On the Hyperbolicity of Evolution Equations for Relativistic Fluids

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Abstract

In this dissertation, systems of partial differential equations describing neutron stars in numerical relativity are investigated concerning their hyperbolicity properties. First, fundamental definitions are given and well-posedness of the initial value problem is explained. The main tool for the hyperbolicity analysis of fluid systems in this thesis is the so-called dual frame formalism in which two different frames can be related to each other. With the help of the formalism, it is shown that strong hyperbolicity is independent of the chosen frame if the appearing speeds are subluminal. Second, with regard to the numerical modeling of neutron stars, the partial differential equation systems of ideal hydrodynamics, ideal magnetohydrodynamics, and resistive magnetohydrodynamics are investigated in general relativity. The system of ideal hydrodynamics serves as a test system for the application of the dual frame formalism. The main focus of this work lies on the investigation of ideal magnetohydrodynamics used in numerical relativity. Two formulations of the system of equations are distinguished, determined by the presence of parametrized combinations of the magnetic field constraint in the evolution equations. The first formulation is strongly hyperbolic. In contrast, the second so-called flux-balance law formulation, which is used in numerical relativity, turns out to be only weakly hyperbolic and hence, no well-posed initial value problem can be found. Finally, the two numerically used systems of equations for resistive magnetohydrodynamics are investigated, and again both systems turn out to be only weakly hyperbolic. The flux-balance law formulation of classical magnetohydrodynamics, as well as the systems of dust and charged dust also turn out to be weakly hyperbolic, for the latter at least in the minimally coupled case. Thus, the results have great impact on the current numerical modeling of neutron stars.

Zusammenfassung

In der vorliegenden Dissertation werden Systeme von partiellen Differentialgleichungen auf ihre Hyperbolizitätseigenschaften untersucht, welche im Zusammenhang mit der Simulation von Neutronensternen in der numerischen Relativitätstheorie Anwendung finden. Dabei werden zunächst grundlegende Definitionen gegeben und Wohldefiniertheit des Anfangswertproblems geschildert. Als Hauptwerkzeug zur Hyperbolizitätsanalyse der Flüssigkeitssysteme wird der sogenannte *Dual Frame Formalismus* benutzt, mit dessen Hilfe zwei verschiedene *Frames* in Verbindung gebracht werden können. Unter Verwendung des Dual Frame Formalismus wird gezeigt, dass starke Hyperbolizität unabhängig vom gewählten Frame ist, solange keine überlichtschnellen Geschwindigkeiten auftauchen. Im Hinblick auf die numerische Modellierung von Neutronensternen werden anschließend Differentialgleichungssysteme von idealer Hydrodynamik, idealer Magnetohydrodynamik und resistiver Magnetohydrodynamik in der Allgemeinen Relativitätstheorie untersucht. Dabei dient die ideale Hydrodynamik als Testsystem für die Anwendung des Dual Frame Formalismus. Das Hauptaugenmerk dieser Arbeit liegt auf der Untersuchung der idealen Magnetohydrodynamik. Dabei wird zwischen zwei Formulierungen der Gleichungssysteme unterschieden, indem verschiedene parametrisierte Kombinationen der Zwangsbedingungen an das magnetische Feld zu den Evolutionsgleichungen addiert werden. Die erste Formulierung ist stark hyperbolisch. Die zweite sogenannte flusserhaltende Formulierung, welche in der numerischen Relativitätstheorie Verwendung findet, stellt sich hingegen als lediglich schwach hyperbolisch heraus, weshalb kein wohldefiniertes Anfangswertproblem gestellt werden kann. Zuletzt werden die beiden numerisch verwendeten Gleichungssysteme für resistive Magnetohydrodynamik untersucht, wobei sich ebenfalls beide Systeme als lediglich schwach hyperbolisch herausstellen. Die flusserhaltende Formulierung von klassischer Magnetohydrodynamik, sowie die Systeme von Staub und geladenem Staub stellen sich ebenfalls als schwach hyperbolisch heraus, für letzteres zumindest im minimal gekoppelten Fall. Die gefundenen Resultate haben daher großen Einfluss auf die aktuelle numerische Modellierung von Neutronensternen.

Notation and Abbreviations

Within this thesis, geometric units with $G = c = 1$ are used throughout, where G is the gravitational constant and c is the speed of light in vacuum. Additionally, Lorentz-Heaviside units for electromagnetic quantities with $\varepsilon_0 = \mu_0 = 1$, where ε_0 is the vacuum permittivity (also called electric constant) and μ_0 is the vacuum permeability (also called magnetic constant) are always employed.

The Einstein summation convention is applied throughout this work. Most of the time the abstract index notation is employed. Small Latin letters a, b, c, d, e and p are taken as abstract indices. They are used to indicate the rank of a tensor rather than writing the tensor in a particular basis. Greek indices run from 0 to 3 and denote spacetime components of tensors in the coordinate basis associated with the coordinates $x^\mu = (t, x^i)$. The small Latin letters i, j, and k run from 1 to 3 and stand for the spatial components of the same basis. Further detailed definitions concerning the index notation can be found in the very beginning of chapter 3. The notation in chapter 2 deviates from the rest of the work and is given when needed at the respective place.

