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FREE VIBRATION FREQUENCIES OF THE CRACKED REINFORCED CONCRETE BEAMS - METHODS OF CALCULATIONS

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Abstract. *The paper presents method of calculation of natural frequencies of the cracked reinforced concrete beams including discreet model of crack. The described method is based on the stiff finite elements method. It was modified in such a way as to take into account local discontinuities (ie. cracks). In addition, some theoretical studies as well as experimental tests of concrete mechanics based on discrete crack model were taken into consideration. The calculations were performed using the author's own numerical algorithm. Moreover, other calculation methods of dynamic reinforced concrete beams presented in standards and guidelines are discussed. Calculations performed by using different methods are compared with the results obtained in experimental tests.*

1 INTRODUCTION

Calculation of reinforced concrete structures requires special attitude because it involves interaction of two materials, such as concrete and steel used in this type of structures. Furthermore, reinforced concrete elements are overloaded and that causes their cracking and stiffness degradation.

There are many theories regarding displacement and redistribution of internal forces in the cracked reinforced concrete beams. The methods proposed describe performance of the reinforced concrete structures including cracks. Typically, the cracking effect and its influence on the distribution of internal forces and deformations is taken into account globally by means of introduction of substitutional stiffness of the cracked element. This kind of approach assures simplicity of calculations by analogy to homogenous structures without cracks.

The experimental tests which were performed [1, 2, 3] proved that appearance of cracks has significant impact not only on deflection and redistribution of the internal forces, but also on the dynamic parameters, such as: natural frequencies and damping. Progressive cracking causes lowering of natural frequencies of the reinforced concrete beams (even by 50%). Moreover, such cracking increases damping properties of element.

Most papers dealing with the dynamics of the cracked reinforced concrete structures try to describe it globally basing on the dynamic substitutional stiffness of the cracked element [1, 2, 4]. Further, this sort of approach makes it possible to apply solutions concerning dynamics of homogenous structures and is characterized by the simplicity of calculations. Nevertheless, it limits observation of structure to the final, summary effects connected with the impact of the element overloading on the dynamic properties. In addition, there are no explicit relations connecting dynamic and static stiffness assumed to calculate deflections. Some experimental tests prove that it is less or equal to effective stiffness [3] while others confirm it is bigger [1,2].

The paper presents alternative approach based on discrete crack model. Calculations were performed using the author's own numerical programme related to Mathematica®. In addition, the obtained records were compared with the existing results acquired in experimental tests. Discussion and comparison of the results was conducted according to the Polish Standard requirements for calculation of support structures for machines [5] and EC 2 directives [6].

2 STIFF FINITE ELEMENT METHOD

2.1 Homogenous beams

Dynamic calculations of most of the structures with continuous mass distribution are connected with discretization. Discretization methods can be divided into two groups : mathematical (with global approximation of displacement state – for ex. the Ritz method with local approximation of displacement state – elastic finite element method) and physical which refers to mass granulation that leads to classical discrete state.

The second group includes stiff finite element method [7]. This method was first applied in naval industry. Later it was used by J. Langer [8] for calculation of bar structures. Beam model consists of stiff mass discs which represent force of inertia of a structure. Discs are connected by elastic constraints (one rotation and two translation) responsible for elastic features of a structure. Movement of each mass discs is described by three general coordinates. In case of transverse vibrations which are considered in this paper, elastic constraints and general coordinates are reduced to two. Example scheme and calculation model of a beam divided into four elements are shown in figure 1.

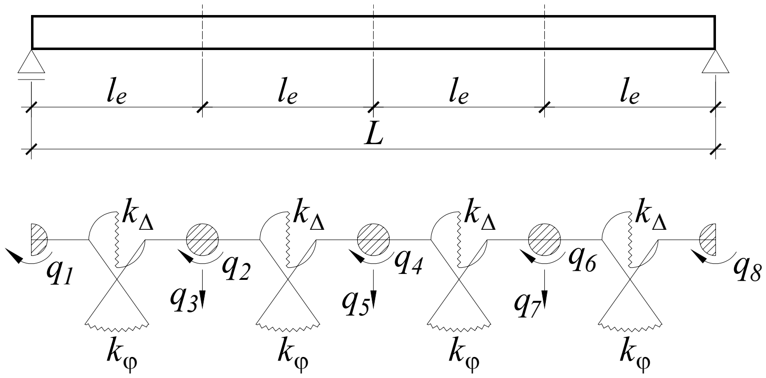

Figure 1: Scheme and numerical model of homogenous beam

Stiffness of constraints connecting mass discs is computed from the following equations (1), (2):

$$k_\varphi = \frac{EI}{l_e}, \tag{1}$$

$$k_\Delta = 12\frac{EI}{l_e^3}, \tag{2}$$

where: EI – beam bending stiffness, l_e – length of finite element.

