

## ALGEBRAICALLY EXTENDED 2D IMAGE REPRESENTATION

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**Abstract.** *This paper presents an algebraically extended 2D image representation. Combining methods of tensor algebra, monogenic signal and quadrature filter, this novel image representation can be derived as the monogenic extension of a curvature tensor. From it, the monogenic signal and the monogenic curvature signal for modeling intrinsically one and two dimensional (i1D/i2D) structures are obtained as special cases. Local amplitude, phase and orientation are correspondingly extracted. Compared with the related work, our approach has the advantage of simultaneous estimation of local phase and orientation. The main contribution is the rotationally invariant phase estimation, which enables many phase-based applications in computer vision research.*

# 1 INTRODUCTION

Model based image representation plays an important role in many computer vision tasks. There are bulk of researches for intensity based image representation, see [7, 16, 15, 4]. However, those approaches suffer from illumination variations. Therefore, this intensively investigated area of research is not adequate to model local structures. Phase information carries most essential structure information of the original image [17]. It is invariant with respect to illumination changes. Consequently, model based image representation should combine both the intensity and phase information.

In 1D signal processing, the analytic signal [9] is an important complex valued signal representation which enables the local amplitude and phase extraction from the original signal. For 2D images, there exist infinite many types of structures which can be classified into three categories according to the intrinsic dimensionality [21] as a local property of a signal. Hence, 2D images can locally belong to the intrinsically zero dimensional (i0D) signals which are constant signals, intrinsically one dimensional (i1D) signals representing straight lines and edges and intrinsically two dimensional (i2D) signals which do not belong to the above two cases. The i2D structures are composed of curved edges and lines, junctions, corners and line ends, etc. The i1D and i2D structures carry most of the important information of the image. Therefore, correct representation of them has great significance for many computer vision applications.

In the literature, there exist many approaches for 2D image representation. The structure tensor [7] and the boundary tensor [15] estimate the main orientation and the energy of the i2D signal. However, the phase is neglected. In [16], a nonlinear image operator for the detection of locally i2D signals was proposed, but it captures no information about the phase. There are lots of papers concerned with applications of the analytic signal for image analysis. But they have serious problems in transferring that concept from 1D to 2D in a rotation-invariant way. The partial Hilbert transform and the total Hilbert transform [11] provide some representations of the phase in 2D. Unfortunately, they lack the property of rotation invariance and are not adequate for detecting i2D features. Bülow and Sommer [2] proposed the quaternionic analytic signal, which enables the evaluation of the i2D signal phase. However, this approach also has the drawback of being not rotationally invariant. For i1D signals, Felsberg and Sommer [5] proposed the monogenic signal as a novel model. It is a rotationally invariant generalization of the analytic signal in 2D and higher dimensions. From the monogenic signal, the local amplitude and a local phase representation can be simultaneously extracted. However, it captures no information of the i2D part. A 2D phase model is proposed in [3], where the i2D signal is split into two perpendicularly superposed i1D signals and the corresponding two phases are evaluated. It delivers a new description of i2D structure by a so-called structure multivector. Unfortunately, steering is needed and only i2D patterns superimposed by two perpendicular i1D signals can be correctly handled.

In this paper, we present a novel 2D image representation. By embedding our problem into a certain geometric algebra, more degrees of freedom can be obtained. Coupling the methods of tensor algebra, monogenic signal and quadrature filter, the 2D image representation can be derived as the monogenic extension of a curvature tensor. From this model, local signal representations for i1D and i2D structures are obtained as the monogenic signal [5] and the monogenic curvature signal, respectively. Thus, local amplitude, phase and orientation are able to be extracted in this unique framework in a rotation invariant manner.

## 2 GEOMETRIC ALGEBRA

The way we intend to derive a representation for the 2D structure is to some extent a generalization of the analytic signal. It cannot be realized in the domain of complex numbers. Instead, a more powerful algebraic system should be taken into consideration. Geometric algebras constitute a rich family of algebras as generalization of vector algebra [12]. Compared with the classical framework of vector algebra, the geometric algebra makes a tremendous extension of modeling capabilities available. For the problem we are concerned, the 2D signal will be algebraically embedded into the Euclidean 3D space. Therefore, in this section, we give a brief introduction to the geometric algebra over 3D Euclidean space ( $\mathbb{R}_3$ ).

