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VECTOR AND BIVECTOR FOURIER TRANSFORMS IN CLIFFORD ANALYSIS

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Abstract. *In the past, several types of Fourier transforms in Clifford analysis have been studied. In this paper, first an overview of these different transforms is given. Next, a new equation in a Clifford algebra is proposed, the solutions of which will act as kernels of a new class of generalized Fourier transforms. Two solutions of this equation are studied in more detail, namely a vector-valued solution and a bivector-valued solution, as well as the associated integral transforms.*

1 INTRODUCTION

The classical Fourier transform is a mathematical tool of the utmost importance in harmonic analysis and has of course an enormous number of applications in virtually all branches of physics and engineering.

The theory of Clifford analysis, in its most basic form, is a refinement of the theory of harmonic analysis in the m-dimensional Euclidean space. By introducing the so-called Dirac operator, the square of which equals the Laplace operator, one introduces the notion of monogenic functions. These are, at the same time, a refinement of harmonic functions and a generalization of holomorphic functions in one complex variable.

As the classical Fourier transform is so important in the study of harmonic analysis, it is a natural question to generalize this type of transform to the setting of Clifford analysis. By now, several authors have presented definitions of new Fourier transforms, all of which preserve some properties of the classical Fourier transform. In this paper, after an overview of the previously introduced transforms, we will introduce a new family of transforms by exploiting an analogy with the Fourier transform in the case of Dunkl operators and the Fourier transform in the case of superspaces. The kernel of the classical Fourier transform as well as of these other two transforms is determined uniquely by a system of partial differential equations. This system can be formulated very compactly in the language of Clifford analysis, and we will call the integral transform associated to every Clifford algebra-valued solution of this system a (generalized) Fourier transform. We will study two of these transforms in some more detail, namely the case of a vector Fourier transform and the case of a bivector Fourier transform.

The paper is organized as follows. In section 2 we introduce a few basic notions of Clifford analysis. In section 3 we give an overview of Fourier transforms in Clifford analysis, as introduced by other authors. In section 4 we define a new class of Fourier transforms. In section 5 we introduce the vector Fourier transform and study some of its properties. Finally, in section 6 we introduce a bivector Fourier transform.

2 CLIFFORD ANALYSIS

Clifford analysis (see a.o. [1, 8]) is a theory that offers a natural generalization of complex analysis to higher dimensions. To $\mathbb{R}^m$, the Euclidean space in m dimensions, we first associate the Clifford algebra $\mathbb{R}_{0,m}$, generated by the canonical basis e_i, $i = 1, \ldots, m$. These generators satisfy the following multiplication rules

$$\begin{aligned} e_i e_j + e_i e_j &= 0, \qquad i \neq j \\ e_i^2 &= -1. \end{aligned}$$

The Clifford algebra $\mathbb{R}_{0,m}$ can be decomposed as follows

$$\mathbb{R}_{0,m} = \oplus_{k=0}^{m} \mathbb{R}_{0,m}^{k}$$

with $\mathbb{R}_{0,m}^{k}$ the space of k-vectors defined by

$$\mathbb{R}_{0,m}^{k} = \text{span}\{e_{i_1 \ldots i_k} = e_{i_1} \ldots e_{i_k}, i_1 < \ldots < i_k\}.$$

More precisely, we have that the space of 1-vectors is given by

$$\mathbb{R}_{0,m}^{1} = \text{span}\{e_i, i = 1, \ldots, m\}$$

and it is obvious that this space is isomorphic with $\mathbb{R}^m$. The space of bivectors is given explicitly by

$$\mathbb{R}^2_{0,m} = \text{span}\{e_{ij} = e_i e_j, i < j\}.$$

We identify the point $(x_1, \ldots, x_m)$ in $\mathbb{R}^m$ with the so-called vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} x_j e_j.$$

