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Computing Tutte paths
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Abstract

Tutte paths are one of the most successful tools for attacking problems on long cycles in planar graphs. Unfortunately, results based on them are non-constructive, as their proofs inherently use an induction on overlapping subgraphs and these overlaps prevent any attempt to bound the running time by a polynomial.

For special cases however, computational results of Tutte paths are known: For 4-connected planar graphs, Tutte paths are in fact Hamiltonian paths and Chiba and Nishizeki [5] showed how to compute such paths in linear time. For 3-connected planar graphs, Tutte paths have a significantly more complicated structure, and it has only recently been shown that they can be computed in polynomial time [24]. However, Tutte paths are defined for general 2-connected planar graphs and this is what most applications need. In this unrestricted setting, no computational results for Tutte paths are known.

We give the first efficient algorithm that computes a Tutte path (in this unrestricted setting). One of the strongest existence results about such Tutte paths is due to Sanders [23], which allows one to prescribe the end vertices and an intermediate edge of the desired path. Encompassing and strengthening all previous computational results on Tutte paths, we show how to compute such a special Tutte path efficiently. Our method refines both, the existence results of Thomassen [29] and Sanders [23], and avoids that the subgraphs arising in the inductive proof intersect in more than one edge by using a novel iterative decomposition along 2-separators. Finally, we show that our algorithm runs in time $O(n^2)$.

Introduction

The question whether a graph $G = (V,E)$ is Hamiltonian, i.e. contains a cycle of length $n := |V|$, is among the most fundamental graph problems. For planar graphs and graphs embeddable on higher surfaces, Tutte paths have proven to be one of the most successful tools for attacking Hamiltonicity problems and problems on long cycles. For this reason, there is a wealth of existential results in which Tutte paths serve as main ingredient; in chronological order, these are [31, 29, 26, 4, 22, 23, 27, 33, 16, 28, 11, 13, 18, 21, 20, 17, 24, 7, 2].

As a historical starting point to these results, Whitney [32] proved that every 4-connected maximal planar graph is Hamiltonian. Tutte extended this to arbitrary 4-connected planar graphs by showing that every 2-connected planar graph $G$ contains a Tutte path [30, 31].
(for a definition of Tutte paths, see Section 2). Thomassen [29] in turn proved the following generalization, which also implies that every 4-connected planar graph is Hamiltonian-connected, i.e. contains a path of length \( n - 1 \) between any two vertices. For a plane graph \( G \), let \( C_G \) be its outer face.

**Theorem 1** (Thomassen [29]). Let \( G \) be a 2-connected plane graph, \( x \in V(C_G) \), \( \alpha \in E(C_G) \) and \( y \in V(G) - x \). Then \( G \) contains a Tutte path from \( x \) to \( y \) through \( \alpha \).

Sanders [23] then generalized Thomassen’s result further by allowing to choose the start vertex \( x \) of the Tutte path arbitrarily.

**Theorem 2** (Sanders [23]). Let \( G \) be a 2-connected plane graph, \( x \in V(G) \), \( \alpha \in E(C_G) \) and \( y \in V(G) - x \). Then \( G \) contains a Tutte path from \( x \) to \( y \) through \( \alpha \).

On top of the above series of fundamental results, Tutte paths have been used in two research branches: While the first deals with the existence of Tutte paths on graphs embeddable on higher surfaces [26, 3, 27, 33, 28, 17], the second [15, 9, 3, 10, 16, 11, 19] investigates generalizations or specializations of Hamiltonicity such as \( k \)-walks, long cycles and Hamiltonian connectedness.

Unfortunately, in all the results mentioned so far, very little is known about the complexity of finding a Tutte path. This is crucial, as the task of finding Tutte paths is almost always the only reason that hinders the computational tractability of the problem. The main obstruction so far is that Tutte paths are found by decomposing the input graph into overlapping subgraphs, on which induction is applied. Although this is enough to prove existence results, these overlapping subgraphs do not allow to bound the running time polynomially (as argued in [12, 24]). The only known computational results on Tutte paths [12, 1, 5, 21, 24] deal therefore with very restricted settings, such as the case that Tutte paths are just Hamiltonian paths: While it is known how to compute Tutte paths for planar 4-connected graphs [5] efficiently (in which case Tutte paths are just Hamiltonian paths), for planar 3-connected graphs a first polynomial-time algorithm was only recently shown [24].

However, no efficient algorithm is published so far that computes Tutte paths in general 2-connected planar graphs (i.e. the ones of Theorem 1 or 2). In fact, the claimed algorithmic results in [26, 27] require polynomial running times for computing such Tutte paths, without giving proofs that such algorithms exist. Given the subtlety of the arguments inherent to Tutte paths, we feel that giving such a proof is necessary. Indeed, history shows that even for the much easier setting that Tutte paths are Hamiltonian paths, an existence result for Tutte paths has been incorrectly claimed to imply a polynomial-time algorithm [29, 4] (again, due to overlapping subgraphs). For finding Tutte paths in certain restrictions of 2-connected and 3-connected planar graphs, the related results in [22, 17] claim polynomial running times as well.

**Our Results**

Our motivation is two-fold. First, we want to make Tutte paths accessible to algorithms. We will show that Tutte paths can be computed in time \( O(n^2) \). This has impact on almost all the applications using Tutte paths listed above.

For several of them, e.g. [26, 22, 27, 17], we immediately obtain polynomial-time algorithms where no efficient algorithms were published before. In addition, Tutte paths were also used in [7, 8] to show that every essentially 4-connected polyhedral graph contains a cycle of length proportional to \( n \). As the existence proofs in this paper are constructive, our result directly implies a efficient (in fact, an \( O(n^2) \)-time) algorithm for the computation of these
cycles. Furthermore, [2] showed that every 3-connected planar graph having at most three 3-separators is Hamiltonian. If a 3-connected planar graph contains at most one 3-separator, our algorithm shows that a Hamiltonian cycle can be computed in $O(n^2)$ time, by using a suitable choice of the intermediate edge $\alpha$.

Second, we aim for computing the strongest possible known variant of Tutte paths, encompassing the many incremental improvements on Tutte paths made over the years. We will therefore develop an algorithm for Sanders’s existence result [23], which was proven to be best possible in many aspects. Sanders result has also an immediate extension to connected planar graphs [20], which can be computed simply and efficiently from our result by using block-cut trees.

