Limit state design of hybrid structures with meshless methods using mathematical optimization[*]

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1   Introduction

The revitalization of existing structures belongs to the frequently tasks in urban reconstruction processes. The adaptation for new requirements will commonly affect substantial changes in the general configuration of structures. The resulting revitalized structures are characterized by a hybrid design, where old and new, identical or diverse materials and members will be coupled in different ways. In the planning stage the treatment of these systems leads to application of complex and hybrid mechanical models respectively. High performance numerical instruments have to be applied for solving not only analysis but also targeted design problems.

The development of the Finite Element Method in the last centuries leads to the availability of a huge amount of numeric tools for the analysis of non-linear structures. Besides dominating technologies performing incremental iterative solving strategies methods based on mathematical optimization are increasingly applied. These optimization algorithms are principally qualified for solving several classes of initial and boundary value problems. The application of these algorithms is beneficial if bounding conditions have to be fulfilled while considering design objectives. Especially the design tasks in civil engineering correspond to this type of numerical interface. That's why mechanical problems can be descriptively formulated as optimization problems. The advantages over strategies basing on the solution of several linear equilibrium systems result from using inequality condition i.e. for the formulation of limit state conditions (i.e. plasticity-, contact conditions) and result from the presence of an objective function for the specification of design intentions. The non-linear behavior of structures can be formulated as a combination of non-linear equality and inequality conditions. The application of that method offers multifaceted possibilities supporting the solution of analysis and design problems in engineering [6,7].

Besides the development of alternative solving strategies several research activities have their focus on the improvement of the quality of the results. A lot of proposals deal with the alternation of the basis polynoms or the amount of nodes belonging to an element. Alternatively methods for the adaptive mesh improvement have been developed. The application of these methods typically leads to an increase in the amount of unknowns. On the other hand most optimization algorithms perform better with less design parameters and subsidiary conditions.

These problems can be avoided while having still a high quality approximation. It can be done by breaking down the traditional finite element boundaries. This strategy has been applied by so-called meshless or element free technologies. One representative is the Element Free Galerkin Method (EFG) proposed by Belytschko et. al. [1,2]. This method was successfully applied in structural engineering solving non-linear analysis problems. The application leads to structural matrices with fewer unknowns but with more density (bandwidth). A graphical comparison of the matrix structure between a similar FE and alternatively EFG modeled structure is given in Fig. 1.1.
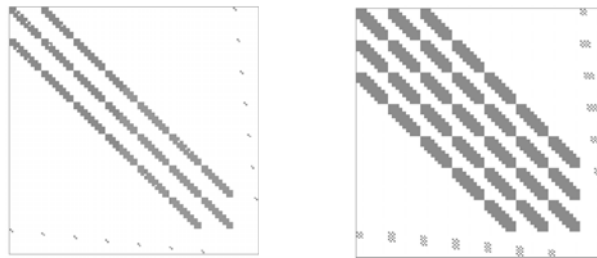

Fig. 1.1 Graphical comparison of the stiffness matrices according to FE (left) and EFG (right)

Because of the hybrid character of mechanical models in revitalization planning processes the use of hybrid technologies is advantageous. In this paper at first these mechanical relations will be formatted for the use with mathematical optimization algorithms. Secondly a simple coupling technique will be used for connecting mixed EFG and FE domains [4]. Thirdly the models derived will be adopted for design purposes of non-linear loaded hybrid structures. The results will be compared with common FE solutions. The elastic and elasto-plastic state of structures according to a given load intensity as well as it's plastic limit state will be considered.

2 Mechanical model

2.1 Derivation of the optimization model for single domains

Optimization models usable for structural design can be directly derived from extremum principles [3]. For analysis of the plastic limit state the static principle of plastic limit state equilibrium can be discretized directly either with FE or EFG methods. One commonly used formulation is the ultimate limit load problem:

$$p \to \text{Max} \tag{1}$$

$$\nabla^T \sigma = p\varphi \qquad \in V \tag{2}$$

$$f(\sigma) \leq 0 \qquad \in S_\lambda \tag{3}$$

$$n^T \sigma = \sigma_0 \qquad \in S_s. \tag{4}$$

The value p is an intensity factor scaling the variable loads. Using a ideal plastic material law and simplifying the yield condition (4) as a linear function, a general problem can be specified that can be solved by means of linear programming [3]. The principle of matrix assembly for a single domain problem can be taken from Table 2.1.

