ON LONGEST CYCLES IN ESSENTIALLY 4-CONNECTED PLANAR GRAPHS

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Abstract

A planar 3-connected graph \(G\) is essentially 4-connected if, for any 3-separator \(S\) of \(G\), one component of the graph obtained from \(G\) by removing \(S\) is a single vertex. Jackson and Wormald proved that an essentially 4-connected planar graph on \(n\) vertices contains a cycle \(C\) such that \(|V(C)| \geq \frac{2n+4}{3}\). For a cubic essentially 4-connected planar graph \(G\), Grünbaum with Malkevitch, and Zhang showed that \(G\) has a cycle on at least \(\frac{3}{2}n\) vertices. In the present paper the result of Jackson and Wormald is improved. Moreover, new lower bounds on the length of a longest cycle of \(G\) are presented if \(G\) is an essentially 4-connected planar graph of maximum degree 4 or \(G\) is an essentially 4-connected maximal planar graph.

Keywords: planar graph, longest cycle.

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1. Introduction and Results

We use standard notation and terminology of graph theory ([1]) and consider a finite simple 3-connected planar graph \(G\) with vertex set \(V(G)\) and edge set \(E(G)\). Let \(N(x), d(x) = |N(x)|\), and \(\Delta(G)\) denote the neighborhood, the degree of

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Let \( x \in V(G) \) in \( G \), and the maximum degree of \( G \), respectively. A subset \( S \subset V(G) \) is an \( s \)-separator of \( G \) if \( |S| = s \) and \( G - S \) is disconnected. It is well-known that \( G - S \) has exactly two components if \( G \) is a 3-connected planar graph and \( S \) is a 3-separator of \( G \). If \( S \) is a 3-separator of a 3-connected planar graph \( G \) and one component of \( G - S \) is a single vertex, then \( S \) is a trivial 3-separator of \( G \). If \( G \) is planar, 3-connected, and each 3-separator \( S \) of \( G \) is trivial, then \( G \) is essentially 4-connected. In the present paper we are interested in the length of longest cycles of an essentially 4-connected planar graph.

Jackson and Wormald [4] proved that every essentially 4-connected planar graph on \( n \) vertices contains a cycle \( C \) such that \( |V(C)| \geq \frac{2n+4}{3} \). For a cubic essentially 4-connected planar graph \( G \), Grünbaum and Malkevitch [3], and Zhang [8] showed that \( G \) has a cycle on at least \( \frac{3}{2}n \) vertices. Given a real constant \( c > \frac{3}{2} \), Jackson and Wormald [4] presented an infinite family of essentially 4-connected planar graphs \( G \) such that \( G \) does not contain a cycle on more than \( c \cdot n \) vertices. This observation is even true for essentially 4-connected maximal planar graphs. To see this, let \( G' \) be a 4-connected maximal planar graph on \( n' \geq 6 \) vertices embedded into the plane and let \( G \) be obtained by inserting a new vertex into each face of \( G' \) and connecting it with all three vertices of that face by an edge. Obviously, \( G \) is an essentially 4-connected maximal planar graph on \( n = n' + (2n' - 4) \) vertices and the \( 2n' - 4 \) vertices in \( V(G) \setminus V(G') \) are pairwise independent. Hence each cycle of \( G \) contains at most \( 2n' = \frac{2}{3}(n + 4) \) vertices. At the end of Section 2 we will show that \( G \) contains a cycle on exactly \( 2n' = \frac{2}{3}(n+4) \) vertices.

It is well-known that a 3-connected planar graph on \( 4 \leq n \leq 10 \) vertices is Hamiltonian. It remains open whether a maximal planar (or even an arbitrary planar) essentially 4-connected graph on \( n \geq 11 \) vertices contains a cycle \( C \) such that \( |V(C)| \geq \frac{2}{3}(n + 4) \).

Our results are presented in the following Theorem 1.

