HYPERMONOGENIC POLYNOMIALS

S.–L. Eriksson* and **J. Kettunen****

*Sirkka-Liisa Eriksson
Department of Mathematics
Tampere University of Technology
P.O. Box 553
FI-33101 Tampere
email: Sirkka-Liisa.Eriksson@tut.fi

**Jarkko Kettunen
Department of Mathematics
Tampere University of Technology
P.O. Box 553
FI-33101 Tampere
email: Jarkko.Kettunen@tut.fi

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Abstract. *Let $C\ell_3$ be the (universal) Clifford algebra generated by e_1, e_2 and e_3 satisfying $e_ie_j + e_je_i = -2\delta_{ij}$, $i, j = 1, 2, 3$. The Dirac operator in $C\ell_3$ is defined by $D = \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i}$, where $e_0 = 1$. The modified Dirac operator is introduced by $Mf = Df + 2\frac{Q'f}{x_3}$, where $'$ is the main involution and Qf is given by the decomposition $f(x) = Pf(x) + Qf(x)\, e_3$ with $Pf(x), Qf(x) \in \mathbb{H}$. A continuously differentiable function $f : \Omega \to C\ell_3$ is called hypermonogenic in an open subset Ω of R^4, if $Mf(x) = 0$ outside the hyperplane $x_3 = 0$. We consider homogenous polynomials in various function spaces. In particular we collect results concerning differentiation and linear independency of the polynomials. We find a basis for homogeneous holomorphic Cliffordian polynomials of degree m.*

1 Introduction

It is well know that the power function x^m is not monogenic. In $\mathcal{C}\ell_3$ there are basically two ways to include the power function into the set of solutions: The hypermonogenic functions satisfying the equation $x_3 Mf = 0$ or holomorphic Cliffordian functions satisfying the equation $\Delta Df = 0$. Hypermonogenic functions are notably studied by H. Leutwiler and the first author for example in [2], [3], [4] and [5] while the holomorphic Cliffordian functions are studied by G. Laville and I. Ramadanoff in [7]. In addition, holomorphic Cliffordian function are in close connection with polyharmonic functions and iterated Dirac operators studied by L. Pernas in [8] and in the complex case by R. Ryan in [9].

L. Pernas has found out the dimension of the space of homogenous holomorphic Cliffordian polynomials of degree m in [8], but his approach did not include a basis. It is known that the hypermonogenic functions are included in the space of holomorphic Cliffordian functions. As our main result we prove the surprising result that the polynomials L_m^α, T_m^β and $\frac{\partial T_{m+1}^\gamma}{\partial x_3}$ form a basis for the right module of homogeneous holomorphic Cliffordian polynomials of degree m. For this task, we first recall the function spaces of monogenic, hypermonogenic and holomorphic Cliffordian functions and give the results needed in the proof of our main theorem. We list some basic polynomials and their properties for the various function spaces. In particular, we consider recursive formulas, rules of differentiation and properties of linear independency for the polynomials.

2 Notations

As general references for the chapter we give [2], [3] and [5]. We consider the universal Clifford algebra $\mathcal{C}\ell_3$, that is the associative algebra over $\mathbb{R}$ generated by the elements e_1, e_2 and e_3 satisfying the relations

$$e_i^2 = -1$$

and

$$e_i e_j = -e_j e_i, \tag{1}$$

for $i, j = 1, 2, 3$. Furthermore, we denote $e_0 = 1$. The general element of the algebra $\mathcal{C}\ell_3$ is of the form

$$a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_{12} e_{12} + a_{13} e_{13} + a_{23} e_{23} + a_{123} e_{123},$$

where the coefficients a_k are real and the abbreviated notations are $e_ie_j = e_{ij}$ and $e_1e_2e_3 = e_{123}$. In general, if $\alpha = \{\alpha_1, ..., \alpha_j\} \subseteq \{1, 2, 3\}$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_j$ we denote $e_\alpha = e_{\alpha_1} \cdots e_{\alpha_j}$, with $e_\emptyset = e_0$, and get the presentation

$$a = \sum a_\alpha e_\alpha.$$

We call the elements $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3$, for $x_i \in \mathbb{R}$, *paravectors.* We identify the set of paravectors with the Euclidean space $\mathbb{R}^4$. By (1) it is clear that $\mathcal{C}\ell_3$ is not a commutative algebra. The elements $\alpha + \beta e_{123}$ with $\alpha, \beta \in \mathbb{R}$ commute with all elements of $\mathcal{C}\ell_3$.

In $\mathcal{C}\ell_3$ we define three involutions. The *main involution* $' : \mathcal{C}\ell_3 \to \mathcal{C}\ell_3$ is the algebra isomorphism defined by $e_0 = 1$ and $e_i = -e_i$, for $i = 1, 2, 3$. The *reversion* is an antiautomorphism $^* : \mathcal{C}\ell_3 \to \mathcal{C}\ell_3$ defined by $e_i^* = e_i$, for $i = 0, ..., 3$ and $(ab)^* = b^*a^*$. The *conjugation* is the antiautomorphism $^- : \mathcal{C}\ell_3 \to \mathcal{C}\ell_3$ defined by $\overline{a} = (a^*)' = (a')^*$.

We can embed the division algebra of quaternions $\mathbb{H}$ into the algebra $\mathcal{C}\ell_3$ by identifying e_1 with i, e_2 with j and e_{12} with k. Furthermore, we can decompose the algebra $\mathcal{C}\ell_3$ into two copies of $\mathbb{H}$ by writing

$$\begin{aligned} a &= a_0 + a_1e_1 + a_2e_2 + a_{12}e_{12} + (a_3 + a_{13}e_1 + a_{23}e_2 + a_{123}e_{12})\, e_3 \\ &= Pa + Qae_3. \end{aligned}$$

The relation defines the operators $P : \mathcal{C}\ell_3 \to \mathbb{H}$ and $Q : \mathcal{C}\ell_3 \to \mathbb{H}$ with the properties

$$P^2 = P,\ PQ = Q,\ QP = 0 \text{ and } Q^2 = 0.$$

We define the involution $\widehat{\ } : \mathcal{C}\ell_3 \to \mathcal{C}\ell_3$ by $\widehat{e_3} = -e_3$, $\widehat{e_i} = e_i$, for $i = 0, 1, 2$ and $\widehat{ab} = \widehat{a}\widehat{b}$. Using this involution we obtain for any $a \in \mathcal{C}\ell_3$ the following formulas

$$Pa = \frac{a - e_3a'e_3}{2} = \frac{a + \widehat{a}}{2} \tag{2}$$

$$Qa = \frac{e_3a' - ae_3}{2} = \frac{\widehat{a} - a}{2}e_3. \tag{3}$$

In proving the formulas (2) and (3) we use the identities

$$a'e_3 = e_3\widehat{a} \text{ and } e_3a' = \widehat{a}e_{3,}$$

for arbitrary $a \in \mathcal{C}\ell_3$ and

$$b'e_3 = e_3b \text{ and } e_3b' = be_{3,}$$

for any $b \in \mathbb{H}$. In addition we can show that

$$P(a') = (Pa)' \text{ and } Q(a') = -(Qa)'.$$

In order to abbreviate the notations we set $(Pa)' = P'a$ and $(Qa)' = Q'a$. For the P- and Q-parts of a product we have

$$\begin{aligned} P(ab) &= (Pa)\,Pb + (Qa)\,Q(b') \\ Q(ab) &= (Pa)\,Qb + (Qa)\,P(b') \qquad (4) \\ Q(ab) &= aQb + (Qa)\,b'. \qquad (5) \end{aligned}$$

