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THE CLIFFORD FOURIER TRANSFORM IN REAL CLIFFORD ALGEBRAS

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Abstract. *We use the recent comprehensive research [15, 17] on the manifolds of square roots of -1 in real Clifford's geometric algebras $Cl(p,q)$ in order to construct the Clifford Fourier transform. Basically in the kernel of the complex Fourier transform the imaginary unit $j \in \mathbb{C}$ is replaced by a square root of -1 in $Cl(p,q)$. The Clifford Fourier transform (CFT) thus obtained generalizes previously known and applied CFTs [7,11,12], which replaced $j \in \mathbb{C}$ only by blades (usually pseudoscalars) squaring to -1. A major advantage of real Clifford algebra CFTs is their completely real geometric interpretation. We study (left and right) linearity of the CFT for constant multivector coefficients $\in Cl(p,q)$, translation ($\boldsymbol{x}$-shift) and modulation (ω-shift) properties, and signal dilations. We show an inversion theorem. We establish the CFT of vector differentials, partial derivatives, vector derivatives and spatial moments of the signal. We also derive Plancherel and Parseval identities as well as a general convolution theorem.*

1 INTRODUCTION

Quaternion, Clifford and geometric algebra Fourier transforms (QFT, CFT, GAFT) [6, 12, 13, 16] have proven *very useful* tools for applications in non-marginal color image processing, image diffusion, electromagnetism, multi-channel processing, vector field processing, shape representation, linear scale invariant filtering, fast vector pattern matching, phase correlation, analysis of non-stationary improper complex signals, flow analysis, partial differential systems, disparity estimation, texture segmentation, as spectral representations for Clifford wavelet analysis, etc.

All these Fourier transforms essentially analyze scalar, vector and multivector signals in terms of sine and cosine waves with multivector coefficients. For this purpose the imaginary unit $i \in \mathbb{C}$ in $e^{i\phi} = \cos\phi + i\sin\phi$ can be replaced by any *square root of* -1 *in a Clifford algebra* $Cl(p,q)$. The replacement by pure quaternions and blades with negative square [6, 13] has already yielded a wide variety of results with a clear geometric interpretation. It is well-known that there are elements other than blades, squaring to -1. Motivated by their special relevance for new types of CFTs, they have recently been studied thoroughly [15, 17, 21].

We therefore tap into these new results on square roots of -1 in Clifford algebras and fully general construct CFTs, with one general square root of -1 in $Cl(p,q)$. Our new CFTs form therefore a more general class of CFTs, subsuming and generalizing previous results. A further benefit is, that these new CFTs become *fully steerable* within the continuous Clifford algebra submanifolds of square roots of -1. We thus obtain a comprehensive *new mathematical framework* for the investigation and application of Clifford Fourier transforms together with *new properties* (full steerability). Regarding the question of the *most suitable* CFT for a certain application, we are only just beginning to leave the terra cognita of familiar transforms to map out the vast array of possible CFTs in $Cl(p,q)$.

This paper is organized as follows. We first review in Section 2 key notions of Clifford algebra, *multivector signal functions*, and the recent results on *square roots of* -1 in Clifford algebras. Next, in Section 3 we define the central notion of *Clifford Fourier transforms* with respect to any square root of -1 in Clifford algebra. Then we study in Section 4 (left and right) linearity of the CFT for constant multivector coefficients $\in Cl(p,q)$, translation ($\boldsymbol{x}$-shift) and modulation (ω-shift) properties, and signal dilations, followed by an inversion theorem. We establish the CFT of vector differentials, partial derivatives, vector derivatives and spatial moments of the signal. We also show Plancherel and Parseval identities as well as a general convolution theorem.