**Theorem 1.** Let \( G \) be an essentially 4-connected planar graph on \( n \geq 11 \) vertices and \( C \) be a longest cycle of \( G \). Then \( |V(C)| \geq \frac{1}{2}(n+4) \), \( |V(C)| \geq \frac{3}{4}n \) if \( \Delta(G) = 4 \), and \( |V(C)| \geq \frac{13}{24}(n+4) \) if \( G \) is maximal planar.

2. **Proofs**

In the remainder of the paper we assume that \( G \) is embedded into the plane. The two open sets into which a cycle \( C \) of \( G \) partitions the plane are the interior \( \text{int}(C) \) and the exterior \( \text{ext}(C) \) of \( C \). Furthermore, let \( B \) be a component of \( G - V(C) \). A vertex \( x \in V(C) \) is a touch vertex of \( B \) if \( x \) is adjacent to a vertex of \( V(B) \). Note that \( B \) has at least 3 touch vertices, if \( G \) is a 3-connected planar graph. In [7], Tutte proved a remarkable and famous result on cycles in 2-connected planar graphs.
graphs implying that a 4-connected planar graph is Hamiltonian. This result has been extended several times ([5, 6]). We will use the following Lemma 2 of Sanders ([5]) as a version of Tutte’s result for 3-connected planar graphs.

**Lemma 2.** Every 3-connected planar graph $G$ with two prescribed edges $a$ and $b$ contains a cycle $C$ through $a$ and $b$ such that each component of $G - V(C)$ has exactly 3 touch vertices.

A cycle $C$ of $G$ is an outer-independent-3-cycle (OI3-cycle), if $V(G) \setminus V(C)$ is an independent set of vertices and $d(x) = 3$ for all $x \in V(G) \setminus V(C)$.

**Lemma 3.** Let $G$ be an essentially 4-connected planar graph, and let $a$ and $b$ be non-adjacent edges of $G$. If $a$ and $b$ belong to a common face of $G$ or all end vertices of $a$ and $b$ have degree at least 4 in $G$, then $G$ contains an OI3-cycle $C$ through $a$ and $b$.

**Proof.** By Lemma 2, let $C$ be a cycle of $G$ through $a$ and $b$ such that each component of $G - V(C)$ has exactly three touch vertices. Since $a$ and $b$ are non-adjacent, $|V(C)| \geq 4$. We will show that $C$ is an OI3-cycle of $G$. Suppose to the contrary that $G - V(C)$ has a component $B$ with at least two inner vertices (w.l.o.g. let $V(B) \subset \text{int}(C)$). Since $G$ is essentially 4-connected and $|V(C)| \geq 4$, the three touch vertices $y, z, u$ of $B$ separate $G$, hence they form the neighborhood of a vertex $x$ of degree 3.

First assume that $x \in V(C)$ as shown in Figure 1 ($C$ is the fat-drawn cycle).

![Figure 1](image1.png)

Let $\alpha$ be the face of $G$ containing $z, u$ and at least one vertex of $V(B)$ and let $P$ be the boundary path of $\alpha$ connecting $u$ and $z$ and containing some vertex of $V(B)$. Furthermore, let $C'$ be the (fat-drawn) cycle with $V(C') = V(P) \cup \{x\}$ as shown in Figure 2. It is clear that $z$ and $u$ are the only vertices of $C'$ which possibly have a neighbor in $\text{int}(C') \cap V(G)$. It follows that $\text{int}(C') \cap V(G) = \emptyset$, because otherwise $\{z, u\}$ forms a 2-separator of $G$ contradicting the 3-connectedness of $G$. Thus $z$ and $u$ are neighbors on $C$ and, by symmetry, $y$ and $u$ are also neighbors on $C$. Consequently, $|V(C)| = 4$, the edges $a$ and $b$ cannot belong to a common
face, and one of them is incident with the vertex $x$ of degree 3 contradicting the choice of $a$ and $b$.

If $x \notin V(C)$ as shown in Figure 3, then, considering the (fat-drawn) cycles $C''$ in Figure 4 and $C'''$ in Figure 5, it follows that $\text{int}(C'') \cap V(G) = \emptyset$ and $\text{int}(C''') \cap V(G) = \emptyset$ with similar arguments, hence $|V(C)| = 3$, also a contradiction.

