On a class of non-Hermitian matrices with positive definite Schur complements

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ON A CLASS OF NON-HERMITIAN MATRICES WITH
POSITIVE DEFINITE SCHUR COMPLEMENTS

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Abstract. Given a positive definite matrix $A \in \mathbb{C}^{n \times n}$ and a Hermitian
matrix $D \in \mathbb{C}^{m \times m}$, we characterize under which conditions there exists
a strictly contractive matrix $K \in \mathbb{C}^{n \times m}$ such that the non-Hermitian
block-matrix
\[
\begin{bmatrix}
A & -AK \\
K^*A & D
\end{bmatrix}
\]
has a positive definite Schur complement with respect to its submatrix $A$. Additionally, we show that $K$ can be chosen such that diagonalizability
of the block-matrix is guaranteed and we compute its spectrum. Moreover, we show a connection to the recently developed frame theory for
Krein spaces.

1. Introduction

Given a matrix $S \in \mathbb{C}^{(n+m)\times(n+m)}$ assume it is partitioned as
\[
S = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix},
\]
where $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}^{m \times m}$. If $A$ is invertible,
then the Schur complement of $A$ in $S$ is defined by
\[
S_{/A} := D - CA^{-1}B.
\]
This terminology is due to Haynsworth [11, 12], but the use of such a con-
struction goes back to Sylvester [15] and Schur [14]. The Schur complement
arises, for instance, in the following factorization of the block matrix $S$:
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
I_n & 0 \\
CA^{-1} & I_m
\end{bmatrix} \begin{bmatrix}
A & 0 \\
0 & D - CA^{-1}B
\end{bmatrix} \begin{bmatrix}
I_n & A^{-1}B \\
0 & I_m
\end{bmatrix},
\]
which is due to Aitken [1]; note that $I_k$ denotes the identity matrix of size
$k \times k$. It is a common argument in the proof of some well-know results in
matrix analysis such as the Schur determinant formula [3]:
\[
\det(S) = \det(A) \cdot \det(S_{/A}),
\]
the Guttman rank additivity formula [10], and the Haynsworth inertia addi-
tivity formula [13].

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The Schur complement has been generalized in numerous ways, for example to the case in which $A$ is non-invertible, where generalized inverses can be used to define it. It is a key tool not only in matrix analysis but also in applied fields such as numerical analysis and statistics. For further details see [16].

If $S$ is a Hermitian matrix, then $C = B^*$ and the Schur complement of $A$ in $S$ is $S_{/A} = D - B^* A^{-1} B$. In this particular case (1.1) reads

\[
\begin{bmatrix}
A & B \\
B^* & D
\end{bmatrix} =
\begin{bmatrix}
I_n & A^{-1} B \\
0 & I_m
\end{bmatrix} *
\begin{bmatrix}
A & 0 \\
0 & D - B^* A^{-1} B
\end{bmatrix} *
\begin{bmatrix}
I_n & A^{-1} B \\
0 & I_m
\end{bmatrix},
\]

which implies the following well-known criteria to determine the positive definiteness of $S$: the block-matrix $S$ is positive definite if and only if $A$ and $S_{/A}$ are both positive definite. This equivalence is not true for positive semidefinite matrices, but Albert [2] showed that $S$ is positive semidefinite if and only if $A$ and $S_{/A}$ are both positive semidefinite and $R(B) \subseteq R(A)$, where $R(X)$ stands for the range of a matrix $X$. Observe that the range inclusion $R(B) \subseteq R(A)$ is equivalent to the existence of a matrix $X \in \mathbb{C}^{n \times m}$ which factorizes $B$ as $B = AX$.

In the present paper, given a positive definite $A \in \mathbb{C}^{n \times n}$ with eigenvalues $0 < \lambda_n \leq \cdots \leq \lambda_1$ and a Hermitian $D \in \mathbb{C}^{m \times m}$ with eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_r \leq 0 < \mu_{r+1} \leq \cdots \leq \mu_m$, we investigate under which conditions there exists a strictly contractive matrix $K \in \mathbb{C}^{n \times m}$ such that

\[
S = \begin{bmatrix}
A & -AK \\
K^* A & D
\end{bmatrix}
\]

has a positive definite Schur complement $S_{/A}$ with respect to the minor $A$, that is, under which conditions there exists a strictly contractive matrix $K \in \mathbb{C}^{n \times m}$ such that

\[
S_{/A} = D + K^* AK
\]

is positive definite.

