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ON MASSLESS FIELD EQUATION IN HIGHER DIMENSIONS

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Abstract. *In the paper, we shall discuss massless field equations in dimensions* 4*, their generalization to higher dimensions, and properties of their polynomial solutions. We shall relate massless field equation to a one parameter family of Bernstein-Gel'fand-Gel'fand resolutions and we shall use their property in a discussion of polynomial solutions of massless field equations. In dimension* 4*, we use spinorial language developed and used often in general relativity. In higher dimensions, we shall need more tools from representation theory of orthogonal group (tensor product decompositions) developed recently by P. Littlemann.*

Classical Clifford analysis is studying properties of solutions of the Dirac operator ∂ acting on functions defined on $\mathbb{R}^m$ with values in the corresponding Clifford algebra $\mathbb{R}_m$. But the Dirac operator D (in mathematics, as well as in physics) should act on spinor fields. It was shown (see [9]) that the corresponding equation for monogenic functions with values in Clifford algebra decomposes into many copies of the equation for spinor valued fields. This is due to the fact that the (complexified) Clifford algebra $\mathbb{C}_m$ can be written (in even dimensions) as the tensor product $\mathbb{S} \otimes \mathbb{S}^*$, where $\mathbb{S}$ denotes the basic irreducible representation of $\mathbb{C}_m$. A similar statement is true also in odd dimensions. Hence the classical Dirac operator ∂ in Clifford analysis should be interpreted as a twisted Dirac operator $D_{\mathbb{V}}$ acting on the product $\mathbb{S} \otimes \mathbb{V}$, where $\mathbb{V} = \mathbb{S}^*$. If we consider $\mathbb{V}$ with the trivial $Spin(m)$ action (the so called L representation based on the left multiplication of values of fields with an element of the spin group), we are back in the traditional Clifford analysis. If we, however, consider the same space $\mathbb{V}$ with the natural action of $Spin(m)$ on the dual of $\mathbb{S}$, (the so called H representation based on the both sides multiplication of values of fields with an element of the spin group), we are identifying the Clifford algebra with the exterior algebra (as $Spin(m)$-modules) and we get identification of $D_{\mathbb{V}}$ with $d + d^*$. For more details on the second possibility, see [3].

It is also possible to investigate interesting operators and equations obtained by restricting values of $D_{\mathbb{V}}$ to invariant subspaces (homogeneous components or their sums) under the H action. In such a way, we get Hodge systems, Mosil-Theodorescu systems and their generalization investigated, e.g., in [11, 1, 5, 6, 4, 2]. In particular, it was possible to treat (higher dimensional analogues of) the Maxwell operator as a system arising from $D_{\mathbb{V}}$ by restricting values of fields to n-forms with n equal to the half of dimension m.

The presented paper has two motivation. Firstly, we want to extend a study of twisted Dirac operators to the case of more complicated $Spin(m)$-modules $\mathbb{V}$. Secondly, we want to pay more attention to higher dimensional versions of massless field equations for higher spin. Massless field equation (on Minkowski space) form a key series of linear PDE's for particle physics. This is a series of equations for fields with spin $\frac{k}{2}$, where k is a non-negative integer. The spin 0 case is just the wave equation, the spin $\frac{1}{2}$ case is the Dirac equation, the spin 1 case is the Maxwell equation, the spin $\frac{3}{2}$ is the equation for the Rarita-Schwinger fields, and the spin 2 case describes equation for linearized gravity.

Higher dimensional analogues of massless field equations were introduced and studied in the framework of Clifford analysis in [21]. Here we return back to study of properties of solutions of massless field equations. In particular, we are going to discuss properties of polynomial solutions of massless field equations for a given homogeneity, i.e., we are looking for analogues of spherical monogenics in higher spin case.

We start with the most important case of dimension 4 (Sect.2). Due to the fact that the tensor algebra in dimension 4 is contained in the spinor algebra, it is efficient to use spinor language and notation (see, e.g., [16]). In higher dimensions, the spinor language is no more appropriate and we have to use various tools from representation theory of orthogonal group. In Sect. 3, we recall the massless field equations in higher dimensions and review their basic properties. In particular, we show that the massless field operator is given by a twisted Dirac operator restricted to functions with values in a suitable invariant subspace. In Sect.4, we shall study homogeneous solutions of the massless field equation. We shall describe the biggest irreducible component of the kernel in a given homogeneity and we conjecture that it is the only component in the kernel. In the paper, we shall restrict for simplicity to the case of even dimensions but similar results holds also in odd dimensions. We shall return back to odd dimensions elsewhere.

1 MASSLESS FIELDS IN DIMENSION 4.

1.1 Quaternionic and Clifford analysis in dimension 4.

Massless fields of spin $\frac{k}{2}$ on Minkowski space M^4 are basic examples of fields considered in theoretical physics. The case $k = 0$ (the wave equation for functions), $k = 1$ (the Dirac equation for fermions), $k = 2$ (the Maxwell equations for electromagnetic fields), $k = 3$ (the Rarita-Schwinger equations) and $k = 4$ (the equations for linearized gravity) are the most common equations in particle physics. We shall describe the whole sequence (for any k) below. We shall do that, however, in Euclidean version. It means that we shall consider these field on the Euclidean space $\mathbb{R}^4$, instead of the Minkowski space M^4.

