SUPERELLIPTIC SHELLS AS NEW CONSTRUCTIVE FORMS.

ABRAMIAN A.K.

St.Petersburg Institute for the Problems in Mechanical Engineering
Russian Academy of Sciences.

ABSTRACT.
In the superelliptic shell joined to a circular cylinder bending stresses are absent when it is subjected to uniform pressure.Some geometrical characteristics have been found. Expressions for determining stresses in the shell crest(in the singular point of plane type) are suggested. The problem of a theoretical critical buckling load of an elongated shell supported by frames is studied. A critical buckling load for two shells with different specifications was found experimentally.

INTRODUCTION.

The present paper is devoted to superelliptic shells which have a uniform thickness and are subjected to uniform pressure. When the circular cylinder is joined with the superelliptic shell in the region of shell's equator ,bending stresses are absent. Due to this property superelliptic shells have some advantages over other shells when one uses them as a head of a pressure vessel. The stability of superelliptic shells is investigated as well. A critical buckling load has been found experimentally for two different shell specifications.

GEOMETRY OF SUPERELLIPTIC SHELLS.

A middle surface of a superelliptic shell is formed by the revolving of the curves around one of the axis of revolution:

$$(y/b)^{n} + (x/a)^{n} = 1, \quad x,y>0, \ n>2 \tag{1}$$

Principal dimensionless radii of curvature and their ratio in curvilinear coordinates are given by:

$$R_1/a = k^{n/n-1}(\sin\theta)^{2n-1/1-n}[1+(k\,\mathrm{ctg}\theta)^{n/n-1}]^{-(n-1)/n}$$
$$R_2/a = (\sin\theta)^{-1}[1+(k\,\mathrm{ctg}\theta)^{n/n-1}]^{-1/n}$$
$$R_2/R_1 = (n-1)k^{-n/n-1}(\cos\theta)^{n-2/n-1}[(\sin\theta)^{-n/n-1} + (k\cos\theta)^{n/n-1}] \tag{2}$$

where k=b/a -is the ratio of the superelliptic shell semiaxis.
The superelliptic shell has a singular point in the crest: $\theta=0$. In this point one has:

$$R_2/R_1 = \infty; \ \lim_{\theta\to 0} R_2/R_1 = n-1$$

It is the singular point of plane type. In the vicinity of the equator where $\theta=\pi/2$ the shell shape turns into the circular cylindrical shell. Varying the exponent of the power and increasing it, the curves fill up the area between the ellipse and the rectangle. A superelliptic shell may be called a supersphere when k=1.

STRESS STATE UNDER THE UNIFORM PRESSURE.

The smooth form of the superelliptic shells and the character of uniform pressure allow us to come to some apriori conclusions.When the circular cylinder is joined to a superelliptic shell in the region of shell's equator,bending stresses are absent because of the smoothness of the junction and the radial equality of the curve . Therefore the membrane stresses occur in the most part of the shell:

$$\sigma_{1m} = pR_2/2\delta$$
$$\sigma_{2m} = pR_2(1 - R_2/2R_1)/\delta$$

The meridian plane stresses may change their sign if : $R_2/2R_1 > 2$ The bending moment stresses occur in the crest. To get the limited solution one uses the general system of differential equations for symmetrical shells of revolution:

$$A_1^{-1} A_2^{-1} dA_2/d\theta\,(T_1-T_2) + A_2^{-1} dT_1/d\theta + N_1/R_1 = 0$$
$$A_1^{-1} A_2^{-1} N_1 dA_2/d\theta + A_1^{-1} dN_1/d\theta\; T_1/R_1 - T_2/R_2 + p = 0$$
$$dA_2/d\theta\; A_1^{-1} A_2^{-1} (M_1-M_2) + A_1^{-1} dM_1/d\theta - N_1 = 0$$
$$(1+\mu)N_1 = A_1^{-1}\, d(M_1+M_2)/d\theta$$
$$(M_1-\mu M_2) R_2^{-1} + (M_2-\mu M_1) R_1^{-1} + A_1^{-1} + A_2^{-1}\delta^2/12\; d[A_1^{-1}A_2 d(T_1 + T_2)/d\theta]/d\theta = 0 \tag{3}$$

