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USING INTERVAL ANALYSIS FOR STRUCTURAL ENGINEERING PROBLEMS

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Abstract.

Interval analysis extends the concept of computing with real numbers to computing with real intervals. As a consequence, some interesting properties appear, such as the delivery of guaranteed results or confirmed global values. The former property is given in the sense that unknown numerical values are in known to lie in a certain interval. The latter property states that the global minimum value of a given function is known to be contained in an interval (or a finite set of intervals). Depending upon the amount of computational effort invested in the calculation, arbitrarily tight bounds on these enclosing intervals can be found.

The downside of interval analysis is, however, that it is mathematically correct, but often too pessimistic. This is in particularly due to the so-called dependency effect, where a single variable is used multiple times in one calculation. When applying interval analysis to structural analysis problems, the dependency effect has a great influence on the quality of numerical results.

In this paper, a brief background of interval analysis is presented and shown how it can be applied to the solution of structural analysis problems. A discussion of possible improvements as well as an outlook to parallel computing is also given.

1 INTRODUCTION

Interval analysis has been a standard mathematical tool for decades since R. E. Moore published his book *Interval Analysis*[4] in 1966. Interval analysis, in its simplest form, is an extension of real analysis where real numbers are replaced by real intervals or, more generally, points in real space are represented by finite boxes. Arithmetic operators that commonly operate on numbers are extended in a suitable fashion by operators that work with intervals. An important consequence of using interval analysis is that numerical methods can guarantee results in the sense that results are as precise as possible within explicitly defined ranges. Also, various types of numerical errors can be handled in a uniform fashion, including rounding errors, truncation errors and input errors, to name a few [2].

In particular, numerically "hard" problems, such as finding all solution sets of non-linear equations and inequalities or determining all global optimizers of multi-model criteria can be solved exactly using interval analysis. In fact, algorithms based on interval analysis are not only able to compute the values of mathematical quantities numerically, they are also able to prove rigorously mathematical statements about quantities or sets of quantities because of the bounding nature of numerical intervals. In contrast, standard numerical methods such as Monte Carlo methods or even a systematic grid search cannot prove simple mathematical properties such as the emptiness of a set of values or that a set is a disconnected union of sets.

Being able to produce guaranteed results for physical quantities is particular useful in engineering problems. For example, depending on the type and topology of a structural system, the results of a structural analysis can be very unreliable if large forces and/or displacements are involved. Often, the only approach to check for numerical error is to switch from single to double precision values and recalculate, observing any notable changes in the result.

2 MATHEMATICAL DEFINITION OF INTERVAL ANALYSIS

Interval analysis is based on the idea of substituting real numbers, which are often known only approximately, by closed intervals, which can be considered as exact representations of numerical values. In the case of multiple dimensions, that is, $\mathbb{R}^n$, "intervals" are defined as axis-aligned boxes.

2.1 Formal definitions

More formally, an interval $[x]$ is a connected subset of $\mathbb{R}$. It has a lower bound $\underline{x}$ defined as

$$\underline{x} = \sup\{a \in \mathbb{R} \cup \{-\infty, \infty\} \mid \forall x \in [x],\ a \leq x\}. \tag{1}$$

Similarly, the upper bound $\overline{x}$ is defined as

$$\overline{x} = \inf\{b \in \mathbb{R} \cup \{-\infty, \infty\} \mid \forall x \in [x], x \leq b\}. \tag{2}$$

Excluding limit cases involving $-\infty$ or $+\infty$, which are not interesting in engineering problems, the quantity $[x]$ is simply the real interval $[\underline{x}, \overline{x}]$.