2 CLIFFORD'S GEOMETRIC ALGEBRA

Definition 2.1 (Clifford's geometric algebra [10, 19]) *Let* $\{e_1, e_2, \dots, e_p, e_{p+1}, \dots, e_n\}$, *with* $n = p+q$, $e_k^2 = \varepsilon_k$, $\varepsilon_k = +1$ *for* $k = 1,\dots,p$, $\varepsilon_k = -1$ *for* $k = p+1,\dots,n$, *be an orthonormal base of the inner product vector space* $\mathbb{R}^{p,q}$ *with a geometric product according to the multiplication rules*

$$e_k e_l + e_l e_k = 2\varepsilon_k \delta_{k,l}, \qquad k,l = 1,\dots n, \tag{1}$$

where $\delta_{k,l}$ *is the Kronecker symbol with* $\delta_{k,l} = 1$ *for* $k = l$, *and* $\delta_{k,l} = 0$ *for* $k \neq l$. *This non-commutative product and the additional axiom of associativity generate the* 2^n*-dimensional Clifford geometric algebra* $Cl(p,q) = Cl(\mathbb{R}^{p,q}) = Cl_{p,q} = \mathscr{G}_{p,q} = \mathbb{R}_{p,q}$ *over* $\mathbb{R}$. *The set* $\{e_A : A \subseteq \{1,\dots,n\}\}$ *with* $e_A = e_{h_1}e_{h_2}\dots e_{h_k}$, $1 \leq h_1 < \dots < h_k \leq n$, $e_\emptyset = 1$, *forms a graded (blade)* *basis of* $Cl(p,q)$. *The grades* k *range from* 0 *for scalars,* 1 *for vectors,* 2 *for bivectors,* s *for* s*-vectors, up to* n *for pseudoscalars. The vector space* $\mathbb{R}^{p,q}$ *is included in* $Cl(p,q)$ *as the subset of 1-vectors. The general elements of* $Cl(p,q)$ *are real linear combinations of basis blades* e_A*, called Clifford numbers, multivectors or hypercomplex numbers.*

In general $\langle A\rangle_k$ denotes the grade k part of $A \in Cl(p,q)$. The parts of grade 0 and $k+s$, respectively, of the geometric product of a k-vector $A_k \in Cl(p,q)$ with an s-vector $B_s \in Cl(p,q)$

$$A_k * B_s := \langle A_k B_s\rangle_0, \qquad A_k \wedge B_s := \langle A_k B_s\rangle_{k+s}, \tag{2}$$

are called *scalar product* and *outer product*, respectively.

For Euclidean vector spaces $(n=p)$ we use $\mathbb{R}^n = \mathbb{R}^{n,0}$ and $Cl(n) = Cl(n,0)$. Every k-vector B that can be written as the outer product $B = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots \wedge \boldsymbol{b}_k$ of k vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in \mathbb{R}^{p,q}$ is called a *simple* k-vector or *blade*.

Multivectors $M \in Cl(p,q)$ have k-vector parts $(0 \le k \le n)$: scalar part $Sc(M) = \langle M\rangle = \langle M\rangle_0 = M_0 \in \mathbb{R}$, vector part $\langle M\rangle_1 \in \mathbb{R}^{p,q}$, bi-vector part $\langle M\rangle_2$, $\ldots$, and pseudoscalar part $\langle M\rangle_n \in \bigwedge^n \mathbb{R}^{p,q}$

$$M = \sum_A M_A \boldsymbol{e}_A = \langle M\rangle + \langle M\rangle_1 + \langle M\rangle_2 + \ldots + \langle M\rangle_n\,. \tag{3}$$

The *principal reverse* of $M \in Cl(p,q)$ defined as

$$\widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle \overline{M}\rangle_k, \tag{4}$$

often replaces complex conjugation and quaternion conjugation. Taking the *reverse* is equivalent to reversing the order of products of basis vectors in the basis blades e_A. The operation $\overline{M}$ means to change in the basis decomposition of M the sign of every vector of negative square $\overline{e_A} = \varepsilon_{h_1} e_{h_1} \varepsilon_{h_2} e_{h_2} \ldots \varepsilon_{h_k} e_{h_k}$, $1 \le h_1 < \ldots < h_k \le n$. Reversion, $\overline{M}$, and principal reversion are all involutions.