Tab: 2.1 Optimization tableau for plastic limit load analysis (single domain)

design variables	s	F_{sup}	p	1		
objective function			-1		$\to$	Min
equilibrium cond.	A^T	A_{sup}	$- f_m$	$- f_d$	=	0
static boundary conditions	$A_S{}^T$			$- s_0$	=	0
plasticity condition	$A_P{}^T$			$- s_{lim}$	$\leq$	0

A_s and A_P are the coefficient matrices and s_0 and s_{lim} the constant parts of the static boundary and plasticity conditions. A_{sup} is the matrix of the support forces F_{sup}. The vector f_m and f_d contain the moving and dead loads applied to the structure. The vector s contains stresses or/and internal forces.

The Matrix A is the gradient matrix of the shape functions for FE

$$u(x) = H_{FEM} u_n \tag{5}$$

or if appropriate for the shape functions according to EFG

$$u(x) = H_{EFG} u_i \quad . \tag{6}$$

The derivation of the shape function matrices H_{FEM} for finite elements is widely known, whereas the derivation of H_{EFG} can be achieved according to [2]:

$$H_{EFG} = p^T B_1{}^{-1} B_2 \tag{7}$$

with

$$B_1 a = B_2 u_i \tag{8}$$

$$B_1 = \sum_{i=1}^{n} w_i p_i p_i^T \quad \text{and} \quad B_2 = w p^T . \tag{9}$$

The matrices B_1 and B_2 are dependencies of the basis vector p and the weight function w that can be chosen i.e. as a cubic spline interpolation function for radius r [5]:

$$w(r) = \begin{cases} 2/3 - 4r^2 + 4r^3 & \text{für} & r \le 0.5 \\ 4/3 - 4r + 4r^2 - 4/3r^3 & \text{für} & 0.5 < r \le 1 \\ 0 & \text{für} & r > 1 \end{cases} \quad . \tag{10}$$

According to the idea of the EFG-method the moving least square sum (MLS) for the unknown coefficients a and u_i will become a minimum

$$J = \sum_{i}^{n} \Phi_i \to \text{Min} . \qquad \Phi_i = B_1 a^2 - 2B_2 a u_i + u_i^2 \tag{11}$$

Equation (5) can be treated according to known finite element achievements. Necessary integrations will be done with help of an underlying (simple, mostly orthogonal) net using the Gaussian approach. That's why the EFG is easily introducible into given computational applications.

2.2 Connecting EFG and FEM

While considering hybrid structures it can be necessary to decompose the structure into different model domains that can be treated alternatively:

- single domain models (either FE or EFG) or

- mixed domain models (FE and EFG).

Because of the non-direct compatibility between the unknowns u_i of the meshless model and the nodal deformations u_n of the FE solution it is necessary to provide a interface or transformation relation. It can be easily done by utilize the shape functions matrices H_{FEM} and H_{EFG} of Eq. (5) and (6). Equating both conditions leads to

$$H_{EFG} u_i = H_{FEM} u_n \tag{12}$$

that can be directly applied for all interface points between two domains.

Tab: 2.2 Tableau for plastic limit load analysis (domains 1 and 2)

design variables	s_1	s_2	$F_{sup,1}$	$F_{sup,2}$	F_c	p	1		
objective function						-1		$\rightarrow$	Min
equilibrium cond.	A_1^T		$A_{sup,1}$		$H_{i,1}{}^{I}$	$- f_{m,1}$	$- f_{d,1}$	=	0
		A_2^T		$A_{sup,2}$	$- H_{i,1}{}^{I}$	$- f_{m,2}$	$- f_{d,2}$	=	0
static bnd. conditions	$A_{S,1}^T$						$- s_{0,1}$	=	0
		$A_{S,2}^T$					$- s_{0,2}$	=	0
plasticity condition	$A_{P,1}^T$						$- s_{lim,1}$	$\le$	0
		$A_{P,2}^T$					$- s_{lim,2}$	$\le$	0

In Tab. 2.2 the optimization tableau for solving the plastic limit load problem is given. The number of unknowns will increase with the vector of coupling forces F_c along then domain interface.

