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NONZONAL WAVELETS ON S^N

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Abstract. *In the present article we will construct wavelets on an arbitrary dimensional sphere S^n due the approach of approximate Identities. There are two equivalently approaches to wavelets. The group theoretical approach formulates a square integrability condition for a group acting via unitary, irreducible representation on the sphere. The connection to the group theoretical approach will be sketched. The concept of approximate identities uses the same constructions in the background, here we select an appropriate section of dilations and translations in the group acting on the sphere in two steps. At First we will formulate dilations in terms of approximate identities and than we call in translations on the sphere as rotations. This leads to the construction of an orthogonal polynomial system in $L^2(SO(n+1))$.*
That approach is convenient to construct concrete wavelets, since the appropriate kernels can be constructed form the heat kernel leading to the approximate Identity of Gauss-Weierstraß. We will work out conditions to functions forming a family of wavelets, subsequently we formulate how we can construct zonal wavelets from a approximate Identity and the relation to admissibility of nonzonal wavelets. Eventually we will give an example of a nonzonal Wavelet on S^n, which we obtain from the approximate identity of Gauss-Weierstraß.

1 INTRODUCTION

During the decades the theory of wavelets became more and more important and is increasingly well investigated. The task of analyzing data, reconstructing functions from measurement data or to save data in an economical way counts to the daily ones.
Hereby, the occurring functions can be very complicated and contain sometimes errors in the high frequency part. This is especially the case if the data came from measurements. Wavelets help us to investigate those functions by splitting it in simpler parts by a wavelet transform. During the reconstruction of the transformed function we can chose the scale (frequency) and can cut the error contained in the high frequency parts. Further we can zoom (change the scale) at arbitrary position, so as well as select the locus of observing a function.
In this setting the concept of dilations and translations in the case of $\mathbb{R}$ has shown to be very useful. The group theoretical approach developed the fact, that dilations, parameterized by $a \in \mathbb{R}_+$ $(f(x) \mapsto \frac{1}{\sqrt{a}} f\left(\frac{x}{a}\right))$ and translations, parameterized by $b \in \mathbb{R}$ $(f(x) \mapsto f(x-b))$ in $\mathbb{R}$ are nothing but a regular representation of the affine linear group G, also called $(ax+b)$-group in Hilbert space $L^2(\mathbb{R})$. The admissibility condition of a wavelet is understood to represent the square integrability of the representation U, i.e. $\int_G |\langle U(f)(g), f\rangle_{L^2(\mathbb{R})}|^2 \, \mathrm{d}\mu(g) < \infty$, where $\mathrm{d}\mu$ denotes the left Haar measure on G.
In the case of the sphere is investigated by J-P. Antoine, P. Vandergheynst, M. Ferreira and others. The appropriate group acting on S^n and containing dilations and translations on the sphere turned out to be the Lorentz group $SO(n+1,1)$. Translations are represented as rotations $T^g \in SO(n+1)$ and dilations as Lorentz boosts $D^a \in SO(1,1)$ (see[1]-[2]). But the set of possible dilations is larger than just taking the Lorentz boosts. All of them are constructed in [8]. However, the dilations on S^n form a one parameter subgroup of $SO(n+1,1)$
In these approach the translation operator on the sphere S^n is given by the rotation group $SO(n+1)$, forming the maximal compact subgroup of $SO(n+1,1)$.
Another approach is based on so-called approximate identities. The basic idea is to consider a family of bounded operators $\{D^\rho,\ \rho > 0\}$ in $L^2(S^n)$, with a reconstruction property for ρ tending to 0. It is coming from the related construction of diffusion wavelets. Here, the family of operators is given as convolution operator with an appropriate kernel. We want to recall that the heat kernel is forming an approximate identity with the time t as dilation parameter. Simultaneously, these operators form a diffusion semigroup and so they are nothing but (continuous) diffusion wavelets on the sphere.
Due the approach of approximate Identities the dilation operator reduces to the choice of the parameter value of a family of wavelets.
This second concept was extensively used by Freeden and others (see e.g. [9]) to construct zonal wavelets, i.e. a type of radial basis functions.
In the preliminaries we will introduce the tools for function spaces on the sphere, will list the orthogonal system of spherical harmonics and that Gegenbauer polynomials in L^2-space for zonal functions on the sphere, of special importance are the connections between them. In the section about approximate identities we call in the reconstruction property of these object and formulate a equivalent condition to corresponding Gegenbauer coefficients of the approximate Identity. This will form the basis of reconstructing functions from there wavelet transformed. As example

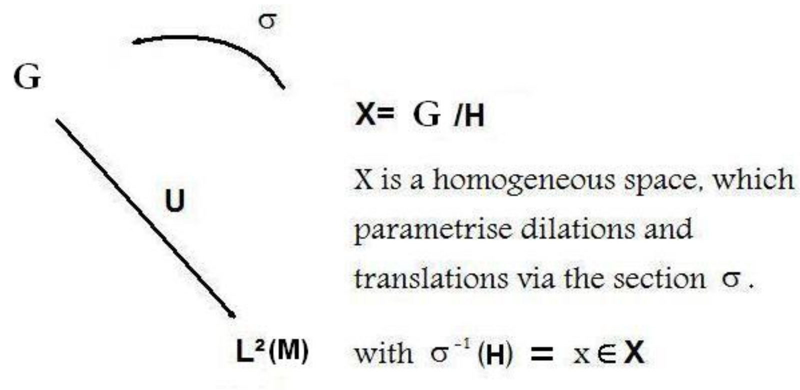
we introduce the fundamental solution of the heat equation on the sphere. Since we will construct also nonzonal wavelets, we need a quasi regular representation of $SO(n+1)$. The introduction of an orthogonal systems of polynomials in $L^2(SO(n+1))$ forms the central in that section. The name Wigner polynomials, comes from the well known case of $SO(3)$. We define a Wavelet by certain admissibility conditions. The first task is than the formula of Wavelet transformed and the reconstruction formula. After the proof of them we show the connection to the group theoretical square integrability condition. Continuing we show the construction of a zonal wavelet from a approximate identity and the connection to nonzonal wavelets. The construction of a nonzonal wavelet from a approximate identity with the help of a weight function will be applied to the heat kernel on S^2. In the weight function we have much freedom and can adapted the wavelet to special requirements.

