Applications of Bergman kernel functions

S. Bock, M. I. Falcão and K. Gürlebeck

Abstract

In this paper we revisit the so-called Bergman kernel method (BKM) for solving conformal mapping problems. This method is based on the reproducing property of the Bergman kernel function. The construction of reproducing kernel functions is not restricted to real dimension 2. Results concerning the construction of Bergman kernel functions in closed form for special domains in the framework of hypercomplex function theory suggest that BKM can also be extended to mapping problems in higher dimensions. We describe a 3-dimensional BKM-approach and present two numerical examples.

1 Introduction

Let Ω be a bounded simply-connected domain with boundary $\partial\Omega$ in the complex z–plane ($z = x + iy$), and let $L^2(\Omega)$ denote the Hilbert space of all square integrable functions which are analytic in Ω. Consider the inner product in $L^2(\Omega)$

$$< g_1(z), g_2(z) >= \int\int_\Omega g_1(z)\overline{g_2(z)}dxdy,$$

assume w.l.o.g. that $0 \in \Omega$ and let $K(.,0)$ be the Bergman kernel function of Ω with respect to 0. Then, the kernel function $K(.,0)$ is uniquely characterized by the *reproducing property*

$$< g, K(.,0) >= g(0), \ \forall g \in L^2(\Omega). \tag{1}$$

2 The Bergman kernel method for numerical conformal mapping

There are several methods for solving conformal mapping problems. In contrast to most conformal mapping techniques, the approximation of the solution obtained by using the Bergman kernel method (BKM) is an analytic function.

The BKM is a method for approximating the mapping f which maps conformally Ω onto the unit disc $D := \{w : |w| < 1\}$, in such a way that $f(0) = 0$ and $f'(0) > 0$. The method is based on the *reproducing property* (1) of the kernel function and on the well known relation of $K(.,0)$ with f,

$$f(z) = \sqrt{\frac{\pi}{K(0,0)}} \int_0^z K(t,0)dt, \tag{2}$$

(see [1, 4, 5]). More precisely, the BKM involves the following four steps:

S1: Choose a complete set of functions $\{\eta_j\}_1^\infty$ for the space $L^2(\Omega)$.

S2: Orthonormalize the functions $\{\eta_j\}_1^n$ by means of the Gram-Schmidt process to obtain an orthonormal set $\{\eta_j^*\}_1^n$.

S3: Approximate the kernel function $K(.,0)$ by the Fourier sum

$$K_n(z,0) = \sum_{j=1}^{n} < K(.,0), \eta_j^* > \eta_j^*(z) = \sum_{j=1}^{n} \overline{\eta_j^*(0)} \eta_j^*(z) \tag{3}$$

S4: Approximate f by

$$f_n(z) = \sqrt{\frac{\pi}{K_n(0,0)}} \int_0^z K_n(t,0)dt. \tag{4}$$

The second step of the BKM involves the use of the Gram-Schmidt process which can be extremely unstable. For this reason we use Maple, as this system provides integration routines so that the inner products involved in the construction of the Gramian matrix can be computed without any loss of accuracy (cf. [7]).

For example, in the case of the squared domain

$$\mathcal{S} := \{z = x + iy : |x| < 1,\ |y| < 1\},$$

the BKM details are as follows:

The usual choice of the basis set in step **S1** is to take the monomials $1,\ z,\ z^2, \cdots$. In this example, because of the symmetry of S it suffices to consider the monomials $1,\ z^4,\ z^8, \cdots$, the other inner products being zero, (see Gaier [4]). Denoting by n the number of monomials used, we have, for example, for $n = 2$,

$$\eta_1 = 1 \qquad \text{and} \qquad \eta_2 = z^4.$$

The corresponding ON functions are

$$\eta_1^* = \frac{1}{2} \qquad \text{and} \qquad \eta_2^* = \frac{1}{76}\sqrt{133} + \frac{15}{304}\sqrt{133} z^4,$$