For easy reference we recall some basic notations of multi-indexes. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. An element $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^4$ is called a *multi-index*. For a multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^4$ and a paravector $x_0 + x_1e_1 + x_2e_2 + x_3e_3$ we have

$$\begin{aligned} x^\alpha &= x_0^{\alpha_0} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \\ \alpha! &= \alpha_0!\alpha_1!\alpha_2!\alpha_3! \text{ (factorial)} \\ |\alpha| &= \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \text{ (length of } \alpha) \\ \binom{m}{\alpha} &= \frac{m!}{\alpha!} = \frac{m!}{\alpha_0!\alpha_1!\alpha_2!\alpha_3!}, \text{ if } |\alpha| = m. \end{aligned}$$

We denote by ε_i the multi-index $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, with $\alpha_i = 1$ and $|\alpha| = 1$.

3 Function spaces

Let Ω be an open subset of $\mathbb{R}^4$. We define the *left Dirac operator* for a mapping $f : \Omega \to \mathcal{C}\ell_3$ with continuously differentiable components by

$$D_l f = \sum_{i=0}^{3} e_i \frac{\partial f}{\partial x_i}$$

and the *right Dirac operator* by

$$D_r f = \sum_{i=0}^{3} \frac{\partial f}{\partial x_i} e_i.$$

The functions satisfying $D_l f = 0$ are called *left monogenic* and the functions satisfying $D_r f = 0$ are called *right monogenic.*

Since the properties $D_l f = 0$ and $D_r f^* = 0$ are equivalent, it is sufficient to consider only the left operator, the properties of the right one being analogous. In particular this is true for paravector valued functions, since in this case $f = f^*$. In what follows, we abbreviate $D_l = D$ and call D the Dirac operator. We define the operator $\overline{D}$ by

$$\overline{D} = \sum_{i=0}^{3} \overline{e_i} \frac{\partial f}{\partial x_i}.$$

A simple calculation shows that we can decompose the Laplace-operator in $\mathbb{R}^4$ as

$$\Delta = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = D\overline{D} = \overline{D}D$$

and therefore monogenic functions are harmonic. However, there are functions, for example $f(x) = x$ that are harmonic but are not monogenic.

The fact that the power function $F(x) = x^m$ is not monogenic even if $m = 1$ was the main reason for H. Leutwiler and S.-L. Eriksson to introduce the modified Dirac operator M defined for functions $f \in \mathcal{C}^1(\Omega, \mathcal{C}\ell_3)$ by

$$Mf = D + \frac{2}{x_3} Q'f$$

and the operator $\overline{M}$ by

$$\overline{M}f = \overline{D} - \frac{2}{x_3} Q'f.$$

Similarly as in the case of monogenic functions we study only the left operator.

A function $f : \Omega \to \mathcal{C}\ell_3$ is called *hypermonogenic*, if $f \in \mathcal{C}^1(\Omega)$ and $x_3 Mf = 0$ for all $x \in \Omega$. Paravector valued hypermonogenic functions are called *H-solutions*. The hypermonogenic functions are closely related to the hyperbolic metric $ds^2 = \frac{1}{x_2^3}(x_0^2 + x_1^2 + x_2^2 + x_3^2)$ and therefore we consider the hyperbolic Laplace operator $\Delta_H = \Delta - \frac{2}{x_3}\frac{\partial}{\partial x_3}$ associated to this metric. The functions satisfying $\Delta_H = 0$ are called *hyperbolic harmonic* functions. The definition of hypermonogenic functions can be written using the P- and Q-parts.

Lemma 1 ([3, Proposition 3]) *Let Ω be an open subset of $\mathbb{R}^4$ and $f : \Omega \to \mathcal{C}\ell_3$ be a mapping with continuous partial derivatives. The equation $Mf = 0$ is equivalent with the system of equations*

$$x_3 \left(D_2(Pf) - \frac{\partial Q'f}{\partial x_3} \right) + 2Q'f = 0, \tag{6}$$

$$D_2(Qf) + \frac{\partial P'f}{\partial x_3} = 0, \tag{7}$$

where $D_2 f = \sum_{i=0}^{2} e_i \frac{\partial f}{\partial x_i}$.

The equations (6) and (7) have some immediate consequences (see [2, Lemma 10]).

Lemma 2 *Let f be hypermonogenic on $\Omega \subset \mathbb{R}^4$ with $\omega = \Omega \cap \mathbb{R}^3 \neq \emptyset$. Then on ω we have*

$$Qf\left(\cdot,0\right) = 0, \tag{8}$$

$$D_2 Pf\left(\cdot,0\right) = -\frac{\partial Q'f}{\partial x_3}\left(\cdot,0\right), \tag{9}$$

$$\frac{\partial Pf}{\partial x_3}\left(\cdot,0\right) = 0. \tag{10}$$

Proof. The proof of (8) is just to evaluate (6) at $x_3 = 0$. Writing (6) as

$$D_2 Pf - \frac{\partial Q'f}{\partial x_3} + 2\frac{Q'f}{x_3} = 0$$

and letting $x_3 \to 0$ yield

$$D_2 Pf\left(\cdot,0\right) - \frac{\partial Q'f}{\partial x_3}\left(\cdot,0\right) + 2\frac{\partial Q'f}{\partial x_3}\left(\cdot,0\right) = 0,$$

since f has continuous partial derivatives. Thus

$$\frac{\partial Q'f}{\partial x_3}\left(\cdot,0\right) = D_2 Pf\left(\cdot,0\right)$$

and we have (9). Combining (9) and (7) we get (10). ■

Lemma 3 *The P- and Q-parts of a hypermonogenic function f satisfy the equations*

$$x_3 \Delta Pf - 2\frac{\partial Pf}{\partial x_3} = 0 \tag{11}$$

$$x_3^2 \Delta Qf - 2x_3\frac{\partial Qf}{\partial x_3} + 2Qf = 0. \tag{12}$$

Proof. Choose $k = 2$ in [6, Lemma 2]. ■

The previous Lemma states that the P-part of a hypermonogenic function is a hyperbolic harmonic function and the Q-component is an eigenfunction of the Laplace–Beltrami operator $x_3^2 \Delta_H$ corresponding to the eigenvalue -2. Furthermore, we obtain for the derivative $\frac{\partial h}{\partial x_3}$ of a hyperbolic harmonic function the following result.

