Some Problems in Obtaining the Green's Function of the Layered Soil

Andrej Strukelj
Faculty of Civil Engineering, University of Maribor
2000 Maribor, Slovenia

1. Introduction

The frame of this paper is the development of methods and procedures for the description of the motion of an arbitrary shaped foundation. Since the infinite half-space cannot be properly described by a model of finite dimensions without violating the radiation condition, the basic problems are infinite dimensions of the half-space as well as its non-homogeneous nature. Consequently, an approach has been investigated to solve this problem indirectly by developing Green's function in which the non-homogeneity and the infiniteness of the half-space has been included. When the Green's function is known, the next step will be the evaluation of contact stresses acting between the foundation and the surface of the half-space through an integral equation. The equation should be solved in the area of the foundation using Green's function as the kernel. The derivation of three-dimensional Green's function for the homogeneous half-space (Kobayashi and Sasaki 1991) has been made using the potential method. Partial differential equations occurring in the problem have been made ordinary ones through the Hankel integral transform. The general idea for obtaining the three-dimensional Green's function for the layered half-space is similar. But in that case some additional phenomena may occur. One of them is the possibility of the appearance of Stonely surface waves propagating along the contact surfaces of layers. Their contribution to the final result is in most cases important enough that they should not be neglected.

2. The Derivation of Green's Function

The surface of the horizontally layered half-space is loaded by the concentrated load $P \cdot H(t)$. As the wave motion generated on such way is axially symmetric, the cylindrical co-ordinate system is introduced. The local co-ordinate systems having their origins on the top surface of the each layer are also defined. The governing equation for the each layer is the known equation of motion:

$$\mu \cdot \nabla^2 \vec{U} + (\lambda + \mu) \cdot \vec{\nabla} \vec{\nabla} \bullet \vec{U} = \rho \cdot \ddot{\vec{U}} \tag{1}$$

which can be separated into two parts:

$$\frac{\mu}{\rho} \cdot \nabla^2 \vec{u} + \frac{\lambda + \mu}{\rho} \cdot \vec{\nabla} \vec{\nabla} \bullet \vec{u} = -\omega^2 \cdot \vec{u} \quad \text{and} \quad \ddot{T} = -\omega^2 \cdot T \tag{2}$$

The displacement vector can be written in potential form as:

$$\vec{u} = \vec{\nabla} \varphi + \vec{\nabla} \times \vec{\psi} \tag{3}$$

After introducing the Eq.(3) into the first of Eq. (2) and obeying the well known expressions for the wave numbers in longitudinal and transversal direction two wave equations for two potentials have been obtained. In cylindrical co-ordinate system they have the following form:

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial z^2}+k_L^2\cdot\varphi=0 \quad \text{and} \quad \frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}-\frac{\psi}{r^2}+k_T^2\cdot\psi=0, \tag{4}$$

Partial differential equations Eq. (4) can be translated into ordinary ones by using the Hankel integral transform $r\rightarrow\xi$, which is defined as:

$$H^n\big(f(r)\big)=\bar{f}^n(\xi)=\int_0^\infty f(r)\cdot J_n(\xi\cdot r)\cdot r\cdot dr \tag{5}$$

Corresponding inverse transform is:

$$f(r)=\int_0^\infty \bar{f}^n(\xi)\cdot J_n(\xi\cdot r)\cdot\xi\cdot d\xi \tag{6}$$

Hankel transform of Eq. (7) is:

$$\frac{d^2\bar{\varphi}^0}{dz^2}-(\xi^2-k_L^2)\cdot\bar{\varphi}^0=0 \quad \text{and} \quad \frac{d^2\bar{\psi}^1}{dz^2}-(\xi^2-k_T^2)\cdot\bar{\psi}^1=0 \tag{7}$$

The fundamental solutions can be written in following form:

$$\bar{\varphi}^0=\Phi_1\cdot e^{\sqrt{\xi^2-k_L^2}\cdot z}+\Phi_2\cdot e^{-\sqrt{\xi^2-k_L^2}\cdot z} \quad \text{and} \quad \bar{\psi}^1=\Psi_1\cdot e^{\sqrt{\xi^2-k_T^2}\cdot z}+\Psi_2\cdot e^{-\sqrt{\xi^2-k_T^2}\cdot z} \tag{8}$$

Since the layered half-space has to be treated as a continuous media, four continuity conditions have to be introduced on each contact surface. On the contact of two layers the equal normal stresses, shear stresses and the displacements in both directions are demanded. Considering that the layered half-space consists of n parallel layers resting on a homogeneous half-space there are $4\cdot n$ of continuity conditions and two boundary conditions on the top of the first layer:

$$\tau_{rz}\big|_{z=0}=0; \qquad \sigma_z\big|_{z=0}=-\frac{P\cdot H(t)\cdot\delta(r)}{2\cdot\pi\cdot r}. \tag{9}$$

