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CONVERGENCE OF A NEW CONSISTENT FOLDED PLATE THEORY

Eva Kasparek

University of Stellenbosch, Department of Civil Engineering
7600 Stellenbosch, South-Africa
E-mail: kasparek@sun.ac.za

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Abstract. *The design of challenging space structures frequently relies on the theory of folded plates. The models are composed of plane facets of which the bending and membrane stiffness are coupled along the folds. In conventional finite element analysis of faceted structures the continuity of the displacement field is enforced exclusively at the nodes. Since approximate solutions for transverse and for in-plane displacements are not members of the same function space, separation occurs in between the common nodes of adjacent elements.*

It is shown that the kinematic assumptions of Bernoulli are accounted for this incompatibility along the edges in facet models. A general answer to this problem involves substantial modification of plate and membrane theory, but a straight forward formulation can be derived for simply folded plates - structures, whose folds do not intersect. A broad class of faceted structures, including models of various curved shells, belong to this category and can be calculated consistently. The additional requirements to assure continuity concern the mapping of displacement derivatives on the edges. An appropriate finite facet element provides node and edge-oriented degrees of freedom, whose transformation to system degrees of freedom, depends on the geometric configuration at each node. The concept is implemented using conform triangular elements.

To evaluate the new approach, the energy norm of representative structures for refined meshes is calculated. The focus is placed on the mathematical convergence towards reliable solutions obtained from finite volume models.

1 INTRODUCTION

A folded plate structure is a spatial assembly of plane shells with polygonal boundaries. The components are called facets and they are also used to approximate the geometry and physical behaviour of curved shells. Finite facet elements comprise independent expressions for bending and membrane strength. The corresponding degrees of freedom are in-plane diplacements for membrane action and transverse displacement plus its first derivatives for bending action. Due to the use of higher order polynomials for the interpolation of transverse displacement, adjacent elements whose outward normals are not colinear, separate along the folds. The discontinuity between two perpendicular facets M and L with outward normals $\mathbf{n}_M$ and $\mathbf{n}_L$ is shown in figure 1. The displacement vector of a point P on the common edge is different for M and L. In particular the coeffients u_2 and u_3 are affected. The discrepancy is shown for u_3 ($u_{3M} \neq u_{3L}$).

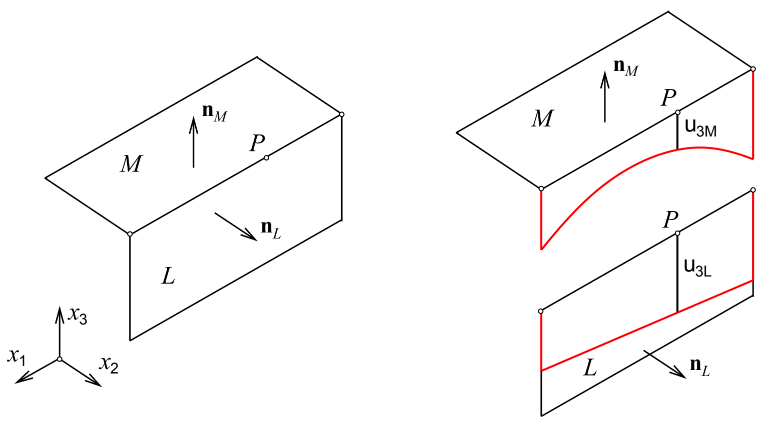


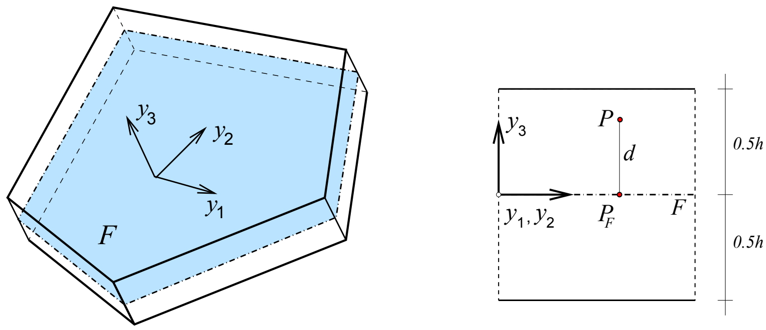
Figure 1: Discontinuity in model of two perpendicular facets

