**Some Harmonic analysis on the Klein bottle in $\mathbb{R}^n$**

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**Abstract**

The aim of this paper we discuss explicit series constructions for the fundamental solution of the Helmholtz operator on some important examples non-orientable conformally flat manifolds. In the context of this paper we focus on higher dimensional generalizations of the Klein bottle which in turn generalize higher dimensional Möbius strips that we discussed in preceding works. We discuss some basic properties of pinor valued solutions to the Helmholtz equation on these manifolds.

# 1 Introduction

Clifford and Harmonic analysis deal with the analysis of the Dirac operator resp. the Laplace operator on manifolds in $n$ variables. A lot of progress has been made for orientable manifolds over the past three decades. In particular, much attention has been paid to orientable conformally flat manifolds with spin structures, such as oriented cylinders and tori as their simplest representants.

In contrast to the cases of the oriented tori and cylinder that we discussed extensively in a series of papers, see for example, [8, 3, 5], which indeed are all examples of spin manifolds, we cannot construct the fundamental solution of the Helmholtz equation on higher dimensional generalizations of the non-oriented Klein bottle in terms of spinor valued sections that are in the kernel of $D - i\alpha$.

One obstacle is the lack of orientability. This does does not allow us to construct spinor bundles over these manifolds. Secondly, it is not possible either to construct non-vanishing solutions in the class Ker $D - i\alpha$ in $\mathbb{R}^n$ that have the additional pseudo periodic property to descend properly to these manifolds. A successful way is to start directly from special classes of harmonic functions that take values in bundles of the $^{+}Pin(n)$ group or $^{-}Pin(n)$ group.

By means of special classes of pseudo-multiperiodic harmonic functions we develop series representations for the Green's kernel of the Helmholtz operator for some $n$-dimensional generalizations of Klein bottle with values in different pin bundles. These functions represent a generalization of the Weierstraß $\wp$-function to the context of these geometries.

This functions that can be used to present Green type integral formulas that provide us with the basic stones for doing harmonic analysis in this geometrical context. This has been worked out in detail in [6].

The case of the Klein bottle has interesting particular features. In contrast to the Möbius strips considered earlier in our paper [7] the Klein bottle is a compact manifold.

The compactness allows us to prove that every solution of the Helmholtz operator having atmost unsessential singularities can be expressed as a finite linear combination of the fundamental solution and a finite amount of its partial derivatives. The proof of this statement represents also a central topic in our forthcoming paper [6].

In this paper here we focus ourselves to establish that the only entire solution of the Helmholtz equation on the Klein bottle is reduced to the constant function $f \equiv 0$.

## 2 Pin structures on conformally flat manifolds

Conformally flat manifolds are $n$-dimensional Riemannian manifolds that possess atlases whose transition functions are conformal maps in the sense of Gauss. For $n > 3$ the set of conformal maps coincides with the set of Möbius transformations. In the case $n = 2$ the sense preserving conformal maps are exactly the holomorphic maps. So, under this viewpoint we may interpret conformally flat manifolds as higher dimensional generalizations of holomorphic Riemann surfaces. On the other hand, conformally flat manifold are precisely those Riemannian manifolds which have a vanishing Weyl tensor.

As mentioned for instance in the classical work of N. Kuiper [10], concrete examples of conformally flat orbifolds can be constructed by factoring out a simply connected domain $X$ by a Kleinian group $\Gamma$ that acts discontinuously on $X$. In the cases where $\Gamma$ is torsion free, the topological quotient $X/\Gamma$, consisting of the orbits of a pre-defined group action $\Gamma \times X \to X$, is endowed with a differentiable structure. We then deal with examples of conformally flat manifolds.

In the case of oriented manifolds it is natural to consider spin structures. In the non-oriented case, this is not possible anymore. However, one can consider pin structures instead. For details about the description of pin structures on manifolds that arise as quotients by discrete groups. we refer the reader for instance to [1]. See also [2] and [4] where in particular the classical Möbius strip and the classical Klein bottle has been considered.