The principal reverse of every basis element $e_A \in Cl(p,q)$, $1 \le A \le 2^n$, has the property

$$\widetilde{e_A} * e_B = \delta_{AB}, \qquad 1 \le A, B \le 2^n, \tag{5}$$

where the Kronecker delta $\delta_{AB} = 1$ if $A = B$, and $\delta_{AB} = 0$ if $A \neq B$. For the vector space $\mathbb{R}^{p,q}$ this leads to a reciprocal basis e^l, $1 \le l,k \le n$

$$e^l := \widetilde{e_l} = \varepsilon_l e_l, \quad e^l * e_k = e^l \cdot e_k = \begin{cases} 1, & \text{for } l = k \\ 0, & \text{for } l \neq k \end{cases}. \tag{6}$$

For $M,N \in Cl(p,q)$ we get $M * \widetilde{N} = \sum_A M_A N_A$. Two multivectors $M,N \in Cl(p,q)$ are *orthogonal* if and only if $M * \widetilde{N} = 0$. The modulus $|M|$ of a multivector $M \in Cl(p,q)$ is defined as

$$|M|^2 = M * \widetilde{M} = \sum_A M_A^2. \tag{7}$$

2.1 Multivector signal functions

A multivector valued function $f:\mathbb{R}^{p,q}\to Cl(p,q)$, has 2^n blade components $(f_A:\mathbb{R}^{p,q}\to\mathbb{R})$

$$f(\boldsymbol{x})=\sum_A f_A(\boldsymbol{x})\boldsymbol{e}_A, \qquad \boldsymbol{x}=\sum_{l=1}^{n} x_l e^l=\sum_{l=1}^{n} x^l e_l. \tag{8}$$

We define the *inner product* of two functions $f,g:\mathbb{R}^{p,q}\to Cl(p,q)$ by

$$(f,g)=\int_{\mathbb{R}^{p,q}} f(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\;d^n\boldsymbol{x}=\sum_{A,B}\boldsymbol{e}_A\widetilde{\boldsymbol{e}_B}\int_{\mathbb{R}^{p,q}} f_A(\boldsymbol{x})g_B(\boldsymbol{x})\;d^n\boldsymbol{x}, \tag{9}$$

with the *symmetric scalar part*

$$\langle f,g\rangle=\int_{\mathbb{R}^{p,q}} f(\boldsymbol{x})*\widetilde{g(\boldsymbol{x})}\;d^n\boldsymbol{x}=\sum_A\int_{\mathbb{R}^{p,q}} f_A(\boldsymbol{x})g_A(\boldsymbol{x})\;d^n\boldsymbol{x}, \tag{10}$$

and the $L^2(\mathbb{R}^{p,q};Cl(p,q))$*-norm*

$$\|f\|^2=\langle(f,f)\rangle=\int_{\mathbb{R}^{p,q}}|f(\boldsymbol{x})|^2 d^n\boldsymbol{x}=\sum_A\int_{\mathbb{R}^{p,q}} f_A^2(\boldsymbol{x})\;d^n\boldsymbol{x}, \tag{11}$$

$$L^2(\mathbb{R}^{p,q};Cl(p,q))=\{f:\mathbb{R}^{p,q}\to Cl(p,q)\mid \|f\|<\infty\}. \tag{12}$$

The *vector derivative* ∇ of a function $f:\mathbb{R}^{p,q}\to Cl(p,q)$ can be expanded in a basis of $\mathbb{R}^{p,q}$ as [23]

$$\nabla=\sum_{l=1}^{n} e^l\partial_l \quad\text{with}\quad \partial_l=\partial_{x_l}=\frac{\partial}{\partial x_l},\; 1\le l\le n. \tag{13}$$

2.2 Square roots of -1 in Clifford algebras

Every Clifford algebra $Cl(p,q)$, $s_8=(p-q)$ mod 8, is isomorphic to one of the following (square) matrix algebras[1] $\mathcal{M}(2d,\mathbb{R})$, $\mathcal{M}(d,\mathbb{H})$, $\mathcal{M}(2d,\mathbb{R}^2)$, $\mathcal{M}(d,\mathbb{H}^2)$ or $\mathcal{M}(2d,\mathbb{C})$. The first argument of $\mathcal{M}$ is the dimension, the second the associated ring[2] $\mathbb{R}$ for $s_8=0,2$, $\mathbb{R}^2$ for $s_8=1$, $\mathbb{C}$ for $s_8=3,7$, $\mathbb{H}$ for $s_8=4,6$, and $\mathbb{H}^2$ for $s_8=5$. For even n: $d=2^{(n-2)/2}$, for odd n: $d=2^{(n-3)/2}$.