The Clifford product of two vectors splits into a scalar part and a bivector part:

$$\underline{x}\underline{y} = \underline{x}.\underline{y} + \underline{x} \wedge \underline{y},$$

with

$$\underline{x}.\underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^{m} x_j y_j$$

and

$$\underline{x} \wedge \underline{y} = \sum_{j<k} e_{jk}(x_j y_k - x_k y_j).$$

It is interesting to note that the square of a vector variable $\underline{x}$ is scalar-valued and equals the norm squared up to a minus sign:

$$\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2.$$

Similarly, we introduce a first order vector differential operator by

$$\partial_{\underline{x}} = \sum_{j=1}^{m} \partial_{x_j} e_j.$$

This operator is the so-called Dirac operator. Its square equals, up to a minus sign, the Laplace operator in $\mathbb{R}^m$:

$$\partial_{\underline{x}}^2 = -\Delta.$$

A function f defined in some open domain $\Omega \subset \mathbb{R}^m$ with values in the Clifford algebra $\mathbb{R}_{0,m}$ is called monogenic if $\partial_{\underline{x}} f = 0$.

Another important operator in Clifford analysis is the Gamma operator, defined by

$$\Gamma_{\underline{x}} = -\underline{x} \wedge \partial_{\underline{x}} = -\sum_{j<k} e_{jk}(x_j \partial_{y_k} - x_k \partial_{y_j}).$$

This operator is bivector-valued.

3 FOURIER TRANSFORMS IN CLIFFORD ANALYSIS: AN OVERVIEW

The classical Fourier transform is given by

$$\mathcal{F}^{+}(.) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{i\langle \underline{x}, \underline{y} \rangle} (.) dV(\underline{x}).$$

This transform is an isomorphism on the space $\mathcal{S}(\mathbb{R}^m)$ of rapidly decreasing functions with inverse given by

$$\mathcal{F}^-(.) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} e^{-i\langle \underline{x}, \underline{y} \rangle} (.) dV(\underline{x}).$$

The properties of this transform are of course well-known, see e.g. [21].

In the sequel, we will need an eigenfunction basis of the Fourier transform. This basis is given by the so-called Hermite functions. They are defined in the following way:

$$\begin{aligned} \psi_{k_1,\ldots,k_m} &= \left(\partial_{x_1}^{k_1} e^{-x_1^2}\right) \ldots \left(\partial_{x_m}^{k_m} e^{-x_m^2}\right) \\ &= H_{k_1}(x_1) \ldots H_{k_m}(x_m) e^{-r^2/2} \end{aligned}$$

for all $\{k_1, \ldots, k_m\} \in \mathbb{N}^m$, with $H_{k_i}(x_i)$ the Hermite polynomial of degree k_i in the variable x_i. The set of all Hermite functions $\{\psi_{k_1,\ldots,k_m}\}$ forms a basis of $\mathcal{S}(\mathbb{R}^m)$ (and also of $L_2(\mathbb{R}^m)$). The action of the Fourier transform and its inverse on this basis is given by

$$\begin{aligned} \mathcal{F}^+(\psi_{k_1,\ldots,k_m}) &= i^{k_1+\ldots+k_m} \psi_{k_1,\ldots,k_m} \\ \mathcal{F}^-(\psi_{k_1,\ldots,k_m}) &= (-i)^{k_1+\ldots+k_m} \psi_{k_1,\ldots,k_m}. \end{aligned}$$

From this result, we see that the classical Fourier transform has 4 different eigenvalues, namely ± 1 and $\pm i$. Hence we easily obtain

$$(\mathcal{F}^+)^4 = 1.$$

As the classical Fourier transform is a scalar transform, it would be interesting to construct a generalization of this transform that does interact with the Clifford algebra $\mathbb{R}_{0,m}$. Three types of such generalizations have received quite some attention in the field of Clifford analysis, namely

- kernels with the imaginary unit replaced by Clifford numbers
- the monogenic extension of the classical Fourier kernel
- the Clifford-Fourier transform.

We discuss them in some more detail. A first generalization is obtained by replacing the kernel $e^{i\langle \underline{x}, \underline{y} \rangle}$ by

$$e^{e_1 x_1 y_1} \ldots e^{e_m x_m y_m},$$

where the role of the imaginary unit is taken over by the generators of the Clifford algebra. This kernel was introduced in [18] and [1] and further studied by Sommen in [19, 20], mostly from a theoretical point of view. In recent years, several related versions of this kernel have been studied. In [4], Bülow and Sommer introduce a quaternionic analogue of this kernel with the aim of establishing a theory of multi-dimensional signal analysis. The two- and three-dimensional case have also been studied by Felsberg (see [13]) and Ebling and Scheuermann (see [11, 12]). Similar transforms have been used by Mawardi and Hitzer in [17] to study uncertainty principles in Clifford analysis.

