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The 1-2-random walk: Some bijections and a central limit theorem

Master Thesis for the achievement of the academic degree
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Zusammenfassung

Ein berühmtes Thema der Stochastik ist die sogenannte zufällige Irrfahrt (random walk), bei der mehrere zufällige Bewegungen um bestimmte Distanzen zu einem Pfad aneinander gereiht werden. Die einfachste Form ist dabei die einfache, symmetrische Irrfahrt, bei der der Pfad auf einer eindimensionalen Zahlenebene in jedem Schritt mit je 50% Wahrscheinlichkeit um 1 in positive oder negative Richtung geht. Viele Probleme wurden dafür bereits untersucht, zum Beispiel die erste Rückkehr in die 0. Die zufällige Irrfahrt konvergiert, wenn die Zeit und ihre Schrittlänge geeignet skaliert werden, auch gegen die Brownsche Bewegung, ein stetiger stochastischer Prozess, der auf der Normalverteilung basiert.

In dieser Arbeit wird jedoch eine modifizierte Irrfahrt behandelt, der 1-2-random walk, dessen Pfade im negativen Bereich Schrittlänge 2 haben statt 1. Dadurch kann die 0 übersprungen werden, wenn der Pfad bei -1 ist, wo Schrittlänge 2 angewendet wird, sodass er danach bei 1 landet. Einige Problemstellungen müssen damit etwas angepasst werden.

Im ersten Kapitel werden die Grundlagen für die einfache zufällige Irrfahrt abgehandelt, darunter auch die Konvergenz gegen die Brownsche Bewegung, die mit dem Donsker-Theorem gezeigt wird.

Im zweiten Kapitel wird zuerst der 1-3-random walk (Schrittlänge 3 im negativen Bereich) behandelt, mit der Fragestellung, wie wahrscheinlich er im nichtnegativen Bereich landet. Danach wird dieselbe Frage für den 1-2-random walk gestellt. Zur Beantwortung wird eine Bijektion zwischen diesen Pfaden und $3 \times n$-Rechtecken aufgestellt, die in Quadrate mit Seitenlänge 1 und 2 zerlegt werden.

Kapitel 3 behandelt eine Konvergenz, die auch in die Richtung der Brownschen Bewegung geht. Dazu ist diesmal etwas mehr Vorarbeit nötig, bevor die Schritte wie beim Donsker-Theorem angewendet werden können. Einige weitere Bijektionen und stochastische Differentialgleichungen werden dafür verwendet.

Zum Schluss kommen noch ein Zusatz zu der Bijektion mit den Rechtecken aus Kapitel 2, und ein paar offene Fragen, die sich um Verallgemeinerungen der behandelten Probleme drehen.

Contents

1 Random Walks

Some basic definitions will be made first. Let $(\Omega, \mathcal{F}, P)$ a probability space. Then a sequence of σ-algebras $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ with $\{\emptyset;\Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ and $\mathcal{F}_n \subseteq \mathcal{F}$ is called *filtration*, and $\left(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, P\right)$ is called *filtered probability space*.

A sequence of random variables $(X_i)_{i\in\mathbb{N}_0}$ is called a *stochastic process* adapted to this filtration when X_i is measurable by $\mathcal{F}_i$ for all $i \in \mathbb{N}_0$. It is called *martingale* when $\mathbb{E}|X_i| < \infty$ and $\mathbb{E}(X_{i+1}|\mathcal{F}_i) = X_i$ are true for all $i \in \mathbb{N}$.

A random variable $\tau : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ is called *stopping time* with respect to the filtration $(\mathcal{F}_i)_{i\in\mathbb{N}_0}$, if $\{\tau = i\} \in \mathcal{F}_i$ is true for all $i \in \mathbb{N}_0$.

A *discrete-time one-dimensional random walk* is a stochastic process where $X_i = \sum_{k=1}^{i} Z_k$ for some independent identically distributed random variables $Z_1, Z_2, \ldots$, which mark the increments. In the simple random walk, we have $P(Z_k = 1) = P(Z_k = -1) = \frac{1}{2}$. In this case, for any time point n and $k \in \mathbb{Z}$ where $n-k$ is even and $|k| \leq n$, we have $P(X_n = k) = \frac{1}{2^n}\binom{n}{\frac{n-k}{2}}$. The numerator consists of the choice of $\frac{n-k}{2}$ negative out of n total steps, the denominator has the total number of different paths until step n, because for every time point there are the two possibilities of going up or down.

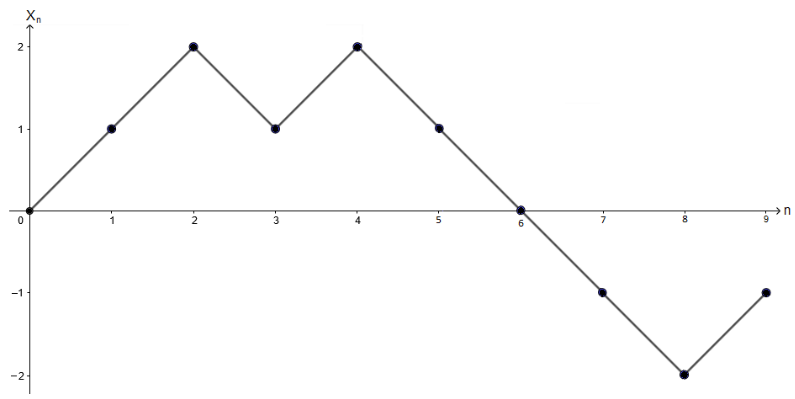

Figure 1: An example path of a discrete-time random walk, $n \leq 9$

This stochastic process is obviously a Martingale because of $\mathbb{E}(X_{i+1}|\mathcal{F}_i) = \mathbb{E}(X_i|\mathcal{F}_i) + \mathbb{E}(Z_{i+1}|\mathcal{F}_i) = X_i + \mathbb{E}(Z_{i+1}) = X_i$. The random variable Z_{i+1} is independent to $\mathcal{F}_i$, so the conditional expectation of Z_{i+1} with respect to $\mathcal{F}_i$ is just its expected value.

1.1 The return to the origin

One considered problem in a one-dimensional random walk is the question if and when the path returns to where it started. That means, to find a time point n where $X_n = 0$.

Obviously n can't be odd because the path must have as many positive as negative steps. So we can write $n = 2m$ with $m \in \mathbb{N}$ and search for time points where $X_{2m} = 0$. It's also clear that in that case, this probability will be $P(X_{2m} = 0) = \frac{\binom{2m}{m}}{2^{2m}}$. A more interesting question is: When is the first time where the path returns to 0? For this purpose, a stopping time τ can be established. Let $\tau := \min\{i > 0 : X_i = 0\}$. For example, in image (1) above, we would have $\tau(\omega) = 6$. What is the probability $P(\tau = 2m)$ for any $m = 0$? The first step for this is our first Proposition.

Proposition 1.1 (one-sided paths with fixed end)**.** *Let* $a, b \in \mathbb{N}_0, a > b$ *and* $a + b = n$. *Then* $P(X_1 > 0, \ldots, X_{n-1} > 0, X_n = a - b) = \frac{a-b}{n \cdot 2^n}\binom{n}{a}$.

Proof. At first, the searched probability will be transformed into a subtraction.

$$\begin{aligned} &P(X_1 > 0, \ldots, X_{n-1} > 0, X_n = a - b) \\ =&P(X_1 = 1, X_n = a - b) - P(X_1 = 1, X_n = a - b, \exists l \in \mathbb{N} : X_l = 0) \end{aligned}$$

The subtrahend is like that because every path that reaches $a - b$ after n steps and isn't completely in positive range has to cross 0 at some time. Furthermore, the last mentioned paths can be reflected up to the time it first reaches 0.

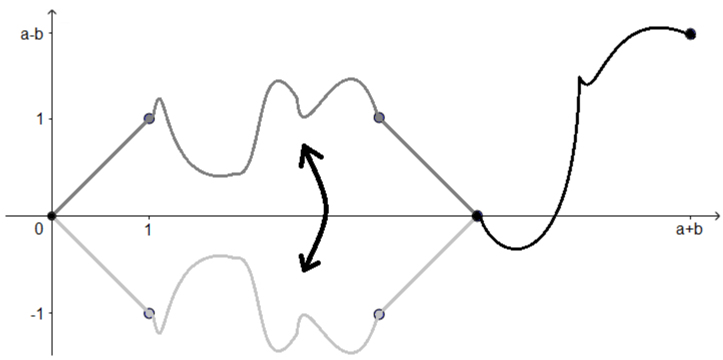

Figure 2: Reflection of the first part until reaching 0

Every path with $X_1 = 1$ that ends in $a - b$ and passes 0 at least once can be mapped to a path that begins with -1 instead. Because $a - b > 0$, a path beginning with -1 always crosses 0 before it can reach $a - b$ anyway, so the remapping is also unique. This implies that there is a bijection between the paths with $X_1 = 1$, $X_n = a - b$ while passing 0 at least once, and the paths with simply $X_1 = -1$, $X_n = a - b$. These probabilities can easily be calculated to conclude the proof.

$$\begin{aligned}
&P(X_1>0,\dots,X_{n-1}>0,X_n=a-b)\\
=&P(X_1=1,X_n=a-b)-P(X_1=1,X_n=a-b,\exists l\in\mathbb{N}:S_l=0)\\
=&P(X_1=1,X_n=a-b)-P(X_1=-1,X_n=a-b)\\
=&\frac{1}{2^n}\left(\binom{n-1}{\frac{n-1+a-b-1}{2}}-\binom{n-1}{\frac{n-1+a-b+1}{2}}\right)=\frac{1}{2^n}\left(\binom{n-1}{a-1}-\binom{n-1}{a}\right)\\
=&\frac{1}{2^n}\cdot\frac{(n-1)!}{(a-1)!\,(n-a-1)!}\left(\frac{1}{n-a}-\frac{1}{a}\right)=\frac{1}{2^n}\cdot\frac{(n-1)!}{(a-1)!\,(b-1)!}\cdot\frac{a-b}{ab}\\
=&\frac{1}{2^n}\cdot\frac{n!}{a!b!}\cdot\frac{a-b}{n}=\frac{1}{2^n}\cdot\binom{n}{a}\cdot\frac{a-b}{n}
\end{aligned}$$

As a side result, we get that the number of those paths in the first n steps is $\frac{a-b}{n}\binom{n}{a}$. □

Before finding the probability of where a random path is 0, a limit is needed.

Lemma 1.2 (odd/even fraction products and a binomial limit). *Let* $n\in\mathbb{N}$. *Then* $\prod\limits_{i=n+1}^{2n}\frac{2i-1}{2i}\leq\frac{3}{4}$ *and* $\lim\limits_{n\to\infty}\frac{\binom{2n}{n}}{4^n}=0$.

Proof. We first prove the inequality.

Induction base: For $n=1$ we have $\prod\limits_{i=2}^{2}\frac{2i-1}{2i}=\frac{3}{4}$.

Induction step: Let the inequality be true for $n=m$. To show it for $n=m+1$, the product is rearranged into a shape where the induction hypothesis can be used:

$$\begin{aligned}
\prod_{i=m+2}^{2(m+1)}\frac{2i-1}{2i}&=\frac{(4m+1)(4m+3)}{(4m+2)(4m+4)}\cdot\frac{2m+2}{2m+1}\prod_{i=m+1}^{2m}\frac{2i-1}{2i}\\
&\overset{\text{I.H.}}{\leq}\frac{(4m+1)(4m+3)}{(4m+2)(4m+2)}\cdot\frac{3}{4}=\frac{3}{4}\cdot\frac{(4m+2)^2-1^2}{(4m+2)^2}<\frac{3}{4}.
\end{aligned}$$

The third binomial formula is used for the last equality sign. This is already the end of the induction step. With it, the induction is complete for $\mathbb{N}$.

With this inequality, we get

$$\frac{\binom{2n}{n}}{4^n}=\frac{(2n)!}{4^n\cdot(n!)^2}=\prod_{i=1}^{n}\frac{2i-1}{2i}\leq\prod_{j=1}^{\log_2(n)}\prod_{k=2^{j-1}+1}^{2^j}\frac{2k-1}{2k}\leq\prod_{j=1}^{\log_2(n)}\frac{3}{4}=\left(\frac{3}{4}\right)^{\log_2(n)},$$

which leads to $0\leq\lim\limits_{n\to\infty}\frac{\binom{2n}{n}}{4^n}=\lim\limits_{n\to\infty}\left(\frac{3}{4}\right)^{\log_2(n)}\leq0$ to conclude the proof. □

Proposition 1.3 (probability of returning to 0)**.** *We have* $P(\forall k \in \mathbb{N} : X_k \neq 0) = 0$ *and* $P(X_1 \neq 0, \ldots, X_{2m-1} \neq 0, X_{2m} = 0) = \frac{1}{m \cdot 2^{2m-1}} \binom{2m-2}{m-1}$ *for any* $m \in \mathbb{N}$.

Proof. The X_i with $1 \leq i < n$ are either all positive or all negative. Because both of these have same probability (the paths can just be completely reflected), it's sufficient to consider the positive case, where the path has to be at 1 after $2m-1$ steps. By using $a = m$ and $b = m-1$ in the formula of Proposition 1.1 and multiplying $\frac{1}{2}$ for the last step from 1 to 0, we obtain

$$\begin{aligned}
&P(X_1 \neq 0, \ldots, X_{2m-1} \neq 0, X_{2m} = 0) \\
=&P(X_1 > 0, \ldots, X_{2m-1} > 0, X_{2m} = 0) + P(X_1 < 0, \ldots, X_{2m-1} < 0, X_{2m} = 0) \\
=&2 \cdot P(X_1 > 0, \ldots, X_{2m-2} > 0, X_{2m-1} = 1, X_{2m} = 0) \\
=&2 \cdot \frac{1}{2} \cdot P(X_1 > 0, \ldots, X_{2m-2} > 0, X_{2m-1} = 1) \\
=&\frac{1}{(2m-1)\, 2^{2m-1}} \cdot \binom{2m-1}{m} = \frac{2m-1}{m(2m-1)\, 2^{2m-1}} \cdot \binom{2m-2}{m-1} = \frac{1}{m \cdot 2^{2m-1}} \binom{2m-2}{m-1}.
\end{aligned}$$

This is already the end of the proof for the second statement. The analog side result to the formula above is that the number of paths with length $2m$ while returning to 0 for the first time there is $\frac{2}{m}\binom{2m-2}{m-1}$.

For the first statement, the following equation is useful.

$$\begin{aligned}
&P(X_{2m-2} = 0) - P(X_{2m} = 0) = \frac{\binom{2m-2}{m-1}}{4^{m-1}} - \frac{\binom{2m}{m}}{4^m} = \frac{\binom{2m-2}{m-1}}{4^{m-1}} \left(1 - \frac{2m(2m-1)}{4m^2}\right) \\
=&\binom{2m-2}{m-1} 4^{m-1} \cdot \frac{1}{2m} = \binom{2m-2}{m-1} m \cdot 2^{2m-1} = P(X_1 \neq 0, \ldots, X_{2m-1} \neq 0, X_{2m} = 0)
\end{aligned}$$

The next equation looks into if the path crosses 0 at any time. Due to telescope sums, it quickly simplifies to what we want. Lemma 1.2 is used to finally calculate the limit.

$$\begin{aligned}
P(\forall k \in \mathbb{N} : X_k \neq 0) &= 1 - \lim_{j \to \infty} \sum_{i=1}^{j} P(X_1 \neq 0, \ldots, X_{2i-1} \neq 0, X_{2i} = 0) \\
=1 - \lim_{j \to \infty} (P(X_0 = 0) - P(X_{2j} = 0)) &= 1 - 1 + \lim_{j \to \infty} \frac{\binom{2m}{m}}{4^m} = 0
\end{aligned}$$

This concludes the proof, and shows that almost every path returns to 0. Therefore, also almost every path returns to 0 infinitely often. □

Another closely related problem is to consider only paths of length $2m$ that end in 0 and don't ever go into negative range, while 0 can be passed this time. However, this is not hard anymore. Imagine going one step up before and one step down after such a path, and the resulting path is bigger than 0 everywhere except for the start (obviously) and the end time $2m+2$, where it is at 0.

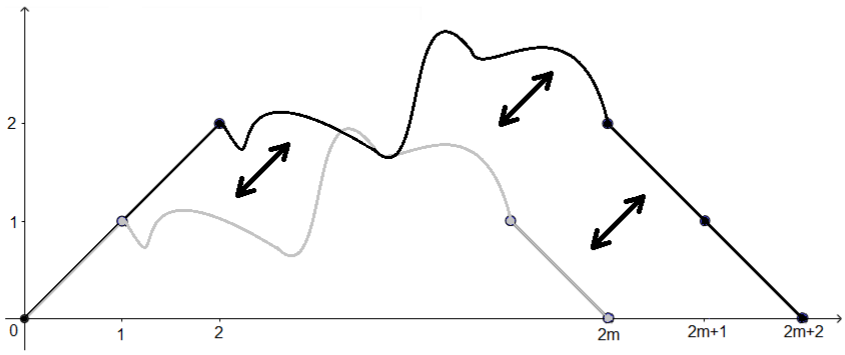

Figure 3: Converting paths between these two problems

The probability for that was just calculated, it just has to be multiplied with $\frac{1}{2}$ because those paths could also be in negative range, and with 4 because the two extra steps with $\frac{1}{2}$ each aren't taken.

$$P\left(X_1 \geq 0, \ldots, X_{2m-1} \geq 0, X_{2m} = 0\right) = 4 \cdot P\left(X_1 > 0, \ldots, X_{2m+1} > 0, X_{2m+2} = 0\right)$$
$$= 4 \cdot \frac{1}{2} \cdot \frac{1}{m \cdot 2^{2m+1}} \binom{2m}{m} = \frac{1}{(m+1) \cdot 2^{2m}} \binom{2m}{m}$$

The number of paths with this property is therefore $\frac{1}{m+1}\binom{2m}{m} =: C_m$. The C stands for *Catalan numbers*, that's how this sequence starting with $1, 1, 2, 5, 14, 42, 132, 429, 1430\ldots$ is called.

Using these random walk paths helps to prove another formula for the Catalan Numbers.

Lemma 1.4 (Segner's recurrence relation)**.** *Let $m \in \mathbb{N}_0$. Then $C_{m+1} = \sum_{l=0}^{m} C_l C_{m-l}$.*

Proof. There are C_{m+1} random paths with length $2m+2$ where $X_{2m+2} = 0$ and $X_k \geq 0$ for $0 \leq k \leq 2m+2$. In such a path, let $2l+2$ be the first time where 0 is reached after starting, $0 \leq l \leq m$. Then the path consists of two subpaths. The first one has length $2l+2$, ends with 0 and is positive everywhere in-between, there are $\frac{1}{l+1}\binom{2l}{l}$ possibilities to create such a subpath. The reason for that is again to ignore the first and the last step and use the formula on the rest. The second one has length $2m-2l$ (which may or may not be 0), also ends with 0 and is non-negative everywhere, and there are clearly $\frac{1}{m-l+1}\binom{2(m-l)}{m-l}$ possible paths for this part.

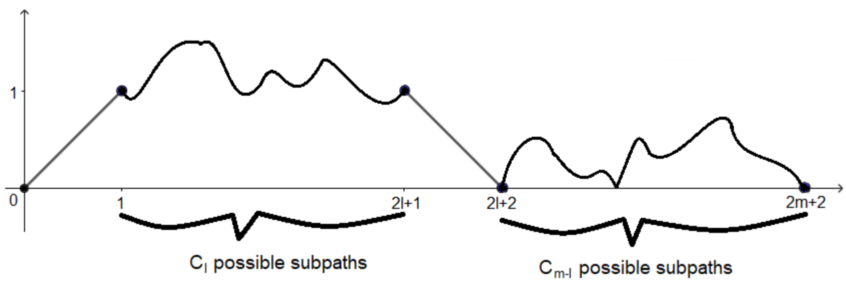

Figure 4: Reflection of the first part until reaching 0

The number of all possible paths is therefore the sum of the product of those numbers of possibilities for every possible l.

$$C_{m+1} = \sum_{l=0}^{m} \frac{1}{l+1}\binom{2l}{l} \cdot \frac{1}{m-l+1}\binom{2(m-l)}{m-l} = \sum_{l=0}^{m} C_l C_{m-l}$$

This is the end of the proof. □

1.2 Limit of random walk

In probability theory, there are some theorems for what happens when the mean of many random variables is constructed. The **Gaussian distribution** plays a very special role there. A real-valued random variable X has a standard Gaussian (or normal) distribution, $X \sim N(0,1)$ when its probability density is $f(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right)$ for $x \in \mathbb{R}$. The values 0 and 1 in the brackets are for the mean value 0 and the variance 1. They can also be modified with $X \sim N(\mu, \sigma^2)$ if X has mean value μ and variance σ^2. Then the density has to be modified to $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$.

This is a fundamental distribution for the **central limit theorem**: Given that there are identically independent distributed (iid) random variables $Z_1, Z_2, \ldots$ with finite expected value μ and variance σ^2. Let S_n be the sum of the first n of those, $S_n = \sum_{i=1}^{n} Z_i$. The central limit theorem implies that S_n adjusted to mean 0 and variance 1 converges in distribution to a normally distributed random variable, $\frac{S_n - n\mu}{\sqrt{n}\sigma} \xrightarrow{\text{d}} X \sim N(0,1)$. The denominator is $\sqrt{n}$ because it is squared for the variance and balances the n random variables in the sum.

With this theorem, we can set Z_i like in the random walk with $P(Z_i = 1) = \frac{1}{2} =$

$P(Z_i = -1)$. In that case, using $X_n = \sum_{i=1}^{n} Z_i$ gives $\frac{S_n}{\sqrt{n}} \xrightarrow{d} X \sim N(0,1)$ because of $\mu = 0$ and $\sigma^2 = 1$. Therefore, the normalized random walk converges in distribution to the normal distribution.

If we want to have a limit for the whole random walk, the concepts have to be generalized. A sequence of σ-algebras $(\mathcal{F}_t)_{t\geq 0}$ with $\{\emptyset; \Omega\} = \mathcal{F}_0, j \leq k \Rightarrow \mathcal{F}_j \subseteq \mathcal{F}_k$ and $\mathcal{F}_t \subseteq \mathcal{F}$ is a filtration, indexed by real numbers this time instead of integers. Again we have $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ its filtered probability space and a sequence of random variables $(X_t)_{t\geq 0}$ is a *continuous-time stochastic process* when for all $t \geq 0$ the random variable X_t measurable by $\mathcal{F}_t$, analog to the *discrete-time* processes we had before. Martingales can be defined in a similar way, stopping times also have their continuous-time definitions.

The reason for this is the construction of a stochastic process that is a generalization for the normal distribution. A continuous-time stochastic process $(B_t)_{t\geq 0}$ with $B_0 = 0$ is called *Brownian motion* (or Wiener process) when the following conditions are met:

- B_t is continuous almost surely,
- $\forall m \in \mathbb{N}, t_1, t_2, \ldots, t_m \in \mathbb{R} : 0 \leq t_1 < t_2 < \cdots < t_m \Rightarrow B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_m} - B_{t_{k-1}}$ are independent,
- $\forall t_1, t_2 \geq 0 : t_1 \leq t_2 \Rightarrow B_{t_2} - B_{t_1} \sim N(0, t_2 - t_1)$.

The t stands for the time that has passed. For every $\omega \in \Omega$, a path can be constructed as function of $t \geq 0$ with $f(t) = B_t(\omega)$.

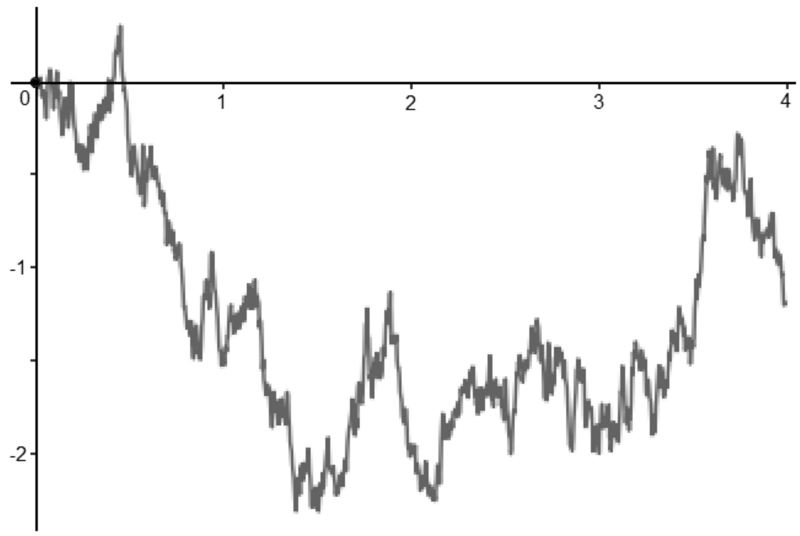

Figure 5: A Brownian motion sample path

For the random walks, we now define certain continuous-time stochastic processes from the original discrete-time stochastic processes. Let $B_{t,n} := \sum_{i=1}^{\lfloor nt \rfloor} \frac{Z_i}{\sqrt{n}}$ for $t, n \in R_0^+$. For any n, the process $(B_{t,n})_{t \geq 0}$ has non-continuous jumps at times $\frac{k}{n}$ with $k \in \mathbb{N}$.

Let's look what happens for large n. Let B_t be a Brownian motion. Let $t > 0$, then $B_{n,t}$ converges in distribution to a $N(0,t)$ distributed random variable with. However, B_t is also $N(0,t)$ distributed, so a random walk might actually converge to the Brownian motion. Setting $t = 0$ also implies $B_{n,0} = 0 = B_0$, so the limit is also always 0. For the other conditions, a stopping time will be used.

For all $n \in \mathbb{N}$ let $\tau_0 := 0$ and $\tau_n := \min\left\{x : x > \tau_{n-1}, B_x \notin]B_{\tau_{n-1}} - 1; B_{\tau_{n-1}} + 1[\right\}$. So for a Brownian motion path, we always look for the first time when the path has changed by 1. Because for $t \to \infty$ the values close to 0 tend to never be chosen, we have $\tau_1 < \infty$ almost surely, and therefore also $\tau_2, \tau_3, \ldots$ are finite almost surely. This implies that we can create a simple random walk with $X_i := B_{\tau_i}$. Obviously all $\tau_i - \tau_{i-1}$ are iid. A similar idea can actually be executed for every other Z_i with $i \in \mathbb{N}$ and $\mathbb{E}Z_i = 0, Var(Z_i) = 1$ and all Z_i are iid, but a few more steps would be needed before that we don't need with the simple case of $P(Z_i = 1) = \frac{1}{2} = P(Z_i = -1)$. The stopping times themselves can be very close to each other or also very far away, depending on the path itself.

