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PROBABILISTIC LIMIT STATE ANALYSIS OF MONUMENTAL STRUCTURE BY MONTE CARLO SIMULATION

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Abstract. *The concept is presented to determine the sensitivity of the limit state of the structure with respect to the selected random variable (or a group of random variables). The sensitivity analysis is performed by a problem-oriented Monte Carlo simulation procedure where the selected variable plays a dominant role. The elementary event is defined as a structural limit state, the sample space consists of limit states. One-dimensional random multiplier is defined on the sample space, composed of limit states. This multiplier refers to the dominant basic variable (group of variables) of the problem. Numerical procedure results in the histogram – the estimator of the PDF of the limit state of the structure. Estimators of reliability, or the probability of failure are statistical characteristics of this histogram. Example of sensitivity analysis of the serviceability limit state of monumental structure illustrates the procedure. Colonnade of Licheń Basilica, situated in central Poland, is examined with respect to the upper deck horizontal deflection. The wind load intensities acting on the lower and on the upper storey of the colonnade, respectively, are identically distributed, but correlated random variables. Three correlation variants of these variables are considered. Relevant limit state histograms are analysed thereafter. The paper ends with the conclusions referring to the method and some general remarks on the fully probabilistic design.*

1 INTRODUCTION

Stochastic problems of structural mechanics are treated as the distinct research domain from several decades. These problems did not lose their importance up till now.

Basic design variables – loads, material and geometrical parameters, imperfections, are frequently recognized as random variables. If random loads are acting on a deterministic structure, the problem is called stochastically linear, whereas the assumption of random structure features makes us classify the problem as stochastically nonlinear.

Probabilistic analysis of structural limit states belongs to the class of problems which deal with the stochastic nonlinear operator. Analytical solutions of these problems are not available at all. Numerical methods are then naturally developed in the field of random limit state analysis.

The present paper considers static problems of elastic – plastic bar structures. The paper introduces the sensitivity analysis of limit states of structures with respect to a selected random variable (or a group of random variables). The sensitivity is presented in the form of the probability distribution of the limit state of the structure.

This concept is presented in the form of a problem-oriented Monte Carlo simulation procedure, where the selected variable plays the dominant role. This procedure was described by the author in the paper [1].

Numerical example concerning the serviceability limit state of the monumental structure illustrates the procedure. In this example dominant is the wind action, in the form of two correlated random variables.

Investigations done within the present paper can be classified into the branch of computational sciences, because the numerical procedure is the core of the presented concept. Generally speaking, modelling and computer simulation is nowadays treated as the third base of contemporary science, complementary to theory and experiment. The method developed in the paper can be therefore included into the field of computational sciences.

2 PROBABILITY DISTRIBUTION OF THE LIMIT STATE

The Monte Carlo simulation method is a general numerical tool with a great many engineering applications. With reference to structural design, the procedure consists of three basic stages:

- generation of random numbers, representing basic variables of the problem,
- performing deterministic operations in every simulation step,
- statistical analysis of the set of results, description of the histogram.

In the simulation procedure to examine the structural limit states it is assumed that the basic random variables, with given probability distribution functions, are represented by sets of random numbers.

The elementary event ω is assumed to be the structural limit state. Thus the sample space Ω consists of the limit states of the structure. The uni-dimensional random variable is defined on the sample space. This variable is the multiplier of the dominant basic variable (group of variables) of the problem. The choice of dominant variables must be done primarily.

The main idea of the procedure lies in the performance of a single simulation step. With reference to the general scheme presented above, the single simulation step consists of the following operations:

- generating loads and characteristics of a particular structure in the form of random numbers - establishing a deterministic structure under deterministic loading,
- uni-parametrical increment of dominant variable (or variables), when the limit state is reached, the limit multiplier of dominant basic variables is recorded.

Consequently, various definitions of limit states may be taken.