We will first give a decomposition that refines the original ones used for Theorems 1 and 2, and allows to decompose $G$ into graphs that pairwise intersect in at most one edge. We then show that this small overlap does not prevent us from achieving a polynomial running time. All graphs will be simple. We proceed by showing how this decomposition can be computed efficiently in order to find the Tutte paths of Theorem 2. Our main result is hence the following, giving the first polynomial-time algorithm for computing Tutte paths.

**Theorem 3.** Let $G$ be a 2-connected plane graph, $x \in V(G)$, $\alpha \in E(C_G)$ and $y \in V(G) - x$. Then a Tutte path of $G$ from $x$ to $y$ through $\alpha$ can be computed in time $O(n^2)$.

Section 3 presents the decomposition with small overlap that proves the existence of Tutte paths. On the way to our main result, we give full algorithmic counterparts of the approaches of Thomassen and Sanders; for example, we describe small overlap variants of Theorem 1 and of the Three Edge Lemma [26, 22], which was used in the purely existential result of Sanders [23] as a black box.

**Our Techniques**

We broadly follow the idea of [5] and construct a Tutte path that is based on certain 2-separators of the graphs constructed during our decomposition. This depends on many structural properties of the given graph. In [5], the necessary properties follow from the restriction to the class of internally 4-connected planar graphs, the restriction on the endpoints of the desired Tutte path, and the fact that the Tutte paths computed recursively are actually Hamiltonian. In contrast, here we give new insights into the much wilder structure of Tutte paths of 2-connected planar graphs, allow $x, y \notin C_G$, and hence extend this technique. We show that based on the prescribed vertices and edge, there are always unique non-interlacing 2-separators that are contained in every possible Tutte path of the given graph. We then use this set of 2-separators to iteratively construct a preliminary Tutte path and use this iterative procedure to avoid overlaps of more than one edge in the decomposition of the input graph.

## 2 Preliminaries

We assume familiarity with standard graph theoretic notations as in [6]. Let $\deg(v)$ be the degree of a vertex $v$. We denote the subtraction of a graph $H$ from a graph $G$ by $G - H$, and the subtraction of a vertex or edge $x$ from $G$ by $G - x$.

A $k$-separator of a graph $G = (V, E)$ is a subset $S \subseteq V$ of size $k$ such that $G - S$ is disconnected. A graph $G$ is $k$-connected if $|V| > k$ and $G$ contains no $(k - 1)$-separator. For a path $P$ and two vertices $x, y \in P$, let $xPy$ be the smallest subpath of $P$ that contains $x$ and $y$. For a path $P$ from $x$ to $y$, let $\text{inner}(P) := V(P) - \{x, y\}$ be the set of its inner vertices. Paths that intersect pairwise at most at their endvertices are called independent.
A connected graph without a 1-separator is called a block. A block of a graph is an inclusion-wise maximal subgraph that is a block. Every block of a graph is thus either 2-connected or has at most two vertices. It is well-known that the blocks of a graph partition its edge-set. A graph is called a chain of blocks if it consists of blocks $B_1, B_2, \ldots, B_k$ such that $V(B_i) \cap V(B_{i+1})$, $1 \leq i < k$, are pairwise distinct 1-separators of $G$ and $G$ contains no other 1-separator. In other words, a chain of blocks is a graph, whose block-cut tree [14] is a path.

A plane graph is a planar embedding of a graph. Let $C$ be a cycle of a plane graph $G$. For two vertices $x, y$ of $C$, let $xCy$ be the clockwise path from $x$ to $y$ in $C$. For a vertex $x$ and an edge $e$ of $C$, let $xCe$ be the clockwise path in $C$ from $x$ to the endvertex of $e$ such that $e \notin xCe$ (define $eCx$ analogously). Let the subgraph of $G$ inside $C$ consist of $E(C)$ and all edges that intersect the open set inside $C$ into which $C$ divides the plane. For a plane graph $G$, let $C_G$ be its outer face.

A central concept for Tutte paths is the notion of $H$-bridges (see [31] for some of their properties): For a subgraph $H$ of a 2-connected plane graph $G$, an $H$-bridge of $G$ is either an edge that has both endvertices in $H$ but is not itself in $H$ or a component $K$ of $G - H$ together with all edges (and the endvertices of these edges) that join vertices of $K$ with vertices of $H$. An $H$-bridge is called trivial if it is just one edge. A vertex of an $H$-bridge $L$ is an attachment of $L$ if it is in $H$, and an internal vertex of $L$ otherwise. An outer $H$-bridge of $G$ is an $H$-bridge that contains an edge of $C_G$.

A Tutte path (Tutte cycle) of a plane graph $G$ is a path (a cycle) $P$ of $G$ such that every outer $P$-bridge of $G$ has at most two attachments and every $P$-bridge at most three attachments. In most of the cases we consider, $G$ will be 2-connected, so that every $P$-bridge has at least two attachments. For vertices $x, y$ and an edge $\alpha \in C_G$, let an $x$-$\alpha$-$y$-path be a Tutte path from $x$ to $y$ that contains $\alpha$. An $x$-$y$-path is an $x$-$\alpha$-$y$-path for an arbitrarily chosen edge $\alpha \in C_G$.

3 Decomposition with Small Overlap

After excluding several easy cases of the decomposition, we prove Thomassen’s Theorem 1 constructively and then show how to use this for proving a proof of the Three Edge Lemma. The Three Edge Lemma, in turn, allows us to give a constructive proof of Sander’s Theorem 2 in which only small overlaps occur in the induction. Due to space constraints, we have to omit this proof; it however derived from [23] in a similar way as Theorem 1 from [29].

We will use induction on the number of vertices. In all proofs about Tutte paths of this section, the induction base is a triangle, in which the desired Tutte path can be found trivially; thus, we will assume in these proofs by the induction hypothesis that graphs with fewer vertices contain Tutte paths. All graphs in the induction will be simple.

The following sections cover different cases of the induction steps of the three statements to prove, starting with some easy cases for which a decomposition into edge disjoint subgraphs was already given [29]. For the remainder of the article, let $G$ be a simple plane 2-connected graph with outer face $C_G$ and let $x \in V(G)$, $\alpha \in E(C_G)$ and $y \in V(G) - x$. If $\alpha = xy$, the desired path is simply $xy$; thus, assume $\alpha \neq xy$. Since $G$ is 2-connected, $C_G$ is a cycle.