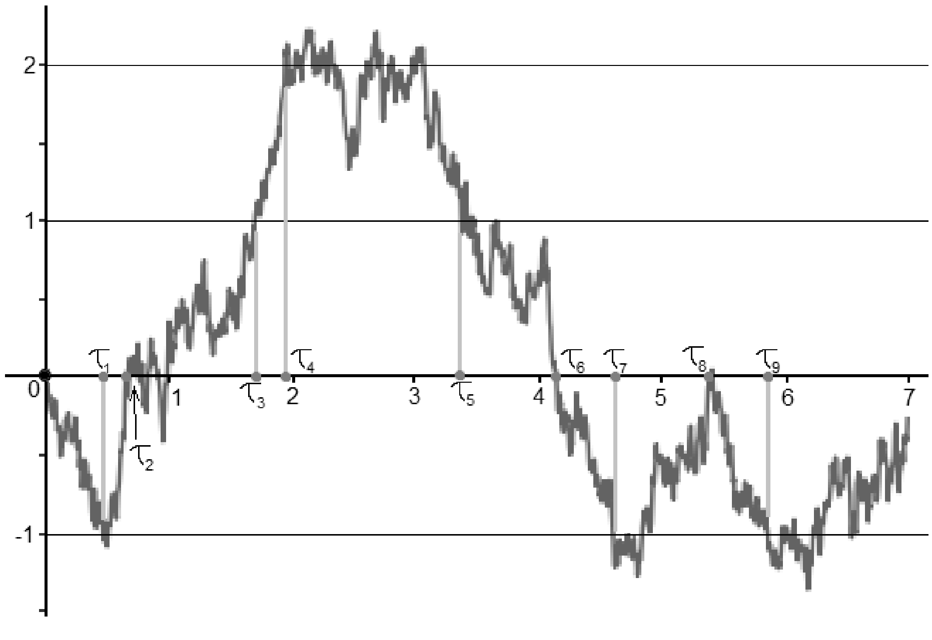

Figure 6: stopping times in a Brownian motion path, $t \leq 7$

As seen, for example τ_1 and τ_2 are only like 0.3 apart, while τ_4 and τ_5 almost have difference 1.5. The mean value for these stopping times can be calculated with help of the following Proposition.

Proposition 1.5 (Brownian motion stopping times first and second moments)**.** *For $a < 0 < b$, let $\tau_{a,b} := \min\{x : B_x \notin]a, b[\}$, then $1 - P\left(B_{\tau_{a,b}} = b\right) = P\left(B_{\tau_{a,b}} = a\right) = \frac{b}{b-a}$ and the mean value of the stopping time is $\mathbb{E}\tau_{a,b} = -ab$. If $-a = b$, the mean of the second moment is $\mathbb{E}\tau_{a,b}^2 = \frac{5a^4}{3}$.*

Using Proposition 1.5, we get $\mathbb{E}(\tau_i - \tau_{i-1}) = \mathbb{E}\tau_1 = \mathbb{E}\tau_{-1,1} = -(-1 \cdot 1) = 1$ for any $i \in \mathbb{N}$. So the fact that in this path the stopping times went up to τ_9 before $t = 6$ shows that they are much more dense than usual, but then τ_{10} doesn't show up before $t = 7$, that makes it up a bit. For the probabilities, we have $P\left(B_{\tau_i} - B_{\tau_{i-1}} = 1\right) = P(B_{\tau_1} = 1) = \frac{1}{2}$ and the same if 1 is replaced with -1. The stochastic process $(B_{\tau_i})_{i \in \mathbb{N}_0}$ has values going 1 up or down with every step with probability $\frac{1}{2}$ each, and the stopping times increase by approximately 1. Because of the independence of each step, this actually is a random walk embedded in the Brownian motion.

The next step uses the already mentioned generalized random variables Z_i that only have same mean and variance as the random walk steps.

Proposition 1.6 (Central limit theorem)**.** *For any $Z_i, i \in \mathbb{N}$ that are iid with mean 0 and variance 1 and $X_n = \sum_{i=1}^{n} Z_i$, the process $\frac{X_n}{\sqrt{n}}$ converges in distribution to a $N(0,1)$ distributed random variable.*

Proof. Let τ_i with $i \in \mathbb{N}_0$ be the stopping times that embed X_i in the Brownian motion. For any $s > 0$ and $t \geq 0$, we have $\frac{B_{st}}{\sqrt{s}} \sim \mathbb{N}\left(0, \frac{st}{\sqrt{s}^2}\right)$. The variance simplifies to t, so $\frac{B_{st}}{\sqrt{s}}$ and B_t have the same distribution. In this case, $\frac{X_n}{\sqrt{n}}$ has the same distribution as $\frac{B_{\tau_n}}{\sqrt{n}}$ for any $n \in \mathbb{N}$. Due to the weak law of large numbers, we also have $\frac{\tau_n}{n} \to 1$ in probability.

Now let $\epsilon > 0$. Choose δ such that $P(|B_t - B_1| > \epsilon, t \in]1-\delta, 1+\delta[) < \frac{\epsilon}{2}$ and N large enough to have $\forall n \geq N : P\left(\frac{\tau_n}{n} - 1 \geq \delta\right) < \frac{\epsilon}{2}$. This is possible because of the already existing convergences. Combining both of these implies

$$P\left(\left|\frac{B_{\tau_n} - B_n}{\sqrt{n}}\right| \geq \epsilon\right) < \epsilon.$$

For $\epsilon \to 0$ and $n \to \infty$, we now get $\frac{B_{\tau_n} - B_n}{\sqrt{n}} \to 0$ in probability, and with that, $\frac{X_n}{\sqrt{n}} - \frac{B_n}{\sqrt{n}} \to 0$ in probability. With $\frac{B_n}{\sqrt{n}} \sim N(0,1)$, the proof is completed. □

This is what we need to prove the convergence.

Theorem 1.7 (Donsker's theorem). *Let $Z_i, i \in \mathbb{N}$ be iid with $EZ_1 = 0$, $Var(Z_1) = 1$ and $X_n = \sum_{i=1}^{n} Z_i$. Then $\left(\frac{X_{nt}}{\sqrt{n}}\right)_{t\geq 0}$ converges to the Brownian motion $(B_t)_{t\geq 0}$ in distribution as $n \to \infty$.*

Proof. Let $Z_{n,m}$ be a triangular array of random variables for $1 \leq m \leq n$ and $X_{n,m} = \sum_{i=1}^{m} Z_{n,i}$. Set τ_m^n such that $X_{n,m} = B_{\tau_m^n}$. For every $u \notin \mathbb{Z}$ with $0 < u < n$, let $X_{n,u}$ be the linear continuation from the rest. So for $m-1 < u < m$ and $q = u - m + 1$, we have $X_{n,u} := qX_{n,m} + (1-q)X_{n,m-1}$.

The Brownian motion is continuous, that means that for all $\epsilon > 0$ there is a $\delta > 0$ such that $\frac{1}{\delta}$ is an integer with

$$P\left(|B_t - B_s| < \epsilon, 0 \leq s \leq 1, |t-s| < 2\delta\right) > 1 - \epsilon.$$

If $\tau_{\lfloor ns \rfloor}^n$ converges to s in probability for any $0 \leq s \leq 1$, an $N \in \mathbb{N}$ can be chosen such that $P\left(\left|\tau_{\lfloor nk\delta \rfloor}^n - k\delta\right| < \delta, k \in \mathbb{N}_0, k \leq \frac{1}{\delta}\right) > 1 - \epsilon$ for any $n > N$. For $s \in](k-1)\delta, k\delta[$, the inequality $\tau_{\lfloor n(k-1)\delta \rfloor}^n - k\delta \leq \tau_{\lfloor ns \rfloor}^n - s \leq \tau_{\lfloor nk\delta \rfloor}^n - (k-1)\delta$, is obviously true because τ_m^n is monotone increasing when increasing m. For $n \geq N$ we now have

$$P\left(sup\left(\left\{\left|\tau_{\lfloor ns \rfloor}^n - s\right| : 0 \leq s \leq 1\right\}\right) < 2\delta\right) \geq 1 - \epsilon.$$

Meanwhile $\left|X_{n,m} - B_{\frac{m}{n}}\right| < \epsilon$ is true for any $n > N$ and $m \leq n$. For u and q like above and $nt = u$, we have

$$\begin{aligned}|X_{n,u} - B_t| \leq (1-q)\left|X_{n,m} - B_{\frac{m}{n}}\right| + q\left|X_{n,m+1} - B_{\frac{m+1}{n}}\right| \\ + (1-q)\left|B_{\frac{m}{n} - B_t}\right| + q\left|B_{\frac{m+1}{n} - B_t}\right|.\end{aligned}$$

The probability of the last two summands combined being greater than ϵ is at worst still smaller than ϵ for each summand, and the first two are combined not greater than ϵ. So $|X_{n,u} - B_t|$ is smaller than 2ϵ with probability greater than $1 - 2\epsilon$. Because ϵ is arbitrary, that means that in this case $|X_{n,nt} - B_t|$ converges to 0 in probability, and all that was needed are the stopping times $\tau_{\lfloor ns \rfloor}^n$ that converge to s for any s in probability.

This is a lot more generalized than we need it. Now set $Z_{n,m} = \frac{Z_m}{\sqrt{n}}$, therefore $X_{n,m} = \frac{X_m}{\sqrt{n}}$ for $m \in \mathbb{Z}$ and its linear continuations at the other places. Then $\tau_1^n, \ldots, \tau_n^n$ are defined to let $X_{n,m}$ and $B_{\tau_m^n}$ have the same distribution. The stopping times τ_i were already created above for the central limit theorem. Then it's implied that τ_m^n and $\frac{\tau_m}{n}$ also have the same distribution, which is what we want. It follows that $\left|\frac{X_{nt}}{\sqrt{n}} - B_t\right|$

converges in distribution to 0 for any $0 \leq t \leq 1$.

Finally, let $f : \mathbb{R} \to \mathbb{R}$ be bounded and continuous and $\epsilon_2 > 0$. For any $\delta_2 > 0$, let $G_{\delta_2} := \{w \in \mathbb{R} : \forall w' \in \mathbb{R} : |w - w'| < \delta_2 \Rightarrow |f(w) - f(w')| < \epsilon_2\}$. Because f is continuous, we have $\lim_{\delta_2 \to 0} G_{\delta_2} = \mathbb{R}$. Then for $t \in [0, 1]$ we get

$$\left| Ef\left(\frac{X_{nt}}{\sqrt{n}}\right) - Ef(B_t)\right| \leq \epsilon + (2 \sup |f(x)|) \left(P(B_t \notin G_{\delta_2}) + P\left(\left|\frac{X_{nt}}{\sqrt{n}} - B_t\right| \leq \delta_2\right)\right).$$

The second summand tends to 0 for $\delta_2 \to 0$ because both summands in the brackets do so. It follows that $\left|Ef\left(\frac{X_{nt}}{\sqrt{n}}\right) - Ef(B_t)\right| \to 0$ for $n \to \infty$, as ϵ_2 is arbitrary.

That's the weak convergence for $t \leq 1$. As for bigger values for t, the proof can be extended from $[0, 1]$ to $[0, M]$ for any $M > 0$ by appropriately shifting the $Z_{n,m}$ to include higher values for m like $m \leq M \cdot n$, and the rest follows analog to what we did for $M = 1$. Then, it naturally follows for $t \in [0, \infty[$, which completes the proof. □

2 Modifications

2.1 The 1-3-random walk

In a 1-3-random walk, we choose to triple the increments of the classic random walk when we are in the negative. Let $Z_1, Z_2, \ldots$ be random variables that are iid with $P(Z_i = 1) = P(Z_i = -1) = \frac{1}{2}$, $X_0 = 0$, and for any $k \in \mathbb{N}$ let

$$X_k = \begin{cases} X_{k-1} + Z_k & X_{k-1} \geq 0 \\ X_{k-1} + 3Z_k & \text{otherwise.} \end{cases}$$

The considered task shall be: What is the probability of such a path to be non-negative? It seems to have the limit $\frac{1}{2}$ in the classic random walk, at least for odd n there are always half of the paths in positive or negative range each. Is there a limit in this modification?

Proposition 2.1 (non-negative 1-3-paths)**.** *We have* $\lim_{n\to\infty} P(X_n \geq 0) = \frac{3}{4}$.

Proof. A combinatorial approach will be used to calculate the number of non-negative 1-3-random walk paths directly. For any $n \in \mathbb{Z}$ and $n \geq 0$, let D_n be the number of paths that are positive at time n and $D(n,k)$ be the number of paths with length n that end in k. We have $D(0,0) = 1, D(0,k) = 0$ for $k \neq 0$ and for $n > 0$.

$$D(n,k) = \begin{cases} D(n-1,1) & k = 0 \\ D(n-1,-1) + D(n-1,1) + D(n-1,3) & k = 2 \\ D(n-1,-4) + D(n-1,0) & k = -1 \\ D(n-1,k-3) + D(n-1,k+3) & k < -1 \\ D(n-1,k-1) + D(n-1,k+1) & \text{otherwise.} \end{cases}$$

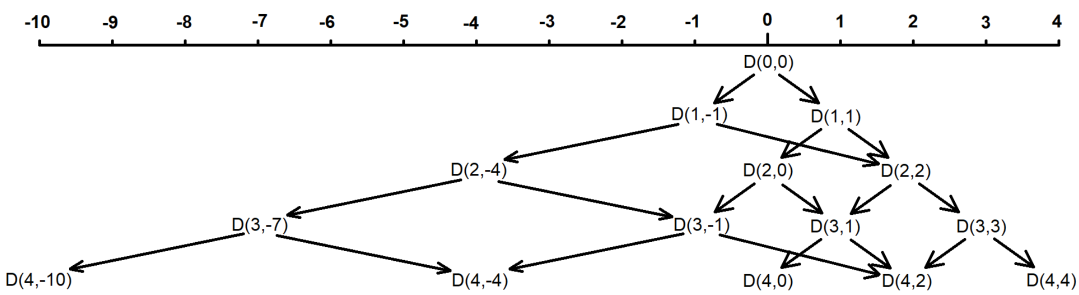

Figure 7: Coefficients for the 1-3 paths

Note that $P(X_n \geq 0) = \frac{D_n}{2^n}$ and $D_n = \sum_{i=0}^{\infty} D(n,i)$.

We will use other coefficients now, to create some kind of modified Pascal triangle. Let $C(n,l)$ for $l \in \{0, \ldots, n\}$ be in the specific spots like in the Pascal triangle, only with the coefficients from the $D(n,k)$. All other spots shall be filled with zeroes.

$$C(n,l) = \begin{cases} D(n, 2-3n+6l) & l < \frac{n}{2} \\ D(n, n-2l) & \text{otherwise} \end{cases}$$

The reason we do this is to make the recursive definition a lot easier.

$$C(n,l) = \begin{cases} C\left(n-1, \frac{n}{2}\right) & l = \frac{n}{2} \\ C\left(n-1, \frac{n}{2}-1\right) + C\left(n-1, \frac{n}{2}\right) + C\left(n-1, \frac{n}{2}+1\right) & l = \frac{n}{2}+1 \\ C(n-1, l-1) + C(n-1, l) & \text{otherwise} \end{cases}$$

It's much less cases here, and the last line almost makes it look like Pascal's triangle.

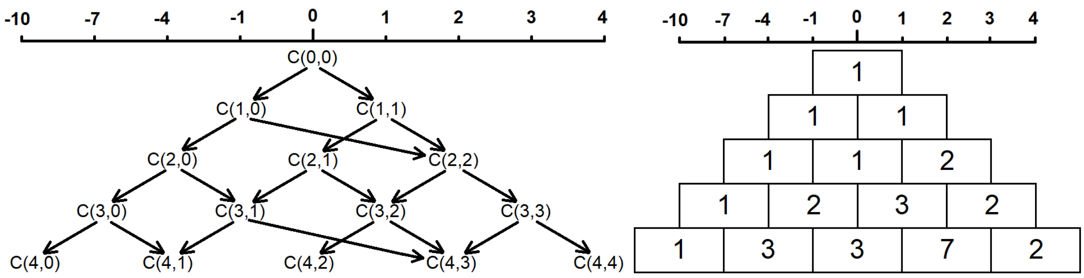

Figure 8: Modified coefficients for 1-3 paths with their values

Now it will be worked out what each of the single values are going to be. We have $C(0,0) = C(1,0) = C(1,1) = 1$. For the rest, there are some more cases. For all $n \geq 1, 0 \leq l \leq n$, complete induction allows us to show the following:

$$C(n,l) = \begin{cases} \binom{n-1}{l} & l \leq \frac{n}{2} \\ \binom{n}{l} & l = \frac{n+1}{2} \\ \binom{n}{l} + \binom{n-1}{l-1} & \text{otherwise} \end{cases}$$

The first case gives a Pascal triangle, shifted by one to the lower left. For $l = \frac{n+1}{2}$, if n is odd because this case only exists there, the coefficient is the exact same as in the Pascal triangle. And the right half is a Pascal triangle added to a lower-right shifted Pascal triangle.

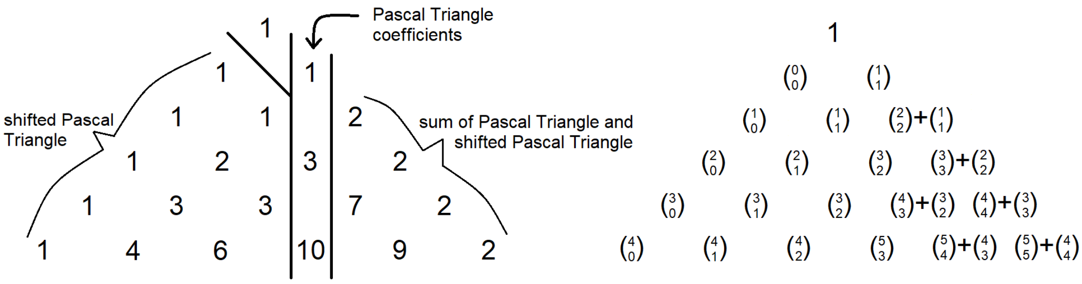

Figure 9: Illustration for the exact coefficients

Induction base: Let $n = 1$: $C(1,0) = 1 = \binom{0}{0}$ and $C(1,1) = 1 = \binom{1}{1}$.

Induction step: If the formula is true for $n = m$, we can show it for $n = m+1$. There are a few different cases for l.

- If $l < \frac{m+1}{2}$, then $C(m+1,l) = C(m,l-1) + C(m,l) = \binom{m-1}{l-1} + \binom{m-1}{l} = \binom{m}{l}$.
- For $l > \frac{m+3}{2}$ we have $C(m+1,l) = C(m,l-1) + C(m,l) = \binom{m}{l-1} + \binom{m-1}{l-2} + \binom{m}{l} + \binom{m-1}{l-1} = \binom{m+1}{l} + \binom{m}{l-1}$.
- If $l = \frac{m+1}{2}$, then $k = 0$. So we have $C\left(m+1, \frac{m+1}{2}\right) = C\left(m, \frac{m+1}{2}\right) = \binom{m}{\frac{m+1}{2}}$.
- For $l = \frac{m}{2} + 1$ the new value is $\binom{n-1}{l-1} = \binom{n-1}{l-2}$. This implies $C\left(m+1, \frac{m}{2}+1\right) = C\left(m, \frac{m}{2}\right) + C\left(m, \frac{m}{2}+1\right) = \binom{m-1}{\frac{m}{2}} + \binom{m}{\frac{m}{2}+1} + \binom{m-1}{\frac{m}{2}} = \binom{m}{\frac{m}{2}} + \binom{m}{\frac{m}{2}+1} = \binom{m+1}{\frac{m}{2}+1}$.
- The last case is $l = \frac{m+3}{2}$ with $C\left(m+1, \frac{m+3}{2}\right) = C\left(m, \frac{m-1}{2}\right) + C\left(m, \frac{m+1}{2}\right) + C\left(m, \frac{m+3}{2}\right) = \binom{m-1}{\frac{m-1}{2}} + \binom{m}{\frac{m+1}{2}} + \binom{m}{\frac{m+3}{2}} + \binom{m-1}{\frac{m+1}{2}} = \binom{m+1}{\frac{m+3}{2}} + \binom{m}{\frac{m+1}{2}}$.

So for all cases, the formula is still correct for $n = m+1$, which completes this induction for this formula.

For $n \geq 2$, we can now use this formula to get the desired probability.

$$P(X_n \geq 0) = \frac{D_n}{2^n} = \frac{\sum\limits_{i \geq \frac{n}{2}} C(n,i)}{2^n}$$

We first consider even n.

$$P(X_n \geq 0) = \frac{\binom{n-1}{\frac{n}{2}} + \sum\limits_{i=\frac{n}{2}+1}^{n} \left(\binom{n}{i} + \binom{n-1}{i-1}\right)}{2^n} = \frac{\binom{n-1}{\frac{n}{2}} + \sum\limits_{i=\frac{n}{2}+1}^{n} \left(\binom{n-1}{i-1} + \binom{n-1}{i} + \binom{n-1}{i-1}\right)}{2^n}$$

$$= \frac{3\sum\limits_{i=\frac{n}{2}}^{n-1} \binom{n-1}{i}}{2^n} = \frac{3 \cdot 2^{n-1-1}}{2^n} = \frac{3}{4}$$

The last sum had the value 2^{n-2} because it's half of the sum of all Binomial Coefficients in row $n-1$. For even n the probability already is $\frac{3}{4}$.

For odd n, this result can already be used. Let $n = 2z+1$ with $z \in \mathbb{N}$. Every value in one row gives itself twice to a value in the next row. All values in the non-negative range are represented twice in this range of the next row like that, except for $X_{n-1} = 0$, which only lands there once. This is also the reason why we set $n > 1$, even if $n = 1$ would work here, too.

$$\begin{aligned} P(X_n \geq 0) =& \frac{\sum_{i=\frac{n+1}{2}}^{n} \left(C(n-1, i-1) + C(n-1, i)\right)}{2^n} \\ =& \frac{2\sum_{i=\frac{n+1}{2}}^{n-1} C(n-1, i-1) - C\left(n-1, \frac{n-1}{2}\right)}{2^n} = \frac{3}{4} - \frac{\binom{n-2}{\frac{n-1}{2}}}{2^n} \end{aligned}$$

Note that $\binom{n-2}{\frac{n-1}{2}} = \binom{n-2}{\frac{n-3}{2}}$, which implies $2\binom{n-2}{\frac{n-1}{2}} = \binom{n-2}{\frac{n-1}{2}} + \binom{n-2}{\frac{n-3}{2}} = \binom{n-1}{\frac{n-1}{2}}$.

$$P(X_n \geq 0) = \frac{3}{4} - \frac{\binom{n-1}{\frac{n-1}{2}}}{2^{n+1}} = \frac{3}{4} - \frac{1}{4}\frac{\binom{n-1}{\frac{n-1}{2}}}{2^{n-1}} = \frac{3}{4} - \frac{1}{4}\frac{\binom{2z}{z}}{2^{2z}}$$

The subtrahend tends to 0 using Lemma 1.2. Then we get $\lim_{z\to\infty} P(X_{2z+1} \geq 0) = \lim_{z\to\infty}\left(\frac{3}{4} - \frac{\binom{2z}{z}}{4^{z+1}}\right) = \frac{3}{4}$, and with $P(X_{2z} \geq 0) = \frac{3}{4}$, we have $\lim_{n\to\infty} P(X_n \geq 0) = \frac{3}{4}$. This completes the proof. □

Considering this problem for the classic random walk, we have a similar situation: For odd n, the probability is $P(X_{2z+1} \leq 0) = \frac{1}{2}$ obviously for every $z \in \mathbb{N}_0$, because every path ending in negative range can be reflected in 0 to get a path in positive range and vice versa, and a path with odd length cannot end in 0. And if n is even, we get $P(X_{2z} \leq 0) = \frac{1}{2} - \frac{\binom{2z}{z}}{2^{2z+1}}$, which tends to $\frac{1}{2}$ using Lemma 1.2.

2.2 The 1-2-random walk

As before, the 1-2-random walk shall have a different increment value when the path is in negative range. Let X_0 and $Z_1, Z_2, \ldots$ be defined as before, and for $k \in \mathbb{N}$ let

$$X_k = \begin{cases} X_{k-1} + Z_k & X_{k-1} \geq 0 \\ X_{k-1} + 2Z_k & \text{otherwise.} \end{cases}$$

We again try to get reasonable values for $P(X_n \geq 0)$. The first idea could be to, again, have a modified Pascal triangle as in the 1-3 case and do it with the combinatorial approach, with maybe the result $\frac{2}{3}$, matching $\frac{3}{4}$. Then we quickly come to the point that this will be very much harder. The difference is the following: In the 1-3-random walk, X_n is always even when n is even, and odd otherwise. That changed in every step, because the value always changed by either 1 or 3. In the 1-2-random walk, this is obviously not the case, as there can be steps with width 2. That makes it impossible to directly find a modified Pascal triangle, because there are way more possible values that can be attained. For the 1-3 random walk, there were some spots that were skipped, which cannot be done now. For $n = 2$, there are already four different possibilities, as $-3, 0, 1$ and 2 can be reached.

Instead, we take another approach here, that, in first glance, might not have to do anything with random walks.

2.2.1 A bijection between non-negative 1-2-random walks and 1-2-squared rectangles of width 3

There are again 2^n different 1-2-random walk paths until time point n. But we are interested in J_n (because the sequence of these is called *Jacobsthal sequence*), the number of those paths that end in the non-negative range, $X_n \geq 0$. Because all 2^n paths have the same probability $\frac{1}{2^n}$, J_n can also be displayed as $J_n = 2^n \cdot P(X_n \geq 0)$.

The other considered part are decompositions of $n \times 3-$rectangles into squares with length 1 or 2. Let n be the number of columns. There are three possibilities for every of those n columns:

1. There are three squares with length 1. Let this be called a *column with normal squares.*

2. It has a square with length 1 at the top and a square with length 2 below. Let it be called a *column with a lower* 2*-square.* In this case, exactly one of the neighboured columns must also have this lower 2-square.

3. Just as in case two, only that the square of length 2 is above of the square with length 1. It will furthermore be called *column with an upper* 2*-square.* Again, it also affects either the column to the left or the one to the right.

The number of possible decompositions shall be denoted as T_n, the T stands for "tiling", the rectangle is tiled into squares.

Lemma 2.2 (number of possible tilings)**.** *Let $n \in \mathbb{N}_0$. Then we have $T_n = \frac{2^{n+1}+(-1)^n}{3}$.*

Proof. We will use complete induction.

Induction base: For $n = 0$, we have a $0 \times 3-$rectangle that can't be tiled at all. So there is exactly one possible decomposition, which fits $\frac{2^{0+1}+(-1)^0}{3} = \frac{2+1}{3} = 1$. For $n = 1$ we have a $1 \times 3-$rectangle, which has one column and is too small to have any 2-square. Therefore, there is also ony one possible decomposition, and $\frac{2^{1+1}+(-1)^1}{3} = \frac{4-1}{3} = 1$.