Most of the results are obtained with the help of Mathematica using xTensor [Martín-García, 2017]. The notebooks can be downloaded from `http://www.tpi.uni-jena.de/~hild/Hydro_DF.tgz`. In appendix A the assignment between the chapters and notebooks is given. In the text it is referred to the respective notebook with a link to appendix A.

The following abbreviations are used across the thesis and most of them will also be introduced when they appear the first time:

BSSNOK	Baumgarte-Shapiro-Shibata-Nakamura-Oohara-Kojima
DF	Dual frame
GR	General Relativity
GRHD	General relativistic hydrodynamics
GRMHD	(Ideal) General relativistic magnetohydrodynamics
HD	Hydrodynamics
HRSC	High resolution shock capturing
IVP	Initial value problem
MHD	(Ideal Newtonian) Magnetohydrodynamics
NR	Numerical relativity
PDE	Partial differential equation
RGRMHD	Resistive general relativistic magnetohydrodynamics
RMHD	(Ideal special) Relativistic magnetohydrodynamics
RRMHD	Resistive (special) relativistic magnetohydrodynamics
SR	Special Relativity

For the sake of clarity, the abbreviations above as well as definitions, quantities, and techniques are sometimes explained or given more than once.

to my parents

„Wenn immer nur unmittelbar anwendungsbezogene Forschung betrieben worden wäre, hätten wir heute eine unglaubliche Vielfalt und Raffinesse an Kerzen;
aber keine Elektrizität.“

– **Anton Zeilinger** –

Originalzitat

"If only direct application-related research had been conducted, today we would have an incredible variety and sophistication in candles;
but no electricity.“

– **Anton Zeilinger** –

English Translation

Contents

Chapter 1

Introduction

The second century in the lifetime of present best known theory of gravity – *General Relativity* (GR) [Einstein, 1915a; Einstein, 1915b] – has started with a short and soft *chirp* on 14th September 2015. With the first measurement of the gravitational wave signal GW150914 [Abbott et al., 2016b], predicted by Einstein himself in 1916 [Einstein, 1916; Einstein, 1918] a new era has begun: the era of gravitational wave astronomy. The chirp signal of the gravitational wave – a small distortion of the spacetime, traveling with the speed of light – was 'heard' by the Laser Interferometer Gravitational-Wave Observatory [LIGO] and was emitted by merging black holes. Since GW150914, further gravitational wave signals from binary black hole mergers were measured [Abbott et al., 2016a; Abbott et al., 2017a; Abbott et al., 2017b; Abbott et al., 2017c], which all confirm GR in its current form. Almost two years after GW150914 and four decades after the discovery of the first neutron star binary by [Hulse and Taylor, 1975], on 17th August 2017 at 12:41:04 UTC the [LIGO] and [VIRGO] gravitational wave detectors measured the most interesting gravitational wave signal so far, namely the signal GW170817 [Abbott et al., 2017d] emitted by a binary neutron star merger. The signal had a total length of $\sim$ 100s and the component masses were identified within the range of 0.86 and 2.26 $M_{\odot}$, which is in accordance with theoretical expectations [Chamel et al., 2013]. Less than 2s after the merger, the gravitational wave signal was followed by a series of electromagnetic counterparts, such as the (short) gamma-ray burst GRB 170817A [Goldstein et al., 2017]. These multi-messenger observations have confirmed the association of short gamma-ray bursts with merging neutron star binaries. This has been expected for many years [Eichler et al., 1989; Narayan et al., 1992].

The measurement of gravitational waves is not only the remarkable result of decades of planning and building large earth based laser interferometers, but also of huge theoretical efforts and research. This includes for instance the analytical solutions to the vacuum Einstein equations describing a black hole by [Schwarzschild, 1916; Reissner, 1916; Nordström, 1918; Kerr, 1963; Newman et al., 1965], the modeling of relativistic stars [Tolman, 1939; Oppenheimer and Volkoff, 1939], the 3+1 decomposition of the field equations [Bruhat, 1952; Arnowitt et al., 1962; York, 1979], the understanding of quasi-normal modes [Vishveshwara, 1970; Press, 1971; Chandrasekhar and Detweiler, 1975], the studies of relativistic two-body dynamics [Buonanno and Damour, 1999; Blanchet, 2014] by high order post Newtonian calculations [Blanchet et al., 1990], and the numerical relativity (NR) breakthroughs [Shibata and Uryū, 2000; Pretorius, 2005; Campanelli et al., 2006; Baker et al., 2006] to simulate compact binaries and extract their gravitational wave signals. Especially in the last phase of inspiraling, strong gravitational fields are present and thus, a full consideration of GR is required. Due to the non-linearity of the field equations, NR is the common key tool to treat those situations properly.

