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REGULAR QUATERNIONIC FUNCTIONS AND THEIR APPLICATIONS

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Abstract. *The theory of regular quaternionic functions of a reduced quaternionic variable is a 3-dimensional generalization of complex analysis. The Moisil-Theodorescu system (MTS) is a regularity condition for such functions depending on the radius vector* $\mathbf{r} = \mathbf{i}x+\mathbf{j}y+\mathbf{k}z$ *seen as a reduced quaternionic variable. The analogues of the main theorems of complex analysis for the MTS in quaternion forms are established: Cauchy, Cauchy integral formula, Taylor and Laurent series, approximation theorems and Cauchy type integral properties. The analogues of positive powers (inner spherical monogenics) are investigated: the set of recurrence formulas between the inner spherical monogenics and the explicit formulas are established. Some applications of the regular function in the elasticity theory and hydrodynamics are given.*

1 INTRODUCTION

In two-dimensional problems of the mathematical physics the methods of complex variable theory are effectively used. As a generalization in multidimensional problems the methods of hypercomplex functions are developed since 1930s papers by R. Fueter, G.C. Moisil and N. Theodorescu (see [1]–[5] and references therein). In [6] one can find a short survey on applications of hypercomplex functions in the three-dimensional theory of elasticity. In a recent paper [7] a new alternative Kolosov-Muskhelishvili formula for the elastic displacement field by means of a (paravector–valued) monogenic, an anti-monogenic and a ψ–hyperholomorphic function is proposed.

In the three-dimensional space the Moisil-Theodorescu system is an analogue of the Cauchy–Riemann system and its theory was developed in three ways: as a theory of partial differential equations, as a part of Clifford analysis [1], and as the theory of regular quaternionic functions of the reduced quaternionic variable [2], [8]– [10]. In this paper we use the last way and give a short survey of our results.

2 PRELIMINARIES AND NOTATIONS

Let i, j, k be the basic quaternions obeying the following rules of multiplication:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

An element q of the quaternion algebra $\mathbb{H}$ we write in the form $q = q_0 + iq_x + jq_y + kq_z = q_0 + \mathbf{q}$, where q_0, q_x, q_y, q_z are the real numbers, q_0 is called the scalar part of the quaternion, $\mathbf{q} = iq_x + jq_y + kq_z$ is called the vector part of the quaternion q. The quaternion conjugation is denoted as $\tilde{q} = q_0 - \mathbf{q}$.

Let x, y, z be the Cartesian coordinates in the Euclidean space $\mathbb{R}^3$. Let Ω be a domain of $\mathbb{R}^3$ with a piecewise smooth boundary. A quaternion-valued function or, briefly, $\mathbb{H}$-valued function f of a reduced quaternionic variable $\mathbf{r} = ix + jy + kz \in \mathbb{R}^3$ is a mapping $f : \Omega \to \mathbb{H}$, such that

$$f(\mathbf{r}) = f_0(\mathbf{r}) + \mathbf{f}(\mathbf{r}) = f_0(x, y, z) + if_x(x, y, z) + jf_y(x, y, z) + kf_z(x, y, z).$$

The functions f_0, f_x, f_y, f_z are real-valued defined in Ω. Continuity, differentiability or integrability of f are defined coordinate-wisely. For continuously real–differentiable functions $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}$, which we will denote for simplicity by $f \in C^1(\Omega, \mathbb{H})$, the operator $\nabla = i\partial_x + j\partial_y + k\partial_z$ is called the generalized Cauchy– Riemann operator.

According to R. Fueter a function f is called left– or right–regular in Ω if $\nabla f = 0$ or $f\nabla = 0$, respectively, for $\mathbf{r} \in \Omega$. From now on in the main part we use only the left-regular functions that, for simplicity, we call regular. With the vectorial notations the regulariry condition is given as follows:

$$\nabla f(\mathbf{r}) = -\nabla \cdot \mathbf{f}(\mathbf{r}) + \nabla f_0(\mathbf{r}) + \nabla \times \mathbf{f}(\mathbf{r}) = 0, \tag{1}$$

where ∇f_0, $\nabla \cdot \mathbf{f}$, $\nabla \times \mathbf{f}$ are the usual gradient, divergence and curl, respectively.

The equations of elastic equilibrium are called the Lame equations:

$$L\mathbf{u} \equiv (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) = 0. \tag{2}$$

If we introduce the next notations

$$(\lambda + 2\mu)\nabla \cdot \mathbf{u} = f_0, \quad -\mu\nabla \times \mathbf{u} = \mathbf{f}, \tag{3}$$

then the Lame equation (2) is transformed into the MTS:

$$\nabla \cdot \mathbf{f} = 0, \quad \nabla f_0 + \nabla \times \mathbf{f} = 0, \tag{4}$$

thus the quaternion function $f = f_0 + \mathbf{f}$ is regular. Such the connection between the Lame equation and quaternion functions was first pointed by G.Moisil.

3 MAIN THEOREMS OF QUATERNIONIC ANALYSIS

Cauchys integral theorem, the formula of Borel-Pompeiu, Cauchys integral formula, Cauchys integral formula for the derivative and some other main properties are established.

