CONSTRUCTIVE ASPECTS OF MONOGENIC FUNCTION THEORY

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As it is well known, the approximation theory of complex valued functions is one of the main fields in function theory. In general, several aspects of approximation and interpolation are only well understood by using methods of complex analysis. It seems to be natural to extend these techniques to higher dimensions by using Clifford Analysis methods or, more specific, in lower dimensions 3 or 4, by using tools of quaternionic analysis.
One starting point for such attempts has to be the suitable choice of complete orthonormal function systems that should replace the holomorphic function systems used in the complex case. The aim of our contribution is the construction of a complete orthonormal system of monogenic polynomials derived from a harmonic function system by using systematically the generalized quaternionic derivative.

Keywords: Monogenic Polynomials, L_2-Approximation, Spherical Harmonics, Orthonormal Systems.

1 Introduction

An important tool in approximation theory in Hilbert spaces is the use of series expansions of functions with respect to a complete orthonormal system. In particular, complete sets of orthonormal polynomials play a crucial role.
In the space of complex functions, the variable $z = x + i\,y$ and its powers z^n, $n \in \mathbb{N}_0$, are the simplest polynomials in x and y that can be used to approximate holomorphic functions. For bounded domains $\Omega \subset \mathbb{C}$, the system of functions

$$\{1, z, z^2, \ldots, z^n, \ldots\}_{n \in \mathbb{N}_0} \tag{1}$$

belong to $L_2(\Omega)$. As it is well known, equipped with the inner product

$$< f, g >= \int_\Omega \overline{f}\, g \, d\Omega \; , \tag{2}$$

where $d\Omega$ is the Lebesgue measure, $L_2(\Omega)$ is a Hilbert space.
Each finite subset of (1) consists of linear independent functions, which can be orthonormalized and lead to polynomials of the form

$$p_n(z) = c_{0,n} + c_{1,n}\, z + \cdots + c_{n,n}\, z^n \; ,$$

with $c_{l,n} \in \mathbb{C}$, $l = 0, 1, \ldots, n-1$ and $c_{n,n} > 0$, $n = 0, 1, 2, \ldots$.
In particular, if $\Omega = \{z \in \mathbb{C} : |z| < 1\}$, the powers z^n, $n \in \mathbb{N}_0$ are automatically orthogonal with respect to the inner product (2) and can be easily normalized to get an orthonormal system (ONS) of polynomials.

Another advantage of the simple basis system (1) is that the derivative is a polynomial with the same structure and one degree lower, i. e,

$$\frac{d}{dz} z^n = n\, z^{n-1} \; .$$

In the case of generalizations to higher dimensions, it seems to be natural to look for systems of polynomials that keep the referred properties, namely, an easy orthonormalization process and a similar behaviour with respect to a derivative. The topic of this paper is to construct a quaternion-valued system, for approximating monogenic functions in the unit ball, that fulfills those properties.

2 Basic definitions and notation

We work in the skew field of quaternions

$$\mathbb{H} := \{z : z = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3, x_i \in \mathbb{R},\ i = 0, 1, 2, 3\},$$

where $e_0 = 1,\ e_1 = i,\ e_2 = j,\ e_3 = k = e_1\, e_2$ are the standard basis elements of $\mathbb{H}$, with the multiplication law $e_i e_j + e_j e_i = -2\delta_{ij},\ \ i, j = 1, 2, 3$, where δ_{ij} is the Kronecker symbol.
We identify each element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the *reduced* quaternion $z = x_0 + x_1 e_1 + x_2 e_2$ (sometimes called paravector) and denote by $Re\ z = \frac{1}{2}(z + \overline{z})$ the real part and by $Im\ z = \frac{1}{2}(z - \overline{z})$ the imaginary part of z. The conjugate is defined by $\overline{z} = x_0 - x_1 e_1 - x_2 e_2$ and the corresponding norm of z is given by $|z| = \sqrt{z\,\overline{z}}$.
Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a sufficiently smooth boundary and the *generalized Cauchy-Riemann* operator

$$D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2$$

with its *conjugate*

$$\overline{D} = \frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1} e_1 - \frac{\partial}{\partial x_2} e_2 \; . \tag{3}$$

A C^1-function f is said to be *left* (resp. *right*) *monogenic* if $Df = 0$ in Ω (resp. $fD = 0$ in Ω).
Here, we work only with left monogenic functions that we call briefly, monogenic.
From now on, let us consider $\Omega := B := B_1(0)$ the unit ball in $\mathbb{R}^3$ and denote by $S = \partial B$ the boundary of B and by $d\sigma$ its surface-element.