A classical way of obtaining pin structures for a given Riemannian manifold is to look for a lifting of the principle bundle associated to the orthogonal group $O(n)$ to a principle bundle for the pin groups ${}^{\pm}Pin(n)$. As described in the above cited works, the group ${}^{+}Pin(n) := Pin(n.0)$ is associated to the Clifford algebra $Cl_{n,0}$ of positive signature $(n,0)$. The Clifford algebra $Cl_{n,0}$ is defined as the free algebra modulo the relation $x^2 = q_{n,0}(x)$ $(x \in \mathbb{R}^n)$ where $q_{n,0}$ is the quadratic form defined by $q_{n,0}(e_i) = +1$ for all basis vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$. For particular details about Clifford algebras and their related classical groups we also refer the reader to [14]. Next we recall that the group ${}^{-}Pin(n) := Pin(0,n)$ is associated to the Clifford algebra $Cl_{0,n}$ of negative signature $(0,n)$. Here the quadratic form $q_{n,0}$ is replaced by the quadratic form $q_{0,n}$ defined by $q_{0,n}(e_i) = -1$ for all $i = 1, \ldots, n$. Topologically both groups are equivalent, however algebraically they are not isomorphic, cf. for example [4]. The more popular $Spin(n)$ group is a subgroup of ${}^{\pm}Pin(n)$ of index 2. Here we have $Spin(n) := Spin(0,n) \cong Spin(n,0)$. $Spin(n)$ consists exactly of those matrices from ${}^{\pm}Pin(n)$ whose determinant equals $+1$. The groups ${}^{\pm}Pin(n)$ double cover the group $O(n)$.

So there are surjective homomorphisms ${}^{\pm}\theta :^{\pm} Pin(n) \to^{\pm} Pin(n)$ with kernel $\mathbb{Z}_2 = \{\pm 1\}$. Adapting from Appendix C of [13], where spin structures have been discussed, this homomorphism gives rise to a choice of two local liftings of the principle $O(n)$ bundle to a principle ${}^{\pm}Pin(n)$ bundle. The number of different global liftings is given by the number of elements in the cohomology group $H^1(M, \mathbb{Z}_2)$. See [11] and elsewhere for details. These choices of liftings give rise to different pinor bundles over $M$. We shall explain their explicit construction on the basis of the examples that we consider in the next section.

# 3 $n$-dimensional generalizations of the Klein bottle

In this main section we present basic series constructions for the fundamental solution of the Helmholtz operator on a class of higher dimensional generalizations of the Klein bottle. For simplicity we consider an $n$-dimensional normalized lattice of the form $\Lambda_n := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n$.

We introduce higher dimensional generalization of the classical Klein bottle by the factorization

$$\mathcal{K}_n := \mathbb{R}^n / \sim^*$$

where the equivalence relation $\sim^*$ is defined by the map

$$(\underline{x} + \sum_{i=1}^{n-1} m_i e_i + (x_n + m_n)e_n) \mapsto (x_1, \cdots, x_{n-1}, (-1)^{m_n} x_n).$$

The manifolds $\mathcal{K}_n$ can be described as the set of orbits of the group action $\Lambda_n \times \mathbb{R}^n \to \mathbb{R}^n$ where the action here is defined by

$$v \circ x := (\sum_{i=1}^{k-1} x_i e_i + \sum_{i=1}^{k-1} m_i e_i + ((-1)^{m_k} x_k + m_k)e_k, x_{k+1}, \ldots, x_{n-1}, x_n),$$

where $v = m_1 e_1 + \cdots + m_n e_n$ is a lattice point from $\Lambda_n$. Here, and in the remaining part of this section, $\underline{x}$ denotes a shortened vector in $\mathbb{R}^{n-1}$. In the case $n = 2$ we re-obtain the classical Klein bottle. Notice that in contrast to the Möbius strips treated in our previous paper [7], here the minus sign switch occurs in one of the component on which the period lattice acts, too. As for the Möbius strips we can again set up distinct pinor bundles. See also [2] where pin structures of the classical four-dimensional Klein-bottle have been considered.

By decomposing the complete $n$-dimensional lattice $\Lambda_n$ into a direct sum of two sublattices $\Lambda_n = \Omega_l \oplus \Lambda_{n-l}$ we can construct in analogy to the oriented torus case treated in [8], $2^n$ distinct pinor bundles by considering the maps

$$(\underline{x} + \sum_{i=1}^{n-1} m_i e_i, x_n + m_n, X) \mapsto (x_1, \cdots, x_{n-1}, (-1)^{m_n} x_n, (-1)^{m_1 + \cdots + m_l} X).$$