3.1 The Easy Cases

We say that $G$ is decomposable into $G_L$ and $G_R$ if it contains subgraphs $G_L$ and $G_R$ such that $G_L \cup G_R = G$, $V(G_L) \cap V(G_R) = \{c, d\}$, $x \in V(G_L)$, $\alpha \in E(G_R)$, $V(G_L) \neq \{x, c, d\}$ and $V(G_R) \neq \{c, d\}$ (or the analogous setting with $y$ taking the role of $x$) (see Figure 1). In
particular, $G_L \neq \{c,d\}$, even if $x \in \{c,d\}$. Hence $\{c,d\}$ is a 2-separator of $G$. There might exist multiple pairs $(G_L, G_R)$ into which $G$ is decomposable; we will always choose a pair that minimizes $|V(G_R)|$. Note that $G_R$ intersects $C_G$ (for example, in $\alpha$), but $G_L$ does not have to intersect $C_G$. In [29], it was shown that every decomposable graph $G$ contains a Tutte path, without using recursion on overlapping subgraphs.

\begin{lemma}[[29]]\right.\ If $G$ is decomposable into $G_L$ and $G_R$, then $G$ contains an $x$-$\alpha$-$y$-path.\end{lemma}

Even if $G$ is not decomposable into $G_L$ and $G_R$, $G$ may contain other 2-separators $\{c,d\}$ that allow for a similar reduction as in Lemma 4 (for example, when modifying its prerequisites to satisfy $\{x,\alpha,y\} \subseteq G_R - \{c,d\}$).

\begin{lemma}[[29]]\right.\ Let $\{c,d\}$ be a 2-separator of $G$ and let $J$ be a $\{c,d\}$-bridge of $G$ having an internal vertex in $C_G$ such that $x$, $y$ and $\alpha$ are not in $J$. Then $G$ contains an $x$-$\alpha$-$y$-path.\end{lemma}

### 3.2 Proof of Theorem 1

We now prove that $G$ contains a Tutte path from $x \in V(C_G)$ to $y \in V(G) - x$ through $\alpha \in E(C_G)$. For simplicity, if $y$ is not in $V(C_G)$ but has degree two and both of its neighbors are in $V(C_G)$, then we change the embedding of $G$ (and therefore $C_G$) such that $y$ belongs to the outer face. If Lemma 4 or 5 can be applied, we obtain such a Tutte path directly, so assume their prerequisites are not met. Let $l_\alpha$ be the endvertex of $\alpha$ that appears first when we traverse $C_G$ in clockwise order starting from $x$, and let $r_\alpha$ be the other endvertex of $\alpha$. If $y \in xC_Gl_\alpha$, we interchange $x$ and $y$ (this does not change $l_\alpha$); hence, we have $y \notin xC_Gl_\alpha$. If $y = r_\alpha$, we mirror the embedding such that $y$ becomes $l_\alpha$ and proceed as in the previous case; hence, $y \notin xC_Gr_\alpha$.

We define two paths $P$ and $Q$ in $G$, whose union will, step by step, be modified into a Tutte path of $G$. Let $Q := xC_Gl_\alpha$ and let $H := G - V(Q)$; in particular, $y \notin Q$ and, if $x$ is an endvertex of $\alpha$, $Q = \{x\}$. Since $G$ is not decomposable, we have $\deg(r_\alpha) \geq 3$, as otherwise the neighborhood of $r_\alpha$ would be the 2-separator of such a decomposition. Since $\deg(r_\alpha) \geq 3$, $r_\alpha$ is incident to an edge $e \notin C_G$ that shares a face with $\alpha$. Let $B_1$ be the block of $H$ that contains $e$. It is straightforward to prove the following about $B_1$ (see Thomassen [29]), which shows that every vertex of $C_G$ is either in $Q$ or in $B_1$.

\begin{lemma}[[29]]\right.\ $B_1$ contains $C_G - V(Q)$ and is the only block of $H$ containing $r_\alpha$.\end{lemma}

Consider a component $A$ of $H$ that does not contain $B_1$. Then the neighborhood of $A$ in $G$ is in $Q$ and must contain a 2-separator of $G$ due to planarity. Hence, either $y \in A$ and we can apply Lemma 4 or $y \notin A$ and we can apply Lemma 5. Since both contradictions our assumptions, $H$ is connected and contains $B_1$ and $y$. Let $K$ be the minimal plane chain of blocks $B_1, \ldots, B_l$ of $H$ that contains $B_1$ and $y$ (hence, $y \in B_l$). Let $v_i$ be the intersection of $B_i$ and $B_{i+1}$ for $1 \leq i \leq l - 1$; in addition, we set $v_0 := r_\alpha$ and $v_l := y$. 

![Figure 1](image-url)

**Figure 1.** a) shows a graph $G$ that is decomposable into $G_L$ and $G_R$. The figures b) to d) show the graphs $G_L', G_R'$ and $G_R''$ (in this order) that are constructed to process $G$ in [29].
Consider any \((K \cup C_G)\)-bridge \(J\). Since Lemma 5 cannot be applied, \(J\) has an attachment \(v_J \in K\). Further, \(J\) cannot have two attachments in \(K\), as this would contradict the maximality of the blocks in \(K\). Let \(C(J)\) be the shortest path in \(C_G\) that contains all vertices in \(J \cap C_G\) and does not contain \(r_a\) as inner vertex (here, \(r_a\) serves as a reference vertex of \(C_G\) that ensures that the paths \(C(J)\) are chosen consistently on \(C_G\)). Let \(l_J\) be the endvertex of \(C(J)\) whose counterclockwise incident edge in \(C_G\) is not in \(C(J)\) and let \(r_J\) be the other endvertex of \(C(J)\).

### 3.2.1 Decomposing along Maximal 2-Separators

At this point we will deviate from the original proof of Theorem 1 in [29], which continues with an induction on every block of \(K\) that leads to overlapping subgraphs in a later step of the proof. Instead, we will show that a \(v_0-v_i\)-path \(P\) of \(K\) can be found iteratively such that the graphs in the induction have only small overlap.