Quaternionic analysis in dimension 4 is a well developed topic with a long history and many applications (see, e.g. [12, 13]. It describes properties of solutions of the Fueter equation for quaternionic valued functions on $\mathbb{R}^4$. It is very closely related to Clifford analysis, which investigates properties of solutions of the Dirac equation for functions with values in the Clifford algebra $\mathbb{C}_4$. To describe the relation between both, we have to consider on Fueter side functions with values in the algebra of complex quaternions $\mathbb{CH}$. (Complex) dimension of $\mathbb{C}_4$ is 16 but we can write is a sum of four spinor spaces, each of them having (complex) dimension 4. The Dirac equation for spinor valued functions is then the same as the Fueter equation for functions with values in $\mathbb{CH}$.

1.2 Spinors in dimension 4.

To write the Dirac (Fueter) equation explicitely, we shall use (nowadays standard and often used) spinor notation. There are two non-isomorphic irreducible basic $Spin(4)$ modules, we shall denote them by $\mathbb{S}_A$ and $\mathbb{S}_{A'}$. As vector spaces, they can be both identified with $\mathbb{C}_2$. The basic fact is that $\mathbb{S}_{AA'} = \mathbb{S}_A \otimes \mathbb{S}_{A'} \simeq \mathbb{C}^4$ as $Spin(4)$-modules. It gives an identification of vectors in (complexified) Euclidean space with the product of both spinor modules. As a consequence, the whole tensor algebra (in dimension 4) can be identified with the tensor algebra.

Any irreducible $Spin(4)$-module can be written as a symmetric power $\odot^k(\mathbb{S}_A)$ of $\mathbb{S}_A$, resp. as a symmetric power $\odot^k(\mathbb{S}'_A)$ of $\mathbb{S}'_A$. Elements in these symmetric powers will be denoted by $\varphi_{A\ldots E}$, resp. $\varphi_{A'\ldots E'}$. Indices are used here as abstract indices (the Penrose abstract index notation, see [16] for more details), but it can as well be used as ordinary (basis dependent) indices $A = 0, 1; A' = 0', 1'$. Rising and lowering spinor indices is possible using elements $\epsilon_{AB} \in \Lambda^2(\mathbb{S}_A)$ and $\epsilon_{A'B'} \in \Lambda^2(\mathbb{S}'_A)$ (again see [16]) for more details). We shall use also the standard notation - round brackets $(\ldots)$ around indices for symmetrisation and the square brackets $[\ldots]$ for anti-symmetrization. A (complex) vector $x_a \in \mathbb{C}^4$ can be (and will be) identified with $x_{AA'} \in \mathbb{S}_A \otimes \mathbb{S}_{A'}$. Similar identification will be made for temsors with more indices. Euclidean space $\mathbb{R}^4$ can be identified with a (real) space of matrices of the form

$$x_a = (x_0, x_1, x_2, x_3);\ x_{AA'} = \begin{pmatrix} x_{00'} & x_{10'} \\ x_{01'} & x_{11'} \end{pmatrix} = \begin{pmatrix} x_0 + ix_3 & ix_1 + x_2 \\ ix_1 - x_2 & x_0 - ix_3 \end{pmatrix}.$$

The Euclidean norm corresponds to the determinant of the corresponding matrix. Hence for derivatives, we get

$$\nabla_a = (\partial_0, \partial_1, \partial_2, \partial_3);\ \nabla_{AA'} = \begin{pmatrix} \partial_{00'} & \partial_{10'} \\ \partial_{01'} & \partial_{11'} \end{pmatrix} = \begin{pmatrix} \partial_0 + i\partial_3 & i\partial_1 + \partial_2 \\ i\partial_1 - \partial_2 & \partial_0 - i\partial_3 \end{pmatrix},$$

where $\partial_0 = \frac{\partial}{\partial x_0}$, etc. We shall rise and lower tensor indices by the metric g_{ab} and spinor indices by ϵ_{AB}, resp. $\epsilon_{A'B'}$. Contraction $\varphi^A \psi_A$ means $\epsilon_{AB}\varphi^A \psi^B$. The Einstein summation convention will be always used (including the case of abstract indices, as above).

The Spin group in dimension 4 is isomorphic to the direct sum $SO(3) \oplus SO(3)$. Hence any decomposition of a tensor product of (complex) irreducible $Spin(4)$-modules reduces to a well known decomposition of irreducible $SL(2)$-modules. We shall use such decomposition below without further comments.

1.3 Massless field equations.

The Dirac equation for a spinor field φ_A have then a form

$$\nabla^A_{A'}\varphi_A = 0.$$

Higher spin massless field equations are written down for fields with values in a k-th symmetric power $\odot^k(\mathbb{S}_A)$ of the basic spinor representation and they have the form

$$\nabla^A_{A'}\varphi_{A\ldots E} = 0,$$

where $\varphi_{A\ldots E}$ is a field with k indices, which is symmetric in these indices. Dimension of $\odot^k(\mathbb{S}_A)$ is equal to $k+1$, the field $\varphi_{A\ldots E}$ has $k+1$ components. A similar equation is used for spinor with primed indices.