The Euclidean space  $\mathbb{R}^3$  is spanned by the orthonormal basis vectors  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ . The geometric algebra,  $\mathbb{R}_3$ , of the 3D Euclidean space consists of  $2^3 = 8$  elements,

$$\mathbb{R}_3 = \text{span}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}, \mathbf{e}_{123} = I_3\} \quad (1)$$

Here  $\mathbf{e}_{23}$ ,  $\mathbf{e}_{31}$  and  $\mathbf{e}_{12}$  are the unit bivectors and the element  $\mathbf{e}_{123}$  is a unit trivector or unit pseudoscalar. The vectors square to one and bivectors and the trivector square to -1. A general combination of these elements is called a multivector, e.g.  $M = a + b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_3 + e\mathbf{e}_{23} + f\mathbf{e}_{31} + g\mathbf{e}_{12} + hI_3$ . The basic product of a geometric algebra is the geometric product, i.e.  $M_1M_2$ , where  $M_1$  and  $M_2$  are two multivectors. Because the square of the bivector or trivector equals -1, the imaginary unit  $i$  of the complex numbers can be substituted by a bivector or a trivector, yielding an algebra isomorphism. The  $k$ -grade part of a multivector is obtained from the grade operator  $\langle M \rangle_k$ . Hence,  $\langle M \rangle_0$  is the scalar part of  $M$ ,  $\langle M \rangle_1$  represents the vector part,  $\langle M \rangle_2$  indicates the bivector part and  $\langle M \rangle_3$  is the trivector part, which commutes with every element of the  $\mathbb{R}_3$ . If only the scalar and the bivectors are involved, the combined result is called a spinor, i.e.  $S = a + e\mathbf{e}_{23} + f\mathbf{e}_{31} + g\mathbf{e}_{12}$ . All spinors form a proper subalgebra of  $\mathbb{R}_3$ , that is the even subalgebra  $\mathbb{R}_3^+$ . A vector-valued signal  $\mathbf{f}$  in  $\mathbb{R}_3$  can be considered as the result of a spinor acting on the  $\mathbf{e}_3$  basis, i.e.  $\mathbf{f} = b\mathbf{e}_1 + c\mathbf{e}_2 + d\mathbf{e}_3 = \mathbf{e}_3S$ . The transformation performed under the action of the spinor delivers access to both the amplitude and phase information of the vector-valued signal  $\mathbf{f}$  [20]. From the logarithm of the spinor representation, two parts can be obtained. They are the scaling which corresponds to the local amplitude and the rotation which corresponds to the local phase representation. The  $\mathbb{R}_3$ -logarithm of a spinor  $S \in \mathbb{R}_3^+$  takes the following form

$$\log(S) = \langle \log(S) \rangle_0 + \langle \log(S) \rangle_2 = \log(|S|) + \frac{\langle S \rangle_2}{|\langle S \rangle_2|} \text{atan} \left( \frac{|\langle S \rangle_2|}{\langle S \rangle_0} \right) \quad (2)$$

where  $\text{atan}$  is the arc tangent mapping for the interval  $[0, \pi)$ . The scalar part  $\langle \log(S) \rangle_0 = \log(|S|)$  illustrates the logarithm of the local amplitude, hence, local amplitude is obtained as the exponential of it,

$$|S| = \exp(\log|S|) = \exp(\langle \log(S) \rangle_0) \quad (3)$$

The bivector part of  $\log(S)$  indicates the local phase representation,

$$\arg(S) = \langle \log(S) \rangle_2 = \frac{\langle S \rangle_2}{|\langle S \rangle_2|} \text{atan} \left( \frac{|\langle S \rangle_2|}{\langle S \rangle_0} \right) \quad (4)$$

## 3 2D SPHERICAL HARMONICS

In order to analyze 2D patterns, we choose the 2D spherical harmonics as basis functions according to the proposal in [3]. Since the angular behavior of a signal can be regarded as