Consequently, $C$ is an OI$_3$-cycle through $a$ and $b$.

Note that a Hamiltonian cycle of a graph is an OI$_3$-cycle. Let $a = yz$ be an edge of an OI$_3$-cycle $C$ of a graph $G$ and assume that $y$ and $z$ have a common neighbor $x \in V(G) \setminus V(C)$. Then let $C'$ be the cycle of $G$ obtained from $C$ by replacing the edge $a$ with the path $(y, x, z)$. In this case, $a$ is an extendable edge of $C$. Note that $C'$ is again an OI$_3$-cycle of $G$, $|V(C')| = |V(C)| + 1$, and that $C'$ has less extendable edges than $C$. Obviously, a longest OI$_3$-cycle of $G$ does not contain an extendable edge.

For the proof of Theorem 1 it suffices to show the following lemma.

**Lemma 4.** Let $G$ be an essentially 4-connected planar graph on $n \geq 11$ vertices.

(i) $G$ contains an OI$_3$-cycle.

(ii) If $C$ is an OI$_3$-cycle of $G$ without extendable edges, then $|V(C)| \geq \frac{1}{2}(n + 4)$.

(iii) If $\Delta(G) = 4$ and $C$ is an OI$_3$-cycle of $G$, then $|V(C)| \geq \frac{5}{8}n$.

(iv) If $G$ is maximal planar and $C$ is a longest OI$_3$-cycle of $G$, then $|V(C)| \geq \frac{13}{21}(n + 4)$.

**Proof.** If $G$ is an essentially 4-connected plane graph without vertices of degree 3, then $G$ is even 4-connected, hence, $G$ contains a Hamiltonian cycle (Lemma 2). Since every Hamiltonian cycle is an OI$_3$-cycle, Lemma 4(i) is true in this case. If $G$ is not maximal planar, then there exist two non-adjacent edges $a$ and $b$ of $G$ belonging to a common face, hence, by Lemma 3, Lemma 4(i) follows.

Thus, for the proof of Lemma 4(i), it remains to deal with the case that $G$ is maximal planar and contains a vertex of degree 3. Let $a = yz$ be an edge connecting two neighbors $y$ and $z$ of a vertex $x$ of degree 3 in $G$. In this case we will show that $d(y) \geq 4$, $d(z) \geq 4$, and that there is an edge $b$ being non-adjacent with $a$, and with both end vertices of degree at least 4. Consequently, the existence of an OI$_3$-cycle in $G$ follows by Lemma 3, and Lemma 4(i) is true.
also in this case. Let \( u \) be the third neighbor of \( x \). The vertices \( y, z, u \) form a separating 3-cycle, hence because \( G \) is 3-connected, all of them have degree at least 4. Let \( w \in N(u) \setminus \{x, y, z\} \) be a fourth neighbor of \( u \). If \( d(u) = 4 \), then \( \{y, z, w\} \) is a 3-separator and both components of \( G - \{y, z, w\} \) contain at least two vertices, a contradiction to the essentially 4-connectedness of \( G \). It follows that \( d(u) \geq 5 \). Let \( v \in N(u) \setminus \{x, y, z, w\} \) such that \( v \in N(w) \). Since \( G \not\cong K_4 \), vertices of degree three are not adjacent in \( G \), thus one of the vertices \( w \) and \( v \) has degree at least four. We are done with \( b = uw \) or \( b = uv \), respectively, and Lemma 4(i) is completely proved.

The following Lemma 5 is proved in [2]. For completeness, we present its short proof here.

**Lemma 5.** If \( C \) is a cycle of a plane graph \( G \) on at least 4 vertices such that \( \text{int}(C) \cap V(G) \) is an independent set of vertices of degree 3 in \( G \) and, for each edge \( xy \) of \( C \), \( x \) and \( y \) do not have a common neighbor in \( \text{int}(C) \cap V(G) \), then \( |\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4) \).

