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A four-node plane EAS-element for stochastic nonlinear materials

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1 Introduction

Iso-parametric finite elements with linear shape functions show in general a too stiff element behavior, called locking. By the investigation of structural parts under bending loading the so-called shear locking appears, because these elements can not reproduce pure bending modes. Many studies dealt with the locking problem and a number of methods to avoid the undesirable effects have been developed. Two well known methods are the "Assumed Natural Strain" (ANS) method (Simo and Hughes 1986) and the "Enhanced Assumed Strain" (EAS) method (Simo and Rifai 1990). A good overview of existing methods and their differences in formulations and results is given in Andelfinger 1991.

For the research activities of the subproject A1 of the Collaborative Research Center 524 "Materials and Structures in Revitalization of Buildings", where a coupling between meshless components and finite elements is used to model stochastic crack evaluation (Most and Bucher 2003), nonlinear materials with stochastic distributed properties are the point of interest. Consequently, an improved finite element is necessary which can be used to model stochastic nonlinear material behavior.

In this study the EAS method is applied to a four-node plane element with four EAS-parameters. A common approach for the enhanced strain modes is given in Andelfinger 1991. The following paper will describe the well-known linear formulation, its extension to nonlinear materials and the modeling of material uncertainties with random fields. The developed element will be verified via a linear and a nonlinear common example. Its applicability for stochastic calculations will be shown on the random hardening behavior of a cantilever beam. The presented element formulation has been implemented in the SLang Software package (Bucher et al. 1995), (Bucher and Schorling 1997), (Bucher et al. 2002), which is available at the Bauhaus-University Weimar for research activities.

2 Finite element modeling

2.1 Linear formulation

The strains of the iso-parametrical formulation are enhanced with the strains $\boldsymbol{\epsilon_{enh}}$. In general these enhanced strains have to satisfy the following equation

$$\int_{\Omega} \boldsymbol{\epsilon_{enh}} \, d\Omega = 0 \qquad (1)$$

in order to let the constant iso-parametric strain modes unchanged. Thus the complete strains are given as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ 2\,\epsilon_{xy} \end{bmatrix} = \boldsymbol{\epsilon_{iso}} + \boldsymbol{\epsilon_{enh}} = \boldsymbol{B_{iso}}\,\boldsymbol{d} + \boldsymbol{B_{enh}}\,\boldsymbol{\alpha}, \tag{2}$$

where $\boldsymbol{d}$ is the vector of the local node displacement $\boldsymbol{d^T} = \begin{bmatrix} u_1 & v_1 & \dots & u_4 & v_4 \end{bmatrix}$ and $\boldsymbol{\alpha}$ is the vector of the additional degrees of freedom $\boldsymbol{\alpha^T} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{bmatrix}$. The matrix $\boldsymbol{B_{iso}}$ is the strain-displacement-matrix of the iso-parametric element and the enhanced matrix $\boldsymbol{B_{enh}}$ is defined as

$$\begin{aligned} \boldsymbol{B_{enh}} &= \frac{det\boldsymbol{J_0}}{det\boldsymbol{J_i}}\,\boldsymbol{T_0}^{-1}\,\boldsymbol{B_{enh,r}}, \\ \boldsymbol{B_{enh,r}} &= \begin{bmatrix} \frac{\phi_1}{\partial r} & 0 & \frac{\phi_2}{\partial r} & 0 \\ 0 & \frac{\phi_1}{\partial s} & 0 & \frac{\phi_2}{\partial s} \\ \frac{\phi_1}{\partial s} & \frac{\phi_1}{\partial r} & \frac{\phi_2}{\partial s} & \frac{\phi_2}{\partial r} \end{bmatrix} = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 0 & s \\ 0 & r & s & 0 \end{bmatrix}, \end{aligned} \tag{3}$$

where $\boldsymbol{J_0}$ and $\boldsymbol{J_i}$ are the Jacobian matrices at the element center with $r = s = 0$ and at the current integration point i, respectively. The matrix $\boldsymbol{T_0}$ is the transformation matrix from the local to the natural coordinate system at element center with $r = s = 0$. Thus, the modified equational system on the element level reads (Bischoff 2001)

$$\begin{bmatrix} \boldsymbol{K_{11}} & \boldsymbol{K_{12}} \\ \boldsymbol{K_{21}} & \boldsymbol{K_{22}} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{bmatrix}, \tag{4}$$