2 GROUP THEORETICAL BACKGROUND- WAVELETS

Before we start to investigate spherical wavelets with the approach of singular integral operators as special diffusion operators, we will recall the group theoretical approach to wavelet theory. During the Paper we will keep the connection to the group theoretical point of view and the corresponding admissibility condition.

2.1 Group theoretical admissibility

The group theoretical approach is investigated J-P. Antoine and P. Vandergheynst. The wavelet admissibility conditions are obtained in the following way.
Classically one forms a frame in $L^2(M)$ by the collection of all dilated and translated versions of a mother wavelet. The mother wavelet has to satisfy an admissibility condition, that guarantees that the operator, which we obtain as wavelet transformation is bounded. Dilations and translations are modeled as actions of a group G on M or more precise are given ar representation unitary U of G in $L^2(M)$. The representation has to be irreducible, so that we obtain a frame in the mentioned way. The irreducibility ensures further that the wavelet transform is invertible. Over and above that the wavelet transform has to be a bounded operator. This is satisfied, if the representation U is square integrable, i.e. there is an admissible vector. In general there is not a irreducible, representation does not need to be square integrable. For instance this is the situation on the sphere. In that case one can choose a subgroup $H \subset G$ and restrict to $G/H =: X$ together with a section $\sigma : X \rightarrow G$. A section is a right inverse of the projection $\pi : G \rightarrow X$ given by $\pi(g) = [g]$, where $[g]$ denotes the restclass of $g \in G$ under factorization with respect to H. The wavelet frame forming dilations and translations in that case are parameterized by X.



Definition 2.1 (Group theoretical admissibility condition)**.** Let U be a irreducible, square integrable modulo a subgroup H representation of G in $L^2(M)$. A nonzero vector $\Psi \in L^2(M)$ is admissible, if it satisfies

$$\int_X \left|\langle f,\, U(\sigma(x))[\Psi]\rangle_{L^2(M)}\right|^2 \, \mathrm{d}\nu(x) < \infty \quad \forall f \in L^2(M)\,. \tag{2.1}$$

The Orbit of Ψ under σ, $\{U(\sigma(x))\psi |\, x \in X\}$ is called a coherent state.

2.2 $SO(n+1,1)$ acting on S^n

On $S^n \subset \mathbb{R}^{n+1}$ the appropriate group, acting on it is the Spherical conformal group. It can be identified by the Lorentz group $SO(n+1,1)$ [5].
Easy to see is the identification of S^n with the light cone C in $\mathbb{R}^{n+1,1}$, where $\mathbb{R}^{n+1,1}$ denotes $\mathbb{R}^{n+2}$ equipped the norm, which is induced from bilinearform of signature $(n+1,1)$. The light cone are all vectors of norm 0 $C = \{x \in \mathbb{R}^{n+1,1} |\; \|x\|_{n+1,1} = 0\}$ and it can be identified with S^n by

$$S^n \ni \xi \mapsto \left\{ \begin{pmatrix} \lambda \\ \lambda\xi \end{pmatrix},\; \lambda \in \mathbb{R} \right\}. \tag{2.2}$$

The action of the lorentz group $SO(n+1,1)$ on C is obvious and gives the action on S^n, together with the identification 2.2
The Iwasawa decomposition partitions $SO_0(n+1,1)$, the subgroup of $SO(n+1,1)$ which leavs the future light cone invariant, into

- $K \sim SO(n+1)$ rotations around the x_{n+2}-axis.
- $A \sim SO_0(1,1)$, the one-dimensional subgroup of Lorentz boosts. (Hyperbolic rotations)
- $N \sim \mathbb{R}^{n-2,1}$ the $n-1$-dimensional Abelian subgroup.

$SO_0(n+1,1)/N = K\,A =: X$ is the parameter space of dilations and translations on the sphere. All possible sections $\sigma : X \to SO_0(n+1,1)$ are developed by M. Ferreira in [8].

We choose σ in that way, that our translations are given as rotations. And Dilations are given as action of Lorentz boosts. Geometrically these action can be identified with the dilation on the sphere, which we obtain by stereographic projection of the dilations in the tangent space at the north pole of S^n [7]. The action of the abelian subgroup can be obtained in the same way by stereographic projection of the translations in the tangent space.

2.3 Wavelet transformation on the Sphere S^n

In general as well as on the sphere, the wavelets transform is defined as L^2-scalar product of the dilated and translated wavelet and the function $f \in L^2(S^n)$, which we want to transform.

Definition 2.2. If Ψ is an admissible vector in $L^2(S^n)$, the wavelet transform $WT(f): \; L^2(S^n) \to L^2(X)$ is defined by

$$WT(f)(x) := \int_{S^n} U(\sigma(x))[\Psi](\xi)\; f(\xi)\; \mathrm{d}\mu(\xi), \quad \forall f \in L^2(S^n).$$

The Reconstruction formula, holds in L^2-sense:

$$f(\xi) := \int_X WT(f)(x)\; U(\sigma(x))[\Psi](\xi)\; \mathrm{d}\nu(\sigma(x)),$$

where $\nu(x)$ is a quasi invariant measure on G/H [4].

3 PRELIMINARY DEFINITIONS AND THEOREMS

3.1 Diffusion semigroups

During the construction of diffusion wavelets we separate the investigations about dilations and translations into the action of an diffusion operator and that of a space operator. Naturally translation on S^n is given by regular representation of $SO(n+1)$ in $L^2(S^n)$.
Let us introduce the diffusion way of dilations:

Definition 3.1. Let $\{D^\rho,\, \rho > 0\}$ be a family of operators, mapping $L^2(M)$ into itself and forming a semigroup

$$\lim_{\rho \to 0} D^\rho = Id \tag{3.1}$$

$$D^{\rho_1} D^{\rho_1} = D^{\rho_1 + \rho_2} \tag{3.2}$$

$$\|D^\rho\|_{L^p} \leq C, \quad \text{independent of } \rho \tag{3.3}$$

Further positivity $D^\rho f \geq 0$ for all $f \geq 0$ in $L^2(S^n)$ is required. Then $\{D^\rho,\, \rho > 0\}$ is called *diffusion semigroup*.