It has been shown [15,17] that $Sc(f)=0$ for every square root of $-\mathbf{1}$ in every matrix algebra $\mathcal{A}$ isomorphic to $Cl(p,q)$. One can distinguish *ordinary* square roots of $-\mathbf{1}$, and *exceptional* ones. All square roots of $-\mathbf{1}$ in $Cl(p,q)$ can be computed using the package CLIFFORD for Maple [1,3,18,20].

In all cases the *ordinary* square roots f of $-\mathbf{1}$ constitute a *unique conjugacy class* of dimension $\dim(\mathcal{A})/2$, which has *as many connected components as the group* $\mathrm{G}(\mathcal{A})$ of invertible elements in $\mathcal{A}$. Furthermore, we have $\mathrm{Spec}(f)=0$ (zero pseudoscalar part) if the associated ring is $\mathbb{R}^2$, $\mathbb{H}^2$, or $\mathbb{C}$. The exceptional square roots of $-\mathbf{1}$ *only* exist if $\mathcal{A}\cong\mathcal{M}(2d,\mathbb{C})$. The manifolds of square roots of -1 in $Cl(p,q)$, $n=p+q=2$, compare Table 1 of [15], are visualized in Fig. 1.

For $\mathcal{A}=\mathcal{M}(2d,\mathbb{R})$, the centralizer (set of all elements in $Cl(p,q)$ commuting with f) and the conjugacy class of a square root f of $-\mathbf{1}$ both have $\mathbb{R}$-dimension $2d^2$ with *two connected components*. For the simplest case $d=1$ we have the algebra $Cl(2,0)$ isomorphic to $\mathcal{M}(2,\mathbb{R})$, pictured in Fig. 2 for $d=1$.

[1] Compare chapter 16 on *matrix representations and periodicity of 8*, as well as Table 1 on p. 217 of [19].

[2] Associated ring means, that the matrix elements are from the respective ring $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{H}^2$.

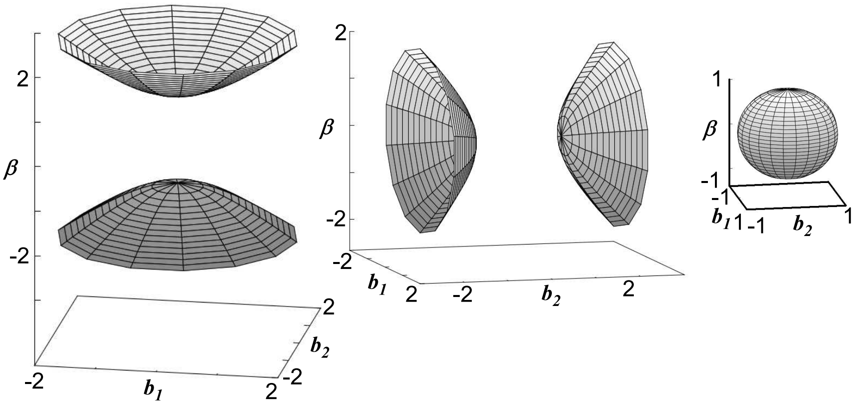

Figure 1: Manifolds [17] of square roots f of -1 in $Cl(2,0)$ (left), $Cl(1,1)$ (center), and $Cl(0,2) \cong \mathbb{H}$ (right). The square roots are $f = \alpha + b_1 e_1 + b_2 e_2 + \beta e_{12}$, with $\alpha, b_1, b_2, \beta \in \mathbb{R}$, $\alpha = 0$, and $\beta^2 = b_1^2 e_2^2 + b_2^2 e_1^2 + e_1^2 e_2^2$.

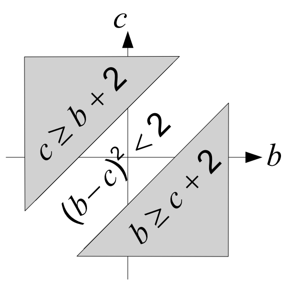

Figure 2: Two components of square roots of $-\mathbf{1}$ in $\mathscr{M}(2,\mathbb{R}) \equiv Cl(2,0)$, see [17] for details.