**Proof.** We proceed by induction on \( c = |V(C)| \). If \( c \leq 5 \), then, obviously, \( |\text{int}(C) \cap V(G)| = 0 \). Now let \( c \geq 6 \), \( d = |\text{int}(C) \cap V(G)| > 0 \), and \( \phi \) be an orientation of \( C \). Consider a fixed vertex \( x \in \text{int}(C) \cap V(G) \) and let \( x_1, x_2, \) and \( x_3 \) be the neighbours of \( x \) on \( C \) met in this order following \( \phi \). For \( i = 1, 2, 3 \), let \( C_i \) be the cycle obtained by the union of the path on \( C \) from \( x_i \) to \( x_{i+1} \) following \( \phi \) and the two edges \( xx_i \) and \( xx_{i+1} \) (where \( x_4 = x_1 \)), \( c_i = |V(C_i)| \), and \( d_i = |\text{int}(C_i) \cap V(G)| \). Obviously, \( c > c_i \geq 4 \) and for each edge \( xy \) of \( C_i \), \( x \) and \( y \) do not have a common neighbor in \( \text{int}(C_i) \cap V(G) \) (\( i = 1, 2, 3 \)). We have \( c_1 + c_2 + c_3 = c + 6, d_1 + d_2 + d_3 = d - 1 \), and, by induction hypothesis, \( d_i \leq \frac{c_i}{2} - 2 \) for \( i = 1, 2, 3 \). This implies \( d \leq \frac{c}{2} - 2 \). \( \square \)

To prove Lemma 4(ii), consider an OI3-cycle \( C \) of \( G \) without an extendable edge. Obviously, \( |V(C)| \geq 4 \) because \( n \geq 4 \). Moreover, for each edge \( xy \) of \( C \), \( x \) and \( y \) do not have a common neighbor in \( (\text{int}(C) \cup \text{ext}(C)) \cap V(G) \). By Lemma 5, \( |\text{int}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4) \) and, by symmetry, \( |\text{ext}(C) \cap V(G)| \leq \frac{1}{2}(|V(C)| - 4) \). Thus \( n = |V(C)| + |\text{int}(C) \cap V(G)| + |\text{ext}(C) \cap V(G)| \leq 2|V(C)| - 4 \) and Lemma 4(ii) is proved.

For the proof of Lemma 4(iii) consider an arbitrary OI3-cycle \( C \) of \( G \). Since \( V(G) \setminus V(C) \) is an independent set and \( d(x) = 3 \) for every \( x \in V(G) \setminus V(C) \), \( 3(n - |V(C)|) \) equals the number \( e \) of edges between \( V(C) \) and \( V(G) \setminus V(C) \). If \( y \in V(C) \), then, because \( d(y) \leq 4 \), \( y \) has at most two neighbors in \( V(G) \setminus V(C) \). It follows \( e \leq 2|V(C)| \) and Lemma 4(iii) is proved.

It remains to prove Lemma 4(iv).

Let \( C \) be a longest OI3-cycle of \( G \). By Lemma 4(ii) and \( n \geq 11 \), we have \( |V(C)| \geq 8 \). Moreover, let \( H = G[V(C)] \) be the graph obtained from \( G \) by removing all vertices of degree 3 which do not lie on \( C \). Obviously, \( H \) is maximal.
planar and $C$ is a Hamiltonian cycle of $H$. A face $\alpha$ of $H$ is an empty face of $H$ if $\alpha$ is also a face of $G$, otherwise $\alpha$ is a non-empty face of $H$. Denote by $\mathcal{F}$ the set of empty faces of $H$. Note that every face of $G$ has at least two (of three) vertices on $C$. The three neighbors of a vertex of $V(G) \setminus V(C)$ induce a separating 3-cycle of $G$ creating the boundary of a non-empty face of $H$.

**Lemma 6.** Let $t = |\mathcal{F}|$ be the number of empty faces of $H$. For a positive real $a$, the inequalities $|V(C)| \leq at$ and $|V(C)| \geq \frac{a}{3a-1}(n+4)$ are equivalent.

