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A NOTE ON THE CLIFFORD FOURIER-STIELTJES TRANSFORM AND ITS PROPERTIES

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Abstract. *The purpose of this article is to provide an overview of the real Clifford Fourier-Stieltjes transform (CFST) and of its important properties. Additionally, we introduce the definition of convolution of Clifford functions of bounded variation.*

1 INTRODUCTION

1.1 Function of bounded variation and its Fourier-Stieltjes transform

The concept of functions of bounded variation plays an important role in probability theory. Among the known attempts made in this direction, the most notable ones are due to Beurling [1], Bochner [2, 6], and Cramér [10].

Let $\sigma(x)$ be a nondecreasing real or complex-valued function of the real variable x, having bounded variation on the whole real axis: $\int_{\mathbb{R}} |d\sigma(x)| < \infty$. It is well known that $\sigma(x)$ has at most an enumerable set of discontinuity points. In such a point we define

$$\sigma(x) = \frac{1}{2}[\sigma(x+0) + \sigma(x-0)].$$

For any function $\sigma(x)$ as above, the expression

$$f(t) = \int_{\mathbb{R}} e^{itx} d\sigma(x), \quad -\infty < t < \infty \tag{1}$$

defines the *Fourier-Stieltjes transform* of $\sigma(x)$. The Fourier-Stieltjes transform (FST) is a well-known generalization of the classical Fourier transform, and is frequently applied in certain areas of theoretical and applied probability and stochastic processes contexts.

There has recently been much interest in the construction of higher dimensional counterparts of the Fourier-Stieltjes transform in the framework of quaternion and Clifford analyses [13, 14]. It is the object of the present paper to give an overview on the (real) Clifford Fourier-Stieltjes transform (CFST), and on some of its important properties [13]. The underlying functions are continuous functions of bounded variation defined in $\mathbb{R}^m$ and taking values in a Clifford algebra. We also introduce the definition of convolution of Clifford functions of bounded variation. The convolution is related to pairs of functions belonging to a certain class in the same way as in the classical case.

The used methods also allow a generalization to the case of Clifford functions that satisfy higher dimensional generalizations of Cauchy-Riemann or Dirac systems. We leave the details of this slight generalization to the interested reader.

1.2 Some basic concepts of Clifford analysis

In the present subsection, we review some definitions and basic algebraic facts of a special Clifford algebra of signature $(0, m)$. For more details, we refer the reader to [7, 16].

Let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^m$ with a product according to the multiplication rules:

$$e_i e_j + e_j e_i = -2\delta_{i,j}, \quad i, j = 1, \ldots, m,$$

where $\delta_{i,j}$ is the Kronecker symbol. Whence, the set $\{e_A : A \subseteq \{1, \ldots, m\}\}$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_r}$, $1 \le h_1 < \ldots < h_r \le m$, and $e_\phi = 1$ forms a basis of the 2^m-dimensional Clifford algebra $Cl_{0,m}$ over $\mathbb{R}$. Any Clifford number a in $Cl_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^m [a]_k$, where $[a]_k = \sum_{|A|=k} e_A a_A$ is the so-called *k-vector part* of a $(k = 0, 1, \ldots, m)$. The real vector space $\mathbb{R}^m$ will be embedded in $Cl_{0,m}$ by identifying the element $(x_1, \ldots, x_m) \in \mathbb{R}^m$ with the Clifford vector $\mathbf{x}$ given by

$$\mathbf{x} := e_1 x_1 + \cdots + e_m x_m.$$

It is worth noting that the square of a vector $\mathbf{x}$ is scalar-valued and equals the norm squared up to a minus sign: $\mathbf{x}^2 = -|\mathbf{x}|^2$. Throughout the paper, we consider $Cl_{0,m}$-valued functions defined in $\mathbb{R}^m$, i.e. functions of the form

$$f : \mathbb{R}^m \longrightarrow Cl_{0,m}, \quad f(\mathbf{x}) = \sum_A f_A(\mathbf{x}) e_A, \tag{2}$$

where f_A are real-valued functions defined in $\mathbb{R}^m$. Properties (like integrability, continuity or differentiability) that are ascribed to f have to be fulfilled by all components f_A.

Let

$$L^1(\mathbb{R}^m; Cl_{0,m}) := \{f \in \mathbb{R}^m \longrightarrow Cl_{0,m} : \int_{\mathbb{R}^m} |f(\mathbf{x})| d\sigma(\mathbf{x}) < \infty\}$$

denote the linear Hilbert space of integrable $Cl_{0,m}$-valued functions defined in $\mathbb{R}^m$. The *left-sided Clifford Fourier transform* (CFT) of $f \in L^1(\mathbb{R}^m; Cl_{0,m})$ is given by [9]

$$\mathcal{F}(f) : \mathbb{R}^m \longrightarrow Cl_{0,m}, \quad \mathcal{F}(f)(\boldsymbol{\omega}) := \int_{\mathbb{R}^m} \mathbf{e}(\boldsymbol{\omega}, \mathbf{x})\, f(\mathbf{x})\, d\sigma(\mathbf{x}), \tag{3}$$

where the kernel function

$$\mathbf{e} : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow Cl_{0,m}, \quad \mathbf{e}(\boldsymbol{\omega}, \mathbf{x}) := \prod_{i=1}^{m} e^{-e_{m+1-i}\, \omega_{m+1-i}\, x_{m+1-i}}.$$

For $i = 1, \ldots, m$, x_i will denote the *space* and ω_i the *angular frequency* variables. It is of interest to remark at this point that the product in (3) has to be performed in a fixed order since, in general, $\mathbf{e}(\boldsymbol{\omega}, \mathbf{x})$ does not commute with every element of $Cl_{0,m}$.