Interest in such non-Hermitian block-matrices arises, for instance, in the recently developed frame theory in Krein spaces, see [6, 8]. There, block-matrices as in (1.3) with a positive definite $A$, a Hermitian $D$ and a positive definite $S_{/A}$ correspond to so-called $J$-frame operators, see Section 5 for more details.

In Theorem 3.3 below we show that this special structured matrix completion problem has a solution if and only if

\[
r \leq n \quad \text{and} \quad \lambda_i + \mu_i > 0 \quad \text{for all} \quad i = 1, \ldots, r.
\]

We stress that $S$ is not diagonalizable in general, not even if $S_{/A}$ is positive definite. Under the above conditions, we construct a particular strictly contractive matrix $K$, which depends on some parameters $\varepsilon_1, \ldots, \varepsilon_r$. In Theorem 4.2 we compute the eigenvalues of the corresponding block matrix $S$ in terms of the eigenvalues of $A$ and $D$ and the parameters $\varepsilon_1, \ldots, \varepsilon_r$. A root locus analysis of the latter reveals that if each $\varepsilon_i$ is small enough,
then $S$ is diagonalizable and has only (positive) real eigenvalues, although $S$ is non-Hermitian.

2. Preliminaries

Given Hermitian matrices $A, B \in \mathbb{C}^{n \times n}$, several relations between the eigenvalues of $A$, $B$ and $A + B$ can be obtained. The following result was first proved by Weyl, see e.g. [4].

**Theorem 2.1.** Let $A, B \in \mathbb{C}^{n \times n}$ be Hermitian matrices. Then,
\[
\lambda_j^+(A + B) \leq \lambda_j^+(A) + \lambda_j^+(B) \quad \text{for } i \leq j;
\]
\[
\lambda_j^+(A + B) \geq \lambda_j^+(A) + \lambda_j^+(B) \quad \text{for } i \geq j;
\]
where $\lambda_j^+(C)$ denotes the $j$-th eigenvalue of $C$ (counted with multiplicities) if they are arranged in nonincreasing order.

Among the numerous consequences of Weyl’s inequalities, it is worthwhile to mention that if $A, B \in \mathbb{C}^{n \times n}$ are Hermitian matrices such that $A \preceq B$ according to Löwner’s order, then
\[
\lambda_j^+(A) \leq \lambda_j^+(B) \quad \text{for } j = 1, \ldots, n.
\]

Another well-known result says that if $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$, then the non-zero eigenvalues of $AB$ and $BA$ are the same (and they have the same multiplicities). Indeed, it is easy to see that
\[
\begin{bmatrix}
I_m & -A \\
0 & I_n
\end{bmatrix}
\begin{bmatrix}
AB & 0 \\
B & 0
\end{bmatrix}
\begin{bmatrix}
I_m & A \\
0 & I_n
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
0 & BA
\end{bmatrix},
\]
and hence the matrices $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ are similar. Therefore, they have the same characteristic polynomial
\[
p(\lambda) = \lambda^n \det(\lambda I_m - AB) = \lambda^m \det(\lambda I_n - BA),
\]
and the assertion follows immediately.

We use the above result to prove the following proposition. For $K \in \mathbb{C}^{n \times m}$ we denote by $\|K\|$ the spectral norm of $K$, i.e., the operator norm induced by the Euclidean vector norm.

