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MODELLING THE PLASTIC HINGE IN THE STATICALLY INDETERMINABLE REINFORCED CONCRETE BAR ELEMENTS

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Abstract. *The paper presents the example numerical model to calculate the reinforced concrete bar structures. Usually applied methods of structure dimensioning do not include the case of plastic hinges occurrence under the limit load of construction. The model represented by A. Borcz is based on the differential equation of deflection line of the beam and it includes the effects of rearrangement of the internal forces and reological effects. The experimental parameters obtained in earlier investigations describe effects resulting from the rise of plastic hinges in the proposed equation.*

1. INTRODUCTION

The methods of dimensioning the elements recommended by the standards are based on the assumption that material can achieve the limit strength values. On the other side, elastic model of the structure must also be taken into account in static calculation. The occurrence of the plastic hinges under the limit loads is then excluded. These assumptions and calculation methods seem to be rather irrelevant.

The situation is caused by the deficiency of computing models which would allow to incorporate the effects of rearrangement of the internal forces in a quick and detailed manner which is an important factor from the engineer's point of view.

The paper presents the suggested method of calculation incorporating experimental parameters known from the previously conducted tests.

2. MODEL FOR CALCULATION OF THE REINFORCED CONCRETE BAR STRUCTURES

The presented model is based on the differential equation of deflection line of the beam, that was proposed by A. Borcz to calculate the reinforced concrete beams in phase I and II [3, 4].

Differential equation for the beam made of elastic material reads:

$$EJ\frac{\partial^4 v(\xi)}{\partial \xi^4} = pl^3 \tag{1}$$

In the case of the reinforced concrete beam the rigidity of the beam depends on level of the load. It can be accepted that before crack the stiffness is approximetely constant. After the crack the coefficient describing stiffness is remarkably reduced.

In most theories regarding the reinforced concrete the value of deflection is calculated by averaging stiffness of the section with a crack. The substitution EJ – stiffness depends on level of the load.

$$\frac{\partial^2}{\partial \xi^2} EJ_{I,II}\left(\frac{M}{M_{cr}}\right)\frac{\partial^2 v(\xi)}{\partial \xi^2} = pl^3 \tag{2}$$

In the equation (2) the effects resulting from creation of plastic hinge do not explicitly occur.

Borcz proposed the alternative way of estimation of characteristic effects of concrete structures, for example the influence of crack, cooperation between concrete and steel, plastification of material. It was proposed that the stiffness, given on the left side of equation (3), should be the same as for ideally elastic rigid structure. All other effects related to material's properieties such as reological, cracks, etc. should be put on the right side of the equation, as the external load. In the equation (3) below it is described by $\mathfrak{D}$ – segment [3].

$$EJ\frac{\partial^4 v(\xi)}{\partial \xi^4} = pl^3 + l^3\mathfrak{D} \tag{3}$$

for perpendicular crack:

$$\mathfrak{D} = \frac{EJ}{l^3}\sum_i \frac{\partial^2 \delta(\xi - \zeta_i)}{\partial \xi^2} r_i \tag{4}$$

where:

$$r_i = -r_{0i} - r_{1i} M(\zeta_i) \tag{5}$$

The physical interpretation of the parameter r_i is the discrete stroke of angle of rotation $\partial v/\partial \xi$ described by Dirac delta function in a point ζ_i.

In the description, residual part r_{0i} and elastic part $r_{1i}M(\zeta_i)$ of the increase of rotation angle on the ordinate where the bar crack occurred were separated. The increase of rotation angle does not relate only to the effects in a crack, but also to all the effects between cracks (e.g. reinforcement concrete slip, disturbance in the structure of concrete, creep, etc.).

The quadruple integration the equation gives the solution for:

- shearing forces:

$$\frac{\partial^3 v(\xi)}{\partial \xi^3} = \frac{pl^3}{EJ} + \frac{l^3}{EJ}\left[\sum_i \frac{\partial}{\partial \xi}\delta(\xi - \zeta_i) + A\right] \tag{6}$$

- bending moments:

$$\frac{\partial^2 v(\xi)}{\partial \xi^2} = \frac{pl^3}{EJ} + \frac{l^3}{EJ}\left[\sum_i \delta(\xi - \zeta_i) + A\xi + B\right] \tag{7}$$

- angles of rotation:

$$\frac{\partial v(\xi)}{\partial \xi} = \frac{pl^3}{EJ} + \frac{l^3}{EJ}\left[\sum_i h(\xi - \zeta_i) + \frac{1}{2}A\xi^2 + B\xi + C\right] \tag{8}$$

- deflections:

$$v(\xi) = \frac{pl^3}{EJ} + \frac{l^3}{EJ}\left[\sum_i (\xi - \zeta_i)h(\xi - \zeta_i) + \frac{1}{6}A\xi^3 + \frac{1}{2}B\xi^2 + C\xi + D\right] \tag{9}$$

The equation (9) is a sum of deflections from solution of the ideally elastic beam and deflection from defects.

$$\begin{gathered} v(\xi) = v_1(\xi) + v_2(\xi, \zeta)\,(\xi - \zeta_i)h(\xi - \zeta_i) \\ h(\xi - \zeta_i) = \begin{cases} 0 & \xi \le \zeta_i \\ 1 & \xi > \zeta_i \end{cases} \end{gathered} \tag{10}$$

This solution can be represented using Green function $G(\xi)$, which is the solution of differential equation from unit specific impulse (crack).