where

$$\begin{aligned} \boldsymbol{K_{11}} &= \int_{\Omega_e} \boldsymbol{B_{iso}}^T\,\boldsymbol{C}\,\boldsymbol{B_{iso}}\,d\Omega, & \boldsymbol{K_{11}} &\in \mathbb{R}^{8\times 8}, \\ \boldsymbol{K_{21}} &= \int_{\Omega_e} \boldsymbol{B_{enh}}^T\,\boldsymbol{C}\,\boldsymbol{B_{iso}}\,d\Omega = \boldsymbol{K_{12}}^T, & \boldsymbol{K_{21}} &\in \mathbb{R}^{4\times 8}, \\ \boldsymbol{K_{22}} &= \int_{\Omega_e} \boldsymbol{B_{enh}}^T\,\boldsymbol{C}\,\boldsymbol{B_{enh}}\,d\Omega, & \boldsymbol{K_{22}} &\in \mathbb{R}^{4\times 4}. \end{aligned} \tag{5}$$

By condensing out the additional internal modes we get the reduced stiffness matrix

$$\boldsymbol{K} = \boldsymbol{K_{11}} - \boldsymbol{K_{12}}\,\boldsymbol{K_{22}}^{-1}\boldsymbol{K_{12}}^T, \qquad \boldsymbol{K} \in \mathbb{R}^{8\times 8}. \tag{6}$$

The vector $\boldsymbol{\alpha}$, which is necessary to calculate the strains according to Eq.(2), is then given as

$$\boldsymbol{\alpha} = -\boldsymbol{K_{22}}^{-1}\,\boldsymbol{K_{12}}^T\,\boldsymbol{d}, \qquad \boldsymbol{\alpha} \in \mathbb{R}^4. \tag{7}$$

2.2 Nonlinear formulation

The application of the EAS method for linear elastic calculations is straight forward as presented in the previous section. For geometrical nonlinear problems we will find solutions in literature (Kasper and Taylor 2000), (Freischläger 2000). In this study the physical nonlinear formulation is the point of interest. For this problem the internal EAS parameters have to be determined for a given displacement state $\boldsymbol{d}$, which is done by using a local implicit calculation at the element level following Freischläger

2000. In Table 1 this local iteration is shown. The qualifier for this iteration is the restoring force vector calculated from the enhanced degrees of freedom, which is theoretically equal to the zero vector if the iteration criterion is fulfilled. For the application of this iteration a tolerance TOL has to be chosen as the iteration criterion, which should be done depending on the harding modulus, concerning the evaluable post decimal positions.

New increment of displacements $\Delta \boldsymbol{d}^{(i)}$

$\Downarrow$

Calculate new nodal displacements

$$\boldsymbol{d}^{(i+1)} = \boldsymbol{d}^{(i)} + \Delta \boldsymbol{d}^{(i)}$$

$\Downarrow$

Calculate $\boldsymbol{\alpha}^{(i+1)}$ from $\boldsymbol{d}^{(i+1)}$ via a local iteration by using the condition $\boldsymbol{r_{enh}}(\boldsymbol{d}^{(i+1)}, \boldsymbol{\alpha}^{(i+1)}) = \boldsymbol{0}$:

Initialize: $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}^{(i)}$ and $v = 0$
DO $v = v + 1$
$\Delta \boldsymbol{\alpha}_v = -\boldsymbol{K_{22}}^{-1}(\boldsymbol{\alpha}_v)\boldsymbol{r_{enh}}(\boldsymbol{\alpha}_v)$
$\boldsymbol{\alpha}_{v+1} = \boldsymbol{\alpha}_v + \Delta \boldsymbol{\alpha}_v$
WHILE $||\boldsymbol{r_{enh}}(\boldsymbol{\alpha}_{v+1})|| >$TOL
Set: $\boldsymbol{\alpha}^{(i+1)} = \boldsymbol{\alpha}_{v+1}$

$\Downarrow$

Calculate strains $\boldsymbol{\epsilon}^{(i+1)}$, stresses $\boldsymbol{\sigma}^{(i+1)}$ and material matrix $\boldsymbol{C}^{(i+1)}$