Under suitable conditions, the original signal f can be reconstructed from $\mathcal{F}(f)$ by the inverse transform. The *inverse (left-sided) Clifford Fourier transform* of $g \in L^1(\mathbb{R}^m; Cl_{0,m})$ is defined as follows:

$$\mathcal{F}^{-1}(g) : \mathbb{R}^m \longrightarrow Cl_{0,m}, \quad \mathcal{F}^{-1}(g)(\mathbf{x}) = \frac{1}{(2\pi)^m} \int_{\mathbb{R}^m} \overline{\mathbf{e}(\boldsymbol{\omega}, \mathbf{x})}\, g(\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}) \tag{4}$$

where $\overline{\mathbf{e}(\boldsymbol{\omega}, \mathbf{x})} := \prod_{i=1}^{m} e^{e_i\, \omega_i\, x_i}$ is called the inverse (left-sided) Clifford Fourier kernel.

2 THE CLIFFORD FOURIER-STIELTJES TRANSFORM AND ITS PROPERTIES

In this section we review the (real) Clifford Fourier-Stieltjes transform (CFST).

2.1 The (real) Clifford Fourier-Stieltjes transform

In the sequel, consider the function

$$\alpha : \mathbb{R}^m \longrightarrow Cl_{0,m}, \quad \mathbf{x} \longmapsto \alpha(\mathbf{x}) := \prod_{i=1}^{m} \alpha^i(x_i)$$

where $\alpha^i : \mathbb{R} \longrightarrow Cl_{0,m}$ are of *bounded variation* on $\mathbb{R}$:

$$\int_{\mathbb{R}} |d\alpha^i(x_i)| := M_i < \infty,$$

and such that $|\alpha^i| \le \delta_i$ for real numbers $\delta_i < \infty$. From here it follows that α is of bounded variation also, since it holds

$$\int_{\mathbb{R}^m} |d\alpha(\mathbf{x})| = \int_{\mathbb{R}^m} \prod_{i=1}^{m} |d\alpha^i(x_i)| = \prod_{i=1}^{m} M_i := M < \infty$$

and, such that

$$|\alpha(\mathbf{x})| \le \prod_{i=1}^{m} \delta_i := \delta < \infty.$$

The class of all such functions is denoted by (V). Unless otherwise stated, throughout this paper the product is meant to be performed in a fixed order:

$$\prod_{i=1}^{m} \alpha^i(x_i) := \alpha^1(x_1)\alpha^2(x_2)\ldots\alpha^m(x_m).$$

For the sets of discontinuity points of each $\alpha^i(x_i)$, we further assume that there exist the limits

$$\lim_{x_i \longrightarrow y_i+} \alpha^i(x_i) = \alpha^i(y_i + 0), \quad \text{and} \quad \lim_{x_i \longrightarrow y_i-} \alpha^i(x_i) = \alpha^i(y_i - 0) \quad (i = 1, \ldots, m)$$

(taken over all directions) for which

$$\alpha^i(y_i) = \frac{1}{2}\Big[\alpha^i(y_i + 0) + \alpha^i(y_i - 0)\Big]$$

holds almost everywhere on $\mathbb{R}$. Each function α^i is said to be a *Clifford distribution*.

The idea behind the construction of a Clifford counterpart of the Stieltjes integral is to replace the exponential function in (1) by a suitable (noncommutative) exponential product. Due to the noncommutativity of the algebra, we recall two different types of CFST [13]:

Definition 2.1. The *CFST* $\mathcal{FS}(\alpha) : \mathbb{R}^m \longrightarrow Cl_{0,m}$ of $\alpha(\mathbf{x})$ is defined as the Stieltjes integrals:

1. Right-sided CFST:

$$\mathcal{FS}_r(\alpha)(\boldsymbol{\omega}) := \int_{\mathbb{R}^m} d\alpha(\mathbf{x})\, \overline{\mathbf{e}(\boldsymbol{\omega}, \mathbf{x})}, \tag{5}$$

2. Left-sided CFST:

$$\mathcal{FS}_l(\alpha)(\boldsymbol{\omega}) := \int_{\mathbb{R}^m} \mathbf{e}(\boldsymbol{\omega}, -\mathbf{x})\, d\alpha(\mathbf{x}). \tag{6}$$

The function $\alpha(\mathbf{x})$ which generates (5) and (6) is essentially unique.