Assumed the limit states investigated with respect to loads, one-dimensional random variable $\Lambda(\omega)$ is defined on the sample space Ω. Its values are limit load multipliers λ_i of the simulation steps, $i = 1, \ldots, N$, where N is the number of realizations. Histogram of the variable $\Lambda(\omega)$ is the estimator of the probability density function of the limit state with respect to loads. The failure probability estimator $\hat{p}_f$ can be calculated by the formula

$$\hat{p}_f = \frac{1}{N}\sum_{i=1}^{N} I(\lambda_i) \tag{1}$$

where the indicator function is defined as follows

$$I(\lambda_i) = \begin{cases} 1 \ \ for \ \ \lambda_i \le 1.0 & (failure\, region) \\ 0 \ \ for \ \lambda_i > 1.0 & (safe\, region) \end{cases} \tag{2}$$

Establishing material parameters as dominant basic variables, one-dimensional random variable $M(\omega)$ is defined on the sample space Ω. The values of $M(\omega)$ are the limit material multipliers μ_i of all simulation steps, $i = 1, \ldots, N$. Histogram of the variable $M(\omega)$ depicts numerically the probability distribution of the limit state with respect to material parameters. Thus the failure probability estimator $\hat{p}_f$ can be calculated by the formula

$$\hat{p}_f = \frac{1}{N}\sum_{i=1}^{N} I(\mu_i) \tag{3}$$

where the indicator function is given

$$I(\mu_i) = \begin{cases} 1 \ \ for \ \ \mu_i \ge 1.0 & (failure\, region) \\ 0 \ \ for \ \mu_i < 1.0 & (safe\, region) \end{cases} \tag{4}$$

3 NUMERICAL EXAMPLE

3.1 General description

The numerical example concerns the probabilistic serviceability limit state analysis of the monumental structure of the Licheń Basilica ([2]). The Basilica was consecrated in the year 2004. Major load-carrying tower part (Fig. 1) consists of the foundation ring, four-column structure supporting the main ring, the two-storey colonnade and the dome. The lower storey of the colonnade consists of two concentric 16-column rings while in the upper storey there are two concentric 16-column rings (Fig. 2). Space frame model of the colonnade is provided (Fig. 3), consisting of 256 elements. Upper deck deflection of the colonnade is examined.

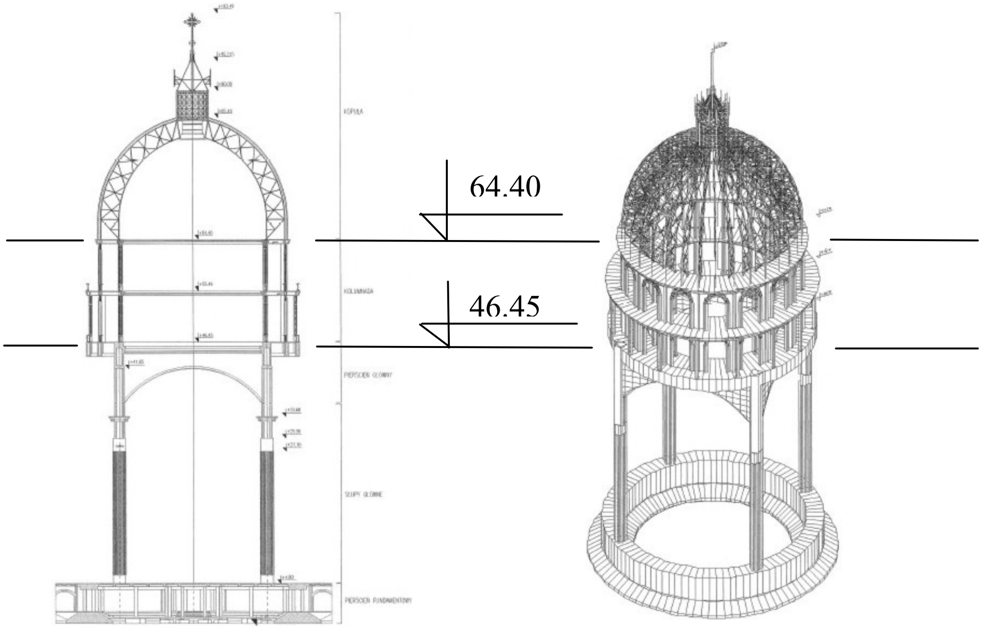

Fig. 1. General view of the tower part of the Basilica, featuring the colonnade

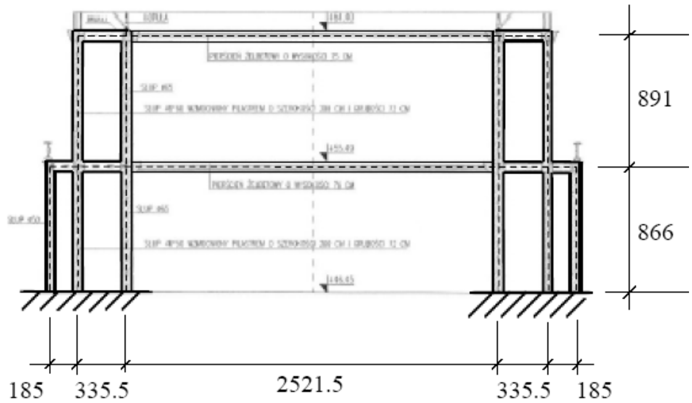

Fig. 2. Cross-section of the colonnade (dimensions in cm)