The dilation operator is given as action of a diffusion semigroup. The concept is introduced for discrete wavelets by R. Coifman in [6]

3.2 Functions on S^n

For integrations on S^n we use polar coordinates, where the euler angles $(\theta_1, \ldots, \theta_n)$ satisfy $\theta_1 \in [0,\, 2\pi)$, $\theta_i \in [0, \pi]$, $i = 2, \ldots, n$ and the usual rotation invariant measure. The measure of the surface of S^n is given by

$$\Omega_n = \int_{S^n} 1 \, \mathrm{d}\mu(\xi) = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)}, \qquad \lambda := \frac{n-1}{2}. \tag{3.4}$$

In $L^2(S^n)$ we have the orthonaormal system $\{Y_k^i,\, i = 1, \ldots, d_k(n)\}$, of spherical harmonics. Hereby, one has

$$d_k(n) = (2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}$$

linearly independent spherical harmonics of degree k. For $f \in L^2(S^n)$ we have the series expansion

$$f(\xi) = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k(n)} \hat{f}(k,i) Y_k^i(\xi),$$

the Fourier coefficients given by $\hat{f}(k,i) = \int_{S^n} f(\xi) Y_k^i(\xi) \, \mathrm{d}\mu(\xi)$.
Of particular importance is the notion of a zonal function.

Definition 3.2. A function $f \in L^p(S^n)$ is called to be zonal with respect to $\xi \in S^n$ if $\eta \cdot \xi = \zeta \cdot \xi$ implies $f(\eta) = f(\zeta)$.

Zonal functions depends only on the angle between the argument η and the point to which they are zonal. In other words, they are constant on the intersection of the sphere with a hyperplane, which is orthogonal to the one dimensional subspace in $\mathbb{R}^{n+1}$ containing ξ. These sections are spheres of dimension $n-1$ with radius $\sin(\arccos \xi \cdot \eta) = \sin\theta$, so they have a measure of $\Omega_{n-1} (\sin\theta)^{2\lambda}$, where θ denotes the angle between ξ and η. Therefore, a function $f \in L^p(S^n)$ being zonal with respect to ξ has L^p-norm

$$\|f\|_{L^p(S^n)}^p = \Omega_{n-1} \int_0^{\pi} |\underline{f}(\theta)|^p (\sin\theta)^{2\lambda} \, \mathrm{d}\theta,$$

where $\underline{f}(\theta) := f(\eta)$ with $\eta \cdot \xi = \cos\theta$. Without loss of generality, if nothing else is said, we will consider zonal functions to be zonal with respect to the north pole N.

Definition 3.3. The space of p-integrable zonal functions on S^n is denoted by L^p_λ. We say $f \in L^p_\lambda$ if the following norm is finite:

$$\|f\|_{p,\lambda} := \left(\frac{\Omega_{n-1}}{\Omega_n} \int_0^{\pi} |f(\theta)|^p (\sin\theta)^{2\lambda} \, \mathrm{d}\theta \right)^{\frac{1}{p}}$$

In $L^2([0,\pi],\sin^{2\lambda}\ \mathrm{d}\theta)$ we have the orthogonal system of Gegenbauer polynomials of order λ. This provides us with a useful connection between $L^2([0,\pi])$ and L^2_λ, since the norms with respect to the measures $\sin^{2\lambda}\ \mathrm{d}\theta$ and $\mathrm{d}\theta$ are equivalent. It is quit natural, that the weight function is the relation between the radius of a sphere and the surface of a sphere in $\mathbb{R}^n$.
In [3] one finds many useful tools about functions on S^n. We will just mention a few which will be used later on.

Theorem 3.4 (Addition theorem)**.**

$$C_k^\lambda(\xi\cdot\eta)=\frac{2\pi^{\lambda+1}}{(k+\lambda)\Gamma(\lambda)}\sum_{i=1}^{d_k}Y_k^i(\xi)Y_k^i(\eta)=\Omega_n\frac{\lambda}{(k+\lambda)}\sum_{i=1}^{d_k}Y_k^i(\xi)Y_k^i(\eta);\quad \xi,\,\eta\in S^n,\tag{3.5}$$

where C_k^λ denotes the Gegenbauer polynomial of degree k and order λ.

In [10] we find the normalization

$$\int_0^\pi C_l^\lambda(\cos\theta)C_k^\lambda(\cos\theta)\,(\sin\theta)^{2\lambda}\ \mathrm{d}\theta=\frac{\pi 2^{1-2\lambda}\Gamma(k+2\lambda)}{k!(\lambda+k)\,(\Gamma(\lambda))^2}\delta_{k,l},\tag{3.6}$$

where $\delta_{k,l}$ denotes the Kroneka symbol.

Theorem 3.5 (Funk- Hecke formula)**.** *For a zonal function $f\in L^1_\lambda(S^n)$ it holds*

$$\int_{S^n}f(\xi\cdot\eta)Y_k^i(\eta)\,\mathrm{d}\mu(\eta)=Y_k^i(\xi)\frac{(4\pi)^\lambda\Gamma(\lambda)\Gamma(k+1)}{\Gamma(k+2\lambda)}\int_0^\pi \underline{f}(\theta)C_k^\lambda(\cos\theta)(\sin\theta)^{2\lambda}\,\mathrm{d}\theta.\tag{3.7}$$

Using the theorem above, for Gegenbauer polynomials we have

$$\int_{S^n}f(\xi\cdot\eta)C_k^\lambda(\zeta\cdot\eta)\,\mathrm{d}\mu(\eta)=C_k^\lambda(\zeta\cdot\xi)\frac{(4\pi)^\lambda\Gamma(\lambda)\Gamma(k+1)}{\Gamma(k+2\lambda)}\int_0^\pi \underline{f}(\theta)C_k^\lambda(\cos\theta)(\sin\theta)^{2\lambda}\,\mathrm{d}\theta\tag{3.8}$$

With these two theorems we find immediately

Proposition 3.6. A zonal function $f\in L^2_\lambda(S^n)$ can be expanded in a series of Gegenbauer polynomials

$$f(\eta)=\sum_{K=0}^\infty\sum_{i=1}^{d_k}\hat{f}(k,i)Y_k^i(\eta)=\sum_{k=0}^\infty\tilde{f}(k)C_k^\lambda(\xi\cdot\eta),$$

where we denote by $\tilde{f}(k)$ the Gegenbauer coefficient of degree k.

$$\tilde{f}(k)=\frac{\Omega_{n-1}}{\Omega_n}\int_0^\pi \underline{f}(\theta)C_k^\lambda(\cos\theta)(\sin\theta)^{2\lambda}\,\mathrm{d}\theta.\tag{3.9}$$

3.3 Approximate identities

In this section we take a closer look into approximate identities on function spaces over the sphere. Hereby, we will make use of the fact, that convolution with a bounded L^1_λ-function forms a continuous, linear and bounded operator from L^p onto itself. This is easily seen by Young's inequality

$$\|f * g\|_r \leq \|f\|_p \, \|g\|_{q,\lambda} \qquad \left(\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0\right), \tag{3.10}$$

using $r = p$.