Lemma 4 *If h is a hyperbolic harmonic function then $\frac{\partial h}{\partial x_3}$ is an eigenfunction of the Laplace–Beltrami operator $x_3^2\Delta_H$ corresponding to the eigenvalue -2.*

Proof. Suppose $\Delta_H h = 0$. Differentiating with respect to the variable x_3 yields

$$0 = \frac{\partial}{\partial x_3}\left(\Delta h - \frac{2}{x_3}\frac{\partial}{\partial x_3}h\right) = \Delta\frac{\partial h}{\partial x_3} - \frac{2}{x_3}\frac{\partial^2 h}{\partial x_3^2} + \frac{2}{x_3^2}\frac{\partial h}{\partial x_3}$$

and the result follows by multiplying by x_3^2. ■

Hypermonogenic functions form a right vectorspace over the quaternions and the derivatives $\frac{\partial f}{\partial x_i}$ of a hypermonogenic function f are hypermonogenic for $i = 0, 1, 2$. But multiplication by e_3 and differentiation with respect to the variable x_3 do not preserve hypermonogenicity.

Lemma 5 *If f is a hypermonogenic function, then the function fe_3 is hypermonogenic if and only if $f = 0$.*

Proof. Assume that f and fe_3 are hypermonogenic. By definition

$$Df + \frac{2}{x_3}Q'f = 0 \tag{13}$$

and

$$D(fe_3) + \frac{2}{x_3}Q'(fe_3) = 0. \tag{14}$$

Since $D(fe_3) = (Df)e_3$ and by (4) $Q(fe_3) = Pf$ multiplying (13) by $-e_3$ from the right and adding up with (14) yields

$$-\frac{2}{x_3}(Q'f)e_3 + \frac{2}{x_3}P'f = 0$$

and thus $\frac{2}{x_3}Qfe_3 + \frac{2}{x_3}Pf = \frac{2}{x_3}f = 0$. If $f = 0$ then trivially fe_3 is hypermonogenic. ■

Lemma 6 *If f is a hypermonogenic function, then the function $\frac{\partial f}{\partial x_3}$ is hypermonogenic if and only if $Qf = 0$.*

Proof. If $Qf = 0$ and f is hypermonogenic then $Df = 0$ and thus $\frac{\partial}{\partial x_3}(Df) = D\left(\frac{\partial}{\partial x_3}f\right) = 0$ and $\frac{\partial}{\partial x_3}f$ is hypermonogenic, since $Q\left(\frac{\partial}{\partial x_3}f\right) = 0$.

Assume that f and $\frac{\partial}{\partial x_3}f$ are hypermonogenic. Then $\frac{\partial}{\partial x_3}\left(Df + \frac{2}{x_3}Q'f\right) = 0$ and $D\left(\frac{\partial}{\partial x_3}f\right) + \frac{2}{x_3}Q'\left(\frac{\partial}{\partial x_3}f\right) = 0$. Subtracting the equations gives $\frac{2}{x_3^2}Q'f = 0$ and therefore $Qf = 0$. ■

Lemma 6 is an equivalent form of the following result.

Lemma 7 ([2, Theorem 4]) *If f is a hypermonogenic function then the function $\frac{\partial f}{\partial x_3}$ is hypermonogenic if and only if $\frac{\partial f}{\partial x_3} = 0$.*

Combining multiplication by e_3 and differentiation with respect to the variable x_3 yields a hypermonogenic function.

Lemma 8 ([4, Theorem 3]) *If f is a hypermonogenic function then the function g defined by*

$$g = \frac{\partial f}{\partial x_3} e_3 + \frac{2Qf}{x_3}$$

is hypermonogenic.

Another way to include the power function into the set of solutions is to consider the *holomorphic Cliffordian functions* defined by the condition $D\Delta f = 0$. The holomorphic Cliffordian functions are studied for example in [7]. The spaces of monogenic, harmonic and hypermonogenic functions are included in the space of holomorphic Cliffordian functions.

4 Polynomial solutions

First we consider the monogenic polynomials. We note that $Dx = -2 \neq 0$ and therefore we cannot use the polynomials x^k to construct monogenic polynomial solutions. Therefore the homogeneous monogenic polynomials of degree m denoted F_m^α and called the *Fueter-polynomials*, (see for example [1], [10] or [11]) are defined by

$$F_m^\alpha(x) = \sum z_{\sigma_1} z_{\sigma_2} \cdots z_{\sigma_{|\alpha|}}, \tag{15}$$

where the sum is over all different permutations $\sigma = \left(\sigma_1, ..., \sigma_{|\alpha|}\right)$ of $|\alpha|$ elements containing α_1 1's α_2 2's and α_3 3's. The variables $z_1 = (x_0 e_1 - x_1)$, $z_2 = (x_0 e_2 - x_2)$ and $z_3 = (x_0 e_3 - x_3)$ in (15) are called the hypercomplex variables. For the Fueter-polynomials it holds (see [11, Lemma 1.6]).

Theorem 9 *([11], Lemma 1.6) The polynomials F_m^α are monogenic and linearly independent over $\mathcal{C}\ell_3$.*

The Fueter-polynomials satisfy the recursion relation

$$F_m^\alpha(x) = z_1 F_m^{\alpha-\varepsilon_1}(x) + z_2 F_m^{\alpha-\varepsilon_2}(x) + z_3 F_m^{\alpha-\varepsilon_3}(x)$$

and the derivatives of the Fueter-polynomials are given by

$$\frac{\partial F_m^\alpha}{\partial x_k} = |\alpha| \, F_{m-1}^{\alpha-\varepsilon_k}$$

(see for example [11] for the proofs).

Next we consider homogeneous hypermonogenic polynomials of degree m. The definitions and results are from [5] if not otherwise stated.

Definition 10 *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. The elementary H-polynomial E_m^α is defined by*

$$E_m^\alpha(x) = \sum_{(\sigma_0,\ldots,\sigma_m)\in\sigma} \sigma_0 x \sigma_1 \cdots x \sigma_m,$$

where σ is the set of all permutations of $m+1$ elements containing α_1 elements equal to e_1 and α_2 elements equal to e_2 and the rest equal to 1. We set $E_0^{(0,0)}(x) = 1$. If $|\alpha| > m+1$ or $\alpha_i < 0$, we set $E_m^\alpha(x) = 0$.

Definition 11 *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. The elementary H-polynomial L_m^α is defined by*

$$L_m^\alpha(x) = \sum_{(\sigma_0,\ldots,\sigma_m)\in\sigma} z_{\sigma_1} z_{\sigma_2} \cdots z_{\sigma_{m+|\alpha|}},$$

where the sum is over all permutations of elements $z_{\sigma_1}, z_{\sigma_2}, \ldots, z_{\sigma_{m+|\alpha|}}$ containing m elements equal to x, α_1 equal to e_1 and α_2 equal to e_2. We set $L_0^{(0,0)}(x) = 1$ and $L_m^\alpha(x)$, if $\alpha_1 < 0$ or $\alpha_2 < 0$.

The polynomials E_m^α and L_m^α satisfy the following recursion formulas (see [11]).