The state of the structure according to a given load intensity can be analyzed using a quadratic optimization routine. The extended model contains material (linear elastic – ideal plastic) and compatibility conditions in the objective function as well as the subsidiary conditions given in Tab. 2.2. The altered objective function for the mixed domain problem is shown in Tab. 2.3. The matrix Q is the flexibility matrix.

Tab. 2.3 Objective function for state analysis

design variables	s_1	s_2	$F_{sup,1}$	$F_{sup,2}$	F_c	p = const	1		
objective function	$s_1^T Q_2$							$\rightarrow$	Min
		$s_2^T Q_2$							

3   Examples

In this chapter the finite element and meshless models presented in this paper will be discussed with help of two simple examples. For comparability of the solutions derived with single and mixed domain models for both FEM and EFG the same configuration of the mesh and Gaussian quadrature is chosen. This results in models with identical number of unknowns. For the finite element modeling of shear walls a standard 4-nodes finite element with bilinear shape function is used. The EFG part applies a bilinear basis and circular support functions.
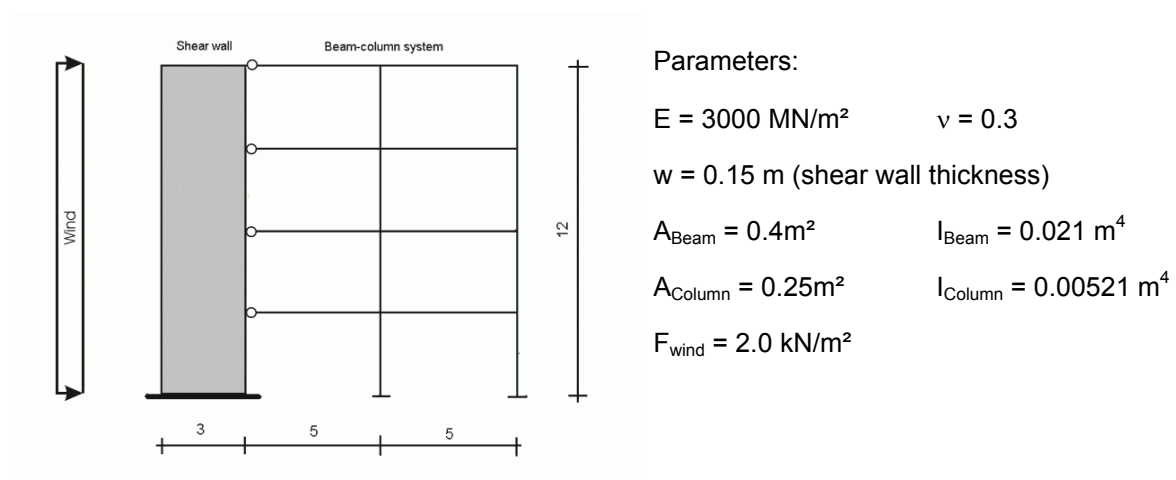

Parameters:

$E = 3000$ MN/m² $\nu = 0.3$

$w = 0.15$ m (shear wall thickness)

$A_{Beam} = 0.4$ m² $I_{Beam} = 0.021$ m⁴

$A_{Column} = 0.25$ m² $I_{Column} = 0.00521$ m⁴

$F_{wind} = 2.0$ kN/m²

Fig. 3.1 Structural System and Parameters

3.1 Coupled shear wall and beam column structure

In Fig. 3.1 a hybrid beam-column-shear wall structures is given. The structure is loaded by dead loads and a variable horizontal load.