Induction step: Let the assumption be true for $n = m - 1$ and $n = m$. Now consider an $(m + 1) \times 3-$rectangle. The last column can have normal squares, in this case there are T_m possible tilings for the other m columns. Or it can have any 2-square, in which case the second-to-last column must have the same (upper or lower) 2-square, and it gives T_{m-1} tilings for both cases. So the exact number of possible decompositions is

$$\begin{aligned} T_{m+1} = T_m + 2T_{m+1} &= \frac{2^{m+1} + (-1)^m + 2 \cdot 2^m + (-1)^{m-1}}{3} \\ &= \frac{2^{m+1} + 2^{m+1} + (-1)^m (1-2)}{3} = \frac{2^{m+2} + (-1)^{m+1}}{3}, \end{aligned}$$

which is exactly the number displayed in the assumption. □

To connect the 1-2-random walks that don't end in the negative range with the considered tilings of $n \times 3-$rectangles, a bijection between those will be made.

Proposition 2.3 (non-negative 1-2-random walk paths)**.** *Let $n \in \mathbb{N}_0$. Then $J_n = T_n$.*

Proof. Just to showcase what is about to be done, at first look at $n \leq 4$.

Case $n \leq 1$.

For $n = 0$, there is only one path because only X_0 is important, which is always 0. This can be connected with the $0 \times 3-$rectangle. If $n = 1$, there are two paths, the one with $X_1 = 1$ and the one with $X_1 = -1$. The second one isn't considered because it ends in the negative range, so only the first one is left, and as discussed above, there is only one possible tiling into squares with length 1 or 2. So the bijection is obvious.

Figure 10: Bijection for $n = 0$ and $n = 1$

For $n > 1$, the general idea should be some kind of composition of the single parts.

Case $n = 2$**.**

There are three rectangle tilings existing. The first one is the one consisting of only squares with length 1. At $n = 1$, we took one step up when we had the column with normal squares. Here we have this column twice, so it's obvious to go two steps up, which means that the path is $X_1 = 1, X_2 = 2$.

Now there are two decompositions left, the first one having one upper 2-square and the second one having one lower 2-square. There are also two of the considered 1-2-random walks left, one of them being $X_1 = 1, X_2 = 0$ and the other one is $-X_1 = X_2 = 1$. Since we have free choice of assignment for these, we can just assign them in mentioned order.

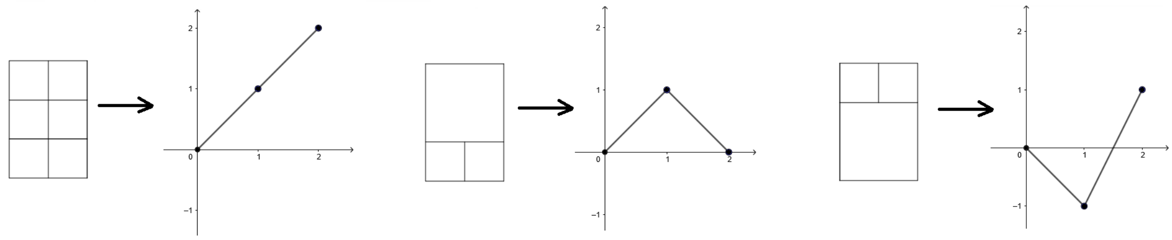

Figure 11: Bijection for $n = 2$

Case $n = 3$**.**

As in $n = 2$, the tiling with only squares with length 1 shall be mapped onto the path that has $X_k = k$. This can also be an idea for any $n \in \mathbb{N}$. Even more, if the last column has normal squares, then a general strat can be to execute the path that resonates to the $(n-1) \times 3-$rectangle before the last column (let it be $Y_0, Y_1, \ldots, Y_{n-1}$) and then at last, make one step up. That means $X_k = Y_k$ for $k < n$ and $X_n = X_{n-1} + 1$. Such a path always ends in positive range because $Y_{n-1} \geq 0$ and $X_n > Y_{n-1}$. Furthermore, if the mapping from the $(n-1) \times 3-$rectangles to the 1-2-random walk paths with length $n-1$ is injective, then with this method, all constructed paths with length n from rectangles with the last column of normal squares are different. Going back to $n = 3$, there are two more paths we immediately get with this strat, belonging to the rectangles that start with an upper or lower 2-square.

The two remaining rectangles are the ones that stop with a 2-square. Before, we mapped the single upper 2-square to the path that goes 1 up first and then goes 1 down. In fact, the outcome is the same as before these two steps, so we basically made a *useless turn*. Composing this with the 1-step up from the first column with normal squares, we geth the path $X_1 = 1, X_2 = 2, X_3 = 1$ for the rectangle that starts with normal squares and then has an upper square. So, the rectangle stopping with a lower square only has the path $X_1 = -1, X_2 = 1, X_3 = 0$ left.

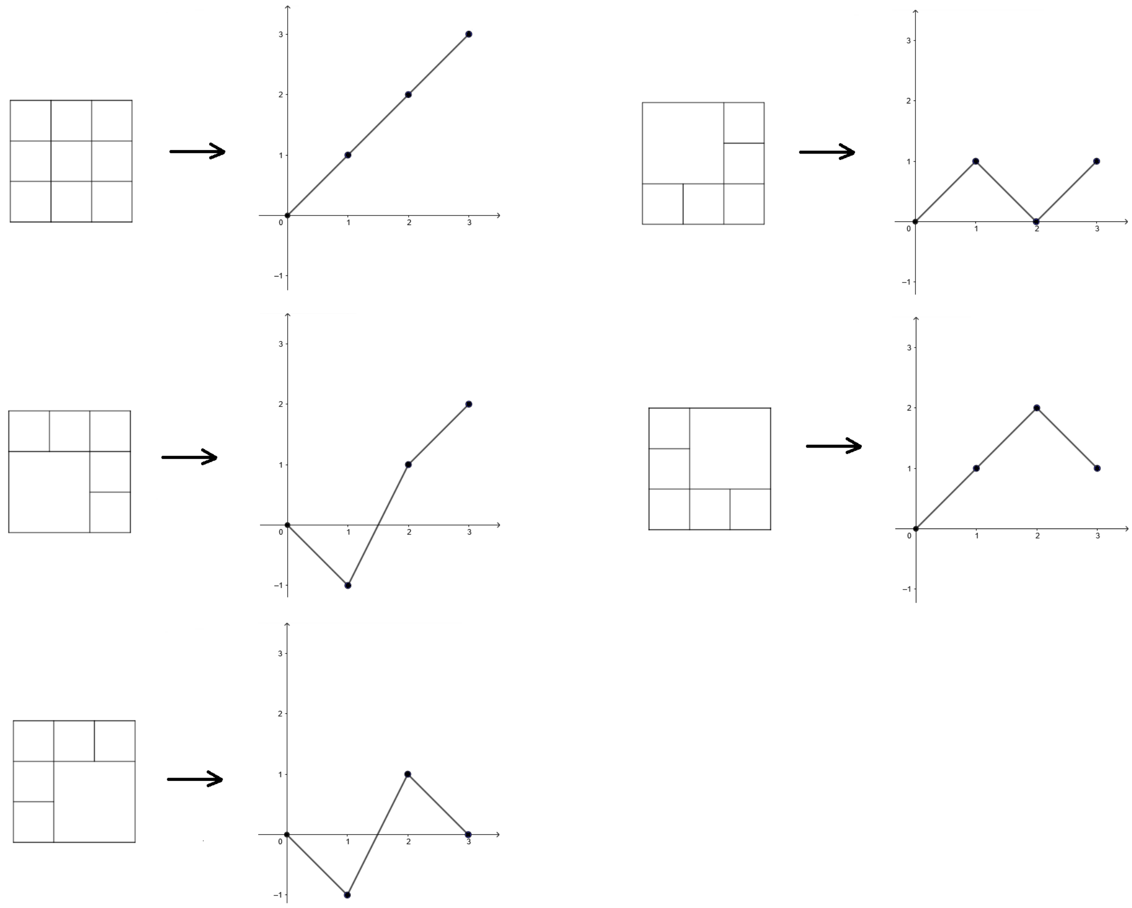

Figure 12: Bijection for $n = 3$

Case $n = 4$**.**

The rectangles with normal squares in the last column can be taken over from $n = 3$ with the last step being $+1$, as mentioned before. The same idea can be transferred to the case that the last two columns have an upper square. Just execute the path with length $n - 2$ before and then do a useless turn, the second-to-last step is $+1$, the last step is -1 in that case. So $X_k = Y_k$ for $k < n-1$ and $X_{n-1} - 1 = X_n = Y_{n-2}$. Again, all the paths generated with this method are different if the mapping for $n - 2$ is injective, they all end in non-negative range because $X_n = Y_{n-2} \geq 0$, and lastly, they are even different to the paths generated from the rectangles with normal squares in the last column, because the last step is definitely different (down for the upper 2-squares, up for the normal squares). For $n = 4$, until now, the bijection stands for all rectangles that don't have a lower square at the last two columns. And because it might be extended to the general case, the normal squares and the upper 2-squares are now very easy to handle. But the general case will not be written down before all the other cases are covered too.

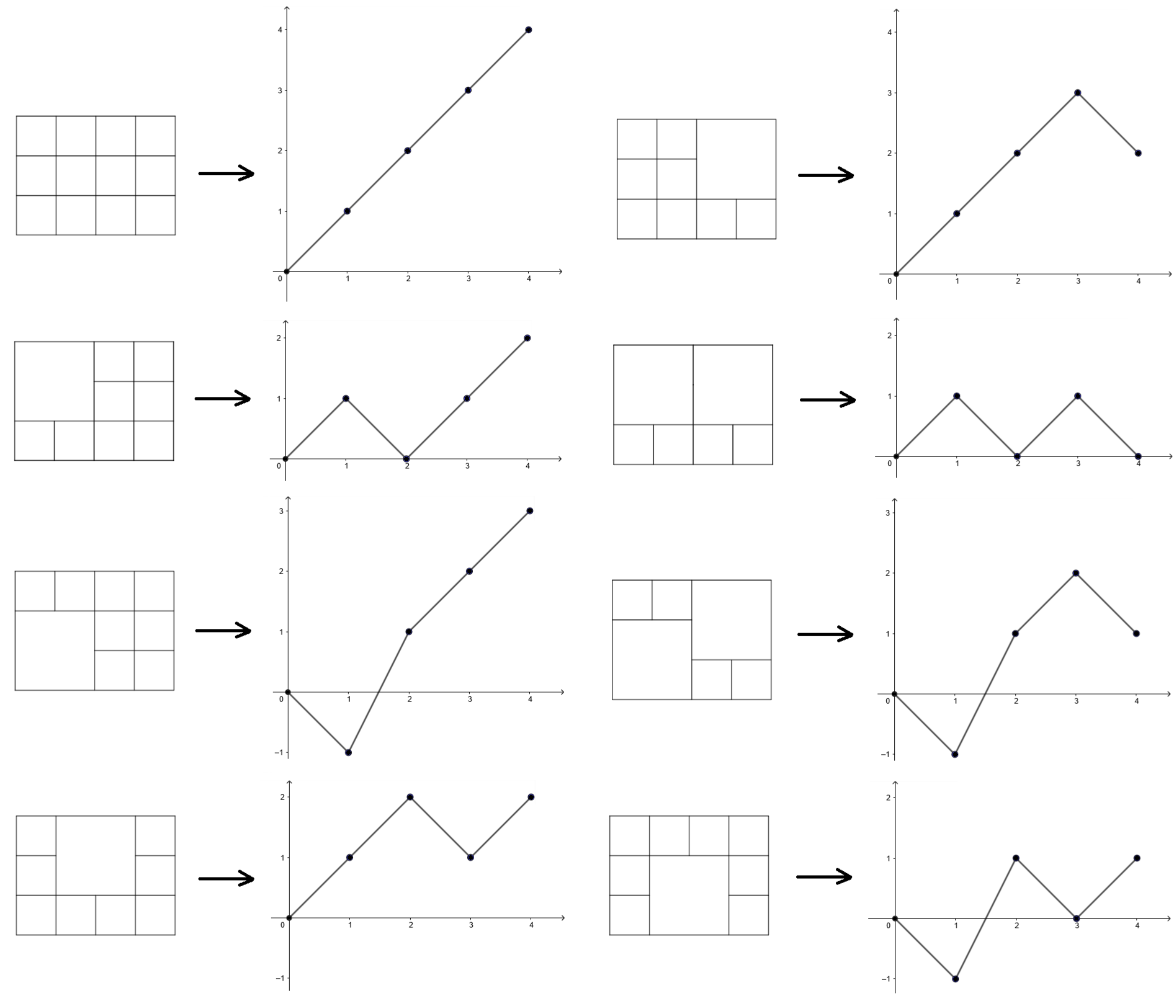

Figure 13: Bijection for $n = 4$, part 1/2

Image 13 shows the bijection as far as it already goes, using the ideas for normal and upper squares. And only three rectangles are left, all of them with a lower 2-square at the last position. We have to think about which of them gets which left path. Since one of the rectangles starts with two columns with normal squares and we have one path left that starts with $X_1 = 1$ and $X_2 = 2$ (it ends with $X_3 = 1$ and $X_4 = 0$), they can be connected. The next rectangle consists of an upper and a lower square in this order. The perfect path it can be mapped onto would be $X_1 = 1, X_2 = 0, X_3 = -1, X_4 = 1$, because it executes both 2-squares as if each of them was the only one. Then the only rectangle left for the path $X_1 = -1, X_2 = -3, X_3 = -1, X_4 = 1$ is the one with two lower 2-squares.

There are now four different cases at the last two turns for a lower square at the last possible position of the rectangle we had for $n \leq 4$ (which is relatively small), so this has to be handled, too. Until now, we just gave these cases to the paths that "were left".

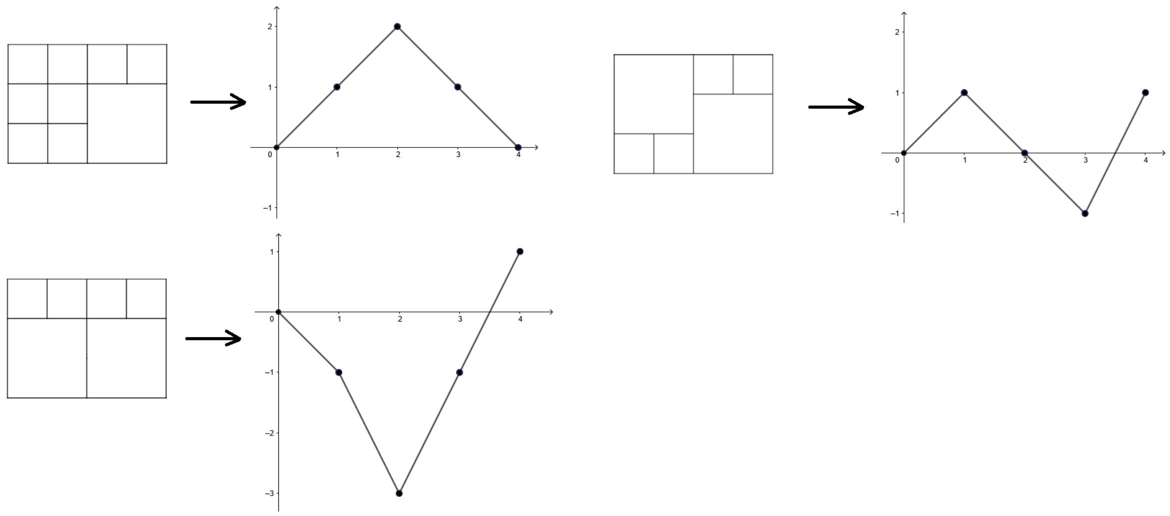

Figure 14: Bijection for $n = 4$, part 2/2

But based on what was done until now, we can now attempt to form a bijection covering all $n \in \mathbb{N}$ by trying a recursive approach. Given a tiling, we pretty much try to execute all of its parts one after another. Let $Y_1, Y_2, \ldots, Y_i, i < n$ be the path that is produced from the whole tiling without the last column, and if it doesn't has normal squares, also without the second-to-last column ($i < n - 1$). In most cases it can just be taken over to the real path, $X_k = Y_k$. For a column of normal squares, we go 1 up, $X_n = X_{n-1} + 1$. For an upper 2-square, we do a useless turn, $X_n = X_{n-1} - 1 = X_{n-2}$.

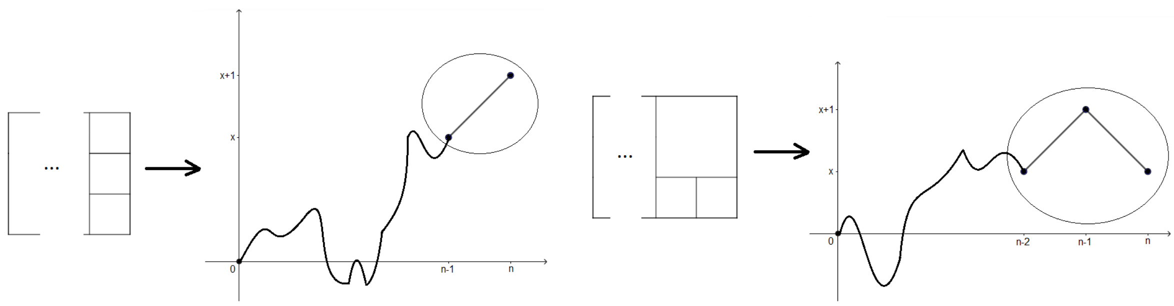

Figure 15: Recursive continuation for normal squares and upper 2-square

But we still don't really know what to do for a lower 2-square. Let's look at its cases one by one. One of the paths was $X_1 = 1, X_2 = 2, X_3 = 1, X_4 = 0$. In this case, the lower square represents two steps down, each with length 1. Taking a closer look, we never covered this with the other parts, because the only paths that end with one step down have a second-to-last step up, for an upper square. So, what needs to be done first, is to try to do two steps down, $X_n = X_{n-1} - 1 = X_{n-2} - 2$. This works when $Y_{n-2} \geq 2$. For $Y_{n-2} = 1$ and $Y_{n-2} = 0$, we need to find something different.

An example path for $Y_{n-2} = 0$ is in $n = 2$ the one for the rectangle only consisting of a lower square. It was $X_1 = -1, X_2 = 1$. This can be transferred to the general case: $X_{n-1} = -1$ and $X_n = 1$. Another case we never covered, because the last step always started in the non-negative range.

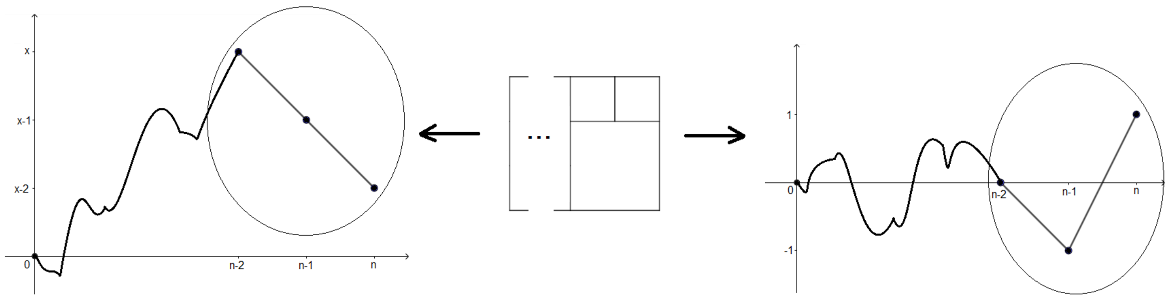

Figure 16: Recursive continuation for lower 2-square, part 1/3

The only case left is $Y_{n-2} = 1$. So far, we've seen two example paths for this case, $X^{(1)}$ and $X^{(2)}$, with $X_1^{(1)} = -1, X_2^{(1)} = 1, X_3^{(1)} = 0$ and $X_1^{(2)} = -1, X_2^{(2)} = -3, X_3^{(2)} = -1, X_4^{(2)} = 1$. Note that both of them have different length, $n^{(1)} = 3$ and $n^{(2)} = 4$. At both of them, the lower square stands for different steps, and the starting points are different, too: We have $Y_{n^{(1)}-2} = 1 = Y_{n^{(2)}-2}$, but we have $X_{n^{(1)}-2}^{(1)} = -1 \neq -3 = X_{n^{(2)}-2}^{(2)}$. So even before this, the executed paths with length $n - 2$ must be modified somehow. We need to find a modification that leads an original path that ends in 1 to -1 in certain cases and to -3 in other cases.

For this purpose, another component is defined. Let $p(Y)$ be the last time before $n-2$ where the path was not positive, $p(Y) := \max\{k \in \{0, 1, \ldots, n-3\} : Y_k \leq 0\}$. Note that $p(Y) \geq 0$ because $Y_0 = 0$. The modification looks like the following: It reverses the executed path $(Y_0, Y_1, \ldots, Y_{n-2})$ after $p(Y)$ and takes over everything before, $X_k = Y_k$ for $k \leq p(Y)$. That means, after time $p(Y)$, it goes down when the Y path goes up and vice versa. It always starts with a step down after $p(Y)$, because in the original path, we must have gone up to get from the non-positive to the positive, and it definitely holds true that $Y_{p(Y)+1} = 1$. In both $X^{(1)}$ and $X^{(2)}$, exactly one step is reversed, the first one in $X^{(1)}$ and the second one in $X^{(2)}$. Now, if a path switches from non-positive to positive, it can do so in two different ways.

The first possibility is $Y_{p(Y)} = 0$, as in the first example path. Then the modification leads to $X_k = 1 - 2Y_k$ for $n-2 \geq k > p(Y)$, so it begins with $X_{p(Y)+1} = 1 - 2Y_{p(Y)+1} = -1$ and ends with $X_{n-2} = 1 - 2Y_{n-2} = -1$. Of course, the modified path mostly goes 2 up and down instead of 1. To conclude the path, set $X_{n-1} = 1$ and $X_n = 0$, as in the example path.

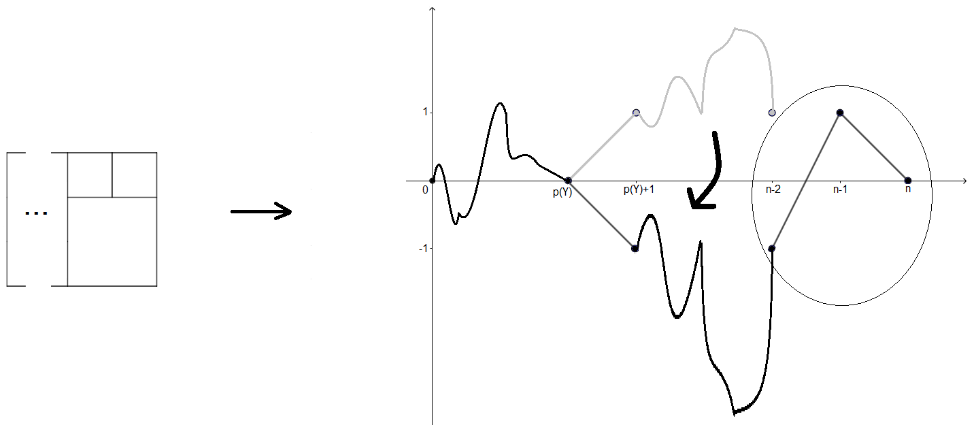

Figure 17: Recursive continuation for lower 2-square, part 2/3

The second possibility is $Y_{p(Y)} = -1$, as in the second example path. Then the modification leads to $X_k = -2Y_k - 1$ for $n-2 \geq k > p(Y)$, so it begins with $X_{p(Y)+1} = -2Y_{p(Y)+1} - 1 = -3$ and ends with $X_{n-2} = -2Y_{n-2} - 1 = -3$. This time, there is even only one entirely possible ending for this path: $X_{n-1} = -1$ and $X_n = 1$.

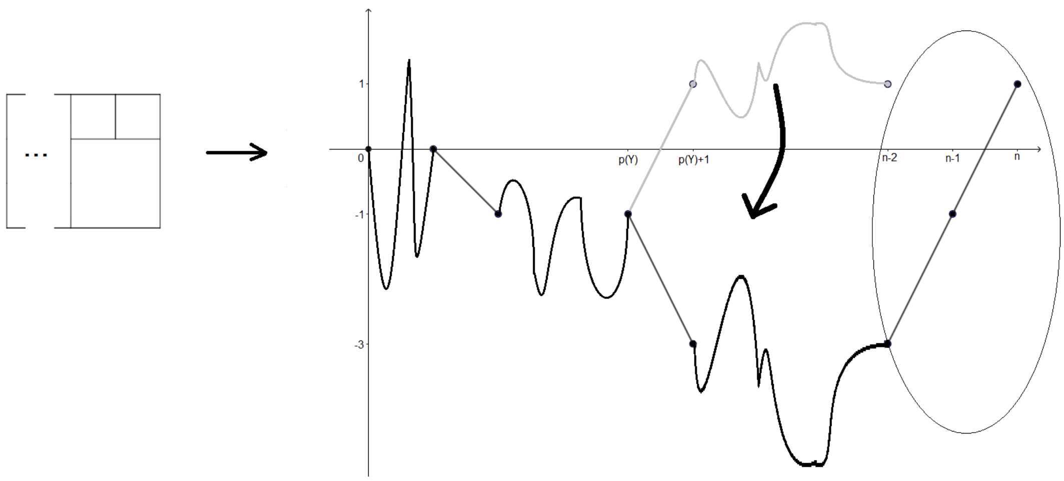

Figure 18: Recursive continuation for lower 2-square, part 3/3

Both of those were never covered by other cases because X_{n-2} could only be negative if the last row has normal squares, and in that case, in the last step we neither go down as we would do if $Y_{p(Y)} = 0$ nor do we go 2 up as in $Y_{p(Y)} = -1$.

So in result, we get a pretty easy mapping of the tilings onto the 1-2-paths, even if it is recursive. We showed that it is injective for n if it is injective for $n-1$ and $n-2$. Because a bijection for $n = 0$ and $n = 1$ already exists, this mapping is definitely injective for all $n \in \mathbb{N}$ via induction.