**Proposition 2.2.** Let $A \in \mathbb{C}^{n \times n}$ be positive definite and $K \in \mathbb{C}^{n \times m}$. Then,
\[
\lambda_j^+(K^*AK) \leq \|K\|^2 \lambda_j^+(A) \quad \text{for } j = 1, \ldots, \min\{n, m\}.
\]

*Proof.* Since $A$ is positive definite it has a well-defined square root $A^{1/2}$. Then, for all $j = 1, \ldots, \min\{n, m\}$,
\[
\lambda_j^+(K^*AK) = \lambda_j^+(K^*A^{1/2}A^{1/2}K) \overset{(2.2)}{=} \lambda_j^+(A^{1/2}KK^*A^{1/2}) \leq \|K\|^2 \lambda_j^+(A),
\]
where the inequality follows from (2.1) because $A^{1/2}KK^*A^{1/2} \preceq \|K\|^2 A$. 

\[\square\]
3. Positive definiteness of the Schur complement

In this section we derive a necessary and sufficient condition for the existence of a strictly contractive matrix $K$ such that the block matrix $S$ in (1.3) has a positive definite Schur complement. Throughout this section we consider the following hypotheses.

**Assumption 3.1.** Assume that $A \in \mathbb{C}^{n \times n}$ is positive definite and $D \in \mathbb{C}^{m \times m}$ is a Hermitian matrix. Let $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_r \leq 0 < \mu_{r+1} \leq \ldots \leq \mu_m$ denote the eigenvalues of $D$ (counted with multiplicities) arranged in nondecreasing order, and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$ denote the eigenvalues of $A$ (counted with multiplicities) arranged in nonincreasing order.

First, we record the following important observation.

**Lemma 3.2.** Let Assumption 3.1 hold and assume that $r > n$. Then, there is no $K \in \mathbb{C}^{n \times m}$ such that $D + K^*AK$ is positive definite.

*Proof.* Let $K \in \mathbb{C}^{n \times m}$ and $S_1 := \ker (K)$ be the nullspace of $K$. Consider the subspace $S_2$ of $\mathbb{C}^m$ spanned by all eigenvectors of $D$ corresponding to non-positive eigenvalues. By Assumption 3.1 we have that $\dim S_2 = r$ and

$$\dim S_1 + \dim S_2 \geq (m - n) + r = m + (r - n) > m.$$  

Thus, $S_1 \cap S_2 \neq \{0\}$ and for any non-trivial vector $v \in S_1 \cap S_2$ we have

$$\langle (D + K^*AK)v, v \rangle = \langle Dv, v \rangle \leq 0.$$  

Therefore, $D + K^*AK$ cannot be positive definite. $\square$

In the following result we focus on a special class of matrices $K$. Recall that $K \in \mathbb{C}^{n \times m}$ is called *strictly contractive*, if its singular values are all smaller than 1. Equivalently, $K$ is strictly contractive if and only if $\|K\| < 1$.

**Theorem 3.3.** Let Assumption 3.1 hold. Then, there exists a strictly contractive matrix $K \in \mathbb{C}^{n \times m}$ such that $D + K^*AK$ is positive definite if and only if

(3.1) \hspace{1em} $r \leq n$ \hspace{1em} and \hspace{1em} $\lambda_i + \mu_i > 0$ \hspace{1em} for all $i = 1, \ldots, r$.

*Proof.* Assume that there exists a strictly contractive matrix $K \in \mathbb{C}^{n \times m}$ such that $D + K^*AK > 0$. By Lemma 3.2, it is necessary that $r \leq n$. On the other hand, by Theorem 2.1,

$$0 < \lambda_{m}^\downarrow (D + K^*AK) \leq \lambda_i^\downarrow (D) + \lambda_{m-i+1}^\downarrow (K^*AK),$$

for $i = 1, \ldots, m$. In particular, for $i = m - r + 1, \ldots, m$ we can combine the above inequalities with Proposition 2.2 and obtain

$$0 < \lambda_i^\downarrow (D) + \|K\|^2 \lambda_{m-i+1}^\downarrow (A) < \mu_{m-i+1} + \lambda_{m-i+1}.$$  

Equivalently, we have that $\mu_j + \lambda_j > 0$ for $j = 1, \ldots, r$. 