The role of NR will become crucial for several reasons. First, a global network of earth based ([LIGO; VIRGO; GEO; KAG; IND; ETU]) and space based ([LISA]) GW detectors will be available in the near future enabling the observation of gravitational waves with unprecedented precision. Second, it is statistically expected that in the future the rate of observed signals will increase due to the technical improvements. For instance, the number of detected binary neutron star events (measured by [LIGO; VIRGO; IND]) is estimated to lie between one in a few years up to hundreds per year [Abbott et al., 2013]. Finally, especially the wave form right after merger contains information about the internal structure determined by the equation of state (EOS) [Read et al., 2009; Radice et al., 2017], which is expected to be influenced by quantum effects [Baiotti and Rezzolla, 2017]. Thus, the accurate measurement could provide new insight into the strong field regime, where quantum effects and gravitation mix. Taken together, one of the main tasks of NR in coming years will be the long-time simulation of neutron stars as well as the accurate construction and modeling of gravitational waveforms, which has already started [Lackey et al., 2017]. As a consequence, sophisticated numerical codes are essential for understanding the physics of the strong field regime.

However, the numerical treatment of neutron stars is not as easy as for binary black hole systems. By the presence of matter, the accuracy suffers in comparison to the vacuum case for several reasons. For example, shocks can form even from smooth initial data (ID). To deal with them, sophisticated methods, such as high resolution shock capturing (HRSC) schemes [Hawke et al., 2005], can be employed, where the set of evolution equations is written in a flux-balance law form [Godunov, 1959; Font, 2008]. Another example is the singular behavior of the fluid equations at the stellar surface and their numerical techniques for treatment [Rezzolla and Zanotti, 2013], which both are a constant source of error [Schoepe et al., 2018]. Ultimately, to guarantee long-lived stable numerical simulations and to have convergence in the first place, it is of great importance to have a well suited mathematical formulation of the initial value problem (IVP). In fact, no numerical approximation can converge if the underlying continuous PDE problem is ill-posed. In fairly general cases, the dynamics of neutron stars in GR are determined by a system of (second order) partial differential equations (PDEs) for the metric tensor coupled to evolution and constraint equations for matter variables. Additionally, the set of equations is augmented depending on the physical situation [Font, 2008]. By reformulation of the equations as a system of first order PDEs, the necessary requirement for well-posedness of the IVP is that the PDE system is strongly hyperbolic [Kreiss and Lorenz, 1989].

The main objective of this thesis is to provide a new tool to treat first order PDE systems and investigate their hyperbolicity structure in the context of neutron stars. The modeling of (binary) neutron stars in NR is currently done with a variety of numerical codes. Different types of matter determined by the proposed energy-momentum tensor are considered. The most common models in use are general relativistic hydrodynamics (GRHD) [Font et al., 2000], (ideal) general relativistic magnetohydrodynamics (GRMHD) [Antón et al., 2006] and resistive general relativistic magnetohydrodynamics (RGRMHD) [Dionysopoulou et al., 2015].

The literature already provides several studies of the hyperbolicity structure for GRHD [Anile, 1990] for GRMHD [Anile and Pennisi, 1987; Anile, 1990; Komissarov, 1999; Antón et al., 2010] and for RGRMHD [Cordero-Carrión et al., 2012], mostly based on the definition of hyperbolicity of [Friedrichs, 1974]. However, these investigations do not always apply directly to the numerically used systems of PDEs, since several subtleties were not taken with due care. For example, the presence of constraints in the evolution equations and the gauge freedom affects the hyperbolicity structure. This can be seen in the different 3+1 formulations of the Einstein equations, for instance by [Arnowitt et al., 1962] and [York, 1979], where for some gauges only the latter one can have a well-posed IVP [Alcubierre, 2008; Sarbach and Tiglio, 2012].

In this thesis, the reexamination of the hyperbolicity structure of the aforementioned popular fluid models in full GR is done. Special focus lies on the system of GRMHD, where magnetic fields are taken into account. In the merger and post-merger phase it is expected that magnetic fields have a significant effect on the behavior of the system. For example, they influence the formation of ultra relativistic jets of ionized matter [Massi and Bernado, 2008; Baiotti and Rezzolla, 2017].

This PhD thesis is structured as follows: Chapter 2 starts with one possible classification of PDEs, followed by the relevant definitions and discussions concerning first order quasi-linear PDE systems. Also, the concept of well-posedness of an IVP in relation to the hyperbolicity properties of the underlying PDE system is explained.

In chapter 3, the dual frame (DF) formalism as the main tool used in this work to analyze the evolution equations of popular fluid models is given. It is proven, that under certain restrictions strong hyperbolicity is independent of the frame.

In chapter 4, the PDE system of GRHD is analyzed in regard to its hyperbolicity structure in two frames. All characteristic quantities, such as eigenvectors, eigenvalues, and characteristic variables, are derived in both frames.

In chapter 5, the model of main interest in this work, namely GRMHD, is investigated. Two different formulations are considered and analyzed regarding their hyperbolicity properties. For the first formulation, a complete characteristic analysis is given and a detailed discussion of the degeneracies is provided. For the second formulation, the breakdown of strong hyperbolicity is explained and suggestions for a numerical test are provided.