The width of a non-empty interval $[x]$ is defined as

$$w([x]) = \overline{x} - \underline{x}, \tag{3}$$

and the midpoint of any bounded and non-empty interval is

$$\mathrm{mid}([x]) = \frac{\underline{x} + \overline{x}}{2}. \tag{4}$$

2.2 Interval arithmetic

As with real numbers, we can do arithmetic with intervals. Extending the concept of addition, subtraction, multiplication and division to the two intervals $[x]$ and $[y]$, we can define

$$[x] \oplus [y] = [\{x \oplus y \in \mathbb{R} \mid x \in [x], y \in [y]\}], \tag{5}$$

where the symbol $\oplus$ stands for each of the above mentioned binary operators. In particular, we can evaluate each arithmetic operator according to the rules

$$\begin{aligned}
[x] + [y] &= [\underline{x} + \underline{y}, \overline{x} + \overline{y}], &(6)\\
[x] - [y] &= [\underline{x} - \overline{y}, \overline{x} - \underline{y}], &(7)\\
[x] * [y] &= [\min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}], &(8)\\
[x]/[y] &= [x] * (1/[y]), &(9)
\end{aligned}$$

where $1/[y]$ is defined as $[1/\overline{y}, 1/\underline{y}]$ if $0 \notin [y]$, undefined otherwise. Also, the multiplication of the interval $[x]$ by the scalar $a \in \mathbb{R}$ is

$$a[x] = [a\underline{x}, a\overline{x}]. \tag{10}$$

To increase the range of allowable numerical computations, elementary functions, such as $\exp, \log, \sin, \cos$, etc., can also be extended to interval functions expressed in terms of bounds. For example, the monotonic function $[\exp]$ can be simply defined as

$$[\exp]([x]) = [\exp(\underline{x}), \exp(\overline{x})]. \tag{11}$$

For non-monotonic functions such as $\sin$, the computation is more complicated, because for "small" intervals the function $\sin$ can be monotonic, for "large" intervals the range of the $\sin$ function is restricted to $[-1, 1]$.

2.3 Dependency effect

An important observation of interval arithmetic, which sets it apart from "ordinary" arithmetic, is the problem of dependency, which can lead to very pessimistic results. For example, in general the expression $[x] - [x] \neq 0$, because

$$[x] - [x] = \{x_1 - x_2 \mid x_1 \in [x],\ x_2 \in [x]\}, \tag{12}$$

and x_1 and x_2 can be chosen independently. If, say, $[x] = [10, 20]$, then $[x] - [x] = [-10, 10] \neq 0$. As a further example of dependency, the square of the interval $[x] = [-2, 3]$ computed according to the basic multiplication rule is

$$[-2, 3] * [-2, 3] = [-6, 9], \tag{13}$$

which is meaningless if the square of a real interval, like a real number, should always be positive. A more reasonable result is the interval $[0, 9]$, which can be calculated by setting the lower bound of the product to 0 if it is negative.

In general, for any equation involving more than one occurrence of an interval variable, pessimistic results will be computed. If possible, expressions should be algebraically transformed to reduce the number of occurrences of each variable. So, although the expressions $(x+1)^2$ and x^2+2x+1 are algebraically equal, the width of the interval $([x]+1)^2$ will usually be less than the width of $[x]^2+2*[x]+1$. If $[x]=[-1,1]$, then the first expression evaluates to $[0,4]$, whereas the value of the second expression evaluates to the larger interval $[-1,4]$.

2.4 Interval vectors and matrixes

An interval vector $[\mathbf{x}]$ is a subset of $\mathbb{R}^n$ and can be defined as the Cartesian product of n closed intervals,

$$[\mathbf{x}] = [x_1] \times [x_2] \times ... \times [x_n], \tag{14}$$

with $[x_i] = [\underline{x}_i, \overline{x}_i]$ for $i = 1, 2, ..., n$. The width of an interval vector is

$$w([\mathbf{x}]) = \max_{1 \le i \le n} w([x_i]), \tag{15}$$

and the midpoint of an interval vector is

$$\mathrm{mid}([\mathbf{x}]) = (\mathrm{mid}([x_1]), \mathrm{mid}([x_2]), ..., \mathrm{mid}[x_n])^T. \tag{16}$$

An $n \times m$ interval matrix $[\mathbf{A}]$ can be defined as

$$[\mathbf{A}] = \begin{pmatrix} [a_{11}] & [a_{12}] & ... & [a_{1m}] \\ [a_{21}] & [a_{22}] & ... & [a_{2m}] \\ ... & & & \\ [a_{n1}] & [a_{n2}] & ... & [a_{nm}] \end{pmatrix} = ([a_{ij}])_{1 \le i \le n, 1 \le j \le m}, \tag{17}$$

where similar rules defining the width and midpoint of an interval matrix apply as in the case of interval vectors.