Remark 2.2. We recall the reader that, the order of the exponentials in (5)-(6) are fixed because of the noncommutativity of the underlying product. It is of interest to remark at this point that in the case $m = 2$ the formulae above reduce to the definitions for the right- and left-sided QFST introduced by the authors in [14]. Detailed information about the QFST and its properties can be found in [14]. For $m = 1$ the CFST is identical to the classical FST.

Remark 2.3. Throughout this text we may investigate the integral (5) only that, for simplicity, we denote by $\mathcal{FS}(\alpha)$. Nevertheless, all computations can be easily converted for (6). In view of (5) and (6), a straightforward calculation shows that:

$$\mathcal{FS}(\alpha)(\boldsymbol{\omega}) = \overline{\int_{\mathbb{R}^m} \mathbf{e}(\boldsymbol{\omega}, \mathbf{x})\, \overline{d\alpha(\mathbf{x})}} = \overline{\mathcal{FS}_l(\overline{\alpha})(-\boldsymbol{\omega})}.$$

From now on, we denote the class of functions which can be represented as (5) by $\mathcal{B}$. Functions in $\mathcal{B}$ are called *(right) Clifford Bochner functions* and $\mathcal{B}$ will be referred to as the *(right) Clifford Bochner set*. It follows that members of $\mathcal{B}$ are entire functions of the real variables ω_i.

It is immediately clear that $\mathcal{B}$ is a linear space, and every element f of $\mathcal{B}$ is a bounded uniformly continuous function:

$$|f(\boldsymbol{\omega})| \leq \int_{\mathbb{R}^m} |d\alpha(\mathbf{x})| = M < \infty. \tag{7}$$

We recall from [13] the following result.

Theorem 2.4. *If a function belongs to $\mathcal{B}$ and is identically equal to zero for all $\omega_i \leq 0$ ($i = 1, \ldots, m$), then it is the Fourier-Stieltjes transform of an absolutely continuous function.*

Proof. Let f be any function in $\mathcal{B}$. By hypothesis,

$$f(\boldsymbol{\omega}) = \int_{\mathbb{R}^m} d\alpha(\mathbf{x})\, \overline{\mathbf{e}(\boldsymbol{\omega}, \mathbf{x})}, \tag{8}$$

where $\int_{\mathbb{R}^m} |d\sigma(\mathbf{x})| = M < \infty$. For $\mathbf{x} = x_1 e_1 + \cdots + x_m e_m \in Cl_{0,m}$ we set

$$Cl_{0,m} \ni \tilde{\mathbf{x}} := \frac{1}{2} \sum_{i=1}^{m} (x_i^2 + 1) e_i.$$

Let $\mathbb{R}_0^{m,+} := \underbrace{\mathbb{R}_0^+ \times \cdots \times \mathbb{R}_0^+}_{m \text{ times}}$. We define the function $G(\mathbf{x})$ as follows

$$G(\mathbf{x}) := \frac{1}{(2\pi)^m} \int_{\mathbb{R}_0^{m,+}} f(\boldsymbol{\omega})\, \overline{\mathbf{e}(\boldsymbol{\omega}, \tilde{\mathbf{x}})} d\boldsymbol{\omega}.$$

Evidently $G(\mathbf{x})$ is analytic for all x_i ($i = 1, \ldots, m$) since it is the product of analytic functions for any fixed x_i. We suppose from now on that this condition is satisfied. From the definition of the function f follows that there exists a constant $M > 0$ so that $|f(\boldsymbol{\omega})| \leq M < \infty$. From (8) and since $f(\boldsymbol{\omega}) = 0$ for all $\omega_i \leq 0$ ($i = 1, \ldots, m$), we may write

$$|G(\mathbf{x})| \leq \frac{1}{(2\pi)^m} \int_{\mathbb{R}_0^{m,+}} |f(\boldsymbol{\omega})| \prod_{i=1}^{m} e^{-\frac{x_i^2+1}{2}\omega_i} d\omega_i \leq \frac{M}{(2\pi)^m} \prod_{i=1}^{m} \frac{2}{x_i^2 + 1}.$$

By straightforward calculation we may show that

$$\int_{\mathbb{R}^m} |G(\mathbf{x})| d\mathbf{x} \leq \frac{M}{(2\pi)^m} \prod_{i=1}^{m} \int_{\mathbb{R}} \frac{2}{x_i^2 + 1} dx_i = M < \infty.$$

This proves the theorem. □

For practical purposes, if $f \in \mathcal{B}$ is given then for any real variables ω_i, and real constants a_i $(i = 1, \ldots, m)$ a direct computation shows that

$$\begin{aligned}
|f(\boldsymbol{\omega})| &= \left| \left(\int_{-\infty}^{a_1} + \int_{a_1}^{\infty} \right) \left(\int_{-\infty}^{a_2} + \int_{a_2}^{\infty} \right) \cdots \left(\int_{-\infty}^{a_m} + \int_{a_m}^{\infty} \right) d\alpha(\mathbf{x}) \, \overline{\mathbf{e}(\boldsymbol{\omega}, \mathbf{x})} \right| \\
&\leq \prod_{i=1}^{m} \Big(\left|\alpha^i(\infty) - \alpha^i(a_{m+1-i})\right| + \left|\alpha^i(a_{m+1-i}) - \alpha^i(-\infty)\right| \Big)
\end{aligned}$$

since $\left|\overline{\mathbf{e}(\boldsymbol{\omega}, \mathbf{x})}\right| = 1$. If $f \in \mathcal{B}$ is given, then

$$f(\mathbf{0}) = \prod_{i=1}^{m} \Big(\alpha^i(\infty) - \alpha^i(-\infty) \Big).$$