Properties of massless field equations were carefully studied, many properties of them are summarized in [17]. We shall now describe their relation with the (twisted) Dirac equation and Clifford analysis.

1.4 Spinor version of Clifford analysis.

Let $\mathbb{S} = \mathbb{S}_A \oplus \mathbb{S}_{A'}$ be the full spinor space written as the sum of two half-spinor representations. It can be embedded in many ways into the Clifford algebra $\mathbb{C}_4$. In general, the Clifford algebra $\mathbb{C}_n$ can be (in even dimensions) identified with the space $\mathrm{End}(\mathbb{S}) \simeq \mathbb{S} \otimes \mathbb{S}^*$ of endomorphisms of $\mathbb{S}$. If we consider $\mathbb{S}^*$ as trivial $Spin(4)$-module, $\mathbb{C}_4 \simeq \mathbb{S} \otimes \mathbb{C}^4$ decomposes into a direct sum of 4 spinor subspaces, isomorphic to $\mathbb{S}$. In the language of Clifford analysis, it means that we consider the Clifford algebra with the L-action of the spin group (given by left mupltiplication). The Dirac equation for functions with values in $\mathbb{C}_4$ hence decomposes in this case into a sum of 4 copies of the Dirac equation (the spin $1/2$ case) for spinor fields with values in $\mathbb{S}$ (which, in turn, can be decomposed into a sum of two Weyl equations for half-spinor fields $\nabla^A_{A'}\varphi_A = 0$ and $\nabla^{A'}_A\varphi'_A = 0$).

1.5 Differential form version of Clifford analysis.

As was described in detail in [3], we can identify the Clifford algebra with the exterior algebra (as vector spaces) and the Dirac operator D and the Hodge operator $d + d^*$ correspond to each other under this identification. In spinor language, it can be described as follows. In the standard language of Clifford analysis ([9]) the Clifford algebra $\mathbb{C}_4$ can be considered with the H-action (given by multiplication by elements of the spin group from both sides). In language of $Spin(4)$ representations, it can be expressed as the isomorphism

$$\mathbb{C}_4 \simeq \mathbb{S} \otimes \mathbb{S},$$

where now both factors are considered as basic $Spin(4)$ representation. Now again the Clifford algebra $\mathbb{C}_4$ decomposes into many irreducible pieces but in a different way. It can be shortly summarized by saying that we have identified it with the exterior algebra $\Lambda^*(\mathbb{C}^4)$ as $Spin(4)$-modules. The exterior algebra decomposes into a sum of homogeneous forms $\Lambda^*(\mathbb{C}^4) = \sum_0^4 \Lambda^j(\mathbb{C}^4)$ and, in the middle dimension we have further splitting $\Lambda^2(\mathbb{C}^4) = \Lambda^2(\mathbb{C}^4)_+ \oplus \Lambda^2_-(\mathbb{C}^4)$ into a sum of self-dual (SD) and anti-self-dual (ASD) two forms. All these spaces fits then together into the de Rham sequence with its de Rham operators d and their Hodge duals d^*.

In spinor language, individual pieces can be described as follows. Using the decomposition $\mathbb{S} = \mathbb{S}_A \oplus \mathbb{S}_{A'}$, we can decompose the full product as

$$\mathbb{S} \otimes \mathbb{S} \simeq (\mathbb{S}_A \otimes \mathbb{S}_A) \oplus (\mathbb{S}_{A'} \otimes \mathbb{S}_{A'}) \oplus (\mathbb{S}_A \otimes \mathbb{S}_{A'}) \oplus (\mathbb{S}_{A'} \otimes \mathbb{S}_A).$$

The first two summand decomposes further as

$$\mathbb{S}_A \otimes \mathbb{S}_A = \odot^2(\mathbb{S}_A) \oplus \Lambda^2(\mathbb{S}_A);\ \mathbb{S}_{A'} \otimes \mathbb{S}_{A'} = \odot^2(\mathbb{S}_{A'}) \oplus \Lambda^2(\mathbb{S}_{A'}).$$

The maps

$$\varphi_{AB} \in \odot^2(\mathbb{S}_A) \mapsto \phi_{ab} = \varphi_{AB}\epsilon_{A'B'} \in \Lambda^2(\mathbb{C}^4);\ \varphi_{A'B'} \in \odot^2(\mathbb{S}'_A) \mapsto \phi_{ab} = \varphi_{A'B'}\epsilon_{AB} \in \Lambda^2(\mathbb{C}^4)$$

shows that these two pieces in the decomposition can be identified with SD, resp. ASD two forms. The trivial representations $\Lambda^2(\mathbb{S}_{A'})$, resp. $\Lambda^2(\mathbb{S}_A)$, can be identified with 0-forms and 4-forms. As described above, the isomorphism

$$\mathbb{C}_4 \simeq \mathbb{S}_A \otimes \mathbb{S}_{A'} \simeq \mathbb{S}_{A'} \otimes \mathbb{S}_A.$$

can be used to describe 1-forms and 3-forms in spinor language.

