MRA on Q_p-spaces

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Abstract

In this paper we want to outline the possibility of using methods of Wavelet analysis to study Q_p-spaces.

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1 Introduction

In recent years a new scale of function spaces emerged from the field of complex analysis, the so-called Q_p-spaces. These spaces are defined in the following way [1]: Let $\Delta = \{z : |z| < 1\}$ be the unit disk in $\mathbb{C}$, $\varphi_a(z) = (a-z)(1-\bar{a}z)^{-1}$ the automorphisms, which map the unit disk onto itself. Then we can define the following semi-norm

$$|f|_{Q_p} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^2(1-|\varphi_a(z)|^2)^p dxdy < \infty$$

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and we have

$$Q_p = \{f \in H(\Delta) : |f|_{Q_p} < \infty\}$$

These Q_p-spaces form a scale of function spaces with the following properties

$$D \subset Q_p \subset Q_q \subset BMOA, \qquad 0 < p < q < 1.$$

Moreover,

$$Q_1 = BMOA,$$

where $BMOA$ denotes the space of all analytic BMO-functions and

$$Q_p = B, \forall p > 1,$$

where B complex Bloch space, i.e. $B = \{f \in H(\Delta) : \sup_{z \in \Delta}(1 - |z|^2)|f'(z)| < \infty\}$. Moreover, this scale of spaces was also generalized in different ways to higher dimensions [GKST] and [CD].

Also, in the last ten years emerged a new method for treating approximation problems in different areas, mainly in signal and image processing, called wavelet or Gabor analysis. Where the continuous part, i.e. the continuous wavelet transform and its applications, are already quite a lot studied in the framework of Clifford analysis, e.g. [C], [BS1], [BS2], [Mi] the same cannot be said about its discrete counterpart, but without this part numerical applications using wavelet analysis are unthinkable.

For the discrete wavelet transform and its corresponding multiresolution analysis there exists an exhaustive theory in the one-dimensional case, but the same cannot be said in higher dimensions. There is, of course, the direct generalization for tensorial domains, i.e. domains which can be obtained as a tensor product of intervals, e.g. rectangles and cubes. For classical problems like image compression this seems to be enough (a photo is always a rectangular domain), but if one wants to consider more general domains, i.e. domains which are not invariant under translations and dilatations, one has to find a new approach. Mainly, Fourier analysis methods, used for defining scaling equations, filters, etc., do not work so easily in these cases.

Nevertheless, a closer look at the above defined Q_p-spaces reveals the possibility to apply wavelet methods in this case. The unit disk has its own group of automorphisms and the Q_p-spaces are invariant under this group making it ideally suited for a wavelet approach.

In this paper we want to outline the possibility to apply Multiresolution analysis (MRA) to the study of Q_p-spaces. We omit here the (in this setting rather complicated) discussion of scaling equations, filter banks and the practical realization of the discrete wavelet transform in form of a fast wavelet transform.

2 Preliminaries

In what follows we will work in $\mathbb{H}$, the skew field of quaternions. This means we can write each element $z \in \mathbb{H}$ in the form

$$z = z_0 + z_1 i + z_2 j + z_3 k, \quad z_n \in \mathbb{R},$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules $i^2 = j^2 = k^2 = -1, ij = -ji = k, kj = -jk = i, ki = -ik = j$. The conjugate element $\bar{z}$ is given by $\bar{z} = z_0 - z_1 i - z_2 j - z_3 k$ and we have the property $z\bar{z} = \bar{z}z = \|z\|^2 = z_0^2 + z_1^2 + z_2^2 + z_3^2$. Moreover, we can identify each vector $\vec{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion x of the form

$$x = x_0 + x_1 i + x_2 j.$$

Also, in what follows we will work in $B_1(0) \subset \mathbb{R}^3$, the unit ball in the real three-dimensional space. $B_1(0)$ is a bounded, simply connected domain with a C^∞-boundary $S_1(0)$. Moreover, we will consider functions f defined on $B_1(0)$ with values in $\mathbb{H}$.

