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Jussi Behrndt, Philipp Schmitz and Carsten Trunk

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Hrsg.: Leiter des Instituts für Mathematik  
Weimarer Straße 25  
98693 Ilmenau

Tel.: +49 3677 69-3621

Fax: +49 3677 69-3270

<http://www.tu-ilmenau.de/math/>

# THE NON-REAL SPECTRUM OF A SINGULAR INDEFINITE STURM–LIOUVILLE OPERATOR WITH REGULAR LEFT ENDPOINT

JUSSI BEHRNDT, PHILIPP SCHMITZ, AND CARSTEN TRUNK

ABSTRACT. We provide bounds on the non-real spectra of indefinite Sturm–Liouville differential operators of the form  $(Af)(x) = \operatorname{sgn}(x)(-f''(x) + q(x)f(x))$  on the interval  $[a, \infty)$ ,  $-\infty < a < 0$ , with real potential  $q \in L^1(a, \infty)$ . The bounds depend only on the  $L^1$ -norm of the negative part of  $q$  and the boundary condition at the regular endpoint  $a$ .

## 1. INTRODUCTION AND MAIN RESULT

We consider the Sturm–Liouville differential operator

$$(A_\alpha f)(x) = \operatorname{sgn}(x)(-f''(x) + q(x)f(x)) \quad (1)$$

with

$$D(A_\alpha) = \left\{ f \in L^2(a, \infty) \left| \begin{array}{l} f, f' \in AC[a, \infty), \\ -f'' + qf \in L^2(a, \infty), \\ \cos(\alpha)f(a) = \sin(\alpha)f'(a) \end{array} \right. \right\} \quad (2)$$

in  $L^2(a, \infty)$  for  $\alpha \in [0, \pi)$ , where  $q \in L^1(a, \infty)$  is a real function and  $-\infty < a < 0$ . Here,  $AC[a, \infty)$  denotes the space of functions which are absolutely continuous on every compact subset of  $[a, \infty)$ . As the weight  $\operatorname{sgn}$  changes the sign the operator  $A_\alpha$  is neither symmetric nor self-adjoint in  $L^2(a, \infty)$  with respect to the usual scalar product  $(\cdot, \cdot)$ . Hence,  $A_\alpha$  may have non-real spectrum. But, equipped with the inner product  $[\cdot, \cdot]$ ,

$$[f, g] = \int_a^\infty f(x)\overline{g(x)} \operatorname{sgn}(x) dx, \quad f, g \in L^2(a, \infty),$$

$L^2(a, \infty)$  is a Krein space with the fundamental symmetry  $J : L^2(a, \infty) \rightarrow L^2(a, \infty)$ ,  $(Jf)(x) = \operatorname{sgn}(x)f(x)$ , where  $A_\alpha$  is self-adjoint with respect to  $[\cdot, \cdot]$ ; for the basic notions in Krein spaces we refer to [1] and [7]. Indeed, while the finite endpoint  $a$  is regular the integrability of  $q$  implies the limit point case at the singular endpoint  $\infty$ , cf. [13, Lemma 9.37]. Hence, the definite Sturm–Liouville operator  $JA_\alpha$  on  $D(JA_\alpha) = D(A_\alpha)$  is self-adjoint in the Hilbert space  $L^2(a, \infty)$  and due to  $(\cdot, \cdot) = [J\cdot, \cdot]$  the self-adjointness of  $A_\alpha$  with respect to  $[\cdot, \cdot]$  follows. By [2, Corollary 3.9] the operator  $A_\alpha$  has nonempty resolvent set and its essential spectrum coincides with the essential spectrum of  $JA_\alpha$ , where  $\sigma_{\text{ess}}(JA_\alpha) = [0, \infty)$ , see Theorem 9.38 and the note below in [13].

Recently, bounds for the non-real spectra of indefinite Sturm–Liouville operators were developed in [3, 8, 9, 10, 11, 12] for operators with two regular endpoints and in [4, 5, 6] for operators with two singular endpoints. The result in Theorem 1.1 addresses singular operators with one regular and one singular endpoint. The proof is based on techniques developed in [5, 6]. In the following let  $q = q_+ - q_-$ , where  $q_+(x) = \max\{q(x), 0\}$  and  $q_-(x) = \max\{-q(x), 0\}$ .

**Theorem 1.1.** *The operator  $A_\alpha$ ,  $\alpha \in [0, \pi)$ , in (1) and (2) is self-adjoint with respect to the inner product  $[\cdot, \cdot]$ . Let  $c_\alpha = 0$  if  $\alpha = 0$  and  $c_\alpha = \cot(\alpha)$  if  $\alpha \in (0, \pi)$ . The essential spectrum of  $A_\alpha$  equals  $[0, \infty)$  and the non-real spectrum of  $A_\alpha$  is purely discrete. Every non-real eigenvalue  $\lambda$  of  $A_\alpha$  satisfies*

$$|\operatorname{Im} \lambda| \leq 24\sqrt{3}(\|q_-\|_1 + |c_\alpha|)^2 \quad (3)$$

and

$$|\lambda| \leq (24\sqrt{3} + 18)(\|q_-\|_1 + |c_\alpha|)^2 + 6|c_\alpha|(\|q_-\|_1 + |c_\alpha|). \quad (4)$$