For $\mathscr{A} = \mathscr{M}(2d,\mathbb{R}^2) = \mathscr{M}(2d,\mathbb{R}) \times \mathscr{M}(2d,\mathbb{R})$, the square roots of $(-\mathbf{1},-\mathbf{1})$ are pairs of two square roots of $-\mathbf{1}$ in $\mathscr{M}(2d,\mathbb{R})$. They constitute a unique conjugacy class with *four connected components*, each of dimension $4d^2$. Regarding the four connected components, the group of inner automorphisms $\mathrm{Inn}(\mathscr{A})$ induces the permutations of the Klein group, whereas the quotient group $\mathrm{Aut}(\mathscr{A})/\mathrm{Inn}(\mathscr{A})$ is isomorphic to the group of isometries of a Euclidean square in 2D. The simplest example with $d = 1$ is $Cl(2,1)$ isomorphic to $M(2,\mathbb{R}^2) = \mathscr{M}(2,\mathbb{R}) \times \mathscr{M}(2,\mathbb{R})$.

For $\mathscr{A} = \mathscr{M}(d,\mathbb{H})$, the submanifold of the square roots f of $-\mathbf{1}$ is a *single connected conjugacy class* of $\mathbb{R}$-dimension $2d^2$ equal to the $\mathbb{R}$-dimension of the centralizer of every f. The easiest example is $\mathbb{H}$ itself for $d = 1$.

For $\mathscr{A} = \mathscr{M}(d,\mathbb{H}^2) = \mathscr{M}(d,\mathbb{H}) \times \mathscr{M}(d,\mathbb{H})$, the square roots of $(-\mathbf{1},-\mathbf{1})$ are pairs of two square roots (f,f') of $-\mathbf{1}$ in $\mathscr{M}(d,\mathbb{H})$ and constitute a *unique connected conjugacy class* of $\mathbb{R}$-dimension $4d^2$. The group $\mathrm{Aut}(\mathscr{A})$ has two connected components: the neutral component $\mathrm{Inn}(\mathscr{A})$ connected to the identity and the second component containing the swap automorphism $(f,f') \mapsto (f',f)$. The simplest case for $d = 1$ is $\mathbb{H}^2$ isomorphic to $Cl(0,3)$.

For $\mathscr{A} = \mathscr{M}(2d,\mathbb{C})$, the square roots of $-\mathbf{1}$ are in *bijection to the idempotents* [2]. First, the *ordinary* square roots of $-\mathbf{1}$ (with $k = 0$) constitute a conjugacy class of $\mathbb{R}$-dimension $4d^2$ of a *single connected component* which is invariant under $\mathrm{Aut}(\mathscr{A})$. Second, there are $2d$ *conjugacy classes* of *exceptional* square roots of $-\mathbf{1}$, each composed of a *single connected component*, characterized by the equality $\mathrm{Spec}(f) = k/d$ (the pseudoscalar coefficient) with $\pm k \in \{1,2,\ldots,d\}$, and their $\mathbb{R}$-dimensions are $4(d^2 - k^2)$. The group $\mathrm{Aut}(\mathscr{A})$ includes conjugation of the pseudoscalar $\omega \mapsto -\omega$ which maps the conjugacy class associated with k to the class associated with $-k$. The simplest case for $d = 1$ is the Pauli matrix algebra isomorphic to the geometric algebra $Cl(3,0)$ of 3D Euclidean space $\mathbb{R}^3$, and to complex biquaternions [21]. See Table 2.2 for representative exceptional ($k \neq 0$) square roots of $-\mathbf{1}$ in conformal geometric algebra $Cl(4,1)$ of three-dimensional Euclidean space [17].

k	f_k	$\Delta_k(t)$
2	$\omega = e_{12345}$	$(t-i)^4$
1	$\frac{1}{2}(e_{23} + e_{123} - e_{2345} + e_{12345})$	$(t-i)^3(t+i)$
0	e_{123}	$(t-i)^2(t+i)^2$
-1	$\frac{1}{2}(e_{23} + e_{123} + e_{2345} - e_{12345})$	$(t-i)(t+i)^3$
-2	$-\omega = -e_{12345}$	$(t+i)^4$

Table 1: Square roots of $-\mathbf{1}$ in conformal geometric algebra $Cl(4,1) \cong \mathscr{M}(4,\mathbb{C})$, $d = 2$, with characteristic polynomials $\Delta_k(t)$. See [17] for details.