The group of Möbius transformations mapping the unit disk onto the unit disk is given by

$$\varphi_a(z) = (a - z)(1 - \overline{a}z), \qquad |a| < 1.$$

We now define the generalized Cauchy-Riemann operator by

$$Df = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2}$$

and its conjugate operator by

$$\overline{D}f = \frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2}.$$

For these operators we have that

$$D\overline{D} = \overline{D}D = \Delta_3,$$

where Δ_3 is the Laplacian for functions defined over domains in $\mathbb{R}^3$. Functions belonging to the kernel of D are called monogenic or regular functions. Let us remark that if we define the action of the above group of Möbius transformations by

$$L(a)f(z) = \frac{1-\overline{z}a}{|1-\overline{a}z|^3} f(\varphi_a(z))$$

then the result will be again a monogenic function [GS2].

For more information about these topics and general quaternionic analysis we refer to [GS1], [KS], [GS2], and [Sud].

3 Multiresolution analysis on Bergman spaces

In the classical one-dimensional setting multiresolution analysis is called a sequence of imbedded subspaces V_j of $L_2(\mathbb{R})$, such that

1. $\cap_{j=0}^{\infty} V_j = \{0\}$
2. $\cup_{j=0}^{\infty} V_j$ is dense in $L_2(\mathbb{R})$
3. For any $j \in \mathbb{Z}$: $f(x) \in V_j$ iff $f(2x) \in V_{j+1}$
4. For any $k \in \mathbb{Z}$: $f(x) \in V_j$ iff $f(x-k) \in V_j$

There exists a scaling function $\phi(x)$ serving to construct a basis in each V_j, via

$$V_j = \overline{\mathrm{span}}\{\phi_{jk}\}_{k\in\mathbb{Z}}$$

with

$$\phi_{jk} = 2^{j/2}\phi(2^j x - k), \qquad j,k \in \mathbb{Z}$$

The main issue of the wavelet approach now is to work with the orthogonal complement spaces W_j defined by

$$V_{j+1} = V_j \oplus W_j$$

Based on the function $\phi(x)$ one can find a function $\psi(x)$, the so-called mother wavelet, of which the translates and dilates constitute orthonormal bases of the spaces W_j:

$$W_j = \overline{\mathrm{span}}\{\psi_{jk}\}_{k\in\mathbb{Z}}$$

generated by the wavelets

$$\psi_{jk} = 2^{j/2}\psi(2^j x - k), \qquad j,k \in \mathbb{Z}$$

Each function $f \in L_2(\mathbb{R})$ can now be expressed as

$$f(x) = \sum_{k\in\mathbb{Z}} c_{j_0k}\phi_{j_0k}(x) + \sum_{j=j_0}^{\infty}\sum_{i\in\mathbb{Z}} d_{jk}\psi_{jk}(x).$$

The transition from $f(x)$ to the coefficients c_{jk} and d_{jk} is called the discrete wavelet transform.

In the case of Clifford-valued functions over the $\mathbb{R}^n$ the corresponding setting was already investigated by M. Mitrea [Mi]. He showed the existence and regularity of a dual pair of wavelet bases for the "Clifford" MRA of $L_2(\mathbb{R}^n)$. Of course, this setting is also not limited to the space L_2, but can be used for any Hilbert or Banach space over $\mathbb{R}^n$ or any tensorial subdomain. We only remark that in case of a Banach space the orthogonal decomposition is substituted by a direct decomposition and the discrete wavelet transformation cannot be obtained via the inner product of the wavelet basis with the function f.

But, when we take a look into domains like the unit disk, the standard setting does not work. Mainly, due to the fact that the groups are different. In the classical setting we use the group consisting of all translations and dilatations. Therefore, conditions 3 and 4 mean nothing else than the space $L_2(\mathbb{R})$ being invariant under translations and dilatations, which mean that in the case of the unit disk we need Hilbert spaces which are invariant under the automorphism group, the group of Möbius transformations, which map the unit disk onto the unit disk.