In particular, a simple argument gives

$$f(-\boldsymbol{\omega}) = \int_{\mathbb{R}^m} d\alpha(\mathbf{x}) \, \overline{\mathbf{e}(-\boldsymbol{\omega}, \mathbf{x})} = \overline{\int_{\mathbb{R}^m} \mathbf{e}(-\boldsymbol{\omega}, \mathbf{x}) \, \overline{d\alpha(\mathbf{x})}} = \overline{g(\boldsymbol{\omega})}, \tag{9}$$

where g is any function which can be represented as $\overline{\mathcal{FS}_l(\overline{\alpha})}(\boldsymbol{\omega})$.

2.2 Lévy's inversion formula

This subsection provides an explicit formula for computing a Clifford function once its Fourier-Stieltjes integral is known.

Theorem 2.5. *For every $\alpha^i : \mathbb{R} \longrightarrow Cl_{0,m}$ $(i = 1, \ldots, m)$, we consider the functions*

$$g_i(\omega_i) = \int_{\mathbb{R}} d\alpha^i(x_i) \, e^{e_i \omega_i x_i}.$$

For any real numbers a and b the following equality holds:

$$\alpha^i(a) - \alpha^i(b) = \lim_{T \longrightarrow \infty} \frac{1}{2\pi} \int_{-T}^{T} g_i(\omega_i) \frac{e^{e_i b \omega_i} - e^{e_i a \omega_i}}{e_i \omega_i} d\omega_i. \tag{10}$$

In particular, it holds

$$\alpha^i(x_i + 0) - \alpha^i(x_i - 0) = \lim_{T \longrightarrow \infty} \frac{1}{2T} \int_{-T}^{T} g_i(\omega_i) e^{-e_i x_i \omega_i} d\omega_i.$$

Proof. We begin with the following observation:

$$\begin{aligned}
g_i(\omega_i)\Big(e^{-e_i a \omega_i} - e^{-e_i b \omega_i}\Big) &= \int_{\mathbb{R}} d\alpha^i(x_i) \, e^{e_i \omega_i (x_i - a)} - \int_{\mathbb{R}} d\alpha^i(x_i) \, e^{e_i \omega_i (x_i - b)} \\
&= \int_{\mathbb{R}} d\alpha^i(x_i + a) \, e^{e_i \omega_i x_i} - \int_{\mathbb{R}} d\alpha^i(x_i + b) \, e^{e_i \omega_i x_i}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\longrightarrow\infty}\Big[\alpha^i(T+a)e^{e_iT\omega_i} - \alpha^i(-T+a)e^{-e_iT\omega_i} \\
&\qquad\qquad -\alpha^i(T+b)e^{e_iT\omega_i} + \sigma^i(-T+b)e^{-e_iT\omega_i} \\
&\qquad\qquad\qquad - \int_{-T}^{T}\Big(\alpha^i(x_i+a) - \alpha^i(x_i+b)\Big)dx_i\, e^{e_i\omega_i x_i}e_i\omega_i\Big] \\
&= -\int_{\mathbb{R}}\Big(\alpha^i(x_i+a) - \alpha^i(x_i+b)\Big)dx_i\, e^{e_i\omega_i x_i}e_i\omega_i.
\end{aligned}
$$

Therefore, it is easy to see that

$$
g_i(\omega_i)\Big(e^{-e_ia\omega_i} - e^{-e_ib\omega_i}\Big)\frac{1}{-e_i\omega_i} = \int_{\mathbb{R}}\Big(\alpha^i(x_i+a) - \alpha^i(x_i+b)\Big)dx_i\, e^{e_i\omega_i x_i}.
$$

From the last equality and from the inverse Fourier transform formula we find

$$
\alpha^i(x_i+a) - \alpha^i(x_i+b) = \frac{1}{2\pi}\int_{\mathbb{R}} g_i(\omega_i)\frac{e^{-e_ia\omega_i} - e^{-e_ib\omega_i}}{-e_i\omega_i}\, e^{-e_ix_i\omega_i}d\omega_i,
$$

and, in particular taking $x_i = 0$ we find

$$
\alpha^i(a) - \alpha^i(b) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\omega_i)\frac{e^{-e_ia\omega_i} - e^{-e_ib\omega_i}}{-e_i\omega_i}d\omega_i,
$$

and this completes the proof. □

Formula (10) is known as the *Lévy's inversion formula*. It is immediately clear if two Clifford functions have the same Fourier-Stieltjes integral, then they are identical up to an additive (Clifford) constant.

3 UNIFORM CONTINUITY

In this section we discuss uniform continuity and its relationship to CFST. We begin by defining uniform continuity.