Lemma 12 *Let $\alpha \in \mathbb{N}_0^2$ and m be a non-negative integer. Then*

$$E_m^\alpha(x) = xE_{m-1}^\alpha(x) + e_1 x E_{m-1}^{\alpha-\varepsilon_1}(x) + e_2 x E_{m-1}^{\alpha-\varepsilon_2}(x) \tag{16}$$

and

$$L_m^\alpha(x) = xL_{m-1}^\alpha(x) + e_1 L_m^{\alpha-\varepsilon_1}(x) + e_2 L_m^{\alpha-\varepsilon_2}(x). \tag{17}$$

Since the mapping

$$x \to \frac{1}{\alpha!}\frac{\partial^{|\alpha|} x^{m+|\alpha|}}{\partial x^\alpha}$$

satisfies the recursion formula (17) with the same initial values as L_m^α, we obtain the second definition for the polynomials L_m^α.

Definition 13 *Let $\alpha \in \mathbb{N}_0^2$ and m be a non-negative integer. Then*

$$L_m^\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|} x^{m+|\alpha|}}{\partial x^\alpha}. \tag{18}$$

The polynomials L_m^α are explicitly known (see [5, Theorem 19]).

Theorem 14 *If* $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ *and* $m \in \mathbb{N}$*, then*

$$L_m^\alpha(x) = \sum_{|\beta|=m} \binom{m+|\alpha|}{\alpha, \beta} c(\beta+\alpha) x^\beta$$

where the coefficients $c(\gamma)$ *for* $\gamma = (\gamma_0, \widetilde{\gamma})$, $\gamma_0 \in \mathbb{N}_0$ *and* $\widetilde{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{N}_0^3$ *are given by*

$$c(\gamma) = \begin{cases} \dfrac{\binom{|\widetilde{\gamma}|/2}{\widetilde{\gamma}/2}}{\binom{|\widetilde{\gamma}|}{\widetilde{\gamma}}} (-1)^{(|\widetilde{\gamma}|/2)}, & \text{if } \widetilde{\gamma} \text{ is even,} \\ \dfrac{\binom{(|\widetilde{\gamma}|-1)/2}{(\widetilde{\gamma}-\varepsilon_i)/2}}{\binom{|\widetilde{\gamma}|}{\widetilde{\gamma}}} (-1)^{((|\widetilde{\gamma}|-1)/2)} e_i, & \text{if } i = 1,2,3 \text{ and } \widetilde{\gamma} - \varepsilon_i \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 14 has an immediate consequence that we list as a Lemma for the proof of our main result Theorem 26.

Lemma 15 *Let* $\alpha \in \mathbb{N}_0^2$ *and* $m \in \mathbb{N}$*. Then*

$$\frac{\partial^k Q L_m^\alpha}{\partial x_3^k}(\cdot, 0) = 0$$

for all even $k \in \mathbb{N}$*.*

Proof. By Theorem 14 e_3 appears in $c(\beta+\alpha)$ only when $\beta_3 - 1$ is even. Thus QL_m^α is odd with respect to the variable x_3. Differentiating an even number of times with respect to x_3 yields terms with odd powers of x_3, which vanish when evaluated at $x_3 = 0$. ■

The derivative of the polynomial E_m^α is given by

$$\frac{\partial E_m^\alpha}{\partial x_i} = (\alpha_i + 1) E_{m-1}^{\alpha+\varepsilon_i} - (2m - \alpha_i + 1) E_{m-1}^{\alpha-\varepsilon_i} + (\alpha_i + 1) E_{m-1}^{\alpha+\varepsilon_i-2\varepsilon_2}$$

$$\frac{\partial E_m^\alpha}{\partial x_0} = (m + |\alpha|) E_{n-1}^\alpha - (m - |\alpha| + 2) \sum_{i=1}^{2} E_{m-1}^{\alpha-2\varepsilon_i},$$

for $i \in \{1,2\}$ and $|\alpha| \geq 2$ (see [5]) while the derivatives of polynomial L_m^α with $|\alpha| \geq 2$ satisfy the equations

$$\frac{\partial L_m^\alpha}{\partial x_0} = (m + |\alpha|) L_{m-1}^\alpha$$

and

$$\frac{\partial L_m^\alpha}{\partial x_k} = (\alpha_k + 1)\, L_{m-1}^{\alpha+\varepsilon_k},$$

for $k \in \{1,2\}$ (see for example [5]). The derivatives of the polynomials L_m^α with respect to x_3 can be found out differentiating the presentation (18). In particular we get

Proposition 16 *For polynomials L_m^α with $|\alpha| = m$*

$$\frac{\partial Q L_m^\alpha}{\partial x_3}(\cdot, 0) = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha}\sum_{l=1}^{m}(-1)^{m-l}\binom{2m}{2l-1}x_0^{2l-1}\left(x_1^2 + x_2^2\right)^{m-l}.$$

Proof. Since x_0 is real we get by the binomial theorem

$$x^{2m} = (x_0 + x_1e_1 + x_2e_2 + x_3e_3)^{2m} = \sum_{k=0}^{2m}\binom{2m}{k}x_0^k\,(x_1e_1 + x_2e_2 + x_3e_3)^{2m-k}$$

$$= \sum_{\substack{k=0\\ k\text{ even}}}^{2m}\binom{2m}{k}x_0^k\,(x_1e_1 + x_2e_2 + x_3e_3)^{2m-k} + \sum_{\substack{k=0\\ k\text{ odd}}}^{2m-1}\binom{2m}{k}x_0^k\,(x_1e_1 + x_2e_2 + x_3e_3)^{2m-k}.$$

Since $(x_1e_1 + x_2e_2 + x_3e_3)^{2l} = (-1)^l\left(x_1^2 + x_2^2 + x_3^2\right)^l$ is real the Q part of x^{2m} comes out from the odd part of the previous sum. Thus writing $k = 2l-1$ we obtain

$$Qx^{2m} = Q\left(\sum_{l=1}^{m}\binom{2m}{2l-1}x_0^{2l-1}\,(x_1e_1 + x_2e_2 + x_3e_3)^{2m-2l+1}\right)$$

$$= Q\left(\sum_{l=1}^{m}\binom{2m}{2l-1}x_0^{2l-1}\,(-1)^{m-l}\left(x_1^2 + x_2^2 + x_3^2\right)^{m-l}(x_1e_1 + x_2e_2 + x_3e_3)\right)$$

$$= \sum_{l=1}^{m}\binom{2m}{2l-1}x_0^{2l-1}\,(-1)^{m-l}\left(x_1^2 + x_2^2 + x_3^2\right)^{m-l}x_3$$

and differentiating with respect to x_3 yields

$$\frac{\partial Qx^{2m}}{\partial x_3} = \sum_{l=1}^{m}\binom{2m}{2l-1}x_0^{2l-1}\,(-1)^{m-l}\left(x_1^2 + x_2^2 + x_3^2\right)^{m-l}$$

$$+\sum_{l=1}^{m-1}\binom{2m}{2l-1}x_0^{2l-1}\,(-1)^{m-l}\,(m-l)\left(x_1^2 + x_2^2 + x_3^2\right)^{m-l-1}2x_3^2$$

and therefore

$$\frac{\partial Qx^{2m}}{\partial x_3}(\cdot,0)=\sum_{l=1}^{m}(-1)^{m-l}\binom{2m}{2l-1}x_0^{2l-1}\left(x_1^2+x_2^2\right)^{m-l}. \tag{19}$$

■

Since $\mathbb{R}^4$ is of even dimension, we need some additional polynomials, called the T-polynomials, to construct a basis for the $\mathbb{H}$-module of homogeneous polynomial H-solutions (see [5, Definition 4]). The homogeneous polynomial T_m^α may be characterized as follows.

