CAD and Discrete Optimization
(Review of Discrete Optimization Techniques for CAD)

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1. Introduction

The role of design optimization in CAD is central. Classical theories of optimization (differential calculus, variational calculus, optimal control theory, mathematical programming) deal with the case when this domain design parameters is infinite. From this angle, the subject of discrete optimization, where the domain design parameters is typically finite, might seem trivial : it is easy to say that «we choose the best from this finite number of prototypes».

1. Discrete optimization in the structure design

Any object to be design consists of a set of functional modules and a systems of actions between them. Synthesis of the object's structure supposes choice of prototypes for each module and a scheme of connections between them. For the enumeration of the polytopes, methods of discrete optimization are used. Two directions for synthesis of the structure are spread, namely, morphological and alternative graph methods.

1.1. ***Morphological method*** is proposed by F.Zwicky [14,15] :

Let by

Functional modules	Prototypes
M_1	$m_1^1, \ldots, m_1^{s_1}$
M_2	$m_2^1, \ldots, m_2^{s_2}$
...	...
M_n	$m_n^1, \ldots, m_n^{s_n}$

and $g(m_1,\ldots,m_n) = 0 \vee 1$ is the characterization function prohibiting some combinations of the modules and making some combinations necessary to present in the structure.

It is problem of discrete optimization :

$$\max f(m_1,\ldots,m_m)$$
$$m_i \in M_i = \{m_i^1,\ldots,m_i^{s_i}\}$$
$$g(m_1,\ldots,m_n) = 1,$$

Methods for enumerating morphological set $M = M_1 \times \ldots \times M_n$ with additive weights of usefulness with elimination of useless variants are proposed in [5].

1.2. ***The alternative graph approach*** for structure synthesis uses an and-or-graph for representing alternatives of the functional modules. For finding the minimal cost structure, enumeration methods with cutting branches of the and-or-graph are used.

1.3. ***Convex discrete optimization without objective function***. Let the objective function $f(m_1,\ldots,m_n)$ be unknown, but one knows that it is discrete-convex. A function $f: M_1 \times \ldots \times M_n \to R$ is called discrete convex if the function $f(m_1,\ldots,m_i^+,\ldots,m_n) - f(m_1,\ldots,m_n)$ is increase, if $m_i = m_i^{k_i}$, then $m_i^+ = m_i^{k_i+1}$.

In many design problems quality of a design object can be estimated by a discrete-convex function that corresponds the property of decreasing the speed of increasing efficiency with increasing the prototype number.

The unknown objective function is given by procedure *sgf-oracle* that says : the function increases or decreases on an edge of lattice $M_1 \times \ldots \times M_n$. $sg_i f - oracle(m_1,\ldots,m_n) = -1(1)$, if design object $(m_1,\ldots,m_n)$ is better (worse) than design object $(m_1,\ldots,m_i^+,\ldots,m_n), m_i^+ = m_i + 1$, $sg_i f - oracle(m_1,\ldots,m_n) = 0$, *if design object* $(m_1,\ldots,m_i^+,\ldots,m_n)$ *is equivalent design object* $(m_1,\ldots,m_n)$.

In continue we assume $sg_i f\text{-}oracle \neq 0$. It means we find locally optimal design objects with deferent values of the objective function.

We now describe a scheme to find locally optimal design objects.

Let $E(M_1 \times \ldots \times M_n)$ be the set of all edges of the Hasse diagram of lattice $(M_1 \times \ldots \times M_n, \leq)$. We decompose $E(M_1 \times \ldots \times M_n)$ in n mutually disjoint subsets $E_i(M_1 \times \ldots \times M_n)$, $i=1,\ldots,n$, where $E_i(M_1 \times \ldots \times M_n)$ is the set of edges that are parallel with axis i. In [7] shown that $E_i(M_1 \times \ldots \times M_n)$ is isomorphic to lattice $M_1 \times \ldots \times M_i^- \times \ldots \times M_n$, where $|M_i^-| = |M_i| - 1$. To find locally optimal design objects, we need to decode *sg$_i$f-oracle* on lattice $M_1 \times \ldots \times M_i^- \times \ldots \times M_n$. To do it, an optimal algorithm for decoding monotonic boolean functions is applied [ ]. The *sg$_i$f-oracle* decoded can be memorized by an ideal. These ideals allows to find locally optimal design objects [6].

