ON INTERPOLATION FUNCTION OF THE BERNSTEIN POLYNOMIALS

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Abstract

The Bernstein polynomials are used for important applications in many branches of Mathematics and the other sciences, for instance, approximation theory, probability theory, statistic theory, number theory, the solution of the differential equations, numerical analysis, constructing Bezier curves, q-calculus, operator theory and applications in computer graphics. The Bernstein polynomials are used to construct Bezier curves. Bezier was an engineer with the Renault car company and set out in the early 1960's to develop a curve formulation which would lend itself to shape design. Engineers may find it most understandable to think of Bezier curves in terms of the center of mass of a set of point masses. Therefore, in this paper, we study on generating functions and functional equations for these polynomials. By applying these functions, we investigate interpolation function and many properties of these polynomials.

Key Words and Phrases. Bernstein polynomials, Bezier curve, Generating function, Interpolation function, Mellin transformation, Gamma function, , Bernoulli polynomials of higher-order, Stirling numbers of the second kind.

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1. Introduction

In this section we can use the following notation:

$$[x:q]=\frac{1-q^{x}}{1-q}.$$

Observe that

$$\lim_{q\to 1}[x:q]=x.$$

If $q\in\mathbb{C}$, we assume that $\mid q\mid<1$. If $q\in\mathbb{R}$, we assume that $0<q<1$.

In this paper, we modify generating functions for the q-Bernstein polynomials, which are many applications: in approximations of functions, in statistics, in numerical analysis, in p-adic analysis and in the solution of differential equations. Using the functional equations for the generating functions and Laplace transform, we derive fundamental properties and some identities of the q-Bernstein polynomials.

The remainder of this paper is summarized as follows:

Section 2: We construct generating function of the q-Bernstein basis functions. Using these generating, some identities and properties of the q-Bernstein basis functions can be given.

Section 3: We give some properties for the q-Bernstein basis functions (Partition of unity, Alternating sum, Subdivision property).

Section 4: We give recurrence retaliations and derivative of the q-Bernstein basis functions.

Section 5: We give application related to the Laplace transform and generating function.
Section 6: We construct interpolation function for the q-Bernstein polynomials.
Section 7: We give further remarks on the q-Bezier curves and integral representation for the q-Bernstein basis functions.

2. Modified the generating function for the q-Bernstein basis type functions

Definition 1. *Let $x \in [0,1]$. Let k and n be nonnegative integers with $n \geq k$. Then we define*

$$\mathfrak{b}_k^n(x;q) = \left(\begin{array}{c} n \\ k \end{array} \right) [x:q]^k q^{(n-k)x} \left[(1-x):q \right]^{n-k}, \tag{2.1}$$

where

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

and $k = 0, 1, 2, ..., n$.

Generating functions for the q-Bernstein basis functions $\mathfrak{b}_k^n(x;q)$ can be defined as follows:

Definition 2. *Let $x \in [0,1]$ and $t \in \mathbb{C}$. Let k be nonnegative integers. Then we define*

$$\mathcal{F}_{k,q}(t,x) = \sum_{n=0}^{\infty} \mathfrak{b}_k^n(x;q) \frac{t^n}{n!}. \tag{2.2}$$

Observe that there is one generating function for each value of k.
We modify generating function for the q-Bernstein type basis functions as follows:

Theorem 1. *Let $x \in [0,1]$ and $t \in \mathbb{C}$. Then we have*

$$\mathcal{F}_{k,q}(t,x) = \frac{1}{k!} t^k [x:q]^k \exp\left(q^x \left[(1-x):q \right] t \right). \tag{2.3}$$