In this structure two different types of models are connected. For the wall structure part the EFG is used, whereas for the beam and columns common FE beam elements are applied. Because of the pure quality of element free results for beam structures this decomposition is always recommended. Fig. 3.2 shows the discretization of the two domains and the interface points.
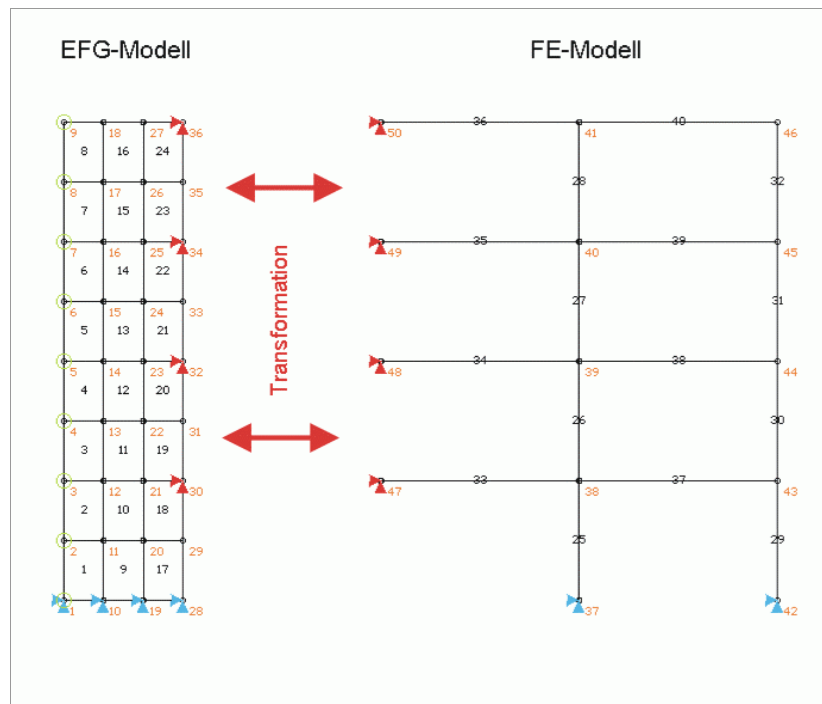
Fig. 3.2 Connections points between EFG and FE part of the model

Following state and limit state analysis configurations will be discussed:

1. Elastic state with load intensity p = 1.0

2. Plastic limit state with equal yield limitations in both tension and pressure direction

3. Plastic limit state with different yield limitations in tension and pressure direction

In this example simple non-interacting uniaxial plasticity conditions for the stresses and internal forces will be used. Selected results of the calculation are given in Fig. 3.3 – 3.6. The plastic limit load is 2.85. Despite the relatively coarse mesh in the wall section the resulting stress distribution is continuous. This applies also for the linear and nonlinear stress distributions.
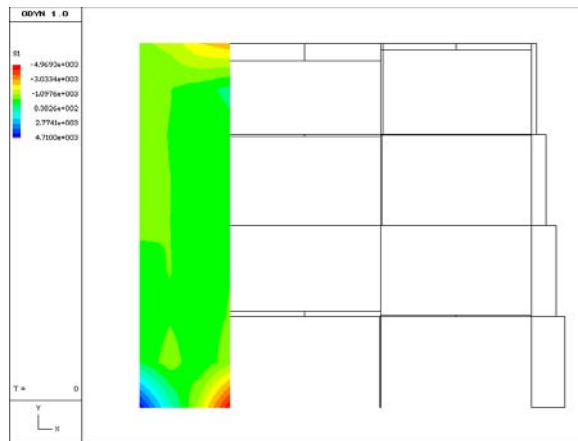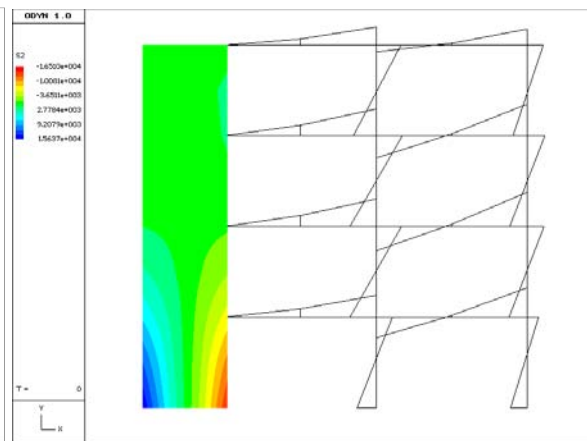
Fig. 3.3 Elastic Response σ_x (case1)