In order to describe the fundamental solution of the Helmholtz operator on the Klein bottlr we recall from standard literature (see for instance [15]) that the fundamental solution to the Klein-Gordon operator $\Delta - \alpha^2$ in Euclidean flat space $\mathbb{R}^n$ is given by

$$E_\alpha(x) = -\frac{i\pi}{2\omega_n \Gamma(n/2)}(\frac{1}{2}i\alpha)^{\frac{1}{2}n-1}|x|^{1-\frac{1}{2}n} H^{(1)}_{\frac{n}{2}-1}(i\alpha|x|). \tag{1}$$

In this formula, $\omega_n$ stands for the surface measure of the unit sphere in $\mathbb{R}^n$ while $H^{(1)}_m$ denotes the first Hankel function with parameter $m$. Furthermore, we choose the root of $\alpha^2$ such that $\alpha > 0$. Then, as proposed in [6] we may introduce the following generalized version of Weierstrass $\wp$-function adapted to this class of Klein bottles by

$$\wp^{\mathcal{K}_n}_\alpha(x) := \sum_{v \in \Lambda_n} E_\alpha(\sum_{i=1}^{n-1}(x_i + m_i)e_i, (-1)^{m_n} x_n + x_m). \tag{2}$$

By similar arguments as applied in the cases of the Möbius strips described in [7] one may establish

**Theorem 3.1.** *Consider the decomposition of the lattice $\Lambda_n = \Lambda_l \oplus \Lambda_{n-l}$ for some $l \in \{1, \ldots, n\}$ and write a lattice point $v \in \Lambda_n$ in the form $v = m_1 e_1 + \cdots + m_l e_l + m_{l+1} e_{l+1} + \cdots + m_n e_n$ with integers $m_1, \ldots, m_n \in \mathbb{Z}$. Let $\mathcal{E}^{(q)}$ be the pinor bundle on $\mathcal{K}_n$ defined by the map*

$$(\underline{x} + \sum_{i=1}^{n-1} m_i e_i, x_n + m_n, X) \mapsto (x_1, \cdots, x_{n-1}, (-1)^{m_n} x_n, (-1)^{m_1 + \cdots + m_l} X).$$

*The fundamental solution of the Helmholtz operator on $\mathcal{K}_n$ (induced by $p_*(\Delta - \alpha^2)$ ) for sections with values in the pinor bundle $\mathcal{E}^{(q)}$ can be expressed by*

$$E'_{\alpha,q}(x') = p_* \left( \sum_{v \in \Lambda_l \oplus \Lambda_{n-l}} (-1)^{m_1 + \cdots + m_l} E_\alpha(\sum_{i=1}^{n-1}(x_i + m_i)e_i, (-1)^{m_n} x_n + x_m) \right), \tag{3}$$

*where $p_*$ denotes the projection from $\mathbb{R}^n$ to $\mathcal{K}_n = \mathbb{R}^n / \sim^*$. The symbol $'$ represents the image under $p_*$.*

The detailed proof is given in our forthcoming paper [6]. To recall the main idea of proof, without loss of generality we may consider the trivial bundle, as the arguments can easily be adapted to the other bundles that we also considered, namely by taking into account the parity factor $(-1)^{m_1 + \cdots + m_l}$.

This parity factor has no influence on the convergence property of the series. Now one can apply the same argumentation as applied for the Möbius strips to estimate each term of the series which turn out to be asymptotically exponentially decreasing, as a consequence of the Bessel functions. Further, it is a simple exercise to establish that the function $\wp_\alpha^{\mathcal{K}_n}$ is an element from Ker $\Delta - \alpha^2$ in $\mathbb{R}^n \backslash \Lambda_n$.

An important feature however is to show:

**Lemma 3.2.** *For all $k := k_1 e_1 + \cdots + k_n e_n \in \Lambda_n$ we have*

$$\wp_\alpha^{\mathcal{K}_n}(x+k) = \wp_\alpha^{\mathcal{K}_n}(x_1, \ldots, x_{n-1}, (-1)^{k_n} x_n).$$

*Proof.* To prove this statement it is important to use the following decomposition

$$\begin{aligned}
\wp_\alpha^{\mathcal{K}_n}(x) &= \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} E_\alpha(x_1+m_1, \ldots, x_{n-1}+m_{n-1}, (-1)^{m_n} x_n + m_n) \\
&= \sum_{(m_1,\ldots,m_{n-1})\in\mathbb{Z}^{n-1}, m_n \in 2\mathbb{Z}} E_\alpha(x_1+m_1, \ldots, x_{n-1}+m_{n-1}, x_n + m_n) \\
&= \sum_{(m_1,\ldots,m_{n-1})\in\mathbb{Z}^{n-1}, m_n \in 2\mathbb{Z}+1} E_\alpha(x_1+m_1, \ldots, x_{n-1}+m_{n-1}, -x_n + m_n).
\end{aligned}$$