Another possibility of generalizing the classical Fourier transform, is by considering the monogenic extension of the Fourier kernel $e^{i\langle \underline{x}, \underline{y} \rangle}$ in $\mathbb{R}^m$ to $\mathbb{R}^{m+1}$. This extension is given by

$$\begin{aligned} e(x, y) &= \sum_{j=0}^{\infty} \frac{1}{j!} (e_0 x_0 \partial_{\underline{x}})^j e^{i\langle \underline{x}, \underline{y} \rangle} \\ &= e^{i\langle \underline{x}, \underline{y} \rangle} \left(\cosh x_0 |\underline{y}| + e_0 i \frac{\underline{y}}{|\underline{y}|} \sinh x_0 |\underline{y}| \right) \end{aligned}$$

and the associated transform has been studied intensively by Li, McIntosh and Qian in [16]. An interesting feature of this transform is that it allows to extend the Paley-Wiener theorem to the framework of Clifford analysis (see [14]) and to study sampling theory in higher dimensions (see [15]).

Finally, in recent years the so-called Clifford-Fourier transform has been introduced by Brackx, De Schepper and Sommen (see [2]). In this case, the kernel is given by

$$\mathcal{H}^{\pm} = e^{\mp i \frac{\pi}{2} \Gamma_{\underline{x}}} e^{-i\langle \underline{x}, \underline{y} \rangle}.$$

The behaviour of the associated integral transform is somewhat peculiar. As an example, it can be noted that in the even dimensional case, the transform has only 2 eigenvalues ± 1, whereas in the odd dimensional case it has 4 eigenvalues, namely ± 1 and $\pm i$.

In the two-dimensional case, the kernel is known in closed form and given by (see [3])

$$e^{\pm \underline{y} \wedge \underline{x}}$$

In higher dimensional cases, obtaining a closed form of the kernel is far from trivial and subject of ongoing research.

4 A NEW CLASS OF FOURIER TRANSFORMS

In this section we will develop a method to define new Fourier kernels in the field of Clifford analysis. We start with the following observation regarding the classical Fourier transform. This transform satisfies the following well-known calculus rules:

$$\begin{aligned} \mathcal{F}^{+}(x_i .) &= -i \partial_{y_i} \mathcal{F}^{+}(.) \\ \mathcal{F}^{+}(\partial_{x_i} .) &= -i y_i \mathcal{F}^{+}(.) \end{aligned}$$

for all $i = 1, \ldots, m$.

In terms of the Fourier kernel, these properties are translated to the system of equations

$$\begin{aligned} \partial_{y_i} e^{i\langle \underline{x}, \underline{y} \rangle} &= i x_i e^{i\langle \underline{x}, \underline{y} \rangle}, \qquad i = 1, \ldots, m \\ \partial_{x_i} e^{i\langle \underline{x}, \underline{y} \rangle} &= i y_i e^{i\langle \underline{x}, \underline{y} \rangle}, \qquad i = 1, \ldots, m. \end{aligned}$$

In particular, it is easily seen that the system

$$\partial_{y_i} K(x,y) = i x_i K(x,y), \qquad i = 1, \ldots, m \tag{1}$$

$$\partial_{x_i} K(x,y) = i y_i K(x,y), \qquad i = 1, \ldots, m \tag{2}$$

has, up to a multiplicative constant, a unique solution, namely

$$K(x,y) = e^{i\langle \underline{x}, \underline{y} \rangle}.$$

We conclude that the kernel of the Fourier transform is uniquely determined by a system of equations. These equations (1), (2) can be reformulated in terms of Clifford analysis, using the Dirac operator and the vector variable. This yields