In chapter 6, the system of RGRMHD is investigated. Two numerically used forms differing in the evolution of the charge density are analyzed.

In chapter 7, a conclusion and future prospects are given.

The PhD thesis is based on the article [Schoepe et al., 2018].

Chapter 2

Basics of PDE Analysis

The ultimate goal of this thesis is to examine whether or not PDE systems of the fluid models GRHD, GRMHD, and RGRMHD are strongly hyperbolic. To investigate the structure of PDE systems, the necessary definitions, notation and tools to do so are provided in this chapter. This will be done in a reporting fashion, mainly based on [Kreiss and Lorenz, 1989; Gustafsson et al., 1995], and furthermore [Reula, 1998; Reula, 2004; Baumgarte and Shapiro, 2010; Hilditch, 2013] were used. The statements given in the literature are slightly adjusted in regard to the application to the fluid models in GR considered in this thesis. Only important results, needed for the analysis of hyperbolic first order PDE systems in the context of GR, will be sketched and explained in a condensed form in the next sections, starting with a classification of second order PDEs.

2.1 Classification of PDEs

At first, a rough overview of how PDEs can be characterized is given and the various attributes used in their names are explained. This section is mainly based on [Kreiss and Lorenz, 1989; Gustafsson et al., 1995; Baumgarte and Shapiro, 2010].

Most of the PDEs in the context of GR are first or second order partial differential equations. The attribute *order* refers to the highest derivative that appears in the PDE. They can be roughly classified into three classes: *elliptic*, *parabolic*, or *hyperbolic*. These classes are inspired by conic sections as follows: Let x^1 and x^2 be two independent variables and denote the partial derivative as $\partial_i \equiv \partial/\partial x^i$, $i = 1, 2$ and let the coefficient functions $A(x^i), B(x^i), C(x^i)$ be sufficiently often differentiable. The second order PDE for a function $U(x^i)$ can be written as

$$A(x^i)\partial_1^2 U(x^i) + 2B(x^i)\partial_1\partial_2 U(x^i) + C(x^i)\partial_2^2 U(x^i) = S(U, \partial U, x^i)\,, \tag{2.1}$$

where the source $S(U, \partial U, x^i)$ does neither have to be linear in U nor in the first derivatives of it, symbolized by ∂U. In the following, the dependence on x^i will be suppressed. The PDE (2.1) at a point x_0^i is called:

$$\begin{aligned} &\textit{elliptic}, \text{ if:} \qquad AC - B^2 > 0\,; \\ &\textit{parabolic}, \text{ if:} \qquad AC - B^2 = 0\,; \\ &\textit{hyperbolic}, \text{ if:} \qquad AC - B^2 < 0\,. \end{aligned} \tag{2.2}$$

If the function U as well as all of its derivatives appear at most to linear order, the PDE is called *linear*. If the coefficients depend on the independent variables x^i, the PDE is called a *variable coefficient* PDE. If all coefficients are constant, the PDE is called a *constant coefficient* PDE.

Elliptic PDEs often arise in the context of time-independent or stationary problems, such as steady-state solutions, or end-state solutions of dynamical systems. To solve them, one needs appropriate boundary conditions. An example of an elliptic equation is the Poisson equation,

$$\partial_1^2 U + \partial_2^2 U = S\,. \tag{2.3}$$

Parabolic PDEs describe diffusive processes. In contrast to elliptic PDEs, they contain a notion of time, expressed by the time coordinate t. They are typically formulated as initial (boundary) value problems. An example is the diffusion equation,

$$\partial_t^2 U - \partial_1(k\partial_1 U) = S\,, \tag{2.4}$$

where k is here the diffusion coefficient. The PDE of main interest in this work is the one of hyperbolic type. Those PDEs have an intrinsic notion of time and signals travel with finite speed. Interestingly but also crucially, discontinuities in initial data (ID) will often be propagated, but can also rise from initially smooth data. The prototype of a hyperbolic equation is the wave equation,

$$\partial_t^2 U - \lambda_0^2 \partial_1^2 U = 0\,, \tag{2.5}$$

with constant wave speed λ_0 (which obeys $0 < \lambda_0 \leq 1$). PDEs of mixed type do also exist.

Taking the second order wave equation (2.5) with $U \equiv \phi$ and introducing the reduction variables $\psi = \partial_t \phi$ and $\varphi = \partial_1 \phi$, one can rewrite the wave equation by the following set of first order PDEs,

$$\partial_t \phi = \psi \,, \qquad \partial_t \psi = \lambda_0^2 \partial_1 \varphi \,, \qquad \partial_t \varphi = \partial_1 \psi, \tag{2.6}$$

or in vector form,

$$\partial_t \mathbf{U} = \mathbf{A} \partial_1 \mathbf{U} + \boldsymbol{\mathcal{S}} \,, \quad \mathbf{U} = \begin{pmatrix} \phi \\ \psi \\ \varphi \end{pmatrix} , \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda_0^2 \\ 0 & 1 & 0 \end{pmatrix} , \quad \boldsymbol{\mathcal{S}} = \begin{pmatrix} \psi \\ 0 \\ 0 \end{pmatrix} . \tag{2.7}$$

Thereby, the second order hyperbolic wave equation is expressed as a coupled system of first order PDEs. The system is symmetric hyperbolic, see below. In the next section hyperbolic PDE systems of first order and their IVP will be discussed.