Fig. 3.4 Elastic Response σ_y (case 1)
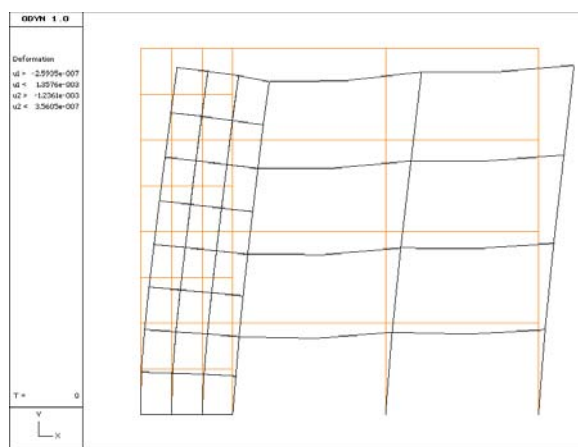

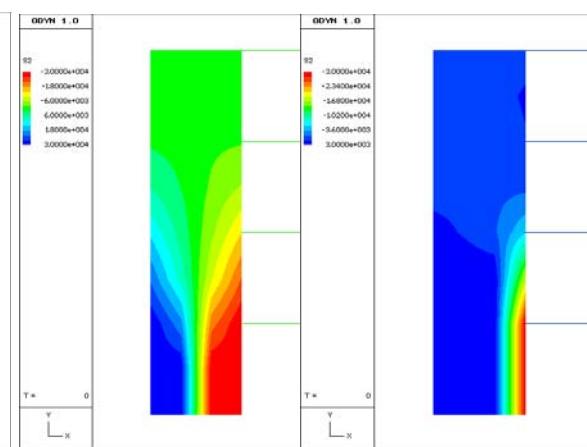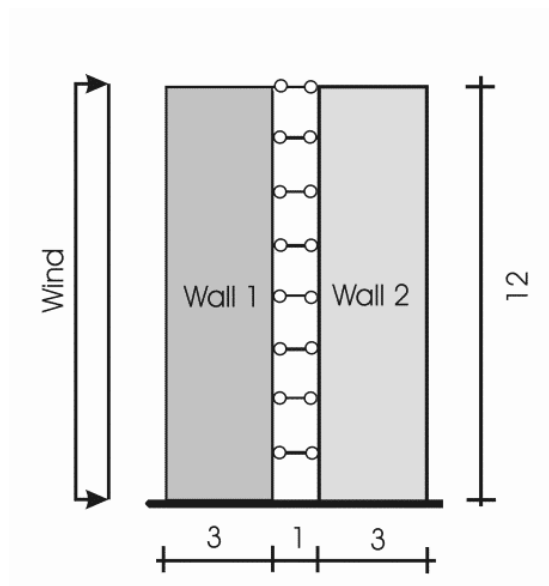Fig. 3.5 Plastic Deformation (case 3)

Fig. 3.6 Plastic Response σ_y (case 2,3)

3.2   Coupled shear wall system

The analysis will be continued with a coupled wall system according to Fig. 3.7.

Parameters:

E = 3000 MN/m² ν = 0.3

w = 0.15 m (shear wall thickness)