Stiffnesses of constraints are grouped in diagonal matrix *{k}*, which for the case shown in Fig. 1 is given below:

$$\{k\} = diag\{k_\varphi, k_\Delta, k_\varphi, k_\Delta, k_\varphi, k_\Delta, k_\varphi, k_\Delta\}\,. \tag{3}$$

Global stiffness matrix **K** is calculated from the following equation:

$$\mathbf{K} = \mathbf{A_k^T}\{k\}\mathbf{A_k}\,, \tag{4}$$

where: $\mathbf{A_k}$ – transformation matrix.

Transformation matrix $\mathbf{A_k}$, transforms general coordinates vector **q** on mutual transposition vector **r**. It has repeatable character and it can be easily generated automatically for optional boundary conditions.

Inertia matrix is a diagonal matrix. Masses of individual discs *m* correspond to translation coordinates while their mass inertia moments J_m correspond to rotational coordinates. For the model shown in Fig.1 inertia matrix is as follows:

$$\mathbf{B} = diag\{J_{m1}, J_{m2}, m, J_{m2}, m, J_{m2}, m, J_{m1}\}\,. \tag{5}$$

Eigen values of matrix **A** being converse product of matrix **B** and matrix **K** are the square of circular frequency ω:

$$\mathbf{A} = \mathbf{B}^{-1}\mathbf{K} = diag\{\omega^2\}\,. \tag{6}$$

2.2 The reinforced concrete cracked beams

The presented approach enables to include local discontinuities (among others cracks) in a discrete way. Adequate division into finite elements allows the introduction of cracks by means of reduction of stiff rotation constraints while calculations are performed as for the homogenous beam.

Stiffnesses of constraints k_φ, k_Δ are commuted using the element stiffness in phase I EI_I. The stiffness of rotation constraints is reduced and has value k_φ^{cr} in the place where the cracks appear. The scheme and calculation model is shown in figure 2.

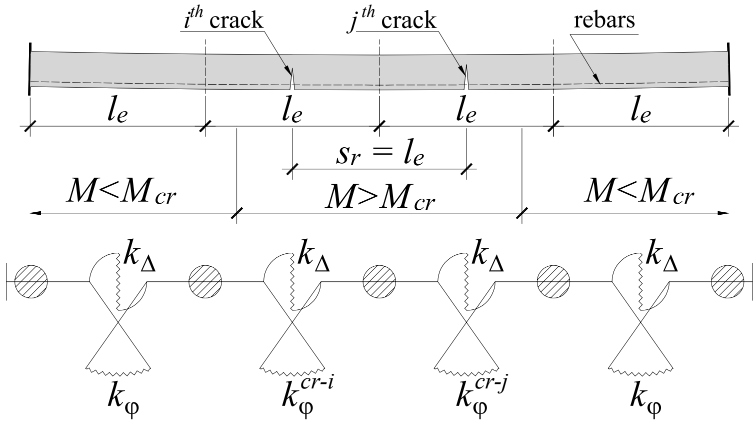

Figure 2: Scheme and numerical model of the reinforced concrete beam with cracks

Calculations are performed using tests and theoretical studies [9], describing work of a beam in phase II with the discrete crack model. According to the above mentioned theory, elastic crack opening can be commuted according to the following equation:

$$\varphi_i^e = r_i\, M(x_i), \tag{6}$$

where: r_i – rotation susceptibility for i^{th} – crack, $M(x_i)$ – bending moment in i^{th} – crack occurence.

Applying general assumptions for the strength of materials and mechanics of concrete structures rotation susceptibility of r_i can be computed according to the following equation:

$$r_i = d_\varphi^{cr-i} = \frac{\psi_z\, s_{rm}}{E_s A_s h^2 (1 - \frac{\alpha_{II}}{3} - \frac{a}{h})(1 - \alpha_{II} - \frac{a}{h})}, \tag{7}$$

where: ψ_z – coefficient describing violation of interaction between steel and concrete calculated according to (8), s_{rm} – average crack spacing, E_s – Young's modulus for steel, A_s – reinforcement cross-sectional area, h – beam height, α_{II} – relative height of the compressed zone in phase II, a – structural concrete cover.

$$\psi_z = 1.3 - s\frac{M_{cr}}{M}, \tag{8}$$

where: s – 1.0 in case of immediate loading and smooth bars, 1.1 in case of immediate loading and ribbed bars, 0.8 in case of long-term loading, M_{cr} – cracking moment, M – maximum moment up to which the cross-section was overloaded.