$\Downarrow$

Calculate a new effective stiffness element matrix via static condensation

$$\boldsymbol{K}^{(i+1)} = \boldsymbol{K_{11}}^{(i+1)} - \boldsymbol{K_{12}}^{(i+1)} \left[\boldsymbol{K_{22}}^{(i+1)}\right]^{-1} \boldsymbol{K_{12}}^{(i+1)^T}$$

and a new effective restoring force vector

$$\boldsymbol{r}^{(i+1)} = \boldsymbol{r_{iso}}^{(i+1)} - \underbrace{\boldsymbol{K_{12}}^{(i+1)} \left[\boldsymbol{K_{22}}^{(i+1)}\right]^{-1} \boldsymbol{r_{eh}}^{(i+1)}}_{\approx 0}$$

Table 1: Implicit computation of the EAS-Parameter for nonlinear materials

The advantages by using this iteration are the exact calculations of the restoring force vector and of the EAS-Parameters. Furthermore it is very stable and efficient, in reason of the iteration at the element level.

2.3 Modeling of material uncertainties

To model material uncertainties a probabilistic model using random fields could be applied. These continuous random fields have to be discretized at certain points. In order to enable a simple computation of the correlation matrix and the matching distribution functions in the discretized and continuous case, in this study point discretization methods are used (Matthies et al. 1997). Two well known types of this methods are the midpoint method and the integration point method, where the random field is discretized at the middle and at each integration point of the finite elements, respectively. The random fields are assumed to be characterized by a given distribution type and a correlation function between the discrete random variables (Brenner 1995).

To model several mutually correlated random field values at each discretization point (e.g. Young's modulus correlated with yield stress) a parameter correlation matrix has to be defined, which will be combined with the computed discretization correlation matrix. The resulting total number of random variables is the product of the number of discretization points and the number of parameters.

To simulate discrete samples of larger random fields, a reduction of the number of random variables could be done by transforming the correlation matrix from a arbitrarily correlated space to the uncorrelated Gaussian space (Brenner 1995). For non-Gaussian distribution types the Nataf transformation (Nataf 1962) has to be applied first. The transformation from the correlated to the uncorrelated Gaussian space will be done by solving the standard eigenvalue problem

$$\mathbf{C_{xx}} = \mathbf{\Psi} \mathbf{C_{yy}} \mathbf{\Psi}^T. \tag{8}$$

For a sufficient representation of the random field only a small number of these eigenvectors are necessary which correspond to the largest eigenvalues.

3 Verification and numerical example

3.1 Cantilever beam with linear elastic material

In this example the stiffener effects with increasing distortions are investigated for the iso-parametric and the enhanced element. The cantilever beam shown in Fig. 1 is discretized with two four-node plane elements and loaded with a constant moment. As we can see in the picture, the iso-parametric element shows a difference of more than 75% to the analytical solution without any distortion. The undistorted enhanced element can reproduce the exact solution. For increasing distortion a increasing deviation from the analytical solution is to be seen.

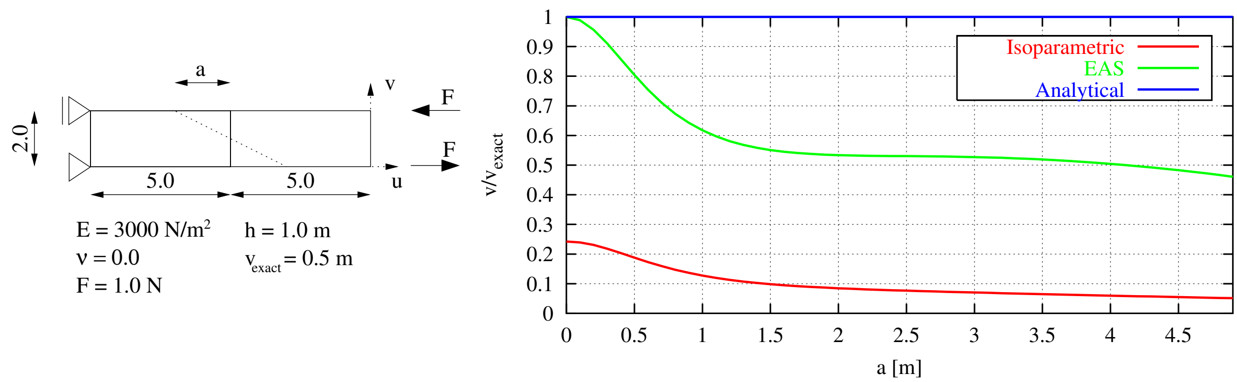

Figure 1: Comparison of the element deformations by increasing element distortions