Definition 3.1. A Clifford function $f : \mathbb{R}^m \longrightarrow Cl_{0,m}$ is *uniformly continuous* on $\mathbb{R}^m$ if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(\boldsymbol{\omega}) - f(\mathbf{t})| < \epsilon$ for all $\boldsymbol{\omega}, \mathbf{t} \in \mathbb{R}^m$ whenever $|\boldsymbol{\omega} - \mathbf{t}| < \delta$.

We now prove some results related to the asymptotic behaviour of the CFST which run:

Proposition 3.2. *Let f be an element of $\mathcal{B}$. For any natural number n, let $f^n : \mathbb{R}^{m-1} \times [-n, n] \longrightarrow Cl_{0,m}$ be the function given by*

$$
f^n(\boldsymbol{\omega}) = \int_{-n}^{n}\int_{\mathbb{R}^{m-1}} d\alpha(\mathbf{x})\,\overline{\boldsymbol{e}(\boldsymbol{\omega},\mathbf{x})}.
$$

Then $f^n(\boldsymbol{\omega}) \longrightarrow_{n\longrightarrow\infty} f(\boldsymbol{\omega})$ uniformly. Also, if f^n are uniformly continuous functions then f is a uniformly continuous function.

Proof. A first straightforward computation shows that

$$\begin{aligned}
\left|f(\boldsymbol{\omega})-f^n(\boldsymbol{\omega})\right| &= \left|\int_{\mathbb{R}^m} d\alpha(\mathbf{x})\,\overline{\mathbf{e}(\boldsymbol{\omega},\mathbf{x})}-\int_{-n}^{n}\int_{\mathbb{R}^{m-1}} d\alpha(\mathbf{x})\,\overline{\mathbf{e}(\boldsymbol{\omega},\mathbf{x})}\right| \\
&= \left|\int_{-\infty}^{-n}\int_{\mathbb{R}^{m-1}} d\alpha(\mathbf{x})\,\overline{\mathbf{e}(\boldsymbol{\omega},\mathbf{x})}+\int_{n}^{\infty}\int_{\mathbb{R}^{m-1}} d\alpha(\mathbf{x})\,\overline{\mathbf{e}(\boldsymbol{\omega},\mathbf{x})}\right| \\
&\leq \prod_{i=1}^{m-1}\left|\alpha^i(\infty)-\alpha^i(-\infty)\right|\left|\alpha^m(-n)-\alpha^m(-\infty)\right| \\
&+ \prod_{i=1}^{m-1}\left|\alpha^i(\infty)-\alpha^i(-\infty)\right|\left|\alpha^m(\infty)-\alpha^m(n)\right|.
\end{aligned}$$

Moreover, having in mind that

$$\alpha^m(-n)-\alpha^m(-\infty)\longrightarrow_{n\longrightarrow\infty}0,\qquad\text{and}\qquad\alpha^m(\infty)-\alpha^m(n)\longrightarrow_{n\longrightarrow\infty}0,$$

and hence, $f^n(\boldsymbol{\omega})\longrightarrow_{n\longrightarrow\infty}f(\boldsymbol{\omega})$ uniformly. In addition, we claim that if $f^n(\boldsymbol{\omega})$ are uniformly continuous functions it follows that $f(\boldsymbol{\omega})$ is an uniformly continuous function. □

In like manner, we have an analogous result.

Proposition 3.3. *Let f be an element of $\mathcal{B}$. For any natural number n, let $f^n:[-n,n]\times\mathbb{R}^{m-1}\longrightarrow Cl_{0,m}$ be the function given by*

$$f^n(\boldsymbol{\omega})=\int_{\mathbb{R}^{m-1}}\int_{-n}^{n} d\alpha(\mathbf{x})\,\overline{\boldsymbol{e}(\boldsymbol{\omega},\mathbf{x})}.$$

Then $f^n(\boldsymbol{\omega})\longrightarrow_{n\longrightarrow\infty}f(\boldsymbol{\omega})$ uniformly. Also, if f^n are uniformly continuous functions then f is a uniformly continuous function.

Proposition 3.4. *Let f be an element of $\mathcal{B}$. For any natural number n, let $f^n:[-n,n]^m\longrightarrow Cl_{0,m}$ be the function given by*

$$f^n(\boldsymbol{\omega})=\int_{[-n,n]^m} d\alpha(\mathbf{x})\,\overline{\boldsymbol{e}(\boldsymbol{\omega},\mathbf{x})}.$$

Then $f^n(\boldsymbol{\omega})\longrightarrow_{n\longrightarrow\infty}f(\boldsymbol{\omega})$ uniformly. Also, if f^n are uniformly continuous functions then f is a uniformly continuous function.

Proof. For simplicity and without loss of generality we will prove the case $m=2$ only. We set

$$A:=d\alpha^1(x_1)d\alpha^2(x_2)\,e^{e_1\omega_1x_1}\,e^{e_2\omega_2x_2}.$$

The key to the proof is the simple observation that:

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A-\int_{-n}^{n}\int_{-n}^{n}A &= \int_{-\infty}^{-n}\int_{-\infty}^{-n}A+\int_{-\infty}^{-n}\int_{-n}^{n}A+\int_{-\infty}^{-n}\int_{n}^{\infty}A \\
&+ \int_{-n}^{n}\int_{-\infty}^{-n}A+\int_{-n}^{n}\int_{n}^{\infty}A \\
&+ \int_{n}^{\infty}\int_{-\infty}^{-n}A+\int_{n}^{\infty}\int_{-n}^{n}A+\int_{n}^{\infty}\int_{n}^{\infty}A.
\end{aligned}$$

For the remaining part of the proof, we use similar arguments as in Proposition 3.2. □

We come now to the main result of this section.