Conversely, assume that \( r \leq n \) and \( \lambda_i + \mu_i > 0 \) for \( i = 1, \ldots, r \). Then, for each \( i = 1, \ldots, r \), let \( 0 < \varepsilon_i < 1 \) be such that \( \varepsilon_i \lambda_i + \mu_i > 0 \) and define \( E \in \mathbb{C}^{n \times m} \) by

\[
E = \begin{bmatrix}
\text{diag}(\sqrt{\varepsilon_1}, \ldots, \sqrt{\varepsilon_r}) & 0_{r,m-r} \\
0_{n-r,r} & 0_{n-r,m-r}
\end{bmatrix},
\]

where \( 0_{p,q} \) stands for the null matrix in \( \mathbb{C}^{p \times q} \). Further, let \( U \in \mathbb{C}^{n \times n} \) and \( V \in \mathbb{C}^{m \times m} \) be unitary matrices such that \( A = UD\lambda U^* \) and \( D = VD\mu V^* \), where

\[
D\lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \quad \text{and} \quad D\mu = \text{diag}(\mu_1, \ldots, \mu_m).
\]

Then, for

\[
(3.2) \quad K := UEV^*,
\]

it is straightforward to observe that \( \|K\| < 1 \) and

\[
D + K^*AK = V(D\mu + E^*U^*AU)E)V^* = V(D\mu + E^*D\lambda E)V^*
\]

\[
= V\begin{bmatrix}
\text{diag}(\varepsilon_1 \lambda_1 + \mu_1, \ldots, \varepsilon_r \lambda_r + \mu_r) & 0_{r,m-r} \\
0_{m-r,r} & \text{diag}(\mu_{r+1}, \ldots, \mu_m)
\end{bmatrix}V^*
\]

is a positive definite matrix.

**Remark 3.4.** Let Assumption 3.1 hold. Observe that if \( \mu_i = 0 \) for some \( i = 1, \ldots, r \), then the condition \( \lambda_i + \mu_i > 0 \) is automatically fulfilled. Hence, if we assume that \( \text{dim ker } D = p \), then \( D \) has only \( r - p \) negative eigenvalues and, in this case, there exists a strictly contractive matrix \( K \in \mathbb{C}^{n \times m} \) such that \( D + K^*AK \) is positive definite if and only if

\[
r \leq n \quad \text{and} \quad \lambda_i + \mu_i > 0 \quad \text{for all } i = 1, \ldots, r - p.
\]

4. **Spectrum of the block matrix**

Throughout this section, we consider the contraction \( K \) constructed in the proof of Theorem 3.3 and investigate the location of the eigenvalues of the block-matrix \( S \) in (1.3) for this particular \( K \). The locations depend on the parameters \( \varepsilon_1, \ldots, \varepsilon_r \) and hence their study resembles a root locus analysis. Before we state the corresponding result we start with a preliminary lemma.

**Lemma 4.1.** Let Assumption 3.1 and (3.1) hold and set

\[
(4.1) \quad \alpha_i := \frac{(\lambda_i - \mu_i)^2}{4\lambda_i^2}, \quad i = 1, \ldots, r.
\]

Then we have that

\[
0 \leq \frac{-\mu_i}{\lambda_i} < \alpha_i < 1, \quad \text{for all } i = 1, \ldots, r.
\]
Proof. Given \( i = 1, \ldots, r \), by (3.1) we find that \((\lambda_i + \mu_i)^2 > 0\), which implies \((\lambda_i - \mu_i)^2 > -4\mu_i \lambda_i\) and hence
\[
\alpha_i > -\frac{\mu_i}{\lambda_i} > 0.
\]
Furthermore,
\[
\lambda_i - \mu_i = -(\lambda_i + \mu_i) + 2\lambda_i < 2\lambda_i,
\]
which implies that \( \alpha_i < 1 \).

We are now in the position to state the main result of this section.

**Theorem 4.2.** Let Assumption 3.1 and (3.1) hold. For \( i = 1, \ldots, r \) choose \( 0 < \varepsilon_i < 1 \) such that \( \varepsilon_i \lambda_i + \mu_i > 0 \).