Lemma 17 ([6, Lemma 9]) *Let $\alpha\in\mathbb{N}_0^3$, with $|\alpha|=m-2$. Then*

$$\frac{\partial^\alpha T_m^\alpha}{\partial x^\alpha}=\alpha! x_3^2 e_3$$

and

$$\frac{\partial^2 T_m^\alpha}{\partial x_3^2}(x)=2x^\alpha e_3$$

for any x with $x_3=0$.

The T_m^a-polynomials are explicitly known (see [11, Theorem 2.14]). The part PT_m^α has the explicit presentation

$$PT_m^\alpha=\sum_{i=0}^{2}\left(\left(-e_i^2\right)\sum_{\beta=0}^{\left[\frac{\alpha}{2}\right]}d_{\beta,i}x^{\alpha-2\beta-\varepsilon_i}x_3^{2|\beta|+3}\right)e_i, \tag{20}$$

for some real coefficients $d_{\beta,i}$ while QT_m^α has the explicit presentation

$$QT_m^\alpha=\sum_{\beta=0}^{\left[\frac{\alpha}{2}\right]}d_\beta x^{\alpha-2\beta}x_3^{2|\beta|+2}, \tag{21}$$

with some real coefficients d_β. The above presentations give us the next result for the proof of our main theorem.

Lemma 18 *For any odd $k\in\mathbb{N}_0$*

$$\frac{\partial^k QT_m^\alpha}{\partial x_3^k}(\cdot,0)=0 \tag{22}$$

and for any even $l\in\mathbb{N}_0$

$$\frac{\partial^l PT_m^\alpha}{\partial x_3^l}(\cdot,0)=0. \tag{23}$$

Proof. By (21) QT_m^α consists of terms containing only even powers of x_3 with degree at least 2 and by (20) PT_m^α consists of terms containing only odd powers of x_3 with degree at least 3 thus after differentiation in both cases x_3 remains in all which vanish evaluating at $x_3 = 0$. ■

Holomorphic Cliffordian polynomials are defined for a multi-index $\alpha \in \mathbb{N}_0^4$ by

$$P^\alpha(x) = \sum_{\sigma \in \sigma(\alpha)} e_{\sigma_1} x e_{\sigma_2} x \cdots e_{\sigma_{|\alpha|-1}} x e_{\sigma_{|\alpha|}}, \tag{24}$$

where the sum is over all distinguishable permutations $\sigma = \left(\sigma_1, \sigma_2, \ldots, \sigma_{|\alpha|}\right)$ of $|\alpha|$ elements of the set $\{0, 1, 2, 3\}$ with α_k elements equal to k $(k = 0, \ldots, 3)$. The polynomial $P^\alpha(x)$ is a homogenous polynomial of degree $|\alpha| - 1$. The polynomials P^α satisfy the same recursion formulas as the polynomials E_n^α. Furthermore the differentiation rules are the same as in the case of the polynomials E_n^α (see [7],[11]).

5 Linear independence of the homogenous polynomials

In this chapter we state results concerning the linear independence of various classes of polynomials. The main result is Theorem 26.

The monogenic polynomials F_m^α are linearly independent over $\mathcal{C}\ell_3$ by Theorem 9. We recall the result

Theorem 19 ([5, Theorem 25]) *The basis of the right $\mathbb{H}$-module generated by the homogeneous polynomial H-solutions of degree m is*

$$\left\{L_m^\alpha \mid \alpha \in \mathbb{N}_0^2,\ |\alpha| \le m\right\} \cup \left\{T_m^\beta \mid \beta \in \mathbb{N}_0^3,\ |\beta| = m-2\right\}.$$

By Theorem 19 the polynomials L_m^α and T_m^β are right linearly independent over $\mathbb{H}$, but even stronger result holds. It is convenient to denote

$$\begin{aligned}
\Lambda_0^m &= \left\{\alpha \in \mathbb{N}_0^2 \mid |\alpha| \le m\right\}, \\
\Lambda_1^m &= \left\{\alpha \in \mathbb{N}_0^3 \mid |\alpha| = m-1\right\}, \\
\Lambda_2^m &= \left\{\alpha \in \mathbb{N}_0^3 \mid |\alpha| = m-2\right\}, \\
\Lambda_3^m &= \left\{\alpha \in \mathbb{N}_0^2 \mid |\alpha| = m\right\}.
\end{aligned}$$

Theorem 20 *The polynomials L_m^α, with $\alpha \in \Lambda_0^m$ and T_m^β, with $\beta \in \Lambda_2^m$ are right linearly independent over $\mathcal{C}\ell_3$ and form a basis of the right Cl_3-module generated by the homogeneous polynomial H-solutions of degree m.*

Proof. It suffices to show that the polynomials are right linearly independent over $\mathcal{C}\ell_3$. Assume that

$$\sum_{\alpha\in\Lambda_0^m} L_m^\alpha a(\alpha) + \sum_{\beta\in\Lambda_2^m} T_m^\beta b(\alpha) = 0$$

for some coefficients $a(\alpha), b(\beta) \in \mathcal{C}\ell_3$. Decomposing the coefficients into their P- and Q-parts yields

$$\sum L_m^\alpha Pa(\alpha) + \sum L_m^\alpha Qa(\alpha) e_3 + \sum T_m^\beta Pb(\beta) + \sum T_m^\beta Qb(\beta) e_3 = 0. \quad (25)$$

Since $\sum L_m^\alpha Pa(\alpha) + \sum T_m^\beta Pb(\beta)$ is a hypermonogenic function, also

$$\sum L_m^\alpha Qa(\alpha) e_3 + \sum T_m^\beta Qb(\beta) e_3$$

must be hypermonogenic and by Lemma 5

$$\sum L_m^\alpha Qa(\alpha) + \sum T_m^\beta Qb(\beta) = 0.$$

By Theorem 19 we know that the polynomials L_m^α and T_m^β are right linearly independent over $\mathbb{H}$ and therefore $Qa(\alpha) = 0$ and $Qb(\beta) = 0$ for all $\alpha \in \Lambda_0^m$ and $\beta \in \Lambda_2^m$. Substituting the zero coefficients into (25) yields $Pa(\alpha) = 0 = Pb(\beta)$. Thus $a(\alpha) = 0 = b(\beta)$ for all $\alpha \in \Lambda_0^m$ and $\beta \in \Lambda_2^m$ and the polynomials L_m^α and T_m^β are right linearly independent over $\mathcal{C}\ell_3$. ■

From the previous proof we obtain a more general result.

