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THE HYPERCOMPLEX SZEGÖ KERNEL METHOD FOR 3D MAPPING PROBLEMS

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Abstract. *In this paper we present rudiments of a higher dimensional analogue of the Szegö kernel method to compute* $3D$ *mappings from elementary domains onto the unit sphere. This is a formal construction which provides us with a good substitution of the classical conformal Riemann mapping. We give explicit numerical examples and discuss a comparison of the results with those obtained alternatively by the Bergman kernel method.*

1 QUATERNIONIC HARDY SPACES AND SZEGÖ KERNELS

In this paper we present the rudiments for a hypercomplex Szegö kernel method to compute $3D$ mappings from elementary domains onto the unit sphere. This provides an extension to the classical method in $2D$ described in [1] and elsewhere.

Let $\{e_1, e_2, e_3\}$ be the imaginary units of the quaternions satisfying the multiplication rules $e_1 e_2 = e_3 = -e_2 e_1$ and $e_j e_0 = e_0 e_j$ and $e_j^2 = -1$ for all $j = 1, 2, 3$.

Each quaternion $z \in \mathbb{H}$ has the form $z = z_0 e_0 + z_1 e_1 + z_2 e_2 + z_3 e_3$. The conjugate of z is defined by $\overline{z} = z_0 - z_1 e_1 - z_2 e_2 - z_3 e_3$. In what follows we identify $\mathbb{R}^3$ with the set $\{z \in \mathbb{H}; z_3 = 0\}$ Real differentiable $\mathbb{H}$-valued functions that satisfy in an open subset of $\mathbb{R}^3$ the Cauchy-Riemann system $Df := \sum_{i=0}^{2} \partial_i e_i f(z) = 0$ are often called left monogenic, cf. [5]. In the sequel suppose that G is a domain with a strongly Lipschitz boundary in $\mathbb{R}^3$. Then, following [5], the quaternionic Hardy space $H^2(\partial G, \mathbb{H})$ of left monogenic functions in G can be introduced as the closure of the set

$$\mathcal{A}^2(\partial G, \mathbb{H}) := \{f \in C^1(\overline{G}) \cap L^2(\partial G); \mid Df(z) = 0 \ \forall z \in G\},$$

endowed with the inner product defined by

$$\langle f, g \rangle := \int_{\partial G} \overline{f(z)} g(z) dS(z).$$

Here $dS(z)$ is the scalar valued surface measure. It is well-known that $H^2(\partial G, \mathbb{H})$ has a uniquely reproducing kernel, the quaternionic Szegö kernel, cf. e.g. [5, 8]. If G is a bounded domain, then the kernel can be approximated by applying the Gram-Schmidt algorithm to the set of the Fueter polynomials. Following [5] and others, the Fueter polynomials are defined by $p_0(z) := 1$, $\mathcal{Z}_i := z_i - z_0 e_i$, $i = 1, 2, 3$, and $p_{l_1,\ldots,l_k}(z) := \frac{1}{k!} \sum_{\pi \in \mathcal{S}_k} \mathcal{Z}_{\pi(l_1)} \ldots \mathcal{Z}_{\pi(l_k)}$, where $\mathcal{S}_k$ is the symmetric group on k elements. These polynomials form a basis for $H^2(\partial G, \mathbb{H})$ if G is a bounded domain containing the origin. The orthonormalization process of Gram-Schmidt applied to the set of the Fueter polynomials then produces an orthonormal set $(h_j)_j$ of $H^2(\partial G, \mathbb{H})$. In view of the non-commutativity the coefficients in the Gram-Schmidt algorithm have to appear on the right hand side. The use of the Fueter polynomials up to degree N corresponds to a total of $n := \frac{(N+1)(N+2)}{2}$ functions. The Szegö kernel is then approximated by the finite Fourier sum

$$S_G^N(0, z) = \sum_{j=1}^{(N+1)(N+2)/2} h_j(z) \overline{h_j(0)}, \quad (N = 0, 1, \ldots),$$

as the kernel can always be developed in such series for any given orthonormal system, see [5]. Then we compute the line integral

$$f_N(z) = \int_0^z {S_G^N}^2(0, u) du, \quad (N = 0, 1, \ldots).$$

Notice that in the three-dimensional case, this line integral is not independent from the choice of the path. Here, we choose the direct line connection from 0 to z as integration path. In view of the non-commutativity this still leads to two different choices of integration. Here we choose the ansatz

$$f_N := z \mapsto \int_0^1 z S^2_{G^N}(tz, 0) dt,$$

which lead so far to the best results. Notice here also that the orthonormalized functions h_j take in general values in $\mathbb{H}$ and not in $\mathbb{R}^3$. In order to obtain a map to $\mathbb{R}^3$, we cut off the e_3-component. In the classical two dimensional complex case, the complex analogue of the function series $(f_N)_N$ converges to the function that maps the given domain G conformally onto the unit disc. In the higher dimensional case we cannot expect f to be conformal in the classical sense of Gauss in general, because the set of conformal maps in the sense of Gauss coincides with the set of Möbius transformations. However, with great astonishment, in the case of 3D rectangular domains, L-pieces, circular cylinders and the double cone we obtained numerically a very good mapping to the unit sphere. The use of MAPLE allowed us to calculate symbolically and exactly, which is necessary due to the numerical instability of the Gram-Schmidt procedure. In this talk we present and discuss the numerical experiments that we obtained by using this hypercomplex Szegö kernel method. We compare our results with the results obtained by the alternative 3D Bergman kernel method described in [4, 2, 3, 7].