$$\frac{\partial^4}{\partial \xi^4} G(\xi) = \delta(\xi - \zeta) \tag{11}$$

For example the solution for deflection is:

$$v(\xi) = [G(\xi, \zeta)] \otimes \left[p^x(\zeta) + \sum_i r_i \frac{\partial^2}{\partial \zeta^2}\delta(\zeta - \zeta_i)\right] \tag{12}$$

It can be concluded from the sub integral multiplication:

$$v(\xi)=\int_0^1 G(\xi,\zeta)\left[p^x(\zeta)+\sum_i r_i \frac{\partial^2}{\partial\zeta^2}\delta(\zeta-\zeta_i)\right]d\zeta \tag{13}$$

And in addition, if the influence of slant cracks is taken into consideration in the pre support area, the equation (13) reads as follows:

$$v(\xi)=\int_0^1 G(\xi,\zeta)\left[p^x(\zeta)+\sum_i r_i \frac{\partial^2}{\partial\zeta^2}\delta(\zeta-\zeta_i)+\sum_k t_k \frac{\partial^3}{\partial\zeta^3}\delta(\zeta-\zeta_k)\right]d\zeta \tag{14}$$

after the integration:

$$v(\xi)=v_1(\xi)+\sum_i r_i \frac{\partial^2}{\partial\zeta^2}G(\xi,\zeta_i)+\sum_k t_k \frac{\partial^3}{\partial\zeta^3}G(\xi,\zeta_k) \tag{15}$$

$v_1(\xi)$ - equation of the deflection line of ideally elastic beam without cracks.

The assumption was made that the skipping increase of rotation angle occurs in each place of the crack appearance, so elastic and residual crack openings and other effects, such as local concrete destruction, disturbance of steel and concrete interaction or reological effects can be modelled.

3. MODELLING THE PLASTIC HINGE

The segment, which describes plastic deformations introduces the stroke of angle of rotation as local deformation. The value of this deformation is the effect resulting from creation of plastic hinge in the beam. The model includes estimates of the load level.

In order to determine the elastic increase of rotation angle in a crack the Borcz's formula was applied [4]. The strength features of materials and geometrical relationships were used to compute the coefficient r_1:

$$r_1=\frac{\psi_a s_{rm}}{E_s A_{s1} h^2\left(1-\frac{\xi^{II}}{3}-\frac{a}{h}\right)\left(1-\xi^{II}-\frac{a}{h}\right)} \tag{16}$$

where: ψ_a – the ratio of average stresses in steel to stresses in the cracked cross section, s_{rm} – average crack spacing , ξ^{II} – relative height of the compressed area in the cracked cross section, h – total height of the rectangular cross section, A_{s1} – cross sectional area of steel in the stretching area, E_s –Young's module for steel.

According to the investigations conducted by Bach [1] the ratio of plastic deflections to elastic ones, depending on the load level was determined.

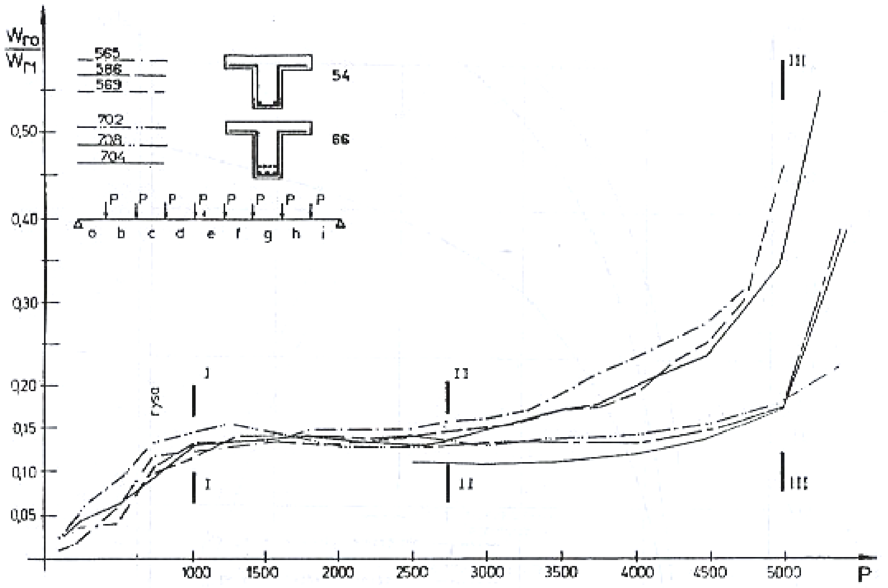

Figure 1. Ratio of plastic deflection to elastic one as a function of the load [6]

This dependence is proposition of calculation r_0 by the following function:

$$r_0 = \begin{cases} 0{,}15 r_1 & M < 0{,}5 M_n \\ 0{,}75 \tan\left(\dfrac{M}{M_n}\right) - 0{,}1 & 0{,}5 M_n < M < 0{,}9 M_n \end{cases} \tag{17}$$

where: M_n – the breaking moment

Numerical parameters are accepted for special types of beams and before they are applied to full scale structures should be proved by some additional researches.

4. FINAL REMARKS

The presented model is a proposition of numerical computing of plastic hinges of the reinforced concrete beams. The linear-elastic model of concrete was established. The model takes into consideration discontinuity in the element caused by the cracks. It is possible due to the application of experimental parameters which are determined using the ratio between the residual and elastic deflections. The cycle of laboratory tests is being currently prepared in order to verify the parameters and the model itself.

Practical objective of the above specified problem is computing model which allows to analyze dislocations and internal forces in the structure when the structure operates both under standard conditions and before failure, after plasticization of the critical cross sections.

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