With respect to any square root $i \in Cl(p,q)$ of -1, $i^2 = -1$, every multivector $A \in Cl(p,q)$ can be split into *commuting* and *anticommuting* parts [17].

Lemma 2.2 *Every multivector $A \in Cl(p,q)$ has, with respect to a square root $i \in Cl(p,q)$ of -1, i.e., $i^{-1} = -i$, the unique decomposition*

$$A_{+i} = \frac{1}{2}(A + i^{-1}Ai), \qquad A_{-i} = \frac{1}{2}(A - i^{-1}Ai)$$
$$A = A_{+i} + A_{-i}, \qquad A_{+i}\, i = i A_{+i}, \qquad A_{-i}\, i = -i A_{-i}. \tag{14}$$

3 THE CLIFFORD FOURIER TRANSFORM

The *general Clifford Fourier transform* (CFT), to be introduced now, can be understood as a generalization of known CFTs [12] to a general real Clifford algebra setting. Most previously known CFTs use in their kernels specific square roots of -1, like bivectors, pseudoscalars, unit pure quaternions, or blades [6]. We will *remove all these restrictions* on the square root of -1 used in a CFT.

Definition 3.1 (CFT with respect to one square root of -1) *Let $i \in Cl(p,q)$, $i^2 = -1$, be any square root of -1. The general Clifford Fourier transform (CFT) of $f \in L^1(\mathbb{R}^{p,q};Cl(p,q))$, with respect to i is*

$$\mathscr{F}^i\{f\}(\omega) = \int_{\mathbb{R}^{p,q}} f(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\omega)} d^n\boldsymbol{x}, \tag{15}$$

where $d^n\boldsymbol{x} = dx_1 \dots dx_n$, $\boldsymbol{x}, \omega \in \mathbb{R}^{p,q}$, and $u : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$.

Since square roots of -1 in $Cl(p,q)$ populate *continuous submanifolds* in $Cl(p,q)$, the CFT of Definition 3.1 is generically *steerable* within these manifolds. In Definition 3.1, the square roots $i \in Cl(p,q)$ of -1 may be from any component of any conjugacy class.

4 PROPERTIES OF THE CFT

We now study important properties of the general CFT of Definition 3.1.

4.1 Linearity, shift, modulation, dilation, and powers of f, g

Regarding *left and right linearity* of the general two-sided CFT of Definition 3.1 we can establish with the help of Lemma 2.2 that for $h_1, h_2 \in L^1(\mathbb{R}^{p,q};Cl(p,q))$, and constants $\alpha, \beta \in Cl(p,q)$

$$\mathscr{F}^i\{\alpha h_1 + \beta h_2\}(\omega) = \alpha \mathscr{F}^i\{h_1\}(\omega) + \beta \mathscr{F}^i\{h_2\}(\omega), \tag{16}$$

$$\begin{aligned}\mathscr{F}^i\{h_1\alpha + h_2\beta\}(\omega) = {} & \mathscr{F}^i\{h_1\}(\omega)\alpha_{+i} + \mathscr{F}^{-i}\{h_1\}(\omega)\alpha_{-i} \\ & + \mathscr{F}^i\{h_2\}(\omega)\beta_{+i} + \mathscr{F}^{-i}\{h_2\}(\omega)\beta_{-i}\,.\end{aligned} \tag{17}$$

For *i power factors* in $h_{a,b}(\boldsymbol{x}) = i^a h(\boldsymbol{x}) i^b$, $a, b \in \mathbb{Z}$, we obtain as an application of linearity

$$\mathscr{F}^i\{h_{a,b}\}(\omega) = i^a \mathscr{F}^i\{h\}(\omega) i^b. \tag{18}$$

Regarding the $\boldsymbol{x}$-*shifted* function $h_0(\boldsymbol{x}) = h(\boldsymbol{x} - \boldsymbol{x}_0)$ we obtain with constant $\boldsymbol{x}_0 \in \mathbb{R}^{p,q}$, assuming linearity of $u(\boldsymbol{x}, \omega)$ in its vector space argument $\boldsymbol{x}$,

$$\mathscr{F}^i\{h_0\}(\omega) = \mathscr{F}^i\{h\}(\omega)\, e^{-iu(\boldsymbol{x}_0,\omega)}. \tag{19}$$