To summarize, the complete mapping will be written down next. Given is an $n \times 3-$rectangle tiled into squares of length 1 and 2. Set $X_0 = 0$. Let $Y_0, Y_1, \ldots, Y_j$ be the path that would be obtained if executed from the rectangle that misses the last column if it has normal squares (in this case $j = n-1$) or that misses the last two columns otherwise ($j = n-2$). Let further $p(Y)$ be the last time before $n-2$ where the path was not positive, $p(Y) := \max\{i \in \{0, 1, \ldots, n-3\} : Y_i \leq 0\}$. At first, the new path before j has to be created. Let $k \in \{0, 1, \ldots, j\}$.

$$X_k = \begin{cases} 1 - 2Y_k & \text{rectangle ends with lower 2-square, } k > p(Y), Y_{n-2} = 1, Y_{p(Y)} = 0 \\ -2Y_k - 1 & \text{rectangle ends with lower 2-square, } k > p(Y), Y_{n-2} = 1, Y_{p(Y)} = -1 \\ Y_k & \text{otherwise} \end{cases}$$

Then the path has to be finished.

$$X_{n-1} = \begin{cases} X_{n-2} + 1 & \text{rectangle ends with upper 2-square} \\ Y_{n-1} & \text{rectangle ends with normal squares (already set)} \\ X_{n-2} - 1 & \text{rectangle ends with lower 2-square, } Y_{n-2} > 1 \\ 1 & \text{rectangle ends with lower 2-square, } Y_{n-2} = 1, Y_{p(Y)} = 0 \\ -1 & \text{otherwise} \end{cases}$$

$$X_n = \begin{cases} X_{n-2} & \text{rectangle ends with upper 2-square} \\ X_{n-1} + 1 & \text{rectangle ends with normal squares} \\ X_{n-2} - 2 & \text{rectangle ends with lower 2-square, } Y_{n-2} > 1 \\ 0 & \text{rectangle ends with lower 2-square, } Y_{n-2} = 1, Y_{p(Y)} = 0 \\ 1 & \text{otherwise} \end{cases}$$

So the proof for $J_n \geq T_n$ is complete. For the other direction, we have a 1-2-path $X_0, X_1, \ldots, X_n$ with $X_n \geq 0$, and try to find a tiling that would be mapped onto this path. But given what we did until now, this isn't hard anymore, as we can try to reverse the steps and build the rectangle from the right.

$$\text{It ends with...}\begin{cases}\text{...normal squares} & X_n = X_{n-1} + 1.\\ \text{...an upper 2-square} & X_n = X_{n-2} = X_{n-1} - 1.\\ \text{...a lower 2-square} & \text{otherwise.}\end{cases}$$

The rest will be constructed from the rest of the path. Of course, the modification that was possibly done in the mapping also has to be reversed. Let $Y_0, Y_1, \dots, Y_j$ be the path that will constructed for the rest of the rectangle. So $j = n-1$ if $X_n = X_{n-1}+1$ and $j = n-2$ otherwise. But we will have to define $z(X) := \max\{k \in \{0, 1, \dots, n-3\} : X_k = 0\}$ and $o(X) := \max\{i \in \{0, 1, \dots, n-3\} : X_i = -1\}$, the last time points where the path passed 0 and -1, respectively. Also, $o(X)$ is bigger than $-\infty$ when needed.

$$Y_k = \begin{cases}\frac{-X_k-1}{2} & X_n = 1, X_{n-1} = -1, X_{n-2} = -3, k > o(X)\\ \frac{1-X_k}{2} & X_n = 0, X_{n-1} = 1, X_{n-2} = -1, k > z(X)\\ X_k & \text{otherwise}\end{cases}$$

When $n = 0$ is reached, the inverse mapping is done. We have to take a look if we really covered every case, which is done if the mapping of the inverse mapping is equal to the identical function. The cases are

- $X_n = X_{n-1} + 1$, covered with the last column of normal squares.
- $X_n = X_{n-1} + 2$, only possible with $X_n = 1$, this can lead to
  - $X_{n-2} = -3$, done with the last lower square and the re-reversing of everything after the last -1.
  - $X_{n-2} = 0$, what we have with the last lower square and not further changing the path.
- $X_n = X_{n-1} - 1$, with possible sub-cases
  - $X_{n-2} = X_{n-1} + 1$, also done with the last lower square and leaving the rest of the path as it is.
  - $X_{n-2} = X_{n-1} - 1$, the only case where the upper square is last.
  - $X_{n-2} = X_{n-1} - 2$, that implies $X_n = 0, X_{n-1} = 1, X_{n-2} = -1$, and the lower square comes last and the path is re-reversed after the last 0.

That makes indeed all cases.

The last thing to do is to show that two different paths really give two different tilings.

Assume that this is false, so there are two paths that give the same tiling with this method. Now assume that these paths are as short as possible and have length m. If the tilings end with normal or upper squares, these can just be left out and we made the paths that disproves the original assumption shorter, which is a contradiction to the shortness of the paths.

So the tilings end with a lower square. After eventually modifying the rest of the paths, they have to be equal, because it would again be a contradiction to the shortness of the paths otherwise. If $X_m = X_{m-1} - 1 = X_{m-2} - 2$, then $Y_{m-2} > 2$. If $X_m = 1, X_{m-1} = -1$ and $X_{m-2} = 0$, then obviously $Y_{m-2} = 0$. In the two other cases, the modification always gives $Y_{m-2} = 1$. So the only case where two of the next paths can be the same is if one path has $X_m = 0, X_{m-1} = 1, X_{m-2} = -1$ and the other one has $X_m = 1, X_{m-1} = -1, X_{m-2} = -3$, so that the modifications are the exact same. But the modification of the first one has a 0 as last non-positive point and the modification of the second one a -1, so that's impossible. So the fake assumption must be wrong and this proves that two different paths give two different tilings.

That also makes the inverse mapping injective for every $n \in \mathbb{N}$, and therefore, the mapping is surjective. We have shown $J_n \leq T_n$, it follows $J_n = T_n$, and we have the bijection between the tilings and the 1-2-paths. □

Having that, there are only few steps to get a limit like in the 1-3-random walks.

$$\lim_{n\to\infty} P\left(X_n \geq 0\right) = \lim_{n\to\infty} \frac{T_n}{2^n} = \lim_{n\to\infty} \frac{2^{n+1} + (-1)^n}{3 \cdot 2^n} = \frac{2}{3} + \lim_{n\to\infty} \frac{(-1)^n}{3 \cdot 2^n} = \frac{2}{3}$$

2.2.2 Skipping 0 in 1-2-random walks

One pretty important thing that happens is when a random 1-2 path goes from -1 up to 1. At the beginning, before the path reaches negative range, X_n is even when n is even, and odd when n is odd. After jumping from -1 to 1, it changes to being the other way around. That is, until it happens again that the path goes 2 up from -1, however. In negative range, the path always has odd numbers at any time point. But when does a path go from -1 to 1? The thing is that not every time point where something like that can happen is equivalent to all the others.

Case 1: The path skips 0 for the first time.

In this case, the time n where $X_{n-1} = -1$ and $X_n = 1$ is even, $n = 2m$ with $m \in \mathbb{N}$.

There is one important time point before, the one where the path moves from 0 to -1. Let $l \in \mathbb{N}_0$ with $X_{2l} = 0$ and $X_{2l+1} = -1$, it might be as small as 0 or as big as $m-1$. After time $2l+1$, the path cannot go higher than -1 before time $2m$, because in that case, $2m$ would not be the first time 0 is skipped.

How many paths can be constructed that way? Before time $2l$, the path does not move below 0. The number of subpaths with this property and length $2l$ is already known, as we have $\frac{1}{l+1} \cdot \binom{2l}{l} = C_l$. After that, the path goes down one step with length 1. The next part that is not fixed are the steps $2l+1$ to $2m-1$. The only thing that is known there is that this subpath doesn't go above -1, but it starts and ends at this point. But such a path can also be achieved by modifying a path of length $2m-2l-2$ starting and ending with 0 and never being in negative range. To do that, this path has to be reflected (reversing the steps), shifted by -1, and of course stretched by factor 2 due to moving in negative range. The number of subpaths there is C_{m-l-1}.

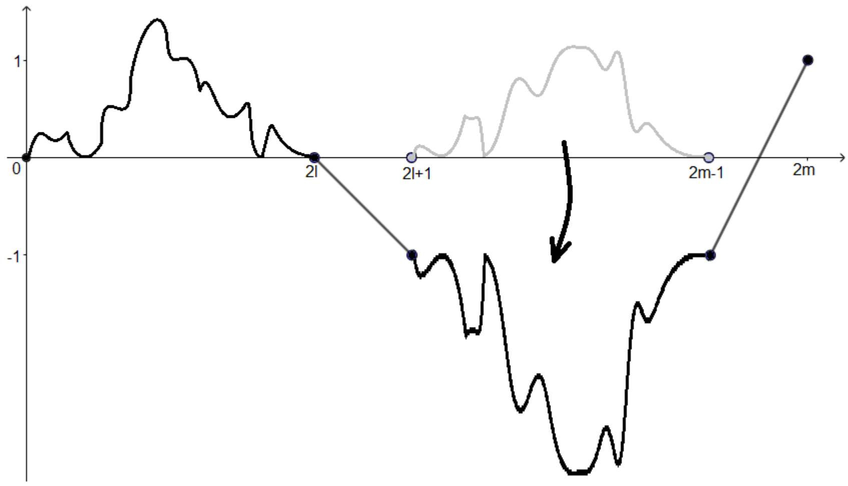

Figure 19: First crossing 0 time

The total number of paths is therefore the product of both of these numbers of subpaths, added for all possible numbers of l. With Lemma 1.4, said number simplifies to $\sum_{l=0}^{m-1} C_l C_{m-l-1} = C_m$. This is the number of paths with length $2m$, starting and ending at 0 and not going into negative range. Could there also be a direct bijection instead of constructing the paths like that?

Let $Y_0, Y_1, \ldots, Y_2m$ be such a classic random path, with $l \in \mathbb{N}_0$ such that $Y_{2l} = 0$ and $Y_k > 0$ for any $2l < k < 2m$, so $2l$ is the last time point where the path touches 0 before $2m$. We will construct a 1-2 random path with the properties we want, $X_0, X_1, \ldots, X_2m$. Set $X_i = Y_i$ for $i \leq 2l$ and $X_k = 1 - 2Y_k$ for $2l < k \leq 2m$. Then the new path is at

-1 after steps $2l+1$ and $2m-1$ and at 1 after the last step. Basically, the method is the same as the one used for the tiling bijection when there is an upper 2-square at the last two columns: The path is the same until the last time 0 is reached before the end, and after that, every step is just reversed. And it also is a bijection because this mapping can also go in the other direction with $Y_k = \frac{X_k - 1}{2}$ for $2l < k \leq 2m$. It was the same reasoning in the upper 2-square case for the tilings, if two paths were different before the mapping, they also are after the mapping. Which concludes the bijection and confirms that the number of $1-2$ paths where $2m$ is the first time with a step from -1 to 1 being exactly C_m.

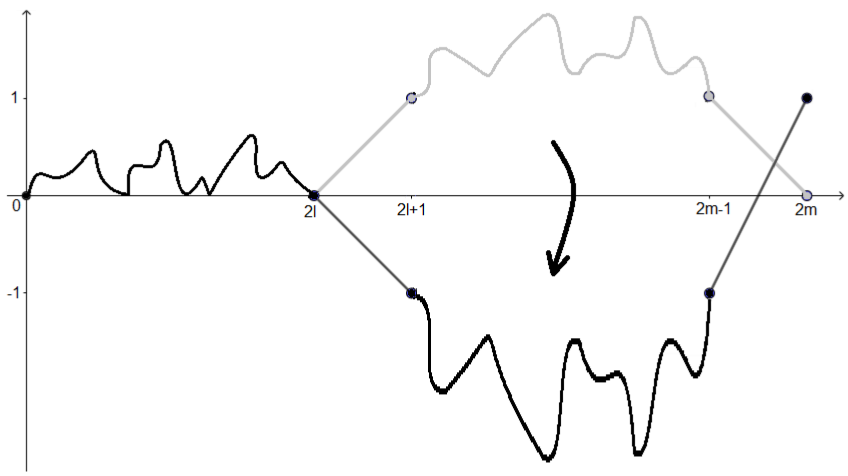

Figure 20: Converting random paths

Case 2: The path did have a step from -1 to 1 at least once before.

Let k be a time point where 0 was just crossed. That means that $X_k = 1$. The next time where that happens shall be $k+o$, so $X_{k+o-1} = -1$ and $X_{k+o} = 1$ in that case, and $\forall i \in \mathbb{N}, i < o : X_{k+i-1} = -1 \Rightarrow X_{k+i} = -1$, to make sure $k+o$ is the first time where 0 is crossed once more. Then o already has to be odd: An odd number of steps is needed to get from 1 to 0 the first time again, and from there, only an even number of steps gets us to skip 0 that way again. So we will investigate the number of paths for o being $2m+1$ with $m \in \mathbb{N}$ now. Also, $o > 2$ is obvious as there is definitely one step needed to get from 1 to 0 and one from 0 to -1, additional to the one from -1 to 1 at time $k+o$.

To build the paths, the strategy is again to use the classic Random paths ending with 0 and being non-negative everywhere. Above, one step from 1 to 0 is mentioned. The first one shall be at time $k+2l+1$, that means $X_n > 0$ for $n \leq k+2l$ and $X_{k+2l+1} = 0$. Then, everything from X_k to X_{k+2l} is a classic Random path ending with 0, just shifted by 1. That means, there are C_l possibilities so far. Then, after step $k+2l+1$, we can just continue at Case 1. The resulting second part has length $2m-2l$, thus having C_{m-l} possibilities for this subpath. Again, these numbers have to be multiplicated, and the products have to be added for every possible l for the chosen m.

What is the difference to Case 1? Well, this time, $m = l$ is forbidden because the second part cannot be a result from the above bijection of the empty path, as there are two necessary steps from 0 to -1 and from -1 to 1 here. That means that it has to be $l < m$. The sum over all possible l is a bit different, but with Lemma 1.4 it is still easy to solve, $\sum_{l=0}^{m-1} C_l C_{m-l} = \sum_{l=0}^{m} C_l C_{m-l} - C_m C_0 = C_{m+1} - C_m$.

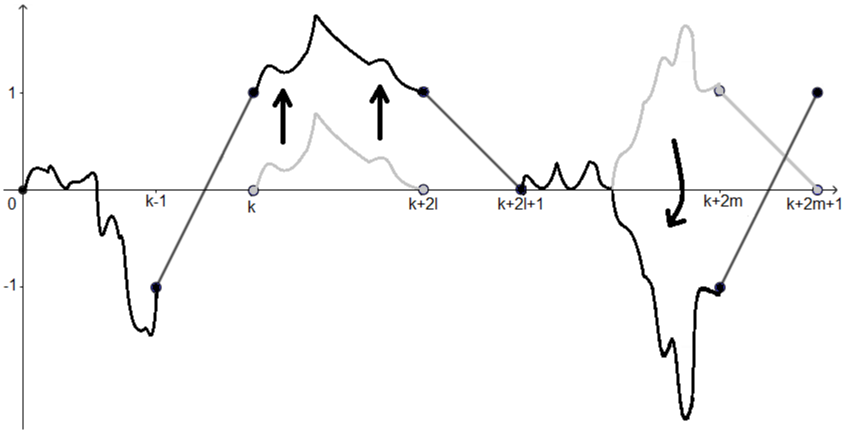

Figure 21: Crossing 0 after the first time doing so

Again, this can be mapped onto the non-negative classic Random paths with length $2m$ ending with 0. The question is if there are exactly C_m of such paths with certain common features that can be ignored for this mapping, to get a bijection again. And there is. As already known, there are $2C_m$ paths with length $2m$ where 0 is first reached again after exactly $2m$ steps ($m > 0$ is important here), that means C_m of them are in non-negative range. Throwing them out of the other pool of paths, there are exactly the paths left that reach 0 at least once before time $2m$. The subpaths of the 1-2 random walk with exactly $2m+1$ steps between two times crossing 0 have therefore a bijection to the classic Random paths with $2m$ steps, $Y_0, \ldots, Y_{2m}$, fulfilling $Y_{2m} = 0$, $Y_i > 0$ for $0 \leq i \leq 2m$ and $\exists l \in \mathbb{N}, l < m : Y_{2l} = 0$. If the smallest possible l is chosen, we have $X_{k+i} = Y_i + 1$ for $i \leq l$ and $X_{k+2l+1}, \ldots, X_{k+2m+1}$ is constructed from the subpath $Y_{2l}, \ldots, Y_{2m}$ exactly as in case 1. The first and last time where the classic path reaches 0 between time 0 and $2m$ are important, and both of them can actually be the same when the 1-2 path immediately goes down to -1 after reaching 0.

3 Convergence of the 1-2-random walk

In the classic random walk, there was the Brownian motion as a limit. It would be nice to have a similar result for the 1-2-random walk. An intuitive idea would be for example to have a Brownian motion above 0 and something like a Brownian motion with twofold rise or fall below 0. The main result for the classic random walk was Donsker's Theorem. This time, there are some more preparations needed.

3.1 Basics for stochastic integration

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ be a filtered probability space. Let $(B_i)_{t\geq 0}$ be an adapted Brownian motion. In this case, we will set $(\mathcal{F}_t)_{t\geq 0}$ as the *augmented Brownian filtration*, which is kind of the filtration generated by the Brownian motion and the P-nullsets to prevent problems regarding random variables that are equal with probability 1. Every equality is also in the sense that there might be inequalities, but only as a nullset.

The thing that we need is a *stochastic integral.* In this case, with respect to a Brownian motion. The easy case is a stochastic process (X_i) with $0 = t_0 < t_1 < \cdots < t_n$ and random variables $A_1, \ldots, A_n$ where A_i is F_{t_i}-measurable. The mentioned Process is then defined by $X_t = \sum_{k=0}^{n-1} A_i \mathbb{1}_{[t_i, t_{i+1}[}(t)$ and is called *Elementary Process.* And the mentioned integral is then defined by $I_t(X) = \sum_{k=0}^{n-1} A_i \left(B_{t\wedge t_{k+1}} - B_{t\wedge t_k}\right)$. The integral process is denoted as $I(X) = (I_t(X))_{t\geq 0}$ It can be imagined like calculating the gain from an amount A_i of shares which is as much worth as the values of a Brownian motion until time t.

We also need another form of measurablility. A stochastic process $(X_i)_{i\geq 0}$ is *progressively measurable* with respect to $(\mathcal{F}_t)_{t\geq 0}$ when it holds true that $\forall t \geq 0 : f : [0,t] \times \Omega \to \mathbb{R}^d, f(s,\omega) = X_s(\omega)$ is $\mathcal{B}([0,t]) \times \mathcal{F}_t$-measurable, in this case with $d = 1$, and $L^2(B)$ is the set of progressively measurable processes that also fulfill $\mathbb{E}\int_0^\infty X_t^2 dt < \infty$.

An adapted stochastic process $(X_i)_{i\geq 0}$ is called $(\mathcal{F}_t)$-*local martingale* when there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ that fulfills the following two conditions:

- The sequence τ_n is non-decreasing and $lim_{n\to\infty}\tau_n = \infty$ almost surely.
- The stopped process X_{τ_n} is a Martingale.

In that case, (τ_n) is called its *localizing sequence.*

We only consider a subset of the local martingales. Let $\mathcal{L}^2_{loc}(B)$ be the set of all $(\mathcal{F}_t)_{t\geq 0}$-progressively measurable stochastic processes $(X_i)_{i\geq 0}$ which have a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ that fulfills $\forall n \in \mathbb{N} : \mathbb{E}\int_0^{\tau_n} X_s^2 ds < \infty$. Obviously, $L^2(B) \subseteq \mathcal{L}^2_{loc}(B)$.

After that, let's continue with the stochastic integral. The discrete-like case with an elementary process was already considered. With non-elementary processes, it would be like that we want to decide really fast and often if it's better to buy or sell some of the shares. Let $X \in L^2(B)$. You can prove that there is a sequence $(X^1), (X^2), \ldots$ of elementary processes that fulfills $lim_{n\to\infty}\mathbb{E}\int_0^\infty (X_s^n - X_s)^2 = 0$. This also implies $lim_{m,n\to\infty}\mathbb{E}\int_0^\infty (X_s^n - X_s^m)^2 = 0$. Because the stochastic integral is linear for elementary processes, the sequence transforms into a Cauchy sequence in that sense. That means there has to exist a limit $lim_{n\to\infty} I(X^n) =: I(X)$, which is also well-defined, hence it is the same for every such sequence. This can even be further generalized to $X \in L^2_{loc}(B)$. A stochastic integral is denoted by $\int_0^t X_s dB_s$.

The stochastic integrals can, apart from the Brownian motion, also be applied for every other stochastic process. But the Brownian motion has some neat properties we want to use. For example the *Ito formula*: For any real-valued function f that is differentiable twice on $\mathbb{R}$, it holds true that

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds.$$

But the Ito Formula can also be used for even more general processes. An *Ito process* is a stochastic process $(X_i)_{i\geq 0}$ that has the form $X_t = X_0 + \int_0^t a_s ds + \int_0^t b_s dB_s$, where $X_0 \in \mathbb{R}$ and a, b are progressively measurable processes that fulfill $\int_0^t |a_s| + b_s^2 ds < \infty$ for any $t \geq 0$ almost surely, where a is called *drift rate* and b the *diffusion rate*. Finally, the solvability of a *stochastical differential equation*

$$dX_t = \mu(t, X_t)\, dt + \nu(t, X_t)\, dB_t \tag{1}$$

can be examined. The first random variable is constant, $X_0 \in \mathbb{R}$.

3.2 Convergence to a modified Brownian motion

How can all of this help us to find a limit for the 1-2-random walk? We want to have a factor 2 in the negative part of the Brownian motion. Setting up $dM_t = \mu(t, M_t)\, dt + \nu(t, M_t)\, dB_t$, we now have to set M_0, μ and ν in appropriate ways. For example, $\mu = 0$ and $\nu = 1$ results in $M_t = B_t$ when $M_0 = 0$, so that is just the standard Brownian motion. The factor 2 can just be implemented by setting $\mu = 0$ and

$$\nu(m) := \nu(t, m) = \begin{cases} 1 & m \geq 0 \\ 2 & \text{otherwise.} \end{cases}$$

Observe that the coefficients don't depend on the time t anymore. From an intuitive point this is also obvious, because in the 1-2-random walk the step length also just depends on the position and not on the time. In such a case, the existence of a *weak solution* (other solutions have the same distribution) can be shown under certain conditions.

Proposition 3.1 (existence of a weak solution)**.** *The Stochastic Differential Equation (1) has a unique weak solution if the Engelbert-Schmidt-conditions are fulfilled: For any $x \in \mathbb{R}$, it holds true that $\nu(x) \neq 0$, and $\frac{1}{\nu^2}$ is locally integrable.*

The Engelbert-Schmidt-conditions are fulfilled here, as ν is nowhere 0, and $\frac{1}{\nu^2} \leq 1$ which when integrated over a finite interval gives back the length of this interval. And all of that even works with any $M_0 \in \mathbb{R}$. It will be important later that the choice is not limited to 0 here.

But how to prove a convergence? The result for the classic random walk was Theorem 1.7. It would be nice to have something similar here, too. And that's where we run into a big problem right at the beginning: The convergence could only happen when the expected value and variance of the step sizes were the same as the one of the Brownian motion, $\mathbb{E}(Z_i) = 0$ and $Var(Z_i) = 1$. Even worse, these steps are not even iid anymore. But we can still try to do everything from back then step by step. For $t, n \geq 0$, let

$$M_{t,n} := \sum_{i=1}^{\lfloor nt \rfloor} \frac{Z_i}{\sqrt{n}}.$$

The next step was the stopping time. In this case, a bit more complicated than before. Let $\tau_0 := 0$. For any $n \in \mathbb{N}$, set the next stopping time as the following:

$$\tau_n = \begin{cases} \min\left\{x : x > \tau_{n-1}, M_x \notin]M_{\tau_{n-1}} - 1; M_{\tau_{n-1}} - 1[\right\} & M_{\tau_{n-1}} \geq 0 \\ \min\left\{x : x > \tau_{n-1}, M_x \notin]M_{\tau_{n-1}} - 2; M_{\tau_{n-1}} + 2[\right\} & \text{otherwise.} \end{cases}$$

Like in Proposition 1.7, the idea is to embed a 1-2-path by setting $X_i := M_{\tau_i}$. This time however, not even the $\tau_i - \tau_{i-1}$ are iid. The main idea is to calculate the expected value for such a stopping time, but it depends on where it starts. To do this, assume first that $M_0^m = m \in \mathbb{R}$ and take the modified Brownian motion from there as $(M_t^m)_{t \geq 0}$.

For this purpose it is needed that there is a unique weak solution for this case too. Set $m_1, m_2 \in \mathbb{R}$ with $m_1 < m < m_2$ and $\tau_{m,m_1,m_2} := \min\{x, M_x^m \notin]m_1; m_2[\}$.

The expected value can be calculated with help of $q_m(x) := \int_m^x \int_m^y \frac{2}{\mu^2(z)} dz dy$. It can be shown with the Ito formula that $q_m(M_t^m) - t$ is a local martingale, which then can be used to calculate $\mathbb{E}\tau_{m,m_1,m_2}$.

Proposition 3.2 (expected stopping time)**.** *The mean value of this stopping time is* $\mathbb{E}\tau_{m,m_1,m_2} = \mathbb{E}q_m\left(M_{\tau_{m,m_1,m_2}}\right)$.

That implies $\tau_{m,m_1,m_2} < \infty$ almost surely. With $\mathbb{E}M_{\tau_{m,m_1,m_2}} = m$, we immediately get $P\left(M_{\tau_{m,m_1,m_2}} = m_1\right) = 1 - P\left(M_{\tau_{m,m_1,m_2}}\right) = \frac{m_2 - m}{m_2 - m_1}$. The expected stopping time then can be transformed to

$$\mathbb{E}\tau_{m,m_1,m_2} = \frac{m_2 - m}{m_2 - m_1} q_m(m_1) + \frac{m - m_1}{m_2 - m_1} q_m(m_2).$$

In the modified Brownian motion, that means that the 1-2-random walk can actually be established by those stopping times.

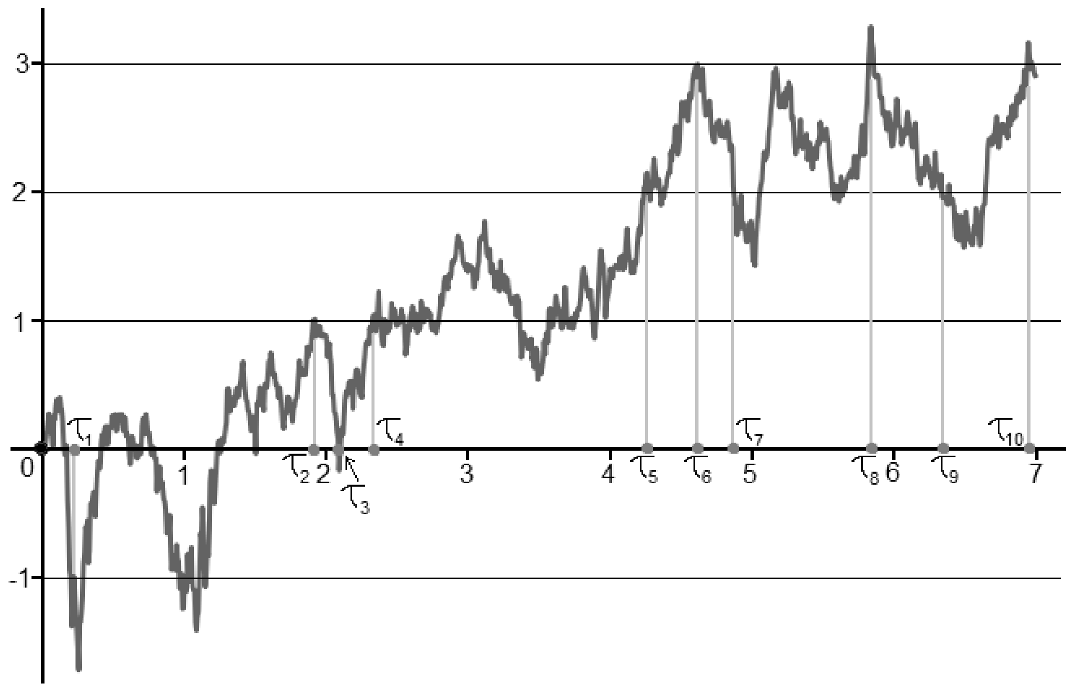

Figure 22: stopping times in a path of the modified Brownian motion

The parts below 0 indeed fall and rise much faster than the normal Brownian motion. Because of the factor 2, the path will in average only spend half of the time in negative range of the time it will be in non-negative range. For τ_2, not the value 0 will be taken at $t \approx 0.4$, but instead the 1 at 1.5 later. After τ_4, it could be like the Brownian motion because 0 isn't reached anymore, at least until $t = 7$.