If \( K \in \mathbb{C}^{n \times m} \) is the strictly contractive matrix defined in (3.2) then the spectrum of the block matrix \( S \in \mathbb{C}^{(n+m)\times(n+m)} \) given in (1.3) consists of the real numbers \( \lambda_{r+1}, \ldots, \lambda_n, \mu_{r+1}, \ldots, \mu_m \) and
\[
\eta^+_i = \frac{\lambda_i + \mu_i}{2} \pm \lambda_i \sqrt{\alpha_i - \varepsilon_i}, \quad i = 1, \ldots, r,
\]
where \( \alpha_i \) is given by (4.1). Moreover, the following conditions hold:

a) if \( 0 < \frac{-\mu_i}{\lambda_i} < \varepsilon_i < \alpha_i \), then \( \eta_i^+ > \eta_i^- > 0 \);

b) if \( \alpha_i < \varepsilon_i < 1 \), then \( \eta_i^+ = \eta_i^- \in \mathbb{C} \setminus \mathbb{R} \);

c) if \( \varepsilon_i = \alpha_i \), then \( \eta_i^+ = \eta_i^- = \frac{1}{2}(\lambda_i + \mu_i) \) and there exists a Jordan chain of length 2 corresponding to this eigenvalue.

Additionally, if \( \varepsilon_i \neq \alpha_i \) for all \( i = 1, \ldots, r \), then \( S \) is diagonalizable.

Proof. First note that by Lemma 4.1 the ranges for \( \varepsilon_i \) in the cases a) and b) are non-empty. Using the notation from the proof of Theorem 3.3 we obtain
\[
S = \begin{bmatrix} A & -AK \\ KA & D \end{bmatrix} = \begin{bmatrix} UD\lambda U^* & -UD\lambda EV^* \\ V E^* D\lambda U^* & V D\mu V^* \end{bmatrix} = \begin{bmatrix} D\lambda & -B \\ B^* & D\mu \end{bmatrix} W^*,
\]
where \( B \in \mathbb{C}^{n \times m} \) is given by
\[
B := D\lambda E = \begin{bmatrix} \text{diag} \left( \lambda_1 \sqrt{\varepsilon_1}, \ldots, \lambda_r \sqrt{\varepsilon_r} \right) & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix},
\]
and \( W := \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \in \mathbb{C}^{(n+m)\times(n+m)} \) is unitary. Then, if \( \{e_1, \ldots, e_{n+m}\} \) denotes the standard basis of \( \mathbb{C}^{n+m} \), it is easy to see that
\[
SW e_i = \lambda_i We_i \quad \text{for } i = r + 1, \ldots, n,
\]
and
\[
SW e_j = \mu_{j-r} We_j \quad \text{for } j = n + r + 1, \ldots, n + m,
\]
which yields that \( \lambda_{r+1}, \ldots, \lambda_n \) and \( \mu_{r+1}, \ldots, \mu_m \) are eigenvalues of \( S \).
Now, define the following $r \times r$ diagonal matrices:

\[ F_\lambda := \text{diag} (\lambda_1, \ldots, \lambda_r), \quad F_\mu := \text{diag} (\mu_1, \ldots, \mu_r), \]

\[ G := \text{diag} (\lambda_1 \sqrt{\varepsilon_1}, \ldots, \lambda_r \sqrt{\varepsilon_r}), \]

and observe that the remaining $2r$ eigenvalues of $S$ coincide with the spectrum of the submatrix $\tilde{S}$ of $W^*SW$ given by

\[ \tilde{S} := \begin{bmatrix} F_\lambda & -G \\ G & F_\mu \end{bmatrix}. \]

In order to calculate the eigenvalues of $\tilde{S}$, we make use of the Schur determinant formula (1.2), by which the characteristic polynomial of $\tilde{S}$ is given by

\[ q(\eta) = \det (\tilde{S} - \eta I_{2r}) = \det (F_\mu - \eta I_r) \det \left( (\tilde{S} - \eta I_{2r})/(F_\mu - \eta I_r) \right). \]