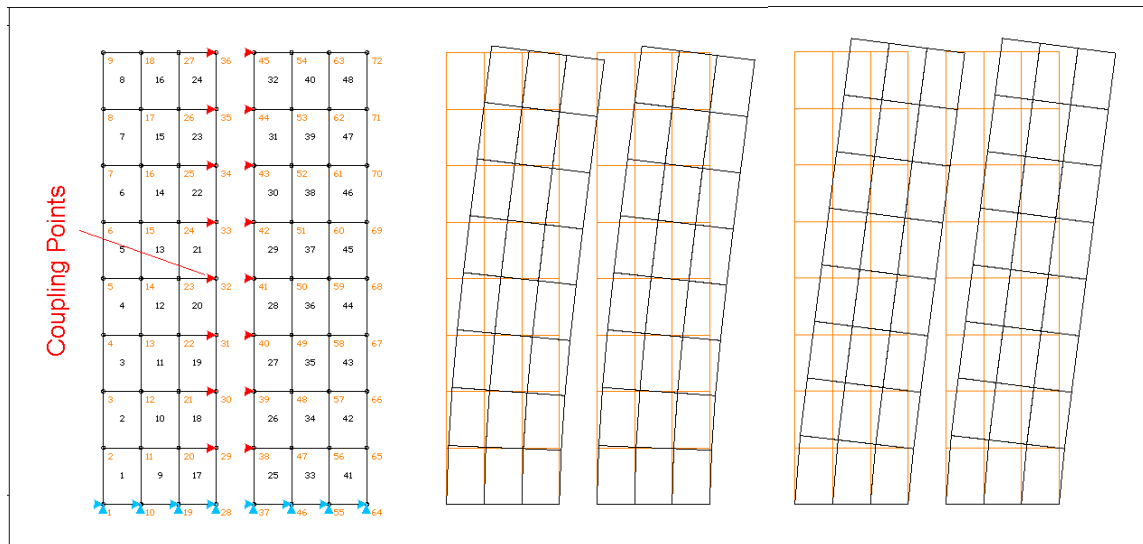F_{wind} = 1.0 kN/m²

Fig. 3.7 Structural System and applied loads

Fig. 3.8 System, Deformation for case 2 elastic p= 1.0 and case2 elasto-plastic p=1.835

For examination purposes three cases will be considered:

1. Both walls are FE-discretized

2. Wall 1 will be EFG and wall 2 FE modeled

3. Both walls will have an EFG discretization

Table 3.1 Results

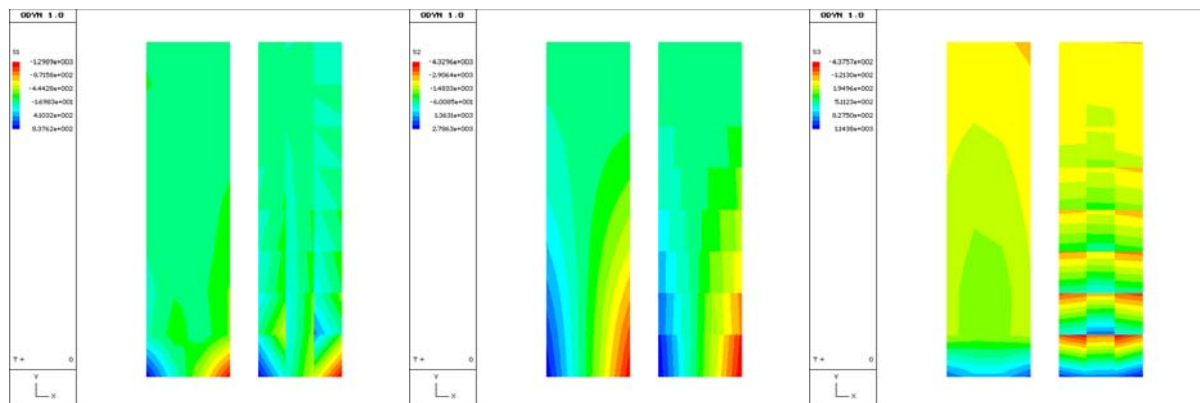| Modell			Elastic calculation		Elastic-Plastic calculation			
case	wall 1	wall2	Intensity factor	max u [mm]	Intensity factor	max u [mm]	Intensity factor	max. u [mm]
1	FE	FE	p=1.0	3.0	p=1.0	3.2	1.835	18.0
2	EFG	FE	p=1.0	3.1	p=1.0	3.3	1.835	28.5
3	EFG	EFG	p=1.0	3.2	p=1.0	3.4	1.835	196.2

