Binary and ternary Clifford analysis

on

Nonion algebra and su(3)

By

J.Larynowicz*, K.Nouno **, D. Nagayama*** and O. Suzuki ***

* *Institute of physics, Lodz University , ul. Pomorska 153/156,Lodz, Poland, Email: jlawryno @uni.lodz, pl*

** *Department of Mathematics, Fukuoka Education University, Munakata Fukuoka, Japan,*

Email:nouno@fukuoka-edu.ac.jp

*** *Department of Compute rand System Analysis College of Humanities and Sciences, Nihon University 156*

Setagaya,Tokyo,Japan E-mail: osuzuki@cssa.chs.nihon-u.ac.jp

Abstract. *A concept of non-commutative Galois extension is introduced and binary and ternary extensions are chosen. Non-commutative Galois extensions of Nonion algebra and su(3) are constructed. Then ternary and binary Clifford analysis are introduced for non-commutative Galois extensions and the corresponding Dirac operators are associated.*

1. BINARY AND TERNARY NON-COMMUTATIVE GALOIS EXTENSIONS

We introduce a concept of non-commutative Galois extension of binary type and ternary type and state some basic facts on the extensions ([6]).

Basic notations on non-commutative Galois extensions

Let A be an algebra and A' be a subalgebra of A. We make the following definition:

DEFINITION 1

(1) We take an element $\tau \in A$ with the following condition $\tau^k = 1$. The following subalgebra $A'[\tau]$ of A is called non-commutative Galois extension of k-nary type:

$$A'[\sqrt[k]{1}] = \{\sum_{\rho=0}^{k-1} \varsigma_\rho \tau^\rho \mid \varsigma_\rho \in A'\}$$

The extension is called proper when $\tau^\rho \notin A'(\rho = 1,..,k-1)$. In this paper we are

concerned with only proper extensions without mentioning it.

(2) We assume that $A_i = A\,[\tau_i](i = 1, 2)$ are subalgebras in a common algebra A_3. When the ismorphism is given by the following multiplication operator: $\theta : A'[\tau_1] \rightarrow A'[\tau_2]$, $\theta(\xi) = \xi (\xi \in A')$, $\theta\,\tau_1 = \tau_2 (\theta \in A_3)$, it is called θ-equivalent.

(3) We assume the same condition in (2). When the isomorphism is given by the Adjoint operator: $Ad_g\xi\,(\exists g \in A_3)$, $Ad_g\xi = g\,\xi\,g^{-1}$., $Ad_g\xi' = \xi'\,(\xi' \in A')$. it is called Ad-equivalent.

(4) When $A'[\tau_1] = A'[\tau_2]$ holds, they are called identical each other. When $\tau_1 = \tau_2^2$, then we have the identical extension: $A'[\tau_1] = A'[\tau_2]$.

REMARKS (1) To define the Galois extension structure, we put some additional condition on the algebra: for example, $\xi\,\tau^l = \sum_\alpha \tau^\alpha \xi_\alpha$ holds with some $\xi_\alpha \in A$ for any $\xi \in A, \alpha, l = 1, 2, .., k-1$. In this paper we are concerned with the algebra with this condition.

(2) The Galois extension is not unique depending on the choice of τ. We are concerned with the Galois extension which does not depend on the choice $\tau(\neq 1)$.

Examples of binary and ternary extensions

Next we proceed to examples of binary and ternary extensions. We obtain binary and ternary Clifford algebras from Galois extensions $A'[\sqrt[k]{1}]$ (see S.4).

Example 1 (Complex numbers)

The first one is the complex number field $R[\sqrt{-1}]$:

$$R[\sqrt{-1}] = \{\theta_1 1 + \theta_2 \sqrt{-1} \mid \theta_1, \theta_2 \in R\}$$
$$= \left\{ \begin{pmatrix} \theta_1 & \theta_2 \\ -\theta_2 & \theta_1 \end{pmatrix} \mid \theta_1, \theta_2 \in R \right\}$$

Example 2 (Quaternion number)