Definition 3.7. For a family $\{K_\rho,\ \rho > 0\} \subset L^1_\lambda(S^n)$ of integrable, zonal functions, with

$$\|K_\rho * f\|_{L^2(S^n)} \leq M\|f\|_{L^2(S^n)} \tag{3.11}$$

$$\lim_{\rho\to 0} \|K_\rho * f - f\|_{L^2(S^n)} = 0 \tag{3.12}$$

the family of convolution operators $\{K_\rho *,\ \rho > 0\}$ forms an approximate identity with kernel $\{K_\rho,\ \rho > 0\}$.

Theorem 3.8. *If* (3.11) *is satisfied, the condition* (3.12) *is equivalent too*

$$\lim_{\rho\to 0} \widetilde{K}_\rho(k) = \frac{k+\lambda}{\lambda}, \quad \forall\, k \in \mathbb{N} \tag{3.13}$$

Proof. Due to density argument it is enough to show the reconstruction property for the system of spherical harmonics. Applying (3.7) and (3.9) one finds

$$(\Phi_\rho * Y^i_k)(\xi) = \frac{1}{\Omega_n} \int_{S^n} \Phi_\rho(\xi\cdot\eta) Y^i_k(\eta)\, \mathrm{d}\mu(\eta) = Y^i_k(\xi) \frac{\lambda}{k+\lambda} \tilde{\Phi}_\rho(k)\,,$$

where the assertion now follows from $\rho \to 0$. □

For us the most important example of an approximate identity is given as fundamental solution of spherical heat equation $(\Delta_* - \partial_t)u = 0$ (where Δ_* denotes the Laplace-Beltrami operator) the spherical heat kernel (Gauss-Weierstrass kernel)

$$e_{heat}(\xi, \rho) = \sum_{k=0}^{\infty} e^{-k(k+2)\rho} \frac{k+\lambda}{\lambda} C^\lambda_k(\eta\cdot\xi),$$

which is zonal with respect to η.
Additionally it forms a diffusion semigroup, that is why we are calling the corresponding wavelets-diffusion wavelets.
Variation of the Parameter ρ corresponds to the action of a dilation.

4 WIGNER POLYNOMIALS ON $SO(N+1)$

For zonal wavelets translations on the sphere are parameterised by S^n: This comes from the fact, that zonal functions are invariant under some rotations in $SO(n+1)$. Let f be a function, zonal with respect to $\eta \in S^n$. For $g \in SO(n+1)$, the regular representation T of $SO(n+1)$ will map it onto a function, zonal with respect to $g^{-1}(\eta)$. For all rotations $g' \in SO(n+1)$ with $g'^{-1}(\eta) = g^{-1}(\eta)$ this will give the same result. All these g' form a so called great circle in $SO(n+1)$, that is a subgroup, isomorphic to $SO(n)$. We can factorize out all these $g' \in SO(n+1)$, which leaves a zonal function invariant. $SO(n+1)/SO(n) \simeq S^n$.

If we are dealing with nonzonal wavelets we have to take into account the whole $SO(n+1)$.

Definition 4.1. The Wigner polynomials $\{T_k^{ij},\, i,j = 1,\dots,d_k(n)\}$ on $SO(n+1)$ are given by the regular representation $SO(n+1)$ in $L^2(S^n)$:

$$Y_k^i(g^{-1}(\xi)) = \sum_{j=1}^{d_k(n)} T_k^{ij}(g) Y_k^j(\xi). \tag{4.1}$$

Remark 4.2. Invariant subspaces: $Harm_k := span\{Y_k^i,\, i = \dots, d_k(n)\}$

Matrix $\left(T_k^{ij}\right)_{i,j=1}^{d_k(n)}$ maps orthogonal systems to orthogonal ones, hence it is unitary and T_k^{ij} are well defined and unique. Moreover, this implies

$$\sum_{j=1}^{d_k(n)} T_k^{ij}(g) T_k^{jl}(g) = \delta_{il} \tag{4.2}$$

for all $g \in SO(n+1)$.

On $SO(n+1)$ we use $\mathrm{d}\nu(g)$, the normalized Haar measure. In a first step we verify, that T_k^{ij} are indeed polynomials.

Lemma 4.3. *For all $i,j = 1,\dots,d_k(n)$, we have that T_k^{ij} is a polynomial of degree k.*

Proof. Since we know that the rotation invariant measure on S^n and the Haar measure on $SO(n+1)$ coincides [12] we have

$$\partial_g^{k+1}\, Y_k^i(g^{-1}(\xi)) = 0.$$

By property (4.1) we have

$$\partial_g^{k+1}\, Y_k^i(g^{-1}(\xi)) = \sum_{j=1}^{d_k(n)} \left(\partial_g^{k+1}\, T_k^{ij}\right)(g) Y_k^j(\xi).$$

Using the linear independence of Y_k^j we get

$$\partial_g^{k+1} T_k^{ij} = 0, \tag{4.3}$$

i.e. T_k^{ij} is a polynomial of degree k. □

Lemma 4.4 (Orthogonality). *Wigner polynomials $\{T_k^{ij},\ i,j = 1,\dots,d_k(n), k \in \mathbb{Z}\}$ are orthogonal in $L^2(SO(n+1))$.*

To verify the orthogonality we recall the orthogonality relation for locally compact group (see [11]). In terms of our group at hand, $SO(n+1)$, we get the following theorem.

Theorem 4.5. *The regular representation $T(g)\ :\ f(\xi) \mapsto f(g^{-1}(\xi))$ of the locally compact group $SO(n+1)$ in $Harm_k$ is irreducible and square integrable. Therefore, there exists a self adjoint operator C over the set of all admissible vectors in $Harm_k$, such that for admissible vectors $v_1,\ v_2 \in Harm_k$ and arbitrary $u_1,\ u_2 \in Harm_k$ it holds:*

$$\int_{SO(n+1)} \langle U(g)v_1, u_1\rangle_{L^2(S^n)} \langle U(g)v_1, u_1\rangle_{L^2(S^n)}\ \mathrm{d}\nu(g) = \langle Cv_1, Cv_2\rangle_{L^2(S^n)} \langle u_1, u_2\rangle_{L^2(S^n)}.$$

Because $SO(n+1)$ is unimodular, C is the identity up to a constant, i.e. $C = c * Id$, $c \in \mathbb{R}$.