This incompatibility in facet models is well known, but there are no solutions or detailed investigations about its impact on the finite element output. To fully understand the problem, the relations and transformations of relevant kinematic variables are analysed in section 2.1. An edge-oriented examination of these quantaties in section 2.2 provides an insight into the causes of incompatibility. It turns out that the establishment of continuity requires substantial modifications of plate and/or membrane theory. However a straight forward, consistent theory can be formulated for a broad subclass of folded plates. Details about the new approach, its implementation in finite elements and its evaluation are presented in sections 3 and 4.

2 EXTENSION OF THE KINEMATIC ASSUMPTION OF FOLDED PLATES

2.1 Incompatibility of membrane and plate kinematics

A plane shell is a body bounded by plane surfaces whose lateral dimensions are large compared to the separation h between these surfaces. The position of a material point P of the body is determined by the coordinates of its projection P_F to the midplane F and its distance d to F. The coordinates of P and all physical quantitites are specified in a local coordinate system whose axis y_3 is perpendicular to F as shown in figure 2.



(a) Midplane F (b) Coordinates of point P

Figure 2: Plane shell

The state variables at P depend only on quantities at P_F and on d. The strain field consists of three independent components ϵ_{11}, ϵ_{22} and ϵ_{12} that are collected in a vector $\boldsymbol{\epsilon}$. This vector can be split into strains $\boldsymbol{\epsilon}_B$ that are caused by bending and strains $\boldsymbol{\epsilon}_M$ that are associated with the membrane behaviour of the structure. The bending strains of thin walled structures are proportional to the second derivatives of the transverse displacement $u_3(y_1, y_2)$ and the membrane strains are determined from the first derivatives of the in-plane displacements $u_i(y_1, y_2)$with $i = 1, 2$.

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix} = -d \begin{bmatrix} u_{3,11} \\ u_{3,22} \\ u_{3,12} \end{bmatrix} + \begin{bmatrix} u_{1,1} \\ u_{2,2} \\ \frac{1}{2}(u_{1,2} + u_{2,1}) \end{bmatrix} \tag{1}$$

The orthogonality of functions $\boldsymbol{\epsilon}_B$ and $\boldsymbol{\epsilon}_M$ leads to independent terms for the bending and membrane behaviour in the variational formulation of plane shells. Valid approaches for these decoupled problems belong to different function spaces. The in-plane displacements can be sufficiently interpolated by linear functions, while a conform interpolation of transverse displacement belongs to Sobolev space H^2. Functions that are members of H^2 possess at least continuous first derivatives and integrable second derivatives. Accordingly the plane deflection gradient, consisting of $u_{3,1}$ and $u_{3,2}$, has to be continuous. Considering folded plates this requirement influences the derivatives of in-plane displacements. The relations between derivatives of transverse and in-plane displacements on a fold are illustrated for the model in figure 1. The quantities on fold k are described with respect to the local coordinate systems y^{Mk} of M and y^{Lk} of L. Unlike general shell systems introduced above, the direction of the first axes y_1^{Mk} and y_1^{Lk} is determined by the orientation of k. The third axes of the edge-oriented system and the corresponding local shell systems point in the same direction ($y_3^{Mk} = y_3^M$ and $y_3^{Lk} = y_3^L$). The local shell system have superscripts M and L respectively to indicate the associated facets. The particular systems are shown in figure 3.

If the structure under consideration is supposed to be continuous, the derivatives $u_{3,1}^{Mk}$ and $u_{2,1}^{Lk}$ have to be equal along the common boundary k. In case of facets that are not perpendicular, but where the axes y_3^{Mk} and y_3^{Lk} form an arbitrary angle α^{ML}, the derivatives $u_{2,1}^{Mk}$ and $u_{3,1}^{Mk}$ are mapped to $u_{2,1}^{Lk}$ and $u_{3,1}^{Lk}$ with a linear transformation depending on α^{ML}. Given a point of intersection of more than two facets, there is a relationship between all local values of $u_{2,1}$ and $u_{3,1}$. One way to assure the continuity of derivatives of adjacent facets at a point is to introduce