On the other hand the fundamental solutions for the potentials φ and ψ for each layer have four integration constants while the fundamental solutions for the underlying half-space, where the radiation conditions should be introduced, have another two of them. For evaluating the values of the $4\cdot n+2$ unknown constants there are thus $4\cdot n+2$ equations. The matrix of this system is band matrix with the band width of maximum 8 terms. The right side of the system is a column matrix where each one except the first term is equal to zero. As the point of interest is only the evaluation of the surface motion, only the solution for the first four integration constants is needed. Potentials $\bar{\varphi}^0$ and $\bar{\psi}^1$ have now the following form:

$$\bar{\varphi}^0=C_1\cdot e^{\sqrt{\xi^2-k_L^2}\cdot z}+C_2\cdot e^{-\sqrt{\xi^2-k_L^2}\cdot z}; \quad \bar{\psi}^1=C_3\cdot e^{\sqrt{\xi^2-k_T^2}\cdot z}+C_4\cdot e^{-\sqrt{\xi^2-k_T^2}\cdot z} \tag{10}$$

Referring to the Hankel transform of Eq. (3) and taking into account that on the surface $z=0$ the relation for the vertical component of Green's function in transformed domain ($\bar{w}^0$) is obtained. Its inverse transform can be obtained by putting it into Eq. (6):

$$w = \int_0^\infty \xi \cdot \overline{w}^0 \cdot J_0(\xi \cdot a) \cdot d\xi \tag{11}$$

where a assigns the dimensionless frequency defined as: $a = r \cdot \omega / c_T$. The product $\xi \cdot \overline{w}^0$ does not vanish in infinity, but it converges to a constant value -(1-ν), as can be proved. Therefore this constant value should be subtracted from the integrand. After this and excluding the singularity 1/r from the integral, Eq. (11) has the following form:

$$w_H = \frac{(1-\nu)}{r} \cdot \left(1 - \frac{a}{1-\nu} \cdot I_{(a)}\right) \tag{12}$$

where:

$$I_{(a)} = \int_0^\infty \left(\xi \cdot \overline{w}^0 + 1 - \nu\right) \cdot J_0(\xi \cdot a) \cdot d\xi \, . \tag{13}$$

It can be seen from the structure of the solution in transformed domain that each layer contributes two couples of conjugate branch points. These branch points as well as Rayleigh and Stonely poles lie along the same lines, which slopes depend on the value of material dumping ratio α. In order to make the solution single valued, the appropriate branch cuts have to be introduced. For α=0 all branch points and poles are lying on the real ξ axis. The usual way to calculate the integral Eq. (13) is to close the integration contour in the complex ξ plane, so the following steps are introduced:

$$2 \cdot J_0(\xi \cdot a) = \frac{2}{\pi} \cdot \int_0^\pi \cos(\xi \cdot a \cdot \sin\zeta) \cdot d\zeta = \frac{1}{\pi} \cdot \int_0^\pi \left[e^{-i \cdot \xi \cdot a \cdot \sin\zeta} + e^{i \cdot \xi \cdot a \cdot \sin\zeta}\right] \cdot d\zeta = h(-\zeta \cdot a) + h(\xi \cdot a) \tag{14}$$

The integral $I_{(a)}$ becomes than $\frac{1}{2} \cdot I_{(h)}$, where:

$$I_h = \int_{-\infty}^\infty \left(\xi \cdot \overline{w}^0 + 1 - \nu\right) \cdot h(\xi \cdot a) \cdot d\xi \tag{15}$$

The integration contour can now be closed with an infinite semicircle lying in the upper half of the ξ plane and the residue theorem can be used for the evaluation of the integral $I_{(h)}$. It can be proved that the contour integral over the semicircle vanishes as the radius R reaches infinity. The integration problem is now reduced on the integration along the branch cut and evaluating of the residues in singular points enclosed by the integration contour. Rayleigh waves appear always when the free surface exists. On the other hand the appearance of the Stonely waves depends on the density and shear modulus ratios of the neighbouring layers. To illustrate this statement the system of two coupled but different half-spaces should be investigated (Fig.1). Obeying the radiation condition for both half-spaces and introducing the substitution $\xi = k_{T1} \cdot \eta$ the fundamental solutions for the potentials can be written in following form:

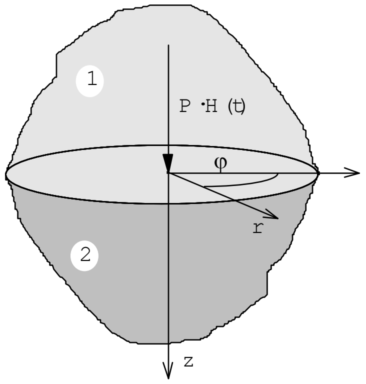

Figure 1. Model of the two half-spaces

$$\overline{\varphi}_1^0 = C_{11} \cdot e^{2\cdot\pi\cdot\sqrt{\eta^2-\gamma_1^2}\cdot\zeta} = C_{11} \cdot e^{2\cdot\pi\cdot\alpha_1\cdot\zeta} \qquad \overline{\psi}_1^1 = C_{12} \cdot e^{2\cdot\pi\cdot\sqrt{\eta^2-1}\cdot\zeta} = C_{12} \cdot e^{2\cdot\pi\cdot\beta_1\cdot\zeta}$$
$$\overline{\varphi}_2^0 = C_{21} \cdot e^{-2\cdot\pi\cdot\sqrt{\eta^2-\gamma_2^2}\cdot\zeta} = C_{21} \cdot e^{-2\cdot\pi\cdot\alpha_2\cdot\zeta} \qquad \overline{\psi}_2^1 = C_{22} \cdot e^{-2\cdot\pi\cdot\sqrt{\eta^2-\upsilon_2^2}\cdot\zeta} = C_{22} \cdot e^{-2\cdot\pi\cdot\beta_2\cdot\zeta} \tag{16}$$

where ζ represents the ratio of the z co-ordinate and the wave length of the transverse waves in the upper half-space. In this case four continuity conditions are needed. The first two of them require the equal values of the normal and shear stresses in the contact surface. The second pair of continuity conditions demands the equality of the vertical and horizontal displacements in the contact surface. The obtained four equations represents the homogeneous system of equations. In order to obtain the non-trivial solutions the system determinant should be equal to zero. The real solutions evaluated on the base of this condition represent the ratios of the first half-space shear wave front velocity and Stonely wave velocities. In the plane, where first of the axes represents the ratio of densities (ρ_1/ρ_2) and the second one represents the ratio of the shear modulus (μ_1/μ_2), two curves (A and B) define the boundaries of the real solutions region, as can be seen in the Fig. 2. The curve A is connecting all points for which the Sonely wave velocity (c_S) coincide with shear wave velocity of the first half-space (c_{T1}) and the curve B represents the boundary for which the Sonely wave velocity (c_S) coincide with shear wave velocity of the second half-space (c_{T2}).

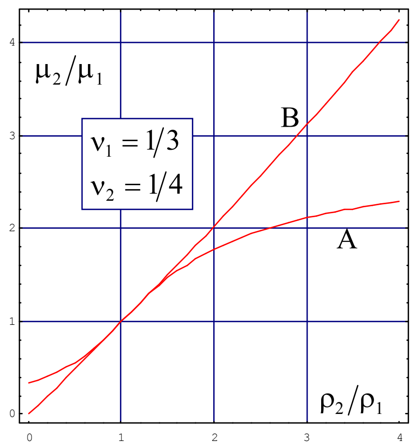

Figure 2. The region of real solutions of the system determinant (between curves A and B)

In the case of the elastic layer resting on a homogeneous half-space (Fig.3) the fundamental solutions for the potentials look like:

$$\overline{\varphi}_1^0 = C_{11} \cdot e^{2\cdot\pi\cdot\alpha_1\cdot\zeta} + C_{12} \cdot e^{-2\cdot\pi\cdot\alpha_1\cdot\zeta} \qquad \overline{\psi}_1 = C_{13} \cdot e^{2\cdot\pi\cdot\beta_1\cdot\zeta} + C_{14} \cdot e^{-2\cdot\pi\cdot\beta_1\cdot\zeta}$$
$$\overline{\varphi}_2 = C_{21} \cdot e^{-2\cdot\pi\cdot\alpha_2\cdot\zeta} \qquad \overline{\psi}_2 = C_{22} \cdot e^{-2\cdot\pi\cdot\beta_2\cdot\zeta} \tag{17}$$

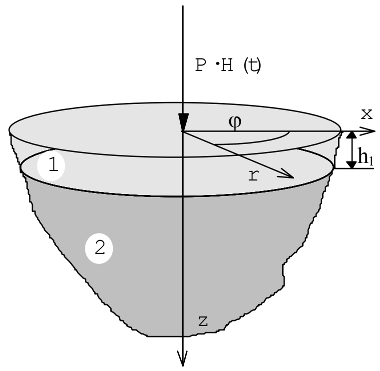

Figure 3. Model of the one layer half-space

The additional two boundary conditions on the free surface should now be introduced. Because of the free surface in this case also the Rayleigh waves occur. The appearance and velocity of the Stonely waves depends now not only on the properties of the neighbouring two materials but also on the thickness of the top layer. In the limit when the thickness of the layer go to infinity the velocity of the Stonely waves becomes more and more similar to the values from the previous example and the velocity of the Rayleigh waves reaches the values for the homogeneous half-space. On the Fig.4 the ratios of the shear velocities and the velocities of Rayleigh and Stonely waves (c_{T1}/c_R and c_{T1}/c_S) depending on the ratio of the layer thickness and the wave length of the shear wave in the layer (Z_1) is shown.