For calculating the expected values for τ_{m,m_1,m_2} here, we definitely have $m \in \mathbb{Z}$ here with $m_1 + \nu(m) = m = m_2 - \nu(m)$. Because m has the same difference to m_1 and m_2, the two probabilities both simplify to $\frac{1}{2}$, and it is easy to calculate that $\mathbb{E}\tau_{m,m-\nu(m),m+\nu(m)} = \frac{1}{2}\left(q_m\left(m-\nu(m)\right) + q_m\left(m+\nu(m)\right)\right)$. There are four cases for m.

Case 1: $m > 0$. Then $m, m-\nu(m), m+\nu(m) \geq 0$, so the range covered completely has $\nu = 1$. We obtain

$$\begin{aligned}\mathbb{E}\tau_{m,m-1,m+1} &= \frac{1}{2}\left(q_m(m-1) + q_m(m+1)\right) = \frac{1}{2}\left(\int_m^{m-1}\int_m^y \frac{2}{1}dzdy + \int_m^{m+1}\int_m^y \frac{2}{1}dzdy\right) \\ &= \frac{1}{2}\left(\int_0^1\int_y^1 2dzdy + \int_0^1\int_0^y 2dzdy\right) = \frac{1}{2}\left(2\cdot\int_0^1 2ydy\right) = 1.\end{aligned}$$

Because the first double integral has switched borders both times, the factor -1 also is multiplied twice when switching them to the correct order. And it also makes sense that the average time here is the same as for a normal Brownian motion.

Case 2: $m < -1$. Then $m, m-\nu(m), m+\nu(m) \leq 0$, that means ν will be 2. Then the stopping time in average is

$$\begin{aligned}\mathbb{E}\tau_{m,m-2,m+2} &= \frac{1}{2}\left(q_m(m-2) + q_m(m+2)\right) = \frac{1}{2}\left(\int_m^{m-2}\int_m^y \frac{2}{4}dzdy + \int_m^{m+2}\int_m^y \frac{2}{4}dzdy\right) \\ &= \frac{1}{2}\left(\int_0^2\int_y^2 \frac{1}{2}dzdy + \int_0^2\int_{\frac{1}{2}}^y 2dzdy\right) = \frac{1}{2}\left(2\cdot\int_0^2 \frac{y}{2}dy\right) = 1.\end{aligned}$$

The factor 2 was chosen for ranges below 0, because the average time is still 1.

Case 3: $m = 0$. Then ν is 2 at the range below m and 1 above. The two double integrals are not equal anymore. The double integrals can partly be taken from the first two cases, and the result is

$$\begin{aligned}\mathbb{E}\tau_{0,-1,1} &= \frac{1}{2}\left(q_0(-1) + q_0(1)\right) = \frac{1}{2}\left(\int_0^{-1}\int_0^y \frac{2}{4}dzdy + \int_0^1\int_0^y \frac{2}{1}dzdy\right) \\ &= \frac{1}{2}\left(\int_{-1}^0 \frac{y}{2}dy + 1\right) = \frac{1}{2}\left(\frac{1}{4} + 1\right) = \frac{5}{8}.\end{aligned}$$

That is no average of 1 anymore. Which wasn't really realizable anyway, but the $\frac{3}{8}$ missing might cause trouble. The worst case, however, will follow now.

Case 4: $m = -1$. Then ν takes the value 2 below 0, so the range from 0 to 1 requires $\nu = 1$. One of the ranges is split up, which leads to splitting up integrals twice,

$$\begin{aligned}\mathbb{E}\tau_{-1,-3,1} &= \frac{1}{2}\left(q_{-1}(-3) + q_{-1}(1)\right) = \frac{1}{2}\left(\int_{-1}^{-3}\int_{-1}^{y}\frac{2}{4}dzdy + \int_{-1}^{1}\int_{-1}^{y}\frac{2}{\nu(z)^2}dzdy\right)\\ &= \frac{1}{2}\left(1 + \int_{-1}^{0}\int_{-1}^{y}\frac{2}{4}dzdy + \int_{0}^{1}\left(\int_{-1}^{0}\frac{2}{4}dz + \int_{0}^{y}\frac{2}{1}dz\right)dy\right)\\ &= \frac{1}{2}\left(1 + \frac{1}{4} + \int_{0}^{1}\frac{1}{2} + 2ydy\right) = \frac{1}{2}\left(\frac{5}{4} + \frac{3}{2}\right) = \frac{11}{8}.\end{aligned}$$

The missing $\frac{3}{8}$ from the third case are too much here. To conclude the four cases, only $m = 0$ and $m = -1$ can really mess things up, the other ones give an average time of 1. But even those two cases together at least have the right mean value of 1.

The turning points of the 1-2-random walk are always the values 0 and -1 anyway: Once in negative range, it can't be exited without going into positive range at least once because the step from -1 to 0 is impossible unlike its counterpart. It would be nice to have some kind of regularity for them. What is the number of paths ending in those numbers with certain numbers of steps? To find out, a Pascal-like triangle for 1-2 paths can be created.

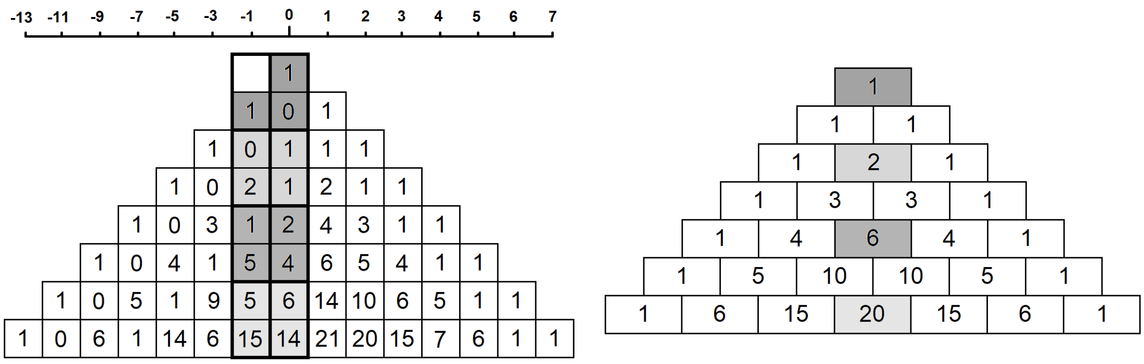

Figure 23: 1-2 triangle (left), Pascal triangle (right)

Looking closer into the triangle, the values 0 and -1 always seem to be apart by only 1. But more important, the sum of two numbers on top of each other in the 0 and -1 columns seems to be equal to the middle values in the Pascal triangle, marked by the domino pieces and the different colors.

Proposition 3.3 (1-2-paths ending on 0). *For any $l \in \mathbb{N}_0$, there are as many classic random walk paths with length $2l$ that end in 0 as 1-2-paths with length $2l$ or $2l+1$ that end in 0.*

Proof. We start by describing a mapping from the set of classic random walk paths with length $2l$ that end in 0 to the set of 1-2-paths with length $2l$ or $2l+1$ that end in 0 and then prove that it is a bijection. Let at first be $Y_0, Y_1, \ldots, Y_{2l}$ be a path of the classic random walk with $Y_{2l} = 0$. The plan is to get a 1-2-path $X_0, X_1, \ldots, X_{2l+a}$ with $X_{2l+a} = 0$ and $a \in \{0, 1\}$. A recursive approach will be used. Indeed, for $l = 0$, the bijection is obvious, because there are only the paths $Y_0 = 0$ and $X_0 = 0$.

Now assume $l > 0$, and all cases with smaller l already can be bijected in a suitable way. Set $m(Y) \in \mathbb{N}_0$ so that $m(Y) < l$ and $2m(Y)$ is the last time where the classic random walk had the value 0, that means $Y_{2m(Y)} = 0$ and $Y_k \neq 0$ for $2m(Y) < k < 2l$. In fact, $m(Y) = max\{i : Y_{2i} = 0, i < l\}$, and that value is always bounded because of $l > 0$ and $Y_0 = 0$. Any subpath $Y_{k_1}, Y_{k_1+1}, \ldots, Y_{k_2}$ with $Y_{k_1} = Y_{k_2} = 0$ and $Y_k \neq 0$ for any $k_1 < k < k_2$ shall be called *segment*. The last segment of Y begins at time $2m(Y)$ and ends at time $2l$.

Now consider the bijection from the same path, just without the last segment. Let $X'_0, X'_1, \ldots, X'_{2m(Y)+a'}$ be the 1-2-path that is mapped from $Y_0, Y_1, \ldots, Y_{2m(Y)}$, which can be done because of the recursive assumption. In this case, $a' \in \{0, 1\}$ is also set. In some cases, we need the second-to-last time where the path was 0, $m^{(2)}(Y) = \max\{l : Y_{2l} = 0, l < m(Y)\}$, and the first time after $2m^{(2)}(Y) + 1$ where 1 is crossed, $m^{(1)}(Y) = \min\left\{i : Y_{2i+1} = 1, i > m^{(2)}(Y)\right\}$. These last two values might be $-\infty$ or ∞, but this shall not matter for now. If $Y_{2m^{(2)}(Y)+1} = 1$, note that the time $2m^{(1)}(Y) + 1$ is the second time after $2m^{(2)}(Y)$ where 1 is reached.

The length of the new path only depends on a. For reasons of applicability, set $Y_k = 0$ for $k < 0$ for the calculation of a.

$$a = \begin{cases} a' & Y_{2l-1} = 1 \text{ or } \left(a' = 1, Y_{2m(Y)-1} = -1\right) \\ a' + (-1)^{a'} & \text{otherwise} \end{cases} \tag{2}$$

Now, the path itself will be constructed. For the most part, the old path should be taken over and be changed as little as possible, but there are cases where things have to be mixed up. Those will be explained later in the proof. We have $k \in \mathbb{N}$ and $k \leq 2l + a$. The case $k < 2m(Y) + a$ is considered first.

$$X_k = \begin{cases} X'_k & a' = 0 \text{ or } a = 1 \text{ or } k \leq 2m^{(2)}(Y) + 1 \\ 3 - 2Y_k & a' = 1, a = 0, 2m^{(2)}(Y) + 1 < k \leq 2m^{(1)}(Y) \\ Y_k & \text{otherwise} \end{cases} \tag{3}$$

For the last segment in $k \geq 2m(Y) + a$, use this equation:

$$X_k = \begin{cases} 0 & k = 2l + a \\ Y_{k-a} & Y_{k-a} \geq 0 \\ 1 + 2Y_k & a = 1, Y_k \leq 0, a' = 0 \\ 1 + 2Y_{k+1} & a = 0, Y_k < 0, m^{(1)}(Y) = \infty \\ 1 - Y_{k-a+1} & \text{otherwise} \end{cases} \tag{4}$$

We will break down when every case will be used next. The general strategy is, as already said, to execute the mapping until the last time before $2l$ where the classic path is 0. Obviously $X_0 = X_{2l+a} = 0$, because the 1-2-path has to end with 0, see the first case of equation (4). The last segment of this path is the subpath from time $2m(Y)$ to $2l$.

Case 1: The path has only one segment.

In this case, $m(Y) = 0$ and $a' = 0$. The path can be completely in positive or negative range.

Case 1.1: The path is in positive range, $Y_1 = 1$.

Then $a = a' = 0$ in the first case of equation (2). By the second case in equation (4) we have $X_k = Y_k$ for $0 \leq k < 2l$.

Figure 24: Bijection for case 1.1

Figure 24 shows that the 1-2-path is the same as the classic path in this case.

Case 1.2: The path is in negative range, $Y_1 = -1$.

Then $a = a' + 1 = 1$. Because of the third case in equation (4), for $0 < k \leq 2l$ we have $X_k = 1 + 2Y_k$.

Figure 25 shows that the 1-2-path still has the same step directions, but all of the steps have factor 2 now except for the first one. After $2l$ steps, the 1-2-path is at 1, and step $2l + 1$ is from 1 to 0.

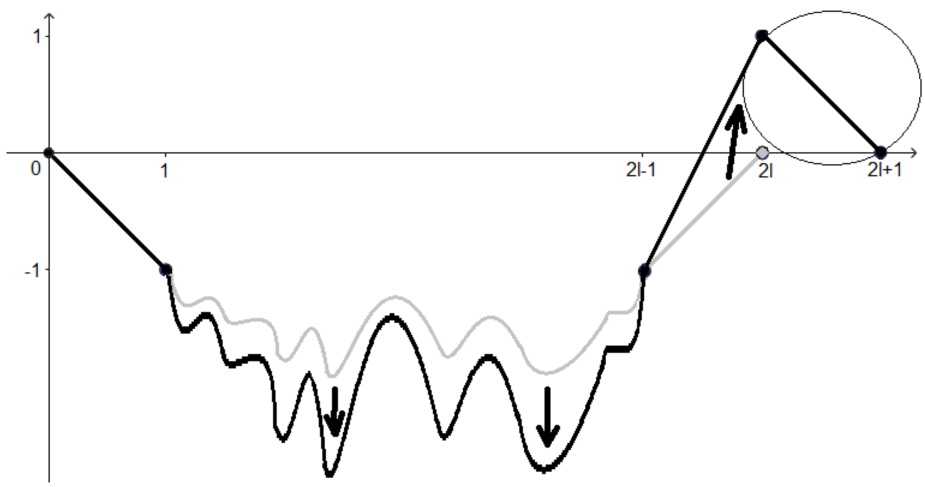

Figure 25: Bijection for case 1.2

Case 2: The path consists of multiple segments.

When $m(Y) > 0$, the idea of splitting up the last segment and taking the previous mapped 1-2-path and adding the last segment really comes into play. At first, consider the cases where the previous path can just be taken over.

Case 2.1: The last segment is in positive range, $Y_{2l-1} = 1$.

Then $a = a'$, and the cases depending on the time are

$$X_k = \begin{cases} X'_k & 1 \leq k < 2m(Y) + a \\ Y_{k-a} & 2m(Y) + a \leq k < 2l + a. \end{cases}$$

After executing the previous path in the first case of equation (3), the last segment is handled like in case 1.1 and just added to the rest of the path, which shows in the second case of equation (4).

Case 2.2: The last segment is in negative range, $Y_{2l-1} = -1$.

The last segment would use case 1.2 if it would just be added to the rest of the path. However, it produces an extra step. This is a problem when already $a' = 1$.

Case 2.2.1: $a' = 0$.

Then the extra step forces the second case in equation (2), therefore $a = 1$. The whole path is

$$X_k = \begin{cases} X'_k & 1 \leq k \leq 2m(Y) \\ 1 + 2Y_k & 2m(Y) < k \leq 2l. \end{cases}$$

This time, the last segment uses the third case of equation (4).

Case 2.2.2: $a' = 1$.

When the last segment can't be just added to the rest of the path because it would result in too many steps, it is important if the second-to-last segment is in positive or negative range. We already have $X'_{2m(Y)} = 1$ because of $X'_{2m(Y)+1} = 0$.

Case 2.2.2.1: The second-to-last segment is in negative range, $Y_{2m(Y)-1} = -1$.

The first case of equation (2) is used again, $a = 1$. The rest is

$$X_k = \begin{cases} X'_k & 1 \le k \le 2m(Y) \\ 1 - Y_k & 2m(Y) < k \le 2l. \end{cases}$$

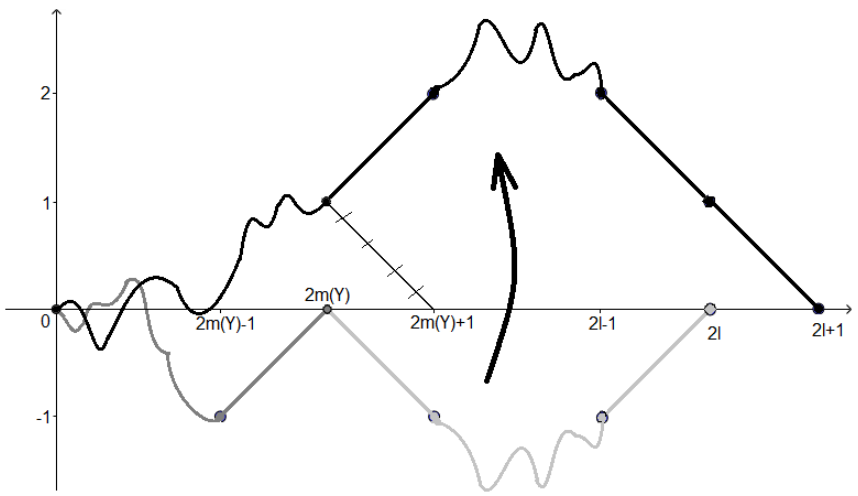

Figure 26: Handling the last segment in case 2.2.2.1

The previous path is taken over except for the last step, then the steps of the last segment are reversed as shown in figure 26. This is represented by the last case of equation (4). Instead of being mapped and added directly, the last segment was integrated into the last part of the 1-2-path. Notice that this is still a 1-2-path in the last segment because of $X_{2m(Y)} = X'_{2m(Y)} = 1 = 1 - Y_{2l} = X_{2l}$ and $X_{2m(Y)+1} = 2$, so the transitions actually work and the rest consists of steps with length 1 in positive range.

Case 2.2.2.2: The second-to-last segment is in positive range, $Y_{2m(Y)-1} = 1$.

This is the only time where a is smaller than a', because $a = 0$ with the second case in equation (2). Compared to just mapping and adding the last segment, we have two less steps. Those are deleted with the first and the last step of the second-to-last segment, and the rest of this segment is reflected and put together with the last segment.

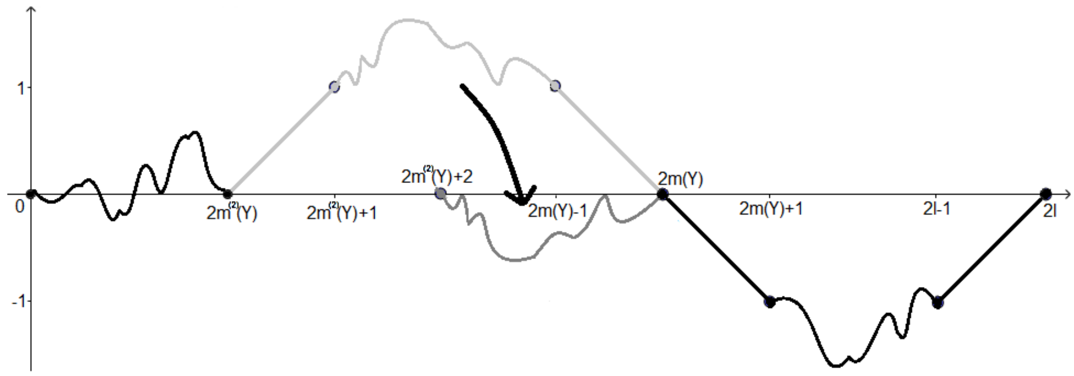

Figure 27: First step in case 2.2.2.2

In figure 27, the second-to-last segment is shortened by two steps and reversed. This new subpath will then be mapped into a 1-2-path by using the cases 1.2 and 2.2.2.1 and then be added to the previous 1-2-path, which is the mapping of Y except for its last two segments. The last case distinction is if the second-to-last segment only consists of two steps from 0 to 1 and back to 0, because in that case, nothing is reflected. For that reason, we consider if the path reaches 1 twice after $2m^{(2)}(Y)$ and before $2m(Y)$, which is shown by $m^{(1)}(Y)$ being bounded in this case because of $Y_{2m^{(2)}(Y)+1} = 1$.

Case 2.2.2.2.1: $m^{(1)}(Y) = \infty$**.**

Because $Y_{2m^{(2)}(Y)+2} = 0$ has to be true when the classic path doesn't cross 1 anymore after time $2m^{(2)}(Y) + 1$, this directly implies $m(Y) = m^{(2)}(Y) + 1$. Then

$$X_k = \begin{cases} X'_k & 1 \leq k < 2m(Y) \\ 1 + 2Y_{k+1} & 2m(Y) \leq k < 2l. \end{cases}$$

After the previously mapped path, there is only a single negative segment left to the mapping, which happens as in case 1.2. The fourth case of equation (4) is used there.

Case 2.2.2.2.2: $m^{(1)}(Y) < \infty$**.**

The cases for the 1-2-path are

$$X_k = \begin{cases} X'_k & 1 \leq k \leq 2m^{(2)}(Y) + 1 \\ 3 - 2Y_k & 2m^{(2)}(Y) + 2 \leq k \leq 2m^{(1)}(Y) \\ Y_k & 2m^{(1)}(Y) < k < 2m(Y) \\ 1 - Y_{k+1} & 2m(Y) \leq k \leq 2l. \end{cases}$$

At first, the three cases of equation (3) are used in this order (although there is the case of $m^{(1)}(Y) = m(Y) - 1$ where the third case isn't used anywhere), and then the path is concluded by the last case of equation (4) again, like in case 2.2.2.1.

We have $X_{2m^{(2)}(Y)+1} = X'_{2m^{(2)}(Y)+1} = 0$. Because of $m^{(1)}(Y) < \infty$, the classic path continues with $Y_{2m^{(2)}(Y)+2} = 2$ and therefore $X_{2m^{(2)}(Y)+2} = 3 - 2 \cdot 2 = -1$. Also, $Y_{2m^{(1)}(Y)} = 1$ implies $X_{2m^{(1)}(Y)} = 3 - 2 \cdot 1 = 1$, and $X_{2m(Y)-1} = Y_{2m(Y)-1} = 1$ is the assumption of case 2.2.2.2 already. All transitions are correct again, making the new path indeed a 1-2-path.

The next step is to show that the mapping is indeed a bijection.

Assume that this mapping is not injective for an $l \in \mathbb{N}$. We can assume $l > 0$ because there is only one classic path with length 0. Choose l with this property to be as small as possible. Then there are two classic paths $Y^{(1)}$ and $Y^{(2)}$ with length $2l$ that are mapped onto the same 1-2-path X with length $2l + a$ for an $a \in \{0, 1\}$. Let $z(X)$ be an indicator for the last time where X is equal to 0, $z(X) := \max\{k \in \{0, 1, \ldots, l-1\} : \exists d \in \{0, 1\} : X_{2k+d} = 0, \}$, and $a' \in \{0, 1\}$ with $X_{2z(X)+a'} = 0$. Note that $X_{2z(X)} \neq X_{2z(X)+1}$, so a' is unique. Let $p(X)$ be an indicator for the last time before $2l$ where X wasn't positive, $p(X) = \max\{t \in \{0, 1, \ldots, l-1\} : X_{2t+a} \leq 0\}$. Then $X_0 = 0$ implies $p(X) \geq 0$ and $z(X) \geq 0$. For $2p(X) + a < k < 2l + a$, we have $X_k > 0$, and either $X_{2p(X)+a} = 0$ or $X_{2p(X)+a} = -1$.

Case $X_{2p(X)+a} = 0$.

The subpath $X_{2p(X)+a}, X_{2p(X)+a+1}, \ldots, X_{2l}$ starts and ends with 0 (especially $p(X) = z(X)$), and is positive everywhere else, which is only possible if the last segment in the original path was the exact same, $Y_k^{(1)} = X_{k+a} = Y_k^{(2)}$ for $2p(X) + a \leq k \leq 2l + a$. Then $Y^{(1)}$ and $Y^{(2)}$ were also different until time $2p(X)$, but these shorter classic paths were also mapped onto the same 1-2-path $X_0, X_1, \ldots, X_{2p(X)+a}$. This is a contradiction to l being as small as possible.

Case $X_{2p(X)+a} = -1$.

The subpath $X_{2z(X)+a'}, X_{2z(X)+a'+1}, \ldots, X_{2l+a}$ starts and ends with 0, is negative from time $2z(X) + a + (-1)^a + 1$ to time $2p(X) + a$ and positive at any other place. Its length has to be odd, $a' = a + (-1)^a$. A side step will be taken: Let Y' be a classic path from time $2z(X) + a'$ to $2l + a$ and

$$Y'_k = \begin{cases} 0 & k \in \{2z(X) + a', 2l + a\} \\ \frac{X_k - 1}{2} & 2z(X) + a' < k \leq 2p(Y) + a \\ 1 - X_k & 2p(Y) + a < k < 2l + a. \end{cases}$$

Note that Y' has length $2l-2z(X)-2a'$. It has only negative segments and is mapped onto $X_{2z(X)+a'},\dots,X_{2l+a}$ using cases 1.2 and 2.2.2.1, because for $k\leq 2p(Y)+a$ we have $X_k=1+2Y'_k$, and $X_k=1-Y'_k$ applies everywhere else, as other cases don't exist. Let Y'' be another path on the same interval. If Y'' has any positive segments, it isn't mapped onto the same path as Y', as the mapped path of Y'' either starts with 1, or it is equal to 0 at least once (because case 2.2.2.2 has to be hit later). However, if Y'' only has negative segments and is different from Y' at a time k', Y' and Y'' are also mapped onto different paths because $1+2Y'_{k'}$ and $1-Y'_{k'}$ aren't equal to both $1+2Y''_{k'}$ and $1-Y''_{k'}$, as for every of those four possibilities, either $Y'_{k'}=Y''_{k'}$ applies or one of $Y'_{k'}$ and $Y''_{k'}$ is positive which contradicts the negative segments.

This implies that Y' is the only path that can be mapped onto $X_{2z(X)+a'},\dots,X_{2l+a}$. However, Y' can only be taken over directly for $a=1$. And because we need a subpath with length $2l-2z(X)$ to conclude the path, $Y_k^{(1)}=Y_k^{(2)}=Y'_k$ for $2z(X)+a+(-1)^a\leq k\leq 2l$ is the only possibility in that case.