For further use, we shall recall the typical shape of the de Rham sequence in dimension 4 with the splitting in the middle dimension. It looks as follows.

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \begin{matrix} \overset{d_+}{\nearrow} & \Lambda^2_+ & \overset{d_+}{\searrow} \\ & \oplus & \\ \underset{d_-}{\searrow} & \Lambda^2_- & \underset{d_-}{\nearrow} \end{matrix} \Lambda^3 \xrightarrow{d} \Lambda^4. \tag{1}$$

The identification of the (Clifford) Dirac operator D with $d + d^*$ is then a typical example of a correspondence between an elliptic complex and an elliptic operator (rolling up the complex). In fact, the Dirac operator D should be interpreted here as a twisted Dirac operator. If $\mathbb{V}$ is any vector space (a bundle with a connection on manifolds), the Dirac operator D acting from the space of spinors $\mathbb{S}$ to itself can be extended to the twisted Dirac operator $d^{\mathbb{V}}$ acting from the product $\mathbb{S} \otimes \mathbb{V}$ to itself. Of course, in the flat case, it just means that we consider many copies of the original Dirac equation. In our case, the twisting vector space is $\mathbb{V} = \mathbb{S}$.

It is well known that the Maxwell equations can be described in terms of two-forms. They can be split into two equations (for SD, resp. ASD two-forms), each one being just spin-1 massles field equation

$$\nabla^A_{A'}\varphi_{AB} = 0;\ \nabla^{A'}_A\varphi_{A'B'} = 0, \tag{2}$$

where φ_{AB}, resp. $\varphi_{A'B'}$ represent (in the way described above) SD, resp. ASD two forms. So the massless field equations for spin-1 fields can be found as particular operators sitting in the middle of the above de Rham sequence. We shall show now that quite similar scheme exists also for higher spin cases.

1.6 Higher spin cases.

The Maxwell equations (2) have a natural generalization for higher spins. The massless field equations for spin $\frac{k}{2}$ looks as follows

$$\nabla^A_{A'}\varphi_{A\dots E}=0;\ \nabla^{A'}_{A}\varphi_{A'\dots E'}=0, \tag{3}$$

where $\varphi_{A\dots E}\in\odot^k(\mathbb{S}_A)$, resp. $\varphi_{A'\dots E'}\in\odot^k(\mathbb{S}_{A'})$.

Let us fix a positive integer $k\geq 2$ and let us consider the twisted Dirac operator $D_{\mathbb{V}}$ with

$$\mathbb{V}=\odot^{k-1}(\mathbb{S}_A)\oplus\odot^{k-1}(\mathbb{S}_{A'}).$$

The twisted Dirac operator $D_{\mathbb{V}}$ is mapping the space $\mathbb{S}\otimes\mathbb{V}$ into itself. We shall decompose this space into irreducible pieces. We get

$$\mathbb{S}_A\otimes\left(\odot^{k-1}(\mathbb{S}_A)\right)\simeq\odot^k(\mathbb{S}_A)\oplus\odot^{k-2}(\mathbb{S}_A)$$

and the corresponding primed version. Moreover, $\mathbb{S}_{A'}\otimes\odot^{k-1}(\mathbb{S}_A)$ and $\mathbb{S}_A\otimes\odot^{k-1}(\mathbb{S}_{A'})$ are irreducible and cannot be decomposed further. Hence the whole product can be written as a sum of 6 irreducible pieces.

If we denote

$$E^0\simeq\odot^{k-2}(\mathbb{S}_{A'}),\ E^1\simeq\mathbb{S}_A\otimes\odot^{k-1}(\mathbb{S}_{A'}),\ E^2_+\simeq\odot^k(\mathbb{S}'_A)$$

$$E^2_-\simeq\odot^k(\mathbb{S}_A),\ E^3\simeq\mathbb{S}_{A'}\otimes\odot^{k-1}(\mathbb{S}_A),\ E^4\simeq\odot^{k-2}(\mathbb{S}_A)$$

we are getting the same scheme of spaces

$$E^0\qquad E^1\qquad \begin{matrix}E^2_+\\ \oplus\\ E^2_-\end{matrix}\qquad E^3\qquad E^4. \tag{4}$$

as above. A natural question is whether there is an exact sequence of operators similar to the de Rham sequence acting among the spaces. In fact, there is such a sequence of conformally invariant operators replacing the de Rham differentials. A substantial difference, however, is that now not all operators are of first order.