In this paper, we will present the results for two domains, on which we gained a remarkable outcome.

The L-piece

In order to make a better comparison of the two approaches, the same L-shaped domain as in [2, 3, 7] was regarded.
Looking at the resulting images and considering the case of the unit cube (see [6]), the figures are very similar to each other: As the regarded L-piece is produced by cutting a cuboid out of the unit cube, the picture looks from all but one side the same as the picture for the unit cube. Only on one side is a deep pit / hole instead of an half-ball-like indentation. This is presented in the following series of pictures.
Comparing now the images with the images in [2, 3, 7], there is clearly a resemblance for the higher grades of N. One notable effect is that when using the Bergman method, we get some points sticking out from the boundary. These effects do not appear when using the Szegö kernel method that we proposed here. So the Szegö method yields, judging from the plots, images which are a bit "smoother".

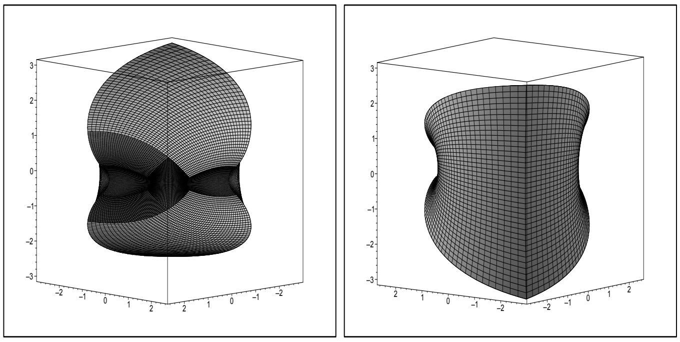

Figure 1: N=2

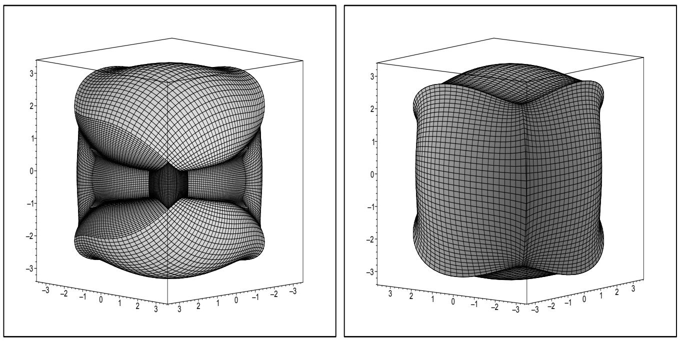

Figure 2: N=4

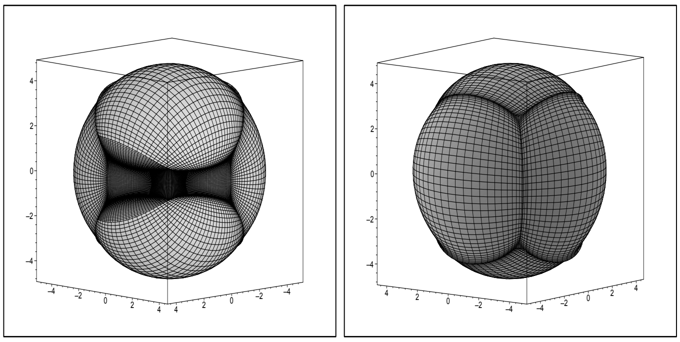

Figure 3: N=6

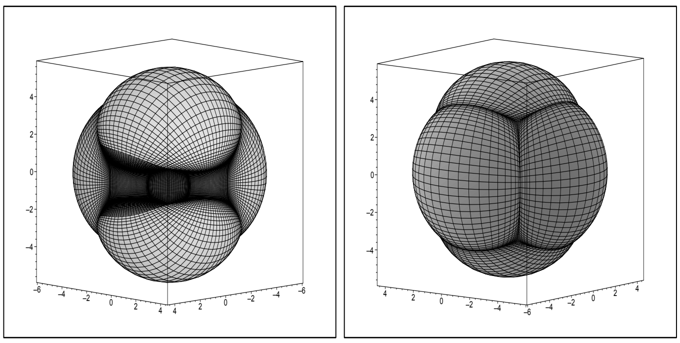

Figure 4: N=14

Unfortunately, as exhibited quantitatively in tables in our forthcoming paper [6], the pictures do not improve significantly any longer from a certain grade on. This phenomenon occurred for all domains that we considered. In the case of the L-piece, the plot does not change very significantly for values of N greater than 6. Since we applied the method for quite many geometrically different domains, the results have to be interpreted as a strong indication that the algorithm that indeed leads to a very ball like domain but does not yield a mapping onto a perfect ball for $N \to \infty$. In [7] the same behavior was also observed when using the Bergman kernel method. That indicates that we still have to incorporate a certain kind of smoothening correcting effect in order to improve the mapping quality.