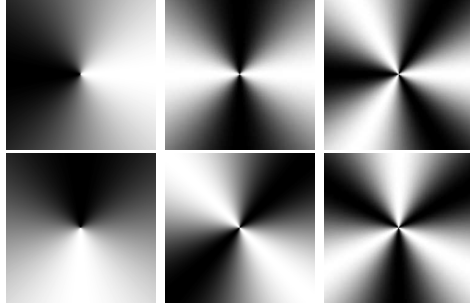


Figure 1: The 2D spherical harmonics from order one to three in the frequency domain. Every 2D spherical harmonic consists of two orthogonal components.

band limited, only spherical harmonics of order zero to three are applied, otherwise, aliasing would occur. To build the signal representation, we are more concerned of the angular portions. Therefore, we use the polar representation of spherical harmonics instead of the Cartesian form stated in [3]. In the frequency domain, spherical harmonics have much simpler forms than that in the spatial domain. Spherical harmonics in the spectral domain read

$$H_n = \exp(n\alpha\mathbf{e}_{12}) = \cos(n\alpha) + \sin(n\alpha)\mathbf{e}_{12} \quad (5)$$

where  $n$  indicates the order of the spherical harmonic, and  $\alpha$  represents the angle in polar coordinates. Every spherical harmonic consists of two orthogonal components. The first order spherical harmonic is basically identical to the Riesz kernel [5]. Figure 1 illustrates 2D spherical harmonics from order one to three, every spherical harmonic consists of two orthogonal components.

In practice, 2D spherical harmonics are combined with radial bandpass filters to form polar separable filters. In this paper, the difference of Poisson (DOP) kernel [3] is employed as bandpass filter. As a result, local signal analysis can be realized in a multi-scale approach in the monogenic scale-space [6]. The DOP is an isotropic bandpass filter which in spectral domain takes the form

$$H_{DOP}(\rho; s) = \exp(-2\pi\rho s_2) - \exp(-2\pi\rho s_1) \quad (6)$$

where  $s_1$  and  $s_2$  represent the fine and coarse scales parameters, respectively.

## 4 ALGEBRAICALLY EXTENDED 2D IMAGE REPRESENTATION

The proposed representation is the monogenic extension of the curvature tensor. Motivated from the differential geometry, this curvature tensor can be constructed. Therefore, a brief introduction to the differential geometry is given.

### 4.1 Differential geometry in image processing

In the image processing field, Koenderink and van Doorn [13, 14] have introduced methods from differential geometry to analyze the local properties of signals. In such case, two dimensional intensity data can be represented as surfaces in 3D Euclidean space. Such surfaces in geometrical terms can be written as Monge patches of the form

$$\mathbf{x} + \mathbf{f}(\mathbf{x}) = x\mathbf{e}_1 + y\mathbf{e}_2 + f(x, y)\mathbf{e}_3 \quad (7)$$

In the following, we will introduce basic concepts of differential geometry and the general 2D signal model in a algebraic framework with more powerful geometric meanings than  $\mathbb{R}_3$ .

Because we are interested in a tensor representation of the image signal, our model will thus be represented in the matrix geometric algebra  $M(2, \mathbb{R}_3)$  which results from the tensor product  $\mathbb{R}_3 \times \mathbb{R}_3$ . The matrix geometric algebra  $M(2, \mathbb{R}_3)$ , see [19], is the geometric algebra of  $2 \times 2$  matrices with entities in  $\mathbb{R}_3$ .

The primary first-order differential quantity for an image is the gradient, which is defined as

$$\nabla \mathbf{f} = \begin{bmatrix} \mathbf{e}_1 \frac{\partial}{\partial x} f(x, y) \mathbf{e}_3 \\ \mathbf{e}_2 \frac{\partial}{\partial y} f(x, y) \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} f_x \mathbf{e}_{13} \\ f_y \mathbf{e}_{23} \end{bmatrix} \quad (8)$$

Analogously, for second-order geometry, the matrix of second derivatives or Hessian  $H$  is given by

$$H = \begin{bmatrix} \mathbf{e}_1 \frac{\partial}{\partial x} f_x \mathbf{e}_{13} & \mathbf{e}_2 \frac{\partial}{\partial y} f_x \mathbf{e}_{13} \\ \mathbf{e}_1 \frac{\partial}{\partial x} f_y \mathbf{e}_{23} & \mathbf{e}_2 \frac{\partial}{\partial y} f_y \mathbf{e}_{23} \end{bmatrix} = \begin{bmatrix} f_{xx} \mathbf{e}_3 & -f_{xy} \mathbf{e}_{123} \\ f_{xy} \mathbf{e}_{123} & f_{yy} \mathbf{e}_3 \end{bmatrix} \quad (9)$$

The Hessian matrix is related to the curvature tensor, which describes the local derivation of the signal  $\mathbf{f}$  on the tangent plane of the surface.