Since $SO(n+1)$ is not only locally compact but compact, every $f \in Harm_k$ is square integrable with respect to representation T, i.e. $\int_{SO(n+1)} \|T(g)f\|_{L^2(S^n)}\ \mathrm{d}\nu(g) < \infty$. Let us choose $v_1 = Y_k^i$, $v_2 = Y_k^j$, $u_1 = Y_k^r$, $u_2 = Y_k^s$. Our orthogonality relation leads to

$$\begin{aligned}&\int_{SO(n+1)} \langle T(g)Y_k^i,\ Y_k^r\rangle_{L^2(S^n)} \langle T(g)Y_k^j,\ Y_k^s\rangle_{L^2(S^n)}\ \mathrm{d}\nu(g) = \int_{SO(n+1)} T_k^{ir}(g) T_k^{js}(g)\ \mathrm{d}\nu(g)\\ =&c\langle Y_k^i,\ Y_k^j\rangle_{L^2(S^n)} \langle Y_k^r,\ Y_k^s\rangle_{L^2(S^n)} = c\delta_{ij}\,\delta_{rs} \qquad \text{with } c \in \mathbb{R}.\end{aligned}$$

This implies

$$\int_{SO(n+1)} \left(T_k^{ij}(g)\right)^2\ \mathrm{d}\nu(g) = \int_{SO(n+1)} \left(T_k^{rs}(g)\right)^2\ \mathrm{d}\nu(g), \quad \forall 1 \le i,j,r,s \le d_k(n)$$

From (4.2) we know that $\sum_{i=1}^{d_k(n)} \int_{SO(n+1)} \left(T_k^{ij}(g)\right)^2\ \mathrm{d}\nu(g) = 1$, and, hence,

$$\|T_k^{ij}\|_{L^2(SO(n+1))} = \frac{1}{d_k(n)}. \tag{4.4}$$

To obtain the orthogonality with respect to k, we extend the space of rotation representing matrices $g \in SO(n+1)$ to those of the form $\{\lambda g,\ g \in SO(n+1),\ \lambda \in \mathbb{R}_+\}$. From the homogeneity of degree k of $Y_k^i(x)$ and the definition of our Wigner functions (4.1) we deduce the homogeneity with respect to g of T_k^{ij}

$$T_k^{ij}(\lambda\, g) = \left(\frac{1}{\lambda}\right)^k T_k^{ij}(g).$$

So T_k^{ij} and T_k^{ij} (for $k \neq l$) are homogeneous of different degrees and hence orthogonal.

Theorem 4.6. *T_k^{ij} are polynomials of degree k. They are orthogonal in i, j, k and homogeneous of degree $-k$. The norm is*

$$\|T_k^{ij}\|_{L^2(SO(n+1))} = \frac{1}{d_k(n)},$$

where $d_k(n)$ is the number of spherical harmonics of degree k on S^n.

For the applications in the next section we have to introduce the notion of the zonal product.

Definition 4.7. For $f, h \in L^2(S^n)$ the zonal product is defined by

$$(f \hat{*} g)(\xi \cdot \eta) = \int_{SO(n+1)} f(g^{-1}(\xi))h(g^{-1}(\eta))\,\mathrm{d}\nu(g), \quad \xi, \eta \in S^n\,. \tag{4.5}$$

Corollary 4.8. Zonal product gives a zonal function on S^n for arbitrary $f, h \in L^2(S^n)$. This can be seen by the characterization in terms of spherical harmonics and Gegenbauer polynomials

$$\begin{aligned}
(f \hat{*} h)(\xi \cdot \eta) &= \sum_{k=0}^{\infty} \sum_{i=1}^{d_k(n)} \sum_{l=0}^{\infty} \sum_{j}^{d_l(n)} \hat{f}(k,i)\hat{h}(l,j) \int_{SO(n+1)} Y_k^i(g^{-1}(\xi))Y_l^j(g^{-1}(\eta))\,\mathrm{d}\nu(g) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^{d_k(n)} \sum_{l=0}^{\infty} \sum_{j=1}^{d_l(n)} \hat{f}(k,i)\hat{h}(l,j) \sum_{i'=1}^{d_k(n)} \sum_{j'=1}^{d_l(n)} Y_l^{i'}(\xi)Y_l^{j'}(\eta) \int\limits_{SO(n+1)} T_k^{ii'}(g)\, T_l^{jj'}(g)\,\mathrm{d}\nu(g) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^{d_k(n)} \hat{f}(k,i)\hat{h}(k,i) \frac{1}{d_k(n)} \sum_{i'=1}^{d_k(n)} Y_k^{i'}(\xi)Y_k^{i'}(\eta) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^{d_k(n)} \hat{f}(k,i)\hat{h}(k,i) \frac{1}{d_k(n)\Omega_n} \frac{k+\lambda}{\lambda} C_k^{\lambda}(\xi \cdot \eta)\,.
\end{aligned} \tag{4.6}$$

Notation: $T(g)f(\xi) = f(g^{-1}(\xi)) =: f_g(\xi)$.

5 WAVELETS ON S^N

5.1 General wavelets conditions

In this section we use the previous introduced tools to formulate the wavelet conditions in the general case, that holds for nonzonal wavelet too. As well the simplification for zonal wavelets well be given. Furthermore we give the connection between the group theoretical approach and that one we used.

Definition 5.1. Let $\alpha(\rho)$ be a positive weight function. A family $\{\Psi_\rho, \rho > 0\} \subset L^2(S^n)$ of functions

$$\Psi_\rho = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k(n)} \hat{\Psi}_\rho(k,i) Y_k^i \in L^2(S^n),$$

satisfying the admissibility conditions

$$\sum_{i=1}^{d_k} \int_0^\infty \hat{\Psi}_\rho(k,i)^2 \alpha(\rho)\, \mathrm{d}\rho = \Omega_n\, d_k(n) \tag{5.1}$$

$$\int_{S^n} \left| \int_R^\infty \left(\Psi_\rho \hat{*} \Psi_\rho\right)(\xi \cdot \eta) \alpha(\rho)\, \mathrm{d}\rho \right| \, \mathrm{d}\mu(\eta) \le M \quad \forall\, \xi \in S^n \tag{5.2}$$

(M independent of R), forms a family of bilinear wavelets.