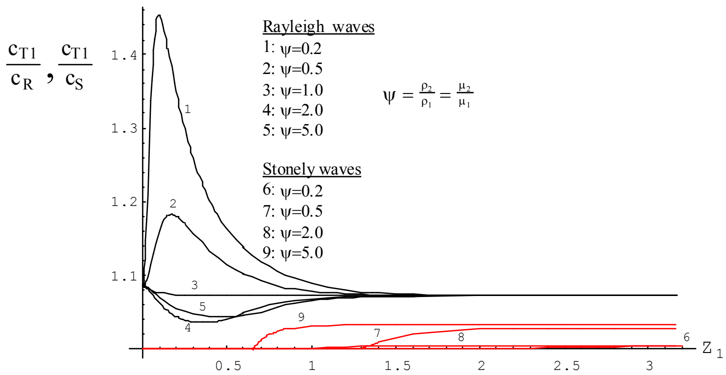

Figure 4. Dispersion curves for one layer half-space

In the following example the vertical component of Green's function for the elastic layer resting on the elastic half-space is calculated. The ratio between the wave length of shear waves in the layer and the layer thickness is considered to be $2\cdot\pi$. No material dumping is considered. Poisson's ratios of the layer and half-space are 1/3 and 1/4, respectively. The density of the half-space is 1.5 times greater and the shear module is 1.6 times greater than corresponding characteristics of the layer. All of the calculation steps in transformed domain were made analytically, only the contour integration for the evaluation of the inverse transform was obtained with numerical algorithms. The results are plotted on Fig. 5 .

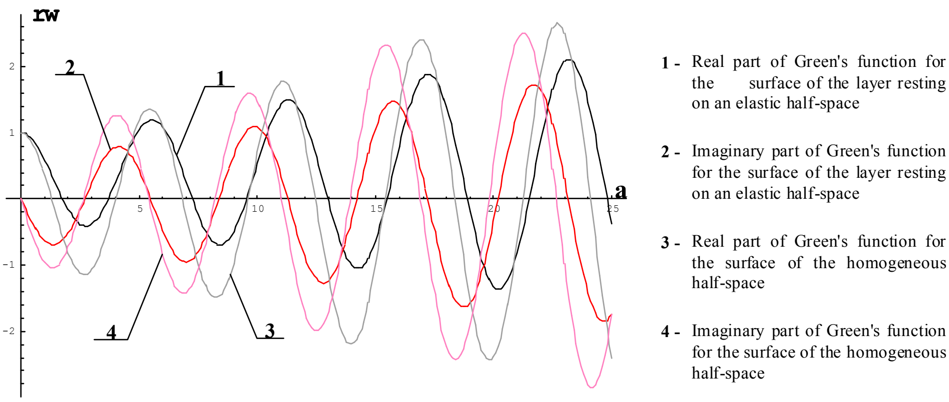

Figure 5. Green's function

3. Conclusions

Until now various numerical solutions for Green's functions of the layered half-space have been given. They are more or less accurate depending on the way of introducing of the radiation and continuity conditions. The main advantage of results presented is therefore their accuracy because all essential steps of Green's function evaluation except of the contour integration along the branch cut are made analytically. On the other hand the disadvantage of this method is that the mathematical effort for obtaining the Green's function is increasing drastically with the increase of the number of layers. Future work will therefore be directed in simplifying of the above described process.

4. References

1. Ewing, W.M., Jardetzky, W.S., Press, F.: Elastic Waves in Layered Media, Mc. Graw-Hill Book Company, 1957
2. Kobayashi, T.; Sasaki, F.: Evaluation of Green's Function on Semi-infinite Elastic Medium,Kajima Technical Research Institute, KICT Report, No.86, 1991
3. Sezawa, K., Kanai K.: The M_2 Seismic Waves, Bull. Earthquake Research Inst. (Tokyo), vol. 13, pp. 750-756, 1935
4. Achenbach, J.D.: Wave Propagation in Elastic Solid, North-Holland Publishing Company - Amsterdam, 1975
5. Wolfram, S.: Mathematica, A System for Doing Mathematics by Computer, Addison-Wesley Publishing Company, 1992
6. Abramowitz, M.; Stegun, I.A.: Handbook of Mathematical Functions, Dover Publications, Inc., 1972