According to the derivative theorem of Fourier theory [18, 1], in the spectral domain, the Hessian matrix reads

$$\mathcal{F}\{H\} = \begin{bmatrix} -4\pi^2 \rho^2 \frac{1+\cos(2\alpha)}{2} \mathbf{F} & (4\pi^2 \rho^2 \frac{\sin(2\alpha)}{2} \mathbf{F}) \mathbf{e}_{12} \\ (-4\pi^2 \rho^2 \frac{\sin(2\alpha)}{2} \mathbf{F}) \mathbf{e}_{12} & -4\pi^2 \rho^2 \frac{1-\cos(2\alpha)}{2} \mathbf{F} \end{bmatrix} \quad (10)$$

where  $\mathbf{F}$  denotes the Fourier transform of the original signal  $\mathbf{f} = f(x, y) \mathbf{e}_3$ . It is obvious that the angular parts of the derivatives are related to spherical harmonics of even orders 0 and 2.

In the heart of differential geometry, we find that the first and second fundamental theorems describe the inner and exterior geometry of a surface. We are here only interested in the exterior geometry represented by the curvature tensor. It is well known that the Gaussian and mean curvatures can be computed according to the determinant and the trace of the curvature tensor, respectively. From these two curvatures, basic types of local geometry (elliptic, hyperbolic, parabolic and planar surfaces) can be decided. We will take advantage from these facts in connection with our algebraic signal embedding.

## 4.2 Curvature tensor and its monogenic extension

In the following, we will extend the ideas of deriving the monogenic signal from a real valued 2D image signal. For a 2D image, every image point is now associated with a curvature tensor which is related to the Hessian matrix. This curvature tensor  $T_e$  indicates the even information of 2D structures and is obtained from a tensor-valued filter  $H_e$  in the frequency domain, i.e.  $T_e = \mathcal{F}^{-1}\{\mathbf{F}H_e\}$ , where  $\mathcal{F}^{-1}$  means the inverse Fourier transform. Since the original 2D signal  $f(x, y)$  is embedded as an  $\mathbf{e}_3$ -valued signal, the tensor-valued filter  $H_e$ , called the even filter, thus takes the following form

$$\begin{aligned} H_e &= \frac{1}{2} \begin{bmatrix} H_0 + \langle H_2 \rangle_0 & -\langle H_2 \rangle_2 \\ \langle H_2 \rangle_2 & H_0 - \langle H_2 \rangle_0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + \cos(2\alpha) & -\sin(2\alpha) \mathbf{e}_{12} \\ \sin(2\alpha) \mathbf{e}_{12} & 1 - \cos(2\alpha) \end{bmatrix} = \begin{bmatrix} \cos^2(\alpha) & -\frac{1}{2} \sin(2\alpha) \mathbf{e}_{12} \\ \frac{1}{2} \sin(2\alpha) \mathbf{e}_{12} & \sin^2(\alpha) \end{bmatrix} \end{aligned} \quad (11)$$

Entities of  $H_e$  are obtained from Eq. (10).

In this filter, the two elements  $\cos^2(\alpha)$  and  $\sin^2(\alpha)$  can be considered as two angular windowing functions which are the same as those of the orientation tensor in [10]. From them, two

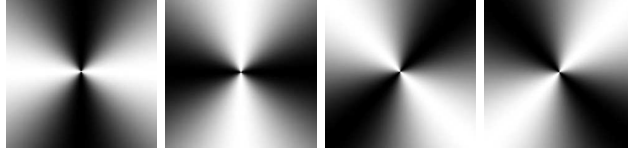


Figure 2: From left to right are the angular windowing functions of  $\cos^2(\alpha)$ ,  $\sin^2(\alpha)$ ,  $\cos^2(\alpha - \frac{\pi}{4})$  and  $\sin^2(\alpha - \frac{\pi}{4})$ . White indicates positive one and black represents zero.