2.2 First Order Hyperbolic PDE Systems

Consider a vector function $\mathbf{U}$ (later called the *state vector*) of arbitrary finite dimension, depending on 3+1 spacetime variables $\{t, x^i\}$ with $x^0 \equiv t$ and $i = 1, 2, 3$. The letters j, k are taken as spatial component indices such as i. The partial derivative with respect to the spacetime variable x^μ is denoted by $\partial_\mu \equiv \partial / \partial x^\mu$ where μ can take the values $\{0, 1, 2, 3\}$. All following statements are in the context of first order PDE systems.

Quasi-linear PDE system. In this thesis, quasi-linear hyperbolic first order systems of evolution PDEs of the form

$$\partial_t \mathbf{U} = \mathbf{A}^p(x^\mu, \mathbf{U}) \partial_p \mathbf{U} + \boldsymbol{\mathcal{S}}(x^\mu, \mathbf{U}) \,, \tag{2.8}$$

are considered. In this section, p stands for a spatial component index such as i and is always placed on the spatial derivative. The system is called *quasi-linear*, because the coefficient matrices, also called the *principal part* of the system, depend not only on the coordinates x^μ but also on the state vector $\mathbf{U} = \mathbf{U}(t, x^i)$. The *source vector* is written as $\boldsymbol{\mathcal{S}}(x^\mu, \mathbf{U})$ and contains all non-principal terms. However, the source terms will not contribute to the PDE analysis. When sources are presented, then only for the sake of completeness. Hereafter the dependence of the principal part on both the solution and the coordinates is suppressed in the notation. The same is performed for the source vector.

The initial value problem. The *initial value problem* (IVP) (sometimes called *Cauchy problem*) is as follows: Specify initial data (ID) for the state vector $\mathbf{U}(t_0, x^i) = \mathbf{U}_0$ at a given time $t = t_0$. The IVP of (2.8) is to find a solution $\mathbf{U}$ to the PDE system (2.8) for initial data $\mathbf{U}_0$.

Well-posedness. The IVP for (2.8) is called *well-posed* if it admits a unique solution that depends continuously, in a suitable norm, on the ID. For a well-posed IVP one can find constants k, α, which are independent of the ID, such that the state vector $\mathbf{U}$ satisfies the inequality

$$||\mathbf{U}(t, x^i)|| \leq k e^{\alpha t} ||\mathbf{U}_0|| \,. \tag{2.9}$$

For linear constant coefficient systems, the L_2 -norm can be used. For variable coefficient problems a Sobolev norm is appropriate [Reula, 1998], but the particular norm is not of importance in the present work.

Strong hyperbolicity. Let s_i be a spatial 1-form normalized so that $(m^{-1})^{ij} s_i s_j = 1$, with $(m^{-1})^{ij}$ an arbitrary symmetric uniformly positive definite matrix which is permitted to depend on the solution. Contracting the principal part $\mathbf{A}^p$ with s_p, the resulting matrix

$$\mathbf{P}^s \equiv \mathbf{A}^s = \mathbf{A}^p s_p \tag{2.10}$$

is called the *principal symbol* (in the s_i-direction) of the PDE system (2.8). At each point in spacetime the system (2.8) is called:

- *weakly hyperbolic*, if for each s_i the eigenvalues of $\mathbf{P}^s$ are real;
- *strongly hyperbolic*, if the system is weakly hyperbolic and for each s_i the principal symbol has a complete set of eigenvectors written as columns in a matrix $\mathbf{T}_s$ and there exists a constant $K > 0$, independent of s_i, such that

$$|\mathbf{T}_s| + |\mathbf{T}_s^{-1}| \leq K; \tag{2.11}$$

- *strictly hyperbolic*, if the system is weakly hyperbolic and for each s_i the eigenvalues are distinct;
- *symmetric hyperbolic*, if there exists a symmetric positive definite *symmetrizer* $\mathbf{H}$, independent of s_i, such that $\mathbf{H}\,\mathbf{A}^p$ is symmetric for each p.

Note that the condition (2.11) with the matrix norm $|\cdot|$ is automatically satisfied if the eigenvectors depend continuously on s_i. In that case, proving strong hyperbolicity at a point is reduced to showing that the principal symbol $\mathbf{P}^s$ is diagonalizable, i.e., that it has real eigenvalues and a complete set of linearly independent eigenvectors. If a system is strictly and/or symmetric hyperbolic, it is also strongly hyperbolic [Kreiss and Lorenz, 1989]. Strong hyperbolicity can also be defined in a similar way to symmetric hyperbolicity, but with the weaker requirement that the symmetrizer $\mathbf{H}(s_k, t, x^i)$ depends continuously on s_p [Kreiss and Lorenz, 1989]. In the general case and especially for the systems considered in this thesis, the principal symbol is solution dependent. Thus, the hyperbolicity of the system depends on the solution, too. The first order reduction of the wave equation (2.7) forms a symmetric hyperbolic PDE system with symmetrizer $\mathbf{H} = \operatorname{diag}(1, \lambda_0^{-1}, \lambda_0)$.