Let us call a $\mathbb{H}$–valued function F as a primitive of the regular function f if $\nabla F = f$, i.e. $\nabla \cdot \mathbf{F} = -f_0, \quad \nabla F_0 + \nabla \times \mathbf{F} = \mathbf{f}$. Obviously, $\nabla(\nabla \mathbf{F}) = -\Delta \mathbf{F} = \nabla f = 0$ and F is a harmonic function. Such the primitive function is not regular. Another notion of a monogenic primitive function is introduced by means of a hypercomplex derivative of a monogenic function (see [11]).

3.1 Analogues of Powers

In higher dimensions analogues of positive powers are constructed by symmetrization of products of the Fueter variables [2] . Now we show a certain other way for introducing the set of regular polynomials. Let us introduce homogeneous polynomials $P^{l,m}(\mathbf{r})$, $l + m = n$, $(l, m = 0, 1, ..., n; n = 0, 1, ...)$ by means of recurrent formulas:

$$P^{l,m}(\mathbf{r}) = P^{1,0}(\mathbf{r})P^{l-1,m}(\mathbf{r}) + P^{0,1}(\mathbf{r})P^{l,m-1}(\mathbf{r}), \tag{5}$$

where $P^{1,0}(\mathbf{r}) = x + jz$, $P^{0,1}(\mathbf{r}) = y - iz$, $P^{0,0} = 1$ and it is assumed that $P^{p,q} = 0$, if at least one of the numbers $p, q < 0$.

It can be proved that these polynomials are linearly independent over $\mathbb{R}$, any finite linear combination $\sum_{l,m} P^{l,m}(\mathbf{r})C_{l,m}$ with $C_{l,m} = const \in \mathbb{H}$ is regular everywhere. Also the polynomials $P^{l,m}(\mathbf{r})$ form a basis of the space of homogeneous regular polynomials of order n. There exist convenient formulas for derivatives:

$$\begin{aligned} &\frac{\partial}{\partial x}P^{l,m}(\mathbf{r}) = (l+m)P^{l-1,m}(\mathbf{r}), \ \frac{\partial}{\partial y}P^{l,m}(\mathbf{r}) = (l+m)P^{l,m-1}(\mathbf{r}), \\ &\frac{\partial}{\partial z}P^{l,m}(\mathbf{r}) = (l+m)\left[-iP^{l,m-1}(\mathbf{r}) + jP^{l-1,m}(\mathbf{r})\right]. \end{aligned} \tag{6}$$

Some other properties of regular polynomials are investigated: the structure of scalar and vector parts, recurrent formulas for scalar and vector parts, explicit formulas for components etc.

Analogues of negative powers are noted $P^{-l-1,-m-1}(\mathbf{r})$ and are introduced in the usual way:

$$P^{-l-1,-m-1}(\mathbf{r}) = \frac{(-1)^{n+1}}{n!}\frac{\partial^n}{\partial x^l \partial y^m}\frac{\mathbf{r}}{r^3} \quad (l, m = 0, 1, ..., n; \ n = 0, 1, ...) \tag{7}$$

The Taylor and Laurent series generalizations by means of introduced powers are proved.

3.2 Approximation Theorems

Runge's Theorem. Each $\mathbb{H}$–valued function f that is regular in an open (not necessarily connected) subset D of $\mathbb{R}^3$ with a connected complement can be uniformly approximated on each compactum $K \Subset D$ arbitrarily closely by regular polynomials.

Lavrent'ev's Theorem. Each $\mathbb{H}$–valued function $f \in C^0$ on a closed subset D of $\mathbb{R}^3$ can be uniformly approximated on this set arbitrarily closely by regular polynomials if and only if this set is a nowhere dense compactum in $\mathbb{R}^3$ that does not separate $\mathbb{R}^3$.

Keldysh's Theorem. Each $\mathbb{H}$–valued function $f \in C^0(\bar{D})$ that is regular in a domain D can be uniformly approximated on a closed domain $\bar{D} \in \mathbb{R}^3$ arbitrarily closely by regular polynomials if and only if the complement of D consists of a single domain G_∞ that contains the point at infinity.

Details can be found in [8].

4 APPLICATIONS

4.1 Three-dimensional Kolosov-Muskhelishvili formulae

In plane problems of theory of elasticity the basis of complex function applications is the representation of the general solution of the equilibrium equations in terms of two arbitrary analytic functions called the Kolosov-Muskhelishvili formulae. In this section a variant of three-dimensional quaternion generalization of the Kolosov-Muskhelishvili formulae is presented, which is effectively applied to solve the basic problems of the theory of elasticity for the ball.

The general solution of the Lame equation (2) in a star–shaped domain Ω^* is expressed in terms of two regular in Ω^* functions $\varphi,\ \psi$ in the form

$$2\mu\mathbf{u}(\mathbf{r}) = \varkappa\Phi(\mathbf{r}) - \widetilde{\mathbf{r}\varphi(\mathbf{r})} - \widetilde{\psi(\mathbf{r})},\ \varkappa = -\frac{3\lambda + 7\mu}{\lambda + \mu}, \tag{8}$$

where as Φ one can take any primitive of function φ, having subordinated ψ to the condition $\varkappa\Phi_0 = \mathbf{r}\cdot\boldsymbol{\varphi} + \psi_0$.