For every block \(B_i \neq B_1\) of \(K\), we choose an arbitrary edge \(\alpha_i = l_{\alpha_i}, r_{\alpha_i}\) in \(C_{B_i}\). In \(B_1\) we choose \(\alpha_1\) such that \(\alpha_1\) is incident to the endvertex of \(C_{B_1} \cap C_G\) that is not \(r_a\). As done for \(G\), we may assume for every \(B_i\) that \(l_{\alpha_i}\) is the endvertex of \(\alpha_i\) that is contained in \(v_{i-1}C_{B_i}\alpha_i\) and that \(v_i \notin v_{i-1}C_{B_i}r_{\alpha_i}\) and (by mirroring the planar embedding and interchanging \(v_i\) and \(v_{i-1}\) if necessary). However, unlike \(G\), every \(B_i\) may satisfy the prerequisites of Lemmas 4 and 5. By the induction hypothesis of Theorem 1, \(B_i\) contains a \(v_{i-1}C_{B_i}\alpha_i\)-path \(P_i\). In [29], the outer \(P_i\)-bridges of \(B_i\) are not only being processed during this induction step, but also in a later induction step when modifying \(Q\). We avoid such overlapping subgraphs by using a new iterative structural decomposition of \(B_i\) along certain 2-separators on \(C_{B_i}\). This decomposition allows us to construct \(P_i\) iteratively such that the outer \(P_i\)-bridges of \(B_i\) are not part of the induction applied on \(B_i\). Eventually, \(P := \bigcup_{1 \leq i \leq l} P_i\) will be the desired \(v_0-v_l\)-path of \(K\).

The outline is as follows. After explaining the basic split operation that is used by our decomposition, we give new insights into the structure of the Tutte paths \(P_i\) of the blocks \(B_i\). These are used in Section 3.2.2 to define the iterative decomposition of every block \(B_i\) into a modified block \(\eta(B_i)\), which will in turn allow to compute every \(P_i\) step-by-step. This gives the first part \(P\) of the desired Tutte path \(x-\alpha-y\) of \(G\). Subsequently, we will show how the remaining path \(Q\) can be modified to obtain the second part.

For a 2-separator \(\{c, d\} \subseteq C_B\) of a block \(B\), let \(B_{cd}^+\) be the \(\{c, d\}\)-bridge of \(B\) that contains \(cC_Bd\) and let \(B_{cd}^-\) be the union of all other \(\{c, d\}\)-bridges of \(B\) (note that \(B_{cd}^+\) contains the edge \(cd\) if and only if \(B_{cd}^+\) is trivial). For a 2-separator \(\{c, d\} \subseteq C_B\), let splitting off \(B_{cd}^+\) (from \(B\)) be the operation that deletes all internal vertices of \(B_{cd}^+\) from \(B\) and adds the edge \(cd\) if \(cd\) does not already exist in \(B\). Our decomposition proceeds by iteratively splitting off bridges \(B_{cd}^+\) from the blocks \(B_i\) of \(K\) for suitable 2-separators \(\{c, d\} \subseteq C_{B_i}\) (we omit the subscript \(i\) in such bridges \(B_{cd}^+\), as it is determined by \(c\) and \(d\)). The following lemma restricts these 2-separators to be contained in specific parts of the outer face.

**Lemma 7.** Let \(P\) be a Tutte path of a block \(B\) such that \(P\) contains an edge \(\alpha'\) and two vertices \(a, b \in C_B\). Then every outer \(P\)-bridge \(J\) of \(B\) has both attachments in \(aC_Bb\) or both in \(bC_Ba\). If additionally \(J\) is non-trivial and \(P' \neq \alpha'\), the attachments of \(J\) form a 2-separator of \(B\).

**Proof.** Let \(e\) be an edge in \(J \cap C_B\) and assume without loss of generalization that \(e \in aC_Bb\). Let \(c\) and \(d\) be the last and first vertices of the paths \(aC_Be\) and \(eC_Bb\), respectively, that are contained in \(P\) (these exist, as \(a\) and \(b\) are in \(P\)). Then \(J\) has attachments \(c\) and \(d\) and no further attachment, as \(P\) is a Tutte path. This gives the first claim. For the second claim, let \(z\) be an internal vertex of \(J\). Since \(P' \neq \alpha'\), \(P'\) contains a third vertex \(e \notin \{a, b\}\). As \(e\) is not contained in \(J\), \(\{c, d\}\) separates \(z\) and \(e\) and is thus a 2-separator of \(B\).
For every block $B_i \neq B_1$ of $K$, let the boundary points of $B_i$ be the vertices $v_i, l_{\alpha_i}, r_{\alpha_i}, v_i$ and let the boundary parts of $B_i$ be the inclusion-wise maximal paths of $C_{B_i}$ that do not contain any boundary point as inner vertex (see Figure 2a; note that boundary parts may be single vertices). Hence, every boundary point will be contained in any possible $v_i-1, l_{\alpha_i}, r_{\alpha_i}, v_i$-path $P_i$, and there are exactly four boundary parts, one of which is $\alpha_i$. Now, if $P_i \neq \alpha_i$, applying Lemma 7 for all boundary points $a, b \in \{v_i-1, l_{\alpha_i}, r_{\alpha_i}, v_i\}$ and $\alpha' := \alpha_i$ implies that the two attachments of every outer non-trivial $P_i$-bridge of $B_i$ form a 2-separator that is contained in one boundary part of $B_i$. For this reason, our decomposition will split off only 2-separators that are contained in boundary parts.

In principle, we will do the same for the block $B_l$. If $v_l \in C_{B_l}$, we define the boundary points of $B_l$ just as before for $i < l$. However, $B_l$ is special in the sense that $v_l$ may not be in $C_{B_l}$. Then we have to ensure that we do not loose $v_l$ when splitting off a 2-separator, as $v_l$ is supposed to be contained in $P_l$ (see Figure 2b). To this end, consider for $v_l \notin C_{B_l}$ the 2-separator $\{w_1, w_p\} \subseteq C_{B_l}$ of $B_l$ such that $B^+_{w_1, w_p}$ contains $v_l$, the path $w_1C_{B_l}w_p$ is contained in one of the paths in $\{v_i-1P_i\alpha_i, \alpha_iC_{B_l}v_i\}$ and $w_1C_{B_l}w_p$ is of minimal length if such a 2-separator exists. The restriction to these three parts of the boundary is again motivated by Lemma 7: If $P_i \neq \alpha_i$ and there is an outer non-trivial $P_i$-bridge of $B_l$, its two attachments are in $P_l$ and thus we only have to split off 2-separators that are in one of these three paths to avoid these $P_l$-bridges in the induction. If the 2-separator $\{w_1, w_p\}$ exists, let $w_1, \ldots, w_p$ be the $p \geq 2$ attachments of the $w_1C_{B_l}w_p$-bridge of $B_l$ that contains $v_l$, in the order of appearance in $w_1C_{B_l}w_p$; otherwise, let for notational convenience $w_1 := \cdots := w_p := l_{\alpha_i}$.