For $a=0$ however, Y' has length $2l-2z(X)-2$, which is 2 steps short to what is needed. So we had to be in case 2.2.2.2 for the original mapping, the only place where two steps were deleted. That's why Y' is just a side step here: Every segment of Y' except for the last one had to be in positive range originally. If k is a time in said last segment, then $Y_k^{(1)}=Y_k^{(2)}=Y'_{k-1}$, as this segment has to be shifted by 1 towards $2l$ to work. For k being any other time greater than $2z(X)$, it has to be $Y_k^{(1)}=Y_k^{(2)}=1-Y'_{k-1}$.

So for every possible a, the two classic subpaths starting at time $2z(X)$ are fixed again, and the previous paths also have to be the same, or else the minimality of l would be violated. But $Y^{(1)}=Y^{(2)}$ is also a contradiction.

Every case of assuming that the mapping is not injective leads to a contradiction. This implies that the mapping is injective, and therefore reversible. The inverse mapping will follow next, also in a recursive approach. We will not show that it actually is the inversion, because that would be much more than needed.

If $l=0$, then X is the empty path. Obviously, Y also has to be the empty path.

If all 1-2-paths with length smaller than $2l$ can be appropriately mapped onto classic paths, let $X_0,\dots,X_{2l+a}$ be a 1-2-path with $X_0=X_{2l+a}=0$ and $a\in\{0,1\}$. The variables $p(X),z(X)$ and a' are already defined, but we also need an indicator for the second-to-last time where 1 is crossed in the last subpath (if that happens at all), $o(X):=\max\{k\in\{z(X),z(X)+1,\dots,l-2\}:X_{2k+a+1}=1\}$. If $X_{2l+a-1}=1$ then $2o(X)+a$ is the second-to-last time where Y has the value 1. Also, $o(X)$ might be $-\infty$. Because only a fully non-negative sequence starting and ending with 0 has even length, we have $a = a'$ exactly when $X_{2p(X)+a} = 0$.

For the path $X_0, X_1, \ldots, X_{2z(X)+a'}$, there is a classic path $Y_0, Y_1, \ldots, Y_{2z(X)}$where the bijection works, $Y_{2z(X)} = 0$. This classic path can directly be taken over without needing to be modified, and only $Y_{2z(X)+1}, \ldots, Y_{2l}$ need to be calculated.

$$Y_k = \begin{cases} X_{k+a} & a = a' \\ 1 - X_k & a = 1, a' = 0, X_k > 0 \\ \frac{X_k - 1}{2} & a = 1, a' = 0, X_k < 0 \\ \frac{X_{k-1} - 1}{2} & a = 0, a' = 1, o(X) = -\infty, k > 2z(X) + 2 \\ 1 + X_{k-1} & a = 0, a' = 1, o(X) = -\infty, k \leq 2z(X) + 2 \\ -1 - X_{k+1} & a = 0, a' = 1, o(X) > -\infty, X_{k+1} > 0, k > 2o(X) \\ X_k & a = 0, a' = 1, X_{k+1} > 0, 2p(X) < k \leq 2o(X) \\ \frac{-X_k + 3}{2} & \text{otherwise} \end{cases} \tag{5}$$

If the section from the last 0 onward is nowhere negative, then the said part is the exact same, maybe shifted by 1 depending on a, in the first case. If $a = 1$ and $a = 0$, then an additional step is inserted and the whole last part is non-positive. This is covered by the third and fourth cases, based on if the part would be reversed into positive or not.

The hard part is again $a = 0, a' = 1$. This time, we have to get two more steps into the classic path. The last part of the 1-2-path either consists of a subpath in almost full negative range except for the last two steps which go from -1 to 1 to 0, then the fourth case is used for that part, except for the additional first two steps in the classic path from 0 to 1 to 0 in the fifth case. Or the last part is in positive range from the second-to-last 1 to the 1 at time $2l - 1$. For that part, the sixth case is used, and in the part before, the two additional steps from 0 to 1 and from 1 to 0 are inserted at beginning and end, where the rest of the classic path is in positive range. Depending on where the 1-2-path is, one of the two last cases is used.

The last step is the proof that the inversion is also injective. Assume that the inversion isn't injective for an $l \in \mathbb{N}$. Choose l as small as possible. Then there are two 1-2-paths $X^{(1)}$ and $X^{(2)}$ with length $2l$ or $2l + 1$ of which the inverse mapping is the same classic path Y with length $2l$. All of those paths end with 0.

Case $Y_{2l-1} = 1$.

Set $k = 2l - 1$. The second and third case of equation (5) only give non-positive values for Y, but $Y_{2l-1} > 0$. The fifth case can't be used for this k, and the last two cases can't be used either. The fourth case would imply $X_{2l-2} = 3$, which doesn't work with $a = 0$ and $X_{2l+a} = 0$. The sixth case also implies $X_{2l} = 2$ which can't be true. So the first case is used, where the last segment from X is just taken over from Y. Removing this segment from $X^{(1)}$ and $X^{(2)}$ makes these paths shorter, but they already are inverse-mapped onto the same path, which contradicts that l is minimal.

Case $Y_{2l-1} = -1$.

The general idea that will not be shown in detail is the following. Take out all consecutive negative segments at the end of Y. The new path is either empty or ends with a positive segment, which is then taken out as well (just to be sure that the last five cases in equation (5) don't mess things up). Then $X^{(1)}$ and $X^{(2)}$ have to be different in the time span that was taken out, because l was chosen as small as possible, and they were equal before this time span. This can be used to consider the cases for a and a' for both paths, and also their z and p values. The possible cases in equation (5) will show that the paths have to be equal in this time span too, which is a contradiction.

Therefore, the mapping and its inversion are injective, which implies that the mapping is indeed a bijection. That proves that indeed there are as many classic paths with length $2l$ ending with 0 as 1-2-paths with length $2l$ or $2l+1$ ending with 0. □

A few examples will follow for illustration. The case $l = 0$ was already mentioned.

Case $l = 1$: There are two paths in the classic random walk with $Y_2 = 0$. With $Y_0 = 0$ always set, only Y_1 has to be considered. We also have $m(Y) = 0$, so for now, only equation (4) will be used. The previous bijection X' is the empty path, $a' = 0$.

For $Y_1 = 1$, we have $a = 0$ using equation (2), as the last segment is is in positive range. That means that $X_1 = Y_1 = 1$ as in the second case in equation (4), and $X_2 = 0$.

For $Y_1 = -1$, the value for a changes from 0 to 1. That means for X_1 and X_2, that the third case will be applied, $X_1 = 1 + 2Y_1 = -1$ and $X_2 = 1 + 2Y_2 = 1$. The 1-2-path is concluded with $X_3 = 0$.

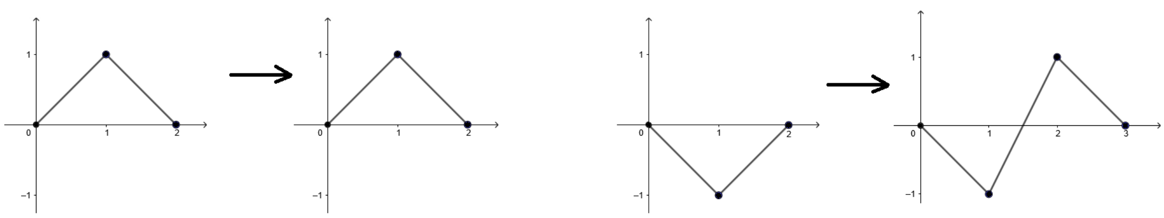

Figure 28: Bijection between classic and 1-2-paths, $l = 1$

Case $l = 2$: There are $\binom{4}{2} = 6$ possibilities to reach 0 with single steps. Two of them don't return to 0 before the final step, hence $m(Y) = 0$ and $a' = 0$. Like before, only equation (4) is needed there, and even for that one only the first 3 cases.

The start and end are $Y_0 = Y_4 = 0$. For $Y_1 = Y_3 = 1, Y_2 = 2$ the 1-2-path stays the same, $X_1 = X_3 = 1, X_2 = 2$ because of $a = a' = 0$. And $Y_1 = Y_3 = -1, Y_2 = -2$ gives $a = 1$. The values are $X_1 = 1 + 2Y_1 = -1 = 1 + 2Y_3 = X_3, X_2 = 1 + 2Y_2 = -3$ and $X_4 = 1 + 2Y_4 = 1$ before the last step, and $X_5 = 0$ for the end.

The other paths where the strategy to just repeat the steps in the classic path works by only using the first three cases of equation (4) start with $Y_1 = 1$ and $Y_2 = 0$. This time we have $X'_1 = 1, X'_2 = 0$ and $m(Y) = 1$. Note that $X_1 = X'_1 = 1$ because of $a' = 0$, regardless of the ending of the path, with the first case in the equation (3). For $Y_3 = 1$, the value for a is still 0, implying $X_2 = Y_2 = 0, X_3 = Y_3 = 1$ and $X_4 = 0$, with analog reasoning to the corresponding case in $l = 1$. We again have $a = 1$ and therefore $X_3 = 1 + 2Y_3 = -1, X_4 = 1 + 2Y_4 = 1$ and $X_5 = 0$.

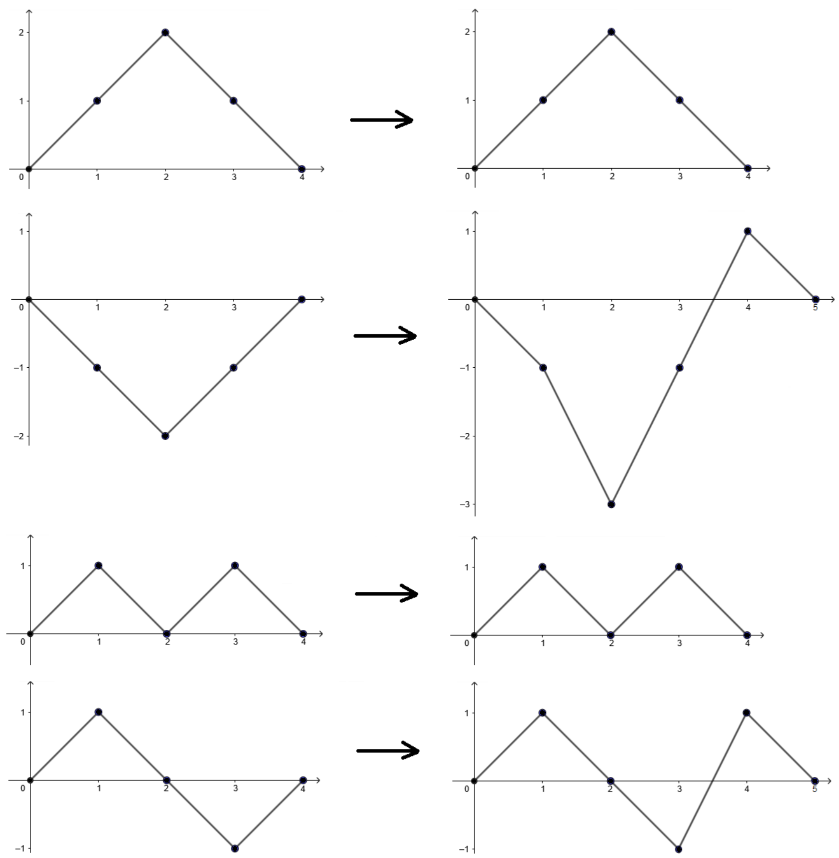

Figure 29: Bijection between classic and 1-2-paths, $l = 2$, part 1/2

For $Y_1 = -1$ and $Y_2 = 0$, we have $a' = 1$, and still $m(Y) = 1$, while the first path part is $X'_1 = -1, X'_2 = 1, X'_3 = 0$. If $Y_3 = 1$, we have $a = a' = 1$ and, $X_1 = X'_1 = -1, X_2 = X'_2 = 1$. In this case, the steps can still be repeated, because the second case of equation (4) is still used: $X_3 = Y_2 = 0, X_4 = Y_3 = 1$ and finally $X_5 = 0$. However, for $Y_3 = -1$, things are a bit different. We still have $a = a' = 1$, as the first case in equation (2) still triggers with $Y_1 = -1$, which means that X_1 and X_2 stay at their values from the X' path again. But the natural continuation $X_4 = -1, X_5 = 1, X_6 = 0$, is impossible, because we only have 5 steps instead of 6. Instead, the special last case in the equation (4) says $X_3 = 1 - Y_3 = 2, X_4 = 1 - Y_4 = 1$ and the conclusion $X_5 = 0$. This is the first time where steps of a classic path are reversed in the 1-2-path, as there are multiple segments of the path next to each other are in negative range in the classic path. The first segment is taken as it is, the other one is reversed and finally a last step to 0 is needed. This is what happens in the third and the last cases of equation (4) respectively.

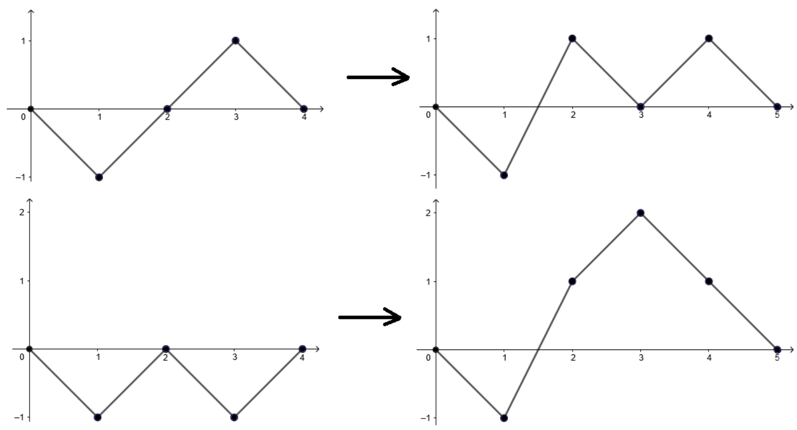

Figure 30: Bijection between classic and 1-2-paths, $l = 2$, part 2/2

Case $l = 3$: The ideas stay the same, so the $\binom{6}{3} = 20$ bijections will not be shown. Split the classic path into its segments and try to execute them one after another. If more than one segment is below 0, reverse all of those except for the first and go to 0 afterwards. That works for every path. Except for one.

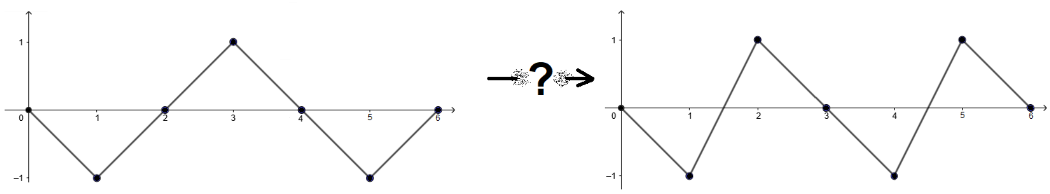

Figure 31: Problem path for $l = 3$

In the classic path $Y_2 = Y_4 = Y_6 = 0, Y_1 = -Y_3 = Y_5 = -1$, there aren't two segments in negative range next to each other, but just doing them in order also doesn't work, because that would result in a path with length 8. But we have to get to $X_1 = X_4 = -1, X_2 = X_5 = 1, X_3 = X_6 = 0$, the path that couldn't be taken for $l = 2$, because the 19 other 1-2-paths are used elsewhere. Actually, the only thing that's different from executing all parts is that there would be another trip to 1 and back to 0 in the middle. These two steps have to be taken out somehow. It will always be the problem when going into negative range twice, that there are two additional steps that have to be reduced. That's where the use for a comes in, it changes from $a' = 1$ back to $a = 0$ in exactly this case where the last segment is in negative range, but cannot be connected with the segment directly before that is is in positive range. Also, the steps done in image 27 come into play.

Using the equation (3) for the first three steps gives out $X_1 = X_1' = -1, X_2 = X_2' = 1, X_3 = X_3' = 0$. Then $X_4' = 1$ and $X_5' = 0$ doesn't help here. Additional values are $m(Y) = 2$ and $m^{(2)}(Y) = 1$ the last two times where 0 is crossed after the double amount of steps in the classic path. Note that $m^{(1)}(Y) = \infty$, because the path doesn't pass 1 twice after time $2m^{(2)}(Y)$ anymore. That means that for the rest, only the fourth case of equation (4) is used aside from $X_6 = 0$, and we get $X_4 = 1+2Y_5 = -1, X_5 = 1+2Y_6 = 1$.

Case $l = 4$: To conclude the small examples, a few paths for $l = 4$ will be considered. Consider the paths Y^1 and Y^2 with $-Y_1 = Y_3 = Y_5 = -Y_7 = 1, Y_0 = Y_2 = Y_6 = Y_8 = 0$ and $Y_j^1 = Y_j^2 = Y_j$ for $j \in \mathbb{N} \setminus \{4\}$. What makes the difference in $Y_4^1 = 2$ compared to $Y_4^2 = 0$?

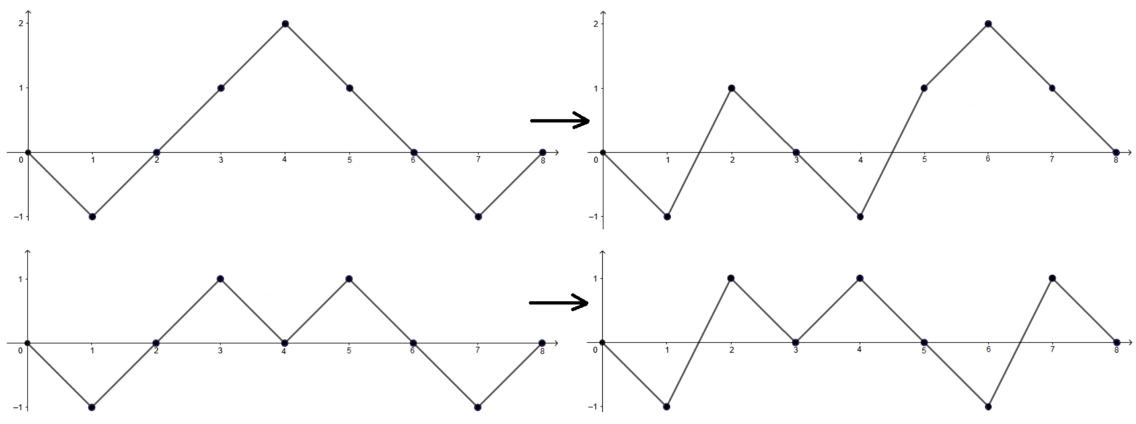

Figure 32: Y^1 to X^1 above, Y^2 to X^2 below, $l = 4$

Note that $a' = 1$ and $a = 0$ both times. For Y^1, we have $m(Y^1) = 3$ and $m^{(2)}(Y^1) = 1$. This time 1 is passed twice after $2m^{(2)}(Y^1)$, which means that $m^{(1)}$ is finite this time with $m^{(1)} = 2$. The only values taken over from $X^{1'}$ are $X_1^1 = -1, X_2^1 = 1, X_3^1 = 0$. Next is $X_4^1 = 3 - 2Y_4^1 = -1, X_5^1 = 3 - 2Y_5^1 = 1$, and after that, equation (4) is used already. Again, the first and the last step from 0 to 1 and back from 1 to 0 in the middle part are removed and the rest is reversed. That's where the second case in equation (3) is used. Then the path is concluded with $X_6^1 = 1 - Y_7^1 = 2, X_7^1 = 1 - Y_8^1 = 1, X_1^8 = 0$. For X_6 and X_7, the last case reverses the steps of the classic path again, that has to be done because the negative part was already covered before.

In Y^2, we also have $m\left(Y^2\right) = 3$, but $m^{(2)}\left(Y^2\right) = 2$. That means that this time, two more values for $X^{2'}$ are taken over: $X_1^2 = -1, X_2^2 = 1, X_3^2 = 0, X_4^2 = 1, X_5^2 = 0$. From the third segment Y_4^2 to Y_6^2, the first and the last step shall be taken out and the rest is reversed, but the rest is an empty path, so there is nothing to reverse. Note that $m^{(1)}\left(Y^2\right) = \infty$. It was the same in the path above in $l = 3$. The rest of the path is just the fourth case of equation (4), so we have $X_6^2 = 1 + 2Y_7^2 = -1, X_7^2 = 1 + 2Y_8^2 = 1$ and finally $X_8^2 = 0$.

That shows the importance of the last positive segment next to a negative segment that forces two steps to be eliminated. But one more thing is important. We more or less put all segments together that are next to each other in negative range. However, there are exceptions to that plan. For example, in $Y_0 = Y_2 = Y_4 = Y_6 = Y_8 = 0, Y_1 = -Y_3 = Y_5 = Y_7 = -1$, the last segment is independent from the others and gives $X_7 = 1 + 2Y_7 = -1, X_8 = 1 + 2Y_8 = 1, X_9 = 0$. Because $a' = 0$ and $a = 1$, the third case of equation (4) is used again. That means that the very last segment has to be treated as individual (like if it would be the only segment) whenever possible, means, exactly when $a' = 0$. In that case, the previous segment can't be changed anymore, no matter what comes after.

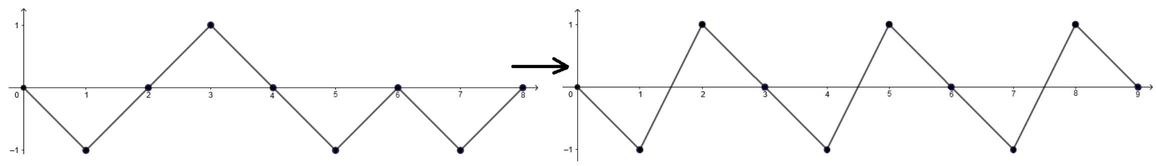

Figure 33: Special bijection for $l = 4$

This shall be a practical illustration for the bijection between classic paths and 1-2-paths ending on 0. But one more number was important. The triangle also had special numbers in the column representing the ending -1. Those will be done next, and the results of Proposition 3.3 and also methods of Proposition 2.3 can already be used.

Proposition 3.4 (1-2-paths ending on -1)**.** *For $l \in \mathbb{N}_0$, there are as many classic* *random walk paths with length $2l$ that end in 0 as 1-2-paths with length $2l$ or $2l+1$ that end in -1.*

Proof. The idea with a bijection can be executed again. This time, the bijection is between 1-2-paths of length $2l$ and $2l+1$. One side are the paths that end with 0, the other side the ones ending on -1.

Consider the paths 1-2-paths that end on 0. If a path $Y_0, \dots, Y_{2l+a}$ with $a \in \{0,1\}$ ends on 0, then the last step had to be from 1 to 0, implying $Y_{2l} = 1$ if $Y_{2l+1} = 0$. So the considered paths are exactly the ones with $Y_{2l} \in \{0,1\}$.

Case 1: $Y_{2l} = 0$.

The obvious idea is to set $X_k = Y_k$ for $k \in \{0, \dots, 2l\}$ and $X_{2l+1} = -1$. That already covers all 1-2-paths with $X_{2l+1} = -1$ and $X_{2l} = 0$.

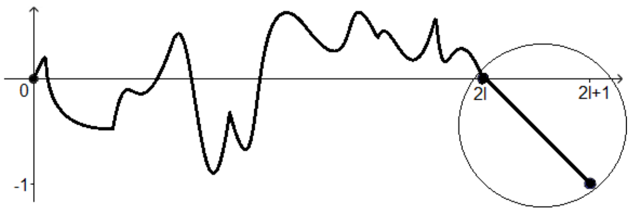

Figure 34: Bijection for case 1

Case 2: $Y_{2l} = -1$.

We have to get to paths that either have $X_{2l} = -1$ already, or $X_{2l+1} = -1$ with $X_{2l+1} = -3$. The idea is to do a reflection again. The last part of a path can be reflected into negative range and maybe we land on -1 at time $2l$ or $2l+1$. It works similar to what has already been done with the rectangles. Let $p(Y)$ be the last time where Y was not positive, $p(Y) = \max\{k \in \{0, \dots, 2l\} : Y_k \leq 0\}$. Because of $Y_0 = 0$, we have $p(Y) \geq 0$. Beginning at time $p(Y)$ every step will be turned around, and $X_k = Y_k$ is set for $k \leq p(Y)$.

Case 2.1: $Y_{p(Y)} = 0$.

After time $p(Y)$, the old path has only steps with length 1, while the steps of the new path have length 2 there, except for the very first one from 0 to -1. Then $X_k = 1 - 2Y_k$ for $k > p(Y)$. That means that X_k will be -1 where Y_k is 1. Because of $Y_{2l} = 1$, we already have $X_{2l} = -1$ and nothing more has to be done. Also, all paths with $X_{2l} = -1$ will be reached, as this reflection can be reversed again: $X_{2l-1} = 0$ is no problem as there will be a single step down after that, and $X_{2l-1} = -3$ has just an extension from $X_{p(Y)+1} = -1$, which is in positive range at path Y. The exact formula is $Y_k = \frac{1-X_k}{2}$, derived from the other equation.

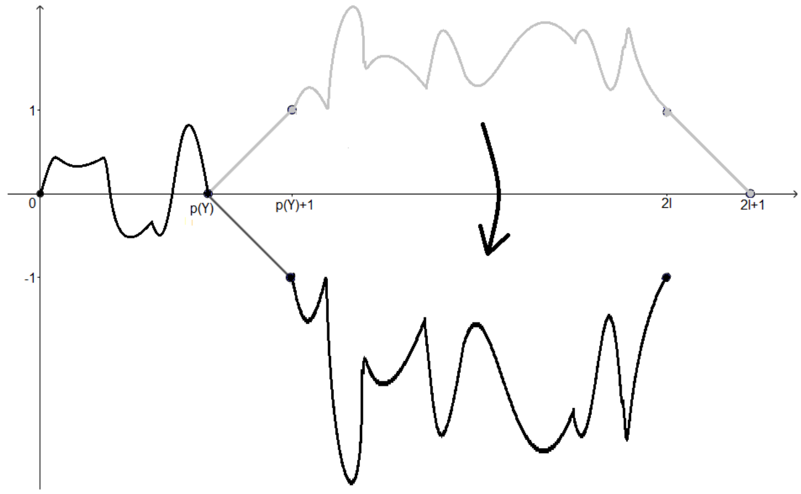

Figure 35: Bijection for case 2.1

Case 2.2: $Y_{p(Y)} = 0$**.**

The last part of the old path this time begins with a step from -1 to 1 and the other steps have length 1, while in the new path, all last steps have length 2. That means we have $X_k = -1 - 2Y_k$ for $k > p(Y)$. Then for $Y_{2l} = 1$, we have $X_{2l} = -3$. But that is okay, because we can just set $X_{2l+1} = -1$. Which also works with $Y_{2l+1} = 0$. The rest of the paths with $X_{2l+1} = -1$ will be reached, and exactly the other ones with $X_{2l} = -3$ that aren't already covered. To find the origin of a mapped path, set $Y_k = \frac{-1-X_k}{2}$.