2.5 Inclusion functions

Given the definitions of arithmetic for manipulating intervals, it is natural to consider functions that map intervals to intervals based on "ordinary" real functions. This leads to the definition of so-called inclusion functions: Consider a function $f : \mathbb{R}^n \to \mathbb{R}^m$. Then the interval function $[f]$ is an inclusion function if

$$\forall [x] \in \mathbb{R}^n,\ f([x]) \subset [f]([x]). \tag{18}$$

Although the image $f([x])$ is not necessarily a box, the inclusion function $[f]([x])$ is a box containing $f([x])$.

The simplest approach in constructing an inclusion function $[f]$ for a given function $f : \mathbb{R}^n \to \mathbb{R}^m$ is to substitute each variable x_i of f by its interval counterpart $[x_i]$ and to substitute each occurrence of $+$, $-$, $*$, $/$, or $\exp$, $\log$, $\sin$, etc., by the corresponding interval operator or interval function.

As an example, consider the two formal expressions

$$f_1(x) = (x+1)^2, \tag{19}$$
$$f_2(x) = x^2 + 2x + 1, \tag{20}$$

presented above. The corresponding natural inclusion functions are

$$[f_1]([x]) = ([x]+1)^2, \quad (21)$$
$$[f_2]([x]) = ([x]^2 + 2 * [x] + 1). \quad (22)$$

Although the expressions for f_1 and f_2 are algebraically equivalent, the same variable $[x]$ occurs once in $[f_1]$ and twice in $[f_2]$, so the dependency effect gives different results, as shown above.

3 INTERVAL ANALYSIS IN STRUCTURAL ANALYSIS

If the parameters that define a structural system, such as nodal coordinates, material properties, forces and loads, cross sections, etc. are given as intervals, and we have interval methods to analyze structural systems at our disposal, then we are able to describe the behavior of a given structural system in terms of interval results. That is, given the range of precision of input values as intervals, we can not only compute the numerical behavior of a structural system numerically, but also state guaranteed, although possibly pessimistic, bounds on the results.

3.1 Solution of a system of linear equations

Since the behavior of structural systems can be computed using well-known methods of structural analysis, such as the matrix displacement methods based on global equilibrium equations and stiffness matrices, a core step in the analysis of structural systems is ability to solve a system of linear equations.

Using the principle of natural inclusion, we can extend standard solvers for systems of equations to interval solvers. For example, if $\mathbf{A}$ is $n \times m$ real matrix and $\mathbf{b}$ is an n real vector, the solution vector $\mathbf{x}$ of the equation $\mathbf{Ax} = \mathbf{b}$ can be found by the Gaussian elimination method by elementary manipulations of $\mathbf{A}$ and $\mathbf{b}$ involving just arithmetic operations on matrix and vector components. Thus, by substituting interval operations for arithmetic operations, an interval Gaussian elimination method can be obtained to solve the corresponding interval equation $[\mathbf{A}][\mathbf{x}] = [\mathbf{b}]$.

As examples show, the results of the interval Gaussian elimination method are often very pessimistic. Results can be improved, however, using the technique of pre-conditioning: If $\mathbf{A}_0$ is an invertible matrix, then the solution of $\mathbf{A}_0^{-1}[\mathbf{A}][\mathbf{x}] = \mathbf{A}_0^{-1}[\mathbf{b}]$ has the same solution set as the original equation. If we chose $\mathbf{A}_0 = \mathrm{mid}([\mathbf{A}])$, then the product $\mathbf{A}_0^{-1}[\mathbf{A}]$ is approximately equal to the interval identity matrix and it can be shown that the interval result improves.

Similarly, the Gauss-Seidel method of iteratively solving a system of linear equations and the extension using pre-conditioning of the coefficient matrix can also be implemented in a straightforward manner.