perpendicular 1D components of the 2D image, oriented along the  $\mathbf{e}_1$  and  $\mathbf{e}_2$  coordinates, can be obtained. The other component of the filter is also the combination of two angular windowing functions, i.e.  $\frac{1}{2}\sin(2\alpha) = \frac{1}{2}(\cos^2(\alpha - \frac{\pi}{4}) - \sin^2(\alpha - \frac{\pi}{4}))$ . These two angular windowing functions yield again two 1D components of the 2D image, which are oriented along the diagonals of the plane spanned by  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . These four angular windowing functions, shown in Fig. 2, can also be considered as four differently oriented filters, which are basis functions to steer a detector for 1D structures [8]. They make sure that 1D components along different orientations are extracted.

The Riesz transform [5] is able to evaluate the corresponding conjugate information of the 1D signal, which is in quadrature phase relation with the 1D signal. Therefore, the odd representation of the curvature tensor, called the conjugate curvature tensor  $T_o$ , is obtained by employing the Riesz transform  $h_R \equiv h_1$  to elements of  $T_e$ , which equals the Riesz transform of the curvature tensor  $T_e$ . Besides, the conjugate curvature tensor  $T_o$  results also from a tensor-valued odd filter  $H_o$ .

$$T_o = h_R * T_e = \mathcal{F}^{-1} \{H_o \mathbf{F}\} \quad (12)$$

The odd filter  $H_o$  equals the Riesz transform of the even filter, i.e.  $H_o = H_R H_e$  with  $H_R \equiv H_1$ . In the spectral domain, the odd filter thus takes the following form

$$H_o \equiv \begin{bmatrix} H_{o11} & H_{o12} \\ H_{o21} & H_{o22} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} H_1(H_0 + \langle H_2 \rangle_0) & H_1(-\langle H_2 \rangle_2) \\ H_1(\langle H_2 \rangle_2) & H_1(H_0 - \langle H_2 \rangle_0) \end{bmatrix} \quad (13)$$

with

$$\begin{aligned} H_{o11} &= (\cos(\alpha) + \sin(\alpha)\mathbf{e}_{12})(\cos^2(\alpha)) \\ &= \frac{1}{4} [(3\cos(\alpha) + \cos(3\alpha)) + (\sin(\alpha) + \sin(3\alpha))\mathbf{e}_{12}] \end{aligned} \quad (14)$$

$$\begin{aligned} H_{o21} &= -H_{o12} = (\cos(\alpha) + \sin(\alpha)\mathbf{e}_{12})(\frac{1}{2}\sin(2\alpha))\mathbf{e}_{12} \\ &= \frac{1}{4} [-(\cos(\alpha) - \cos(3\alpha)) + (\sin(\alpha) + \sin(3\alpha))\mathbf{e}_{12}] \end{aligned} \quad (15)$$

$$\begin{aligned} H_{o22} &= (\cos(\alpha) + \sin(\alpha)\mathbf{e}_{12})(\sin^2(\alpha)) \\ &= \frac{1}{4} [(\cos(\alpha) - \cos(3\alpha)) + (3\sin(\alpha) - \sin(3\alpha))\mathbf{e}_{12}] \end{aligned} \quad (16)$$

It is obvious that this tensor-valued filter consists of odd order spherical harmonics. Hence, the Riesz transform of the curvature tensor  $T_e$  gives its corresponding odd representation  $T_o$ . Combing the curvature tensor and its conjugate representation forms the algebraically extended 2D image representation, i.e.  $T = T_e + T_o$ . This representation can be regarded as the monogenic extension of the curvature tensor. Hence, it is called the monogenic curvature tensor.

### 4.3 Local representations for i1D and i2D image structures

Analogous with the Hessian matrix, since the non-zero determinant indicates the existence of i2D structure, the even and odd parts of i2D structures are obtained from the determinants of the curvature tensor and its odd representation, respectively. The even part of i2D structures reads

$$\mathbf{d}_e = \det(T_e)\mathbf{e}_3 = A\mathbf{e}_3 \quad (17)$$

Since the determinant of the curvature tensor is scalar valued, similar as the monogenic signal, the even part of i2D structures is embedded as the  $\mathbf{e}_3$  component in the 3D Euclidean space. The odd part of i2D structures is

$$\mathbf{d}_o = \det(T_o)\mathbf{e}_2 = B\mathbf{e}_1 + C\mathbf{e}_2 \quad (18)$$