3 A system of polynomials based on a permutational product

For a basis of monogenic polynomials in $\mathbb{H}$, the expected candidate would be the function $f(z) = z = x_0 + e_1 x_1 + e_2 x_2$ and its powers z^n, $n \in \mathbb{N}_0$. Unfortunately, these functions are neither left nor right monogenic. Instead of them, the left and right monogenic hypercomplex variables $z_k = x_k - e_k x_0 = -\frac{1}{2}(z e_k + e_k z)$, $k = 1, 2$, can be used (c.f [5], [2], [7] and [8]). However, their usual product $z_1 z_2$ is not

monogenic.
Following [8], one considers the functions (often called *generalized powers*)

$$\overrightarrow{z}^{\nu} := z_1^{\nu_1} \times z_2^{\nu_2} = \underbrace{z_1 \times z_1 \times \cdots \times z_1}_{\nu_1 \ times} \times \underbrace{z_2 \times z_2 \times \cdots \times z_2}_{\nu_2 \ times}$$
$$= \frac{1}{|\nu|!} \sum_{\pi(i_1,\ldots,i_{|\nu|})} z_{i_1} \cdots z_{i_{|\nu|}}$$

where $\nu = (\nu_1, \nu_2)$ is a multi-index, $|\nu| = \nu_1 + \nu_2$, $\overrightarrow{z} = (z_1, z_2)$ and the sum is taken over *all* permutations of $(i_1, \ldots, i_{|\nu|})$.
Taking into account that all functions of the form $\overrightarrow{z}^{\nu}$ are monogenic, they were used in [8] to construct Taylor series of monogenic functions in a way similar to the case of several complex variables.
Note that for each $n \in \mathbb{N}_0$, with $|\nu| = n$, the polynomials $\overrightarrow{z}^{\nu}$ belong to $span\{1, e_1, e_2\}$ and are homogeneous of degree n.

The importance of the generalized powers is reflected by

Theorem 3.1 [2], [8] *For each $n \in \mathbb{N}_0$ and $|\nu| = n$, the $n + 1$ polynomials $\overrightarrow{z}^{\nu}$ form a basis in $\mathbb{H}$.*

Using the $\mathbb{H}$-valued inner product

$$< f, g >_{L_2(S)} = \int_S \bar{f}\, g \, d\sigma \ ,$$

we can establish the following result:

Theorem 3.2 [3] *Let $\nu = (\nu_1, \nu_2)$ and $\mu = (\mu_1, \mu_2)$, $n = |\nu| = |\mu|$ be the degree of homogeneity and*

$$< z_1^{\nu_1} \times \ z_2^{\nu_2}, z_1^{\mu_1} \times z_2^{\mu_2} >_{L_2(S)} = a \, ,$$

where $a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \in \mathbb{H}$.
Then, it holds
i) $a_0 = a_1 = a_2 = 0$ and $a_3 \neq 0$ if $|\nu_1 - \mu_1|$ and $|\nu_2 - \mu_2|$ are both odd numbers;
ii) $a_1 = a_2 = a_3 = 0$ and $a_0 \neq 0$ if $|\nu_1 - \mu_1|$ and $|\nu_2 - \mu_2|$ are both even numbers.

Examples
For $n = 5$, we have

$$< z_1^4 \times \ z_2, z_1^3 \times z_2^2 >_{L_2(S)} = -\frac{64}{2475} \pi \, e_3 \ ,$$

illustrating $i)$ and

$$< z_1^3 \times \ z_2^2, z_1 \times z_2^4 >_{L_2(S)} = \frac{128}{1925} \pi \ ,$$

illustrating ii).

As a consequence of i), the use of the real-valued inner product

$$< f, g >_{0,L_2(S)} = \int_S Re(\bar{f}\, g)\, d\sigma \ , \tag{4}$$

implies a large number of automatically orthogonal polynomials, for the same degree of homogeneity.