In [7] proved that this scheme yields an Shannon optimal algorithm for the boolean lattice, i.e. $|M_i| = 2, i = 1,\ldots,n$. The complexity of this algorithm is $n \; x \begin{pmatrix} n \\ \lfloor n/2 \rfloor \end{pmatrix}$ measured in calls of the *sgf-oracle*. The classes of discrete-convex function coincides with the class of submodular function on the boolean lattice.

One should point out that the sgf-oracle is cheaper for some applications then a procedure computing the value of the objective function.

The problem is that the number of locally optimal structures may be very large and thus, the Shannon complexity of the best algorithm for the worst example is exponential.

2. Matroidal Decomposition

Decomposition is one of the main tools for rousing the dimension of practical problems structural design solvable in acceptable time. In this section we briefly the decomposition technique which hoes been developed in [8,11,12].

Let N be a finite set. A set-system $R \subseteq 2^N$ is called a ring family (lattice), of $X,Y \in R$ implies that $X \cap Y,\ X \cup Y \in R$. The minimum element of the ring family (with respect to inclusion) will be denoted by R_{min} an R_{max}. The sequence

$$\tilde{N} = \left\{ R_{\min} = C_0 \subset C_1 \subset \ldots \subset C_k = R_{\max} \right\}$$

of elements from R is called a R-chain.

The length of the chain is equal to R and denoted by $|C|$. The chain of the maximal length is called maximal. The set of all R-chains is denoted by $\mathcal{C}_R$. We use the tern chain for 2^N -chains, but denote the family of all chains by $\mathcal{C}$.

On $\mathcal{C}$ we define the functional

$$\psi\ (\tilde{N}) = \sum_{i=0}^{|\tilde{N}|-1} \left(f(\tilde{N}_i) + f(\tilde{N}_{i-1}) \right) \left(g(\tilde{N}_i) - g(\tilde{N}_{i-1}) \right),$$

and consider the problem

$$\min_{c \in \boldsymbol{C}} \psi(\boldsymbol{C}) \qquad \textbf{(1)}$$

Theorem 1 [12]. *There exists the ring family R(f,g) such that any maximal chain in R(f,g) is an optimal solution to (1).*

For any ring family $R \subseteq 2^N$ with $\varnothing, N \in R$, there is a directed graph $G_R = (N,E)$ such that members of R are exactly closed subsets in G (a subset $X \subseteq N$ is closed in G if there is no are $(v,w) \in E$ with $v \in X$ and $w \notin X$). It should be noted that the ring family $R(f,g)$ can be represented by graph $G_{R(f,g)}$ in time polynomial in $|N|$ and number of calls of the oracle computing f and g.

2.1. ***Decomposition of layered matrices.*** The both functions, f and g, are modular (linear) on *R(f,g)*. This property can be used to construct algorithms for a number of decomposition problems. Here, as an example, we apply this approach to block-trangularization of layered matrices.

Consider the system of equations of the following form

$$Qx = b, \quad f(x) = 0 \tag{2}$$

A Typical examples are systems of equations describing electrical networks, elastic structures, hydrodynamic systems. The structural equations (linear equations in (2)) are made up of equations for conservation laws (systems of actions between functional modules; Kirchhoff's lows, the conditions of «equilibrium of forces and moments» and etc.). The constitutive equations (nonlinear equations in (2)) describe characteristics of functional modules.