Proof. By substituting (2.1) into the right hand side of (2.2), we obtain

$$\begin{aligned} \mathcal{F}_{k,q}(t,x) &= \sum_{n=0}^{\infty} \left(\left(\begin{array}{c} n \\ k \end{array} \right) [x:q]^k q^{(n-k)x} \left[(1-x):q \right]^{n-k} \right) \frac{t^n}{n!} \\ &= \frac{t^k [x:q]^k}{k!} \sum_{n=k}^{\infty} \frac{\left(q^x \left[(1-x):q \right] t \right)^{n-k}}{(n-k)!}. \end{aligned}$$

The right hand side of the above equation is a Taylor series for

$$\exp\left(q^x \left[(1-x):q \right] t \right),$$

thus we arrive at the desired result. $\square$

3. Some properties for the q-Bernstein basis functions are given as follows

In [13] and [14], Simsek present much background material on computations functional equation of the generating function for the Bernstein basis functions. We give some functional equations which are used to find some new identities related to the q-Bernstein basis functions. Our method is similar to that of Simsek's [13].

3.1. **Partition of unity.** The polynomials $\mathfrak{b}_k^n(x;q)$ have **partition of unity**, which is given by the following theorem.

Theorem 2. *(Sum of the polynomials $\mathfrak{b}_k^n(x;q)$)*

$$\sum_{k=0}^{n} \mathfrak{b}_k^n(x;q) = 1.$$

Proof. By using (2.3), we have

$$\sum_{k=0}^{\infty} \mathcal{F}_{k,q}(t,x) = \exp\left(q^x\left[1-x:q\right]t\right) \sum_{k=0}^{\infty} \frac{1}{k!} t^k [x:q]^k.$$

The right hand side of the above equation is a Taylor series for

$$\exp\left(\left[x:q\right]t\right),$$

thus we obtain

$$\sum_{k=0}^{\infty} \mathcal{F}_{k,q}(t,x) = \exp\left(\left(q^x\left[1-x:q\right]+\left[x:q\right]\right)t\right). \tag{3.1}$$

If we substitute the following identity

$$[a+b:q] = [a:q] + q^a\left[b:q\right],$$

into the right-hand side of (3.1), we find that

$$\sum_{k=0}^{\infty} \mathcal{F}_{k,q}(t,x) = \exp(t).$$

By using (2.2) and Taylor expansion of $\exp(t)$ in the above equation, we get

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathfrak{b}_k^n(x;q) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we arrive at the desired result. □

Remark 1. *Simsek and Acikgoz* [15] *defined the q-Bernstein type basis functions as follows:*

$$Y_n(k,x;q) = \left(\begin{array}{c} n \\ k \end{array} \right) [x:q]^k \left[1-x:q\right]^{n-k}. \tag{3.2}$$

*The polynomials $Y_n(k,x;q)$ have not **partition of unity**. That is*

$$\sum_{k=0}^{n} Y_n(k,x;q) = \left(\left[x:q\right]+\left[1-x:q\right]\right)^n \neq 1. \tag{3.3}$$

By using (2.1) and (3.2), one can easily see that

$$\mathfrak{b}_k^n(x;q) = q^{x(n-k)} Y_n(k,x;q).$$

Thus generating functions of the polynomials $\mathfrak{b}_k^n(x;q)$ give us modification that of the polynomials $Y_n(k,x;q)$.

Remark 2. *In the special case when $q \to 1$, Definition 1 immediately yields the corresponding well known results concerning the classical Bernstein basis functions $B_k^n(x)$:*

$$B_k^n(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}, \tag{3.4}$$

where $k = 0, 1, \cdots, n$ and $x \in [0,1]$ cf. ([1]-[15]*).*

Since

$$q^x \left[(1-x) : q\right] = 1 - [x : q],$$

we modify Definition 1 as follows:

$$\mathfrak{b}_k^n(x;q) = \begin{pmatrix} n \\ k \end{pmatrix} [x : q]^k \left(1 - [x : q]\right)^{n-k}$$

or

$$\mathfrak{b}_k^n(x;q) = B_k^n\left([x : q]\right).$$

3.2. **Alternating sum.** By using (2.3), we obtain the following functional equation:

$$\sum_{k=0}^{\infty} (-1)^k \mathcal{F}_{k,q}(t,x) = \exp\left(\left(q^x [1-x : q] - [x : q]\right) t\right). \tag{3.5}$$