Because  $\det(T_o)$  is spinor valued, by multiplying the  $\mathbf{e}_2$  basis from the right,  $\mathbf{d}_o$  takes a vector valued representation. A local representation for i2D structures is obtained by combining the even and odd parts of i2D structures. This local representation for i2D structures is called the monogenic curvature signal and it takes the following form

$$\mathbf{f}_{i2D} = \mathbf{d}_e + \mathbf{d}_o = A\mathbf{e}_3 + B\mathbf{e}_1 + C\mathbf{e}_2 \quad (19)$$

The parabolic and planar surface patches, corresponding to i1D and i0D structures, have zero determinants of the monogenic curvature tensor. In order to separate them with each other, the trace of the tensor pair  $T_e$  and  $T_o$  is computed. Non-zero trace illustrates the existence of i1D structure. Therefore, combination of traces of  $T_e$  and  $T_o$  can be considered as the local representation of i1D structures, that is

$$\mathbf{f}_{i1D} = \text{trace}(T_e) + \text{trace}(T_o) = \mathcal{F}^{-1} \{ \text{trace}(H_e\mathbf{F}) + \text{trace}(H_o\mathbf{F}) \} \quad (20)$$

$$\begin{aligned} &= \mathcal{F}^{-1} \{ (\text{trace}(H_e) + \text{trace}(H_o))\mathbf{F} \} \\ &= \mathcal{F}^{-1} \{ (1 + H_1)\mathbf{F} \} = \mathbf{f} + h_1 * \mathbf{f} = \mathbf{f} + h_R * \mathbf{f} \end{aligned} \quad (21)$$

where  $h_R$  is the spatial representation of the Riesz kernel. This indicates that the local representation for i1D structures, obtained from the proposed general signal model, is the combination of the original signal and its Riesz transform. This means, the derived i1D structure representation is just the monogenic signal as proposed in [5]. Hence, the proposed signal representation includes the monogenic signal and the monogenic curvature signal as special cases to represent i1D and i2D image structures.

### 4.4 Local features of the monogenic curvature signal

Since the monogenic signal has been investigated in detail in [5], this section mainly discusses the local features of the monogenic curvature signal.

From the monogenic curvature signal, three independent local features can be extracted. In the light of the introduction in Section 2, local features of the monogenic curvature signal can be defined using the logarithm of  $\mathbb{R}_3^+$ . The spinor field which maps the  $\mathbf{e}_3$  basis vector to the monogenic curvature signal  $\mathbf{f}_{i2D}$  is given by  $\mathbf{f}_{i2D}\mathbf{e}_3$ . According to equations (3) and (4), the local amplitude and local phase representation are obtained as

$$|\mathbf{f}_{i2D}| = \exp(\langle \log(\mathbf{f}_{i2D}\mathbf{e}_3) \rangle_0) = \exp(\log(|\mathbf{f}_{i2D}\mathbf{e}_3|)) \quad (22)$$

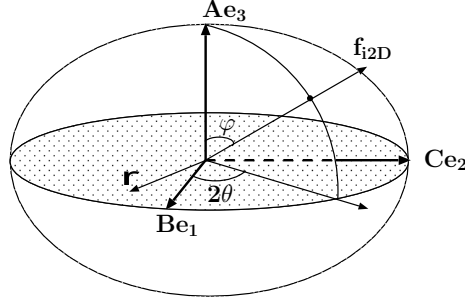


Figure 3: The geometric model for the monogenic curvature signal. Here,  $Ae_3$  indicates the even information of the i2D structure,  $Be_1$  and  $Ce_2$  are two components of the odd part. Phase is represented by  $\varphi$ ,  $2\theta$  denotes the main orientation in terms of double angle representation,  $\mathbf{r}$  indicates the rotation vector.