3.2 Patch-test for nonlinear material

To check the element behavior under a constant strain state a Patch-test (Tayler et al. 1986) could be applied. If the enhanced modes do not affect the constant stresses the test is fulfilled. To control the behavior for nonlinear materials the Patch-test was done by using von Mises plasticity with linear isotropic hardening. Fig. 2 shows the results of the enhanced and the iso-parametric elements which agree exactly. This means that the EAS modes of the enhanced element are not activated with this loading type, therefor the EAS element has the qualities concerning the convergence like the iso-parametric element, but without locking effects.

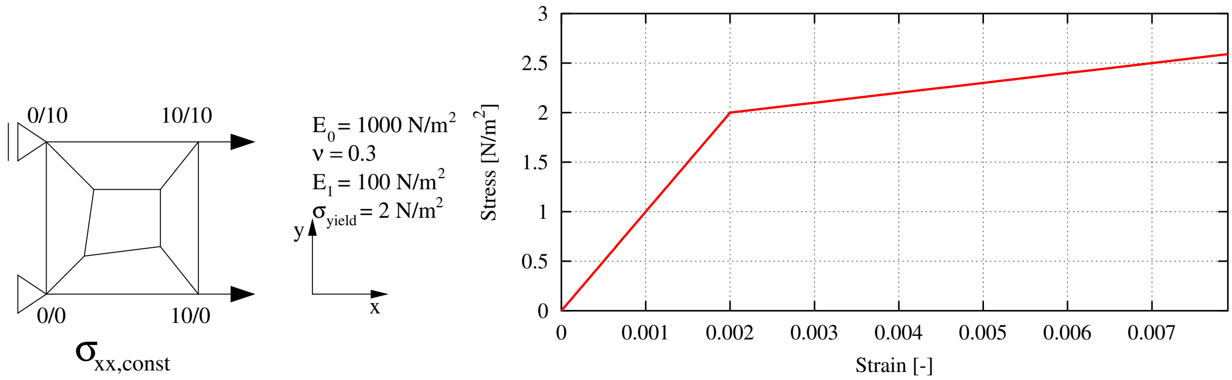

Figure 2: Patch-test for the enhanced and the iso-parametric element using a nonlinear constitutive law

3.3 Stochastic hardening behavior of a cantilever beam

To demonstrate the applicability of the presented element for stochastic calculations, the cantilever beam shown in Fig.1 is investigated again. The structure is descretized with 4x20 elements. The material behavior is modeled with von Mises plasticity with linear isotropic hardening. The Young's modulus of the linear elastic part is assumed with $E_0 = 3000N/m^2$, the Poisson ratio with $\nu = 0$, the yield stress with $\sigma_y = 5N/m^2$ and the hardening modulus with $E_1 = 600N/m^2$. The Young's modulus and the yield stress are modeled with lognormally distributed random fields by using the integration point method with 0.2 as coefficient of variation and $10m$ as correlation length, which corresponds to the beam length. The first 40 eigenvectors are used, which represents the random field with a quality of 97.31%. The correlation coefficient between the two parameters is assumed with $C_{par} = 0.8$. In Fig.3 one sample of the random field is shown. For a statistical evaluation of the assumed material