The whole scheme looks as follows.

$$E^0\xrightarrow{D(k)^0}E^1\ \begin{matrix}\overset{D(k)^1_+}{\nearrow}\\ \underset{D(k)^1_-}{\searrow}\end{matrix}\ \begin{matrix}E^2_+\\ \oplus\\ E^2_-\end{matrix}\ \begin{matrix}\overset{D(k)^2_+}{\searrow}\\ \underset{D(k)^2_-}{\nearrow}\end{matrix}\ E^3\xrightarrow{D(k)^3}E^4. \tag{5}$$

All these sequences (for any k) are special cases of the so called Bernstein-Gel'fand-Gel'fand (BGG) complexes (for more details, see, e.g., [19]). It is possible to say more about individual operators. Most of them are of the first order, hence easy to describe. Using spinor language, they look as follows (we shall use a shorter notation $D(k)^i\equiv D^i$).

$$D^0(\varphi_{C'\dots E'})=\nabla_{B(B'}\varphi_{C'\dots E')};\ D^1_+(\varphi_{B\,B'\dots E'})=\nabla_{B(A'}\varphi^B_{B'\dots E')} \tag{6}$$

$$D^2_-(\varphi_{A\dots E})=\nabla^A_{A'}\varphi_{A\,B\dots E},\ D^3(\varphi_{A'B\dots E})=\nabla_{A'(A}\varphi^{A'}_{B\dots E)} \tag{7}$$

All fields in the equations are symmetric in indices of the same type, round bracket around indices indicates symmetrization in indices of the same type (for further details concerning spinor notation, see [16]). Note that the operator D^2_- is exactly the operator appearing in the massless field equation for spin $k/2$ introduced above. It apears at exactly same place in the diagram as ASD part of the Maxwell equations in the case of spin 1 field.

Now the other two operators are higher order differential operators, they have order $k-1$. They can be described in the following way.

$$D^1_-(\varphi_{EA'\dots D'}) = \nabla^{A'}_{(A}\dots\nabla^{D'}_{D}\varphi_{E)A'\dots D'} \tag{8}$$

$$D^2_+(\varphi_{A'\dots E'}) = \nabla^{B'}_{B}\dots\nabla^{E'}_{E}\varphi_{A'B'\dots E')} \tag{9}$$

The exactness of the BGG complex described above is a very strong property, which can help substantially in investigation of properties of massless fields.

It is important to realize that the solutions of massless field equations are, in fact, solutions of a suitable (twisted) Dirac equation of a special type. Indeed, we can consider a suitable subcomplex of the corresponding BGG complex defined as follows. Consider the representation $\mathbb{V} = \odot^{k-1}(\mathbb{S}_A)$ and the corresponding twisted Dirac operator $D_{\mathbb{V}}$. It maps the space $(\mathbb{S}_A \oplus \mathbb{S}_{A'}) \otimes \odot^{k-1}(\mathbb{S}_A)$ to itself. As was described above, the product $\mathbb{S}_A \otimes \odot^{k-1}(\mathbb{S}_A)$ contains the space $\odot^k(\mathbb{S}_A)$ as an irreducible piece in the decomposition. If we consider fields with special values in this subspace of the whole space, the twisted Dirac operator reduces to the massless field operator.

Lemma 1. *Let $\mathbb{V} = \odot^{k-1}(\mathbb{S}_A)$. The product $\mathbb{S}_A \otimes \mathbb{V}$ decomposes into irreducible pieces as*

$$(\mathbb{S}_A \oplus \mathbb{S}_{A'}) \otimes \odot^{k-1}(\mathbb{S}_A) \simeq E^2_- \oplus E^3 \oplus E^4, \tag{10}$$

with

$$E^2_- \simeq \odot^k(\mathbb{S}_A),\ E^3 \simeq \mathbb{S}_{A'} \otimes \odot^{k-1}(\mathbb{S}_A),\ E^4 \simeq \odot^{k-2}(\mathbb{S}_A).$$

These spaces, together with the corresponding maps from the BGG sequence form a complex

$$E^2_- \overset{D(k)^2_-}{\longrightarrow} E^3 \overset{D(k)^3}{\longrightarrow} E^4, \tag{11}$$

which is a subcomplex of the BGG complex.

The twisted Dirac operator $D_{\mathbb{V}}$ restricted to E^2_- coincides with the massless field operator.

Proof. The decomposition (10) was described above and it makes clear that (14) is indeed a subcomplex of the BGG complex.

On the other hand, the twisted Dirac operator has the form

$$\begin{matrix} E^2_- & \overset{D^2_-}{\searrow} & \\ \oplus & & E^3. \\ E^4 & \underset{(D^4)^*}{\nearrow} & \end{matrix} \tag{12}$$

The form of the operator follows immediately from the classification of the first order operators ([10, 20]). The operator $(D^3)^*$ is dual to the operator D^3 and can be written explicitely as

$$(D^3)^*(\varphi_{C\dots E}) = \nabla^{B'}_{(B}\varphi_{C\dots E)}.$$

If we consider only fields with values in E^2_-, we get clearly massless field operator. □

It is possible to show that this subcomplex is not exact at all places. In particular, $\operatorname{Im} D(k)^2_-$ is a proper subset of $\operatorname{Ker} D(k)^3$ and the map $D(k)^2_-$ is not injective. We shall not discuss details here.