$$\Phi = \arg(\mathbf{f}_{i2D}) = \langle \log(\mathbf{f}_{i2D}\mathbf{e}_3) \rangle_2 = \frac{\langle \mathbf{f}_{i2D}\mathbf{e}_3 \rangle_2}{|\langle \mathbf{f}_{i2D}\mathbf{e}_3 \rangle_2|} \operatorname{atan} \left( \frac{|\langle \mathbf{f}_{i2D}\mathbf{e}_3 \rangle_2|}{\langle \mathbf{f}_{i2D}\mathbf{e}_3 \rangle_0} \right) \quad (23)$$

where  $\arctan(\cdot) \in [0, \pi)$ ,  $\arg(\cdot)$  denotes the argument of the expression and  $\frac{\langle \mathbf{f}_{i2D}\mathbf{e}_3 \rangle_2}{|\langle \mathbf{f}_{i2D}\mathbf{e}_3 \rangle_2|}$  indicates the local main orientation vector. As the bivector part of the logarithm of the spinor field  $\mathbf{f}_{i2D}\mathbf{e}_3$ , this local phase representation describes a rotation from the  $\mathbf{e}_3$  axis by a phase angle  $\varphi$  in the oriented complex plane spanned by  $\mathbf{f}_{i2D}$  and  $\mathbf{e}_3$ , i.e.  $\mathbf{f}_{i2D} \wedge \mathbf{e}_3$ . The orientation of this complex plane indicates the local orientation. Therefore, the local phase representation combines local phase and local orientation of i2D structures. The dual of the local phase representation  $\Phi$  is a rotation vector

$$\mathbf{r} = (\arg(\mathbf{f}_{i2D}))^* = \langle \log(\mathbf{f}_{i2D}\mathbf{e}_3) \rangle_2^* \quad (24)$$

The rotation vector  $\mathbf{r}$  is orthogonal to the local orientation and its absolute value represents the phase angle of the i2D structure. With the algebraic embedding, a geometric model for the monogenic curvature signal can be visualized, see Figure 3.

Given a synthetic image shown in Figure 4, local features can be simultaneously extracted from the monogenic curvature signal. Those blobs in this test image are considered as i2D structures. The square of the local amplitude is the local energy which illustrates the existence of i2D structures. The estimated local energy also indicates the rotation invariant property of the monogenic curvature signal. Local main orientation denotes the main orientation of the i2D structure, its minor orientation is simply perpendicular to the main orientation. The evaluated local phase contains structure information with respect to the occurrence of even and odd symmetric i1D structures meeting at the location selected by the monogenic curvature operator.

## 5 CONCLUSIONS

In this paper, an algebraically extended 2D image representation is presented. A 2D image is embedded into a certain geometric algebra to obtain more degrees of freedom. Coupling methods of differential geometry, tensor algebra, monogenic signal and quadrature filter, this novel image representation can be derived as the monogenic extension of the curvature tensor. Based on it, local representations for i1D and i2D structures are obtained as the monogenic signal and the monogenic curvature signal. As in the i1D case, also for i2D structures, from the monogenic curvature signal, local features of amplitude, phase and orientation can be simultaneously estimated in a unique framework. Compared with other related work, the proposed representation



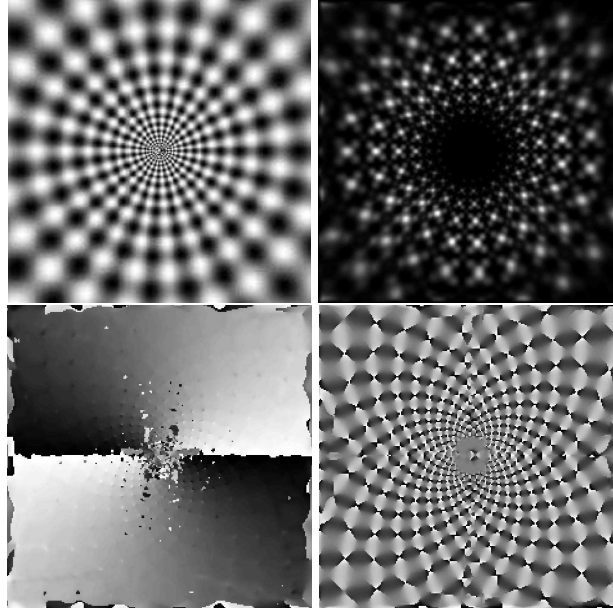


Figure 4: Top row: a synthetic image and the energy output from the monogenic curvature signal. Bottom row: local main orientation and phase estimation from the monogenic curvature signal.

has the advantage of enabling local features evaluation from a unique framework in a rotation invariant manner.

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