Assuming that the susceptibility of finite elements connections is a sum of susceptibilities resulting from the beam deformation (for phase II) and susceptibility resulting from the crack appearance the following relationship can be written:

$$d_{\varphi}^{II-i} = (k_{\varphi}^{I})^{-1} + d_{\varphi}^{cr-i}, \tag{9}$$

where: k_{φ}^{I} – stiffness of rotation constraints calculated according (1) for the phase I. (EI_I).

Knowing susceptibility (9) stiffness of rotation constraints for the cracked cross-section can be computed:

$$k_{\varphi}^{cr\text{-}i} = (d_{\varphi}^{II\text{-}i})^{-1}. \tag{10}$$

3. OTHER CALCULATION METHODS

The approach proposed in Polish Standards [5] referring to the support structures for machines recommends calculation of global stiffness of the bended element according to the Young's modulus for concrete and inertia moment for the whole concrete cross-section but does not take into consideration reinforcement. While this sort of approach seems to be quite correct in calculations performed for phase I, it is less convincing in case of the cracked beam. Assuming constant stiffness for the total scope of element work may cause errors.

It is more reasonable to calculate frequency using the relationship given below which is recommended in EC2 [6].

$$\alpha = \zeta\alpha_{II} + (1-\zeta)\alpha_{I}. \tag{11}$$

Parameter α is the one which is considered (for example cross-section deformation, curvature, rotation or deflection) and α_I and α_{II} are the values of this parameter calculated under the assumption that cracks do not occur and for the completely cracked objects respectively, while ζ is coefficient of distribution. It is assumed in the paper that the parameter to be considered is natural frequency. It should be noted that element overloading is accompanied by the decrease of bending stiffness and resulting from this natural frequency.

The approach based on calculation of dynamic stiffness different from the static one is presented in ACI Journal [1]. Dynamic stiffness is expressed by the following formula (12):

$$EI_D = E_C\left[\left(\frac{\alpha M_{cr}}{M}\right)I_I + \left(1-\frac{\alpha M_{cr}}{M}\right)I_{II}\right], \tag{12}$$

where: E_C – Young' modulus for concrete , α – constant parameter (according to [1] $\alpha = 0.6 – 0.8$), I_I – inertia moment in phase I, I_{II} – inertia moment in phase II.

Similarly as in case of the dependency (11) overloading is accompanied by the decrease of element stiffness. A draft character of the natural frequency change in the function of the overloading history is shown in figure 3.

As literature studies proved, the final approach presented gained the most popularity. Empirical dependencies are drawn in order to include estimation of substitutional dynamic stiffness of element. Thus, application of closed solutions of the structure dynamics can be considered in calculations of the cracked reinforced concrete structures.

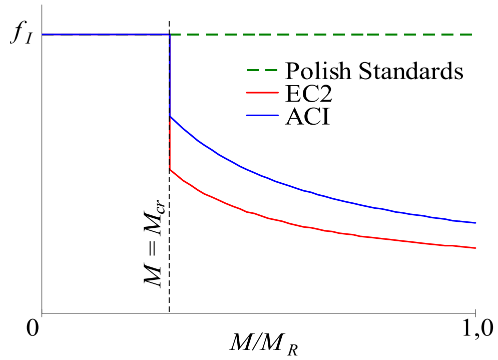

Figure 3: Change of natural frequency in function of overloading history

4 NUMERICAL ANALYSIS – EXAMPLE AND COMPARISON

4.1 Input data

In order to verify numerical analysis some experimental results were applied [3]. Experimental tests were performed on the beam elements such as shown in figure 4.

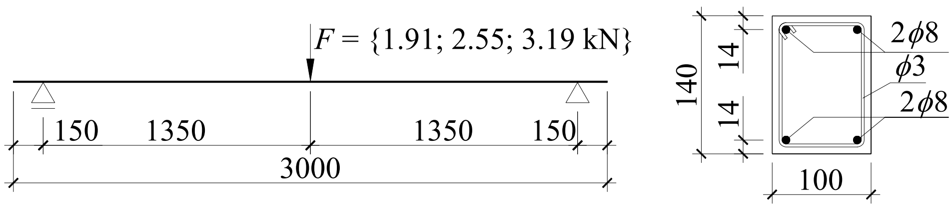

Figure 4: Scheme of the analysed beam (dimensions in mm)

The beam was loaded using force F in three stages. Each time sudden removal of the load caused the element to vibrate. Using the registered vibrogram the frequency of free vibrations was determined which approximately complies with the first natural frequency. Increasing loading caused degradation of the beam stiffness (progressive crack propagation). Other input data is included in the table 1.