For the purpose of *modulation* we make the special assumption, that the function $u(\boldsymbol{x}, \omega)$ is linear in its frequency argument ω. Then we obtain for $h_m(\boldsymbol{x}) = h(x)\, e^{-iu(\boldsymbol{x},\omega_0)}$, and constant $\omega_0 \in \mathbb{R}^{p,q}$ the modulation formula

$$\mathscr{F}^i\{h_m\}(\omega) = \mathscr{F}^i\{h\}(\omega + \omega_0). \tag{20}$$

Regarding *dilations*, we make the special assumption, that for constants $a_1, \dots, a_n \in \mathbb{R} \setminus \{0\}$, and $\boldsymbol{x}' = \sum_{k=1}^n a_k x^k \boldsymbol{e}_k$, we have $u(\boldsymbol{x}', \omega) = u(\boldsymbol{x}, \omega')$, with $\omega' = \sum_{k=1}^n a_k \omega^k \boldsymbol{e}_k$. We then obtain for $h_d(\boldsymbol{x}) = h(\boldsymbol{x}')$ that

$$\mathscr{F}^i\{h_d\}(\omega) = \frac{1}{|a_1 \dots a_n|} \mathscr{F}^i\{h\}(\omega_d), \qquad \omega_d = \sum_{k=1}^n \frac{1}{a_k} \omega^k \boldsymbol{e}_k. \tag{21}$$

For $a_1 = \dots = a_n = a \in \mathbb{R} \setminus \{0\}$ this simplifies under the same special assumption to

$$\mathscr{F}^i\{h_d\}(\omega) = \frac{1}{|a|^n} \mathscr{F}^i\{h\}(\frac{1}{a}\omega). \tag{22}$$

Note, that the above assumption would, e.g., be fulfilled for $u(\boldsymbol{x}, \omega) = \boldsymbol{x} * \widetilde{\omega} = \sum_{k=1}^n x^k \omega^k = \sum_{k=1}^n x_k \omega_k$.

4.2 CFT inversion , moments, derivatives, Plancherel, Parseval

For establishing an inversion formula, moment and derivative properties, Plancherel and Parseval identities, certain *assumptions* about the phase function $u(\boldsymbol{x}, \omega)$ need to be made. One possibility is, e.g., to assume

$$u(\boldsymbol{x}, \omega) = \boldsymbol{x} * \widetilde{\omega} = \sum_{l=1}^n x^l \omega^l = \sum_{l=1}^n x_l \omega_l, \tag{23}$$

which will be assumed for the current subsection.

We then get the following *inversion* formula

$$h(\boldsymbol{x}) = \mathscr{F}^i_{-1}\{\mathscr{F}^i\{h\}\}(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathscr{F}^i\{h\}(\omega)\, e^{iu(\boldsymbol{x},\omega)} d^n\omega, \tag{24}$$

where $d^n\omega = d\omega_1 \dots d\omega_n$, $\boldsymbol{x}, \omega \in \mathbb{R}^{p,q}$. For the existence of (24) we need $\mathscr{F}^i\{h\} \in L^1(\mathbb{R}^{p,q};\ Cl(p,q))$.

Additionally, we get the transformation law for *partial derivatives* $h_l'(\boldsymbol{x}) = \partial_{x_l} h(\boldsymbol{x})$, $1 \le l \le n$, for h piecewise smooth and integrable, and $h, h_l' \in L^1(\mathbb{R}^{p,q};\ Cl(p,q))$ as

$$\mathscr{F}^i\{h_l'\}(\omega) = \omega_l \mathscr{F}^i\{h\}(\omega) i, \quad \text{for } 1 \le l \le n. \tag{25}$$

The *vector derivative* of $h \in L^1(\mathbb{R}^{p,q};\ Cl(p,q))$ with $h_l' \in L^1(\mathbb{R}^{p,q};\ Cl(p,q))$ gives therefore

$$\mathscr{F}^i\{\nabla h\}(\omega) = \mathscr{F}^i\{\sum_{l=1}^n e^l h_l'\}(\omega) = \omega \mathscr{F}^i\{h\}(\omega) i. \tag{26}$$