the approximation K_2 to the Bergman kernel function is

$$K_2(z,0) = \frac{83}{304} + \frac{105}{1216} z^4$$

and finally, the approximation f_2 to the conformal mapping function is

$$f_2(z) = \frac{1}{76}\sqrt{1577\pi} z + \frac{21}{25232}\sqrt{1577\pi} z^5.$$

Denote by ε_n the error estimate obtained by sampling the function $|1 - |f_n(z)||$ at a number of test points on $\partial\mathcal{S}$. The following table contains the values of ε_n and the errors E_n corresponding to results presented in [7], for several values of n.

n	2	9	18	26	28
ε_n	$2.2E-2$	$5.2E-9$	$1.5E-17$	$4.0E-25$	$5.0E-27$
E_n	–	$1.4E-8$	$1.5E-17$	$1.0E-24$	–

TABLE 1. Errors estimates for the square

The results E_9 and E_{26} were obtained by Levin et al [8] and Papamichael et al [9], respectively, and are the best possible. The result E_{18} was obtained by Jank [7] by using the Maple system. At that time it was not possible to reach values of $n > 18$. Now it is clear that by using the Maple system and thus avoiding, whenever it is possible, the numeric Gram-Schmidt process, it is possible to obtain better results.

3 From $\mathbb{C}$ to $\mathbb{H}$

The construction of reproducing kernel functions is not restricted to real dimension 2. Indeed, the two complex variable case has been already considered by Bergman himself (c.f.[1]). Moreover, results concerning (and restricted to) the construction of Bergman kernel functions in closed form for special domains in the framework of hypercomplex function theory (which not supposes the consideration of spaces corresponding to *even* real dimensions) can be found in [2, 3, 10, 11]. They suggest that BKM can also be extended to mapping problems in higher dimensions, particularly 3 or 4-dimensional cases.

We describe such a generalized BKM-approach and present numerical examples obtained by the use of specially developed software packages for quaternions. A general and more rigorous exposition with more technical details and examples will be published elsewhere. We will use the notations of [6] without repeating them here.

3.1 The Bergman Kernel Method

Let Ω be a bounded simply-connected domain in $\mathbb{R}^3$ and consider the $\mathbb{H}$-valued functions defined in Ω:

$$f : \mathbb{R}^3 \to \mathbb{R}^4 \cong \mathbb{H}$$

$$f(x) = e_0 f_0(x) + e_1 f_1(x) + e_2 f_2(x) + e_3 f_3(x),$$

where $x = (x_0, x_1, x_2) \in \mathbb{R}^3$, $e_0 := 1,\ e_1,\ e_2,\ e_3$ are the canonical quaternionic units and f_k are real valued in Ω functions. On the set $C^1(\Omega, \mathbb{H})$ define the quaternionic Cauchy-Riemann operator

$$D = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2},$$

and recall that a C^1-function f is called *left-monogenic* (resp. *right-monogenic*) in a domain Ω if

$$Df = 0, \text{ in } \Omega \qquad (\text{ resp. } fD = 0 \text{ in } \Omega).$$

Now denote by $L_r^2(\Omega, \mathbb{H})$ the right-Hilbert space of all square integrable $\mathbb{H}$-valued functions, endowed with the inner product:

$$< f(z), g(z) >= \int_\Omega \overline{f(z)} g(z)\ dV. \tag{5}$$

The right linear set $L_r^2(\Omega, \mathbb{H}) \cap \ker D$ is a subspace in $L_r^2(\Omega, \mathbb{H})$ and has also a unique reproducing kernel $K(z, \zeta)$, i.e

$$< K(.,\zeta), f >= f(\zeta),\ \forall f \in L_r^2(\Omega, \mathbb{H}) \cap \ker D. \tag{6}$$

and if we now take an orthonormal complete system of functions $\{\eta_j^*\}$ then it can be proved a Fourier series expansion for all functions $f \in L_r^2(\Omega, \mathbb{H}) \cap \ker D$

$$f(z) = \sum_{j=1}^{\infty} \eta_j^*(z) < \eta_j^*, f >$$

and therefore

$$K(z,\zeta) = \sum_{j=1}^{\infty} \eta_j^*(z) < \eta_j^*, K(z,\zeta) >= \sum_{j=1}^{\infty} \eta_j^*(z)\overline{\eta_j^*(\zeta)}. \quad (7)$$