However, the second property of the basis system that we want to keep is lost. In fact, the hypercomplex derivative $(\frac{1}{2})\overline{D}$ (c.f [6]) of the generalized powers $\overrightarrow{z}^{\nu},\ |\nu| = n$, for a fixed $n \in \mathbb{N}$, given by [8],

$$(\frac{1}{2}\bar{D})(z_1^{\nu_1} \times z_2^{\nu_2}) = -\nu_1(z_1^{\nu_1-1} \times z_2^{\nu_2})e_1 - \nu_2(z_1^{\nu_1} \times z_2^{\nu_2-1})e_2$$

is now a linear combination of two polynomials of degree $n-1$.

4 A special system of homogeneous monogenic polynomials

It is possible to construct, by another approach, basis systems of homogeneous monogenic polynomials that keep both desired properties.
The main idea of such a construction is based on the decomposition of the Laplace operator $\Delta = \overline{D}\, D = D\, \overline{D}$. This property allow us to obtain homogeneous monogenic polynomials as a result of the application of $\overline{D}$ to real homogeneous harmonic polynomials.

An $\mathbb{H}$-valued function is harmonic in Ω if and only if each of its components is harmonic in Ω.
Evidently, any monogenic function is a harmonic function in all its components. The natural way to deal with homogeneous harmonic polynomials on the sphere is the use of spherical coordinates

$$x_0 = r\cos\theta,\ x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi,$$

where $0 \le r < \infty$, $0 \le \theta \le \pi$, $0 \le \varphi \le 2\pi$. Each $x = (x_0, x_1, x_2) \in \mathbb{R}^3$, can be written as

$$x = r\omega\ ,\ |\omega| = 1$$

where

$$\omega = \omega_0 + \omega_1 e_1 + \omega_2 e_2$$

with $\omega_i = \dfrac{x_i}{r},\ i = 0, 1, 2$. Of course, $\overline{x} = r\overline{\omega}$, where $\overline{\omega} = \omega_0 - \omega_1 e_1 - \omega_2 e_2$.

Consider H_n as an $\mathbb{H}$-valued *homogeneous polynomial of degree* n, i.e, an $\mathbb{H}$-valued polynomial such that

$$H_n(tx) = t^n H_n(x),\ t \in \mathbb{R},\ x \in \mathbb{R}^3.$$

From this formulation, it is clear that a homogeneous polynomial is determined by its restriction to S.
Now we can rewrite the homogeneous harmonic polynomial H_n in spherical coordinates as

$$H_n(x) = r^n H_n(\omega)$$

and its restriction to the boundary of the ball is called *spherical harmonic of order* n. Spherical harmonics play an important role in several fields of mathematics and physics, namely, in celestial mechanics, terrestrial magnetism, earthquakes and in a general way in all fields of geosciences. Indeed, spherical harmonics are essential for the analysis of any phenomena with spherical symmetry. For a detailed study on spherical harmonics, we refer [1] and [9], for example.
Analogously, denote by H_ν^n a homogeneous monogenic polynomial of degree $n \in \mathbb{N}$, $|\nu| = n$. In spherical coordinates,

$$H_\mu^n(x) = r^n H_\mu^n(\omega)$$

and the restriction to the boundary of the ball is called *spherical monogenic of order* n.

For each $n \in \mathbb{N}_0$, we take the $2n+3$ linearly independent functions in the space of real-valued spherical harmonics of order $n+1$,

$$\begin{cases} U_{n+1}^0(\theta,\varphi) = P_{n+1}(\cos\theta) \\ U_{n+1}^m(\theta,\varphi) = P_{n+1}^m(\cos\theta)\cos m\varphi \\ V_{n+1}^m(\theta,\varphi) = P_{n+1}^m(\cos\theta)\sin m\varphi\,, \qquad m = 1,\dots,n+1, \end{cases} \tag{5}$$

where P_{n+1} is the *Legendre polynomial of degree* $n+1$,

$$\begin{cases} P_{n+1}(t) = \displaystyle\sum_{k=0}^{[\frac{n+1}{2}]} a_{n+1,k}\; t^{n+1-2k} \\ P_0(t) = 1 \end{cases}$$

(the upper bound $[s]$ in the sum denote, as usual, the largest integer contained in $s \in \mathbb{R}$),
with

$$a_{n+1,k} = (-1)^k \frac{1}{2^{n+1}} \frac{(2n+2-2k)!}{k!\,(n+1-k)!\,(n+1-2k)!}\,,$$

and

$$P_{n+1}^m(t) := (1-t^2)^{\frac{m}{2}}\, \frac{d^m}{dt^m} P_{n+1}(t), \qquad m = 1,\dots,n+1$$

are the *associated Legendre functions* [1], associated to Legendre polynomials of degree $n+1$.