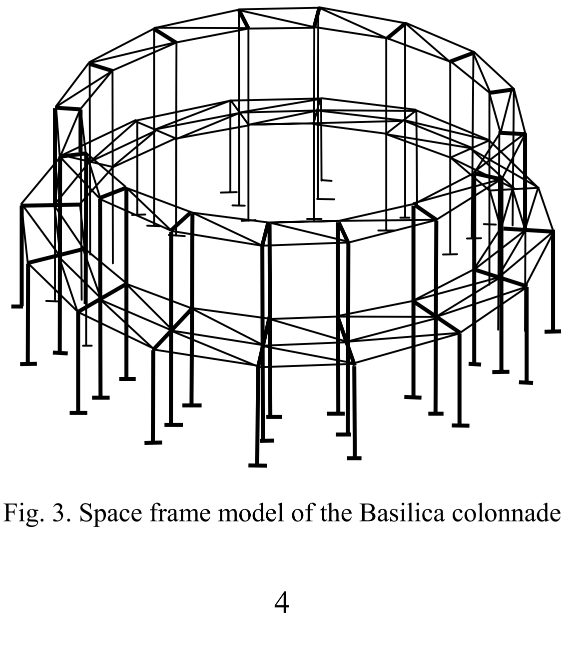
Fig. 3. Space frame model of the Basilica colonnade

Main loads acting on the model are: dead load, wind acting on the colonnade walls and the forces on the upper deck of the colonnade, which come from the dome's weight and the wind acting on the dome. The wind direction is shown in Fig. 3. The wind load is assumed as the uniformly distributed load acting on the columns of lower storey middle ring and on the columns of the upper storey outer ring (rigid plates are provided between the columns). Wind load intensity, as the function of the horizontal angle is shown graphically in Fig. 4.

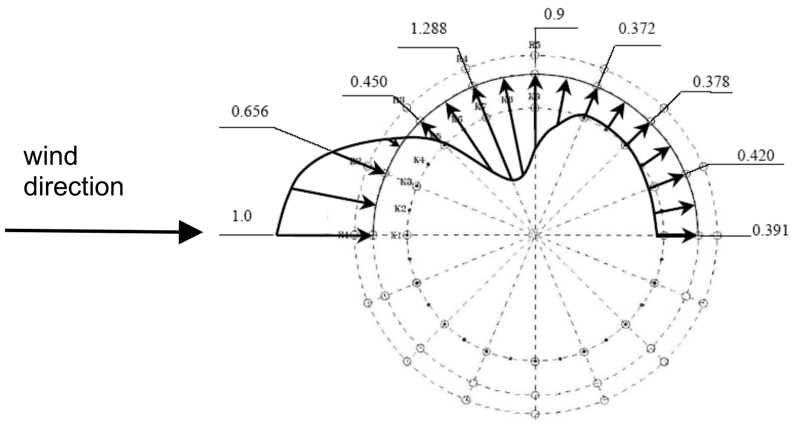

Fig. 4. Situation of the columns of the colonnade on the altitude 46.45 m (three – ring lower colonnade), relative wind load intensities, referring to the maximum value

Basic random variables of the problem were assumed of the following probability distributions (in the form of bounded histograms):

- Dead load of colonnade and dome: Gaussian, N(1.0; 0.0333), the range (0.9, 1.1) – variable $D(\omega)$,
- Young's modulus of concrete: uniform, the range (0.8; 1.0) – variable $E(\omega)$
- Wind load – the variables: $W_1(\omega)$ referring to the lower storey of the colonnade and $W_2(\omega)$ to the colonnade's upper storey and the dome. Both variables are quadratic transforms of the variables V_1 and V_2, which depict wind velocities in appropriate intervals. Assumption is made that the two variables V_1 and V_2 are correlated, Gumbel distributed (extreme value, type I).

Assumption is made that the dominant variables are both the wind actions W_1 and W_2, thus structural sensitivity to wind actions is examined throughout the example.