Proposition 21 *Any set of hypermonogenic polynomials that is right linearly independent over* $\mathbb{H}$ *is right linearly independent over* $\mathcal{C}\ell_3$ *as well.*

Laville and Ramadanoff have proved in [7] that the holomorphic Cliffordian polynomials P^α in (24) form a generating set for the homogeneous polynomials satisfying the equation $D\Delta f = 0$. However, the polynomials P^α are not linearly independent (see for example [11, Example 3.4.]). The dimension of the space of homogeneous holomorphic Cliffordian polynomials of degree m is

$$1 + 3\frac{m(m+1)}{2} \quad (26)$$

as stated in [8, Theorem 14]. We know by Lemma 5 that the polynomials $L_m^\alpha e_3$ and $T_m^\beta e_3$ are not hypermonogenic, but they are holomorphic Cliffordian. In addition they are linearly independent over $\mathcal{C}\ell_3$, when $\alpha \in \Lambda_0^m$ and $\beta \in \Lambda_2^m$. We try to extend the set

$$\{L_m^\alpha \mid \alpha \in \Lambda_0^m\} \cup \{T_m^\beta \mid \beta \in \Lambda_2^m\}$$

in order to find a basis for the homogeneous polynomials of degree m satisfying the equation $D\Delta f = 0$.

For technical reasons we define a set of new polynomials S_m^α.

Lemma 22 *For every $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}_0^3$ with $|\alpha| = m - 1$ there exists a homogeneous polynomial H-solution of degree m denoted by S_m^α satisfying the properties*

$$\frac{\partial Q S_m^\alpha}{\partial x_3}(\cdot, 0) = x^\alpha \text{ and } \frac{\partial^2 Q S_m^\alpha}{\partial x_3^2}(\cdot, 0) = 0.$$

Proof. If S_m^α is hypermonogenic, the function

$$f = \frac{Q S_m^\alpha}{x_3}$$

is harmonic. Note that the harmonicity of the function f implies that f is determined by the values $f(\cdot, 0)$ and $\frac{\partial f}{\partial x_3}(\cdot, 0)$ (see for example [11, Lemma 1.3]). Since $Q S_m^\alpha$ is odd and thus f even, we only need the values of f in the plane $x_3 = 0$. Since f is a polynomial f is smooth enough for us to obtain

$$f(\cdot, 0) = \lim_{x_3 \to 0} \frac{Q S_m^\alpha}{x_3}(x) = \frac{\partial Q S_m^\alpha}{\partial x_3}(\cdot, 0) = x^\alpha,$$

we get

$$\frac{Q S_m^\alpha}{x_3} = x^\alpha + \sum_{k=0}^{\left[\frac{m-1}{2}\right]} \frac{(-1)^k \Delta_3^k x^\alpha x_3^{2k}}{(2k)!},$$

where $\Delta_3 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian in $\mathbb{R}^3$. By [3, Corollary 16] the polynomial S_m^α is determined by the Q-part

$$Q S_m^\alpha = x^\alpha x_3 + x_3 \sum_{k=0}^{\left[\frac{m-1}{2}\right]} \frac{(-1)^k \Delta_3^k x^\alpha x_3^{2k}}{(2k)!}.$$

■

Theorem 23 *The polynomials S_m^α, T_m^β and F_m^γ are right linearly independent over $\mathcal{C}\ell_3$ for $\alpha \in \Lambda_1^m$, $\beta \in \Lambda_2^m$ and $\gamma \in \Lambda_3^m$ and therefore form a basis of the right Cl_3-module generated by the homogeneous polynomial H-solutions of degree m.*

Proof. By Prosition 21 it suffices to prove the linear independency over $\mathbb{H}$. To that end assume that

$$\sum_{\alpha\in\Lambda_1^m} S_m^{\alpha} a(\alpha) + \sum_{\beta\in\Lambda_2^m} T_m^{\beta} b(\beta) + \sum_{\gamma\in\Lambda_3^m} F_m^{\gamma} c(\gamma) = 0 \tag{27}$$

for some coefficients in $\mathbb{H}$. Differentiating (27) with respect to x_3 and evaluating at $x_3 = 0$ yield $\sum_{\alpha\in\Lambda_1^m} x^{\alpha} a(\alpha) = 0$ and thus $\alpha(\alpha) = 0$ for all $\alpha \in \Lambda_1^m$. Evaluating at $x_3 = 0$ yields $b(\gamma) = 0$ for all $\gamma \in \Lambda_3^m$. Finally, since the polynomials T_m^{β} with $\beta \in \Lambda_2^m$ are linearly independent we obtain $b(\beta) = 0$ for all $\beta \in \Lambda_2^m$. We are considering

$$\binom{m+1}{2} + \binom{m}{2} + \binom{m+1}{1} = \binom{m+2}{2} + \binom{m}{2}$$

polynomials. Since by [3, Theorem 43] the dimension of the right Cl_3-module generated by the homogeneous polynomial H-solutions of degree m is

$$\binom{m+2}{2} + \binom{m}{2}$$

we have a basis. ■

Proposition 24 *The polynomial L_m^{α} has a presentation*

$$L_m^{\alpha} = \sum_{\beta\in\Lambda_1^m} S_m^{\beta} a(\beta) + \sum_{\gamma\in\Lambda_3^m} F_m^{\gamma} b(\gamma),$$

where $a(\beta) \in \mathbb{H}$ and $b(\gamma) \in \mathbb{H}$.

Proof. The polynomial L_m^{α} has a presentation in terms of the polynomials S_m^{δ}, T_m^{β} and F_m^{γ} for $\delta \in \Lambda_1^m$, $\beta \in \Lambda_2^m$ and $\gamma \in \Lambda_3^m$ by Theorem 23. Since by Lemma 15 $\frac{\partial^2 QL_m^{\alpha}}{\partial x_3^2}(\cdot, 0) = 0$ and by Lemma 17 $\frac{\partial^2 QT_m^{\beta}}{\partial x_3^2}(\cdot, 0) = 2x^{\beta} \neq 0$ the presentation does not contain polynomials T_m^{β}. ■

Proposition 25 *Let R_m be a homogenous polynomial H-solution of degree $m \in \mathbb{N}$. If*

$$\frac{\partial QR_m}{\partial x_3}(\cdot, 0) = 0 \text{ and } \frac{\partial^2 QR_m}{\partial x_3^2}(\cdot, 0) = 0$$

then R_m is a homogenous monogenic polynomial H-solution independent of x_3.

Proof. We can present R_m in the basis S_m^δ, T_m^β and F_m^γ for $\delta \in \Lambda_1^m$, $\beta \in \Lambda_2^m$ and $\gamma \in \Lambda_3^m$. Since $\frac{\partial^2 T_m^\beta}{\partial x_3^2}(\cdot,0) = 2x^\beta \neq 0$ by Lemma 17 and $\frac{\partial S_m^\alpha}{\partial x_3}(\cdot,0) = x^\alpha \neq 0$ by Lemma 22 the presentation can contain only polynomials F_m^γ and thus R_m is a homogenous monogenic polynomial H-solution independent of x_3. ■

Theorem 26 *Let $\alpha \in \Lambda_1^m$, $\beta \in \Lambda_2^m$, $\gamma \in \Lambda_3^m$ and $\delta \in \Lambda_2^{m+1}$. Then the polynomials S_m^α, T_m^β, F_m^γ and $\frac{\partial T_{m+1}^\delta}{\partial x_3}$ are right linearly independent over $\mathcal{C}\ell_3$ and form a basis for the right module of homogeneous holomorphic Cliffordian polynomials of degree m.*