**Proof.** Since every face of $G$ which is not an empty face of $H$ has exactly one vertex in $V(G) \setminus V(C)$, calculating the number of faces of $G$ leads to $2n - 4 = t + 3(n - |V(C)|)$. It follows $t = 3|V(C)| - n - 4$ and directly the equivalence of $|V(C)| \leq at$ and $|V(C)| \geq \frac{a}{3a-1}(n+4)$. $\Box$

Using Lemma 6, it suffices to prove $|V(C)| \leq \frac{13}{15}t$.

Let $H_1$ and $H_2$ be the spanning subgraphs of $H$ consisting of the cycle $C$ and of its chords lying in the interior and in the exterior of $C$, respectively. Note that $E(H_1) \cap E(H_2) = E(C)$ and $H_1$ and $H_2$ are maximal outerplanar graphs.

An empty face $\varphi$ of $H$ is a $j$-face if exactly $j$ of its three incident edges belong to $E(C)$. Since $|V(C)| \geq 8$, it follows $j \in \{0, 1, 2\}$ for any $j$-face $\varphi$ of $H$. Note that $C$ and a non-empty face of $H$ do not have an edge in common because otherwise such an edge would be an extendable edge of $C$ in $G$.

Since $C$ does not contain extendable edges, every face of $H$ incident with an edge of $C$ is an empty face. An edge $e$ of $C$ incident with the faces $\varphi$ and $\psi$ is a $(j, k)$-edge for $1 \leq j, k \leq 2$, if $\varphi$ is a $j$-face and $\psi$ is a $k$-face.

For every edge $e \in E(C)$ we define the weight $w_0(e) = 1$. Obviously, $\sum_{e \in E(C)} w_0(e) = |V(C)|$.

**First redistribution of weights**

If $x, y$, and $z$ are the vertices incident with a face $\varphi$ of $H$, then we write $\varphi = [x, y, z]$. Let $(u, x, y, v)$ be a subpath of $C$, $xy$ be a $(2, 2)$-edge of $C$, and $\alpha = [u, x, y]$ and $\sigma = [x, y, v]$ be two adjacent 2-faces of $H$. Moreover, let $\beta$ and $\tau$ be the faces of $H$ incident with $uy$ and $xv$ and distinct from $\alpha$ and $\sigma$, respectively (see Figure 6). The cycle $\tilde{C}$ obtained from $C$ by replacing the path $(u, x, y, v)$ by the path $(u, y, x, v)$ is also a longest OI3-cycle of $G$, hence both $uy$ and $xv$ are not extendable edges of $\tilde{C}$ and therefore $\beta$ and $\tau$ are also empty faces of $H$.

The weight of all edges of $C$ will be completely redistributed to empty faces of $H$ by the following rules.

**Rule R1.** A $(2, 2)$-edge $xy$ of $C$ (Figure 6) sends weight $\frac{1}{3}$ to both incident 2-faces $\alpha$ and $\sigma$ and weight $\frac{1}{6}$ to $\beta$ (through the edge $uy$) and to $\tau$ (through the edge $xv$).
Rule R2. A (1, 2)-edge of $C$ sends weight $\frac{2}{3}$ to the incident 1-face and weight $\frac{1}{3}$ to the incident 2-face.

Rule R3. A (1, 1)-edge of $C$ sends weight $\frac{1}{2}$ to both incident 1-faces.

For an empty face $\varphi$, let $w_1(\varphi)$ be the total weight obtained by $\varphi$ (in first redistribution). Obviously, $\sum_{\varphi \in \mathcal{F}} w_1(\varphi) = |V(C)|$.

Every empty face gets weight from (or through) at most two of its three incident edges (otherwise $|V(C)| \leq 6$, a contradiction). An empty face $\varphi$ of $H$ is good if $w_1(\varphi) \leq \frac{2}{3}$, otherwise it is bad.