[1]These functions are also called *Ferrers functions* by some authors because they were introduced in 1877 by N. Ferrers. Sometimes they are defined with the factor $(-1)^m$ as, for example, Olver [10]. Here, we follow Sansone in [11].

Taking $n = 0, 1, 2, \ldots$ the system (5) is complete in $L_2(S)$ (c.f. [11]).

The main idea of the construction of a system of desired monogenic polynomials is the application of the hypercomplex derivative $(\frac{1}{2})\overline{D}$ to the harmonic homogeneous polynomials

$$\{r^{n+1}\, U^0_{n+1}, r^{n+1}\, U^m_{n+1}, r^{n+1}\, V^m_{n+1}\}, \qquad m = 1, \ldots, n+1\ .$$

Their restriction to the boundary of the ball give us the *spherical monogenics*

$$\begin{aligned} X^0_n &= \frac{1}{2}\overline{D}\,(r^{n+1}\, U^0_{n+1})|_{r=1} \\ &= A^{0,n} + B^{0,n}\cos\varphi\, e_1 + B^{0,n}\sin\varphi\, e_2, \end{aligned}$$

where

$$\begin{aligned} A^{0,n} &= A^{0,n}(\theta) = \frac{1}{2}\left(\sin^2\theta\,\frac{d}{dt}[P_{n+1}(t)]_{t=\cos(\theta)} + (n+1)\cos\theta P_{n+1}(\cos\theta)\right), \\ B^{0,n} &= B^{0,n}(\theta) = \frac{1}{2}\left(\sin\theta\cos\theta\,\frac{d}{dt}[P_{n+1}(t)]_{t=\cos(\theta)} - (n+1)\sin\theta\, P_{n+1}(\cos\theta)\right), \end{aligned}$$

and

$$\begin{aligned} X^m_n &= \frac{1}{2}\overline{D}\,(r^{n+1}\, U^m_{n+1})|_{r=1} \\ &= A^{m,n}\cos m\varphi + \\ &\quad +(B^{m,n}\cos\varphi\cos m\varphi - C^{m,n}\sin\varphi\sin m\varphi)\, e_1 + \\ &\quad +(B^{m,n}\sin\varphi\cos m\varphi + C^{m,n}\cos\varphi\sin m\varphi)\, e_2, \end{aligned}$$

$$\begin{aligned} Y^m_n &= \frac{1}{2}\overline{D}\,(r^{n+1}\, V^m_{n+1})|_{r=1} \\ &= A^{m,n}\sin m\varphi + \\ &\quad +(B^{m,n}\cos\varphi\sin m\varphi + C^{m,n}\sin\varphi\cos m\varphi)\, e_1 + \\ &\quad +(B^{m,n}\sin\varphi\sin m\varphi - C^{m,n}\cos\varphi\cos m\varphi)\, e_2, \end{aligned}$$

with the notations

$$\begin{aligned} A^{m,n} &= A^{m,n}(\theta) = \frac{1}{2}\left(\sin^2\theta\,\frac{d}{dt}[P^m_{n+1}(t)]_{t=\cos\theta} + (n+1)\cos\theta\, P^m_{n+1}(\cos\theta)\right) \\ B^{m,n} &= B^{m,n}(\theta) = \frac{1}{2}\left(\sin\theta\cos\theta\,\frac{d}{dt}[P^m_{n+1}(t)]_{t=\cos\theta} - (n+1)\sin\theta\, P^m_{n+1}(\cos\theta)\right) \\ C^{m,n} &= C^{m,n}(\theta) = \frac{1}{2}\, m\frac{1}{\sin\theta}\, P^m_{n+1}(\cos\theta) \end{aligned}$$

$m = 1 \ldots n+1$.