The quaternion number field can be obtained by the non-commutative Galios extension of the complex number field $R[\sqrt{-1}]$:

$$C[\sqrt{-1_2}] = \{\theta_1 1 + \theta_2 \sqrt{-1_2} \mid \theta_1, \theta_2 \in C\}$$
$$= \left\{ \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & -\theta_1 \end{pmatrix} \mid \theta_1, \theta_2 \in C \right\}$$
$$= \left\{ \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 \\ -\theta_2 & \theta_1 & \theta_4 & -\theta_3 \\ \theta_3 & \theta_4 & -\theta_1 & -\theta_2 \\ \theta_4 & -\theta_3 & +\theta_2 & -\theta_1 \end{pmatrix} \mid \theta_1, \theta_2, \theta_3, \theta_4 \in R \right\}$$

Example 3 (Cubic root numbers)

We give a basic ternary Galois extension. The simplest example is the complex cubic numbers $R[\sqrt[3]{1}]$

$$R[\sqrt[3]{1}] = \{\theta_1 1 + \theta_2 j + \theta_{32} j^2 \mid \theta_1, \theta_2, \theta_3 \in R\}$$

$$= \left\{ \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 \\ \theta_3 & \theta_1 & \theta_2 \\ \theta_2 & \theta_3 & \theta_1 \end{pmatrix} \mid \theta_1, \theta_2, \theta_3 \in R \right\}$$

In the next section we give ternary extensions in Nonion algebra.

Successive extensions

We consider successive Galois extensions. We take an extension: $A_1 = A_0[\tau_1]$ and make an extension $A_2 = A_1[\tau_2]$. Then we have the successive extension $A_2 = (A_0[\tau_1])[\tau_2]$ as follows: $A_2 = \{\sum x_{i,j} \tau^i_1 \tau^j_2 \mid x_{i,j} \in A_0\}$ .We can also make the tensor product extension. Namely we can define $A_2 = A_0[\tau_1 \otimes \tau_2]$ by $A_2 = \{\sum x_{i,j} \tau^i_1 \otimes \tau^j_2 \mid x_{i,j} \in A_0\}$. The example 2 is the tensor product extension.

2. THE GALOIS EXTENSION STRUCTURE ON NONION ALGEBRA

We introduce a concept of Nonion algebra N and discuss ternary Galois extension structures on it. We begin with the definition of Nonion algebra ([1],[4]):

DEFINITION 2

(1)The matrix algebra which is generated by the following 3 matrices over $R[\sqrt[3]{1}]$ is called Nonion algebra:

$$Q_1 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(2) The matrix algebra which is generated by the following 3 matrices over the real field R is called basic algebra B:

$$T_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad T_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(3)The algebra generated by T_2 (or T_3) is called cubic algebra and is denoted by B':

$$T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(4) The algebra $\tilde{N}$ generated by the following four elements over $R[\sqrt[3]{1}]$ is called the binary extension of N:

$$Q_1 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad T_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

Then we can prove the following proposition:

PROPOSITION 3

(1)The following 9 elements constitute linear basis of Nonion algebra:

$$Q_1 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix} \quad Q_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\overline{Q}_1 = \begin{pmatrix} 0 & 0 & 1 \\ j^2 & 0 & 0 \\ 0 & j & 0 \end{pmatrix} \quad \overline{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ j & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix} \quad \overline{Q}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}$$

(2) The following 6 elements are linear basis of B:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad T_5 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(3) The following 3 elements are linear basis of B'

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(4) B' is a subalgebra of N, and B is a subalgebra of $\tilde{N}$.

PROOF

The proofs are direct calculations by use of the following product tables.