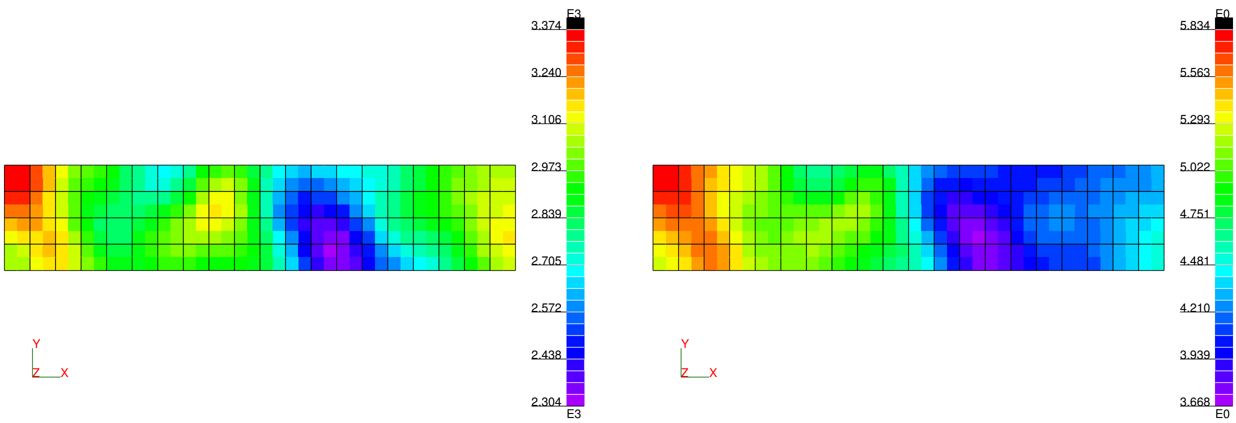

Figure 3: Correlated Young's modulus and yield stress distribution for one random field sample

uncertainties, the external forces F are increased until $F = 5N$. The resulting force-displacement curves of all random field samples are used for the statistical analysis. To reduced the numerical effort instead of Plain Monte Carlo Simulation Latin Hypercube Sampling was applied, whose applicability was shown in Novàk et al. 2001 and Ebert 2002 for linear and nonlinear random field problems. The obtained force-displacement curve $F - u$ by averaging 1000 Latin Hypercube Samples is shown in Fig.4. The deterministic finite element solution and the results of an analytical analysis are displayed

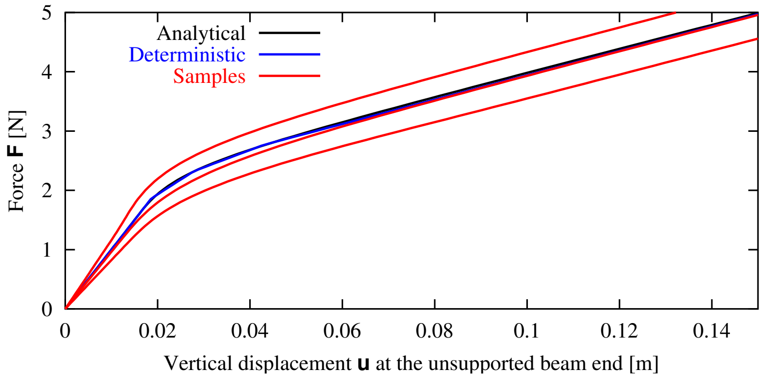

Figure 4: Obtained force-displacement curves of the analytical, deterministic and stochastic analysis

additionally in the picture. By assuming Bernoulli-hypothesis the analytical function reads

$$\begin{aligned} &\text{elastic:} \quad F = 100\frac{N}{m} \cdot u; && 0 \leq u \leq \frac{1}{60}m, \\ &\text{plastic:} \quad F = 2N + 20\frac{N}{m} \cdot u - \frac{1}{5400}Nm^2 \cdot \frac{1}{u^2}; && u > \frac{1}{60}m. \end{aligned} \tag{9}$$

As shown in Fig.4 the analytical and the deterministic curves agree excellent, the sharp bends in the deterministic curve are caused by the synchronous beginning of the hardening of these integration points which have the same distance to the beam axis. Due to the asynchronous behavior of these integration points by randomly distributed material properties, the stochastic force-displacement curves becomes more smooth.

4 Conclusion

In this paper a plane element is presented which uses an enhanced formulation to avoid shear locking effects. Therefore the standard iso-parametric finite element formulation is enlarged with four ”Enhanced Assumed Strain” modes. For linear elastic materials this enhancement can be done straight forward. For nonlinear material behavior the EAS parameters can not be determined directly. In this study the problem is solved by using an internal iteration at the element level, which is much more efficient and stable than the determination via a global iteration. To verify the deterministic element behavior the results of common test examples are presented for linear and nonlinear materials. The modeling of material uncertainties is done by point-discretized random fields. To show the applicability of the element for stochastic finite element calculations Latin Hypercube Sampling was applied to investigate the stochastic hardening behavior of a cantilever beam with nonlinear material. The enhanced linear element can be applied as an alternative to higher-order finite elements where more nodes are necessary. The presented element formulation can be used in a similar manner to improve stochastic linear solid elements.

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