Examples:
For $n = 3$, some of the 9 homogeneous monogenic polynomials are

$$X^0_3 = 2x_0^3 - 3x_0x_1^2 - 3x_0x_2^2 + (-\frac{3}{4}x_1x_2^2 + 3x_0^2x_1 - \frac{3}{4}x_1^3)e_1 + (-\frac{3}{4}x_1^2x_2 - \frac{3}{4}x_2^3 + 3x_0^2x_2)e_2$$

$$
\begin{aligned}
X_3^3 &= \frac{105}{2}x_1^3 - \frac{315}{2}x_1x_2^2 + (-\frac{315}{2}x_0x_1^2 + \frac{315}{2}x_0x_2^2)e_1 + 315x_0x_2x_1e_2 \\
X_3^4 &= (630x_1x_2^2 - 210x_1^3)e_1 + (-210x_2^3 + 620x_1^2x_2)e_2 \\
Y_3^1 &= 15x_0^2x_2 - \frac{15}{4}x_1^2x_2 - \frac{15}{4}x_2^3 + \frac{15}{2}x_0x_2x_1e_1 + (-5x_0^3 + \frac{15}{4}x_0x_1^2 + \frac{45}{4}x_0x_2^2)e_2 \\
Y_3^3 &= \frac{315}{2}x_1^2x_2 - \frac{105}{2}x_2^3 - 315x_0x_2x_1e_1 + (-\frac{315}{2}x_0x_1^2 + \frac{315}{2}x_0x_2^2)e_2
\end{aligned}
$$

An important fact is that the derivatives of these polynomials are related to the original ones, like $n\,z^{n-1}$ to z^n in the complex case:

Theorem 4.1 [3] *For the polynomials X_n^m, $m = 0, \ldots, n$ and Y_n^m, $m = 1, \ldots, n$, the following identities are true:*

$$
\begin{aligned}
(\frac{1}{2}\overline{D})X_n^m &= (n+m+1)X_{n-1}^m, \quad m = 0, \ldots, n \\
(\frac{1}{2}\overline{D})Y_n^m &= (n+m+1)Y_{n-1}^m, \quad m = 1, \ldots, n\,.
\end{aligned}
$$

Considering the set of $2n+3$ polynomials constructed above and the set of these polynomials multiplied by e_3 (the polynomials $X_n^{n+1}\,e_3$ and $Y_n^{n+1}\,e_3$ are linearly dependent of Y_n^{n+1} and X_n^{n+1}, respectively, so we remove them), for each $n \in \mathbb{N}_0$, we have the set of $4n+4$ spherical monogenics

$$
\{X_{n,0}^0,\, X_{n,0}^m,\, Y_{n,0}^m,\, X_{n,3}^0,\, X_{n,3}^l, Y_{n,3}^l,\ m = 1, \ldots, n+1,\, l = 1, \ldots, n\}\,. \tag{6}
$$

Here, we use the notation $X_{n,0}^m := X_n^m$ and $X_{n,3}^m := X_n^m\,e_3$, $m = 0, \ldots, n+1$ (analogously, $Y_{n,0}^m := X_n^m$ and $Y_{n,3}^m := X_n^m\,e_3$).

Based on the set (6), we construct an orthonormal basis for the space of quaternion-valued polynomials with real coefficients, with respect to the inner product considered in (4). First, we observe that for the same degree of homogeneity, we have a very large number of automatically orthogonal polynomials. In fact, within the group of $2n+3$ polynomials $\{X_{n,0}^0,\, X_{n,0}^m,\, Y_{n,0}^m\,, m = 1, \ldots, n+1\}$ we have orthogonality, as well as within the group of $2n+1$ polynomials $\{X_{n,3}^0,\, X_{n,3}^l, Y_{n,3}^l,\, l = 1, \ldots, n\}$.
The norms of these polynomials are

$$
\begin{aligned}
||X_{n,0}^0||_{0,L_2(S)} &= ||X_{n,3}^0||_{0,L_2(S)} = \sqrt{\pi(n+1)} \\
||X_{n,0}^m||_{0,L_2(S)} &= ||Y_{n,0}^m||_{0,L_2(S)} = ||X_{n,3}^l||_{0,L_2(S)} = ||Y_{n,3}^l||_{0,L_2(S)} = \\
&= \sqrt{\frac{\pi}{2}(n+1)\frac{(n+1+m)!}{(n+1-m)!}}\,, \quad m = 1, \ldots, n+1.
\end{aligned}
$$

Considering the normalized set

$$\{\tilde{X}^0_{n,0},\, \tilde{X}^m_{n,0},\, \tilde{Y}^m_{n,0},\, \tilde{X}^0_{n,3},\, \tilde{X}^l_{n,3}, \tilde{Y}^l_{n,3},\; m=1,\ldots,n+1,\, l=1,\ldots,n\}\;,$$

resulting from (6), after taking normalization, we easily arrive to the following result:

Theorem 4.2 [3] *For $n = 0, 1, 2, \ldots$,*
i) The system

$$\tilde{X}^0_{n,0},\, \tilde{X}^m_{n,0},\, \tilde{Y}^m_{n,0},\; m=1,\ldots,n+1 \tag{7}$$

is an orthonormal system.
ii) The same is true for

$$\tilde{X}^0_{n,3},\, \tilde{X}^l_{n,3},\, \tilde{Y}^l_{n,3}\; l=1,\ldots,n\;. \tag{8}$$

iii) Between the two groups of orthonormal polynomials (7) and (8), we have orthogonality, except

$$< \tilde{X}^m_{n,0},\, \tilde{Y}^l_{n,3} >_{0,L_2(S)} = - < \tilde{Y}^m_{n,0}, \tilde{X}^l_{n,3} >_{0,L_2(S)} = \begin{cases} 0, & m \neq l \\ \frac{l}{n+1}, & m = l \end{cases}$$

For each $n \in \mathbb{N}_0$, the system (6) is a linearly independent set and can be easily orthonormalized to get the ONS

$$\{X^{0,*}_{n,0},\, X^{m,*}_{n,0},\, Y^{m,*}_{n,0},\, Y^{l,*}_{n,3},\, X^{0,*}_{n,3},\, X^{l,*}_{n,3},\, m=1,\ldots,n+1,\, l=1,\ldots,n\}, \tag{9}$$

where

$$\begin{aligned} X^{0,*}_{n,0} &= \tilde{X}^0_{n,0} \\ X^{m,*}_{n,0} &= \tilde{X}^m_{n,0} \\ Y^{m,*}_{n,0} &= \tilde{Y}^m_{n,0} \\ Y^{l,*}_{n,3} &= \sqrt{s_{n,l}}\left((n+1)\;\tilde{Y}^l_{n,3} - l\;\tilde{X}^l_{n,0}\right) \\ X^{0,*}_{n,3} &= \tilde{X}^0_{n,3} \\ X^{l,*}_{n,3} &= \sqrt{s_{n,l}}\left((n+1)\;\tilde{X}^l_{n,3} + l\;\tilde{Y}^l_{n,0}\right) \end{aligned}$$

with

$$s_{n,l} = \frac{1}{(n+1+l)(n+1-l)}\;,$$

$m = 1, \ldots, n+1$, $l = 1, \ldots, n$.

Let $\{\phi^*{}_{n,j} : j = 1, \ldots, 4n+4\}_{n\in\mathbb{N}_0}$ be the ONS of spherical monogenics of degree n in $L_2(S)$, constructed in (9).

Denoting by $\phi_{n,j} = r^n \, \phi^*{}_{n,j}$, $j = 1, \ldots, 4n+4$, the extensions of these polynomials into the ball, we know from [4] that

$$< \phi_{n,j}, \phi_{k,l} >_{0,L_2(B)} = \frac{1}{n+k+3} < \phi^*{}_{n,j}, \phi^*{}_{k,l} >_{0,L_2(S)}$$

and, consequently, $\{\phi_{n,j} : j = 1, \ldots, 4n+4\}_{n \in \mathbf{N}_0}$ is an orthogonal system of homogeneous monogenic polynomials in $L_2(B)$. As

$$||\phi_{n,j}||^2_{0,L_2(B)} = \frac{1}{2n+3} ||\phi^*{}_{n,j}||^2_{0,L_2(S)}$$

the system

$$\{\sqrt{2n+3} \; \phi_{n,j} : j = 1, \ldots, 4n+4\}_{n \in \mathbf{N}_0} \tag{10}$$

forms a ONS of homogeneous monogenic polynomials in $L_2(B)$.
It is possible to prove that

Theorem 4.3 [3] *The system of homogeneous monogenic polynomials (10) form a complete ONS of homogeneous monogenic polynomials in $L_2(B)$.*

Acknowledgements

Research (partially) supported by Unidade de Investigação Matemática e Aplicações of Universidade de Aveiro through Programa Operacional "Ciência, Tecnologia, Inovação" (POCTI) of the Fundação para a Ciência e a Tecnologia (FCT), co-financed by the European Community fund FEDER.
The author would like also to thank the organization committee of IKM2003 for the financial support.

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