*Proof.* Consider an eigenvalue  $\lambda \in \mathbb{C} \setminus \mathbb{R}$  of  $A_\alpha$  and a corresponding eigenfunction  $f$  with  $\|f\|_2 = 1$ . Let

$$U(x) = \int_x^\infty |f|^2 \operatorname{sgn}, \quad V(x) = \int_x^\infty (|f'|^2 + q|f|^2), \quad (5)$$

for  $x \in [a, \infty)$ . One can show that  $f$  satisfies  $\lim_{x \rightarrow \infty} f'(x)\overline{f(x)} = 0$ ,  $f' \in L^2(a, \infty)$  and  $q|f|^2 \in L^1(a, \infty)$ , cf. [6, Appendix A]. Hence, the functions  $V$  and  $U$  given by (5) are well-defined on  $[a, \infty)$  with values in  $\mathbb{R}$ . Moreover,  $\lim_{x \rightarrow \infty} U(x) = 0$  and  $\lim_{x \rightarrow \infty} V(x) = 0$ . Integration by parts together with the eigenvalue equation  $\lambda f = A_\alpha f$  yields

$$\lambda U(x) = \int_x^\infty (A_\alpha f)\overline{f} \operatorname{sgn} = V(x) + f'(x)\overline{f(x)}. \quad (6)$$

As  $f \in D(A_\alpha)$  we have  $f'(a)\overline{f(a)} = c_\alpha |f(a)|^2 \in \mathbb{R}$ . Evaluating (6) at  $x = a$  and comparing the imaginary parts we obtain  $U(a) = 0$  and  $V(a) = -c_\alpha |f(a)|^2$ . Hence,

$$0 \leq \|f'\|_2^2 = - \int_a^\infty (q_+ - q_-)|f|^2 - c_\alpha |f(a)|^2 \leq (\|q_-\|_1 + |c_\alpha|)\|f\|_\infty^2. \quad (7)$$

Furthermore, this implies

$$\int_a^\infty q_+ |f|^2 \leq (\|q_-\|_1 + |c_\alpha|)\|f\|_\infty^2, \quad \|qf^2\|_1 \leq 2(\|q_-\|_1 + |c_\alpha|)\|f\|_\infty^2. \quad (8)$$

Here, the norm  $\|f\|_\infty$  can be estimated as follows. Since  $f \in L^2(\mathbb{R})$  is continuous there exists a sequence  $(y_n)_{n \in \mathbb{N}}$  in  $(a, \infty)$  with  $y_n \rightarrow \infty$  and  $f(y_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,

$$|f(x)|^2 = |f(y_n)|^2 + 2 \operatorname{Re} \int_{y_n}^x f' \overline{f}, \quad \|f\|_\infty^2 \leq 2\|f'\|_2,$$

where we used the Cauchy–Schwarz inequality and  $\|f\|_2 = 1$  in the last estimate. This together with (7) leads to

$$\|f'\|_2^2 \leq 4(\|q_-\|_1 + |c_\alpha|)^2 \quad \text{and} \quad \|f\|_\infty^2 \leq 4(\|q_-\|_1 + |c_\alpha|). \quad (9)$$

Observe, that it is no restriction to consider  $\|q_-\|_1 + |c_\alpha| > 0$  since otherwise  $f$  is constantly zero. We define an absolutely continuous function  $g$  by

$$g(x) = \begin{cases} \frac{x}{\delta} & \text{if } x \in (-\delta, \delta), \\ \operatorname{sgn}(x) & \text{if } x \in [a, -\delta] \cup [\delta, \infty), \end{cases} \quad \text{where } \delta = \frac{1}{24(\|q_-\|_1 + |c_\alpha|)}.$$

Here, the interval  $[a, -\delta]$  is considered to be empty if  $-\delta < a$ . We have  $\|g\|_\infty = 1$  and  $\|g'\|_2 = \sqrt{2/\delta}$ . Then

$$\begin{aligned} \int_a^\infty g'U &= \int_a^\infty g|f|^2 \operatorname{sgn} \geq \int_{(a, \infty) \setminus (-\delta, \delta)} |f|^2 = 1 - \int_{-\delta}^\delta |f|^2 \\ &\geq 1 - 2\delta\|f\|_\infty^2 \geq 1 - 8\delta(\|q_-\|_1 + |c_\alpha|) \geq \frac{2}{3}. \end{aligned} \quad (10)$$

Further, we obtain with (8) and (9)

$$\begin{aligned} \left| \int_a^\infty g'V \right| &= \left| \int_a^\infty g(|f'|^2 + q|f|^2) - g(a)V(a) \right| \leq \|f'\|_2^2 + \|qf^2\|_1 + |c_\alpha| \|f\|_\infty^2 \\ &\leq 12(\|q_-\|_1 + |c_\alpha|)^2 + 4|c_\alpha|(\|q_-\|_1 + |c_\alpha|) \end{aligned} \quad (11)$$

and with (9)

$$\begin{aligned} \left| \int_a^\infty g'\bar{f}f' \right| &\leq \|f\|_\infty \|f'\|_2 \|g'\|_2 \leq 4\sqrt{2/\delta}(\|q_-\|_1 + |c_\alpha|)^{\frac{3}{2}} \\ &\leq 16\sqrt{3}(\|q_-\|_1 + |c_\alpha|)^2. \end{aligned} \quad (12)$$

By (6) we have

$$\lambda \int_a^\infty g'U = \int_a^\infty g'(V + f'\bar{f}). \quad (13)$$

A comparison of the imaginary parts and the absolute values in (13) together with the estimates (10)–(12) shows (3) and (4).  $\square$

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INSTITUT FÜR ANGEWANDTE MATHEMATIK, TECHNISCHE UNIVERSITÄT GRAZ, STEYRERGASSE 30, A-8010 GRAZ, AUSTRIA

*E-mail address:* behrndt@tugraz.at

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT ILMENAU, POSTFACH 100565, D-98684 ILMENAU, GERMANY

*E-mail address:* philipp.schmitz@tu-ilmenau.de

INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT ILMENAU, POSTFACH 100565, D-98684 ILMENAU, GERMANY

*E-mail address:* carsten.trunk@tu-ilmenau.de