Since the matrix $(\tilde{S} - \eta I_{2r})/(F_\mu - \eta I_r) = (F_\lambda - \eta I_r) + G(F_\mu - \eta I_r)^{-1}G$ is diagonal and has the form

\[
\begin{bmatrix}
\lambda_1 - \eta + \varepsilon_1 \frac{\lambda_1^2}{\mu_1 - \eta} & 0 & \cdots & 0 \\
0 & \lambda_2 - \eta + \varepsilon_2 \frac{\lambda_2^2}{\mu_2 - \eta} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_r - \eta + \varepsilon_r \frac{\lambda_r^2}{\mu_r - \eta}
\end{bmatrix},
\]

we have that

\[ q(\eta) = \prod_{i=1}^{r} (\mu_i - \eta) \prod_{i=1}^{r} \left( \lambda_i - \eta + \varepsilon_i \frac{\lambda_i^2}{\mu_i - \eta} \right) = \prod_{i=1}^{r} ((\mu_i - \eta)(\lambda_i - \eta) + \varepsilon_i \lambda_i^2). \]

Thus, $\eta \in \mathbb{C}$ is a root of $q(\eta)$ if and only if

\[ \eta^2 - (\lambda_i + \mu_i)\eta + \lambda_i(\mu_i + \varepsilon_i \lambda_i) = 0 \]

for some $i \in \{1, \ldots, r\}$. This leads to the following eigenvalues of $\tilde{S}$:

\[ \eta_i^\pm = \frac{\lambda_i + \mu_i}{2} \pm \frac{1}{2} \sqrt{(\lambda_i - \mu_i)^2 - 4\varepsilon_i \lambda_i^2} \]

for $i = 1, \ldots, r$. Hence, (4.2) follows and statement b) holds. For statement a) we additionally observe that if $\varepsilon_i > \frac{\mu_i}{\lambda_i}$ then

\[ \eta_i^- > \frac{1}{2} (\lambda_i + \mu_i) - \frac{1}{2} \sqrt{(\lambda_i - \mu_i)^2 + 4\lambda_i \mu_i} = 0. \]
To prove c), assume that \( \varepsilon_i = \alpha_i \) for some \( i \in \{1, \ldots, r\} \). Since \( \eta_i^+ = \eta_i^- = \frac{1}{2}(\lambda_i + \mu_i) \) and \( \sqrt{\varepsilon_i} = \frac{\lambda_i - \mu_i}{2\lambda_i} \), it is straightforward to compute that

\[
\left( \bar{S} - \frac{1}{2}(\lambda_i + \mu_i)I_{2r} \right) \left( 1 + \frac{2}{\lambda_i - \mu_i} \right) f_i = \left( f_i \right),
\]

\[
\left( \bar{S} - \frac{1}{2}(\lambda_i + \mu_i)I_{2r} \right) f_i = 0,
\]

using the standard basis \( \{f_1, \ldots, f_r\} \) of \( \mathbb{C}^r \). The vectors above form a Jordan chain of length 2 of \( \bar{S} \) corresponding to the eigenvalue \( \frac{1}{2}(\lambda_i + \mu_i) \). Hence, a Jordan chain of \( S \) can be constructed corresponding to the eigenvalue \( \frac{1}{2}(\lambda_i + \mu_i) \).

Finally, assume that \( \varepsilon_i \neq \alpha_i \) for all \( i = 1, \ldots, r \). In this case, the space \( \mathbb{C}^{n+m} \) has a basis consisting of eigenvectors of \( S \). Indeed, this follows from (4.3) together with

\[
\left( \bar{S} - \eta_i^+ I_{2r} \right) \left( f_i \right) = 0, \quad \left( \bar{S} - \eta_i^- I_{2r} \right) \left( \frac{f_i}{\lambda_i\sqrt{\varepsilon_i}} \right) = 0
\]

for \( i = 1, \ldots, r \).

We emphasize that if for all \( i = 1, \ldots, r \) the parameter \( \varepsilon_i \) in Theorem 4.2 is chosen such that a) holds, then the block matrix \( S \) in (1.3) is diagonalizable and has only positive eigenvalues. This is possible because of Lemma 4.1.

**Example 4.3.** We illustrate Theorem 4.2 with a simple example. Let \( n = m = 1, D = [0] \) and \( A = [a] \) with \( a > 0 \). Then \( r = 1 \) and choosing \( K \) as in (3.2) with \( 0 < \varepsilon < 1 \) gives \( K = [\sqrt{\varepsilon}] \). In this case \( \alpha = \frac{1}{4} \).