Table 1: Input data

Analysed properties	Value	Unit
Element dimensions l_{eff} x b x h	2700x100x140	mm
Average material density ρ_m	2452	kg/m^3
Cross-section area of reinforcement $A_{s1} = A_{s2}$	1.00	cm^2
Young's modulus for concrete E_c	29.8	GPa
Tensile strength of concrete f_{ct}	2.33	MPa
Young's modulus for steel E_s	200	GPa
Yield point of steel f_y	272	MPa
Average spacing of cracks	100	mm

Unfortunately not all the properties were measured or given in the paper [3] while some of them were assumed arbitrarily (for ex. spacing of cracks). The beam was divided into 27 finite elements (l_e = 270 mm).

4.2 Results of numerical analysis

Numerical analysis were carried out for the data as In 4.1. Calculations were performed using three methods:

- according to EC2,
- according to ACI,
- using author's own algorithm based on stiff finite elements method related to *Mathematica*®.

The obtained results are shown by means of diagram of natural frequency in function of loading history (figure 5).

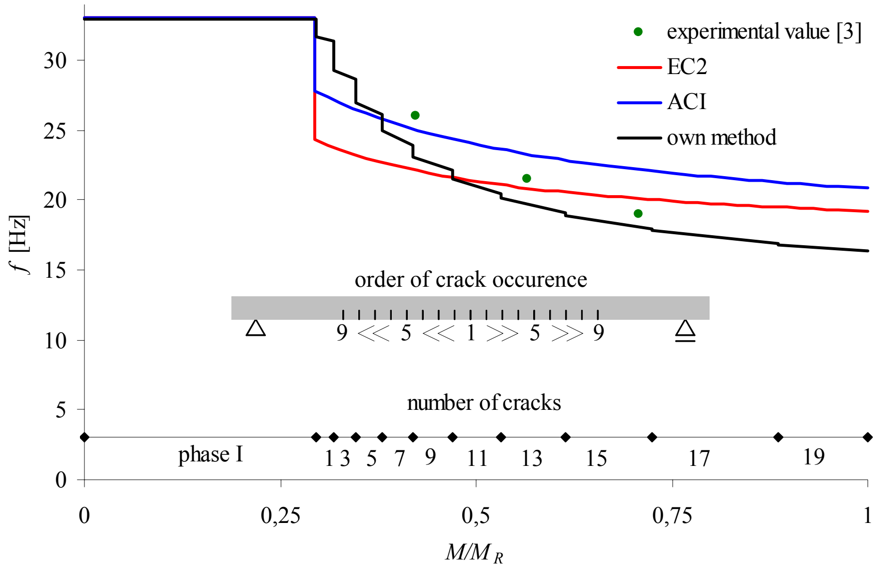

Figure 5: Natural frequencies in function of loading history

Horizontal axis illustrates relative overloading level of element. In addition , a number of cracks resulting from loading procedure is given. For simplicity reasons it was assumed that cracks appear symmetrically on both sides of a beam (when $M \geq M_{cr}$).

4.3 Review of results

Diagram 5 shows quite noticeable decrease of frequency depending on the level of beam overloading. The obtained curves have similar character and in more or less precise way they resemble actual element work. Curves calculated according to EC2 and ACI have continuous character. Effects connected with the crack occurrence and violence of interaction between concrete and steel are concurrent. Author's own method separates these two effects.

Quite visible abrupt value decreases can be noticed caused by appearance of successive cracks. Besides, there occurs continual decrease of frequency which is a result of violation of interaction between concrete and steel.

4.4 SUMMARY

The experimental tests carried out so far prove that element overloading causes changes of dynamic characteristics (natural frequencies, damping parameters). Thus including this factor in calculations seems to be quite reasonable.

The paper presents different methods of calculation of natural frequencies of the cracked reinforced concrete beams. It can be noticed according to some literature studies that the most popular approach is based on the global description of the effect (substitutional element stiffness). This sort of attitude makes it possible to use the closed solutions of the structure dynamics for simple static schemes.

The author's own method is the alternative approach which considers the crack morphology in a detailed way. It allows to follow processes connected with the influence of overloading on the natural frequencies of the cracked reinforced concrete beams.

The results obtained so far can only prove that the assumptions made may be correct. At the moment the method is being developed. The authors planned their own experimental investigations to be carried out.

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