	Q_1	Q_2	Q_3	$\overline{Q}_1$	$\overline{Q}_2$	$\overline{Q}_3$	R_1	R_2	R_3
Q_1	$\overline{Q}_1$	$j^2\overline{Q}_3$	$j\,\overline{Q}_2$	R_1	j^2R_3	$j\,R_2$	Q_1	Q_2	Q_3
Q_2	$j\,\overline{Q}_3$	$\overline{Q}_2$	$j^2\overline{Q}_1$	$j\,R_2$	R_1	j^2R_3	Q_2	Q_3	Q_1
Q_3	$j^2\overline{Q}_2$	$j\,\overline{Q}_1$	$\overline{Q}_3$	j^2R_3	$j\,R_2$	R_1	Q_3	Q_1	Q_2
$\overline{Q}_1$	R_1	R_2	R_3	Q_1	j^2Q_3	$j\,Q_2$	$\overline{Q}_1$	$j^2\overline{Q}_3$	$j\,\overline{Q}_2$
$\overline{Q}_2$	R_3	R_1	R_2	$j\,Q_3$	Q_2	j^2Q_1	$\overline{Q}_2$	$j^2\overline{Q}_1$	$j\,\overline{Q}_3$
$\overline{Q}_3$	R_2	R_3	R_1	j^2Q_2	$j\,Q_1$	Q_3	$\overline{Q}_3$	$j^2\overline{Q}_2$	$j\,\overline{Q}_1$
R_1	Q_1	Q_2	Q_3	$\overline{Q}_1$	$\overline{Q}_2$	$\overline{Q}_3$	R_1	R_2	R_3
R_3	j^2Q_2	j^2Q_3	j^2Q_1	$\overline{Q}_3$	$\overline{Q}_1$	$\overline{Q}_2$	R_2	R_3	R_1
R_3	$j\,Q_3$	$j\,Q_1$	$j\,Q_2$	$\overline{Q}_2$	$\overline{Q}_3$	$\overline{Q}_1$	R_3	R_1	R_2

	T_1	T_2	T_3	T_4	T_5	T_6
T_1	T_1	T_2	T_3	T_4	T_5	T_6
T_2	T_2	T_3	T_1	T_5	T_6	T_4
T_3	T_3	T_1	T_2	T_6	T_4	T_5
T_4	T_4	T_6	T_5	T_1	T_3	T_2
T_5	T_5	T_4	T_6	T_2	T_1	T_3
T_6	T_6	T_5	T_4	T_3	T_2	T_1

The explicit construction of binary extension of Nonion algebra

The binary extension $\tilde{N}$ of N is given as follows:

$$\tilde{N} = \{x + yT_4 \mid x, y \in N\}$$

Then we can give the linear basis of $\tilde{N}$ as follows:

$$Q_1 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix} \qquad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} (=T_3)$$

$$\overline{Q}_1 = \begin{pmatrix} 0 & 0 & 1 \\ j^2 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \qquad \overline{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ j & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \qquad \overline{Q}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} (=T_2)$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \qquad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}$$

$$Q'_1 = \begin{pmatrix} 0 & 0 & 1 \\ j^2 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \qquad Q'_2 = \begin{pmatrix} 0 & 0 & 1 \\ j & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \qquad Q'_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overline{Q}'_1 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix} \qquad \overline{Q}'_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix} \qquad \overline{Q}'_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R'_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad R'_2 = \begin{pmatrix} 0 & 0 & j^2 \\ 0 & j & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad R'_3 = \begin{pmatrix} 0 & 0 & j \\ 0 & j^2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

As for the non-commutative Galois structure of Nonion algebra, we can prove the following theorem:

THEOREM I

(1) Nonion algebra is the Galois extension of the algebra B' : $N = B'[\tau]$ by $\tau = R_i\,(i=2,3), Q_i, \overline{Q}_i\,(i=1,2,3)\,(\tau^3 = 1)$.

(2) The Galois extension $\widetilde{N} = N[\sqrt[2]{1}]$ can be expressed as $\widetilde{N} = B[\sqrt[3]{1}]$.

Hence we have the following commutative diagram:

$$
\begin{array}{ccc}
 & \widetilde{N} & \\
\nearrow & & \nwarrow \\
N & & B \\
\nwarrow & & \nearrow \\
 & B' &
\end{array}
$$
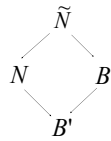
PROOF

(1) We notice that B' is the commutative Galois extension: $B' = R[\sqrt[3]{1}]$.
Choosing $\tau = R_i\,(i=2,3), Q_i, \overline{Q}_i\,(i=1,2,3)$, we make the Galois extension $B'[\sqrt[3]{1}_3]$. Then we see that this is identical with N.

(2) We notice that B is the non-commutative Galois extension of B' : $B = B'[\sqrt[2]{1}]$, where $\sqrt[2]{1}_3 = T_4$. Choosing $\tau = R_2$, we make the Galois extension . Then we see that it is identical with $\widetilde{N}$: $\widetilde{N} = B[\sqrt[3]{1}]$.