$$\partial_{\underline{y}} K(x,y) = i K(x,y) \underline{x} \tag{3}$$

$$(K(x,y)) \partial_{\underline{x}} = i \underline{y} K(x,y). \tag{4}$$

The unique (up to a multiplicative constant) *scalar* solution to this system is the classical Fourier kernel. Every other bounded and Clifford-algebra valued solution $K(x, y)$ gives rise to a new Fourier transform in Clifford analysis, by means of

$$\mathcal{F}_K(.) = \int_{\mathbb{R}^m} K(x, y)(.) dV(\underline{x}),$$

with as main property that it intertwines the Dirac operator with the vector variable. Indeed, using (3), (4) we obtain that

$$\begin{aligned} \mathcal{F}_K(\underline{x}.) &= -i\partial_{\underline{y}}\mathcal{F}_K(.) \\ \mathcal{F}_K(\partial_{\underline{x}}.) &= -i\underline{y}\mathcal{F}_K(.). \end{aligned}$$

Before discussing a few special solutions to (3), (4) leading to new Fourier transforms in the following sections, we first present some more evidence for the appropriateness of the proposed system.

First of all, it is possible to construct a deformation of the classical partial derivatives in $\mathbb{R}^m$ to a set of operators (called Dunkl operators) that are only invariant under a certain finite reflection group $\mathcal{G}$ and not under the whole orthogonal group (see [10]). Also in this case there exists a Fourier transform which is now not orthogonally invariant, but only invariant under the group $\mathcal{G}$ (see [9] and [7] for a thorough study of this so-called Dunkl transform). The kernel of this integral transform is given by the unique solution of a generalization of equations (1), (2). Again this set of equations can be formulated in terms of Clifford algebras as

$$\begin{aligned} \mathcal{D}_{k,y}K(x, y) &= iK(x, y)\underline{x} \\ (K(x, y))\mathcal{D}_{k,x} &= i\underline{y}K(x, y) \end{aligned}$$

with $\mathcal{D}_k$ the Dunkl-Dirac operator (see [5]).

Similarly, in the study of superspaces it is also possible to introduce a Fourier transform (see [6]), which is now symplectically invariant. Again the kernel of this transform arises as the unique scalar solution of the system

$$\begin{aligned} \partial_{\underline{\grave{y}}}K(x, y) &= iK(x, y)\underline{\grave{x}} \\ (K(x, y))\partial_{\underline{\grave{x}}} &= i\underline{\grave{y}}K(x, y) \end{aligned}$$

with $\partial_{\underline{\grave{x}}}$ the fermionic Dirac operator and $\underline{\grave{x}}$ the corresponding vector variable.

Finally, also the Clifford-Fourier transform $\mathcal{H}^{\pm} = e^{\mp i\frac{\pi}{2}\Gamma_{\underline{x}}}e^{-i\langle \underline{x},\underline{y}\rangle}$ in $\mathbb{R}^m$ can be cast in a similar form. Indeed, we have that

$$\begin{aligned} \mp(\mp i)^m\partial_{\underline{y}}\mathcal{H}^{\mp} &= \mathcal{H}^{\pm}\underline{x} \\ (\mathcal{H}^{\pm})\partial_{\underline{x}} &= \pm(\mp i)^m\underline{y}\mathcal{H}^{\mp}. \end{aligned}$$

These three examples show that it is indeed a good idea to consider each solution of the system (3), (4) as a new Fourier transform. In the next two sections, we construct a vector and a bivector solution to this system and discuss a few properties of the associated integral transforms.

5 THE VECTOR FOURIER TRANSFORM

We will determine a kernel

$$K(x, y) = \sum_{i=0}^{m} K_i(x, y) e_i$$

that satisfies the system (3), (4) and that is as close as possible to the classical Fourier transform.

We begin by considering a term $K_j e_j$. Such a term satisfies the mentioned system if

$$\begin{aligned} \partial_{x_j} K_j &= i y_j K_j \\ \partial_{x_k} K_j &= -i y_k K_j, \qquad k \neq j. \end{aligned}$$

Hence, we obtain, up to a constant,

$$K_j(x, y) = e^{i x_j y_j - \sum_{k \neq j} i x_k y_k}.$$

Although the function $K_j e_j$ is a solution to the system, it is not very symmetrical. Extension by cyclic permutation yields

$$\begin{aligned} K(x, y) &= \sum_{j=0}^{m} e^{i x_j y_j - \sum_{k \neq j} i x_k y_k} e_j \\ &= e^{-i\langle \underline{x}, \underline{y} \rangle} \sum_{j=0}^{m} e^{2 i x_j y_j} e_j. \end{aligned}$$

In other words, we obtain a new Fourier kernel that is the product of the (inverse) classical Fourier kernel with a vector.