Theorem 3.5. *Let $f \in \mathcal{B}$ be given, and $g : \mathbb{R}^m \longrightarrow Cl_{0,m}$ be a continuous and absolutely integrable function. For any $\alpha : \mathbb{R}^m \longrightarrow Cl_{0,m}$ the following relations hold:*

1. $\displaystyle\int_{\mathbb{R}^m} f(\mathbf{t})g(\boldsymbol{\omega}-\mathbf{t})d\mathbf{t} = \int_{\mathbb{R}^m} d\alpha(\mathbf{x}) \int_{\mathbb{R}^m} \overline{\boldsymbol{e}(\boldsymbol{\omega},\mathbf{x})}\, g(\boldsymbol{\omega}-\mathbf{t})d\mathbf{t}$;

2. $\displaystyle\int_{\mathbb{R}^m} f(\mathbf{t})g(\mathbf{t})d\mathbf{t} = (2\pi)^m \int_{\mathbb{R}^m} d\alpha(\mathbf{x})\,\mathcal{F}^{-1}(g)(\mathbf{x})$.

Proof. Assume $g : \mathbb{R}^m \longrightarrow Cl_{0,m}$ to be a continuous and absolutely integrable function. For any real variables ρ_i $(i = 1, \ldots, m)$ let $\Omega_{\rho_i} := [-\rho_1, \rho_1] \times \cdots \times [-\rho_m, \rho_m] \subset \mathbb{R}^m$. We now define the function

$$R : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow Cl_{0,m}, \quad (\mathbf{x}, \boldsymbol{\rho}) \longmapsto R(\mathbf{x}, \boldsymbol{\rho}) := \int_{\Omega_{\rho_i}} \prod_{i=1}^{m} e^{e_i t_i x_i} g(\mathbf{t})d\mathbf{t}.$$

Take

$$f^n(\mathbf{t}) = \int_{[-n,n]^m} d\alpha(\mathbf{x}) \prod_{i=1}^{m} e^{e_i t_i x_i}.$$

Using the fact that g is an absolutely integrable function it follows

$$\begin{aligned} \int_{\Omega_{\rho_i}} f^n(\mathbf{t})g(\mathbf{t})d\mathbf{t} &= \int_{\Omega_{\rho_i}} \left(\int_{[-n,n]^m} d\alpha(\mathbf{x}) \prod_{i=1}^{m} e^{e_i t_i x_i} \right) g(\mathbf{t})d\mathbf{t} \\ &= \int_{[-n,n]^m} d\alpha(\mathbf{x}) \int_{\Omega_{\rho_i}} \prod_{i=1}^{m} e^{e_i t_i x_i} g(\mathbf{t})d\mathbf{t} \\ &= \int_{[-n,n]^m} d\alpha(\mathbf{x}) R(\mathbf{x}, \rho). \end{aligned} \tag{11}$$

From the last proposition we know that $\lim\limits_{n\longrightarrow\infty} f^n(\boldsymbol{\omega}) = f(\boldsymbol{\omega})$ uniformly. Moreover, since $g(\mathbf{t})$ is an absolutely integrable function it follows that $R(\mathbf{x}, \boldsymbol{\rho})$ is a uniformly continuous function. Hence

$$\lim_{n\longrightarrow\infty} \int_{\Omega_{\rho_i}} f^n(\mathbf{t})g(\mathbf{t})d\mathbf{t} = \int_{\Omega_{\rho_i}} f(\mathbf{t})g(\mathbf{t})d\mathbf{t}. \tag{12}$$

With this argument at hand, and from (11) we conclude that

$$\lim_{n\longrightarrow\infty} \int_{\Omega_{\rho_i}} f^n(\mathbf{t})g(\mathbf{t})d\mathbf{t} = \int_{\mathbb{R}^m} d\alpha(\mathbf{x}) R(\mathbf{x}, \boldsymbol{\rho}). \tag{13}$$

From the last equality and from (12) we obtain

$$\int_{\mathbb{R}^m} d\alpha(\mathbf{x}) R(\mathbf{x}, \boldsymbol{\rho}) = \int_{\Omega_{\rho_i}} f(\mathbf{t})g(\mathbf{t})d\mathbf{t}.$$