THEOREM II

We can prove the following assertions for N :

(1) We have the following ternary Galois extensions which are called basic extension:

$$
\left\{
\begin{aligned}
& A[R] = \{xR_1 + yR_2 + zR_3 \mid x,y,z \in R[j]\} \\
& A[Q_i] = \{xR_1 + yQ_i + z\overline{Q}_i \mid x,y,z \in R[j]\}(i=1,2,3) \\
& (A[\overline{Q}_i] = \{xR_1 + y\overline{Q}_i + zQ_i \mid x,y,z \in R[j]\}(i=1,2,3))
\end{aligned}
\right.
$$

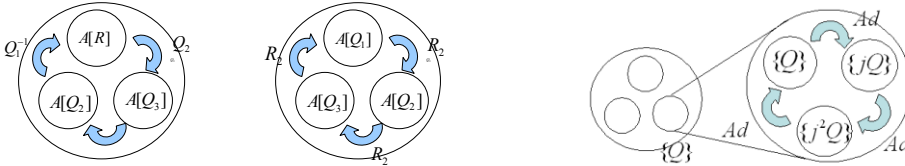We notice that the extension is unique . Namely we have

$$N = A[R] = A[Q_1] = A[Q_2] = A[Q_3]$$

(2) $Q_i, \overline{Q}_j (i,j=1,2,3)$ give a part of generators of the Galois group of $N : N = B'[\sqrt[3]{1}]$:
Namely putting $A_U[R] = \{xR_1 + yUR_2 + z\overline{U}R_3 \mid x,y,z \in R[j]\}$, where $U = Q_i, \overline{Q}_j (i,j=1,2,3)$,
we have Galois extensions (θ-equivalent):

$$
\left\{
\begin{aligned}
& (1) \quad A_{Q_1}[R] = A[Q_2], A_{Q_2}[R] = A[Q_3], \\
& \qquad A_{\overline{Q}_1}[R] = A[Q_3], A_{\overline{Q}_2}[R] = A[Q_1], \\
& (2) \quad A_{R_2}[Q_1] = A[Q_2], A_{R_2}[Q_2] = A[Q_3], A_{R_2}[Q_3] = A[Q_1], \\
& \qquad A_{\overline{Q}_1}[R] = A[Q_3], A_{\overline{Q}_2}[R] = A[Q_1], A_{\overline{Q}_3}[R] = A[Q_2],
\end{aligned}
\right.
$$

(3) The Ajoint operation gives a part of generators of Galois group of $N = \sqrt[3]{I_n}[B']$
(Ad-equivalent):

$$
\left\{
\begin{aligned}
& Ad_{Q_i}R_1 = R_1, Ad_{Q_i}R_2 = jR_2, Ad_{Q_{i1}}R_3 = j^2R_3 \ (i=1,2,3), \\
& Ad_{Q_i}Q_1 = Q_1, Ad_{Q_i}Q_2 = jQ_2, Ad_{Q_{i1}}Q_3 = j^2Q_3 \ (i=1,2,3), \\
& Ad_{Q_i}\overline{Q}_1 = \overline{Q}_1, Ad_{Q_i}\overline{Q}_2 = j^2\overline{Q}_2, Ad_{Q_{i1}}\overline{Q}_3 = j\overline{Q}_3 \ (i=1,2,3),
\end{aligned}
\right.
$$

3.THE GALOIS EXTENSION STRUCTRE ON su(3)

In this section we discuss the structure of the Galois extension on su(3).
(1) At first we write up the basis of the algebra ([5]).

$$
f_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad
f_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad
f_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

$$
f_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad
f_5 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad
f_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \quad
f_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad
f_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
$$

(3) We consider the linear subspace L_1 generated by the following 3 elements:

$$L_1: \qquad e_1 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Also we introduce the following two linear subspaces L_2 and L_3:

$$L_2: \qquad e'_1 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad e'_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad e'_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}$$

$$L_3: \qquad e''_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \qquad e''_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad e''_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

REMARK We notice the following relation $f_8 = 1/\sqrt{3}(e'_3 + e''_3)$. Hence we see that $e_1, e_2, .., e''_3$ constitute the basis of su(3) omitting one of e_3, e'_3, e''_3 .