Characteristic variables. Given a strongly hyperbolic system in the form of (2.8) with principal symbol $\mathbf{P}^s$ and matrix of right eigenvectors $\mathbf{T}_s$, the diagonalized form of $\mathbf{P}^s$ with its eigenvalues on the diagonal is given by

$$\mathbf{\Lambda}^s = \mathbf{T}_s^{-1}\mathbf{P}^s\mathbf{T}_s\,. \tag{2.12}$$

The orthogonal projector to s_i is introduced, that is ${}^m\!\perp^j{}_i = \delta^j{}_i - (m^{-1})^{jk}s_k s_i$ and, in this section, capital letters A, B, C are used to denote projected component indices. The components of the transformed state vector $d_\mu\hat{\mathbf{U}} = \mathbf{T}_s^{-1}\partial_\mu\mathbf{U}$ are called the *characteristic variables* in direction s_i. The 'd' symbol is used here to illustrate the fact that the matrix $\mathbf{T}_s^{-1}$, which is generally both position and solution dependent, is not to be commuted with the partial derivative. Therefore, when presenting quantities like the characteristic variables in terms of state vector components, or the state vector in a particular (space-time dependent) basis, a δ notation is used. Thus, $\delta\varphi$ is written to denote some derivative of a component φ of the state vector $\mathbf{U}$.

The characteristic variables have the property that they satisfy particularly simple equations of motion if the derivatives transverse to $\hat{s}^i = (m^{-1})^{ij}s_j$ and the lower order source terms are ignored,

$$d_t\hat{\mathbf{U}} = \mathbf{\Lambda}^s d_{\hat{s}}\hat{\mathbf{U}} + (\mathbf{T}_s^{-1}\mathbf{A}^A\mathbf{T}_s)d_A\hat{\mathbf{U}} + \mathbf{T}_s^{-1}\boldsymbol{\mathcal{S}}\,. \tag{2.13}$$

In the linear constant coefficient approximation, where the coefficients are supposed to be frozen, the matrices are constant and can be commuted with the partial derivative. Dropping the aforementioned terms and taking the approximation, one arrives at decoupled advection equations with speeds given by the eigenvalues of $\mathbf{P}^s$.

Strong hyperbolicity is a necessary requirement for well-posedness of the IVP. A first order PDE system must be strongly hyperbolic to be able to obtain a well-posed IVP [Kreiss and Lorenz, 1989]. To study this statement phenomenologically, the following proof is sketched. For the sake of simplicity, the source vector is set to zero and a linear constant coefficient PDE system in one space dimension ($x^1 \equiv x$) is considered:

$$\partial_t \mathbf{U} = \mathbf{A}\partial_x \mathbf{U}\,. \tag{2.14}$$

For the particular choice of an initial condition $\mathbf{U}(0,x) = e^{i\omega x}\hat{\mathbf{f}}(\omega)$, $\omega \in \mathbb{R}$, one can make the separation ansatz $\mathbf{U}(t,x) = e^{i\omega x}\hat{\mathbf{U}}(t,\omega)$. The solution to the IVP (2.14) is then

$$\mathbf{U}(t,x) = e^{i\omega\mathbf{A}t}\mathbf{U}(0,x)\,, \qquad \mathbf{U}(0,x) = e^{i\omega x}\hat{\mathbf{f}}(\omega)\,. \tag{2.15}$$

Taking now an appropriate norm such as the L^2-norm and writing $\mathbf{U}(0,x) \equiv \mathbf{U}_0$, one can estimate

$$||\mathbf{U}(t)|| = ||e^{i\omega\mathbf{A}t}\mathbf{U}_0|| \leq |e^{i\omega\mathbf{A}t}|\,||\mathbf{U}_0||\,. \tag{2.16}$$

Suppose that the principal symbol $\mathbf{A}$ is diagonalizable. Then, by using the matrix of right eigenvectors $\mathbf{T}$, estimating gives

$$|e^{i\omega\mathbf{A}t}| = |\mathbf{T}e^{i\omega\mathbf{\Lambda}t}\mathbf{T}^{-1}| \leq |\mathbf{T}||\mathbf{T}^{-1}|\,, \tag{2.17}$$