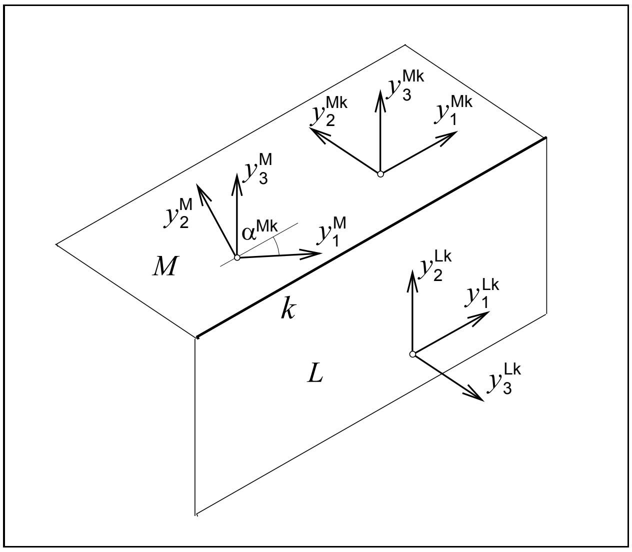


Figure 3: Loacal coordinates in two-facet model

the complete displacement gradient tensor with respect to a given coordinate system x. The local derivatives $u_{2,1}$ and $u_{3,1}$ of each facet intersecting at this point can then be deduced from this tensor by spatial transformation.

In the following equations the transformation rule for the displacement gradient **U**, a second order tensor, from x to a rotated system $\tilde{x}$ is shown. The matrix **R** contains columnwise the base vectors **b**j of system $\tilde{x}$, the matrix $\tilde{\mathbf{U}}$ contains the displacement derivatives in terms of $\tilde{x}$. The displacement gradient consists of only seven independent coefficients, since in plate theory $u_{3,1} = -u_{1,3}$ and $u_{3,2} = -u_{2,3}$.

$$\tilde{\mathbf{U}} = \mathbf{R}^T \mathbf{U}\, \mathbf{R} \qquad \text{with} \tag{2}$$

$$\mathbf{R} = \left[\ \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3\ \right] = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & -u_{3,1} \\ u_{2,1} & u_{2,2} & -u_{3,2} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \tag{3}$$

The calculations of the coefficients $\tilde{u}_{3,1}$ and $\tilde{u}_{1,3}$ are given in detail below:

$$\begin{aligned} \tilde{u}_{1,3} &= a + b_{13}b_{21}\, u_{2,1} + b_{23}b_{11}\, u_{1,2} + (b_{13}b_{31} - b_{22}b_{11})\, u_{3,1} + (b_{23}b_{31} - b_{33}b_{21})\, u_{3,2} \\ \tilde{u}_{3,1} &= a + b_{23}b_{11}\, u_{2,1} + b_{13}b_{21}\, u_{2,1} - (b_{13}b_{31} - b_{22}b_{11})\, u_{3,1} - (b_{23}b_{31} - b_{33}b_{21})\, u_{3,2} \\ &\text{with} \quad a = b_{11}b_{13}\, u_{1,1} + b_{21}b_{23}\, u_{2,2} + b_{31}b_{33}\, u_{3,3} \end{aligned}$$

The equality of $\tilde{u}_{3,1}$ and $-\tilde{u}_{1,3}$ impose several restrictions on the independence of coefficients of **U**. It is satisfied for arbitrary base vectors $\mathbf{b}_j$ only if $u_{1,1} = u_{2,2} = u_{3,3}$ and $u_{2,1} = -u_{1,2}$. The evaluation of the terms $\tilde{u}_{3,2}$ and $\tilde{u}_{2,3}$ leads to the same conclusion: The independence of the quantities $u_{2,1}$ and $u_{1,2}$ as well as the independence of normal strains ($\epsilon_{ii} = u_{i,i}$) is contradictory to the kinematic assumptions of plate theory.