For the transformation of the *spatial moments* with $h_l(\boldsymbol{x}) = x_l h(\boldsymbol{x})$, $1 \le l \le n$, $h, h_l \in L^1(\mathbb{R}^{p,q};\ Cl(p,q))$, we obtain

$$\mathscr{F}^i\{h_l\}(\omega) = \partial_{\omega_l} \mathscr{F}^i\{h\}(\omega) i, \tag{27}$$

and for the *spatial vector moment*

$$\mathscr{F}^i\{\boldsymbol{x}h\}(\omega) = \nabla_\omega \mathscr{F}^i\{h\}(\omega) i, \tag{28}$$

Moreover, for the functions $h_1, h_2, h \in L^2(\mathbb{R}^{p,q};\ Cl(p,q))$ we obtain the *Plancherel* identity

$$\langle h_1, h_2 \rangle = \frac{1}{(2\pi)^n} \langle \mathscr{F}^i\{h_1\}, \mathscr{F}^i\{h_2\} \rangle, \tag{29}$$

as well as the *Parseval* identity

$$\|h\| = \frac{1}{(2\pi)^{n/2}} \left\| \mathscr{F}^i\{h\} \right\|. \tag{30}$$

4.3 Convolution

We define the *convolution* of two multivector signals $a, b \in L^1(R^{p,q}; Cl(p,q))$ as

$$(a \star b)(\boldsymbol{x}) = \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b(\boldsymbol{x}-\boldsymbol{y}) d^n \boldsymbol{y}. \tag{31}$$

We assume that the function u is linear with respect to its first argument. The *CFT of the convolution* (31) can then be expressed as

$$\mathscr{F}^i\{a \star b\}(\omega) = \mathscr{F}^{-i}\{a\}(\omega)\mathscr{F}^i\{b_{-i}\}(\omega) + \mathscr{F}^i\{a\}(\omega)\mathscr{F}^i\{b_{+i}\}(\omega) \tag{32}$$

5 CONCLUSIONS

We have established a comprehensive *new mathematical framework* for the investigation and application of Clifford Fourier transforms (CFTs) together with *new properties*. Our new CFTs form a more general class of CFTs, subsuming and generalizing previous results. We have applied new results on square roots of -1 in Clifford algebras to fully general construct CFTs, with a general square root of -1 in real Clifford algebras $Cl(p,q)$. The new CFTs are *fully steerable* within the continuous Clifford algebra submanifolds of square roots of -1. We have thus left the terra cognita of familiar transforms to outline the vast array of possible CFTs in $Cl(p,q)$.

We first reviewed the recent results on *square roots of* -1 in Clifford algebras. Next, we defined the central notion of the *Clifford Fourier transform* with respect to any square root of -1 in Clifford algebra. Finally, we investigated important *properties* of these new CFTs: linearity, shift, modulation, dilation, moments, inversion, partial and vector derivatives, Plancherel and Parseval formulas, as well as a convolution theorem.

Regarding numerical implementations, usually 2^n complex Fourier transformations (FTs) are sufficient. In some cases this can be reduced to $2^{(n-1)}$ complex FTs, e.g., when the square root of -1 is a pseudoscalar. Further algebraic studies may widen the class of CFTs, where $2^{(n-1)}$ complex FTs are sufficient. Numerical implementation is then possible with 2^n (or $2^{(n-1)}$) discrete complex FTs, which can also be fast Fourier transforms (FFTs), leading to fast CFT implementations.

A well-known example of a CFT is the quaternion FT (QFT) [4, 5, 8, 9, 13, 16, 22], which is particularly used in applications to partial differential systems, color image processing, filtering, disparity estimation (two images differ by local translations), and texture segmentation. Another example is the spacetime FT, which leads to a multivector wave packet analysis of spacetime signals (e.g. electro-magnetic signals), applicable even to relativistic signals [13, 14].

Depending on the choice of the phase functions $u(\boldsymbol{x}, \omega)$ the multivector basis coefficient functions of the CFT result carry information on the symmetry of the signal, similar to the special case of the QFT [4].

The convolution theorem allows to design and apply multivector valued filters to multivector valued signals.

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