The Jacobian of systems (2) has the form

$$A = \begin{bmatrix} Q \\ T \end{bmatrix}, \tag{3}$$

where T is the Jacobian of the vector-function $f(x)$. Assume that we intend to the solve systems (2) by Newton's method. Then at each iteration we have to solve a system of linear equations with constraint matrix A, in which only part T changes from iteration to iteration, where as part Q remains invariable. There fore, some of the computations will be repeated from iteration to iteration. The problem is to perform all the computations common to all iterations before beginning the iterative process.

The matrix A of the form (3) is said to be layered if the element of matrix Q belong to some field K (say, $K=R$) and non-zero of the matrix T are algebraically independent over K. Any elementary matrix operations over Q and all possible permutations of rows and columns are taken to be feasible operations on the matrix A. The category of feasible operations does not include elementary matrix operations on T, since these are symbolic calculations. It is required to reduce the matrix A to block-triangulow form with the maximum possible number of block. This problem of the decomposition of a layered matrix was first formulated in [11].

Let N denote the set of columns of the matrix A.

On 2^N we define the function $g(I) = (N \backslash I)$ and $f(I) = j(I) + rank\, Q^I$, where $j(I)$ is the number of rows in T having non-zero entries in rows I, Q^I is the submatrix of Q with the set of columns. Let $R \subseteq 2^N$ a ring family such that the bath functions, f and g, are modular on R. Then we can associate with R a decomposition of A such that reduce A to block-triangular form A with the following properties (see [8,12] for details).

a) blocks of $\overline{A}$ are in one to one correspondence with the strongly connected components of graph G_R;

b) blocks are ordered in accordance with the partial order induced bye G_R, i.e. a block B_1 is proceeded by a block B_2 if in f_{GR} there is a path from B_1 to B_2.

It can be proved that we get the «finest» decomposition if $R = R(f,g)$.

In [11], the lattice $L(\rho)$ of minimal of the function $\rho(I) = f(I) + |I|$ was used to produce the decomposition. It should be notet that, in the case when $rank\, A = n$, the decompositions with respect to $L(\rho)$ and $R(f,g)$ are the same.

If *rank A* $< n$, then the decomposition with respect to *R(f,g)* is more finest than with respect to *L(ρ)*.

3. Discrete Optimization in Designing

Placement problem is considered as one of the general design problem and formulated as follows. We are given n modules and m possible position for their placement. Let *Aij* be a distance between positions i and j (in any metric) and B_{kl} be a degree of connectivity of modules *k* and *l*. The goal is to find a placement in order to minimize

$$min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} A_{ip} B_{jq} X_{ij} X_{pq}$$

$$\sum_{i=1}^{n} X_{ij} = 1, \qquad \sum_{j=1}^{n} X_{ij} = 1$$

$$X_{ij} = 0 \quad or\ 1,\ i = 1,\dots,n,\ \ j = 1,\dots,n$$

or in other form :

Considering problem is well-known as Quadratic Programming Problem and it is *NP-hard*. Real models have as a rule a range of additional restrictions such as size or volume of modules, connections between them have to be realize in a specific way. Therefore traditional procedures have to be modified in order to satisfy these additional restrictions. In [4,6,17] there was developed an approach for solving similar problems based on compound of algorithms and hybrid algorithms.

Some particular placement problems such as placement in line :

$$\min_{\pi}\ 1/2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\left|\pi(i)-\pi(j)\right|;$$

$$\min_{\pi}\ \max_{1\le k\le n-1} \sum\nolimits_{\{i,j:\pi(i)\le k\le \pi(j)\}} c_{ij};$$

$$\min_{\pi}\ \max\left\{c_{ij}\left|\pi(i)-\pi(j)\right|\right\}.$$

3.1. ***Packing problem*** is combinatorial - geometrical problems of discrete design optimization. These problems often arise in integrated CAD Systems when many items of small details must be optimally cut from large materials, in distribution when many items of small products must be optimally packed in transports as well as in layout design when many electronic devices must be optimally placed on a chip. In architecture and building these problems often arise in buildings design (compounding) and in optimal cutting (waste minimization) of the building materials such as window glass, wood, metal, plastic and so on.