2 MASSLESS FIELD EQUATIONS IN HIGHER DIMENSIONS

2.1 Tools from representation theory.

There is a natural generalization of massless fields equations from dimension 4 to higher dimensions. From point of view of representation theory, there is always difference between the case of even or odd dimension. For simplicity, we shall consider here only the case of even dimensions. But the same questions and problems can be investigated in odd dimensions as well and similar results should be expected. We shall return to the case of odd dimensions elsewhere. Hence we shall suppose from now on that our fields are defined on $\mathbb{R}^m$ with $m = 2n$.

First we have to specify values for our fields. There are two basic $Spin(m)$ modules, we shall denote them $\mathbb{S}_A$, resp., $\mathbb{S}_{A'}$. In higher dimensions, it is necessary to deal with more complicated representation. They can be realized as tensor fields with a particular symmetry but the symmetry needed is more complicated than symmetrization or antisymmetrization. Moreover, representations with highest weights with half-integral components can be realized only inside the tensor algebra tensored with the basic spin module. Hence we need a language how to described such more complicated representations.

All tools needed from representation theory for the group $Spin(m)$, $m = 2n$ can be found in [14, 5.2.2]. The classification of irreducible $Spin(m)$ modules is standardly given in term of the highest weight of the module. In even dimension ($m = 2n$), a highest weight of an irreducible $Spin(m)$ module is a vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ of integers (or half-integers) satisfying the relation $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq |\lambda_n|$. Using the Weyl dimensional formula, we can compute dimensions of all these spaces. We shall now extend the discussion of massless fields in dimension 4 to higher dimensions.

We shall denote irreducible representations by their highest weights. So $S_A \simeq (\frac{1}{2}, \ldots, \frac{1}{2})$. The other basic spin module has the label $S_{A'} \simeq (\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$. The space of harmonic polynomials of homogeneity j forms an irreducible representation with the highest weight $(j, 0, \ldots, 0)$. Hence the Fischer decomposition of the space $\mathcal{P}_j$ of homogeneous polynomials of degree j has the form

$$\mathcal{P}_j \simeq (j, 0, \ldots, 0) \oplus (j-2, 0, \ldots, 0) \oplus (j-4, 0, \ldots, 0) \oplus \ldots,$$

where the sum ends either with the trivial or the vector representation. The Cartan power $\boxtimes^j(S_A)$ has the highest weight $(\frac{j}{2}, \ldots, \frac{j}{2})$; similarly, $\boxtimes^j(S_{A'})$ has the highest weight equal to $(\frac{j}{2}, \ldots, \frac{j}{2}, -\frac{j}{2})$.

2.2 Decomposition of tensor products.

Let us now summarize a few facts on massless field equations in higher dimensions from [21].

Lemma 2. *Let us fix a positive integer $k > 1$. The weights*

$$\mu^j = (\underbrace{\frac{k}{2}, \ldots, \frac{k}{2}}_{j}, \underbrace{\frac{k}{2} - 1, \ldots, \frac{k}{2} - 1}_{n-j-1}, -\frac{k}{2});\ j = 0, \ldots, n-1,$$

are highest weights of irreducible representations, which will be denoted by F^j. The weight $\mu_+^n = (\frac{k}{2}, \ldots, \frac{k}{2}, -\frac{k}{2})$ will be the highest weight of the module denoted by $F_+^n \simeq \boxtimes^k(\mathbb{S}_{A'})$.

Similarly, the weights

$$\mu^j = (\underbrace{\frac{k}{2},\dots,\frac{k}{2}}_{2n-j},\underbrace{\frac{k}{2}-1,\dots,\frac{k}{2}-1}_{j-n});\ j=n+1,\dots,2n,$$

are highest weights of irreducible representations, which will be denoted by F^j. The weight $\mu^n_- = (\frac{k}{2},\dots,\frac{k}{2})$ will be the highest weight of the module denoted by $F^n_- \simeq \boxtimes^k(\mathbb{S}_A)$.

We have the following tensor product decompositions.

(1)

$$\mathbb{S}_A \otimes \boxtimes^{k-1}(\mathbb{S}_A) \simeq F^n_- \oplus F^{n+2} \oplus \dots,$$

where the sum ends with F^{2n}, or with F^{2n-1}.

(2)

$$\mathbb{S}_{A'} \otimes \boxtimes^{k-1}(\mathbb{S}_A) \simeq F^{n+1} \oplus F^{n+3} \oplus \dots,$$

where the sum also ends with F^{2n}, or with F^{2n-1}.

Similarly,

(3)

$$\mathbb{S}_A \otimes \boxtimes^{k-1}(\mathbb{S}_{A'}) \simeq F^n_+ \oplus F^{n-2} \oplus \dots,$$

where the sum ends with F^0, or with F^1.

(4)

$$\mathbb{S}_{A'} \otimes \boxtimes^{k-1}(\mathbb{S}_{A'}) \simeq F^{n-1} \oplus F^{n-3} \oplus \dots,$$

where the sum also ends with F^0, or with F^1.