Then we can prove the following theorem:

THEOREM III

We have the binary and ternary Galois extension structures on su(3):

(1) We have the following Adjoint strucutre on $L_i (i = 1,2,3)$.

$$\left\{ \begin{array}{l} He_1 H^{-1} = -e_2, \ He_2 H^{-1} = e_1, \ He_3 H^{-1} = e_3, \\ H' e'_1 H'^{-1} = -e'_2, \ H' e'_2 H'^{-1} = -e'_1, \ H' e_3 H'^{-1} = e'_3, \\ H' e''_1 H'^{-1} = e''_2, \ H' e''_2 H'^{-1} = e''_1, \ H' e''_3 H'^{-1} = e''_3, \end{array} \right.$$

where

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix},$$

(2) We can obtain the following commutation relation:

$$\left\{ \begin{array}{l} e_1^2 = e_2^2 = e_3^2 = -1_2 \\ e_1 e_2 = -e_2 e_1 = e_3, e_2 e_3 = -e_3 e_2 = e_1, e_3 e_1 = -e_1 e_3 = e_3, \end{array} \right.$$

where $1_2 = diag[1,1,0]$. After the central extension, we have the Clifford algebra which is isomorphic to Quaternion algebra. For the case of e'_i and $e''_i (i = 1,2,3)$, we have the same assertions. Hence we can define the binary non-commutative Galois structure on $L_i (i = 1,2,3)$. We notice that we can introduce three Dirac operators . This is directly connected the three quarks for the Gell-Mann quark model ([5]).

(3) $\{e_i, e'_i, e''_i\}$ $(i = 1,2,3)$ constitute the ternary Galois extensions by use of the following Adjoint operators:
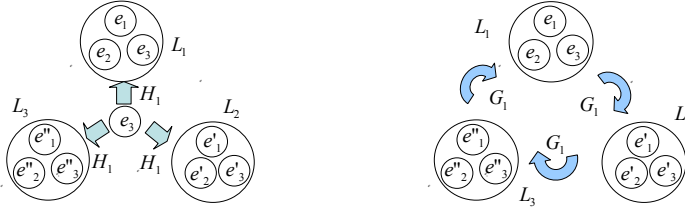
$$\left\{ \begin{array}{l} G_1 e_k G_1^{-1} = e''_k (k = 1,2,3), \ G_1 e'_k G_1^{-1} = e_k (k = 1,2,3), \\ G_1 e''_k G_1^{-1} = e'_k (k = 1,2), \ G_1 e''_3 G_1^{-1} = -e'_3 \end{array} \right.$$

where

$$G_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} (= T_2 \text{ in Proposition 4})$$

(4) Hence su(3) has the following non-commutative Galois extension:

(1) $su(3) = L_1 \cup L_2 \cup L_3$

(2) $L_i (i = 1,2,3)$ is isomorphic to su(2) and it is a binary Galois extension $L_i = B_0[\sqrt[2]{1_3}]$ over $B_0 = R[e_3.]$

(3) su(3) is a ternary Galois extension $B[\sqrt[3]{1_3}]$ over $B = su(2)$

PROOF : The assertions follow from the direct calculations and may be omitted.

4. A METHOD OF NON-COMMUTATIVE GALOIS EXTENSION TO BINARY AND TERNARY CLIFFORD ANALYSIS

In this section we introduce concepts of binary and ternary Clifford algebras and discuss the relationship between the Clifford analysis and non-commutative Galois extensions. We introduce Dirac operators and Klein-Gordon operators for the both Clifford algebras.