It is well known that the monogenic Fueter polynomials are a complete set of functions (see p.e. [6]) and thus steps **S1 - S3** can be rewritten easily in order to obtain a numerical procedure to construct approximations to K similar to the complex case.

All these results underline that the Clifford analysis and one complex variable analysis are closely connected. Thus, if we go further and introduce

S4 Compute

$$f_n(z) = C_n \int_0^z K_n(t,0)dt; \; n = 1, 2, \cdots ,$$

where C_n denotes some constant (depending on $K_n(0,0)$), shall we get a "mapping" function from the domain Ω to a sphere?

Before attempting to answer this question, we should make some remarks.

Remark 1. The polynomials η_j are in $\Omega \subset \mathbb{R}^3 \cong \mathcal{A} := \text{span}_{\mathbb{R}}\{1, e_1, e_2\}$, but the corresponding ON polynomials η_j^* are, in general, in $\mathbb{H} \cong \mathbb{R}^4$. This means that the kernel function K and the mapping function f are, in fact, functions from Ω in $\mathbb{R}^4$.

Remark 2. From the geometric and practical point of view, we would like f to map domains $\Omega \subset \mathbb{R}^3$ to a sphere (for the moment, not necessarily the unit sphere).

Remark 3. It can be proved easily that if a function f of the form $f = f(z) = f_0(z) + f_1(z)e_1 + f_2(z)e_2$, is left-monogenic then f is also right-monogenic. Conversely, if a function of the form $f = f(z) = f_0(z) + f_1(z)e_1 + f_2(z)e_2 + f_3(z)e_3$, is monogenic from both sides and is such that $\exists a \in \Omega : \; f(a) = 0$, then, $f_3 = 0$, i.e. $f : \mathbb{H}^2 \to \mathcal{A} \cong \mathbb{R}^3$.

Remark 4. We do not expect f to be right monogenic from both sides. We recall that Möbius transformations are the only conformal mappings in $\mathbb{R}^{m+1}$, $(m \geq 2)$, but quaternionic Möbius transformations themselves are neither left nor right monogenics. However, the results presented in Remark 3 give the motivation for the numerical procedure we propose for computing f in step **S4** of BKM.

S4.1 Approximate the mapping function $g : \Omega \to \mathbb{H}$ by

$$g_n(z) = \int_0^z K_n(t,0)dt; \; n = 1, 2, \cdots \quad (8)$$

S4.2 Approximate the mapping function f by "cutting" the "e_3-part" in (8), i.e. if g_n is of the form

$$g_n(z) = g_n^{\{0\}}(z) + g_n^{\{1\}}(z)e_1 + g_n^{\{2\}}(z)e_2 + g_n^{\{3\}}(z)e_3, \quad (9)$$

then construct the function f_n from Ω into $\mathcal{A} \cong \mathbb{R}^3$ by means of

$$f_n(z) = g_n^{\{0\}}(z) + g_n^{\{1\}}(z)e_1 + g_n^{\{2\}}(z)e_2. \quad (10)$$

3.2 Numerical examples

We apply now the above technique to a cube and a L-shaped domain. Consider first the cube

$$\mathcal{C} := \{(x, y, z) \in \mathbb{R}^3 : |x| < 1, \ |y| < 1, \ |z| < 1\}.$$