This kernel gives rise to the following new integral transform

$$\mathcal{F}_K(.) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} K(x, y)(.) dV(x).$$

We calculate the action of this transform on the basis $\{\psi_{k_1,\ldots,k_m}\}$ of $\mathcal{S}(\mathbb{R}^m)$. We obtain for a basis element that

$$\begin{aligned} \mathcal{F}_K(\psi_{k_1,\ldots,k_m}) &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} K(x, y) \psi_{k_1,\ldots,k_m} dV(x) \\ &= (2\pi)^{-m/2} \sum_{j=0}^{m} e_j \int_{\mathbb{R}^m} e^{i x_j y_j - \sum_{k \neq j} i x_k y_k} \psi_{k_1,\ldots,k_m} dV(x) \\ &= \sum_{j=0}^{m} e_j (-i)^{k_1} \ldots (i)^{k_j} \ldots (-i)^{k_m} \psi_{k_1,\ldots,k_m} \\ &= (-i)^{\sum_{j=1}^{m} k_j} \left(\sum_{j=0}^{m} e_j (-1)^{k_j} \right) \psi_{k_1,\ldots,k_m}. \end{aligned}$$

Note that every $\psi_{k_1,\ldots,k_m}$ is an eigenfunction of the new Fourier transform. The eigenvalues are no longer real numbers, but 1-vectors in the Clifford algebra $\mathbb{R}_{0,m}$.

The transform $\mathcal{F}_K(.)$ is invertible. Indeed, putting $G = K(-x, y)$, we have

$$\begin{aligned}\mathcal{F}_G(\psi_{k_1,\ldots,k_m}) &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} K(-x,y)\psi_{k_1,\ldots,k_m} dV(x) \\ &= i^{\sum_{j=1}^m k_j} \left(\sum_{j=0}^m e_j(-1)^{k_j} \right) \psi_{k_1,\ldots,k_m}.\end{aligned}$$

Hence we obtain

$$\mathcal{F}_G(\mathcal{F}_K(\psi_{k_1,\ldots,k_m})) = (-m)\psi_{k_1,\ldots,k_m}.$$

We also have that

$$\mathcal{F}_k^4 = m^2,$$

which is a similar relation as satisfied by the classical Fourier transform.

Now, let us consider the action of the vector Fourier transform on a vector function, i.e. a function of the form

$$f(x) = \sum_{i=1}^m f_i(x)e_i, \qquad f_i(x) \in L_1(\mathbb{R}^m).$$

It is easily seen that $\mathcal{F}_K(f)$ consists of a scalar part and a bivector part. The scalar part is given by

$$[\mathcal{F}_K(f)]_0 = -(2\pi)^{m/2} \sum_{j=0}^m \int_{\mathbb{R}^m} e^{-i\langle \underline{x}, \underline{y} \rangle} e^{2ix_j y_j} f_j(x) dV(\underline{x})$$

and the bivector part by

$$[\mathcal{F}_K(f)]_2 = (2\pi)^{m/2} \sum_{j<k} e_{jk} \int_{\mathbb{R}^m} e^{-i\langle \underline{x}, \underline{y} \rangle} \left(e^{2ix_j y_j} f_k(x) - e^{2ix_k y_k} f_j(x) \right) dV(\underline{x}).$$

6 THE BIVECTOR FOURIER TRANSFORM

We begin by considering a bivector of the form $K_{jk} e_{jk}$ with $j < k$. Such a term satisfies the system (3), (4) if

$$\begin{aligned}\partial_{x_p} K_{jk} &= -iy_p K_{jk}, \qquad p \in \{j,k\} \\ \partial_{x_q} K_{jk} &= iy_q K_{jk}, \qquad q \notin \{j,k\}.\end{aligned}$$