This result is not surprising. The main hint in spatially combining in-plane and transverse displacements is caused by the normal-hypothesis of plate theory: A fiber that is normal to the midplane in the undeformed configuration remains normal to the deformed midplane after load application. This means that this fiber and an infinitesimal fiber in the midplane possess the same rotation angle at their point of intersection ($u_{3,j} = -u_{j,3}$ for $j = 1, 2$). This statement can not be transferred to the theory of membranes. Two in-plane fibers do in general not rotate by the same angle and $u_{2,1} \neq -u_{1,2}$. In early developments of membrane elements for folded plates this fact was neglected and rotational degrees of freedom β_3 were incorporated at the nodes. These elements could not present the shear behaviour of membranes and are therefore unsatisfactory. The introduction of β_3 proved to be sucessful if interpreted as an average quantity along the edge.

2.2 Edge-oriented investigation of incompatibility

For a better insight into the compatibility problem, and to be able to find and assess solutions, the domain of a facet is decomposed into three sets as shown in figure 4. The inner set is denoted with Ω, the union of edges except for the endpoints forms set C_e and the set of endpoints is denoted with C_n.

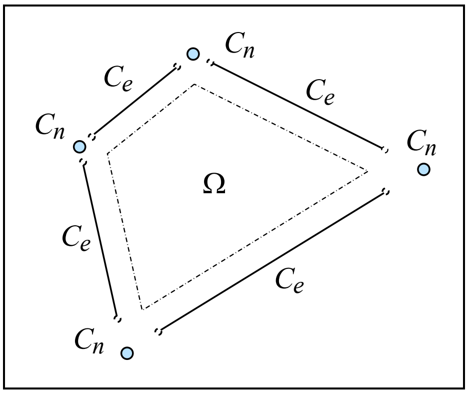


Figure 4: Decomposition of a facet

The compatibility problem arises only for the last two sets, for points in Ω the theory of plane shells is valid. To formulate the transformation between displacement derivatives of adjacent facets at points in C_e and C_n, local edge-oriented coordinate systems as introduced in the previous section are useful. At points in set C_e at most two facets M and L intersect on a fold k. The rule to transform the displacement vector from a system y^{Lk} to y^{Mk} is given below:

$$\mathbf{u}_{Mk} = \mathbf{R}_{ML}\mathbf{u}_{Lk} \qquad \text{with} \qquad \mathbf{R}_{ML} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_{ML} & \sin\alpha_{ML} \\ 0 & -\sin\alpha_{ML} & \cos\alpha_{ML} \end{bmatrix} \tag{4}$$

The same transformation matrix $\mathbf{R}_{ML}$ is applicable for the first derivatives with respect to the local y_1 axis, since $y_1^{Lk} = y_1^{Mk}$.

$$\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix}_{(Mk)} = \mathbf{R}_{ML} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix}_{(Lk)} \tag{5}$$

In the theory of plates and membranes there are no restrictions on variation of $u_{1,1}$. The continuity of $u_{3,1}$ at points in set C_E demands satisfaction of the last two rows of equation (5). These can be rewritten as:

$$\begin{bmatrix} u_{2,1} \\ u_{3,1} \end{bmatrix}_{(Mk)} = \mathbf{T}_{ML} \begin{bmatrix} u_{2,1} \\ u_{3,1} \end{bmatrix}_{(Lk)} \tag{6}$$

It is not sufficient to prescribe the mapping between derivatives with respect to the edge. The complete plane deflection gradient is continuous and therefore also the images of the derivatives $u_{3,2}^{Mk}$ and $u_{3,2}^{Lk}$ have to be taken into account. The derivatives are equivalent to the local rotations β_1^{Mk} and β_1^{Lk} and can be transformed using the first row of matrix $\mathbf{R}_{ML}$:

$$u_{3,2}^{Mk} = \beta_1^{Mk} = \beta_1^{Lk} = u_{3,2}^{Lk} \tag{7}$$

The equations (4), (6) and (7) provide the basis for the new consistent folded plate theory. The mappings implied between variables of adjacent facets cause no contradictions with either membrane or plate theory and ensure continuous displacement fields along the folds. The only restriction is the fact that they are valid only for points in C_e. In the next section (2.3) it will be shown that similar expressions can not be derived for points in C_n.