Every 2-face $\varphi$ gets weight only by rules R1 or R2, thus $w_1(\varphi) \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ and $\varphi$ is good.

A 0-face $\varphi$ can get weight only by rule R1. It can get weight $\frac{1}{6}$ from two distinct edges of $C$ through the same incident edge, thus $w_1(\varphi) \leq \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) = \frac{2}{3}$ and $\varphi$ is good.

Every 1-face $\varphi$ gets weight $\frac{2}{3}$ (by R2) or weight $\frac{1}{2}$ (by R3) from the incident edge lying on $C$. Furthermore, $\varphi$ can get weight also through one of the remaining two incident edges (by R1). Thus $w_1(\varphi) \leq \frac{2}{3} + \left(\frac{1}{6} + \frac{1}{6}\right) = 1$. Moreover, if $\varphi$ is bad, then $w_1(\varphi) = \frac{5}{6}$ or $w_1(\varphi) = 1$.

Now we describe all possible neighborhoods of bad faces.

Lemma 7. Let $\beta \in F(H_i), i \in \{1, 2\}$, be a bad face of $H$ and let $\alpha$ and $\gamma$ be the two faces of $H_i$ adjacent to $\beta$, where $\alpha$ is a 2-face of $H$. The face $\beta$ is of one of the following four types (Figure 7):

(B1) $w_1(\beta) = \frac{5}{6}$ and $\gamma$ is an empty face,

(B2) $w_1(\beta) = 1$ and $\gamma$ is an empty 0-face,

(B3) $w_1(\beta) = 1$ and $w_1(\gamma) = \frac{1}{2}$,

(B4) there is a 2-face $\sigma$ of $H_{3-i}$ adjacent (in $H$) to $\alpha$, $\beta$, and $\tau$, where $\tau$ is an empty 0-face of $H$.

Proof. If $\beta \in F(H_i), i \in \{1, 2\}$, is a bad face of $H$, then there is a 2-face $\alpha$ of $H_i$ adjacent to $\beta$. Let $\gamma (\gamma \neq \alpha)$ be the second face of $H_i$ adjacent to $\beta$ (Figure 8).
Case 1. Let $w_1(\beta) = \frac{5}{6}$ and $ux$ be a $(2, 2)$-edge (i.e., $zx \in E(H_{3-i})$, see Figure 9). The cycle $\tilde{C}$ obtained from $C$ by replacing the path $(z, u, x, y, v)$ by the path $(z, x, y, u, v)$ is a longest OI3-cycle of $G$ and contains the edge $uv$, thus $\gamma$ is an empty face of $H$ (and $\beta$ is of type B1).

Case 2. Let $w_1(\beta) = \frac{5}{6}$ and $xy$ be a $(2, 2)$-edge (i.e., $xv \in E(H_{3-i})$). The face $\sigma = [x, y, v]$ is a 2-face of $H_{3-i}$. Let $\tau$ ($\tau \neq \sigma$) be the second face of $H_{3-i}$ incident with $xv$. Since $|V(C)| \geq 8$, it follows $u \neq w$, hence $\tau$ cannot be a 2-face of $H_{3-i}$.

Case 2.1. If $\tau$ is a 0-face (Figure 10), then the cycle $\tilde{C}$ obtained from $C$ by replacing the path $(u, x, y, v)$ by the path $(u, y, x, v)$ is a longest OI3-cycle of $G$ and contains the edge $xv$, thus $\tau$ is an empty face of $H$ (and $\beta$ is of type B4).

Case 2.2. If $\tau$ is a 1-face (Figure 11), then $\tau = [x, v, w]$ (since $uv \in E(H_i) \setminus E(C)$, $uv$ is not an edge of $H_{3-i}$). The cycle $\tilde{C}$ obtained from $C$ by replacing the path $(u, x, y, v, w)$ by the path $(u, v, y, x, w)$ is a longest OI3-cycle of $G$ and contains the edge $uv$, thus $\gamma$ is an empty face of $H$ (and $\beta$ is of type B1).
Case 3. Let \( w_1(\beta) = 1 \). Now both \( ux \) and \( xy \) are \((2, 2)\)-edges (i.e., \( zx, xv \in E(H_{3-i}) \)). The face \( \sigma = [x, y, v] \) is a 2-face of \( H_{3-i} \). Let \( \tau (\tau \neq \sigma) \) be the second face of \( H_{3-i} \) incident with \( xv \). Again, \( \tau \) cannot be a 2-face of \( H_{3-i} \) and we consider two subcases.