Proof. Suppose that

$$\sum S^\alpha a(\alpha) + \sum T_m^\beta b(\beta) + \sum F^\gamma c(\gamma) + \sum \frac{\partial T_{m+1}^\delta}{\partial x_3} d(\delta) = 0$$

for some coefficients in $\mathcal{C}\ell_3$. Decomposing the coefficients yields

$$\begin{aligned} 0 = {} & \sum S^\alpha Pa(\alpha) + \sum T_m^\beta Pb(\beta) + \sum F^\gamma Pc(\gamma) \\ & + \sum S^\alpha Qa(\alpha) e_3 + \sum T_m^\beta Qb(\beta) e_3 + \sum F^\gamma Qc(\gamma) e_3 + \sum \frac{\partial T_{m+1}^\delta}{\partial x_3} d(\delta), \end{aligned} \tag{28}$$

where $\sum S^\alpha Pa(\alpha) + \sum T_m^\beta Pb(\beta) + \sum F^\gamma Pc(\gamma)$ is a hypermonogenic function. Hence

$$\begin{aligned} h_1 = {} & \sum S^\alpha Qa(\alpha) e_3 + \sum T_m^\beta Qb(\beta) e_3 + \sum F^\gamma Qc(\gamma) e_3 \\ & + \sum \frac{\partial T_{m+1}^\delta}{\partial x_3} Pd(\delta) + \sum \frac{\partial T_{m+1}^\delta}{\partial x_3} Qd(\delta) e_3 \end{aligned} \tag{29}$$

for some hypermonogenic function h_1. Since by Lemma 8 the function

$$\sum \frac{\partial T_{m+1}^\delta Qd(\delta)}{\partial x_3} e_3 + \sum \frac{2Q\left(T_{m+1}^\delta Qd(\delta)\right)}{x_3} = h_2$$

is hypermonogenic, the function

$$\begin{aligned} h_1 - h_2 = {} & \sum S^\alpha Qa(\alpha) e_3 + \sum T_m^\beta Qb(\beta) e_3 + \sum F^\gamma Qc(\gamma) e_3 \\ & + \sum \frac{\partial T_{m+1}^\delta}{\partial x_3} Pd(\delta) - \sum \frac{2QT_{m+1}^\delta}{x_3} Q'd(\beta) \end{aligned}$$

is hypermonogenic. Applying (8) to

$$Q\left(h_1-h_2\right)=\sum PS^{\alpha}Qa\left(\alpha\right)+\sum PT_m^{\beta}Qb\left(\beta\right)$$
$$+\sum PF^{\gamma}Qc\left(\gamma\right)+\sum\frac{\partial QT_{m+1}^{\delta}}{\partial x_3}P'd\left(\beta\right)$$

we infer

$$0=\sum PS^{\alpha}\left(\cdot,0\right)Qa\left(\alpha\right)+\sum PT_m^{\beta}\left(\cdot,0\right)Qb\left(\beta\right)$$
$$+\sum PF^{\gamma}\left(\cdot,0\right)Qc\left(\gamma\right)+\sum\frac{\partial QT_{m+1}^{\delta}}{\partial x_3}\left(\cdot,0\right)P'c\left(\beta\right).$$

Using (22) and (23) we obtain

$$\sum PS^{\alpha}\left(\cdot,0\right)Qa\left(\alpha\right)+\sum PF^{\gamma}\left(\cdot,0\right)Qc\left(\gamma\right)=0.$$

and thus

$$\sum D_2PS^{\alpha}\left(\cdot,0\right)Qa\left(\alpha\right)+\sum D_2PF^{\gamma}\left(\cdot,0\right)Qc\left(\gamma\right)=0. \tag{30}$$

For hypermonogenic functions S^{α} and F^{γ} it holds by (9)

$$D_2PS^{\alpha}\left(\cdot,0\right)=-\frac{\partial Q'S^{\alpha}}{\partial x_3}\left(\cdot,0\right) \tag{31}$$

and

$$D_2PF^{\gamma}\left(\cdot,0\right)=-\frac{\partial Q'F^{\gamma}}{\partial x_3}\left(\cdot,0\right)=0, \tag{32}$$

where the last equality holds since F^{γ} does not depend on x_3. Thus the equation (30) becomes

$$\sum\frac{\partial Q'S^{\alpha}}{\partial x_3}\left(\cdot,0\right)Qa\left(\alpha\right)=\sum x^{\alpha}Qa\left(\alpha\right)=0$$

and we obtain

$$Qa\left(\alpha\right)=0 \tag{33}$$

for all α. Substituting (33) to (28) we infer

$$0=\sum S^{\alpha}Pa\left(\alpha\right)+\sum T_m^{\beta}Pb\left(\beta\right)+\sum T_m^{\beta}Qb\left(\beta\right)e_3+\sum F^{\gamma}Pc\left(\gamma\right)$$
$$+\sum F^{\gamma}Qc\left(\gamma\right)e_3+\sum\frac{\partial T_{m+1}^{\delta}}{\partial x_3}Pd\left(\delta\right)+\sum\frac{\partial T_{m+1}^{\delta}}{\partial x_3}Qd\left(\delta\right)e_3.$$

Separating the P-part we find out the equation

$$0=\sum PS^{\alpha}Pa(\alpha)+\sum PT_{m}^{\beta}Pb(\beta)-\sum QT_{m}^{\beta}Q'b(\beta)+\sum PF^{\gamma}Pc(\gamma)$$
$$-\sum QF^{\gamma}Q'c(\gamma)+\sum\frac{\partial PT_{m+1}^{\delta}}{\partial x_{3}}Pd(\delta)-\sum\frac{\partial QT_{m+1}^{\delta}}{\partial x_{3}}Q'd(\delta).$$

Hence we have

$$0=\sum\frac{\partial PS^{\alpha}}{\partial x_{3}}Pa(\alpha)+\sum\frac{\partial PT_{m}^{\beta}}{\partial x_{3}}Pb(\beta)-\sum\frac{\partial QT_{m}^{\beta}}{\partial x_{3}}Q'b(\beta)\tag{34}$$
$$+\sum\frac{\partial^{2}PT_{m+1}^{\delta}}{\partial x_{3}^{2}}Pd(\delta)-\sum\frac{\partial^{2}QT_{m+1}^{\delta}}{\partial x_{3}^{2}}Q'd(\delta),$$

since F^{γ} is independent of x_3. By (20) PT_{k}^{α} has only terms with odd powers of x_3 with degree at least 3 and therefore

$$\frac{\partial^{2}PT_{m+1}^{\delta}}{\partial x_{3}^{2}}(\cdot,0)=0\text{ and }\frac{\partial PT_{m}^{\beta}}{\partial x_{3}}(\cdot,0)=0.\tag{35}$$

Furthermore, by (10) and (22) we have

$$\frac{\partial PS^{\alpha}}{\partial x_{3}}(\cdot,0)=0\text{ and }\frac{\partial QT_{m}^{\beta}}{\partial x_{3}}(\cdot,0)=0.$$

The equation (34) evaluated at $x_3=0$ reduces to

$$\sum\frac{\partial Q^{2}T_{m+1}^{\delta}}{\partial x_{3}^{2}}(\cdot,0)\,Q'd(\delta)=\sum 2x^{\delta}Q'd(\delta)=0$$

and therefore

$$Qd(\delta)=0$$

for all δ.