The compatibility equations can nevertheless be used to get a consistent theory for various types of folded plates. To specify this category we build a subset $C_{\tilde{n}} \subset C_n$. This set contains endpoints where at most two facets intersects. The new theory is valid for structures that comprise only points in Ω, C_e and $C_{\tilde{n}}$. They are subsequently denoted as *simply* folded plates because its folds do not intersect. The physical behaviour of points in Ω is described by the theory of plane shells. For points in C_e and $C_{\tilde{n}}$ the additonal transformation rules (4,6,7) have to be satisfied.

In spite of the new theory being restricted to simply folded plates, it has a broad field of application. Nearly all folded plates and also facet models of various types of curved shells can be classified in this category.

2.3 Kinematic contradictions for general folded plates

The continuous transformation of the local derivatives $u_{3,1}$ and $u_{3,2}$ between more than two coordinate systems implies violations of the kinematic assumptions of either plate or membrane theory. To demonstrate these contradictions a system consisting of three perpendicular facets M, L and K is shown in figure 5. The edge-oriented coordinate systems are shown in figure 6.

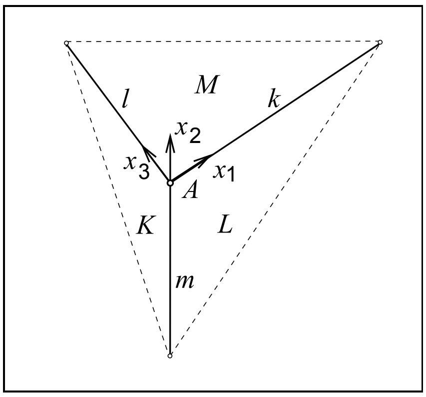


Figure 5: Model of three perpendicular facets

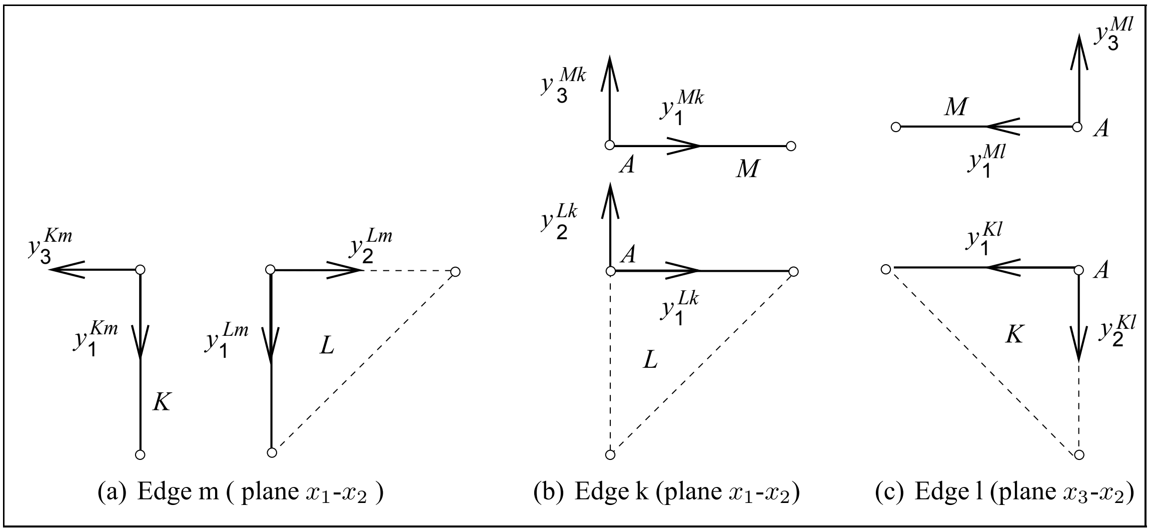


Figure 6: Edge-oriented coordinante systems of three-facet model

We consider the local partial derivatives $u_{2,1}^{Lm}$ and $u_{2,1}^{Lk}$ of element L. Its values depend on derivatives of the transverse displacement on edges of elements K and M:

$$u_{2,1}^{Lm} = -u_{3,1}^{Km} \tag{8}$$

$$u_{2,1}^{Lk} = +u_{3,1}^{Mk} \tag{9}$$

Because axis y_1^{Km} is equal to y_2^{Kl}, the value of $u_{3,1}^{Km}$ is equal to $u_{3,2}^{Kl}$. Because axis y_1^{Mk} is equal to $-y_2^{Ml}$, the value of $u_{3,1}^{Mk}$ is equal to $-u_{3,2}^{Ml}$. These expressions are substituted in equations (8) and (9):

$$u_{2,1}^{Lm} = -u_{3,2}^{Kl} \tag{10}$$

$$u_{2,1}^{Lk} = -u_{3,2}^{Ml} \tag{11}$$

As a result of continuous mapping of rotations β_1, defined in equation (7), the right hand sides of equations (10) and (11) are equal. Consequently:

$$u_{2,1}^{Lm} = u_{2,1}^{Lk} \tag{12}$$

Axis y_2^{LM} correspond to y_1^{Lk} and therefore $u_2^{LM} = u_1^{Lk}$. As y_1^{LM} is equal to $-y_2^{Lk}$ it follows that $u_{2,1}^{LM} = -u_{1,2}^{Lk}$. This expression is substituted into equation (12):

$$-u_{1,2}^{Lk} = u_{2,1}^{Lk} \tag{13}$$

If this expression holds, the membrane shear ϵ_{12} at the point of intersection A is zero. This is inconsistent with the theory of membranes and can only be resolved by renouncing the continuity of plate rotations β_1. A novel approach to this problem is subject of current research.

3 FINITE FACET ELEMENT

3.1 Element degrees of freedom

To prove the impact of the theory of simply folded plates an appropriate facet element is needed. The element has to provide suitable degrees of freedom and shape functions to satisfy the additional kinematic requirements. For every edge k of the element the following displacement derivatives are required:

$$u^k_{2,1} \quad u^k_{3,1} \quad u^k_{3,2} \tag{14}$$

It is convient to place these degree of freedoms directly at the nodes. At each node of an element M two edges k and l intersects. Because of the imposed continuity of the deflection gradient, nodal values for $u^M_{3,1}$ and $u^M_{3,2}$ in terms of coordinate system y^M can be used. The edge specific values $u^{Mk}_{3,1}$ and $u^{Ml}_{3,1}$ can then be determined by plane transformations. The incorporation of the complete membrane displacement gradient $u_{i,j}$, $i, j = 1, 2$ is not advisable. The coefficients can not be related to respective values of an adjacent element for want of compatible spatial transformation rules. Consequently two independent values $u^l_{2,1}$ and $u^k_{2,1}$ are introduced at the intersection of k and l. The freedom vector of element M at a node intersecting k and l is:

$$\mathbf{u}_n = [\quad u^M_1 \quad u^M_2 \quad u^M_3 \quad u^M_{3,1} \quad u^M_{3,2} \quad u^{Mk}_{2,1} \quad u^{Ml}_{2,1} \quad] \tag{15}$$

To avoid discontinuity of the displacement field along edge k, the interpolation functions for the in-plane displacement u^{Mk}_2 and for the transverse displacement u^{Mk}_3 has to coincide. As this requirement holds also for the adjacent element L at k no gap between M and L emerges. The conclusion is illustrated at the two-facet model introduced in figure 1(a). For transparency reasons again only the variation of one displacement is shown in figure 7.