3.2 Generation of correlated random variables

Three variants of calculations are performed. They differ in the correlation coefficients of the variables $W_1(\omega)$ and $W_2(\omega)$.

The technique to generate correlated random variables of a given covariance matrix is based on the following theorem, formed by Devroye [3]:

Theorem. Let $\mathbf{X} \equiv \{X_i\}$, $i = 1,2,...,d$ be the random vector composed of the i.i.d. random variables of zero mean and unit variance. There exists a nonsingular matrix **H**, that fulfils the equation

$$\mathbf{Y} = \mathbf{HX} \tag{5}$$

where **Y** is the random vector of a given covariance matrix **C**. The matrix **H** may be derived from the equation:

$$\mathbf{H}\mathbf{H}^{\mathrm{T}} = \mathbf{C}. \tag{6}$$

Indeed, the statistical moments of Y satisfy the assumptions:

$$\begin{aligned} E(\mathbf{Y}) &= \mathbf{H}E(\mathbf{X}) = 0 \\ E\left(\mathbf{Y}\mathbf{Y}^{T}\right) &= \mathbf{H}E\left(\mathbf{X}\mathbf{X}^{T}\right)\mathbf{H}^{T} = \mathbf{H}\mathbf{H}^{T} = \mathbf{C} \end{aligned} \tag{7}$$

where $E(\cdot)$ is the expectation operator.

It is worth pointing out that no restrictions are introduced on the type of probability distributions of vectors involved. The problem is to find the matrix **H** from (6), given the matrix **C**. It is possible to build a lower triangular matrix **H** satisfying the equations (7). In the case of two-dimensional random vectors, given the correlation coefficient *h*, the matrix **H** may be derived as:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \\ h & \sqrt{1-h^2} \end{bmatrix}. \tag{8}$$

The following steps are distinguished in the algorithm:

- Generation of the vector **X** consisting of two uncorrelated random variables, uniformly distributed in the range <0, 1> (Fig. 5). Let us assume that all the histograms included in the paper present relative frequencies on their ordinates.

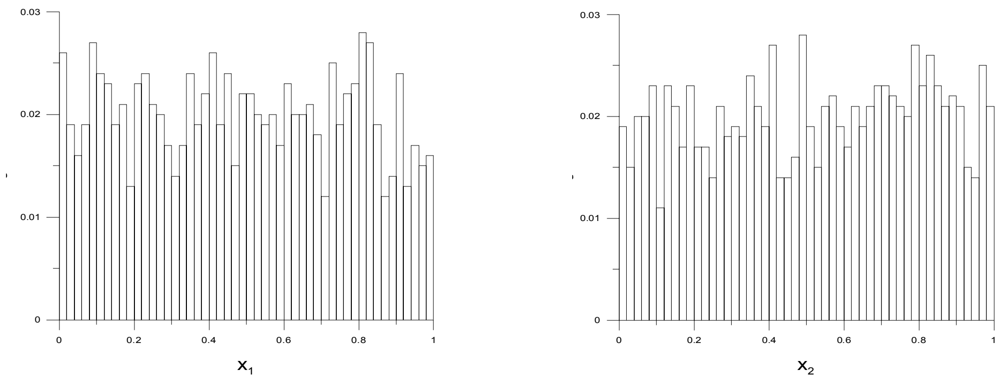

Fig. 5. Histograms of variables X_1 and X_2, uncorrelated, uniformly distributed, the range <0, 1>

- Linear mapping $\mathbf{X} \rightarrow \mathbf{Y}$; **Y** is the vector of uncorrelated uniformly distributed variables of zero mean and unit variance (standardized), according to the formula:

$$Y_i = \left(2X_i - 1\right)\sqrt{3}, \qquad i = 1,2. \tag{9}$$

- Matrix operation $\mathbf{T} = \mathbf{H}\mathbf{Y}$, resulting in the vector **T** of a covariance matrix **C**. Taking the matrix **H** in the form (8), we get the following relations:

$$\begin{cases} T_1 = Y_1 \\ T_2 = hY_1 + Y_2\sqrt{1-h^2} \end{cases} \tag{10}$$

In general case types of pairs of random variables: $\{\, Y_1,\, Y_2 \,\}$ and $\{\, T_1,\, T_2 \,\}$ may be different. In the case considered T_1 is uniformly distributed in the range $\langle -\sqrt{3}, \sqrt{3} \rangle$, but T_2, as a linear combination of uniform random variables, is the variable of triangular (Simpson) distribution in the range $\langle -g,\, g \rangle$, where $g = \left(h + \sqrt{1-h^2} \right)\sqrt{3}$. The histograms of variables T_1 and T_2, assumed h = 0.8, are shown in Fig. 6.