Separating the Q-part in (28) and setting $Qd(\delta)=0$ we obtain

$$0=\sum QS^{\alpha}P'a(\alpha)+\sum QT_{m}^{\beta}P'b(\beta)+\sum QF^{\gamma}P'c(\gamma)$$
$$+\sum PT_{m}^{\beta}Qb(\beta)+\sum PF^{\gamma}Qc(\gamma)+\sum\frac{\partial QT_{m+1}^{\delta}}{\partial x_{3}}P'd(\delta).$$

Differentiating twice with respect to the variable x_3 and evaluating at $x_3=0$ we infer using Lemma 15, (35) and (22)

$$0=\sum\frac{\partial^{2}QS^{\alpha}}{\partial x_{3}^{2}}(\cdot,0)\,P'a(\alpha)+\sum\frac{\partial^{2}QT_{m}^{\beta}}{\partial x_{3}^{2}}(\cdot,0)\,P'b(\beta)$$
$$+\sum\frac{\partial^{2}PT_{m}^{\beta}}{\partial x_{3}^{2}}(\cdot,0)\,Qb(\beta)+\sum\frac{\partial^{3}QT_{m+1}^{\delta}}{\partial x_{3}^{3}}(\cdot,0)\,P'd(\delta)$$
$$=\sum\frac{\partial^{2}QT_{m}^{\beta}}{\partial x_{3}^{2}}(\cdot,0)\,P'b(\beta)=\sum 2x^{\alpha}P'b(\beta).$$

Thus $Pb(\beta) = 0$ for all β.

Collecting all the previous information to (28) we have

$$\begin{aligned}0 = \sum S^\alpha Pa(\alpha) + \sum F^\gamma Pc(\gamma) + \sum T_m^\beta Qb(\beta)\, e_3 \\ + \sum F^\gamma Qc(\gamma)\, e_3 + \sum \frac{\partial T_{m+1}^\delta}{\partial x_3} Pd(\delta)\,.\end{aligned} \tag{36}$$

This means that the function

$$\begin{aligned}h = -\sum S^\alpha Pa(\alpha) - \sum F^\gamma Pc(\gamma) \\ = \sum T_m^\beta Qb(\beta)\, e_3 + \sum F^\gamma Qc(\gamma)\, e_3 + \sum \frac{\partial T_{m+1}^\delta}{\partial x_3} Pd(\delta)\end{aligned}$$

is hypermonogenic and the function

$$Ph = -\sum QT_m^\beta Q'b(\beta) - \sum QF^\gamma Q'c(\gamma) + \sum \frac{\partial PT_{m+1}^\delta}{\partial x_3} Pd(\delta)$$

is hyperbolic harmonic. Since QT_m^β, QF^γ and $\frac{\partial PT_{m+1}^\delta}{\partial x_3}$ are eigenfunctions of the Laplace–Beltrami operator, applying $x_3^2 \triangle_H Ph = 0$ we obtain

$$\begin{aligned}0 = -\sum x_3^2 \triangle_H \left(QT_m^\beta\right) Q'b(\beta) - \sum x_3^2 \triangle_H \left(QF^\gamma\right) Q'c(\gamma) \\ + \sum x_3^2 \triangle_H \left(\frac{\partial PT_{m+1}^\delta}{\partial x_3}\right) Pd(\delta) \\ = -\sum -2QT_m^\beta Qb(\beta)\, e_3 - \sum -2QF^\gamma Q'c(\gamma) + \sum -2\frac{\partial PT_{m+1}^\delta}{\partial x_3} Pd(\delta) \\ = -2Ph.\end{aligned}$$

Hence $Ph = 0$, which by (9) implies

$$-\frac{\partial Q'h}{\partial x_3}(\cdot, 0) = \sum D_2 Ph(\cdot, 0) = 0.$$

On the other hand

$$\begin{aligned}-\frac{\partial Q'h}{\partial x_3}(\cdot, 0) = \sum \frac{\partial QS^\alpha}{\partial x_3}(\cdot, 0)\, Pa(\alpha) + \sum \frac{\partial QF^\gamma}{\partial x_3}(\cdot, 0)\, Pc(\gamma) \\ = \sum \frac{\partial QS^\alpha}{\partial x_3}(\cdot, 0)\, Pa(\alpha) = \sum x^\alpha Pa(\alpha) = 0\end{aligned}$$

and we deduce

$$Pa(\alpha) = 0. \tag{37}$$

After (37) the sum (36) reduces to

$$\sum F^{\gamma} Pc(\gamma) + \sum T_m^{\beta} Qb(\beta)\, e_3 + \sum F^{\gamma} Qc(\gamma)\, e_3 + \sum \frac{\partial T_{m+1}^{\delta}}{\partial x_3} Pd(\delta) = 0. \tag{38}$$

Differentiating (38) with respect to the variable x_3 and evaluating at $x_3 = 0$ we infer

$$\begin{aligned} 0 &= \sum \frac{\partial P T_m^{\beta}}{\partial x_3}(\cdot, 0)\, Qb(\beta)\, e_3 - \sum \frac{\partial Q T_m^{\beta}}{\partial x_3}(\cdot, 0)\, Q' b(\beta) \\ &+ \sum \frac{\partial^2 P T_{m+1}^{\delta}}{\partial x_3^2}(\cdot, 0)\, Pd(\delta) - \sum \frac{\partial^2 Q T_{m+1}^{\delta}}{\partial x_3^2}(\cdot, 0)\, P'd(\delta) \\ &= -\sum \frac{\partial^2 Q T_{m+1}^{\delta}}{\partial x_3^2}(\cdot, 0)\, P'd(\delta) = -\sum 2x^{\delta} P'd(\delta), \end{aligned}$$

and $Pd(\delta) = 0$ for all δ. Since the polynomials F^{γ} are linearly independent over $\mathcal{C}\ell_3$ and do not depend on x_3 when $\gamma \in \Lambda_3^m$ evaluating at $x_3 = 0$ we infer $c(\gamma) = 0$ for all $\gamma \in \Lambda_3^m$. Since the polynomials T_m^{β} are linearly independent over $\mathcal{C}\ell_3$, the only coefficients remaining in (38) must vanish and therefore $Qb(\beta) = 0$ for all β. The number of polynomials

$$\binom{m+2}{2} + \binom{m}{2} + \binom{m+1}{2}$$

is the same as the dimension in 26 completing the proof. ■

Our main result, Theorem 26, can be written in an equivalent form using Theorem 24.

Theorem 27 *Let $\alpha \in \Lambda_0^m$, $\beta \in \Lambda_2^m$, and $\gamma \in \Lambda_2^{m+1}$. The polynomials L_m^{α}, T_m^{β} and $\frac{\partial T_{m+1}^{\gamma}}{\partial x_3}$ are right linearly independent over $\mathcal{C}\ell_3$ and form a basis for the right module of homogeneous holomorphic Cliffordian polynomials of degree m.*

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