Let us consider the simplest case, the case of a weighted Bergman space defined by the norm

$$||f||_p^2 = \int_{B_1(0)} |f(z)|^2(1-|z|^2)^p dB_z, \quad p \geq -1.$$

This space is a Hilbert space endowed with the inner product

$$< f, g >_p = \int_{B_1(0)} \overline{f(z)} g(z)(1-|z|^2)^p dB_z$$

We already knew that the automorphism group of $B_1(0)$ consists of mappings of the form

$$\varphi_a(z) = (a - z)(1 - \overline{a}z)^{-1}, \qquad |a| < 1.$$

The problems now consists in finding good replacements for the translations by $\mathbb{Z}$ and the dilatations by 2^j in our automorphism group. Principally, the translations are of major importance because they correspond to the construction of a basis in V_j or W_j.

Let us take a look at two possibilities.

In the first one we shall consider a in polar coordinates $a = r\xi$, where $\xi \in S_1(0)$. Then we take $r = 2^{-j}$ as a replacement of the dilatation part and $\xi = \xi_k$ generated by the Euler angles $\left(\frac{2\pi k_1}{j}, \frac{2\pi k_2}{j}, \frac{2\pi k_3}{j}\right), k_i = 0, \ldots, j$, as a replacement for the translation part. Thus we can consider the spaces $V_j = V_j = \overline{\text{span}}\{\phi_{jk}\}_{k=\{0,\ldots,j\}^3}$ generated by

$$\phi_{jk}(z) = \frac{1 - 2^{-j}\overline{z}\xi_k}{|1 - 2^{-j}\overline{\xi}_k z|^3}\phi(\varphi_{2^{-j}\xi_k}(z))$$

as well as the spaces $W_j = W_j = \overline{\text{span}}\{\psi_{jk}\}_{k=\{0,\ldots,j\}^3}$ generated by

$$\psi_{jk}(z) = \frac{1 - 2^{-j}\overline{z}\xi_k}{|1 - 2^{-j}\overline{\xi}_k z|^3}\psi(\varphi_{2^{-j}\xi_k}(z)),$$

where ϕ and ψ are suitably chosen.

For the second approach we rewrite our Möbius transformation $\varphi_a(z)$ in the form

$$\varphi_a(z) = \varphi_{r,p}(z) = p(r - \overline{p}zp)(1 - rpz\overline{p})^{-1}\overline{p}$$

by choosing $a = pr\overline{p}$. Taking $p = p_k$ the rotations defined by Euler angles $\left(\frac{2\pi k_1}{j}, \frac{2\pi k_2}{j}, \frac{2\pi k_3}{j}\right), k_i = 0, \ldots, j$ we have the spaces $V_j = \overline{\text{span}}\{\phi_{jk}\}_{k=\{0,\ldots,j\}^3}$ and $W_j = \overline{\text{span}}\{\psi_{jk}\}_{k=\{0,\ldots,j\}^3}$ now generated by

$$\phi_{jk}(z) = \frac{1 - 2^{-j}p_k\overline{z}\,\overline{p}_k}{|1 - 2^{-j}p_k z\overline{p}_k|^3}\phi(\varphi_{2^{-j},p_k}(z))$$

and

$$\psi_{jk}(z) = \frac{1 - 2^{-j}p_k\overline{z}\,\overline{p}_k}{|1 - 2^{-j}p_k z\overline{p}_k|^3}\psi(\varphi_{2^{-j},p_k}(z))$$

again with a suitable chosen ϕ and ψ.

Let us remark that in both cases the translations are replaced by rotations.

Due to the fact that the original Bergman space is invariant under our Möbius transformations $\varphi_a(z)$ and the fact that the rotations p_k form a discrete subgroup of the group of all rotations we get that in both cases our spaces V_j satisfy conditions 1 to 4.