3.2 The sign accord method

The sign accord method was developed by J. Rohn ([5], [1]) and is currently considered as one of the best methods for solving systems of linear equations. In general, the exact solution set X_s,

$$X_s = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{A} \in [\mathbf{A}],\ \mathbf{b} \in [\mathbf{b}]\} \quad (23)$$

is not an $n-$dimensional interval, in fact, it is usually not even convex.

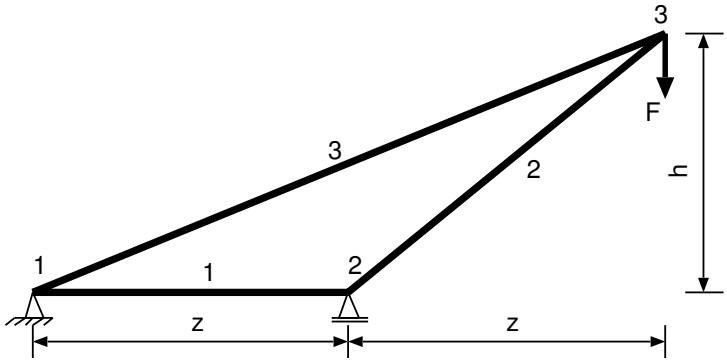

Figure 1: a cantilever

Among the elements in the set X_s, so-called corner solutions are of particular interest. The subset $X_c \subset X_s$ consists of the vectors $\mathbf{A}^{-1}\mathbf{b}$ where all the elements in $\mathbf{A}$ and $\mathbf{b}$ are chosen as one of the end points of the corresponding interval element in $[\mathbf{A}]$ and $[\mathbf{b}]$, respectively. Because of the combinational effects, there are a total of 2^{n^2+n} corner solutions, which is a large number even for small n.

However, the set X_c can be reduced even further to a set X_e by considering only so-called extreme solutions,

$$X_e = \{\mathbf{x} \in \mathbb{R}^n \mid \ |\text{mid}([\mathbf{A}])\mathbf{x} - \text{mid}([\mathbf{b}])| = r([\mathbf{A}])|\mathbf{x}| + r([\mathbf{b}])\}, \tag{24}$$

where the radius $r([\mathbf{x}])$ of an interval is defined as half its width $w([\mathbf{x}])$. Since it can be shown that the convex hulls of X_e and X_s are equal, we basically get the same solution set. To find solutions of X_e, this set can be reformulated as

$$X_e = \{\mathbf{x} \in \mathbb{R}^n \mid \text{mid}([\mathbf{A}])\mathbf{x} - \text{mid}([\mathbf{b}]) = \mathbf{D}_y(r([\mathbf{A}])|\mathbf{x}| + r([\mathbf{b}])\}, \tag{25}$$

for some

$$\mathbf{y} \in S^n = \{\mathbf{y} \in \mathbb{R}^n \mid y_i \in \{-1, 1\},\ i = 1, ..., n\}, \tag{26}$$

and the diagonal matrix

$$\mathbf{D}_y = \text{diag}(y_1, y_2, ..., y_n). \tag{27}$$

Carrying through the analysis, it can be shown that the total number of elements in X_e is reduced from 2^{n^2+n} to only 2^n.

3.3 Cantilever example

As an example, in the structural system shown in Fig. 1, the force F_2 in member 2 can be determined analytically as

$$F_2 = 2\ F\ \frac{\sqrt{h^2 + z^2}}{h}. \tag{28}$$

If $z = 2000$ mm, $h = 2000$ mm and $F = -10000$ N, then $F_2 = -28284.27$ N. If, however, the coordinates of nodes 1 to 3 are known only approximately, say to ± 1 mm, then the value of F_2 will be an interval. In this case, $[z] = [1998, 2002]$ mm, $[h] = [1998, 2002]$ mm and $[F_2]$ is

$$[F_2] = 2[F]\frac{\sqrt{[h]^2 + [z]^2}}{[h]} = [-28340.89, -28227.76] \text{ mm}. \tag{29}$$

Table 1: The force $[F_2]$ computed using various methods for systems of linear equations