In addition, we have

$$R(\mathbf{x},\boldsymbol{\rho}) \underset{\substack{\rho_i \longrightarrow \infty \\ (i=1,\ldots,m)}}{\longrightarrow} \int_{\mathbb{R}^m} \prod_{i=1}^{m} e^{e_i t_i x_i} g(\mathbf{t}) d\mathbf{t}$$

and hence, for any fixed a_i it follows

$$\int_{\Omega_{a_i}} d\alpha(\mathbf{x}) R(\mathbf{x},\boldsymbol{\rho}) \underset{\substack{\rho_i \longrightarrow \infty \\ (i=1,\ldots,m)}}{\longrightarrow} \int_{\Omega_{a_i}} d\alpha(\mathbf{x}) \int_{\mathbb{R}^m} \prod_{i=1}^{m} e^{e_i t_i x_i} g(\mathbf{t}) d\mathbf{t}. \tag{14}$$

For the sake of simplicity, in the considerations to follow we will often omit the argument and write simply R instead of $R(\mathbf{x},\boldsymbol{\rho})$. Since R is uniformly bounded then there exists a positive constant M so that $|R| \leq M$ for all real numbers x_i and ρ_i. Without loss of generality, we will prove the case $m = 2$ only. A direct computation shows that

$$\begin{aligned} I \; := \; & \Big| \int_{-\infty}^{-a_1} \int_{-\infty}^{-a_2} d\alpha^1(x_1) d\alpha^2(x_2) R + \int_{-a_1}^{a_1} \int_{-\infty}^{-a_2} d\alpha^1(x_1) d\alpha^2(x_2) R \\ & + \int_{a_1}^{\infty} \int_{-\infty}^{-a_2} d\alpha^1(x_1) d\alpha^2(x_2) R + \int_{-\infty}^{-a_1} \int_{-a_2}^{a_2} d\alpha^1(x_1) d\alpha^2(x_2) R \\ & + \int_{a_1}^{\infty} \int_{-a_2}^{a_2} d\alpha^1(x_1) d\alpha^2(x_2) R + \int_{-\infty}^{-a_1} \int_{a_2}^{\infty} d\alpha^1(x_1) d\alpha^2(x_2) R \\ & + \int_{-a_1}^{a_1} \int_{a_2}^{\infty} d\alpha^1(x_1) d\alpha^2(x_2) R + \int_{a_1}^{\infty} \int_{a_2}^{\infty} d\alpha^1(x_1) d\alpha^2(x_2) R \Big| . \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} I \; \leq \; & M \Big[\big|\alpha^1(-a_2) - \alpha^1(-\infty)\big| \, \big|\alpha^2(-a_1) - \alpha^2(-\infty)\big| \\ + \; & \big|\alpha^1(-a_2) - \alpha^1(-\infty)\big| \, \big|\alpha^2(a_1) - \alpha^2(-a_1)\big| + \big|\alpha^1(-a_2) - \alpha^1(-\infty)\big| \, \big|\alpha^2(\infty) - \alpha^2(a_1)\big| \\ + \; & \big|\alpha^1(a_2) - \alpha^1(-a_2)\big| \, \big|\alpha^2(-a_1) - \alpha^2(-\infty)\big| + \big|\alpha^1(a_2) - \alpha^1(-a_2)\big| \, \big|\alpha^2(\infty) - \alpha^2(a_1)\big| \\ + \; & \big|\alpha^1(\infty) - \alpha^1(a_2)\big| \, \big|\alpha^2(-a_1) - \alpha^2(-\infty)\big| + \big|\alpha^1(\infty) - \alpha^1(a_2)\big| \, \big|\alpha^2(a_1) - \alpha^2(-a_1)\big| \\ + \; & \big|\alpha^1(\infty) - \alpha^1(a_2)\big| \, \big|\alpha^2(\infty) - \alpha^2(a_1)\big| \Big] \longrightarrow_{a_1,a_2 \longrightarrow \infty} 0. \end{aligned}$$

Extending the last inequality to a total of 2^m terms ($m > 2$), and using (14) we get

$$\lim_{a_i \longrightarrow \infty} \int_{\Omega_{a_i}} d\alpha(\mathbf{x}) R(\mathbf{x},\boldsymbol{\rho}) = \int_{\mathbb{R}^m} d\alpha(\mathbf{x}) \int_{\mathbb{R}^m} \prod_{i=1}^{m} e^{e_i t_i x_i} g(\mathbf{t}) d\mathbf{t}.$$

From here and (13) we find

$$\int_{\mathbb{R}^m} f(\mathbf{t}) g(\mathbf{t}) d\mathbf{t} = \int_{\mathbb{R}^m} d\alpha(\mathbf{x}) \int_{\mathbb{R}^m} \prod_{i=1}^{m} e^{e_i t_i x_i} g(\mathbf{t}) d\mathbf{t} = (2\pi)^m \int_{\mathbb{R}^m} d\alpha(\mathbf{x}) \, \mathcal{F}^{-1}(g)(\mathbf{x}).$$

Making the change of variables $t_i \longrightarrow \omega_i - t_i$ $(i = 1, \ldots, m)$ in the definition of g we finally find

$$\int_{\mathbb{R}^m} f(\mathbf{t}) g(\boldsymbol{\omega} - \mathbf{t}) d\mathbf{t} = \int_{\mathbb{R}^m} d\alpha(\mathbf{x}) \int_{\mathbb{R}^m} \prod_{i=1}^{m} e^{e_i t_i x_i} \, g(\boldsymbol{\omega} - \mathbf{t}) d\mathbf{t}.$$

□

4 CONVOLUTION

In this section we introduce the definition of convolution of Clifford functions. The convolution is related to pairs of Clifford functions belonging to (V).