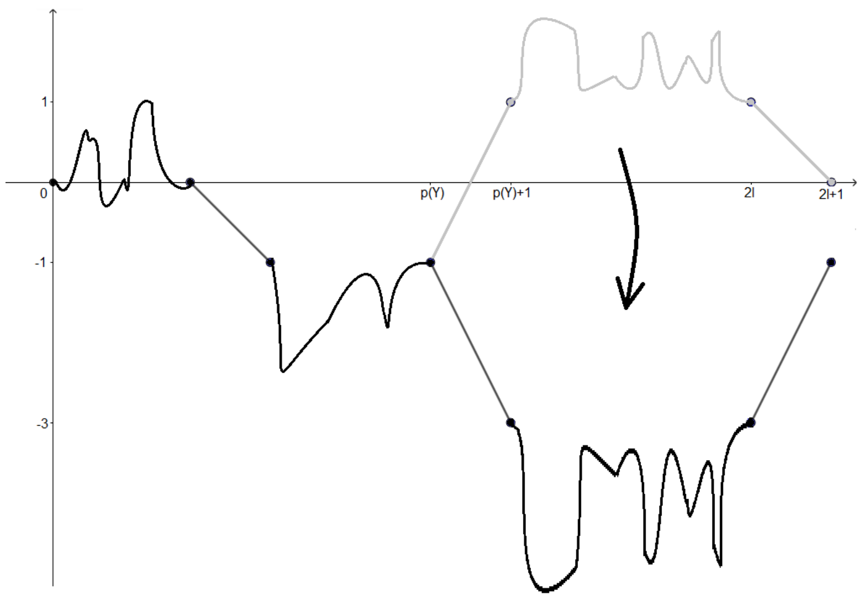

Figure 36: Bijection for case 2.2

This is the complete mapping. Let $k \leq 2l+1$, then this is the equation to get a -1-ending 1-2-path from a 0-ending one.

$$X_k = \begin{cases} Y_k & k \leq p(Y) \\ -1 & k = 2l+1, Y_{2l} = 0 \\ 1 - 2Y_k & p(Y) < k \leq 2l, Y_{p(Y)} = 0 \\ -1 - 2Y_k & p(Y) < k, Y_{p(Y)} = -1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

And this is the other direction. Let $z(X) := \max\{i \in \{0, 1, \ldots, 2l-1\} : X_i = 0\}$ and $o(X) := \max\{i \in \{0, 1, \ldots, 2l-1\} : X_i = -1\}$ be the last time points where the path passed 0 and -1, respectively. When $o(X)$ is needed, it is not $-\infty$.

$$Y_k = \begin{cases} 0 & k = 2l+1, X_{2l} = -1 \\ \text{undefined} & k = 2l+1, X_{2l} = 0 \\ \frac{1-X_k}{2} & z(Y) < k \leq 2l, X_{2l} = -1 \\ \frac{-1-X_k}{2} & o(Y) < k, X_{2l} = -3 \\ X_k & \text{otherwise} \end{cases}$$

This completes the bijection, which means that for any $n \in \mathbb{N}$ there are as many 1-2-path with length $2l$ or $2l+1$ ending with -1 as with 0, and therefore as many as there are classic paths ending with 0 with length $2l$. $\square$

In order to continue with the modified Brownian motion, we need one last bijection.

Proposition 3.5 (1-2-paths in 0 or -1 probabilities)**.** *For $l \in \mathbb{N}_0$, let $X_0, X_1, \ldots$ be the 1-2-random walk. Then $P(X_n = 0) = \frac{(-1)^n}{2^n} + P(X_n = -1)$ for $n \in \mathbb{N}_0$.*

Proof. Induction base: Because of $P(X_0 = 0) - \frac{1}{2^0} = P(X_0 = -1) = 0 = P(X_1 = 0) = P(X_1 = -1) - \frac{1}{2^1}$, the equation is fulfilled for $n = 0$ and $n = 1$.

Induction step: Assume that the equation is proven for $n = m$. Next thing to show is that the equation also works for $n = m + 2$.

If $X_{m+2} = 0$, then $X_{m+1} = 1$, and therefore $X_m \in \{-1, 0, 2\}$. Because from each of those three values there is only one way to get to 0 in two steps, we have $P(X_{m+2} = 0) =$

$\frac{1}{4}P(X_m \in \{-1,0,2\})$. On the other hand, for $X_{m+2} = -1$ we have $X_{m+1} \in \{-3,0\}$ and then $X_m \in \{-5,-1,1\}$, which leads to $P(X_{m+2} = -1) = \frac{1}{4}P(X_m \in \{-5,-1,1\})$. The equation that has to be shown can be transformed, also using said equation for $n = m$, which is already known to be true, and the result is

$$\begin{aligned}
& P(X_{m+2}=0) = \frac{(-1)^{m+2}}{2^{m+2}} + P(X_{m+2}=-1) \\
\Leftrightarrow \quad & \frac{1}{4}\left(P(X_m=-1)+P(X_m=0)+P(X_m=2)\right) \\
& = \frac{1}{4}\left(\frac{(-1)^m}{2^m} + P(X_m=-5)+P(X_m=-1)+P(X_m=1)\right) \\
\overset{\text{I.H.}}{\Leftrightarrow} \quad & P(X_m=-1)+P(X_m=2) = P(X_m=-5)+P(X_m=1).
\end{aligned}$$

Therefore, it is sufficient to show that there are as many paths with length m that end with 2 or -1 as there are paths of the same length ending with 1 or -5. Let w.l.o.g. $X_m \in \{-1,2\}$. The goal is to create a bijection onto a path $Y_0, Y_1, \ldots, Y_m$ with $Y_m \in \{-5,1\}$. For this purpose, let $p(X) = \max\{k \in \mathbb{N} : X_k \in \{-1;0\}\}$ be the last time before m where X is -1 or 0.

Case 1: $X_m = 2$ and $X_{p(X)} = -1$.

The steps after $p(X)$ can be reflected, and due to the factor 2 in negative range, this implies $Y_m = -5$. This can also be reversed, from any path with $Y_m = -5$ the steps after $p(Y)$ can be reflected to get a path with $X_m = 2$ and $X_{p(X)} = -1$.

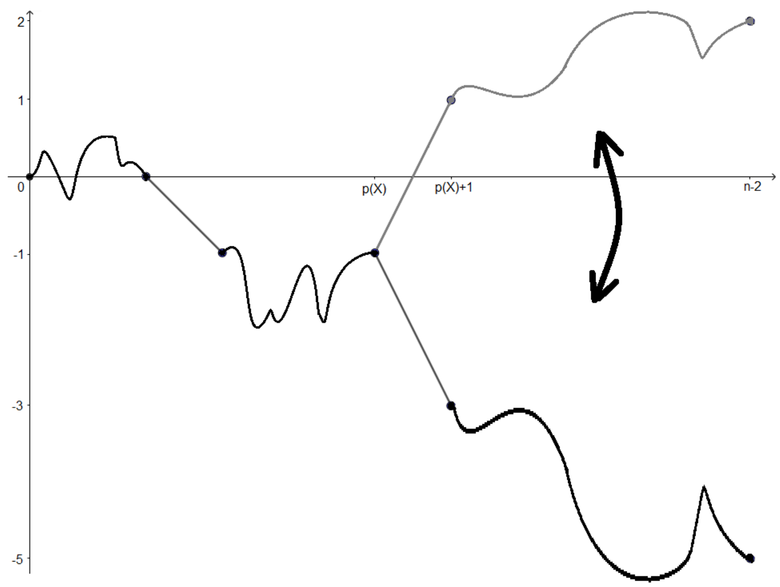

Figure 37: Bijection between first part of 1-2-paths ending on 0 and -1, case 1

Case 2: $X_m = 2$ and $X_{p(X)} = 0$.

Everything after $p(X)$ can be reflected again until some certain part. After the point where it doesn't pass 1 anymore before n, at a time $o(X)$, the new path does the same steps as the old path again, and is always 1 below the other path, which leads to $Y_m = X_m - 1 = 1$. The reverse steps from a path with $Y_m = 1$ and $Y_{p(Y)} = -1$ would be to reverse everything after a time $z(Y)$ where 0 was crossed the last time before m until $p(Y)$, where the steps are the same again. The distinction with $o(X)$ and $z(Y)$ is necessary, but in this case, these values are indeed not $-\infty$.

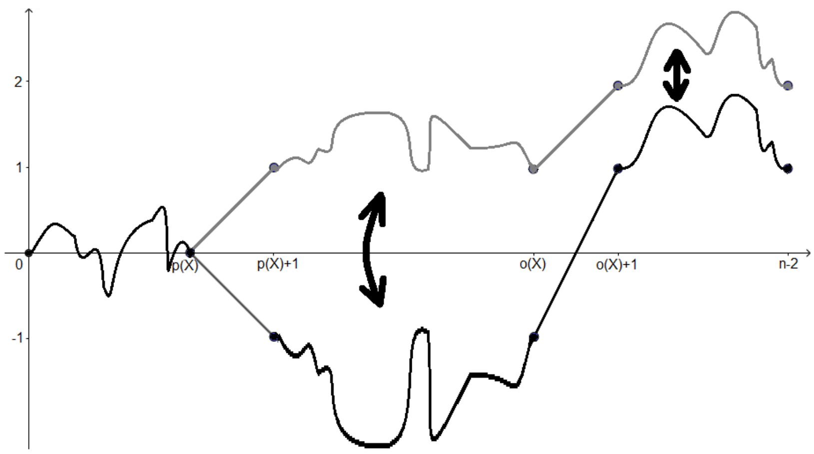

Figure 38: Bijection between first part of 1-2-paths ending on 0 and -1, case 2

Case 3: $X_m = -1$.

The only paths left are the ones with $Y_m = 1$ with $Y_{p(Y)} = 0$. This can work with reflecting every step after $z(X)$. Also note that $z(X) = p(Y)$. The steps after $z(X)$ have length 2 again, but it will be $Y_m = 1$. And the other way around, we also have $X_m = 1$ when reflecting every step after $p(Y)$.

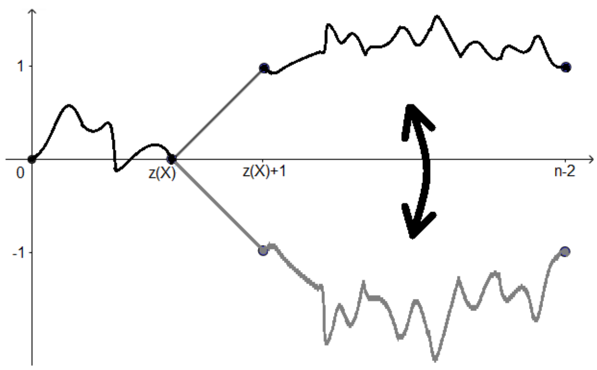

Figure 39: Bijection between first part of 1-2-paths ending on 0 and -1, case 3

That's all cases for both paths covered. To sum it up, here is the case distinction for $k \in \{0, 1, \ldots, m\}$. Let $X_m \in \{-1; 2\}$ with $p(X) = \max\{i \in \{0, 1, \ldots, m-1\} : X_i \leq 0\}$, and for Y respectively. The same thing for $z(X) := \max\{i \in \{0, 1, \ldots, m-1\} : X_i = 0\}$ and $o(X) := \max\{i \in \{0, 1, \ldots, m-1\} : X_i = 1\}$. It will $o(X) > \infty$ in the cases it is needed, which is exactly when $X_m = 2$. The cases here are in the same order as considered in the mapping, except for the last line where the first part of the path is covered that is just taken over as it is.

$$Y_k = \begin{cases} -1 - 2X_k & X_m = 2, X_{p(X)} = -1, k > p(X) \\ X_k - 1 & X_m = 2, X_{p(X)} = 0, k > o(X) \\ 1 - 2X_k & X_m = 2, X_{p(X)} = 0, p(X) < k \leq o(X) \\ \frac{1 - X_k}{2} & X_m = -1, k > z(X) \\ X_k & \text{otherwise} \end{cases}$$

And this is the other direction, where $Y_0, Y_1, \ldots, Y_m$ is given with $X_m \in \{-5; 1\}$ and $X_0, X_1, \ldots, X_m$ shall be calculated, again in the same order as in the mapping from X to Y.

$$X_k = \begin{cases} \frac{-1 - Y_k}{2} & Y_m = -5, k > p(Y) \\ Y_k + 1 & Y_m = 1, Y_{p(Y)} = -1, k > p(Y) \\ \frac{1 - Y_k}{2} & Y_m = 1, Y_{p(Y)} = -1, z(Y) < k \leq p(Y) \\ 1 - 2Y_k & Y_m = 1, Y_{p(Y)} = 0, k > p(Y) \\ Y_k & \text{otherwise} \end{cases}$$

Both equations use the first line in case 1, the fourth line in case 3 and the second and third lines in case 2 for the parts in the two paths that are different from each other. The last line is in all three cases for the part that isn't reflected.

Because these mappings inverse each other, we indeed have a bijection. This implies there are as many 1-2-paths ending on -5 or 1 as 1-2-paths ending on -1 or 2 for a fixed length. In conclusion, the equation we want to show is true for $n = m + 2$. Because the induction began with $n = 0$ and $n = 1$, the equation can be followed for all $n \in \mathbb{N}_0$. □

We can finally continue with the modified Brownian motion $(M_t)_{t \geq 0}$. The point before these three bijections was that the expected value of the stopping times is sometimes $\frac{3}{8}$ off of 1. But what we want is that $\frac{\tau_n}{n} \to 1$ in probability. The plan there is to show that the expected value converges to 1 and the variance converges to 0.

At first let's compute $\mathbb{E}\frac{\tau_n}{n}$. For every path and time point, we have to look at which number it is and then add the expected value of the specific stopping time at that number. Because every path has the same probability, the formula for the expected value of every single stopping time can just be taken over using Proposition 3.5.

$$\begin{aligned}
\mathbb{E}\frac{\tau_n}{n} &= \mathbb{E}\sum_{i=1}^{n}\frac{\tau_i-\tau_{i-1}}{n} = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\tau_{X_{i-1},X_{i-1}-\nu(X_{i-1}),X_{i-1}+\nu(X_{i-1})} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\left(P\left(X_i\notin\{-1,0\}\right)+\frac{5}{8}P\left(X_i=0\right)+\frac{11}{8}P\left(X_i=-1\right)\right) \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\left(1+\frac{3}{8}P\left(X_i=-1\right)-\frac{3}{8}P\left(X_i=0\right)\right) = \frac{n}{n}+\frac{1}{n}\sum_{i=0}^{n-1}\frac{3}{8}\cdot\frac{-\left(-1\right)^i}{2^i} \\
&= 1-\frac{3}{8n}\sum_{i=0}^{n-1}\left(\frac{-1}{2}\right)^i = 1-\frac{3}{8n}\cdot\frac{1-\left(\frac{-1}{2}\right)^n}{1-\frac{-1}{2}} = 1-\frac{3}{8n}\cdot\frac{2}{3}\cdot\left(1-\left(\frac{-1}{2}\right)^n\right) \\
&= 1-\frac{1}{4n}\cdot\left(1-\left(\frac{-1}{2}\right)^n\right)
\end{aligned}$$

The expected value tends to 1 for $n\to\infty$: Both of the fractions tend to 0, so the value in the brackets tends to 1 and therefore the whole subtrahend tends to 0.

Recall $X_i = M_{\tau_i}$ for any $i\in\mathbb{N}$. Define $f\left(x_{i-1},x_i\right) := \mathbb{E}\left(\tau_i-\tau_{i-1}|X_{i-1}=x_{i-1},X_i=x_i\right)$ and $\sigma^2\left(x_{i-1},x_i\right) := \mathbb{E}\left(\left(\tau_i-\tau_{i-1}-f\left(x_{i-1},x_i\right)\right)^2|X_{i-1}=x_{i-1},X_i=x_i\right)$ for any x_{i-1},x_i with $P\left(X_{i-1}=x_{i-1},X_i=x_i\right)>0$. Then $f\left(x_{i-1},x_i\right)$ is the expected time that $\left(M_t\right)_{t\geq0}$ needs to go from x_{i-1} to x_i under the condition that these specific two values are hit at time $i-1$ and i, and $\sigma^2\left(x_{i-1},x_i\right)$ the corresponding variance. However, they just depend on the values of x_{i-1} and x_i, and not on i itself, because $\left(M_t\right)_{t\geq0}$ is also time-independent.

For $x_{i-1}\notin\{-1,0\}$ we have $f\left(x_{i-1},x_i\right) = \mathbb{E}\tau_{x_{i-1},x_{i-1}-\nu(x_{i-1}),x_{i-1}+\nu(x_{i-1})} = 1$, because every path starting at x_{i-1} and ending at $x_{i-1}-\nu\left(x_{i-1}\right)$ or $x_{i-1}+\nu\left(x_{i-1}\right)$ can just be reflected to hit the other of those two values. For $x_{i-1}>0$ Proposition 1.5 implies $\sigma^2\left(x_{i-1},x_i\right) = \mathbb{E}\left(\left(\tau_i-\tau_{i-1}\right)^2|X_{i-1}=x_{i-1},X_i=x_i\right)-f\left(x_{i-1},x_i\right)^2 = \frac{5\cdot1^4}{3}-1^2 = \frac{2}{3}$, the variance there is finite. The same argument can be used for $x_{i-1}<-1$, and the variance is also finite, $\sigma^2\left(x_{i-1},x_i\right)<\infty$. For $x_{i-1}=0$ however, a distinction for the possible x_i is necessary, as we have $\frac{1}{2}\left(f\left(0,1\right)+f\left(0,-1\right)\right)=\frac{5}{8}$, but they don't need to be equal. The variances $\sigma^2\left(0,-1\right)$ and $\sigma^2_{0,-1}$ can also be different from each other, but they are finite, because the variance of $\tau_{0,-1,1}$ is not greater than $\sigma^2\left(1,2\right)+\sigma^2\left(-3,-5\right)$ and therefore bounded. Finally, if $x_{i-1}=-1$, then $f\left(-1,-3\right)$ and $f\left(-1,1\right)$ must have the same difference to $\frac{11}{8}$. And the variances $\sigma^2(-1,1)$ and $\sigma^2(-1,-3)$ are also bounded, as the variance of $\tau_{-1,-3,1}$ is bounded:

$$\begin{aligned} Var(\tau_{-1,-3,1}) &= Var(\tau_{-1,-3,0}) + \frac{2}{3} Var(\tau_{0,-3,1}) \\ &= Var(\tau_{-1,-3,0}) + \frac{2}{3}\left(Var(\tau_{0,-1,1}) + \frac{1}{2} Var(\tau_{-1,-3,1})\right) \\ \Leftrightarrow \quad Var(\tau_{-1,-3,1}) &= \frac{3}{2} Var(\tau_{-1,-3,0}) + Var(\tau_{0,-1,1}) \leq \frac{3}{2}\sigma_{-2}^2 + Var(\tau_{0,-1,1}) < \infty. \end{aligned}$$

The $\frac{2}{3}$ and $\frac{1}{2}$ are the probabilities to reach 0 before -3 from -1 and to reach -1 before 1 from 0, respectively. All the variances of these specific stopping times are finite.

Set $\sigma_{max}^2 := \max\{\sigma^2(1,2), \sigma^2(-3,-5), \sigma^2(0,1), \sigma^2(0,-1), \sigma^2(-1,1), \sigma^2(-1,-3)\}$ and $T(n) := \mathbb{E}(\tau_n | X_0, X_1, \ldots, X_n)$. Because of $\tau_0 = 0$ and the independent increments of those stopping times, we have $T(n) = \sum_{i=1}^{n} f(X_{i-1}, X_i)$ for any $n \in \mathbb{N}_0$. This is a discrete random variable, and its number of different values is at most the number 1-2-paths with length n, which is 2^n, and every path has the same probability $\frac{1}{2^n}$.

Lemma 3.6 (stopping time tending to their expected values)**.** *Let $n \to \infty$. Then $\frac{\tau_n - T(n)}{n} \to 0$ in probability.*

Proof. Define $D(n) := (\tau_n - T(n))^2$. Use telescope sums and $\tau_0 = T(0) = 0$ to get

$$\begin{aligned} \mathbb{E}(D(n)) &= \mathbb{E}\left(\tau_n - \sum_{i=1}^{n} f(X_{i-1}, X_i)\right)^2 = \mathbb{E}\left(\sum_{i=1}^{n} (\tau_i - \tau_{i-1} - f(X_{i-1}, X_i))\right)^2 \\ &= \sum_{i=1}^{n} \mathbb{E}(\tau_i - \tau_{i-1} - f(X_{i-1}, X_i))^2 \leq \sum_{i=1}^{n} \sigma_{max}^2 = n\sigma_{max}^2. \end{aligned}$$

All $\tau_k - \tau_{k-1} - f(X_{k-1}, X_k)$ have expected value 0 (hence the expectation of their squares is equal to their variances), and they are uncorrelated to each other. Because of that, it doesn't matter if all of those summands are added or squared first.

Now it's just a short step to

$$\mathbb{E}\left(\frac{\tau_n - T(n)}{n}\right)^2 = \mathbb{E}\left(\frac{D(n)}{n}\right)^2 \leq \frac{n\sigma_{max}^2}{n^2} = \frac{\sigma_{max}^2}{n},$$

which tends to 0 for $n \to \infty$, implying $\frac{\tau_n - T(n)}{n} \to 0$ in probability. □

Proposition 3.7 (limit expected value of stopping times)**.** *For $n \to \infty$, the random variables $\frac{T(n)}{n}$ converge in probability to 1.*

Proof. A rough estimation is the following:

$$\mathbb{E}\left|1-\frac{T(n)}{n}\right| = \frac{\mathbb{E}\left|n-\sum_{i=1}^{n}F(i)\right|}{n} \leq \frac{\mathbb{E}\sum_{i=1}^{n}|1-F(i)|}{n}$$
$$= \frac{\sum_{i=1}^{n}\sum_{j=-2i+2}^{i-1} P(X_{i-1}=j)\cdot\frac{1}{2}\left(|1-f(j,j-\nu(j))|+|1-f(j,j+\nu(j))|\right)}{n}$$

The second sum has to start at $-2i+2$ instead of $-2i+3$ because for $i=1$ we have to start with 0 and not with 1. All summands outside $j\in\{-1,0\}$ can be ignored because either the expected value of the stopping time with specific outcome is 1, which also reduces to 0 in the absolute, or the probability of attaining a negative even value is 0. Let $a_0 := |1-f(0,1)|+|1-f(0,-1)|$ and $a_{-1} := |1-f(-1,1)|+|1-f(-1,-3)|$. Then

$$\mathbb{E}\left|1-\frac{T(n)}{n}\right| = \frac{\sum_{i=1}^{n}\left(a_0\cdot P(X_{i-1}=0)+a_{-1}\cdot P(X_{i-1}=-1)\right)}{2n}.$$

If n is odd, the sum is greater than the sum for $n-1$ and smaller than the sum for $n+1$. So we can just assume that n is even, $n=2m$. The reason is that we can combine the amount of paths with length $2k$ and $2k+1$ for any $k\in\mathbb{N}_0$ that end on 0 or -1, of which the number is already known because of Propositions 3.3 and 3.4. We have

$$\mathbb{E}\left|1-\frac{T(n)}{n}\right| \leq \frac{\sum_{i=1}^{2m} P(X_{i-1}=0)\,a_0+P(X_{i-1}=-1)\,a_{-1}}{4m}$$
$$= a_0\frac{\sum_{i=1}^{m}\left(P(X_{2i-2}=0)+P(X_{2i-1}=0)\right)}{4m}$$
$$+ a_{-1}\frac{\sum_{i=1}^{m}\left(P(X_{2i-2}=-1)+P(X_{2i-1}=-1)\right)}{4m}$$

For any $k\in\mathbb{N}_0$, Proposition 3.3 says that the number of 1-2-paths with length $2k$ or $2k+1$ ending on 0 is equal to the number of classic paths with length $2k$ ending on 0. Proposition 3.4 yields the same, only that the 1-2-paths end on -1 instead. Therefore, $2^{2k}P(X_{2k}=0)+2^{2k+1}P(X_{2k+1}=0)=\binom{2k}{k}=2^{2k}P(X_{2k}=-1)+2^{2k-1}P(X_{2k+1}=-1)$

is always true. This leads us to

$$\begin{aligned}\mathbb{E}\left|1-\frac{T(n)}{n}\right| \leq & a_0\frac{\sum_{i=1}^{m}\left(P\left(X_{2i-2}=0\right)+2P\left(X_{2i-1}=0\right)\right)}{4m} \\ & + a_{-1}\frac{\sum_{i=1}^{m}\left(P\left(X_{2i-2}=-1\right)+2P\left(X_{2i-1}=-1\right)\right)}{4m} \\ = & a_0\sum_{i=1}^{m}\frac{\binom{2i-2}{i-1}}{4m\cdot 4^{i-1}}+a_{-1}\sum_{i=1}^{m}\frac{\binom{2i-2}{i-1}}{4m\cdot 4^{i-1}}=\frac{a_0+a_{-1}}{4m}\sum_{i=0}^{m-1}\frac{\binom{2i}{i}}{4^i}.\end{aligned}$$

The main point is to estimate $\sum_{i=0}^{k}\frac{\binom{2i}{i}}{4^i}$ for any $k\in\mathbb{N}_0$. Let $l\geq 2$. Lemma 1.2 implies $\prod_{i=l+1}^{2l}\frac{2i-1}{2i}\leq\frac{3}{4}$, and we get

$$\begin{aligned}\frac{\binom{4l-2}{2l-1}}{4^{2l-1}}+\frac{\binom{4l}{2l}}{4^{2l}} &= \left(1+\frac{2l}{2l-1}\right)\frac{\binom{4l}{2l}}{4^{2l}}=\left(2+\frac{1}{2l-1}\right)\cdot\frac{\binom{2l}{l}}{4^l}\cdot\prod_{i=l+1}^{2l}\frac{2i\cdot(2i-1)}{4i^2} \\ &= \left(2+\frac{1}{2l-1}\right)\cdot\frac{\binom{2l}{l}}{4^l}\cdot\prod_{i=l+1}^{2l}\frac{2i-1}{2i}\leq\left(2+\frac{1}{2l-1}\right)\cdot\frac{3}{4}\cdot\frac{\binom{2l}{l}}{4^l} \\ &\leq\frac{7}{3}\cdot\frac{3}{4}\cdot\frac{\binom{2l}{l}}{4^l}=\frac{7}{4}\cdot\frac{\binom{2l}{l}}{4^l}.\end{aligned}$$

Let $V(k):=\sum_{i=2^{k-1}+1}^{2^k}\frac{\binom{2i}{i}}{4^i}$ for any $k\in\mathbb{N}$. Then $V(1)=\frac{3}{8}$ and

$$V(k+1)=\sum_{i=2^k+1}^{2^{k+1}}\frac{\binom{2i}{i}}{4^i}=\sum_{i=2^{k-1}+1}^{2^k}\left(\frac{\binom{4i-2}{2i-1}}{4^{2i-1}}+\frac{\binom{4i}{2i}}{4^{2i}}\right)\leq\frac{7}{4}\cdot\sum_{i=2^{k-1}+1}^{2^k}\frac{\binom{2l}{l}}{4^l}=\frac{7}{4}V(k).$$

Writing this in an explicit way implies $V(k)\leq\frac{3}{8}\cdot\left(\frac{7}{4}\right)^{k-1}$. Then the geometric formula can be used.

$$\begin{aligned}\sum_{i=0}^{m-1}\frac{\binom{2i}{i}}{4^i} &\leq\frac{\binom{0}{0}}{1}+\frac{\binom{2}{1}}{4}+\sum_{k=1}^{\lceil log_2 m\rceil}V(k)\leq 1+\frac{1}{2}+\frac{3}{8}\sum_{k=1}^{\lceil log_2 m\rceil}\left(\frac{7}{4}\right)^{k-1} \\ &=\frac{3}{2}+\frac{3}{8}\cdot\frac{\left(\frac{7}{4}\right)^{\lceil log_2 m\rceil-1}-1}{\frac{7}{4}-1}\leq\frac{3}{2}+\frac{\left(\frac{7}{4}\right)^{log_2 m}}{2}\end{aligned}$$

Finally, we get

$$\mathbb{E}\left|1-\frac{T(2m)}{2m}\right| \leq \frac{a_0+a_{-1}}{4m}\cdot\left(\frac{3}{2}+\frac{\left(\frac{7}{4}\right)^{log_2 m}}{2}\right)$$
$$=\frac{a_0+a_{-1}}{8}\cdot\left(\frac{3}{m}+\frac{\left(\frac{7}{4}\right)^{log_2 m}}{2^{log_2 m}}\right)=\frac{a_0+a_{-1}}{8}\cdot\left(\frac{3}{m}+\left(\frac{7}{8}\right)^{log_2 m}\right).$$

Therefore, $\lim_{m\to\infty}\mathbb{E}\left|1-\frac{T(2m)}{2m}\right|=0$, as both summands in the bracket tend to 0. This is it for even n, but for odd n, there is only one more summand in the sum of the binomial coefficients that also tends to 0 when divided by n. □

Lemma 3.6 and Proposition 3.7 now yield that $\frac{\tau_n-T(n)}{n}\to 0$ in probability and $\frac{T(n)}{n}\to 1$, also in probability. That implies $\frac{\tau_n}{n}\to 1$ in probability.