(1) Binary Clifford algebras and Galois extensions

We show that a special class of binary Galois extensions introduces binary Clifford algebras. We call the usual Clifford algebra as binary Clifford algebra. Namely we put the following definition:

DEFINITION 4

An algebra with generators $\{T_1, T_2, .., T_n\}(n = 2^p)$ is called binary Clifford algebra, when we have the following commutation relations:

$$T_i T_j + T_j T_i = \pm 2\delta_{ij} 1 \, (i, j = 1,2,..,n) \,.$$

Then we can introduce the following operators on the n-dimensional Euclidean space:

$$\begin{cases} D = T_1 \dfrac{\partial}{\partial x_1} + T_2 \dfrac{\partial}{\partial x_2} + ... + T_n \dfrac{\partial}{\partial x_n} \\ D^* = T^*_1 \dfrac{\partial}{\partial y_1} + T^*_2 \dfrac{\partial}{\partial y_2} + ... + T^*_n \dfrac{\partial}{\partial y_n} \, (T^*_j = -T_j (j = 1,2,..,n)) \end{cases}$$

The operator is called Dirac operator and its conjugate operators when they satisfy the following condition:

$$\Delta = D*D = DD^*, \quad \Delta = \mp(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \dots + \frac{\partial^2}{\partial x_n{}^2}) \otimes 1_n$$

The operator is called the binary Lapalce operator.

Next we proceed to the connections between non-commutative Galois extensions and binary Clifford algebras. At first we notice that non-commutative Galois extensions do not necessarily define a Clifford algebra (see example below). Hence we can make the following definition:

DEFINITION 5
We take a successive binary non-commutative Galois extension : $\{T_1, T_2, \dots, T_n\}(n = 2^p)$. A pair $\{T_a, T_b\}$ is called Clifford pair, when they satisfy the following condition:

$$T_a T_b + T_b T_a = \pm 2\delta^{ab} I_n$$

EXAMPLE: We see that we have only one Clifford pair $\{e_1, e_4\}$ for $C \times C$ Also we see that each pair $\{e_i, e_j\}(i \neq 1, j \neq 1, i \neq j)$ of H is a Clifford pair.

$$C \times C = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4$$
$$= \left\{ x_1 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + x_2 \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix} + x_3 \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} + x_4 \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\}$$

Then we can prove the following theorem:

THEOREM V
When a Clifford algebra A with generators $\{T_1, T_2, \dots, T_n\}(n = 2^p)$ is given, then there exists a sequence of successive non-commutative binary Galois extensions which defines the Clifford algebra. Namely we have the following:

$$T_i T_j + T_j T_i = -2\delta_{ij} I_n \implies A_k = A_{k-1}[\sqrt[2]{-I_n}] \ (k = 1, 2, .., m)(A = A_m, A_0 = B)$$

PROOF: We prove the assertion by the induction. The quaternion numbers are obtained by the non-commutative Galois extension from the complex numbers. Next we choose a Clifford algebra with generators: $\{T_1, T_2, \dots, T_n\}(n = 2^p)$. Putting

$$\hat{T}_i = \begin{pmatrix} T_i & 0 \\ 0 & -T_i \end{pmatrix} (i = 1, 2, .., n), \ \hat{T}_{n+1} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \ \hat{T}_{n+2} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix},$$

we can make a successive binary Galois extension: $A_{n+1} = A_n[\hat{T}_{n+1}], A_{n+2} = A_{n+1}[\hat{T}_{n+2}]$ which also defines a Clifford algebra with the commutation relations: $\hat{T}_i \hat{T}_j + \hat{T}_j \hat{T}_i = -2\delta_{ij} I_{2m}$

(2) Ternary Clifford algebras and Ġalois extensions
Next we proceed to the construction of the ternary Clifford analysis by Galois extensions.

DEFINITION 6

An algebra which is generated by $\{T_1, T_2, T_3\}$ is called ternary Clifford algebra when it satisfies the following commutation relations:

$$\begin{cases} T_a T_b T_c + T_b T_c T_a + T_c T_b T_a = 3\eta^{abc} E_3 \\ \eta^{abc} = \eta^{bca} = \eta^{cab} \\ \eta^{111} = \eta^{222} = \eta^{333} = 1, \eta^{123} = \eta^{231} = \eta^{321} = j^2, \\ \eta^{321} = \eta^{213} = \eta^{132} = j \end{cases}$$

Next we proceed to the derivation of field operators from a ternary Galois extension.