By Theorem 4.2, for \( \varepsilon = \frac{1}{4} \) there is a Jordan chain of length 2 corresponding to the only eigenvalue \( \frac{a}{2} \), and indeed we find that

\[
\left( \frac{1}{a} \right), \left( 1 \right)
\]

form a Jordan chain of \( S \), hence \( S \) is not diagonalizable.

On the other hand, for \( \varepsilon \neq \frac{1}{4} \) the block matrix \( S \) has eigenvalues \( \eta^+ = \frac{a}{2} + a\sqrt{\frac{1}{4} - \varepsilon} \) and \( \eta^- = \frac{a}{2} - a\sqrt{\frac{1}{4} - \varepsilon} \). They are positive if \( \varepsilon < \frac{1}{4} \), and they are non-real if \( \frac{1}{4} < \varepsilon < 1 \). In these last two cases \( S \) is diagonalizable.

### 5. Application to J-frame operators

In this section, we exploit Theorems 3.3 and 4.2 to investigate whether a block matrix \( S \) as in (1.3) represents a so-called \( J \)-frame operator and when it is similar to a Hermitian matrix. In the following we briefly recall the concept of \( J \)-frame operators, which arose in [6, 8] in the context of frame theory in Krein spaces.
In a finite-dimensional setting, every indefinite inner product space is a (finite-dimensional) Krein space, see [9]. A map \( \langle \cdot, \cdot \rangle : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C} \) is called an indefinite inner product in \( \mathbb{C}^k \), if it is a non-degenerate Hermitian sesquilinear form. The indefinite inner product allows a classification of vectors: \( x \in \mathbb{C}^k \) is called positive if \( \langle x, x \rangle > 0 \), negative if \( \langle x, x \rangle < 0 \) and neutral if \( \langle x, x \rangle = 0 \). Also, a subspace \( \mathcal{L} \) of \( \mathbb{C}^k \) is positive if every \( x \in \mathcal{L} \setminus \{0\} \) is a positive vector. Negative and neutral subspaces are defined analogously. A positive (negative) subspace of maximal dimension will be called maximal positive (maximal negative, respectively).

It is well-known that there exists a Gramian (or Gram matrix) \( G \in \mathbb{C}^{k \times k} \), which is invertible and represents \( \langle \cdot, \cdot \rangle \) in terms of the usual inner product in \( \mathbb{C}^k \), i.e., \( \langle x, y \rangle = \langle Gx, y \rangle \) for all \( x, y \in \mathbb{C}^k \). The positive (resp. negative) index of inertia of \( \langle \cdot, \cdot \rangle \) is the number of positive (resp. negative) eigenvalues of the Gramian \( G \), and it equals the dimension of any maximal positive (resp. negative) subspace of \( \mathbb{C}^k \). It is clear that the sum of the inertia indices equals the dimension of the space.

A finite family of vectors \( \mathcal{F} = \{f_i\}_{i=1}^q \) in \( \mathbb{C}^k \) is a frame for \( \mathbb{C}^k \), if
\[
\text{span} \{\{f_i\}_{i=1}^q\} = \mathbb{C}^k,
\]
see e.g. [5] and the references therein. Roughly speaking, a \( J \)-frame is a frame, which is compatible with the indefinite inner product \( \langle \cdot, \cdot \rangle \).

**Definition 5.1.** Let \( \langle \mathbb{C}^k, \langle \cdot, \cdot \rangle \rangle \) be an indefinite inner product space. Then, a frame \( \mathcal{F} = \{f_i\}_{i=1}^q \) in \( \mathbb{C}^k \) is called a \( J \)-frame for \( \mathbb{C}^k \), if
\[
\mathcal{M}_+ := \text{span} \{ f \in \mathcal{F} \mid \langle f, f \rangle \geq 0 \}
\]
and
\[
\mathcal{M}_- := \text{span} \{ f \in \mathcal{F} \mid \langle f, f \rangle < 0 \}
\]
are a maximal positive and a maximal negative subspace of \( \mathbb{C}^k \), respectively.