In the case $v_l \notin C_{B_l}$, let the boundary points of $B_l$ be $v_l-1, l_{\alpha_i}, r_{\alpha_i}, w_1, \ldots, w_p$ and let the boundary parts of $B_l$ be the inclusion-wise maximal paths of $C_{B_l}$ that do not contain any boundary point as inner vertex.

**Lemma 8.** If the 2-separator $\{w_1, w_p\}$ exists, it is unique and every $v_{i-1}\alpha_i, v_i$-path $P_i$ of $B_l$ contains the vertices $w_1, \ldots, w_p$.

**Proof.** Let $J \subseteq B^+_{w_1, w_p}$ be the $w_1C_{B_l}w_p$-bridge of $B_l$ that contains $v_l$ and has attachments $w_1, \ldots, w_p$. For the first claim, assume to the contrary that there is a 2-separator $\{w'_1, w'_p\} \neq \{w_1, w_p\}$ of $B_l$ having the same properties as $\{w_1, w_p\}$. By the connectivity of $J$ and the property that restricts $\{w'_1, w'_p\}$ to the three parts of the boundary of $B_l$, $\{w'_1, w'_p\}$ may only split off a subgraph containing $v_l$ if $w_1C_{B_l}w_p \subseteq w'_1C_{B_l}w'_p$. This however contradicts the minimality of the length of $w'_1C_{B_l}w'_p$.

For the second claim, let $P_i$ be any $v_{i-1}\alpha_i, v_i$-path of $B_l$. Assume to the contrary that $w_j \notin P_i$ for some $j \in \{1, \ldots, p\}$. Then $w_j$ is an internal vertex of an outer $P_l$-bridge $J'$ of $B_l$. By Lemma 7, both attachments of $J'$ are in $C_{B_l}$. However, since $J$ contains a path from $w_j \notin P_i$ to $v_l \in P_i$ in which only $w_j$ is in $C_{B_l}$, at least one attachment of $J'$ is not in $C_{B_l}$, which gives a contradiction.
Lemma 8 ensures that the boundary points of any $B_i$ are contained in every Tutte path $P_i$ of $B_i$. Every block $B_i \neq B_j$ has exactly four boundary parts and $B_i$ has at least three boundary parts (three if $v_i \notin C_{B_i}$ and $\{w_1, w_p\}$ does not exist), some of which may have length zero. For every $1 \leq i \leq l$, the boundary parts of $B_i$ partition $C_{B_i}$, and one of them consists of $\alpha_i$. This implies in particular that $B_i$ has at least two boundary parts of length at least one unless $B_i = \alpha_i$. We need some notation to break symmetries on boundary parts. For a boundary part $Z$ of a block $B$, let $\{c, d\}^* \subseteq Z$ denote two elements $c$ and $d$ (vertices or edges) such that $cC_{B}d$ is contained in $Z$ (this notation orders $c$ and $d$ consistently to the clockwise orientation of $C_{B}$); if $cC_{B}d$ is contained in some boundary part of $B$ that is not specified, we just write $\{c, d\}^* \subseteq C_{B}$.

We now define which 2-separators are split off in our decomposition. Let a 2-separator $\{c, d\}^* \subseteq C_{B}$ of $B$ be maximal in a boundary part $Z$ of $B$ if $\{c, d\} \subseteq Z$ and $Z$ does not contain a 2-separator $\{c', d'\}$ of $B$ such that $cC_{B}d \subseteq c'C_{B}d'$. Let a 2-separator $\{c, d\}^* \subseteq C_{B}$ of $B$ be maximal if $\{c, d\}^*$ is maximal with respect to at least one boundary part of $B$. Hence, every maximal 2-separator is contained in a boundary part, and 2-separators that are contained in a boundary part are maximal if they are not properly “enclosed” by other 2-separators on the same boundary part.

Let two maximal 2-separators $\{c, d\}^*$ and $\{c', d'\}^*$ of $B$ interlace if $\{c, d\} \cap \{c', d'\} = \emptyset$ and their vertices appear in the order $c, c', d, d'$ or $c', c, d', d$ on $C_{B}$ (in particular, both 2-separators are contained in the same boundary part of $B$). In general, maximal 2-separators of a block $B_i$ of $K$ may interlace; for example, consider the two maximal 2-separators when $B_i$ is a cycle on four vertices in which $v_{i-1}$ and $v_i$ are adjacent. However, the following lemma shows that such interlacing is only possible for very specific configurations.

Lemma 9. Let $\{c, d\}^*$ and $\{c', d'\}^*$ be interlacing 2-separators of $B_i$ in a boundary part $Z$ such that $c' \in cC_{B}d$ and at least one of them is maximal. Then $d'C_{B}c = v_{i-1}v_i = \alpha_i$.

Proof. Since $\{c, d\}$ is a 2-separator, $B_i - \{c, d\}$ has at least two components. We argue that there are exactly two. Otherwise, $B_i - \{c, d\}$ has a component that contains the inner vertices of a path $P'$ from $c$ to $d$ in $B_i - (C_{B_i} - \{c, d\})$. Then $B_i - \{c', d'\}$ has a component containing $(P' \cup C_{B_i}) - \{c', d'\}$ and no second component, as this would contain the inner vertices of a path from $c'$ to $d'$ in $B_i - ((P' \cup C_{B_i}) - \{c', d'\})$, which does not exist due to planarity. Since this contradicts that $\{c', d'\}$ is a 2-separator, we conclude that $B_i - \{c, d\}$, and by symmetry $B_i - \{c', d'\}$, have exactly two components.

