - f_2(x_n-) = -\frac{i\alpha_n}{c} f_1(x_n),\ n \in \mathbb{N} \end{matrix} \right\}. \tag{4.53}$$

As was shown in [37, 22] the Gesztesy-Šeba realization $D_{X,\alpha}$ is always selfadjoint.

Proposition 4.21. Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, let $D_{X,\alpha}$ be the Gesztesy-Šeba realization of the Dirac operator given by (4.53), let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be the boundary triple defined by (4.41) and let

$$M(\lambda) = \bigoplus_{n=1}^{\infty} M_n(\lambda), \quad \gamma(\lambda) = \bigoplus_{n=1}^{\infty} \gamma_n(\lambda)$$

with $M_n(\lambda)$ and $\gamma_n(\lambda)$ given by (4.39) and [22, (3.11)], respectively. Then:
(4.54)

$$\operatorname{dom} D_{X,\alpha} = \ker(\Gamma_1 - B_\alpha \Gamma_0), \quad \text{where} \quad B_\alpha = \begin{pmatrix} 0_{1\times 1} & 0 \\ 0 & \bigoplus_{n=1}^{\infty} B_n \end{pmatrix}, \quad B_n = \begin{pmatrix} 0_{1\times 1} & 1 \\ 1 & \alpha_n \end{pmatrix}.$$

Moreover,

$$\lambda \notin \sigma_p(D_{X,\alpha}) \Longleftrightarrow 0 \notin \sigma_p(B_\alpha - M(\lambda)), \tag{4.55}$$

and the following Kreĭn-type formula holds

$$(D_{X,\alpha} - \lambda)^{-1} = (D_0 - \lambda)^{-1} + \gamma(\lambda)\big(B_\alpha - M(\lambda)\big)^{-1}\gamma(\bar{\lambda})^*, \qquad \lambda \in \rho(D_{X,\alpha}) \cap \rho(D_0). \tag{4.56}$$

Proof. The equality (4.54) is implied by (4.53) and (4.41). The formulas (4.55) and (4.56) follow from [26, Theorem 5.8]. □

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