The solution of the system is of the form :

$$T_1 = T_{10} + \sum_{\lambda=0}^{\infty} a_\lambda \theta^{k1+\lambda};$$
$$T_2 = T_{20} + \sum_{\lambda=0}^{\infty} b_\lambda \theta^{k2+\lambda};$$
$$M_1 = M_{10} + \sum_{\lambda=0}^{\infty} g_\lambda \theta^{k4+\lambda};$$

$$M_2= M_{10}+\sum_{\lambda=0}^{\infty} f_\lambda \theta^{k5+\lambda};$$
$$N_1= N_{10}+\sum_{\lambda=0}^{\infty} c_\lambda \theta^{k3+\lambda};$$

(4)

In the system (4) k1, k2, k3, k4, k5- are unknown quantities. As a result one obtains, the system of algebraic equations. Each equation of the system (3) to be correct in all points including 0 , if one satisfies the following conditions:

$$T_{10}= T_{20;} M_{10}= M_{20;} N_{10}=0;$$
$$k1=k2=n(n-1)^{-1} ; k3=(n-1)^{-1} ; k4=2(n-1)^{-1;} k5=2(n-1)^{-1};$$
$$c_0=-0.5pd_1^{-1;} ; g_0=-(1.5+0.5\mu)pd_1^{-2}/8$$
$$a_0=-0.5(n-1)(n+2)^{-1;}pd_1^{-1}-12(1-\mu)(n+2)^{-1} n^{-1} d_1^{-1}\delta^2 M_{10}$$
$$b_0=-0.5(n-1)(n+2)^{-1;}pd_1^{-1}-12(1-\mu)(n+1) n^{-1}(n+2)^{-1} d_1^{-1}\delta^2 M_{10}$$
$$f_0=-0.125(0.5+1.5\mu)pd_1^{-2}$$
$$a_1 = b_1 = c_1 =g_1=f_1 =0$$

(5)

Unknown quantities a_2, b_2, c_2, g_2, f_2 can be found by the same method but with more terms. For determining stresses only two terms will be enough in the solution. It is true for $\delta/b\leq 0$; To solve the problem of using only two terms in the Taylor series expansion of forces and moments we should compare these terms with the next ones.For this purpose using above given equation (4) and (3). I have calculated the following magnitudes : a_2, b_2, c_2, g_2, f_2. It occurs that, for example, for n=2,4; k=1 magnitudes $a_2\theta^{n/m-1+2}$ in 3 orders less than $a_0\theta^{n/n-1}$. Similar results have been achieved for magnitudes f_2. From above mentioned results one can conclude that for practical purpose two terms may be used in the solution. Analysis of convergence has been done numerically. One can find the magnitudes of the main terms of the solution in the shell vertex using the conditions:

$$M_1 (\theta=\theta_0)=0$$
$$T_1 (\theta=\theta_0)=0.5pR_2$$
$$T_2 (\theta=\theta_0)=0.5pR_2 (2-R_2/ R_1)$$

(6)

Knowing forces and moments in the vertex it is possible to define corresponding stresses.

BUCKLING OF SUPERELLIPTIC SHELL UNDER UNIFORM PRESSURE.