Definition 5.2. The corresponding Wavelet transform $(WT)(f) : L^2(S^n) \to L^2(\mathbb{R} \times SO(n+1))$ is defined by

$$(WT)(f)(\rho, g) := \frac{1}{\Omega_n} \int_{S^n} \Psi_\rho(g^{-1}(\eta)) f(\eta)\, \mathrm{d}\mu(\eta), \quad f \in L^2(S^n) \tag{5.3}$$

Theorem 5.3. *The following reconstruction formula holds in L^2-sense:*

$$f(\xi) = \int_{SO(n+1)} \int_0^\infty (WT)(f)(\rho, g) \Psi_{\rho,g}(\xi) \alpha(\rho)\, \mathrm{d}\rho\, \mathrm{d}\nu(g) \tag{5.4}$$

Proof.

$$\begin{aligned}
&\lim_{R\to 0} \int_{SO(n+1)} \int_R^\infty \Psi_{\rho,g}(\xi) WT(f)(\rho,g) \alpha(\rho)\, \mathrm{d}\rho\, \mathrm{d}\nu(g) \\
&= \frac{1}{\Omega_n} \lim_{R\to 0} \int_{SO(n+1)} \int_R^\infty \Psi_{\rho,g}(\xi) \int_{S^n} \Psi_{\rho,g}(\eta) f(\eta)\, \mathrm{d}\mu(\eta) \alpha(\rho)\, \mathrm{d}\rho\, \mathrm{d}\nu(g) \\
&= \frac{1}{\Omega_n} \lim_{R\to 0} \int_{S^n} \int_R^\infty \underbrace{\int_{SO(n+1)} \Psi_\rho(g^{-1}(\xi)) \Psi_\rho(g^{-1}(\eta))\, \mathrm{d}\nu(g)}_{=(\Psi_\rho \hat{*} \Psi_\rho)(\xi\cdot\eta)} \alpha(\rho)\, \mathrm{d}\rho f(\eta)\, \mathrm{d}\mu(\eta)
\end{aligned}$$

We call in the result (4.6) of calculation of $(\Psi_\rho \hat{*} \Psi_\rho)(\xi \cdot \eta)$ and obtain

$$= \lim_{R\to 0} \frac{1}{\Omega_n} \int_{S^n} \underbrace{\sum_{k=0}^\infty \frac{k+\lambda}{\lambda} \frac{1}{\Omega_n\, d_k(n)} \sum_{i=1}^{d_k} \int_R^\infty \left(\hat{\Psi}_\rho(k,i)\right)^2 \alpha(\rho)\, \mathrm{d}\rho\, C_k^\lambda(\xi \cdot \eta)}_{:=\Xi_\rho(\xi\cdot\eta)} f(\eta)\, \mathrm{d}\mu(\eta)$$

We have to identify $\{\Xi_\rho,\, \rho > 0\}$ as approximate identity. With $\Xi_R(\xi \cdot \eta) = \int_R^\infty (\Psi_\rho \hat{*} \Psi_\rho)(\xi \cdot \eta) \alpha(\rho)\, \mathrm{d}\rho$ from admissibility condition(5.2) we can deduce, that $\{\Xi_\rho,\, \rho > 0\}$ is uniformly bounded in $L^1_\lambda(S^n)$. Admissibility condition (5.1) is ensured by

$$\begin{aligned}
\lim_{R\to 0} \tilde{\Xi}_R(k) &= \frac{k+\lambda}{\lambda} \frac{1}{\Omega_n\, d_k(n)} \sum_{i=1}^{d_k} \int_R^\infty \left(\hat{\Psi}_\rho(k,i)\right)^2 \alpha(\rho)\, \mathrm{d}\rho \\
&= \frac{k+\lambda}{\lambda}
\end{aligned}$$

□

5.2 Zonal wavelets

A Zonal function can be expanded in a Fourier series as well as in a Gegenbauer Series. Between Frourier and Gegenbauer coefficients of a zonal function Φ_ρ we have the relation

$$\hat{\Phi}_\rho(k,i) = \Omega_n \widetilde{\Phi}_\rho(k) \frac{\lambda}{k+\lambda} Y_k^i(\eta). \tag{5.5}$$

Applying that to the admissibility conditions in Definition 5.1, we obtain the following

Remark 5.4. Let $\alpha(\rho)$ be a positive weight function. A family $\{\Phi_\rho,\ \rho > 0\} \subset L_1^2(S^n)$ of zonal functions $\Phi_\rho = \sum_{k=0}^{\infty} \widetilde{\Phi}_\rho(k) C_k^\lambda$, satisfying the admissibility conditions

$$\int_0^\infty \widetilde{\Phi}_\rho^2(k)\alpha(\rho)\,\mathrm{d}\rho = \left(\frac{k+\lambda}{\lambda}\right)^2 \tag{5.6}$$

$$\int_{S^n} \left| \int_R^\infty \left(\Phi_\rho * \Phi_\rho\right)(\xi\cdot\eta)\alpha(\rho)\,\mathrm{d}\rho \right| \mathrm{d}\mu(\eta) \le M \quad \forall\, \xi,\, \eta \in S^n \tag{5.7}$$

(M independent of R), forms a family of zonal Wavelets.

Similar to the nonzonal case the corresponding Wavelet transform $(WT)(f) : L^2(S^n) \to L^2(\mathbb{R}\times S^n)$ is defined by

$$(WT)(f)(\rho,\xi) := \frac{1}{\Omega_n} \int_{S^n} \Phi_\rho(\xi\cdot\eta)) f(\eta)\,\mathrm{d}\mu(\eta), \quad f \in L^2(S^n), \tag{5.8}$$

and is further invertible on its range:

$$f \;=\; \frac{1}{2\pi^2} \int_{S^n} \int_0^\infty (WT)(f)(\rho,\,\eta)\Psi_{\rho,\eta}(\cdot)\alpha(\rho)\,\mathrm{d}\rho\,\mathrm{d}\mu(\eta), \quad \forall f \in L^2(S^n).$$

5.3 Relation to group theoretical admissibility

As wavelet forming actions on S^n we have translations $\gamma \in SO(n+1)$ and dilations, parameterized by $\mathbb{R}_+$. The representation of the corresponding section over $SO(n+1), \mathbb{R}^+$ is:

$$\begin{aligned}
U(\sigma(\rho,\gamma))[\Psi_1](\xi) &= (T(\gamma)\circ D(\rho))[\Psi_1](\xi) \\
D(\rho)[\Psi_1](\xi) &= \Psi_\rho(\xi) \\
T(\gamma)[\Psi_\rho](\xi) &= \Psi_{\rho,\gamma}(\xi) = \Psi_\rho(\gamma^{-1}(\xi)).
\end{aligned}$$