Definition 4.1. Let $\alpha, \beta : \mathbb{R}^m \longrightarrow Cl_{0,m}$ belong to (V). The convolution $\alpha \odot \beta$ of α and β is the uniquely determined function $\gamma : \mathbb{R}^m \longrightarrow Cl_{0,m}$ given by

$$\gamma \,=\, \alpha \odot \beta \,:=\, \int_{\mathbb{R}^m} \alpha(\mathbf{x}-\mathbf{y})\, d\beta(\mathbf{y}) \tag{15}$$

for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$.

We underline that due to the noncommutativity of the quaternionic product, $\alpha \odot \beta$ does not coincide with $\beta \odot \alpha$ in general. We notice that the function γ given by (15) is well defined since it obviously holds

$$|\gamma(\mathbf{x})| \leq \int_{\mathbb{R}^m} |\alpha(\mathbf{x}-\mathbf{y})||d\beta(\mathbf{y})| \leq \delta \int_{\mathbb{R}^m} |d\beta(\mathbf{y})| < \infty.$$

Let α, β, ζ be elements of (V), and λ a Clifford constant. In particular, the convolution retains the following properties:

1. $(\alpha \odot \beta) \odot \zeta = \alpha \odot (\beta \odot \zeta)$;
2. $\alpha \odot (\beta \pm \zeta) = \alpha \odot \beta \pm \alpha \odot \zeta$;
3. $\lambda(\alpha \odot \beta) = (\lambda\alpha) \odot \beta, \quad (\alpha \odot \beta)\lambda = \alpha \odot (\beta\lambda)$;
4. $\lambda(\alpha \odot \beta) \neq \alpha \odot (\lambda\beta)$ in general.

Next we formulate the results of this section.

Proposition 4.2. *If α and β are elements of (V) then the function γ defined by (15) belongs to (V).*

Proof. Let $\mathbf{x}, \mathbf{z} \in \mathbb{R}^m$. Since α is a continuous function then for every $\epsilon > 0$ there exists a real $\delta = \delta(\epsilon) > 0$ such that

$$|\alpha(\mathbf{x}) - \alpha(\mathbf{z})| < \frac{\epsilon}{M}$$

whenever $|\mathbf{x} - \mathbf{z}| < \delta$. The constant $M > 0$ is chosen so that $\int_{\mathbb{R}^m} |d\beta(\mathbf{y})| = M$ for every $\mathbf{y} \in \mathbb{R}^m$. Hence, it holds

$$|\gamma(\mathbf{x}) - \gamma(\mathbf{z})| = \left| \int_{\mathbb{R}^m} (\alpha(\mathbf{x}-\mathbf{y}) - \alpha(\mathbf{z}-\mathbf{y}))\, d\beta(\mathbf{y}) \right| < \epsilon.$$

Consequently γ is a continuous function in $\mathbf{x}$. Since $\mathbf{x} \in \mathbb{R}^m$ is arbitrarily chosen it follows that γ is also continuous. $\square$

Proposition 4.3. *Let α and β be elements of (V). If $\alpha \in L^1(\mathbb{R}^m; Cl_{0,m})$ then it holds*

$$\|\gamma\|_{L^1(\mathbb{R}^m;Cl_{0,m})} \leq M\|\alpha\|_{L^1(\mathbb{R}^m;Cl_{0,m})}.$$

Proof. Let α and β be any functions in (V). A first straightforward computation shows that

$$\begin{aligned}
\|\gamma\|_{L^1(\mathbb{R}^m;Cl_{0,m})} &= \int_{\mathbb{R}^m} |\gamma(\mathbf{x})| d\sigma(\mathbf{x}) \\
&= \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} \alpha(\mathbf{x}-\mathbf{y}) d\beta(\mathbf{y}) \right| d\sigma(\mathbf{x}) \\
&\leq \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} |\alpha(\mathbf{x}-\mathbf{y})| d\sigma(\mathbf{x}) \right) |d\beta(\mathbf{y})| \\
&= M\|\alpha\|_{L^1(\mathbb{R}^m;Cl_{0,m})}.
\end{aligned}$$

□

In consequence, the following result holds:

Corollary 4.4. *Let α, β and α^n be elements of (V). If $\alpha, \alpha^n \in L^1(\mathbb{R}^m; Cl_{0,m})$ then it holds*

$$\|\gamma^n - \gamma\|_{L^1(\mathbb{R}^m;Cl_{0,m})} \longrightarrow 0$$

when $n \longrightarrow \infty$, where $\gamma^n = \alpha^n \odot \beta$ and $\gamma = \alpha \odot \beta$.

Proof. From the previous proposition a direct computation shows that

$$\|\gamma^n - \gamma\|_{L^1(\mathbb{R}^m;Cl_{0,m})} \leq M\|\alpha^n - \alpha\|_{L^1(\mathbb{R}^m;Cl_{0,m})},$$

and from this follows our assertion. □

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