Fig. 3.9 Elastic Response (case 2, p=1.0)
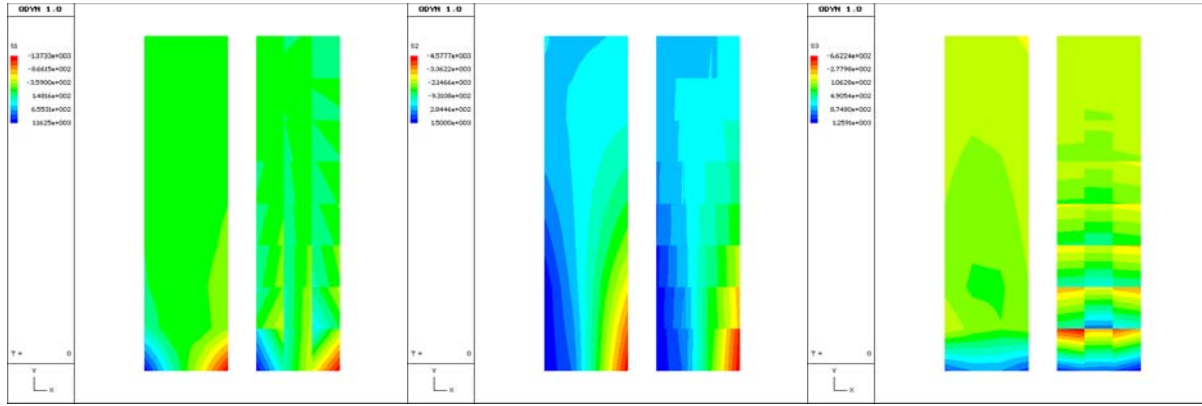
Fig 3.10 Elasto-Plastic Response (case 2, p=1.0)

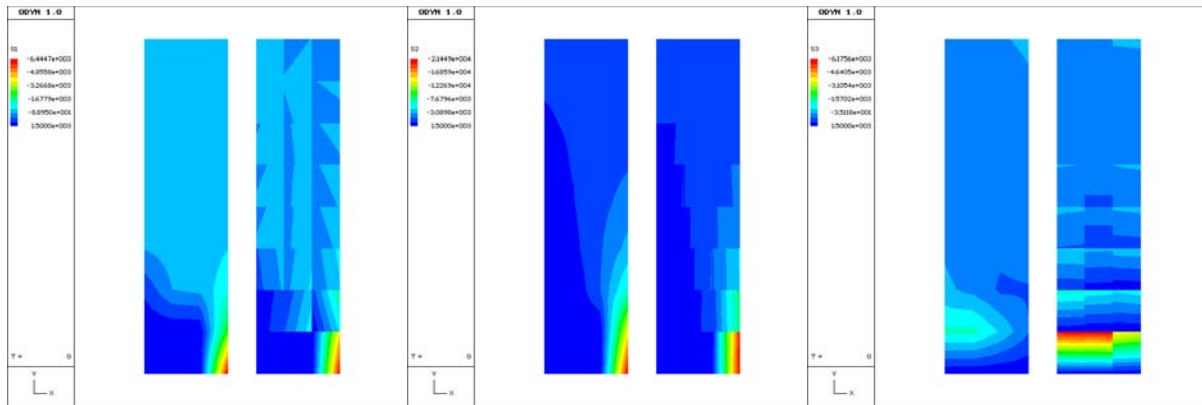

Fig 3.11 Elasto-Plastic Response (case 2, p=1.853)

For all cases an elastic state and plastic limit state analysis will be performed. The results are listed in Table 3.1. As visible in the mixed structure results (Fig. 3.8 to 3.11) the EFG-discritization has an advantage in representing the stress distribution in the structure. In limit state analysis the difference in modeling is not so important for the determination of the ulti-mate limit load itself. But it is essential for the approximation of the resulting stresses and deformations. It should be stated that the FE solution can be approved by refining the mesh or using better element formulations.

4   Conclusions

The investigations show a good adaptability of the meshless methods to the design of hybrid structures by using optimization strategies. As well as single domain models, mixed domain models can be used. With this method the advantages of both finite element and meshless methods can be utilized most suitable. With the property of a minimum amount of unknowns by maintaining an adequate quality of the results the application of mixed finite element and meshless methods is a promising alternative to traditional methods in structural analysis and optimization.

5   Literature

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