By using same method with the author [14] and (3.5), we derive a formula for the alternating sum which is given the following Theorem:

Theorem 3. *(Alternating sum)*

$$\sum_{k=0}^{n} (-1)^k \mathfrak{b}_k^n(x;q) = \left(1 - 2[x : q]\right)^n. \tag{3.6}$$

Remark 3. *If we let $q \to 1$ in (3.6), then we arrive at the well-known Goldman's results* [4]-[3, Chapter 5, pages 299-306] *and see also* [14]*:*

$$\sum_{k=0}^{n} (-1)^k B_k^n(x) = (1-2x)^n.$$

3.3. **Subdivision property.** By using similar method of Simsek's [13], we define the following functional equation:

$$\mathcal{F}_{k,q}(t,xy) = \mathcal{F}_{k,q}(t\,[y : q^x], x) \exp\left(q^{xy} [1-y : q^x]\, t\right). \tag{3.7}$$

By using the above functional equation, we derive subdivision property for the q-Bernstein basis functions by the following theorem:

Theorem 4. *Then the following identity holds:*

$$\mathfrak{b}_j^n(xy;q) = \sum_{k=j}^{n} \mathfrak{b}_j^k(x;q)\mathfrak{b}_k^n(y;q^x).$$

Remark 4. *If we let $q \to 1$ in Theorem 4, we have*

$$B_j^n(xy) = \sum_{k=j}^{n} B_j^k(x) B_k^n(y). \tag{3.8}$$

The above identity is fundamental in subdivision property for the Bernstein basis functions cf. ([4]-[3, Chapter 5, pages 299-306], [14], [13]*).*

4. Recurrence retaliations and derivative of the q-Bernstein basis functions:

In this section, we give higher order derivatives of the Bernstein basis functions. We define

$$\mathcal{F}_{k,q}(t,x) = g_{k,q}(t,x)h_q(t,x), \tag{4.1}$$

where

$$g_{k,q}(t,x) = \frac{t^k \left[x:q\right]^k}{k!}$$

and

$$h_q(t,x) = \exp\left(q^x \left[1-x\right] t\right).$$

In this section we are going to differentiate (4.1) with respect to t to derive a recurrence relation for the Bernstein basis functions.

Using Leibnitz's formula for the vth derivative, with respect to t, we obtain the following *higher order partial differential equation*:

$$\frac{\partial^v \mathcal{F}_{k,q}(t,x)}{\partial t^v} = \sum_{j=0}^{v} \left(\begin{array}{c} v \\ j \end{array} \right) \left(\frac{\partial^j g_{k,q}(t,x)}{\partial t^j} \right) \left(\frac{\partial^{v-j} h_q(t,x)}{\partial t^{v-j}} \right). \tag{4.2}$$

From the above equation, we have the following theorem:

Theorem 5.

$$\frac{\partial^v \mathcal{F}_{k,q}(t,x)}{\partial t^v} = \sum_{j=0}^{v} \mathfrak{b}_j^v(x;q) \mathcal{F}_{k-j,q}(t,x). \tag{4.3}$$

By same method in [14] and [13], Theorem 5 is proved by induction on v using (4.2).