where $|e^{i\omega\mathbf{\Lambda}t}| = 1$ since $\mathbf{\Lambda}$ is diagonal with real entries. Hence, the estimate (2.9) holds with $\alpha = 1$ and $k = |\mathbf{T}||\mathbf{T}^{-1}|$ and the IVP is well-posed. Suppose instead that $\mathbf{A}$ has at least one complex eigenvalue $\lambda = \zeta + i\xi$ with real ζ, ξ and $\xi \neq 0$. Then $|e^{i\omega\mathbf{A}t}| \geq |e^{i\omega\lambda t}| = e^{-\omega\xi t}$. Thus, one can never find a bound independent of ω and therefore α and/or k in (2.9) become dependent on the ID. This IVP is ill-posed. Suppose now that the system is weakly hyperbolic, and the Jordan form $\mathbf{J}[\mathbf{A}]$ of $\mathbf{A}$ contains at least one nontrivial Jordan block. For simplicity take $\mathbf{A}$ to be

$$\mathbf{A} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad e^{i\omega\mathbf{A}t} = e^{i\omega\lambda t}\begin{pmatrix} 1 & i\omega t \\ 0 & 1 \end{pmatrix}, \tag{2.18}$$

where the norm goes like $|e^{i\omega\mathbf{A}t}| \sim (1+|\lambda\omega|t)$. Again one arrives at an ill-posed IVP. In the linear constant coefficient case considered here, strong hyperbolicity and well-posedness of the IVP are equivalent. In the more general case of a quasi-linear system (2.8), strong hyperbolicity at each point is a necessary condition for well-posedness. Additionally, smoothness conditions are needed to guarantee well-posedness, namely that the symmetrizer $\mathbf{H}(s_k, t, x^i)$ as well as the state vector $\mathbf{U}(t, x^i)$ must depend smoothly on their arguments. For further explanations and proofs concerning the general case, the interested reader is referred to [Kreiss and Lorenz, 1989; Gustafsson et al., 1995].

PDE systems with constraints. A system of evolution equations supplemented by constraints is called a *constrained PDE system.* Constraints reduce freedom in the choice of physical initial conditions. One can show that if the constraints are initially satisfied, they remain satisfied at later times. However, in NR one is faced with rounding errors such that this statement is only true for the continuum PDE system, but there exist techniques to keep them nearly fulfilled (see also chapter 6). Adding multiples of constraints to the evolution equations can of course change the principal symbol and thus the level of hyperbolicity. The models of GRMHD and RGRMHD form constrained PDE systems by the presence of electromagnetic fields and the residual gauge freedom of Maxwell equations. Taking the evolution equations and adding a parametrized combination of the constraints to each evolution equation leads to different *formulations* for different choices of the constraint addition parameters. In chapter 5 this is considered in more detail.

Summary. To formulate a well-posed IVP it is indispensable to have a strongly hyperbolic PDE system. If the continuum PDE system is not strongly hyperbolic and therefore the IVP ill-posed, no estimates for the time evolution of the variables can be given. Also, for ill-posed IVPs one has to worry about existence and/or uniqueness of the solution in the first place. In the context of NR *no* numerical code can converge if the IVP of the underlying continuous PDE system is ill-posed. Numerical hacks cannot prevent this, since it is a property of the PDE system itself. A first step to obtain stable numerical codes is thus to consider a strongly hyperbolic PDE system. This crucial property became already visible in the beginning of NR, where the vacuum Einstein equations were considered numerically as ill-posed IVPs [Alcubierre, 2008].

2.3 The Einstein Equations with Matter

In this thesis, different types of matter in full GR are studied and the governing PDE systems are examined for strong hyperbolicity. More precisely, the Einstein equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{2.19}$$

are considered, which contain derivatives up to second order in space and time for the metric components $g_{\mu\nu}$ on the left-hand side, with the energy-momentum tensor $T_{\mu\nu}$ as a source term on the right-hand side. These equations are supplemented with additional evolution equations for the matter variables. The latter may be fluid and/or electromagnetic variables, depending on the physical system which is considered. To treat the metric variables one may make a first order reduction by introducing reduction variables (as for the wave equation (2.7)) and construct a suitable first order hyperbolic reformulation of the Einstein equations. In this way one can write the principal symbol schematically as

$$\mathbf{P}^s = \begin{pmatrix} \mathbf{P}^s_{\mathrm{g}} & \mathbf{P}^s_{\mathrm{g}\times\mathrm{m}} \\ \mathbf{P}^s_{\mathrm{m}\times\mathrm{g}} & \mathbf{P}^s_{\mathrm{m}} \end{pmatrix}, \tag{2.20}$$