The buckling of superelliptic shells under uniform external pressure is investigated in this part of the paper.The experiment has shown that local loss of cupola-shaped shell stability takes place under external hydrostatic pressure. As a result a dent occurs. Before the loss of stability the shell has an axially symmetrical equilibrium.Assume that under some critical load the shell loses the axially symmetrical equilibrium and frequent waves with infinitesimal amplitude appear on the shell surface. Geometrical parameters are changed insignificantly along one half wave. Therefore to determine theoretical critical buckling load one can use the Mushtari equations [1]. Using these equations
the author one obtains a theoretical critical buckling load in [3] :

$$q_k=2E\delta^2/(3-3\mu^2)^{1/2} R_2^2(\theta_k)$$

The value of the angle θ_k obtained in [3] is defined as follows :

$$R_2/R_1=(n-1)(\cos\theta_k)^{(n-2)/(n-1)}[(\sin\theta_k)^{n/(n-1)} +(k \cos\theta_k)^{n/(n-1)}]k^{-n/(n-1)} =1$$

(7)

The expression (7) shows that the region of the dent is located close to the equator. While parameter increases, the region moves to the vertex.For parameter close to 2,the vertex is a possible region of instability. If bending and tensile stresses are located in the vertex compressive stresses are located in the vicinity of the equator. Therefore the critical buckling load is given by:

$$q_k=1.21 E\delta^2R_2^{-2}(\theta_k)$$

where μ=0.3.

Let's study the problem of critical buckling load of the elongated superelliptic shells.This shell is supported by a rigid bulkhead and assisted by frames located in the equal distance apart.

When elongated superelliptic shells have the same ratio as the larger axis to the less one from 4 to 10 they may be used as ship hulls. The parameter will be more , the nearer to the form of a circular cylinder their main parts are.

The specific features of such shells are a very small and smooth changes of angle and curvature radii along 0.9 shell length.

In the transverse direction such a shell will have bending rigidity which is corresponded to frame rigidity distributed along the frame spacing .One considers the shell to be constructive anisotropic.Longitudinal stiffness corresponds to shell stiffness.

One considers tensile and compressive stiffness to be equal to the stiffness of the shell without frames in both directions.
Assume that the shell has only membrane stresses before buckling. When solving the problem one should use equations similar to the equations of semimomentless theory of cylindrical shells.The author of [3] notes the following equation for displacement :

$$(\tau^{-2}\partial^2/\partial\Psi^2 + R_2^{-2})\nabla\Delta_D(W)+E\delta\nabla_k^2(\nabla_k^2)W=\nabla_*^2 q_k$$

Designate

$$\lambda_*=m\pi\tau^{-}L^{-2}$$

Now one can have equation of high critical buckling load from

$$q_k=E/(0.5\lambda_*^2+(1-R_2/2R_1)(n_1^2-\tau^2/ R_2^2))(J(n_1^2-\tau^2/R_2^2)^2/R_2^2\tau^2 1 + (\delta\tau^2(\lambda_*^2+ R_2/2R_1N_1^2)^2/R_2^3(\lambda_*^2+n_1^3)^2] \quad (8)$$

Maximum value of $q_k$will be if m=1. To find the critical buckling load one should choose such n_l when q_k is minimum. If and one can obtain the well known Novogilov formula [2] from equation (8). It means that formula (8) is correct for a critical buckling load.

EXPERIMENTAL RESULTS AND DISCUSSION .
In order to verify the theory, a careful experimental investigation has been carried out in which the normal deflection,stresses and buckling loads were measured.To make test specimens special punches were applied.Aluminum alloy AMG-3M was used as a material . 24 specimens were produced .The specimens in the shape of superelliptic shells joined tangentially to cylinders.The dimensions of the shells were as following :