First we note that

$$\wp_\alpha^{\mathcal{K}_n}(x_1+k_1, \ldots, x_{n-1}+k_{n-1}, x_n) = \wp_\alpha^{\mathcal{K}_n}(x_1, \ldots, x_n)$$

for all $(k_1, \ldots, k_{n-1}) \in \mathbb{Z}^{n-1}$. This follows by the direct series rearrangement

$$\begin{aligned}
& \wp_\alpha^{\mathcal{K}_n}(x_1+k_1, \ldots, x_{n-1}+k_{n-1}, x_n) \\
= & \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} E_\alpha(x_1+k_1+m_1, \ldots, x_{n-1}+k_{n-1}+m_{n-1}, (-1)^{m_n} x_n + m_n) \\
= & \sum_{(p_1,\ldots,p_n)\in\mathbb{Z}^n} E_\alpha(x_1+p_1, \ldots, x_{n-1}+p_{n-1}, (-1)^{p_n} x_n + p_n)
\end{aligned}$$

where we put $p_i := m_i + k_i \in \mathbb{Z}$ for $i = 1, \ldots, n-1$ and $p_n := m_n$. Notice that rearrangement is allowed because the series converges normally on $\mathbb{R}^n \backslash \Lambda_n$.

It thus suffices to show

$$\wp_\alpha^{\mathcal{K}_n}(x_1, \ldots, x_{n-1}, x_n + 1) = \wp_\alpha^{\mathcal{K}_n}(x_1, \ldots, x_{n-1}, -x_n).$$

We observe that

$$\begin{aligned}
& \wp_\alpha^{\mathcal{K}_n}(x_1,\ldots,x_{n-1},x_n+1) \\
= & \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}^n} E_\alpha(x_1+m_1,\ldots,x_{n-1}+m_{n-1},(-1)^{m_n}(x_n+1)+m_n) \\
= & \sum_{(m_1,\ldots,m_{n-1})\in\mathbb{Z}^{n-1},m_n\in 2\mathbb{Z}} E_\alpha(x_1+m_1,\ldots,x_{n-1}+m_{n-1},x_n+\underbrace{m_n+1}_{odd}) \\
+ & \sum_{(m_1,\ldots,m_{n-1})\in\mathbb{Z}^{n-1},m_n\in 2\mathbb{Z}+1} E_\alpha(x_1+m_1,\ldots,x_{n-1}+m_{n-1},-x_n+\underbrace{m_n-1}_{even}) \\
= & \sum_{(p_1,\ldots,p_{n-1})\in\mathbb{Z}^{n-1},p_n\in 2\mathbb{Z}+1} E_\alpha(x_1+p_1,\ldots,x_{n-1}+p_{n-1},x_n+p_n) \\
= & \sum_{(p_1,\ldots,p_{n-1})\in\mathbb{Z}^{n-1},q_n\in 2\mathbb{Z}} E_\alpha(x_1+p_1,\ldots,x_{n-1}+p_{n-1},-x_n+q_n) \\
= & \wp_\alpha^{\mathcal{K}_n}(x_1,\ldots,x_{n-1},-x_n).
\end{aligned}$$

The fact that

$$\wp_\alpha^{\mathcal{K}_n}(x_1,\ldots,x_{n-1},x_n+k_n) = \wp_\alpha^{\mathcal{K}_n}(x_1,\ldots,x_{n-1},(-1)^{k_n}x_n)$$

is true for all $k_n \in \mathbb{Z}$ now follows by a direct induction argument on $k_n$. $\square$

With this property we may infer that $\wp_\alpha^{\mathcal{K}_n}$ descends to a well-defined pinor section on $\mathcal{K}_n$ by applying the projection $p_*(\wp_\alpha^{\mathcal{K}_n})$. The result of this projection will be denoted by ${E'}_\alpha(x')$. ${E'}_\alpha(x')$ is the canonical skew symmetric periodization of $E_\alpha(x)$ that is constructed in such a way that it descends to the manifold. Therefore, the reproduction property of ${E'}_\alpha(x'-y')$ on $\mathcal{K}_n$ follows from the reproduction property of the usual Green's kernel $E_\alpha(x-y)$ in Euclidean space, where we apply the usual Green's integral formula for the Helmholtz operator.