Case 3.1. If \( \tau \) is a 0-face (see Figure 12, possibly \( \tau = [z, x, v] \)), then, for a similar reason as in Case 2.1, \( \tau \) is an empty face of \( H \) (and \( \beta \) is of type \( B_4 \)).

Case 3.2. If \( \tau \) is a 1-face, then \( \tau = [x, v, w] \). Since \( |V(C)| \geq 8 \), it follows \( z \neq w \), hence \( \gamma \) is not a 2-face of \( H_i \). We consider the last two subcases.

Case 3.2.1. If \( \gamma \) is a 0-face (see Figure 13), then, for a similar reason as in Case 1, \( \gamma \) is an empty face of \( H \) (and \( \beta \) is of type \( B_2 \)).

Case 3.2.2. If \( \gamma \) is a 1-face, then \( \gamma \neq [z, u, v] \) (otherwise \( \{z, x, v\} \) is a non-trivial 3-separator, a contradiction). Thus \( \gamma = [u, v, w] \) (see Figure 14) and \( vw \) is an \((1, 1)\)-edge (and \( \beta \) is of type \( B_3 \)). □

For a better overview, we list the current weights of all faces considered in Lemma 7:

(B1) \( w_1(\alpha) = \frac{2}{3} \), \( w_1(\beta) = \frac{5}{6} \), and \( w_1(\gamma) \leq \frac{2}{5} \);

(B2) \( w_1(\alpha) = \frac{2}{3} \), \( w_1(\beta) = 1 \), and \( w_1(\gamma) \leq \frac{1}{3} \), because \( \gamma \) obtains no weight through its common edge with \( \beta \) and at most \( \frac{1}{6} + \frac{1}{6} \) through at most one of its remaining two edges;

(B3) \( w_1(\alpha) = \frac{2}{3} \), \( w_1(\beta) = 1 \), and \( w_1(\gamma) = \frac{1}{2} \);

(B4) \( w_1(\alpha) = \frac{2}{3} \), \( \frac{5}{6} \leq w_1(\beta) \leq 1 \), \( w_1(\sigma) = \frac{2}{3} \), and \( w_1(\tau) \leq \frac{1}{2} \), because \( \tau \) obtains weight \( \frac{1}{6} \) through its common edge with \( \sigma \) and at most \( \frac{1}{6} + \frac{1}{6} \) through at most one of its remaining two edges.

Second redistribution of weights

The weight of all bad faces exceeded \( \frac{13}{18} \) will be redistributed to good faces in their neighborhoods.

Rule R4. A bad face \( \beta \) of type \( B_1 \) sends weight \( \frac{1}{18} \) to \( \alpha \) and to \( \gamma \) (through the common edge).

Rule R5. A bad face \( \beta \) of type \( B_2 \) or \( B_3 \) sends weight \( \frac{1}{18} \) to \( \alpha \) and weight \( \frac{2}{9} \) to \( \gamma \) (through the common edge).
Rule R6. A bad face $\beta$ of type B4 sends weight $\frac{1}{18}$ to $\alpha$ and to $\sigma$ (through the common edge) and the weight $\frac{1}{6}$ to $\tau$ (through the edge $xv$, see Figure 10).

For an empty face $\varphi$, let $w_2(\varphi)$ be the total weight of $\varphi$ (after second redistribution). Obviously, $\sum_{\varphi \in F} w_2(\varphi) = \sum_{\varphi \in F} w_1(\varphi) = |V(C)|$.