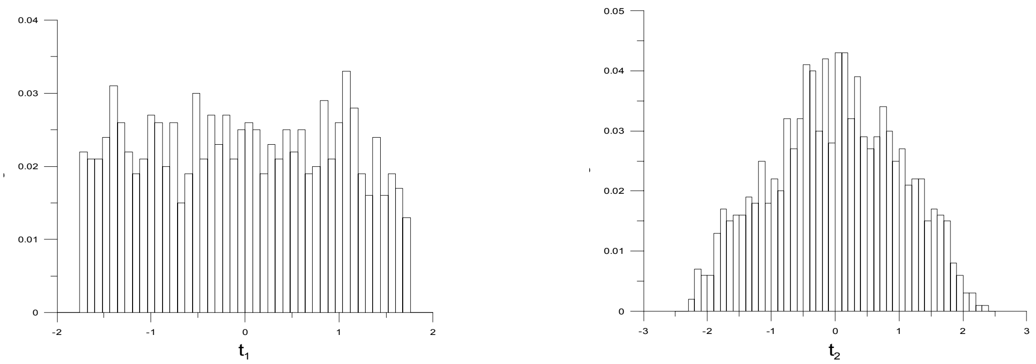

Fig. 6. Histograms of random variables T_1 and T_2, correlated, h = 0.8.The variable T_1 uniformly distributed in the range $\left(-\sqrt{3}, \sqrt{3} \right)$, the variable T_2 triangularly distributed

- Transformation of the vector **T** into **Z**, which consists of two uniformly distributed random variables in the range <0, 1>, with given covariance matrix **C**. The variable Z_1 is taken by the formula:

$$Z_1 = \frac{T_1}{2\sqrt{3}} + 0.5 . \tag{11}$$

while the transformation $Z_2 \rightarrow T_2$ is performed using the cumulative probability distribution function of the Simpson distribution:

$$F_T\left(t\right) = 0.5 + \frac{1}{g}\left(sign\left(t\right)\frac{t^2}{2g} + t \right). \tag{12}$$

- Transforming the vector $\mathbf{Z} = \{Z_1 \quad Z_2\}^T$ into $\mathbf{V} = \{V_1 \quad V_2\}^T$ using the inverse Gumbel distribution function:

$$V_i = F_V^{-1}\left(Z_1\right) = u - \frac{1}{\alpha}\ln\left(-\ln Z_1\right), \quad i = 1,2 . \tag{13}$$

In the above equation $F_V\left(\cdot\right)$ symbolizes the Gumbel distribution function of parameters α and u . In the worked example the following values were assumed: u = 0.2, α = 8.0.

- Creating the random wind load vector $\mathbf{W} = \{W_1 \quad W_2\}^T$, by the formula $W_i = V_i^2$, i = 1, 2

The procedure presented above results in the variables W_1 and W_2 of the correlation coefficient $r = \rho_{W_1W_2}$, which may differ from the correlation coefficient of the variables Z_1 and Z_2 (i.e. the value $h = \rho_{Z_1Z_2}$. It is possible to obtain in an iterative way, variables W_1 and W_2 of a correlation coefficient approximately equal to the arbitrary value r.

The following operations make up the single simulation step:

- Analysis of the space framed structure, calculating the initial value of the horizontal upper deck deflection $u(\omega)$ in the wind direction(Fig. 3),
- Uni-parametrical load increment, while the structural characteristics are fixed, until the allowable upper deck deflection u_0 is reached. In numerical calculations it was assumed $u_0 = H/400$, where H = 17.95 m is the colonnade height.

The single simulation step produces the limit load multiplier λ_i of this realization – a single value of the variable $\Lambda(\omega)$.