By the same argument, $inner(cC_{B}d)$ and $inner(dC_{B}c)$ are contained in different components of $B_i - \{c, d\}$ and the same holds for $inner(c'C_{B}d')$ and $inner(d'C_{B}c')$ in $B_i - \{c', d'\}$. Hence, the component of $B_i - \{c, d\}$ that contains $inner(cC_{B}d) \neq \emptyset$ does not intersect $inner(dC_{B}c)$. If $inner(dC_{B}c) \neq \emptyset$, this implies that $\{c, d\} \subseteq Z$ is a 2-separator of $B_i$, which contradicts the maximality of $\{c, d\}$ or of $\{c', d'\}$. Hence, $inner(dC_{B}c) = \emptyset$, which implies that $d'C_{B}c$ is an edge. As $Z$ is not an edge, $d'C_{B}c = \alpha_i$. Since $c$ and $d'$ are the only boundary points of $B_i$, either $\{c, d'\} = \{v_{i-1}, v_i\}$ or $B_i = B_i, v_i \notin C_{B_i}, \{c, d'\} = \{v_{i-1}, w_2\}, v_{i-1} - w_1$ and $w_2 = w_p$. However, the latter case is impossible, as then $\{c, d'\}$ would be a 2-separator that separates $inner(cC_{B}d') \neq \emptyset$ and $v_i$, which contradicts the maximality of $\{c, d\}$ or of $\{c', d'\}$. This gives the claim.

If two maximal 2-separators interlace, Lemma 9 thus ensures that these two are the only maximal 2-separators that may contain $v_{i-1}$ and $v_i$, respectively. This gives the following direct corollary.

Corollary 10. Every block of $K$ has at most two maximal 2-separators that interlace.
Note that any boundary part may nevertheless contain arbitrarily many (pairwise non-interlacing) maximal 2-separators. The next lemma strengthens Lemma 7.

**Lemma 11.** Let \(P_t\) be a \(v_{i-1} - \alpha_i, v_i\)-path of \(B_t\). Let \(J\) be a non-trivial outer \(P_t\)-bridge of \(B_t\) and let \(e\) be an edge in \(J \cap C_{B_t}\). Then the attachments of \(J\) are contained in the boundary part of \(B_t\) that contains \(e\).

**Proof.** Let \(c\) and \(d\) be the attachments of \(J\) such that \(e \in cC_{B_t}d\) and let \(Z\) be the boundary part of \(B_t\) that contains \(e\). If \(P_t = \alpha_i, v_{i-1} = l_i, v_i = r_i\), are the only boundary points of \(B_t\). Then \(c\) and \(d\) are the endvertices of \(Z = v_iC_{B_t}v_{i-1} \ni e\), which gives the claim.

Otherwise, let \(P_i \neq \alpha_i\). By applying Lemma 7 with \(a = l_{\alpha_i}\) and \(b = r_{\alpha_i}\), \(\{c, d\}\) is a 2-separator of \(B_t\) that is contained in \(C_{B_t}\). By definition of \(w_1, \ldots, w_p\), there are at least three independent paths between every two of these vertices in \(B_t\); thus, \(\{c, d\}\) does not separate two vertices of \(\{w_1, \ldots, w_p\}\). Since all other possible boundary points \(\{v_{i-1}, l_{\alpha_i}, r_{\alpha_i}, v_i\}\) are contained in \(P_t\), applying Lemma 7 on these implies that \(\{c, d\}\) does not separate two vertices of these remaining boundary points. Hence, if \(\{c, d\} \not\subseteq Z\), we have \(B_t = B_t\) and \(v_i \not\subseteq C_{B_t}\) such that \(\{c, d\}\) separates \(\{w_1, \ldots, w_p\}\) from the remaining boundary points. Since the \(P_t\)-bridge \(J\) does not contain \(\alpha_i \in P_t, cC_{B_t}d \subseteq J\) contains \(\{w_1, \ldots, w_p\}\), but inner \(cC_{B_t}d\) does not contain any other boundary point. As \(v_i \in P_t\), at least one of \(\{w_1, w_p\}\) must be in \(P_t\), say \(w_p\) by symmetry. Then \(d = w_p\), as \(w_p \in P_t\) cannot be an internal vertex of \(J\). Now, in both cases \(p = 2\) (which implies \(c \neq w_1\), as \(\{c, d\} \not\subseteq Z = w_1C_{B_t}w_2\)) and \(p \geq 3\), \(J\) contains the edge of \(P_t\) that is incident to \(v_i\). As this contradicts that \(J\) is a \(P_t\)-bridge, we conclude \(\{c, d\} \subseteq Z\).

Now we relate non-trivial outer \(P_t\)-bridges of \(B_t\) to maximal 2-separators of \(B_t\). In the next section, we will use this lemma as a fundamental tool for a decomposition into subgraphs having only small overlaps, which will eventually construct \(P\).

**Lemma 12.** Let \(P_t\) be a \(v_{i-1} - \alpha_i, v_i\)-path of \(B_t\) such that \(P_t \neq \alpha_i\). Then the maximal 2-separators of \(B_t\) are contained in \(P_t\) and do not interlace pairwise. If \(J\) is a non-trivial outer \(P_t\)-bridge of \(B_t\), there is a maximal 2-separator \(\{c, d\}^*\) of \(B_t\) such that \(J \subseteq B_t^+\).

### 3.2.2 Construction of \(P\)

We do not know \(P_t\) in advance. However, Lemma 12 ensures under the condition \(P_t \neq \alpha_i\) that we can split off every non-trivial outer bridge \(J\) of \(P_t\) by a maximal 2-separator, no matter how \(P_t\) looks like. This allows us to construct \(P_t\) iteratively by decomposing \(B_t\) along its maximal 2-separators. Since maximal 2-separators only depend on the graph \(B_t\) (in contrast to the paths \(P_t\), which depend for example on the \(K \cup C_G\)-bridges), we can access them without knowing \(P_t\) itself. We now give the details of such a decomposition.

**Definition 13.** For every \(1 \leq i \leq l\), let \(\eta(B_t) = \alpha_i\) if \(\alpha_i = v_{i-1}v_i\) and otherwise the graph obtained from \(B_t\) as follows: For every maximal 2-separator \(\{c, d\}^*\) of \(B_t\), split off \(B_t^+\). Moreover, let \(\eta(K) := \eta(B_t) \cup \cdots \cup \eta(B_t)\).