For $N = 2$ the ON polynomials in step **S2** are

$$\eta_1^* = \tfrac{3}{56}\sqrt{70}(x_1^2 - x_0^2 - 2x_1x_0e_1),$$

$$\eta_2^* = \tfrac{3}{224}\sqrt{14}(14x_1x_2 - 14x_2x_0e_1 - 4x_1x_0e_2 + (5x_1^2 - 5x_0^2)e_3),$$

$$\eta_3^* = \tfrac{3}{32}\sqrt{10}(-x_1^2 - x_0^2 + 2x_2^2 - 2x_2x_0e_2 + 2x_1x_2e_3).$$

and the image of $\mathcal{C}$ by the BKM approximation f_{12} is illustrated in Figure 1(a).

The analysis of the "e_3-part" in (9), i.e. $g_N^{\{3\}}(z)$ shows some evidence that as N grows this function gets smaller. However we did not go further than $N = 14$, as our program becomes very time consuming. Figure 1(b) corresponds to the plot of $g_{14}^{\{3\}}(z)$, where $z \in \{(x, y, z) \in \mathbb{R}^3 : x = 1, |y| < 1, |z| < 1\}$.

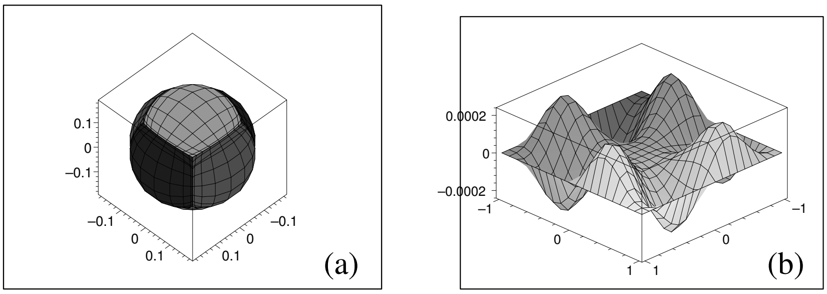

FIGURE 1. The mapping functions $f_{12}(\mathcal{C})$ and $g_{14}^{\{3\}}(z)$

Consider now the L-shaped domain presented in Figure 2(a). The BKM result for $N = 8$ is illustrated in Figure 2(b).

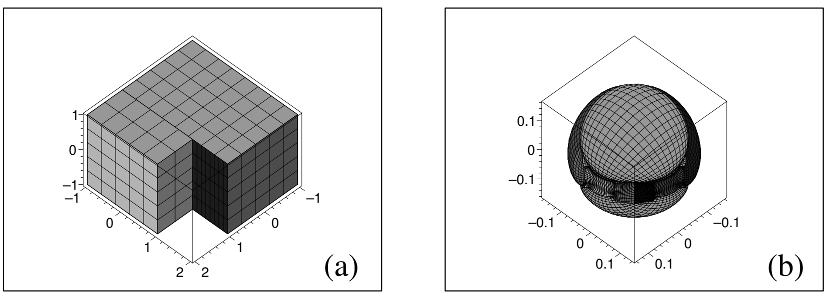

FIGURE 2. An L-shaped domain

It is well known that for a 2-dimensional L-shapped domain, the classical BKM gives very poor approximations to the conformal mapping function f as this function has a serious branch point singularity (see [9] for all the details). Although we do not have for the moment a theoretical justification for the remarkable results achieved by the BKM proposed, even for small values of N and a “difficult” domain, we are convinced that this BKM-approach for 3 dimensional cases works and it is useful to continue the investigation in this direction.

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S. Bock
Bauhaus-Universität Weimar
D-99421 Weimar
Germany

bastian.bock@web.de

M. I. Falcão
Departamento de Matemática
Universidade do Minho
4470 Braga
Portugal
mif@math.uminho.pt

K. Gürlebeck
Bauhaus-Universität Weimar
Institut Mathematik/Physik
D-99421 Weimar
Germany
guerlebe@fossi.uni-weimar.de