Choosing $\{T_1, T_2, T_3\}$, we introduce the following three operators on the 3-dimensional Euclidean space:

$$\begin{cases} D = T_1 \dfrac{\partial}{\partial x_1} + T_2 \dfrac{\partial}{\partial x_2} + T_3 \dfrac{\partial}{\partial x_3} \\ D^* = T_1 \dfrac{\partial}{\partial x_1} + j^2 T_2 \dfrac{\partial}{\partial x_2} + j T_3 \dfrac{\partial}{\partial x_3} \\ D** = T_1 \dfrac{\partial}{\partial x_1} + j T_2 \dfrac{\partial}{\partial x_2} + j^2 T_3 \dfrac{\partial}{\partial x_3} \end{cases}$$

The operators are called Dirac operator and its conjugate operators when they satisfy the following condition:

$$\Delta = D D^* D**, \quad \Delta = \left(\dfrac{\partial^3}{\partial x_1^{\,3}} + \dfrac{\partial^3}{\partial x_2^{\,3}} + \dfrac{\partial^3}{\partial x_3^{\,3}} - 3 \dfrac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} \right) \otimes 1_3$$

The operator is called the ternary Klein Gordon operator.

(3) Binary and ternary Dirac operators for Nonion algebra:

We begin with introducing the following concept of ternary Clifford triple:

DEFINITION 7

We take a successive ternary non-commutative Galois extension : $\{T_1, T_2, ..., T_n\}$ $(n = 3^p)$.
A triple $\{T_a, T_b, T_c\}$ is called Clifford triple, when it generate the ternary Clifford algebra:

At first we are concerened with the binary and ternary Dirac operators on B.

PROPOSITION 8

From the linear basis $\{T_1, T_2, T_3\}$ of the algebra B', we can introduce the binary and ternary Dirac operators:

$$\begin{cases} D_t = T_1 \dfrac{\partial}{\partial y_1} + T_3 \dfrac{\partial}{\partial y_2} + T_2 \dfrac{\partial}{\partial y_3} \\ D_t = T_1 \dfrac{\partial}{\partial y_1} + j T_3 \dfrac{\partial}{\partial y_2} + j^2 T_2 \dfrac{\partial}{\partial y_3} \\ D_t = T_1 \dfrac{\partial}{\partial y_1} + j^2 T_3 \dfrac{\partial}{\partial y_2} + j T_2 \dfrac{\partial}{\partial y_3} \end{cases}$$

PROOF: The proof is a direct calculation by use of the table and may be omitted.

We can prove the following theorem:

THEOREM V

(1) The ternary triples $\{X_1, X_2, X_3\}$ which are generated by the linear basis can be listed as follows:

$$\{Q_1,Q_1,Q_1\} \ \{Q_2,Q_2,Q_2\} \ \{Q_3,Q_3,Q_3\} \ \{Q_1,Q_2,Q_3\} \ \{R_2,R_2,R_2\}$$
$$\{R_1,Q_1,\overline{Q}_1\} \ \{R_1,Q_2,\overline{Q}_2\} \ \{R_1,Q_3,\overline{Q}_3\} \ \{R_1,R_1,R_1\} \ \{R_1,R_2,R_3\}$$
$$\{\overline{Q}_1,\overline{Q}_1,\overline{Q}_1\} \ \{\overline{Q}_2,\overline{Q}_2,\overline{Q}_2\} \ \{\overline{Q}_3,\overline{Q}_3,\overline{Q}_3\} \ \{\overline{Q}_1,\overline{Q}_2,\overline{Q}_3\} \ \{\overline{R}_2,\overline{R}_2,\overline{R}_2\}$$

Hence the ternary Dirac operator is defined by the Clifford triple $\{X_1, X_2, X_3\}$:

$$\left\{ \begin{array}{l} D_t = X_1 \dfrac{\partial}{\partial y_1} + X_3 \dfrac{\partial}{\partial y_2} + X_2 \dfrac{\partial}{\partial y_3} \\[2mm] D_t = X_1 \dfrac{\partial}{\partial y_1} + jX_3 \dfrac{\partial}{\partial y_2} + j^2 X_2 \dfrac{\partial}{\partial y_3} \\[2mm] D_t = X_1 \dfrac{\partial}{\partial y_1} + j^2 X_3 \dfrac{\partial}{\partial y_2} + jX_2 \dfrac{\partial}{\partial y_3} \end{array} \right.$$

(2) Binary and ternary Dirac operators on su(3)