If \(\alpha_i \neq v_{i-1}v_i\), \(\alpha_i\) cannot be a \(v_{i-1} - \alpha_i, v_i\)-path of \(B_t\); hence, the maximal 2-separators of \(K\) that were split in this definition do not interlace due to Lemma 12. This implies that the order of the performed splits is irrelevant. In any case, we have \(V(C_{\eta(B_t)}) \subseteq V(C_{B_t})\) and the only 2-separators of \(\eta(B_t)\) must be contained in some boundary part of \(B_t\), as there would have been another split otherwise. See Figure 3 for an illustration of \(\eta(B_t)\). The following lemma highlights two important properties of every \(\eta(B_t)\).
Theorem 1. Constructed such that no non-trivial outer 2-separators in 2-separators on the same boundary part. b) The graph \( \eta(B_i) \).

**Lemma 14.** Every \( \eta(B_i) \) is a block. Let \( P_i^0 \) be a \( v_{i-1}\alpha_i\alpha_iv_i \)-path of some \( \eta(B_i) \) such that \( P_i^0 \neq \alpha_i \). Then every outer \( P_i^0 \)-bridge of \( \eta(B_i) \) is trivial.

The next lemma shows how we can construct a Tutte path \( P \) of \( K \) iteratively using maximal 2-separators. We will provide the details of an efficient implementation in Section 4.

**Lemma 15 (Construction of \( P \)).** For every \( 1 \leq i \leq l \), a \( v_{i-1}\alpha_i\alpha_iv_i \)-path \( P_i \) of \( B_i \) can be constructed such that no non-trivial outer \( P_i \)-bridge of \( B_i \) is part of an inductive call of Theorem 1.

**Proof.** The proof proceeds by induction on the number of vertices in \( B_i \). If \( B_i \) is just an edge or a triangle, the claim follows directly. For the induction step, we therefore assume that \( B_i \) contains at least four vertices. If \( \alpha_i = v_{i-1}v_i \), we set \( P_i := \alpha_i \), so assume \( \alpha_i \neq v_{i-1}v_i \).

In particular, \( \eta(B_i) \neq \alpha_i \) and \( \alpha_i \) is no \( v_{i-1}\alpha_i\alpha_iv_i \)-path of \( \eta(B_i) \). As \( |V(\eta(B_i))| < N \), we may apply an inductive call of Theorem 1 to \( \eta(B_i) \), which returns a \( v_{i-1}\alpha_i\alpha_iv_i \)-path \( P_i^0 \neq \alpha_i \) of \( \eta(B_i) \). This does not violate the claim, since \( \eta(B_i) \) does not contain any non-trivial outer \( P_i^0 \)-bridge by Lemma 14.

Now we extend \( P_i^0 \) iteratively to the desired \( v_{i-1}\alpha_i\alpha_iv_i \)-path \( P_i \) of \( B_i \) by restoring the subgraphs that were split off along maximal 2-separators one by one. For every edge \( cd \in C_{\eta(B_i)} \) such that \( \{c,d\} \) is a maximal 2-separator of \( B_i \) (in arbitrary order), we distinguish the following two cases: If \( cd \notin P_i^0 \), we do not modify \( P_i^0 \), as in \( B_i \) the subgraph \( B_{cd}^+ \) will be a valid outer bridge. If otherwise \( cd \in P_i^0 \), we consider the subgraph \( B_{cd}^+ \) of \( B_i \). Clearly, \( B := B_{cd}^+ \cup \{cd\} \) is a block. Define that the boundary points of \( B \) are \( c, d \) and the two endpoints of some arbitrary edge \( \alpha_B \neq cd \) in \( C_B \). This introduces the boundary parts of \( B \) in the standard way, and hence defines \( \eta(B) \). Note that \( B \) may contain several maximal 2-separators in \( cC_Bd \) that in \( B_i \) were suppressed by \( \{c,d\} \), as \( \{c,d\} \) is not a 2-separator of \( B \). In consistency with Lemma 12, which ensures that no two maximal 2-separators of \( B_i \) interlace, we have to ensure that no two maximal 2-separators of \( B \) interlace in our case \( \alpha_i \neq v_{i-1}v_i \), as otherwise \( \eta(B) \) would be ill-defined. This is however implied by Lemma 9, as \( \alpha_B \neq cd \). Since \( |V(\eta(B))| < |V(B_i)| \), a \( c-\alpha_B-d \)-path \( P_B \) of \( B \) can be constructed such that no non-trivial outer \( P_B \)-bridge of \( B \) is part of an inductive call of Theorem 1. Since \( \alpha_B \neq cd \), \( P_B \) does not contain \( cd \). We now replace the edge \( cd \) in \( P_i^0 \) by \( P_B \). This gives the desired path \( P_i \) after having restored all subgraphs \( B_{cd}^+ \).

Applying Lemma 15 on all blocks of \( K \) and taking the union of the resulting paths gives \( P \). In the next step, we will modify \( Q \) such that \( P \cup \{\alpha\} \cup Q \) becomes the desired Tutte path of \( G \). By Lemma 15, no non-trivial outer \( P \)-bridge of \( K \) was part of any inductive call of Theorem 1 so far, which allows us to use these bridges inductively for the following modification of \( Q \) (the existence proof in [29] used these arbitrarily large bridges in inductive calls for both constructing \( P \) and modifying \( Q \)).
3.2.3 Modification of Q

We show how to modify $Q$ such that $P \cup \{\alpha\} \cup Q$ is an $x$-$y$-path of $G$. To this end, consider a $(P \cup \{\alpha\} \cup Q)$-bridge $J$ of $G$. Since Lemma 5 cannot be applied, $J$ does not have all of its attachments in $Q$. On the other hand, if $J$ has all of its attachments in $P \subseteq K$, $J \subseteq K$ follows from the maximality of blocks and therefore $J$ satisfies all conditions for a Tutte path of $G$. Hence, it suffices to consider $(P \cup \{\alpha\} \cup Q)$-bridges that have attachments in both $P$ and $Q$. The following lemma showcases some of their properties.

Lemma 16. Let $J$ be a $(P \cup \{\alpha\} \cup Q)$-bridge of $G$ that has an attachment in $P$. Then $J \cap K$ is either exactly one vertex in $P$ or exactly one non-trivial outer $P$-bridge $L$ of $K$. In particular, $J$ has at most two attachments in $P$.