2.3 The BGG complex.

All spaces in the decompositions above are related by various operators in the corresponding BGG complex (for details and proofs, see [19]). There is many parameter system of various BGG complexes, the ones used here form a one-parameter subfamily. The complexes look as follows.

Theorem 1. *(A family of BGG complexes)*

For any $k > 1$, we get the following exact complex

$$F^0 \xrightarrow{D^0} F^1 \dots F^{n-2} \xrightarrow{D^{n-1}} F^{n-1} \begin{matrix} \overset{D^{n-1}_+}{\nearrow} \\ \\ \underset{D^{n-1}_-}{\searrow} \end{matrix} \begin{matrix} F^n_+ \\ \oplus \\ F^n_- \end{matrix} \begin{matrix} \overset{D^n_+}{\searrow} \\ \\ \underset{D^n_-}{\nearrow} \end{matrix} F^{n+1} \xrightarrow{D^{n+1}} F^{n+2} \dots F^{2n-1} \xrightarrow{D^{2n-1}} F^{2n}. \tag{13}$$

All operators are of the first order with exception of D^{n-1}_- and D^n_+, which are of order $k-1$. All operators are conformally invariant (for a proper conformal weight, which depends on a choice of an individual operator). The case $k=2$ coincides with the de Rham sequence.

All needed facts concerning the family of the BGG complexes stated in the theorem above are well known and can be found, e.g., at [19]. Note that the massless field operator forms a part of the complex, it is just the operator D^n_- in the middle splitting of the sequence. Information coming from the exactness of the sequence will help to understand polynomial solutions of the massless field equation.

As before, we can show that the solutions of the massless field equations are solutions of the a suitable twisted Dirac equation.

Lemma 3. *Let $k > 1$. There is a subcomplex in the the BGG sequence of the form*

$$F_{-}^{n} \xrightarrow{D_{-}^{n}} F^{n+1} \xrightarrow{D^{n+1}} F^{n+2} \dots F^{2n-1} \xrightarrow{D^{2n-1}} F^{2n}. \tag{14}$$

Let $\mathbb{V} := \boxtimes^{k-1}(\mathbb{S}_A)$. The twisted Dirac operator $D_{\mathbb{V}}$ restricted to F_{-}^{n} coincides with the massless field operator.

Proof. The first claim is clear from the description of the BGG complex.

On the other hand, the twisted Dirac operator has the form

$$(F_{-}^{n} \oplus F^{n+2} \oplus \dots) \xrightarrow{D_{\mathbb{V}}} (F^{n+1} \oplus F^{n+3} \oplus \dots). \tag{15}$$

From the classification of the first order operators ([10, 20]), it follows immediately that the operator $D_{\mathbb{V}}$ restricted to fields with values in F_{-}^{n} coincides with the massless field operator. □

It can be proved by methods used below that the subcomplex in the above lemma is not exact at all places. In particular, $\mathrm{Im}\, D_{-}^{n}$ is a proper subset of $\mathrm{Ker}\, D^{n+1}$ and the map D_{-}^{n} is not injective. The complex is exact at all other places. We shall not go into details here.

3 POLYNOMIAL SOLUTIONS

We shall now concentrate on properties of homogeneous solutions of the massless field equation. Recall that the massless field operator is acting from the space of functions on $\mathbb{R}^n$ with values in the modules F_{-}^{n} into the space of functions on $\mathbb{R}^n$ with values in F^{n+1}. The operator itself is conformally invariant (for a suitable conformal weight).

Values of the field φ itself, as well as values of its image $D_{-}^{n}(\varphi)$ are $Spin(m)$-modules, hence the corresponding function spaces decompose under the $Spin(m)$ action into (an infinite) sum of finite dimensional irreducible representations. The action preserves homogeneity, hence the subspaces of homogenous polynomials of a given degree (with appropriate values) are invariant under the action and they have finite dimension. The space of homogeneous solutions of the given degree is also invariant (due to the invariance of the operator). It is a finite dimensional space and it decomposes hence into a sum of irreducible pieces. To understand basic properties of homogeneous solutions of the given degree, we have to characterize these irreducible components. Note that for the classical Dirac operator (on spinor valued polynomials), the corresponding space of polynomial solutions of homogeneity k (called usually spherical monogenics of degree k) is an irreducible $Spin(m)$-module with the highest weight $(\frac{2k+1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. We propose the following conjecture for homogeneous solutions of the massless field equation.

Conjecture 1. *Fix a positive integer $k > 1$ and consider the massless field operator*

$$D_{-}^{n} : \mathcal{C}^{\infty}(\mathbb{R}^m, F_{-}^{n}) \mapsto \mathcal{C}^{\infty}(\mathbb{R}^m, F^{n+1})$$

The space of polynomial solutions of the equation $D_{-}^{n}(\varphi) = 0$ of degree j is an irreducible representation of the group $Spin(m)$ with the highest weight $(\frac{k}{2} + j, \frac{k}{2}, \dots, \frac{k}{2})$. Its dimension can be computed from the Weyl dimension formula.

At the moment, it is possible to prove the following statement on homogeneous solutions of the massless field equation.