1.1 The cylinder with height 3 and radius 1

An excellent result was achieved for the cylinder with radius 1 and height 3. For this domain, the resulting image is really very close to a ball. The only divergence is that the figure is visibly flattened at the poles; carvings, which have approximately the shape of a circle, can be seen.
The numerical evaluation also yielded very good results. The variance for the highest grade which could be calculated (N=10) is in the size of 10^{-4}, the variance of the fourth component is the size of 10^{-23}. So we really seem to get a mapping into the three-dimensional space here, and the image also comes very close to an actual ball. Note also that the results for this domain outdo the results of the Bergman method. This is quantitatively more explicated in [6].

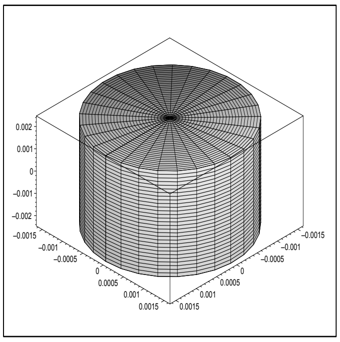

Figure 5: N=1

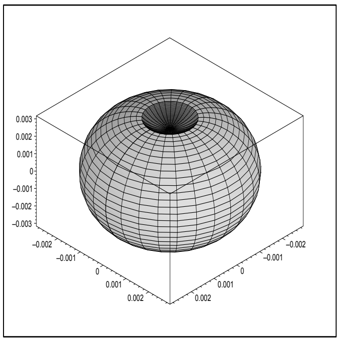

Figure 6: N=2

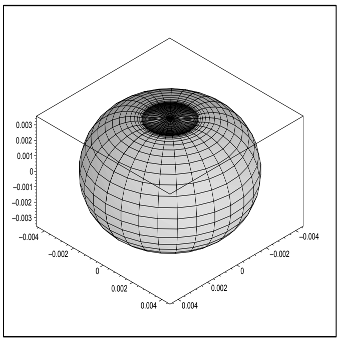

Figure 7: N=6

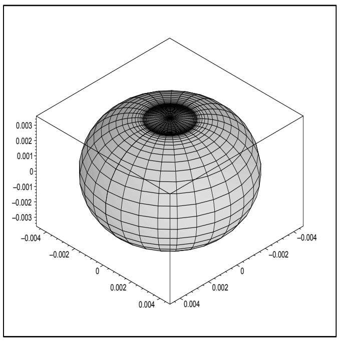

Figure 8: N=10

As in the previous case mentioned, the images stop getting closer to the shape of a ball from a certain grade on. The astonishing thing for this cylindrical domain is that the image for $N = 2$ equals, up to minor differences (the pits at the poles of the figure broaden a bit), already the final image for the highest regarded grade, in this case $N = 10$, and is thus also the best approximation to a ball we will probably get with the algorithm we used.

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REFERENCES

[1] S. Bell. *The Cauchy Transform, Potential Theory and Conformal Mapping*. CRC Press, Boca Raton, 1992.

[2] S. Bock, M.I. Falcão and K. Gürlebeck. Applications of the Bergman kernel functions. In: *Internationales Kolloquium über Anwendungen der Informatik und Mathematik in Architektur und Bauwesen*, Bauhaus Universität Weimar, 2003.

[3] S. Bock, M.I. Falcão, K. Gürlebeck and H. Malonek. A 3-dimensional Bergman kernel method with applications to rectangular domains. *Journal of Compuational and Applied Mathematics* **189** (2006), 67–79.

[4] B. Boone. *Bergmankern en conforme afbeelding*. Diplomarbeit, Lehrstuhl II für Mathematik RWTH Aachen, Aachen, 1992.

[5] F. Brackx, R. Delanghe and F. Sommen. *Clifford Analysis*. Pitman Res. Notes **76**, Boston-London-Melbourne, 1982.

[6] D. Constales, D. Grob, R.S. Kraußhar. Constructing 3D mappings onto the unit sphere with the hypercomplex Szego kernel, *to appear*

[7] J. Rüsges. *Bergmankerne und Abbildungen auf die Einheitskugel*, Diplomarbeit, Lehrstuhl II für Mathematik RWTH Aachen, Aachen, 2007.

[8] Shapiro, M. and N. Vasilevski: *On the Bergmann kernel function in hyperholomorphic analysis.* Acta Appl. Math. 46 (1997) No. 1, 1 – 27.