Let $J$ be a $(P \cup \{\alpha\} \cup Q)$-bridge of $G$ that has attachments in both $P$ and $Q$ and recall that $C(J) = l_JG_{r_J}$. Because Lemma 5 is not applicable to $G$, there is no other $(P \cup \{\alpha\} \cup Q)$-bridge than $J$ that intersects $(J \cup C(J)) - P - \{l_J, r_J\}$; in other words, $J \cup C(J)$ is everything that is enclosed by the attachments of $J$ in $G$. In order to obtain the Tutte path of Theorem 1, we will thus replace the subpath $C(J)$ with a path $Q_J \subseteq (J \cup C(J)) - P$ from $l_J$ to $r_J$ such that any $(Q_J \cup P)$-bridge of $G$ that intersects $(J \cup C(J)) - P - \{l_J, r_J\}$ has at most three attachments and at most two if it contains an edge of $C_G$. Since $l_J$ and $r_J$ are contained in $Q$, no other $(P \cup \{\alpha\} \cup Q)$-bridge of $G$ than $J$ is affected by this “local” replacement, which proves its sufficiency for obtaining the desired Tutte path.

We next show how to obtain $Q_J$. If $C(J)$ is a single vertex, we do not need to modify $Q$ at all (hence, $Q_J := C(J)$), as then $J \cup C(J)$ does not contain an edge of $C_G$ and has at most three attachments in total (one in $Q$ and at most two in $P$ by Lemma 16). If $C(J)$ is not a single vertex, we have the following lemma.

Lemma 17 ([29, 4]). Let $J$ be a $(P \cup \{\alpha\} \cup Q)$-bridge of $G$ that has an attachment in $P$ and at least two in $Q$. Then $(J \cup C(J)) - P$ contains a path $Q_J$ from $l_J$ to $r_J$ such that any $(Q_J \cup P)$-bridge of $G$ that intersects $(J \cup C(J)) - P - \{l_J, r_J\}$ has at most three attachments and at most two if it contains an edge of $C_G$.

By Lemma 16, any $(P \cup \{\alpha\} \cup Q)$-bridge $J$ of $G$ intersects $K$ in at most one non-trivial $P$-bridge of $K$ having attachments $c$ and $d$. By Lemma 15, this non-trivial $P$-bridge was never part of an inductive call of Theorem 1 before (in fact, at most its edge $cd$ was). Replacing $C(J)$ with $Q_J$ for every such $J$, as described in Lemma 17 and before, therefore concludes the constructive proof of Theorem 1.

4 A Quadratic Time Algorithm

In this section, we give an algorithm based on the decomposition shown in Section 3 (see Algorithm 1). It is well known that there are algorithms that compute the blocks of a graph and the block-cut tree of $G$ in linear time, see [25] for a very simple one. Using this on $G - Q$, we can compute the blocks $B_1, \ldots, B_t$ of $K$ in time $O(n)$.

We now check if Lemma 4 or 5 is applicable at least once to $G$; if so, we stop and apply the construction of either Lemma 4 or 5. Checking applicability involves the computation of special 2-separators $\{c, d\}$ of $G$ that are in $C_G$ (e.g., we did assume minimality of $|V(G_R)|$ in Lemma 4). In order to find such a $\{c, d\}$ in time $O(n)$, we first compute the weak dual $G^*$ of $G$, which is obtained from the dual of $G$ by deleting its outer face vertex, and note that such pairs $\{c, d\}$ are exactly contained in the faces that correspond to 1-separators of $G^*$. Once more, these faces can be found by the block-cut tree of $G^*$ in time $O(n)$ using the above
Algorithm 1 TPATH\((G, x, \alpha, y)\) \>

1. if \(G\) is a triangle or \(\alpha = xy\) then return the trivial \(x-\alpha-y\) path of \(G\) \>

2. if Lemma 4 or 5 is applicable at least once to \(G\) then \> weak dual block-cut tree, \(O(n)\)

3. apply TPATH on \(G_L\) and \(G_R\) as described and return the resulting path \(\triangleright O(1)\)

4. if there is a 2-separator \(\{c, d\} \in C_G\) of \(G\) then

5. do simple case 2

6. Compute the minimal plane chain \(K\) of blocks of \(G\) \> block-cut tree of \(G - Q\), \(O(n)\)

7. Compute \(\eta(K)\) \> dyn. progr. on weak dual block-cut tree, \(O(n)\)

8. Compute \(P\) by the induction of Lemma 15 \> dyn. progr. precomputes all possible \(B_{cd}^+\), \(O(n)\)

9. Modify \(Q\) by the induction of Lemma 17 \> traversing outer faces of bridges, \(O(n)\)

10. return \(P \cup \{\alpha\} \cup Q\)

algorithm. Since the block-cut tree is a tree, we can perform dynamic programming on all these 1-separators bottom-up the tree in linear total time, in order to find one desired \(\{c, d\}\) that satisfies the respective constraints (e.g. minimizing \(|V(G_R)|\), or separating \(x\) and \(\alpha\)).

Now we compute \(\eta(K)\). Since the boundary points of every \(B_i\) are known from \(K\), all maximal 2-separators can be computed in time \(O(n)\) by dynamic programming as described above. We compute in fact the nested tree structure of all 2-separators on boundary parts due to Lemma 12, on which we then apply the induction described in Lemma 15. Hence, no non-trivial outer \(P\)-bridge of \(K\) is touched in the induction, which allows to modify \(Q\) along the induction of Lemma 17.

In our decomposition, every inductive call is invoked on a graph having less vertices than the current graph. The key insight is now to show a good bound on the total number of inductive calls to Theorem 2. In order to obtain good upper bounds, we will restrict the choice of \(\alpha_i\) for every block \(B_i\) of \(K\) such that \(\alpha_i\) is an edge of \(C_{B_i} - v_{i-1}v_i\). This prevents several situations in which the recursion stops because of the case \(\alpha = xy\), which would uncase the following arguments. The next lemma shows that only \(O(n)\) inductive calls are performed. Its argument is, similarly to one in [5], based on a subtle summation of the Tutte path differences that occur in the recursion tree.

\textbf{Lemma 18.} The number of inductive calls for TPATH\((G, x, \alpha, y)\) is at most \(2n - 3\).

Hence, Algorithm 1 has overall running time \(O(n^2)\), which proves our main Theorem 3.

\textbf{Corollary 19.} Let \(G\) be a 2-connected plane graph and let \(\alpha, \beta, \gamma\) be edges of \(C_G\). Then a Tutte cycle of \(G\) that contains \(\alpha, \beta\) and \(\gamma\) can be computed in time \(O(n^2)\).

\textbf{References}