One could now argue about the existence of "suitably chosen" functions ϕ and ψ, where "suitably chosen" means that ϕ_{jk} and ψ_{jk} form a Riesz basis in V_j resp. W_j. For that we can take as a mother wavelet $\psi(z)$ the functions $\psi(z) = 1$ or the Cauchy kernel $\psi(z) = \frac{z-b}{|z-b|^3}$ with a arbitrarily chosen b such that $|b| > 1$. Let us remark that if we take $b = 0$ then we get for each space W_j some kind of Eisenstein series.

Let us finish the section with the following theorem:

Theorem 3.1 *The discrete Wavelet transform L_ψ of a function f with $||f||_p < \infty$ is given by the vector*

$$L_\psi f = \left(\int_{B_1(0)} \overline{f(z)} \psi_{jk}(z)(1 - |z|^2)^p dB_z \right)_{jk}.$$

4 Multiresolution analysis on Q_p spaces

As we already mentioned in the introduction there are different definitions of Q_p-spaces, depending on the differential operator. Using the gradient we can consider the following definition by J. Cnops and R. Delanghe [CD]:

Definition 4.1 *Let $f : B_1(0) \mapsto \mathbb{H}$ a function defined over the unit ball in $\mathbb{R}^3$. Then the Q_p-space is the space of all monogenic functions, such that the semi-norm*

$$|f|_{Q_p} = \sup_{a \in B_1(0)} \int_{B_1(0)} \sum_{k=0}^{2} |\frac{\partial f}{\partial x_k}|^2 (1 - |\varphi_a(z)|^2)^p dB,$$

is finite, i.e. $Q_p = \{f \in \ker D : |f|_{Q_p} < \infty\}$.

Using $\overline{D}$ we obtain the Q_p-scales defined by K. Gürlebeck, U. Kähler, M.V. Shapiro, and L.M. Tovar [GKST]:

Definition 4.2 *Let $f : B_1(0) \mapsto \mathbb{H}$ again a function defined over the unit ball in $\mathbb{R}^3$. Then the Q_p-space is the space of all monogenic functions, such that the semi-norm*

$$|f|_{Q_p} = \sup_{a \in B_1(0)} \int_{B_1(0)} |\overline{D}f|^2 (1 - |\varphi_a(z)|^2)^p dB,$$

is finite, i.e. $Q_p = \{f \in \ker D : |f|_{Q_p} < \infty\}$.

For the sake of simplicity we will restrict ourselves to the case of the first definition. Let us remark that in the second definition $\overline{D}f$ corresponds to $2\partial_{x_0} f$ which allows a similar treatment as in the first case from the point of view of MRA. The Q_p-spaces defined via the gradient can be easily transformed into Hilbert spaces endowed with the inner product

$$< f, g >= \overline{f(0)} g(0) + \sup_{a \in B_1(0)} \int_{B_1(0)} \left[\overline{\nabla f(z)} \cdot \nabla g(z) \right] (1 - |\varphi_a(z)|^2)^p dB,$$

where $\left[\overline{\nabla f(z)} \cdot \nabla g(z) \right]$ denotes the Euclidean inner product of two vectors. This allows us to use the same approach as in the previous section.

Due to the fact that these spaces are invariant under the group of automorphisms we can define our wavelet base ψ_{jk} in the same way as before. The main difference resides in the discrete Wavelet transform:

Theorem 4.1 *The discrete Wavelet transform L_ψ of a function $f \in Q_p$ is given by the supremum over all vectors*

$$L_{\psi,a} f = \left(\int_{B_1(0)} \overline{f(z)} \psi_{jk}(z) (1 - |\varphi_a(z)|^2)^p dB_z \right)_{jk},$$

therefore,

$$L_\psi f = \sup_{a \in B_1(0)} L_{\psi,a} f.$$

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