Assuming, that our admissibility conditions (5.1) and (5.2) are satisfied, in terms of square integrability condition we write

$$\begin{aligned}
&\int_X \left|\langle f,\, U(\sigma(x))[\Psi_1]\rangle_{L^2(S^n)}\right|^2 \,\mathrm{d}\nu(x) \\
&= \int_{SO(n+1)} \int_0^\infty \left|\langle f,\, \Psi_{\rho,g}\rangle_{L^2(S^n)}\right|^2 \alpha(\rho)\,\mathrm{d}\rho\,\,\mathrm{d}\nu(g) \\
&= \int_{SO(n+1)} \int_0^\infty \left|\sum_{k=0}^{\infty}\sum_{i=1}^{d_k(n)} \hat{f}(k,i) \sum_{j=1}^{d_k(n)} T_k^{ji}(g)\hat{\Psi}_\rho(k,j)\right|^2 \alpha(\rho)\,\mathrm{d}\rho\,\mathrm{d}\nu(g) \\
&= \sum_{k=0}^{\infty}\sum_{i=1}^{d_k(n)} |\hat{f}(k,i)|^2 \sum_{j=1}^{d_k(n)} \int_0^\infty \hat{\Psi}_\rho^2(k,j)\alpha(\rho)\,\mathrm{d}\rho \int_{SO(n+1)} T_k^{ji}(g)T_k^{ji}(g)\,\mathrm{d}\nu(g) \\
&= \Omega_n \sum_{k=m+1}^{\infty}\sum_{i=1}^{d_k(n)} |\hat{f}(k,i)|^2 = \Omega_n\,\|f\|_{L^2(S^n)} < \infty, \qquad \forall f \in L^2(S^n).
\end{aligned}$$

6 CONSTRUCTION OF WAVELETS

6.1 Construction of zonal wavelets

Theorem 6.1. *We suppose* $\{\Phi_\rho,\, \rho > 0\}$ *to be a non negative kernel of an approximate identity. Further the Gegenbauer coefficients* $\widetilde{\Phi}_\rho$ *shall be differentiable with respect to* ρ *and* $\lim_{\rho\to\infty} \widetilde{\Phi}_\rho(k) = 0$*, for all* $k \geq 1$*. The corresponding wavelet is defined due its Gegenbauer coefficients, which are given by*

$$\widetilde{\Psi}_\rho(k) \;=\; \left(-\alpha(\rho)^{-1}\frac{\mathrm{d}}{\mathrm{d}\rho}\widetilde{\Phi}_\rho^2(k)\right)^{\frac{1}{2}}, \quad \forall\,\rho \in (0,\,\infty) \text{ and } k = 0, 1, \ldots$$

By the previous assumptions the Gegenbauer-coefficients are well-defined and non-negative.

The wavelet admissibility condition (5.6) is satisfied, since

$$\lim_{R\to 0}\int_R^\infty -\alpha(\rho)^{-1}\frac{\mathrm{d}}{\mathrm{d}\rho}\widetilde{\Phi}_\rho^2(k)\alpha(\rho)\,\mathrm{d}(\rho) = \lim_{R\to 0}\widetilde{\Phi}_R^2(k) \qquad = \left(\frac{k+\lambda}{\lambda}\right)^2. \tag{6.1}$$

As well the second condition (5.7) is satisfied: Expanding Ψ_ρ in Gegenbauer series and applying the orthogonality (3.6) of Gegenbauer polynomials under consideration of Fuck-Hecke theorem we calculate

$$\int_R^\infty (\Psi_\rho * \Psi_\rho)(\eta\cdot\xi)\,\mathrm{d}\rho = \sum_{k=0}^{\infty}\widetilde{\Phi}_\rho^2(k)\frac{\lambda}{k+\lambda}C_k^\lambda(\eta\cdot\xi).$$

Utilizing again Fuck-Hecke theorem and (3.6) we deduce that this is equal to

$$(\Phi_R * \Phi_R)(\xi \cdot \eta).$$

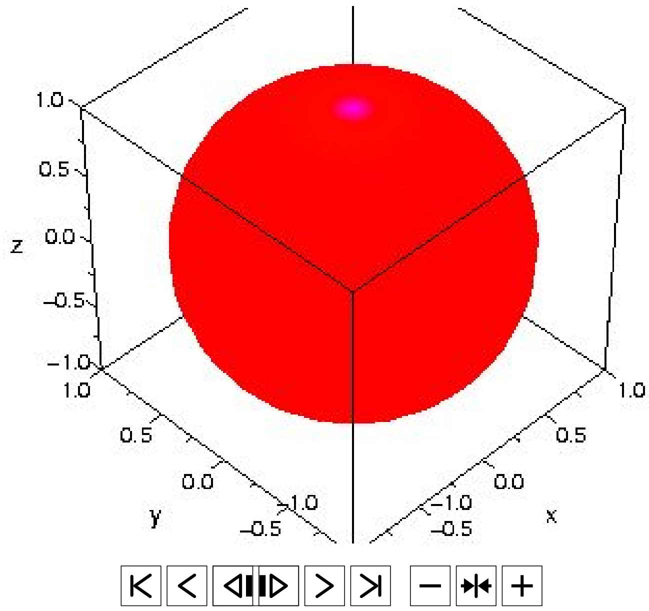
Because $\{\Phi_\rho,\, \rho > 0\}$ as approximate identity is uniformly bounded in $L^1(S^n)$, by Young's inequality (3.10) we find

$$\int_{S^n} |(\Phi_R * \Phi_R)(\xi \cdot \eta)| \,\mathrm{d}\mu(\eta) \le \|\Phi_R\|_{L^1} \|\Phi_R\|_{L^1_1} < T', \quad \text{independent of } R.$$

In the case of S^3 the reader finds detailed calculations in [7], for S^n under consideration of (3.6) the calculation are similar.
For our example of the heat kernel on S^n we obtain the following Gegenbauer coefficients for the corresponding zonal wavelet:

$$\widetilde{\Psi}_\rho(k) = \sqrt{\alpha(\rho)^{-1}\, 2\,(k+n-1)}\frac{2k+n-1}{n-1} e^{-k(k+n-1)\rho}.$$

The following animation shows the wavelet corresponding to the approximate identity of Gauss-Weierstrßfor growing values of ρ. We can observe the localization property for ρ tending to 0.