The *Packing problem* have a long history. The first *Packing problem* was formulated 1878 by well-known Russian scientist P.I.Chebushev. The first optimization method for a *Packing problem* was proposed 1939 by L.V.Kantorovich.

Unfortunately the majority of *Packing problem* are *NP* complete. The resent bibliography on *Packing problem* counts more than one half thousand publications for last five years. We have examined several types of *Packing problem*. The most important results we have received for guillotine *Packing problem* by using some discrete optimizations approaches.

Let us consider guillotine pallet loading problem *(GPL)*. It is necessary to pack a number of rectangular boxes of two given dimensions *a*b*h* or *c*d*h* with fixed orientation into a single big rectangular box *L*W*H* such that unused volume of the big box is minimal. Using the classification of Prof. Dr. H. Dyckhoff (RWTH Aachen) the *GPL* problem belongs to the class *3/B/O/**.

We turn the *GPL* problem into shortest path problem in a simple a cyclic directed graph (grid). It allows us to introduce an exact polynomial time algorithm for solving *GPL* problem. This algorithm is based on the decomposition of the initial problem into three subproblems. Two of these subproblems can be solved by special greedy algorithm exactly and the third subproblem can be solved during constant time. The exact solution of *GPL* problem can be obtained by this algorithm in the polynomial time $O(n^2)$, where $n=log_2 \max\{L, W, H, a, b, c, d, h\}$ is the length of the largest number in the binary representation.

This algorithm requires a minimization of linear function into residue ring. A polynomial time algorithm is obtained for this purpose.

These problems was discussed and approved by the following specialists: Prof. Dr. P.Brucker (TU Osnabrück), Prof. Dr. H.Dyckhoff and Prof. Dr. W.Oberschelp (RWTH Aachen), Prof. Dr. E.Girlich (Universitaet Magdeburg), Prof. Dr. A. Hoffmann (TU Ilmenau), Prof. Dr. R.Lang (Universitaet Hamburg), Prof. Dr. G.Reinelt (Universitaet Heidelberg), Prof. Dr. J.Terno (TU Dresden), Prof. Dr. G.Wäscher (Universitaet Halle).

3.2. *Optimal arrangement of rectangles and shortest paths in L_1-metrics.* The geometry of rectangles has much applications in VLSI-design. Some theoretical results were obtained [8].

3.3. *Partition problems.* This problems arise when items have to be partitioned into some groups in order to minimize a weight function. This type of problems very important for decomposition approach for design.

4. Discrete optimization in computational geometry and computer graphics

The methods of discrete optimization have wide application in the fields of computational geometry and computer graphics. Among a great variety of problems we distinct the ones for which the d.o. algorithms give the maximal effect.

The computational geometry problems presenting special interest for discrete optimization investigators are the following :

- ***Maxima of a point set on the plane.*** The problem has a number of applications in computer graphics [13]. Some results were obtained for general (n-dimensional) case with respect to greedy algorithms for solving this problem.

- ***Triangulation.*** Special types of triangulations serve as a base for constructing surfaces in geometric design and computer graphics. Finding Algorithms for constructing triangulation with minimal sum of edge length is still an open problem [1]. We have investigated the dependence between types of triangulations and their quality characteristics for practical applications (the accurancy of surface interpolation and partial derivatives retrieval) [14]. Special techniques for minimal triangulation construction were also investigated using the partial order method. Another problem concerns finding optimal triangulations with Steiner points.

- ***One of the main problems in computer graphics is removing hidden lines and surfaces.*** One approach consists in the reduction of this defining a partial order on set of the plain faces and problem to maintaining then a linear order for further sequential rendering. The techniques for optimal ordering of 3-dimensional objects present special interest for discrete optimization investigators.

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