**Remarks**. The fundamental solution of the Helmholtz operator on the usual Klein bottle in two real variables (for pinor sections with values in the trivial bundle) has the form

$$p_*\left(\sum_{v\in\Lambda_2} E_\alpha((x_1+m_1),(-1)^{m_2}x_2+m_2)\right).$$

In terms of this formula for the fundamental solution of the Helmholtz operator on the manifolds $\mathcal{K}_n$ we can deduce similar representation formulas for the solutions to the general inhomogeneous Helmholtz problem with prescribed boundary conditions on these manifolds as presented in the context of the Möbius strips in [7]. This is also a topic treated in detail in [6].

The fact that the manifolds $\mathcal{K}_n$ are compact manifolds has some interesting special function theoretical consequences. As also shown in our forthcoming paper [6], one can express any arbitrary solution of the Helmholtz equation with unessential singularities on these manifolds as a finite sum of linear combinations of the fundamental solution ${E'}_\alpha$ and its partial derivatives.

In this paper we restrict ourselves to show that there are no non-vanishing entire solutions to the Helmholtz equation on the Klein bottle. To establish the latter statement one first has to show

**Lemma 3.3.** *Let $\alpha \neq 0$. Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ is an entire solution of $(\Delta - \alpha^2)f = 0$ on the whole $\mathbb{R}^n$. If $f$ additionally satisfies*

$$f(x_1 + m_1, \cdots, x_n + m_n) = f(x_1, \ldots, x_{n-1}, (-1)^{m_n} x_n) \tag{4}$$

*for all $(m_1, \ldots, m_n) \in \mathbb{Z}^n$, then $f$ vanishes identically on $\mathbb{R}^n$.*

*Proof.* Since $f$ satisfies the relation

$$f(x_1 + m_1, \cdots, x_n + m_n) = f(x_1, \ldots, x_{n-1}, (-1)^{m_n} x_n),$$

it takes all its values in the $n$-dimensional period cell $[0, 1]^{n-1} \times [0, 2]$, because

$$f(x_1 + m_1, x_2 + m_2, \ldots, x_{n-1} + m_{n-1}, x_n + 2m_n) = f(x_1, x_2, \ldots, x_{n-1}, x_n)$$

for all $(m_1, \ldots, m_n) \in \mathbb{Z}^n$. The set $[0, 1]^{n-1} \times [0, 2]$ is compact. Since $f$ is an entire solution of $(\Delta - \alpha^2)f = 0$ on the whole $\mathbb{R}^n$, it is in particular continuous on $[0, 1]^{n-1} \times [0, 2]$. Consequently, $f$ must be bounded on $[0, 1]^{n-1} \times [0, 2]$ and therefore it must be bounded over the whole $\mathbb{R}^n$, too.

Since $f$ is an entire solution of $(\Delta - \alpha^2)f = 0$, it can be expanded into a Taylor series of the following form, compare with [3],

$$f(x) = \sum_{q=0}^{\infty} |x|^{1-q-n/2} J_{q+n/2-1}(\alpha|x|) H_q(x).$$

This Taylor series representation holds in the whole space $\mathbb{R}^n$.

Here, $H_q(x)$ are homogeneous harmonic polynomials of total degree $q$. These are often called spherical harmonics, cf. for example [12].

Since the Bessel $J$ functions are exponentially unbounded away from the real axis, $f$ can only be bounded if all spherical harmonics $H_q$ vanish identically. Hence, $f \equiv 0$. □

Notice that all constant functions $f \equiv C$ with $C \neq 0$ are not solutions of $(\Delta - \alpha^2)f = 0$. As a direct consequence we obtain

**Corollary 3.4.** *There are no non-vanishing entire solutions of* $(\Delta - \alpha^2)f = 0$ *on the manifolds* $\mathcal{K}_n$ *(in particular on the Klein bottle* $\mathcal{K}_2$*).*

This is a fundamental consequence of the compactness of the manifolds $\mathcal{K}_n$. Notice that this argument cannot be carried over to the context of the manifolds that we considered in the previous section, since those are not compact.

**Remark**. The statement can be adapted to the harmonic case $\alpha = 0$. In this case one has a Taylor series expansion of the simpler form

$$f(x) = \sum_{q=0}^{\infty} H_q(x),$$

where only the spherical harmonics of total degree $q = 0, 1, \ldots$ are involved. The only bounded entire harmonic functions are constants. Applying the same argumentation leads to the fact that the only harmonic solutions on $\mathcal{K}_n$ are constants.

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