6.2 Construction of nonzonal wavelets

A fundamental fact, that we utilize is(5.5). There occurs $Y_k^i(\eta)$ asweight to distribute the Gegenbauer coefficient of the zonalfunction to $d_k(n)$ Fourier coefficients of the same degree. Fornonzonal functions the weights can be changed.

Definition 6.2. Let w_k be a vector in $\mathbb{R}^{d_k(n)}$ with components$w_k(i)$. w_k is called to be an admissible weight vector, if it satisfies

$$\sum_{k=i}^{d_k(n)} (w_k(i))^2 = \frac{d_k(n)}{\Omega_n}. \tag{6.2}$$

We will see, that the concept of admissible weight vectors allows usto construct nonzonal wavelets with a construction, adapted from thezonal case.

Theorem 6.3. *Let $\{w_k,\ k \in \mathbb{N}\}$ be a family of admissible weightvectors. Assuming an approximate identity $\{\Phi_\rho,\ \rho > 0\}$,satisfying the assumption of Theorem 6.1, the Fourier expansion of anonzonal Wavelet Ψ_ρ on S^n corresponding to thatapproximate identity has the form*

$$\Psi_\rho(\eta) = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k(n)} \left(-\alpha(\rho)^{-1} \frac{\mathrm{d}}{\mathrm{d}\rho} \widetilde{\Phi}_\rho^2(k) \right)^{\frac{1}{2}} \frac{\Omega_n \lambda}{k+\lambda} w_k(i)\ Y_k^i(\eta). \tag{6.3}$$

The zonal case is included as special case, where$w_k(i) = Y_k^i(\eta)$ for an $\eta \in S^n$.

Let us briefly verify, that the wavelet admissibilityconditions are satisfied. (5.1) weobtain by

straight calculation

$$\lim_{R\to 0} \sum_{i=1}^{d_k(n)} \int_R^\infty -\frac{\mathrm{d}}{\mathrm{d}\rho} \widetilde{\Phi}_\rho^2(k)\ \mathrm{d}\rho \left(\frac{\Omega_n \lambda}{k+\lambda} w_k(i) \right)^2 = \lim_{R\to 0} \widetilde{\Phi}_\rho^2(k) \left(\frac{\lambda}{k+\lambda} \right)^2 \Omega_n d_k(n)$$

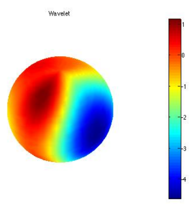
which is equal to $\Omega_n d_k(n)$ because $\{\Phi_\rho,\ \rho\}$is the kernel of an approximate identity.The second admissibility condition(5.2) follows like in the zonalcase, from the fact that

$$\begin{aligned}\int_R^\infty (\Psi_\rho \hat{*} \Psi_\rho)(\xi\cdot\eta)\alpha(\rho)\ \mathrm{d}\rho &= \sum_{k=0}^{\infty} \sum_{i=1}^{d_k(n)} \int_R^\infty -\frac{\mathrm{d}}{\mathrm{d}\rho} \widetilde{\Phi}_\rho^2(k)\ \mathrm{d}\rho\ w_k(i)^2 \frac{\Omega_n}{d_k(n)} \frac{\lambda}{k+\lambda} C_k^\lambda(\xi\cdot\eta) \\ &= \sum_{k=0}^{\infty} \widetilde{\Phi}_R^2(k) \frac{\lambda}{k+\lambda} C_k^\lambda(\xi\cdot\eta).\end{aligned}$$

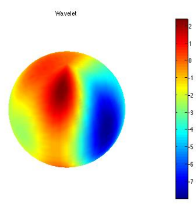
The simplest choice of $w_k(i)$ is the constant one, given by$w_k(i) = \frac{1}{\sqrt{\Omega_n}}$. If we choose it in that way, forthe example on S^2 with $\alpha(\rho) = 1\rho^3$ we obtainthe following fourier expansion of the wavelet corresponding to theWeierstraß kernel:

$$\Psi_\rho(\eta) = \sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} \sqrt{2k(k+1)\rho^3} 2\sqrt{\pi} e^{-k(k+1)\rho} Y_k^i(\eta). \tag{6.4}$$

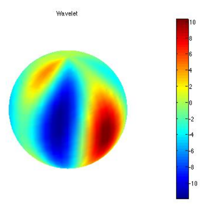
In the following figures you find this wavelet visualized fordifferent values of for ρ.



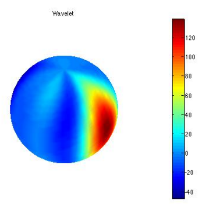
$\rho = 1$



$\rho = 0.7$



$\rho = 0.5$



$\rho = 0.3$

7 OUTLOOK

The concept of construction of diffusion wavelets from theheat kernel is a very powerful one. It can be applied to manyother manifolds. For example to the torusLet us recall that the fundamental solution to the diffusionequation of the full space $\mathbb{R}^n \times \mathbb{R}^+$ has the form

$$e^H_{heat}(\mathbf{x};t) = \frac{H(t)}{(2\sqrt{\pi t})^n} e^{-\frac{\|\mathbf{x}\|_2^2}{4t}}$$

where $H(\cdot)$ stands for the usual Heavy-side function.Let $1 \leq k \leq n$. Let $\omega_1, \ldots, \omega_k$ be $k\mathbb{R}$-linearly independent vectors. The associated generated$k$-dimensional lattice in $\mathbb{R}^n$ then is the set of points

$$\Omega_k = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_k.$$

Theorem 7.1. *The series*

$$\wp^H_{heat;k}(\mathbf{x};t) := \sum_{\omega \in \Omega_k} e^H_{heat}(\mathbf{x}+\omega;t) \tag{7.1}$$

is normally convergent and represents a non-vanishing k-foldperiodic function in $\mathbb{R}^n \times \mathbb{R}^+$ satisfying in each point of$\mathbb{R}^n \times \mathbb{R}^+$ the diffusion equation $(\Delta_{\mathbf{x}} - \partial_t)\wp_{0,\cdots,0}(\mathbf{x};t) = 0$.

This is the starting point to construct diffusionwavelets on conformally flat torus and cylinders in athe same way we have done on S^n.

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