After that, the rest works just like the proof of Donsker's Theorem. It is proven in an analog way that $\frac{X_n}{\sqrt{n}}$ converges in distribution to a random variable that is distributed like M_1. This is the reason why ν is defined the way it is, being equal on the whole positive and the whole negative range: $\forall u_1,u_2>0:\nu(u_1)=\nu(u_2),\nu(-u_1)=\nu(-u_2)$. In any other case, it wouldn't be true that $\frac{M_s t}{\sqrt{s}}$ has the same distribution as M_t for any $s,t>0$. And $\nu(1)=1$ and $\nu(-1)=2$ follows from the fact that the stopping time differences must have a mean value of 1, or else the normed stopping times wouldn't converge to 1 in probability. And the rest can be directly taken over from the proof of Donsker's Theorem, only with M_t instead of B_t at every place with a Brownian motion. This shows that $(M_t)_{t\geq 0}$ is indeed the limit of the 1-2-random walk.

The final result is the following theorem.

Theorem 3.8 (limit of the 1-2-random walk)**.** *Let $Z_i, i\in\mathbb{N}$ be iid with $P(Z_i=1)=\frac{1}{2}=P(Z_i=-1)$. Let $X_0=0$ and $X_k=X_{k-1}+Z_i$ when $X_{k-1}\geq 0$ and $X_k=X_{k-1}+2Z_i$ when $X_{k-1}<0$ for $k\in\mathbb{N}$. Then $\left(\frac{X_{nt}}{\sqrt{n}}\right)_{t\geq 0}$ converges in distribution to $(M_t)_{t\geq 0}$ for large n, where $(M_t)_{t\geq 0}$ is a weak solution of the Stochastic Differential Equation*

$$dY_t=\nu(Y_t)\,dB_t,$$
$$Y_0=0,$$
$$\nu(m)=\begin{cases}1 & m\geq 0\\ 2 & \textit{otherwise.}\end{cases}$$

4 Miscellaneous

This isn't everything that was found out during the process of the creation. Another bijection with the $n \times 3$-rectangles can be made with the paths that weren't considered before, this will be looked into first. Also, some follow-up questions can be raised that will follow in the very last part.

4.1 Negative 1-2-random walk

Now that the non-negative paths were considered, the ones that end in negative range are left over.

Proposition 4.1 (number of negative 1-2 random walks)**.** *For $n \in \mathbb{N}$, there exists a bijection between the 1-2-random walks of length n that end in negative range, and the tilings of the $(n-1) \times 3-$rectangles.*

Proof. The total number of paths of length n is 2^n, obviously, as in each step, you can go up or down. Therefore, the number of paths ending in the negative, is

$$\begin{aligned} J_n^- &= 2^n - J_n = 2^n - \frac{2^{n+1} + (-1)^n}{3} = \frac{3 \cdot 2^n - (2 \cdot 2^n + (-1)^n)}{3} \\ &= \frac{2^n - (-1)^n}{3} = \frac{2^n + (-1)^{n-1}}{3} = J_{n-1}. \end{aligned}$$

The number is equal to the number of 1-2-random walk paths ending in non-negative range with length $n-1$. But $J_{n-1} = T_{n-1}$. That means that the number of negative paths with length n is equal to the number of tilings of $(n-1) \times 3-$rectangles, so there must exist a bijection between those. □

To find such a bijection, we can do a similar recursive approach to the one used before. Even the methods can mostly be transferred. The beginning is $n = 1$. The only tiling with length 0 is the empty rectangle, and the only negative path with length 1 is the one with $X_1 = 1$. So, there isn't much of a choice.

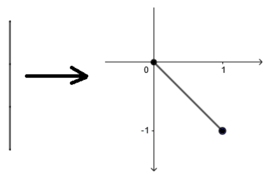

Figure 40: Bijection for $n = 1$ and negative paths

Having this as a base, we can again look what could be done when adding columns of squares. At first, the normal squares can be considered once more. Last time, the previous path was executed and the last step was 1 up. That can be done this time too, but the last step must be downwards, obviously, and it goes 2 down instead of 1 due to operating in negative range. Note that because of this, the only values that can actually be reached, are the odd numbers.

The upper 2-square made a useless turn of going 1 up and 1 down as last two steps in the non-negative case. This time, the useless turn is to go down first, and then going up as last step. Again, these two steps change the value by 2 instead of 1.

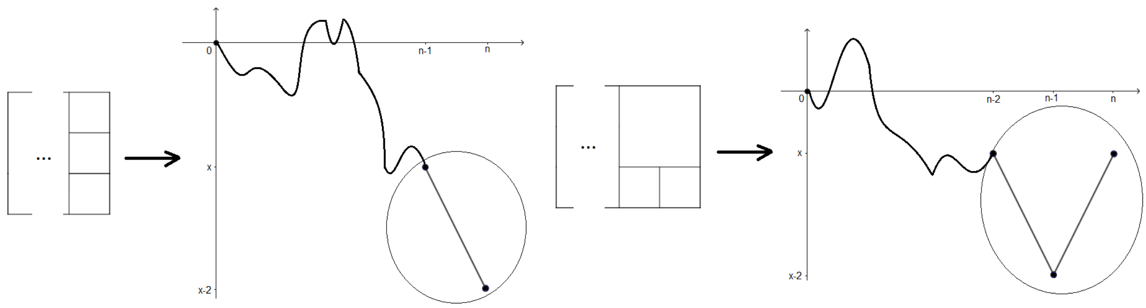

Figure 41: Recursive continuation for normal squares/upper 2-squares in negative paths

The lower 2-squares have multiple cases again, but we will see that it will be a bit easier this time, as there are only few cases crossing the 0. The first case for the non-negative paths was to go two steps down after executing the path before, if possible. That means that this time, we try to go two steps up at the very end. Because that means the path ascends by 4 at the end, that only works for $Y_{n-2} < -4$.

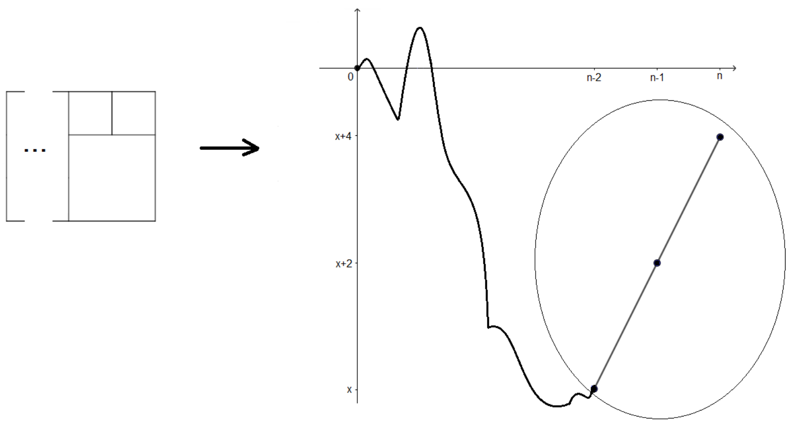

Figure 42: Recursive continuation for lower 2-squares in negative paths part 1/3

Which cases are left? Every $X_n = X_{n-1} - 2$ is covered with normal squares, and

$X_n = X_{n-1} + 2$ is also completely done with upper or lower 2-squares, depending on $X_{n-2} = X_n$ or $X_{n-2} = X_n - 4$, respectively. That means, the only possibility for the last step is, to have only 1 length, so it has to be $X_n = -1$ and $X_{n-1} = 0$, because the path has to end below 0. But for X_{n-1} to be 0, X_{n-2} has to be 1, which already fixes the last part of the path: the last two executed steps have to be two steps down from 1. Obviously, 1 cannot be reached by a path of length $n-2$, so the path before has to be modified again. We try to do the same reflections used in the non-negative case, just the other way around.

Looking at the previously executed paths with length $n-2$ again. Because all paths with $Y_{n-2} < -4$ were covered and Y_{n-2} is odd, it has to be $Y_{n-2} \in \{-1; -3\}$. To modify such a path so that the new path ends at 1, another component will be defined, an equivalent to $p(Y)$ for the non-negative case, but a bit more complicated. The reversion cannot happen at a certain point crossed, so the new variable has to be directly dependent on Y, or more accurately, on Y_{n-2} itself. Let $r(Y)$ be the last time point where the path was at a bigger value than at its end, $r(Y) := \max\{k \in \{0, 1, \ldots, n-3\} : Y_k\, Y_{n-2}\}$. Note that $r(Y) \geq 0$, because Y_{n-2} is always negative and $Y_0 = 0 > Y_{n-2}$, so it can always be used due to not being $-\infty$. Now, the path will be reflected from $Y_{r(Y)}$ to Y_{n-2} again, the modified path goes up when the original path goes down and vice versa.

For $Y_{n-2} = -1$, the path is reflected at the last time point where a 0 was crossed. At $r(Y)$, we do a step from 0 to 1, and every step after that goes only 1 up or down. It is not a real reflection because the original path is stretched by a factor of 2, but we still have the exact opposite steps.

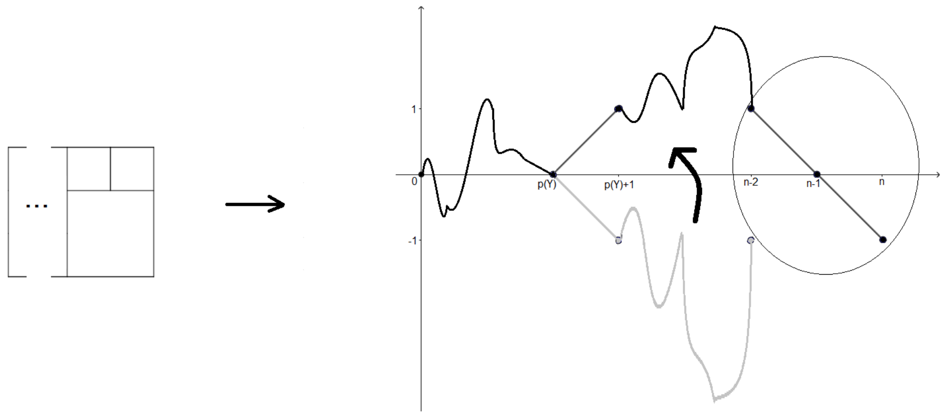

Figure 43: Recursive continuation for lower 2-squares in negative paths part 2/3

For $Y_{n-2} = -3$, the reflection happens at the last time -1 is reached. One step before $r(Y)$ has to be from 0 to -1, at $r(Y)$, we go from -1 to 1 and after that, there are again steps with height 1 instead of 2.

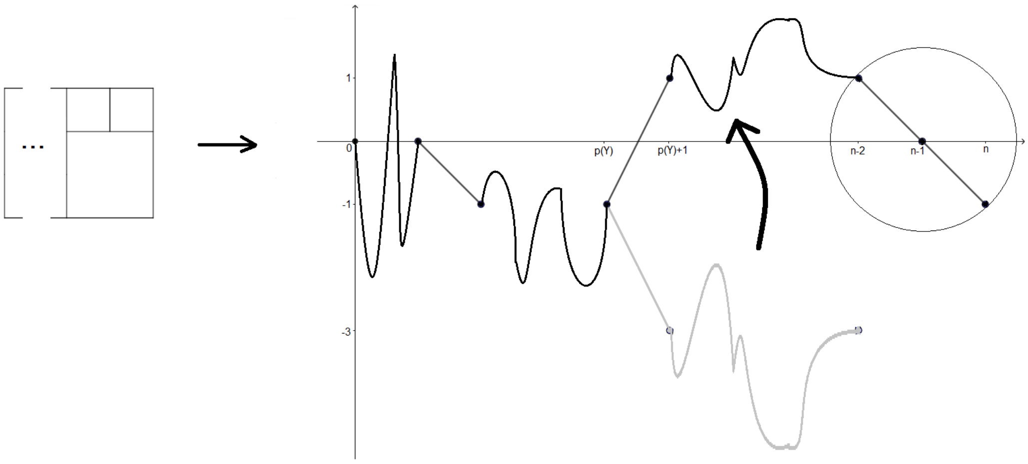

Figure 44: Recursive continuation for lower 2-squares in negative paths part 3/3

That ends the projection, as every case for the end of the previous path is covered, along with every last column possibility of the rectangle. The injectivity will be proven later this time. At first, the bijection will be shown again.

Again, we have an $n \times 3-$rectangle tiled into squares of length 1 and 2 and $X_0 = 0$. For $Y_0, Y_1, \dots, Y_j$ being the path obtained by the same rectangle without the last column if it has normal squares ($j = n-1$) or without the last two columns otherwise ($j = n-2$). Let further $r(Y)$ be the last time point where this previous path was at a bigger value than at time j, $r(Y) := \max\{k \in \{0, 1, \dots, j-1\} : Y_k\, Y_j\}$. For the new path $X_0, \dots, X_j$, let $k \in \{0, 1, \dots, j\}$.

$$X_k = \begin{cases} \frac{1-Y_k}{2} & \text{rectangle ends with lower 2-square, } k > r(Y), Y_{n-2} = -1 \\ \frac{-1-Y_k}{2} & \text{rectangle ends with lower 2-square, } k > r(Y), Y_{n-2} = -3 \\ Y_k & \text{otherwise} \end{cases}$$

Now it's down to the last step(s).

$$X_{n-1} = \begin{cases} X_{n-2} - 2 & \text{rectangle ends with upper 2-square} \\ Y_{n-1} & \text{rectangle ends with normal squares (already set)} \\ X_{n-2} + 2 & \text{rectangle ends with lower 2-square, } Y_{n-2} < -3 \\ 0 & \text{otherwise} \end{cases}$$

$$X_n = \begin{cases} X_{n-2} & \text{rectangle ends with upper 2-square} \\ X_{n-1} - 2 & \text{rectangle ends with normal squares} \\ X_{n-2} + 4 & \text{rectangle ends with lower 2-square, } Y_{n-2} < -3 \\ -1 & \text{otherwise} \end{cases}$$

To prove that this is actually a bijection, we will try to reverse this projection. Let $X_0, \ldots, X_n$ be a 1-2-path with $X_n < 0$. For $n = 1$, we have the empty rectangle.

$$\text{For } n > 1\text{, the rectangle ends with...} \begin{cases} \text{...normal squares} & X_n = X_{n-1} - 2. \\ \text{...an upper 2-square} & X_n = X_{n-2} = X_{n-1} + 2. \\ \text{...a lower 2-square} & \text{otherwise.} \end{cases}$$

Then, the rest of the path might have to be reversed again. Let $Y_0, Y_1, \ldots, Y_j$ be the path that will constructed for the rest of the rectangle, $j = n - 1$ if $X_n = X_{n-1} - 2$ and $j = n - 2$ otherwise. Let $k \in \{1, \ldots, n\}$ and $p(X)$ as in the non-negative case.

$$Y_k = \begin{cases} 1 - 2X_k & X_n = -1, X_{n-1} = 0, X_{n-2} = 1, p(X) = 0, k > p(X) \\ -1 - 2X_k & X_n = -1, X_{n-1} = 0, X_{n-2} = 1, p(X) = -1, k > p(X) \\ X_k & \text{otherwise} \end{cases}$$

For two different paths, if two of them get the same end for the rectangle, the previous paths are still different. The only case where this can go wrong is if $X_n = -1, X_{n-1} = 0$ and $X_{n-2} = 1$. But it doesn't, because the reversed paths are then different too, or they were different before $p(X)$.

With that, the bijection is complete, because we already know that the number of negative 1-2-paths with length n is equal to the number of $3 \times (n-1)$ tilings, and the remapping is injective. That is all we need to have a bijection.

Using the tilings, we now also have a bijection between positive 1-2-paths of length n and negative 1-2-paths of length $n + 1$ for all $n \geq 0$.

4.2 Open questions

From the known results, there are some following questions that can be raised.

Is there a reasonable limit of the $n \times 3$-rectangles?

That might or might not be the case. The thing is that only paths ending in non-

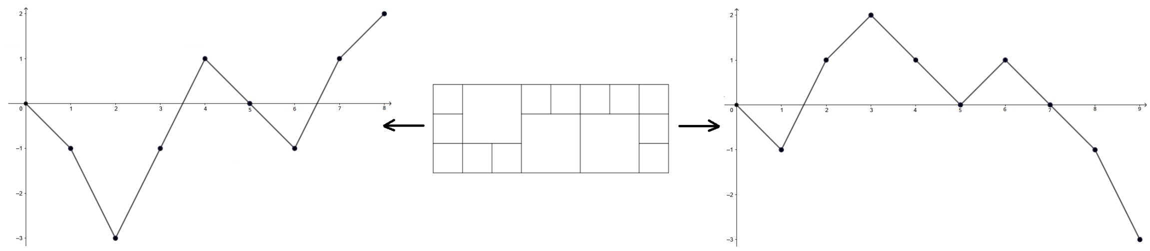

Figure 45: 1-2 random walks double bijection

negative range are considered (or paths that actually end in negative range, depending on which case is taken). To find a limit of one of these cases alone can be very hard, although they can be also taken together, because a bijection between the 1-2-paths of length n and $k \times 3$-rectangles with $k \in \{n-1\}$ exists. But the bijections considered here also can be different from the ones that are needed to answer such a question.

Are there other bijections between special random walks and rectangles of a similar form?

There are some cases that even look hard right from the beginning. For example, for the 1-3-random walk, a bijection is needed that covers exactly $3 \cdot 2^{n-2}$ paths for even and positive n and a few less for the odd cases. This is already a case where such a bijection could be impossible. It's very unclear if it's none, some or all other random walks that can be covered like that.

Does another 1-u-random walk also have a limit? If so, which one?

Let $u \in \mathbb{N}, u > 1, X_k = \sum_{i=1}^{k} Z_i$ and $P(Z_k = 1) = P(Z_k = -1) = \frac{1}{2}$ for $X_{k-1} \geq 0$ and $P(Z_k = u) = P(Z_k = -u) = \frac{1}{2}$ elsewhere. The approach would be the same as with the 1-2-random walk. The obvious idea is again a solution to a stochastic differential equation, with $X_0 = \mu = 0$ and $\nu(x) = 1$ for $x \geq 0$ and $\nu(x) = u$ for $x < 0$. stopping times τ_n for $n \in \mathbb{N}_0$ would be defined to embed the possible paths again, and the only conditioned mean stopping time differences without direct mean value 1 are those that start at value 0 or -1. The calculation of the mean values give

$$\mathbb{E}\tau_{0,-1,1} = \frac{1}{2}\left(\int_0^{-1}\int_0^y \frac{2}{u^2}dzdy + \int_0^1\int_0^y \frac{2}{1}dzdy\right) = \frac{1}{2}\left(\int_{-1}^0 \frac{2y}{u^2}dy + 1\right) = \frac{1}{2} + \frac{1}{2u^2}$$

and

$$\begin{aligned}
\mathbb{E}\tau_{-1,-u-1,u-1} &= \frac{1}{2}\left(\int_{-1}^{-u-1}\int_{-1}^{y}\frac{2}{u^2}dzdy + \int_{-1}^{u-1}\int_{-1}^{y}\frac{2}{\nu(z)^2}dzdy\right)\\
&= \frac{1}{2}\left(1+\int_{-1}^{0}\int_{-1}^{y}\frac{2}{u^2}dzdy + \int_{0}^{u-1}\left(\int_{-1}^{0}\frac{2}{u^2}dz + \int_{0}^{y}\frac{2}{1}dz\right)dy\right)\\
&= \frac{1}{2}\left(1+\frac{1}{u^2}+\int_{0}^{u-1}\frac{2}{u^2}+2ydy\right) = \frac{1}{2}\left(1+\frac{1}{u^2}+\frac{2u-2}{u^2}+(u-1)^2\right)\\
&= \frac{1}{2}\left(1+\frac{2u-1}{u^2}+u^2-2u+1\right) = 1+\frac{u^4-2u^3+2u-1}{2t^2}.
\end{aligned}$$

It would've been good to get $\frac{3}{2}-\frac{1}{2u^2}$ to have at least mean value 1 between these two conditioned stopping times. In this case, the mean value is $1+\frac{u^3-2u^2-u+2}{4u} = 1+\frac{(u-2)(u-1)(u+1)}{4u}$, which is only 1 when $u=2$. This was exactly the case for the 1-2-random walk. The next question is if it matters, since the stopping times themselves only have their specific probability. But looking at $u=3$ already, we notice that $P(X_2n=0) = P(X_{2n+1}=-1) = \frac{\binom{2n}{n}}{2^{n+1}}$ for $n\in\mathbb{N}$, which would've perfectly worked if $u^3-2u^2-u+2=0$. This does not work to prove $\mathbb{E}\frac{\tau_n}{n}\to 1$ in probability.

But that doesn't need to be the end. It might $P(X_n\in\{-1,0\})\in\mathcal{O}\left(\frac{\binom{2n}{n}}{4^n}\right)$ be true, and then we can take similar steps as already done before. The probabilities for values close to 0 in general have to be small enough to make a geometric approach possible. And the variances for the specific stopping times have to be finite. Then the odds of the convergence of the random walk to a weak solution of the stochastic differential equation are at the very least existing.

This can be driven even further. What if $u>0$, but u does not have to be an integer? Then the same stochastic differential equation maybe can be used again. For the solution of that, $P(-u<X_i<1)$ has to be considered for the stopping times. A good side result would again be $P(-u<X_n<1)\in\mathcal{O}\left(\frac{1}{4^n}\binom{2n}{n}\right)$ for the stopping times, which again should have finite variance. The case of an u-v-random walk, $u,v>0$, with $P(Z_k=u) = P(Z_k=-u) = \frac{1}{2}$ for $X_k\geq 0$ and $P(Z_k=v) = P(Z_k=-v) = \frac{1}{2}$ for $X_k<0$ could then just be transferred from the 1-$\frac{v}{u}$-random walk.

A very far-fetched version could be this one: Let $f:\mathbb{R}\to\mathbb{R}^+$ be any real-valued function with positive values. Let $(X_i)_{i\in\mathbb{N}_0}$ be a random walk with independent $(Z_k)_{k\in\mathbb{N}}$ with $X_k = X_{k-1}+Z_k$ and $P(Z_k=f(X_{k-1})) = P(Z_k=-f(X_{k-1})) = \frac{1}{2}$. That is also the reason why f has positive values, because negative values wouldn't change anything at these probabilities and can therefore be ignored, and 0 as a value could lead to the random walk standing still forever. What could be a convergence stochastic process of this random walk? It might be the case that $P(a < X_n < b) \in \mathcal{O}\left(\frac{1}{4^n}\binom{2n}{n}\right)$ for any $0 < a < b$, and all stopping times that are possible have finite variance. And that might lead to a convergence of $\left(\frac{X_{nt}}{\sqrt{n}}\right)_{t\geq 0}$ to a weak solution $(M_t)_{t\geq 0}$ of the following stochastic differential equation:

$$\begin{aligned} dY_t &= \mu(t, Y_t)\,dt + \nu(t, Y_t)\,dB_t, \\ \mu &= Y_0 = 0, \\ \nu(t,x) &= \begin{cases} \lim\limits_{y\to\infty} f(y) & x > 0 \\ \lim\limits_{y\to-\infty} f(y) & x < 0 \\ f(0) & x = 0 \end{cases} \end{aligned}$$

Of course, those two limits must exist and be finite, if that's not the case, the limit might not exist. If one of those limits is 0, the Engelbert-Schmidt-Conditions in Proposition 3.1 are not fulfilled. Also, for every $x \in \mathbb{R}$ and $x_1, x_2, \ldots$ with $\lim\limits_{n\to\infty} x_n = x$, the partial limit of the function values $\lim\limits_{n\to\infty} f(x_n)$ must not be 0 to avoid a convergence to a certain value. Although, if the random walk itself converges to a finite value, $P\left(\lim\limits_{n\to\infty} X_n \in \{c_1, c_2, \ldots, c_m\}\right)$ for $m \in \mathbb{N}$ and some $c_1, \ldots, c_m \in \mathbb{R}$, then the limit always converges to 0, so this case can be considered too. But for now, it should just be set that $f : \mathbb{R} \to [a, b]$ for some $a, b > 0$ with $a \leq b$ to be safe. Another decision here was made to let $\nu(0) \neq 0$ to assure that the stochastic process can actually exit 0.

This is just a speculative analysis. If it is even true, the proof of this might need the work of the best mathematicians of the century.

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Declaration

Independence declaration

I hereby declare that the thesis was written on my own and only with use of the mentioned sources. It hasn't been used as a means of another exam yet or was published in English or any another language.

The author does not have something against the use of this paper for public purposes.

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Jena, November 11th, 2020