Another form of general solution of the Lame equation in Ω^* is given in terms of two regular in Ω^* functions $f, \nabla g_0$ by V. V. Naumov (see [12]) in the form:

$$\mathbf{u}(\mathbf{r}) = \frac{\mathbf{r}}{\mu} \times I^1\mathbf{f} + \nabla\left\{r^2\left[\frac{3\lambda + 7\mu}{4\mu(\lambda + 2\mu)}I^{1/2} - \frac{1}{\mu}I^1\right]f_0\right\} + \nabla g_0, \tag{9}$$

where $f = (\lambda + 2\mu)\nabla\cdot\mathbf{u} - \mu\nabla\times\mathbf{u}$; I^α is the operator of radial integration:

$$I^\alpha f(\mathbf{r}) = \int_0^1 t^\alpha f(\mathbf{r}t)dt.$$

It can be shown that both representations (8) and (9) are equivalent, in particular cases of plane and axially symmetric deformations both representations go into the Kolosov-Muskhelishvili and Solovyev formulae. Some details can be found in [6].

4.2 Equilibrium problems for elastic ball

Using the representation (9) it is shown that the main problems of elastic ball equilibrium can be obtained in a closed form as analogues of the Poisson and Neumann formulas. Solutions of these problems are also expressed in terms of solutions of the Dirichlet and Neumann problems for three independent harmonic functions in a ball. For example, let us consider the equilibrium of a ball U with a radius R when on its boundary S purely normal displacements are given:

$$\begin{cases} L\mathbf{u}(\mathbf{r}) = 0, \quad \mathbf{r}\in U, \quad \mathbf{u}\in C^2(U)\cap C^1(\overline{U}) \\ u_r|_{\partial U} = u(\theta,\varphi)\in C^0(\partial U); \quad u_\theta|_{\partial U} = u_\varphi|_{\partial U} = 0. \end{cases} \tag{10}$$

By mean of the quaternion representation (9) the solution of this problem is obtained in the form

$$\mu\mathbf{u}(\mathbf{r}) = \frac{\mathbf{r}}{R}F + \frac{R^2 - r^2}{2R}\nabla\left[æ(2æ-1)I^{æ} - 2æ - 1\right]F, \quad æ = \frac{2(\nu-1)}{3-4\nu} \in \left(-1, \, -\frac{2}{3}\right), \tag{11}$$

where F is a solution of the next Dirichlet problem:

$$\begin{cases} \Delta F(\mathbf{r}) = 0, \quad \mathbf{r} \in U, \\ F|_{r=R} = \mu u(\vartheta, \varphi) \in C^0(S). \end{cases} \tag{12}$$

Then using the Poisson formula for the solution of the Dirichlet problem (12) and radial integration operator I^{α} properties, we have the solution of the problem (10) in the closed form:

$$u_r(\mathbf{r}) = \frac{1-t^2}{4\pi R^2}\oint u(2æ+1)\left[(2-æ)\frac{t}{s^3} + \frac{2æ^2 - æ}{2æ+1}\cdot\frac{c}{s^3} + 3(1-t^2)\frac{t-c}{2s^5} + \right.$$

$$\left. + \frac{æ}{2t}(1-2æ)\left(\frac{1}{s} - (æ+1)I^{æ}\frac{1}{s}\right)\right] dS;$$

$$\begin{Bmatrix} u_\theta(\mathbf{r}) \\ u_\varphi(\mathbf{r}) \end{Bmatrix} = \frac{1}{4\pi R^2}\oint u \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}\left(\frac{1}{2} + æ\right)(1-t^2)\left[3\frac{t^2-1}{s^5} + \frac{4æ^2 - 2æ}{(2æ+1)s^3} + \right.$$

$$\left. + (æ - 2æ^2)I^{æ+1}\frac{1}{s^3}\right] dS,$$

where $s^2 = 1 - 2tc + t^2$, $t = r/R$, $c = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$, $\xi = c_{,\theta}$, $\eta = c_{,\varphi}(\sin\theta)^{-1}$. Integrals $I^{\omega}s^{-\alpha}$ are integral representations of the Appell hypergeometric function [6].

Analogous results are obtained for the case of the Stokes flow [13].

5 CONCLUSION

In this paper the theory of the Moisil-Theodorescu system in terms of regular quaternionic functions of reduced quaternionic variable is used. The analogues of positive powers (inner spherical monogenics) are investigated: the set of recurrence formulas between the inner spherical monogenics and the explicit formulas are established in Cartesian coordinates. Unlike [11], we used another notion of primitive of regular function. Therefore, we have another version of three-dimensional quaternionic analogue of the complex Kolosov-Muskhelishvili formulae. As applications the problem of elastic sphere equilibrium in the case of normal displacements is solved. The solution is expressed in terms of one harmonic function, which is the solution of the Dirichlet problem with the boundary condition as in the original problem. This solution is also expressed in terms of quadratures of elementary functions and the Appell hypergeometric function.

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