Hence, we obtain, up to a constant,

$$K_{jk}(x,y) = e^{-ix_j y_j - ix_k y_k + \sum_{l \neq j,k} ix_l y_l}.$$

Although the function $K_{jk} e_{jk}$ is a solution to the system, it is not very symmetrical. Extension by cyclic permutation yields

$$\begin{aligned}K(x,y) &= \sum_{j<k} e^{-ix_j y_j - ix_k y_k + \sum_{l \neq j,k} ix_l y_l} e_{jk} \\ &= e^{i\langle \underline{x}, \underline{y} \rangle} \sum_{j<k} e^{-2ix_j y_j - 2ix_k y_k} e_{jk}.\end{aligned}$$

In other words, we obtain a new Fourier kernel that is the product of the classical Fourier kernel with a bivector.

This kernel gives rise to the following new integral transform

$$\mathcal{F}_K(.) = (2\pi)^{-m/2} \int_{\mathbb{R}^m} K(x,y)(.)dV(x).$$

We calculate again the action of this transform on the basis $\{\psi_{k_1,\ldots,k_m}\}$ of $\mathcal{S}(\mathbb{R}^m)$. We obtain for a basis element that

$$\begin{aligned}
\mathcal{F}_K(\psi_{k_1,\ldots,k_m}) &= (2\pi)^{-m/2} \int_{\mathbb{R}^m} K(x,y)\psi_{k_1,\ldots,k_m} dV(x) \\
&= (2\pi)^{-m/2} \sum_{j<k} e_{jk} \int_{\mathbb{R}^m} e^{-ix_j y_j - ix_k y_k + \sum_{l\neq j,k} ix_l y_l} \psi_{k_1,\ldots,k_m} dV(x) \\
&= \sum_{j<k} e_{jk} i^{k_1} \ldots (-i)^{k_j} \ldots (-i)^{k_k} \ldots i^{k_m} \psi_{k_1,\ldots,k_m} \\
&= i^{\sum_{j=1}^m k_j} \left(\sum_{j<k} e_{jk} (-1)^{k_j + k_k} \right) \psi_{k_1,\ldots,k_m}.
\end{aligned}$$

The eigenvalues of the transform are now bivectors. This makes inverting the transform more complicated, as one does not have a general formula for the inverse of a bivector. In low dimensional cases ($m = 2$, 3), the transform is more simple and we can obtain an inverse. Indeed, if $m = 2$, then the transform takes the following form

$$\mathcal{F}_K(.) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\langle \underline{x}, \underline{y} \rangle} e_{12}(.)dV(\underline{x}).$$

The inverse of this transform is clearly given by

$$-\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i\langle \underline{x}, \underline{y} \rangle} e_{12}(.)dV(\underline{x}).$$

The kernel of the inverse transform is thus given by $-K(-x,y)$.

If $m = 3$, then the transform is given by

$$\mathcal{F}_K(.) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\langle \underline{x}, \underline{y} \rangle} \left(e^{2ix_3 y_3} e_{12} + e^{2ix_2 y_2} e_{13} + e^{2ix_1 y_1} e_{23} \right) (.)dV(\underline{x}).$$

Again, we can calculate the inverse transform. This transform is given by

$$-\frac{1}{3}(2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i\langle \underline{x}, \underline{y} \rangle} \left(e^{-2ix_3 y_3} e_{12} + e^{-2ix_2 y_2} e_{13} + e^{-2ix_1 y_1} e_{23} \right) (.)dV(\underline{x}).$$

7 CONCLUSIONS AND OUTLOOK

In this paper we have introduced a new equation in a Clifford algebra. Every solution of this equation gives rise to a generalized Fourier transform in Clifford analysis. We have studied two special types of solutions of this equation, leading to the vector and bivector Fourier transform.

In the vector case, we have obtained the inverse of the transform. In the bivector case this is still an open problem.

In future work we will further study the solutions to the introduced equation, with focus on k-vector solutions. We are also interested in the behaviour of the new class of transforms with respect to spherical monogenics. Finally, we would also like to study solutions which have certain symmetry properties, such as rotational or translational invariance.

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