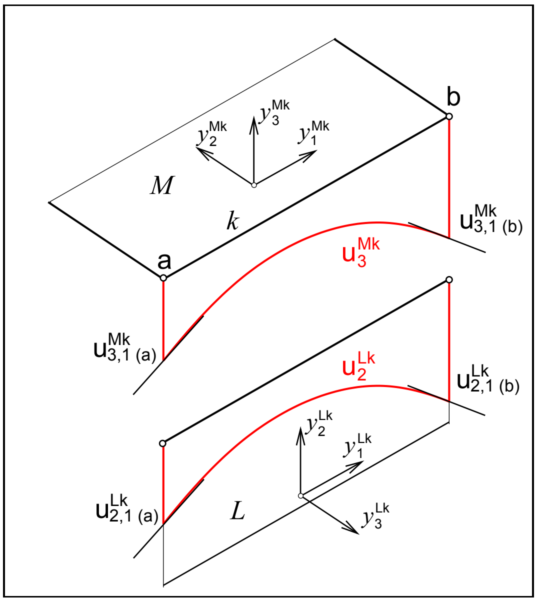


Figure 7: ,

Consistent displacement variation along edge k

Suitable interpolation approaches for triangular elements were presented in [1]: Both local displacements u_2 and u_3 are interpolated cubically along the edges, while the displacement u_1 varies linear along the edges. The proposed element has the advantage that the approach for u_3 belongs to Sobolev space H^2 and the approaches for the in-plane displacements are member of Sobolev space H^1. Consequently it is a conform finite element for plane shell problems.

3.2 System degrees of freedom

The total number of system degrees of freedom at a node, as well as the applicable nodal coordinate system, depend on the number of elements and their mutual geometric orientation. It is necessary to distinguish between *plane* nodes, where all elements have the same outward normal, and nodes on folds. The treatment of the three system displacement freedoms u_j is not affected by this classification of a node: every element possess the complete displacement tensor and its coefficients can be transformed into any nodal coordinate system. Hence the displacement freedoms are not enclosed in the presentation of the different cases below.

In case of plane nodes the nodal system freedom vector contains the deflection gradient and a value $u_{2,1}$ for each edge. The derivatives $u_{3,1}$ and $u_{3,2}$ are related to a coordinate system, whose axis y_3 is normal to the midplane of the elements. The deflection gradient of an element can be transformed into this system with a standard rotation matrix. The freedoms $u_{2,1}$ of adjacent elements on their common boundary refer to colinear coordinate edge-oriented systems. Each pair of degrees of freedom can be mapped to corresponding system freedoms without any transformations.

If the node is on a fold, here denoted with k, the nodal system vector contains the derivatives $u^k_{2,1}$, $u^k_{3,1}$ and $u^k_{3,2}$ that refer to an edge-oriented, element-independent coordinate system y^k, and values $u_{2,1}$ for each edge that do not coincide with k. The treatment of freedoms $u_{2,1}$ that are not associated with edge k correspond to the mapping in plane problems. The transformation of the other derivatives requires two steps. Consider an element M with a coordinate system y^M. First the derivatives of transverse displacement are transformed from y^M to y^{Mk}. The local freedom $u_{2,1}$ is per definition related to system y^{Mk}. The nodal coordinate system y^k does not necessarly coincide with y^{Mk}. The axes y^k_1 and y^{Mk}_1 are parallel, but y^k_3 and y^{Mk}_3 form an arbitrary angle α. The transformation between y^k and y^{Mk} is therefore similar to the transformation between y^{Mk} and y^{Lk}. The applicable transformation rules for the coefficients $u_{3,1}$ and $u_{2,1}$ can be deduced from equation (6), and for $u_{3,2}$ from equation (7). Special attention is drawn to elements K who have only one node on k and none of its edges. To peform the second step of the described transformation process a fictitious element freedom $u^{Kk}_{2,1} = 0$ has to be introduced at the current node.

4 CONVERGENNCE

The behaviour of a structure can be characterised by a single value, a norm. A very reliable measurement is the energy norm, which is calculated by integrating the underlying bilinear form $\boldsymbol{\epsilon}^T \mathbf{E} \boldsymbol{\epsilon}$ over the whole body. The analytical value is not known for relevant structures. The convergence of the new method therefore has to be compared with reference solutions of well defined models. Conventional facet models are not considered because of two major limitations:

- Conventional facet elements are not conform: The discontinuity along the edges results in degradation of energy that can not be determined exactly.
- The theory underlying conventional facet models is not consistent: The results do not guarantee a good approximation of the real structure. It was shown already by Bernadou and Trouve [2] that calculations based on conventional facet elements, with and without drilling degrees of freedom, do not necessarly converge towards exact solutions. The study focused on facet models of curved shells.

The mathematical convergence of the new method can be quantified reliable in comparison with volume models. Representative folded plate structures are discretised with volume elements and with the new facet elements. The results will be presented at the conference.

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