Next we proceed to the Dirac operators for su(3). From the Clifford structure

$$\left\{ \begin{array}{l} e_1^{\ 2} = e_2^{\ 2} = e_3^{\ 2} = -1_2 \\[1mm] e_1 e_2 = -e_2 e_1 = e_3, e_2 e_3 = -e_3 e_2 = e_1, e_3 e_1 = -e_1 e_3 = e_3, \end{array} \right.$$

we can introduce the binary Dirac operators: Making the central extension by e_0 , we have the Dirac operators for $\{e_0, e_1, e_2, e_3\}$:

$$\left\{ \begin{array}{l} D = e_0 \dfrac{\partial}{\partial x_0} + e_1 \dfrac{\partial}{\partial x_1} + e_2 \dfrac{\partial}{\partial x_1}. + e_3 \dfrac{\partial}{\partial x_3} \\[2mm] \overline{D} = \overline{e}_0 \dfrac{\partial}{\partial x_0} + \overline{e}_1 \dfrac{\partial}{\partial x_1} + .\overline{e}_2 \dfrac{\partial}{\partial x_1} + \overline{e}_3 \dfrac{\partial}{\partial x_3} \end{array} \right.$$

We can obtain the binary Dirac operators for $\{e'_0, e'_1, e'_2, e'_3\}$ in a similar manner.   .
Next we proceed to the introduction of the ternary Dirac operator for $\{e_0, e_1, e_2, e_3\}$.

$$\left\{ \begin{array}{l} D = T_1 \dfrac{\partial}{\partial \theta_1} + G_1 \dfrac{\partial}{\partial \theta_2} + G_1^2 \dfrac{\partial}{\partial \theta_3} \\[2mm] D* = T_1 \dfrac{\partial}{\partial \theta_1} + j^2 G_1 \dfrac{\partial}{\partial \theta_2} + jG_1^2 \dfrac{\partial}{\partial \theta_3} \\[2mm] D** = T_1 \dfrac{\partial}{\partial \theta_1} + jG_1 \dfrac{\partial}{\partial \theta_2} + j^2 G_1^2 \dfrac{\partial}{\partial \theta_3} \end{array} \right.$$

For $\{e'_0, e'_1, e'_2, e'_3\}$ and $\{e''_0, e''_1, e''_2, e''_3\}$, we can define the ternary Dirac operator, replacing G_1 with G_2 and G_3 respectively:

$$G_2 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & j \\ 1 & 0 & 0 \end{pmatrix}, \qquad G_3 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & j^2 \\ 1 & 0 & 0 \end{pmatrix},$$

APPLICATION TO THE THEORY OF ELEMENTARY PARTICLE

We give two applications of a method of non-commutative Galois theory to the theory of elementary particles. The details will be given in another paper.

(1) The generation of elementary particles can be described by use of the Galois extensions. At the very beginning of the universe, there exists only one photon. This can be given the identity matrix. Then particles and anti-particles are produced and mesons are created. This process can be descried by binary extensions. Then the quark-baryon phase transitions happened and baryons are born. This process can be described by the successive binary and ternary Galois extensions . We notice that the following corresponding between the binary and ternary extensions.

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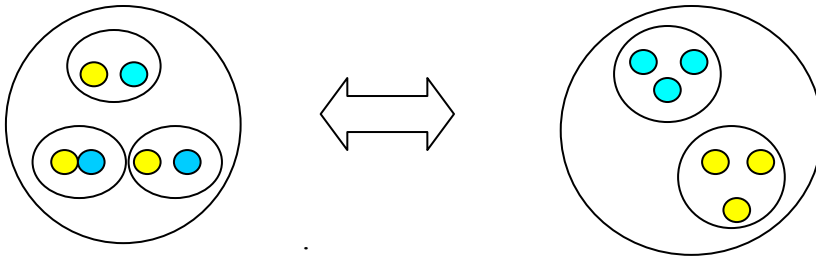
(2) The second application is the construction of quark models. We can realize the Gell-Mann model by use of the Galois extension structure on su(3). In fact we can introduce three quarks by $\{e_0, e_1, e_2, e_3\}$, $\{e'_0, e'_1, e'_2, e'_3\}$ and $\{e''_0, e''_1, e''_2, e''_3\}$. Then we can realize the Gell-Mann model by use of the binary and ternary Galois extensions.

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