A bad face $\varphi$ of type B1 sends weight $2 \times \frac{1}{18}$ to good faces, thus $w_2(\varphi) = \frac{5}{6} - 2 \times \frac{1}{18} = \frac{13}{18}$. A bad face $\varphi$ of type B2 or B3 sends weight $\frac{1}{18} + \frac{2}{9} = \frac{13}{18}$. Finally, a bad face $\varphi$ of type B4 sends weight $2 \times \frac{1}{18} + \frac{1}{6}$ to good faces, thus $w_2(\varphi) \leq 1 - 2 \times \frac{1}{18} - \frac{1}{6} = \frac{13}{18}$.

If a 2-face $\varphi$ gets weight by the rules R4, R5, or R6, then either by exactly one of the rules R4 and R5 ($\varphi = \alpha$ is adjacent to a 1-face $\beta$ in this case) or by R6 ($\varphi = \sigma$ is adjacent to a 0-face $\tau$ in this case). Thus $w_2(\varphi) \leq \frac{2}{3} + \frac{1}{18} = \frac{13}{18}$.

A good 1-face $\varphi$ has at most one adjacent bad face (otherwise $|V(C)| \leq 7$ by Lemma 7, a contradiction). If $w_1(\varphi) = \frac{1}{2}$, then $w_2(\varphi) \leq \frac{2}{3} + \frac{2}{9} = \frac{13}{18}$ (by R5). If $w_1(\varphi) = \frac{2}{3}$, then $w_2(\varphi) \leq \frac{2}{3} + \frac{1}{18} = \frac{13}{18}$ (by R4).

A 0-face $\varphi$ gets through at least one of its incident edges no weight in first redistribution (1RD) and also in second redistribution (2RD). Let $e$ be an edge incident with $\varphi$. If $\varphi$ gets weight $\frac{2}{3}$ through $e$ (by R5) in 2RD, then $\varphi$ obtained no weight through $e$ in 1RD. If $\varphi$ gets weight $\frac{1}{6}$ through $e$ (by R6) in 2RD, then $\varphi$ has already obtained weight $\frac{1}{6}$ through $e$ in 1RD. Finally, if $\varphi$ gets no weight through $e$ in 2RD, then $\varphi$ has obtained weight at most $\frac{1}{3}$ through $e$ in 1RD. Thus $\varphi$ obtain through $e$ weight at most $\frac{1}{3}$ (in 1RD and 2RD in total) and $w_2(\varphi) \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ follows. Thus, Lemma 4 is completely proved.

It remains to show that the essentially 4-connected maximal planar graph $G$ on $n = n' + 2(n' - 4)$ vertices constructed in Section 1 from the 4-connected maximal planar graph $G'$ on $n' \geq 6$ vertices contains a cycle on exactly $2n'$ vertices. To see this, let $a$ and $b$ be two adjacent edges of $G'$ which do not belong to a common face of $G'$. Note that $a$ and $b$ exist since $n \geq 6$ implies that each vertex of $G'$ has degree at least 4. Consider a Hamiltonian cycle $C'$ of $G'$ through $a$ and $b$ (apply Lemma 2). Let $a = e_1, e_2, \ldots, e_{n'-1}, e_n' = b$ be the edges of $C'$ met in this order along $C'$. For $j = 1, \ldots, n'$, consider the common neighbors $x_j \in (V(G) \setminus V(G')) \cap \text{int}(C')$ and $y_j \in (V(G) \setminus V(G')) \cap \text{ext}(C')$ of the end vertices $u_j$ and $v_j$ of $e_j$. It is easy to see that the vertices in $\{x_1, \ldots, x_{n'}, y_1, \ldots, y_{n'}\}$ are pairwise distinct (if $n'$ is odd, then note that $a$ and $b$ do not belong to a common face of $G'$). Eventually, let the cycle $C$ of $G$ be obtained by replacing $e_j$ in $C'$ with the path $(u_j, x_j, v_j)$ if $j$ is odd and $(u_j, y_j, v_j)$ if $j$ is even ($j = 1, \ldots, n'$).

References

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