If \( \langle \cdot, \cdot \rangle \) is an indefinite inner product with positive and negative index of inertia \( n \) and \( m \), respectively, then the maximality of \( \mathcal{M}_+ \) and \( \mathcal{M}_- \) is equivalent to
\[
\dim \mathcal{M}_+ = n \quad \text{and} \quad \dim \mathcal{M}_- = m.
\]
Note that if \( \mathcal{F} \) is a \( J \)-frame for \( \mathbb{C}^k \), then there are no (non-trivial) \( f \in \mathcal{F} \) with \( \langle f, f \rangle = 0 \).

Given a \( J \)-frame \( \mathcal{F} = \{f_i\}_{i=1}^q \) for \( \mathbb{C}^k \), its associated \( J \)-frame operator \( S : \mathbb{C}^k \to \mathbb{C}^k \) is defined by
\[
Sf = \sum_{i=1}^q \sigma_i \langle f, f_i \rangle f_i,
\]
where \( \sigma_i = \text{sgn} \langle f, f_i \rangle \) is the signature of the vector \( f_i \). \( S \) is an invertible symmetric operator with respect to \( \langle \cdot, \cdot \rangle \), i.e.,
\[
[Sf, g] = [f, Sg] \quad \text{for all} \quad f, g \in \mathbb{C}^k.
\]
Its relevance follows from the indefinite sampling-reconstruction formula: given an arbitrary $f \in \mathbb{C}^k$,

$$f = \sum_{i=1}^{q} \sigma_i [f, S^{-1}f_i] f_i = \sum_{i=1}^{q} \sigma_i [f, f_i] S^{-1}f_i.$$ 

In the following, we aim to apply the results from Sections 3 and 4, hence we restrict ourselves to the following inner product on $\mathbb{C}^k = \mathbb{C}^{n+m}$,

$$[(x_1, \ldots, x_{n+m}), (y_1, \ldots, y_{n+m})] = \sum_{i=1}^{n} x_i y_i - \sum_{j=1}^{m} x_{n+j} y_{n+j}.$$ 

In [6, Theorem 3.1] a criterion was provided to determine if an (invertible) symmetric operator is a $J$-frame operator. In our setting it says that an invertible operator $S$ in $(\mathbb{C}^k, [\cdot, \cdot])$, which is symmetric with respect to $[\cdot, \cdot]$, is a $J$-frame operator if and only if there exists a basis of $\mathbb{C}^k$ such that $S$ can be represented as a block-matrix

$$(5.1) \quad S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix},$$

where $A \in \mathbb{C}^{n \times n}$ is positive definite, $K \in \mathbb{C}^{n \times m}$ is strictly contractive, and $D \in \mathbb{C}^{m \times m}$ is a Hermitian matrix such that $D + K^*AK$ is also positive definite. Any block-matrix $S \in \mathbb{C}^{(n+m) \times (n+m)}$ of the form (5.1), which satisfies these conditions will be called $J$-frame matrix.

Therefore, Theorem 3.3 can be restated in the following way.

**Theorem 5.2.** Let $A \in \mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{m \times m}$ be matrices satisfying Assumption 3.1. Then there exists $K \in \mathbb{C}^{n \times m}$ with $\|K\| < 1$ such that $S$ as in (5.1) is a $J$-frame matrix if and only if

$$r \leq n \quad \text{and} \quad \lambda_i + \mu_i > 0 \quad \text{for} \quad i = 1, \ldots, r.$$ 

We mention that the study of the spectral properties of a $J$-frame operator is quite recent, see [6, 7]. In the case of $J$-frame matrices, for given $A$ and $D$, we always find conditions such that a strictly contractive $K$ exists which turns $S$ into a matrix similar to a Hermitian one. The following result is a direct consequence of Theorem 4.2 and Lemma 4.1.

**Theorem 5.3.** Let Assumption 3.1 and (3.1) hold. Then, there exists a strictly contractive matrix $K$ such that the matrix $S$ given in (5.1) is a $J$-frame matrix which is similar to a Hermitian matrix. In this case, all eigenvalues of $S$ are positive and there exists a basis of $\mathbb{C}^{n+m}$ consisting of eigenvectors of $S$.

**References**

ON A CLASS OF NON-HERMITIAN MATRICES


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