Using (2.2) and (3.4) in Theorem 5, we obtain a recurrence relation for the Bernstein basis functions:

Theorem 6.

$$\mathfrak{b}_k^n(x;q) = \sum_{j=0}^{v} \mathfrak{b}_j^v(x;q) \mathfrak{b}_{k-j}^{n-v}(x;q). \tag{4.4}$$

Proof. By substituting right hand side of (2.2) into (4.3), we get

$$\frac{\partial^v}{\partial t^v} \left(\sum_{n=0}^{\infty} \mathfrak{b}_k^n(x;q) \frac{t^n}{n!} \right) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{v} \mathfrak{b}_j^v(x;q) \mathfrak{b}_{k-j}^{n-v}(x;q) \right) \frac{t^n}{n!}.$$

Therefore

$$\sum_{n=v}^{\infty} \mathfrak{b}_k^n(x;q) \frac{t^{n-v}}{(n-v)!} = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{v} \mathfrak{b}_j^v(x;q) \mathfrak{b}_{k-j}^{n-v}(x;q) \right) \frac{t^n}{n!}.$$

From the above equation, we get

$$\sum_{n=v}^{\infty} \mathfrak{b}_k^n(x;q) \frac{t^{n-v}}{(n-v)!} = \sum_{n=v}^{\infty} \left(\sum_{j=0}^{v} \mathfrak{b}_j^v(x;q) \mathfrak{b}_{k-j}^{n-v}(x;q) \right) \frac{t^{n-v}}{(n-v)!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result. □

Remark 5. *If we let $q \to 1$ in (4.5), then we arrive at Theorem 9 in* [14].

By using (2.3), we derive derivative of the q-Bernstein basis functions for in the next theorem:

Theorem 7. *Let $x \in [0,1]$. Let k and n be nonnegative integers with $n \geq k$. Then we have*

$$\frac{d}{dx}\mathfrak{b}_k^n(x;q) = \frac{q^x \log(q^n)}{q-1}\left(\mathfrak{b}_{k-1}^{n-1}(x;q) - \mathfrak{b}_k^{n-1}(x;q)\right). \tag{4.5}$$

Remark 6. *If we let $q \to 1$ in (4.5), then we arrive at Corollary 1 in* [14].

5. Applications

In this section we apply Laplace transform to the generating function for the q-Bernstein basis function. We derive new identity.

From (2.3), we get the following generating functions:

$$e^{[x]t}\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^n}{n!} = \frac{[x:q]^k}{k!}t^k e^t. \tag{5.1}$$

$$e^{-t}\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^n}{n!} = \frac{[x:q]^k}{k!}t^k e^{-[x]t}. \tag{5.2}$$

$$e^{-q^x[1-x]t}\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q)\frac{t^n}{n!} = \frac{[x:q]^k}{k!}t^k. \tag{5.3}$$

Theorem 8.

$$\sum_{n=0}^{\infty}[x]\,\mathfrak{b}_k^n(x;q) = 1. \tag{5.4}$$

Proof. Integrate equation (5.2) (by parts) with respect to t from zero to infinity, we have

$$\sum_{n=0}^{\infty}\frac{\mathfrak{b}_k^n(x;q)}{n!}\int_0^{\infty}e^{-t}t^n dt = \frac{[x:q]^k}{k!}\int_0^{\infty}t^k e^{-[x:q]t}dt. \tag{5.5}$$

We here assume that

$$x > 0.$$

of the following Laplace transform of the function $f(t) = t^k$:

$$\mathcal{L}(t^k) = \frac{k!}{[x:q]^{k+1}},$$

on the both sides of (5.5), we find that

$$\sum_{n=0}^{\infty}\mathfrak{b}_k^n(x;q) = \frac{1}{[x:q]}.$$

Thus we arrive at the desired result. □

Remark 7. *If we let $q \to 1$ in (5.4), then we arrive at Theorem 15 in* [14].

6. Interpolation Function

In this section, we construct interpolation function for the q-Bernstein polynomials. This function interpolates the q-Bernstein polynomials at negative integers.

Let $s \in \mathbb{C}$, and $x \in R$ with $x \neq 1$. By applying the Mellin transformation to (2.3), we give integral representation of the interpolation function $I_q(s,k;x)$ as follows:

$$I_q(s,k;x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathcal{F}_{k,q}(-t,x)dt,$$

where $\Gamma(s)$ denotes the Euler gamma function. By using the above integral representation, we are now ready to define interpolation function of the q-Bernstein polynomials.