Method	$[F_2]$	$w([F_2])$
Gauss elimination	[-124293.02, 63295.17]	187588.19
Gauss elimination with pre-conditioning	[-83789.93, 24500.41]	108290.34
Gauss-Seidel with pre-conditioning	[-83814.24, 25069.10]	108883.34
The sign accord method	[- 78956.37, 20451.73]	99408.10

The width of $[F_2]$ is about 113 mm, or about 0.4% of the mean interval value. As noted above, we can tighten the bounds of $[F_2]$ if we rearrange Eqn. 29 to remove dependency effects. We can rewrite the equation for $[F_2]$ as

$$[F_2] = 2[F]\sqrt{1 + \frac{[z]^2}{[h]^2}} \tag{30}$$

and get a better value of $[F_2] = [-28312.60, -28256.03]$, where the width $w([F_2])$ is now only half as wide as the previous interval. If we further assume that the cross sectional area $A = 5,000$ mm^2 and the modulus of elasticity $E = 210,000$ N/mm^2, then we can use a general structural analysis program to calculate $[F_2]$. Using the methods Gauss elimination, Gauss elimination with pre-conditioning, Gauss-Seidel with pre-conditioning and sign accord to solve the resulting system of linear equations, we get the results shown in Table 1.

As can be seen in the table, pre-conditioning in the Gauss elimination method improves it considerably. The basic Gauss-Seidel method does not work very well, however, the preconditioned Gauss-Seidel method compares to the preconditioned Gaussian elimination method. Finally, this example seems to confirm the strength of the sign accord method for solving systems of linear equations as this method delivers the tightest bounds.

Although the results in Table 1 are mathematically rigorous and exact, from an engineering point of view they are not very convincing. Basically, they imply that even a manufacturing precision of only ± 1 mm in the geometry of nodal coordinates make the results meaningless, because we can't even determine if the force F_2 is compressive or tensile. To further see the degree of sensitivity in this interval analysis example, Table 2 shows the interval of $[F_2]$ and its width in steps of ± 0.1 mm for all node positions. The method for solving linear equations is restricted to sign accord, because it is expected that other methods will only be worse.

Table 2: The force $[F_2]$ computed with the sign accord method for various degrees of nodal coordinate errors.

node tol. $\pm$d [mm]	$[F_2]$	$w([F_2])$	node tol. $\pm$d [mm]	$[F_2]$	$w([F_2])$
0.0	[-28284.27, -28284.27]	0.00	0.6	[-57680.34, 432.07]	58113.03
0.1	[-33069.25, -23517.90]	9551.30	0.7	[-62825.09, 5328.18]	68154.08
0.2	[-37884.23, -18758.81]	19125.41	0.8	[-68077.05, 10287.51]	78364.56
0.3	[-42740.78, -13995.72]	28745.07	0.9	[-73448.61, 15324.00]	88772.60
0.4	[-47650.76, -9217.02]	38433.56	1.0	[-78956.37, 20451.73]	99408.10
0.5	[-52626.38, -4411.68]	48214.70			

4 GLOBAL OPTIMIZATION

Many engineering problems can be formulated as mathematical optimization problems, in particular, finding the minimum value of a continuous function $f : D \to \mathbb{R}^n$, where $D \subset \mathbb{R}^n$. The function f, often called an objective or design function, may have many local minima, but usually only the smallest one, the so-called global minimum f^*, is of particular interest,

$$f^* = \inf\{f(x) \mid x \in D\}. \tag{31}$$

An iterative algorithm of the branch and bound type can find the global minimum value without the use of any types of derivatives.

Let $[f]$ be an interval extension of the function f and $[\mathbf{x}_0]$ be the initial domain. At each stage of the iteration, there is a working set W of intervals to be inspected and a result set R of intervals which are known to contain the global minimum. In each iteration, the following steps are carried out as long as the working set is not empty:

1. Get an interval entry $[x]$ from the working set and split it into two pieces $[x_1]$ and $[x_2]$. If $D \subset \mathbb{R}$, i.e. $[x]$ is just a real interval, then $[x_1]$ and $[x_2]$ are just two halves of the interval $[x]$. In the multi-dimensional case, $[x]$ is split along one dimension, usually the one where the width of the interval has the largest value.