Theorem 2. *Fix a positive integer $k > 1$ and consider the massless field operator*

$$D_-^n : \mathcal{C}^\infty(\mathbb{R}^m, F_-^n) \mapsto \mathcal{C}^\infty(\mathbb{R}^m, F^{n+1})$$

The space of polynomial solutions of the equation $D_-^n(\varphi) = 0$ of degree j contains an irreducible component with the highest weight $(\frac{k}{2}+j, \frac{k}{2}, \dots, \frac{k}{2})$. Dimension of the component can be computed by the Weyl dimensional formula.

To prove the theorem, we have to understand a Fischer decomposition of the spaces of polynomials with values in F_-^n, resp. F^{n+1}. Denoting $\mathcal{P}_j$ the space of homogeneous polynomials on $\mathbb{R}^n$ of degree j, we want to understand a decomposition of the product $\mathcal{P}_j \otimes F_-^n$ and $\mathcal{P}_j \otimes F^{n+1}$. Note first that

$$\mathcal{P}_j \simeq (j, 0, \dots, 0) \oplus (j-2, 0, \dots, 0) \oplus (j-4, 0, \dots, 0), \dots,$$

ending with either $(1, 0, \dots, 0)$ or $(0, \dots, 0)$. Hence we have to understand behaviour of the products $(j', 0, \dots, 0) \otimes (\frac{k}{2}, \dots, \frac{k}{2})$, resp. $(j', 0, \dots, 0) \otimes (\frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2}-1)$ for all j'. The results are given in the following lemma.

Lemma 4.
(1)

(i) for $j \geq k$,

$$(j, 0, \dots, 0) \otimes (\frac{k}{2}, \dots, \frac{k}{2}) \simeq \oplus_{i=0}^{k} (\frac{k}{2} + j - i, \frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2} - i) \tag{16}$$

(ii) for $k \geq j$,

$$(j, 0, \dots, 0) \otimes (\frac{k}{2}, \dots, \frac{k}{2}) \simeq \oplus_{i=0}^{j} (\frac{k}{2} + j - i, \frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2} - i). \tag{17}$$

(2)

(i) for $j \geq k$, the decomposition of the product

$$(j, 0, \dots, 0) \otimes (\frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2} - 1)$$

contains three different groups of summands:

$$\oplus_{i=0}^{k-1} (\frac{k}{2} + j - i, \frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2} - i - 1) \tag{18}$$

$$\oplus_{i=0}^{k-1} (\frac{k}{2} + j - i - 1, \frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2} - i) \tag{19}$$

$$\oplus_{i=0}^{k-2} (\frac{k}{2} + j - i - 1, \frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2} - 1, \frac{k}{2} - i - 1) \tag{20}$$

(ii) for $k > j \geq 1$, the decomposition of the product

$$(j, 0, \dots, 0) \otimes (\frac{k}{2}, \dots, \frac{k}{2}, \frac{k}{2} - 1)$$

contains three different groups of summands:

$$\oplus_{i=0}^{j}(\frac{k}{2}+j-i,\frac{k}{2},\ldots,\frac{k}{2},\frac{k}{2}-i-1) \tag{21}$$

$$\oplus_{i=0}^{j-1}(\frac{k}{2}+j-i-1,\frac{k}{2},\ldots,\frac{k}{2},\frac{k}{2}-i) \tag{22}$$

$$\oplus_{i=0}^{j-1}(\frac{k}{2}+j-i-1,\frac{k}{2},\ldots,\frac{k}{2},\frac{k}{2}-1,\frac{k}{2}-i-1) \tag{23}$$

Proof. For the proof, it is sufficient to use theorems on the decomposition of the tensor product of the group $Spin(m)$ proved by P. Littlemann ([15]). Further information on the Littlemann theorem can be found in the PhD thesis of M. Plechšmíd ([18]), where they were used for the same purpose as here. □

Proof of the theorem. To prove the theorem, we have to use the fact that the massless field operator

$$D_-^n : \mathcal{P}_j \otimes F_-^n \mapsto \mathcal{P}_{j-1} \otimes F^{n+1}$$

is an intertwining operator for the action of $Spin(m)$. Hence it can map (in a nontrivial way) an irreducible component of a certain weight in the decomposition of the source only to an irreducible component of the same weigth in the decomposition of the target. Now it is possible to check the for a given homogeneity $j \geq 1$, a component with the weight $(\frac{k}{2}+j,\frac{k}{2},\ldots,\frac{k}{2})$ is contained in the decomposition of the product $\mathcal{P}_j \otimes F^n-$, while it is not contained in the decomposition of the product $\mathcal{P}_{j-1} \otimes F^{n+1}$. Hence these components belongs to the kernel of the operator D_-^n on the space of polynomials of homogeneity j. For homogeneity $j = 0$, the whole space of constant polynomials with values in F_-^n forms an irreducible component with the highest weight $(\frac{k}{2},\frac{k}{2},\ldots,\frac{k}{2})$ and it belongs clearly to the kernel of the operator D_-^n. □

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