Definition 3. *Let k be a nonnegative integer. Let $s \in \mathbb{C}$, and $x \in R$ with $x \neq 1$. The interpolation function $I_q(s,k;x)$ is defined by*

$$I_q(s,k;x) = (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)\Gamma(k+1)} \frac{[x:q]^k}{q^{x(k+s)}[1-x:q]^{k+s}}.$$

Theorem 9. *Let n be a positive integer. Then we have*

$$I_q(-n,k;x) = \mathfrak{b}_k^n(x).$$

Proof of this theorem is same as that of Theorem 12 in [12]. So we omit it.

7. Further remarks

7.1. **Bezier curve.** The Bezier curves are constructed by the Bernstein polynomials and control points. The Bezier curves are widely used in computer graphics to model smooth curves. The history of the Bezier curves can be traced back to Pierre Bezier, who was an engineer with the Renault car company and set out in the early 1960's to develop a curve formulation which would lend itself to shape design. Engineers may find it most understandable to think of the Bezier curves in terms of the center of mass of a set of point masses.

q-Bezier curves $B(x:q)$ with control points $P_0, \cdots, P_n$ is defined by

$$B(x:q) = \sum_{k=0}^n P_k \mathfrak{b}_k^n(x).$$

Observe that if $q \to 1$, we have the standard Bezier curves

$$B(x:1) = B(x) = \sum_{k=0}^n P_k B_k^n(x) \text{ cf. [2]}.$$

If we substitute $\mathfrak{b}_k^n(x;q) = B_k^n([x:q])$ into the above equation, then q-Bezier curves have same properties as standard Bezier curves. Because the q-Bernstein basis functions are parametrization of the standard Bernstein basis functions. The the q-Bernstein basis functions might be the **affect** of q on the **shape of the curves**.

7.2. **Integral representation for the q-Bernstein basis functions.** In this section we derive very powerful result related to integral representation for the q-Bernstein basis functions, which can be obtained from generating function.

Integral representation for the q-Bernstein basis functions is given as follows:

$$\mathfrak{b}_k^n(x;q) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \mathcal{F}_{k,q}(z,x) \frac{dz}{z^{n+1}}, \tag{7.1}$$

where $\mathcal{C}$ is a circle around the origin and the integration is in positive direction, $z \in \mathbb{C}$, $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ and $x \in [0, 1]$.

In [12], we give integral representation for the q-Bernstein basis functions. Here we give in detail about this representation as follows:

By substituting (2.3) into (7.1) and using Cauchy Residue Theorem, we obtain

$$\frac{n!}{2\pi i}\int_{\mathcal{C}} \mathcal{F}_{k,q}(z,x)\frac{dz}{z^{n+1}} = \frac{n!}{2\pi i}\left(2\pi i Res\left(\frac{\mathcal{F}_{k,q}(t,x)}{z^{n+1}}, 0\right)\right).$$

We now compute residue of $\frac{\mathcal{F}_{k,q}(t,x)}{z^{n+1}}$ at $z = 0$ by Laurent series as follows:

$$\mathfrak{b}_0^n(x;q)\frac{1}{z^{n+1}} + \mathfrak{b}_k^n(x;q)\frac{1}{z^n} + \cdots + \frac{\mathfrak{b}_k^n(x;q)}{n!}\frac{1}{z} + \mathfrak{b}_k^{n+1}(x;q) + \cdots .$$

By using the above Laurent series, we have

$$Res\left(\frac{\mathcal{F}_{k,q}(t,x)}{z^{n+1}}, 0\right) = \frac{\mathfrak{b}_k^n(x;q)}{n!}.$$

Consequently, one can obtain easily arrive at (7.1).

We note that our method same as of that of Lopez and Temme' [9] and Kim et al [7].

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