Three variants of calculations were provided, with respect to the correlation coefficient r of the wind load variables. The following cases were considered:

a) variables W_1 and W_2 uncorrelated – the coefficient $r = 0$

b) variables W_1 and W_2 correlated – the coefficient $r = 0.62$

c) variables W_1 and W_2 fully correlated – the coefficient $r = 1$

The histograms of the serviceability limit state of the structure are presented in Fig. 7

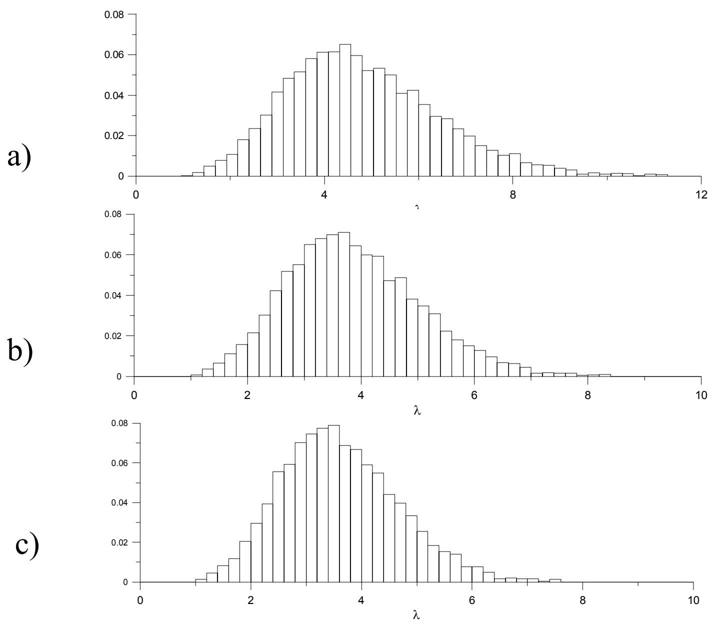

Fig. 7. Relative histograms of the serviceability limit state of the colonnade of the Licheń Basilica, three variants of calculations, with respect to the correlation of the wind load

3.3 Results

The results of the three variants of calculations are the relative histograms of the limit state of the structure (Fig. 7), statistical characteristics are collected in the table (Tab. 1). It is worth pointing out that in the assumed structural and stochastic model each variant of calculation results in the probability of exceeding the allowable deflection lower than the accuracy of the method (the reciprocal of the number of realizations). On the basis of probabilistic limit state analysis it can be stated that the examined part of the structure is stiff enough to assure the proper structural service.

Variant of calculations	a)	b)	c)
Correlation coefficient of the wind load variables W_1 and W_2 : $r = \rho_{W_1W_2}$	0	0.62	1
Mean value	4.8133	3.8755	3.6128
Standard deviation	1.6106	1.1748	1.0540
Minimum value λ_{min}	1.1804	1.0427	1.0135
Maximum value λ_{max}	11.2338	8.3317	7.5443
Probability of failure	$<10^{-4}$	$<10^{-4}$	$<10^{-4}$

Tab. 1 Statistical parameters of the limit state of Basilica colonnade, three variants of calculations

4 CONCLUSIONS

4.1 Remarks on the procedure

Sensitivity analysis of limit states of structures is proposed in the paper, by means of the specific Monte Carlo algorithm. It leads to the third level probabilistic information about the structure, i.e. the limit state histogram. The procedure also enables us to solve the problem limited to the reliability, or the probability of failure estimation.

Simulation-based limit state analysis usually means creating a population of structural states and choosing the failed cases, which determine the failure probability. The procedure proposed is modified and therefore developed. A group of dominant basic variables is chosen, in every simulation step. These variables increase uni-parametrically, to reach finally the structural limit state. Thus every simulated case is led to the limit state. The set of non-dimensional multipliers of dominant variables is the result of simulation. Its histogram serves as the estimator of the PDF of the structural limit state.

The proposed procedure makes it possible to perform the reliability assessment only. In this case techniques to reduce the number of simulations may be used (see [1])

4.2 The fully probabilistic design

The semi-probabilistic design procedures (for instance LRFD) make use of partial factors to depict random scatter of basic variables. Values of loads and resistance coefficients are to be calibrated on the basis of statistical data.

The fully probabilistic design is the subject of a great number of present day's publications. Several international codes (e.g. ISO 2394: General principles of structural reliability, and EN 1990: Basis of structural design) present the overall design scheme exploring random analysis in greater extent. These documents serve as the code formats only (according to the JCSS nomenclature). They form the very basis of fully random design. No specific design codes exist up till now, referring to particular civil engineering branches (metal, concrete, timber structures, etc.) which really represent the fully probabilistic point of view. The transformation, described in numerous papers, seems to be a long-time process, requiring a huge effort of the code-writing committees and a population of professional designers, deeply educated the semi-probabilistic methods.

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