$$\delta=1mm; R=70mm; n=2.4$$
$$n=4; k=0.5; k=1$$

where R - cylinder radius.
During the experiment the specimens were subjected to inside and outside hydrostatic pressure. The experimental investigation has shown that superelliptic shells may lose their stability either in the vertex or in the vicinity of the equator in multiwaves form where: $R_l/ R_2=1$. When N is close to 2 ,the critical buckling load magnitudes for multiwaves form buckling and for the dent in the vertex may be near in value.When N increases the buckling load,under which the vertex dent occurs,decreases and this critical buckling load determines the shell stability,when N is close to 4.
During the stability tests the normal deflections have been measured. The shell loading continued until large permanent dents appeared.Thus the turning inside out of the shall took place and pressure reduced. Then when the process neared its end the pressure began to rise. In that case the deformation had a nonlinear character and deflection values were greater than the shell thickness.Before the total shell failure ratio of the deflection to the shell thickness was from 4 to 20.
The comparison of the test results with equations shows that when N=2.4 theoretical values of are less than the experimental ones for the linear range.It means the shell is more deformable than the theory shows,when N=4 everything is vice versa the linear theory is true when the ratio of
The investigation shows that the specimens 7,8,9,24 have first multiwave buckling, and only then the dent in the vertex appears with a snap and the turning inside out of the shell.The dent has a ring form at first and the pressure drops as a rule.After the pressure stabilization and the stabilization of the dent dimension the dent form is changing,from the semi-spherical form it is becoming multangular. From this moment the load begins to increase.
The shell 4 and the shell 10 do not achieve a multangular buckling form.
The shell with N=4 has a different way of deformation .Buckling takes place with the appearance of a semi-spherical dent in the vertex and the process of the turning inside out of the shell is without a snap.Further the shell deformation takes place in the same way as the deformation for the shell with N=2.4
Summarizing the result we can conclude: 1.
Mushtari shows that convex shells of revolution lose their stability in the region where θ has values which ensure the fulfillment of $R_1/ R_2=1$.Our tests have shown that the superelliptic shells lost their stability either in the area where $R_1/ R_2=1$ or in the vertex.The test results agree with Mushtary's theoretical conclusion.
When N=2.4,the shell loses its stability with a snap.
When N=4 the shell loses its stability (as a rule) without a snap . A microsnap probably takes place.
When the load is removed from the superelliptic shell with N=4 the shell takes its initial form.
2. Critical buckling load is given by :

$$q_k=1.21\alpha\ E\delta^2/R_1R_2(\theta_k)$$

where α- is the coefficient determined by the test. It takes into account initial distortion of the shell form and the shell material behavior under loading not according to the Hooke law.The author recommends taking the coefficient asfollows:

$$\alpha=0.4 \text{ for the shell when } N=2.4$$

α=0.3 for the shell when N=4

3.The critical buckling load of the shells when N= 2.4 is greater than the critical buckling load of the cylindrical parts of the specimens.

The experiment investigations of stresses have shown :

1.There are moment stresses in the crest.

2.The region of bending stresses is:

$$S=ak^{n-1/2n}\theta^{*}$$

When θ and rigidity increase the shell tends to take the plate form .In this case the area subjected to the bending moment increases, but the shell effect (curvature effect) vanishes. The region dimension is not more than the dimension of the edge effect region, i.e. the magnitude of the order of ~ $(R_2^2 \delta)^{1/2}$

3.Conditions (6) of determining moments and forces in the crest is satisfactory for :

n=2.4. The investigation shows a very good agreement between the theoretical value of total stresses in the crest of the shell when and experimental ones. But the author hasn't achieved an agreement for values of stresses in the crest between theory and experiment when n=4. There are membrane stresses on the other parts of the shell. In the vicinity of the shell and cylinder joint, bending stresses are absent.

The difference of theoretical and experimental results can be explained by following reasons : -probably the author failed to get the singular point of plane type in practice; -the author would have used the nonlinear theory of thin shells to compare the results of tests for the parameter .

In the work [4] the author has shown:

1. the superelliptic shells when n=2,4 have significant advantages over the compound bottoms.Using them one can decrease stresses in the area of cylinder and cone junction.

2.Using the part of superelliptic shell instead of torus in a compound bottom one obtains significant advantages in weight.

We should choose the shell curvature radii closed to sphere curvature radii in the area of the junction. The given method of determining shell stresses may be used for designing heads of pressure vessels and hull structures in shipbuilding and cupolas in architecture .

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