2. For each interval $[x_1]$ and $[x_2]$, check to see if the width of the image of each interval is less than some given bound ϵ,

$$[f]([x_i]) \leq \epsilon, \quad i = 1, 2. \tag{32}$$

   If so, put the interval $[x_i]$ into the result set.

3. If the width of the image is greater than ϵ, add the interval to the working set.

Finally, reduce the result set by eliminating all those elements which do definitely not contain the global minimum. This can be done in two steps: First, find the lowest upper bound $\hat{f}$ of all images of the intervals in R,

$$\hat{f} = \min_{\overline{y}}\{[y] = [f]([x]),\ [x] \in R\}. \tag{33}$$

Then, remove all those elements of R where the lower bound of the image is greater than $\hat{f}$,

$$R := R \setminus \{\ y \mid \underline{y} \geq \hat{f}\ \}. \tag{34}$$

Note that it might be tempting to include a reduction step during the iteration to decrease the size of the working set. However, due once again to the dependency effect, this can lead to a loss of solution values, with a possible empty result set, even though the minimum value is well-defined.

For example, the images of the intervals $[10, 20]$ and $[20, 30]$ might both possibly contain the global minima. If the second interval is eliminated because all its values are larger than those in the first interval, we can then iterate by splitting the first interval into the intervals $[10, 15]$ and $[15, 20]$ and processing both intervals. At this stage we can discover that, although the global minimum is in the image of $[10, 20]$, is not contained in the image of either the interval $[10, 20]$

or $[20, 30]$. Now we are left with an empty result set. This can happen if the value of the global minimum is, in fact, in the second interval, say, $\hat{x} = 21$. Because of the dependency effect, the image of the "large" interval $[10, 20]$ contains the global minimum, but each "smaller" image $[10, 15]$ and $[15, 20]$ does not.

Because we have to be careful about deleting "superfluous" intervals in the working set, the size of the working set can increase rapidly. However, since each element of the working set can (and should) be evaluated independently, this lends itself to a course grain parallelization approach.

4.1 Cantilever optimization example

To give a brief example of a minimization problem in structural analysis, the cross section of element 1 in the cantilever shown in Fig. 1 is to be optimized. The constraint imposed on the element is that the maximum compressive strength should not exceed 250 N/mm^2. Since the structure is statically determinate, the force F_1 in element 1 is 20,000 N, if we assume numerical values are exact. Therefore, the cross sectional area A of element 1 is expected to be

$$A = \frac{F}{\sigma} = \frac{20,000 \text{ mm}}{250 \text{ N/mm}^2} = 80 \text{ mm}^2. \tag{35}$$

For simplicity, we can chose the objective function to be equal to A. If a complete structural analysis program is employed in the optimization, we get a final result of $[A] = [79.84, 79.92]$ mm. Here, the maximum width of a candidate interval image is set to $\epsilon = 0.1$ mm. The maximum size of the working set is 130 elements.

Note that although $80 \notin [79.84, 79.92]$, this is the "lowest" interval of image width less than 0.1 that contains the global minimum: The interval of compressive strength corresponding to this interval is, in fact, $[250, 250.73]$, an interval which includes the value 250.

5 CONCLUSION

Interval analysis can be extended to solve specific problems often found in engineering design, in particular calculations involving vectors and matrixes. However, the so-called dependency effect can force the development of methods that must be more sophisticated than otherwise necessary.

Given the response of a structural system defined with a parameter whose precise value is unknown, we can find an interval for this parameter so that the calculated response of the system agrees with the observed response.

Further, given an objective function for a structural system, we find the variant of the structure which minimizes the objective function. Because the search space can be highly irregular, non-convex or even disconnected, finding a global minima in such cases can be very difficult. Using interval analysis to "box in" allowable values can provide definite assertions on the existence and accuracy of possible minima.

Although solving for solutions using branch and bound interval search methods may lead to exponential complexity in general, branch and bound methods do generate highly independent subtasks which, in combination with well-chosen heuristics, can be solved efficiently in parallel [3]. Thus, interval analysis has the potential for developing highly parallel global optimization methods.

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