with the principal symbols for the metric $\mathbf{P}^s_{\mathrm{g}}$ and matter variables $\mathbf{P}^s_{\mathrm{m}}$. If the evolution equations for the matter variables contain neither second nor higher order derivatives of the metric, the matrix $\mathbf{P}^s_{\mathrm{m}\times\mathrm{g}}$ can be set to zero by replacing first derivatives with reduction variables. Furthermore, if the energy-momentum tensor contains no derivatives of the fluid variables whose equations of motion are assumed to be first order, then one can take $\mathbf{P}^s_{\mathrm{g}\times\mathrm{m}} = 0$. Under these circumstances, $T_{\mu\nu}$ is not only a physical source term but also in the sense of PDEs, as defined above. For $\mathbf{P}^s_{\mathrm{m}\times\mathrm{g}} = \mathbf{P}^s_{\mathrm{g}\times\mathrm{m}} = 0$, the system is said to be *minimally coupled*. In such a case the characteristic analysis can be performed separately for $\mathbf{P}^s_{\mathrm{g}}$ and $\mathbf{P}^s_{\mathrm{m}}$. Thus, taking a strongly hyperbolic first order formulation for the metric variables (e.g., BSSNOK [Nakamura et al., 1987; Shibata and Nakamura, 1995; Baumgarte and Shapiro, 1998] using reduction variables [Alcubierre, 2008]) one needs to study only the properties of $\mathbf{P}^s_{\mathrm{m}}$. Minimal coupling is assumed throughout the work. In the following the subscript 'm' will be dropped. Note that the choice of a strongly hyperbolic first order formulation for the metric variables also serves as a restriction for the eligible coordinates.

In the following chapters, the evolution PDEs are written in various forms similar to (2.8). For convenience, instead of the partial derivative ∂, the spacetime covariant derivative ∇ and various other operators are used, which are introduced in the next chapter. The assumption of minimal coupling allows to ignore first derivatives of the metric that appear in these expressions by assuming implicitly that they are replaced by the metric reduction variables. This approach is appropriate for any minimally coupled metric-based theory of gravity. Please note that care is sometimes needed to avoid a violation of the condition. However, in the Cowling approximation [Cowling, 1941] where the background is a priori given, only the matter variables need to be evolved and the coupling does not affect the analysis.

Chapter 3

The Dual Frame Formalism

In this chapter the dual frame formalism (DF formalism) is introduced. The DF approach is based on the dual foliation formalism presented in [Hilditch, 2015; Hilditch et al., 2018]. This formalism relates two different frames to each other, allowing to make statements about unknown quantities in one frame through knowledge of certain quantities in the other one. It will become the key tool to obtain characteristic quantities in the Eulerian frame, especially for GRMHD.

In the first section, a short introduction to the DF approach is given. Some of the relations are explicitly calculated. For deeper insights and to obtain a full understanding of the construction it is required to read [Hilditch, 2015]. In this thesis it is only worked in frames, in contrast to [Hilditch, 2015; Hilditch et al., 2018], where coordinates are introduced. In the present work, only one of the frames necessarily defines a coordinate tensor basis. The advantage of a frame formalism is the possibility to translate quantities easily between two frames just by 3+1 decomposing the tensors. Note that the choice of coordinates does only affect the form of the principal symbol $\mathbf{P}_{\mathrm{g}}^{s}$ of the metric variables, since this fixing of gauge freedom leads to a particular occurrence of constraints in the evolution equations. The analyses of the matter principal symbols below is in this sense independent of the choice of variables.

In the second part of this chapter, the frame independence of strong hyperbolicity is shown in section 3.2 and an algorithm to recover characteristic quantities in the unknown frame from the known one is given in section 3.3. Before starting with the basics, the index notation valid for the rest of the thesis is given now.

Index notation. The index notation is now introduced which is in use in the rest of this work. Small Latin letters a, b, c, d, e are used as abstract indices. The index p is also used as an abstract index, and is always placed on the spatial derivative appearing on the right-hand side of the first order PDE systems. The metric tensor g_{ab} is the only object permitted to raise and lower indices. Greek indices run from 0 to 3 and denote the components of tensors in the coordinate basis associated with the coordinates $x^\mu = (t, x^i)$. The metric tensor has the signature $(-1, 1, 1, 1)$. Latin indices i, j, k run from 1 to 3 and stand for the spatial components in the same basis. The symbol ∂_a stands for the flat covariant derivative naturally defined by x^μ [Wald, 1984]. The covariant derivative to g_{ab} is denoted by ∇_a and the connected Christoffel symbols are denoted in the coordinate basis by $\Gamma^\mu_{\nu\sigma}$.

Various tensors will be introduced in this chapter. To provide a point of reference some of them are mentioned now without giving them a meaning. Indices n, N, u, V, S, Q_1, Q_2, s, q_1, q_2, s, q_1, q_2, $\hat{\mathsf{s}}$, $\hat{\mathsf{q}}_1$, $\hat{\mathsf{q}}_2$, and z label contraction in that slot with n^a or n_a and so on, respectively. Capital Latin letters A, B, C are abstract indices denoting application of the projection operators ${}^{\mathrm{Q}}\!\perp$ or ${}^{\mathsf{q}}\!\perp$. Similarly indices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\hat{\mathbb{A}}$, $\hat{\mathbb{B}}$, $\hat{\mathbb{C}}$ are used to denote the application of the projection operator ${}^{\mathsf{q}}\!\perp$ over a vector or dual-vector, respectively. The summation convention applies also for abstract capital letters of different type but same letter. For products of different projectors it is written for instance ${}^{\mathsf{q}}\!\perp^a{}_{\hat{\mathbb{B}}}\,{}^{\mathrm{Q}}\!\perp^B{}_c \equiv {}^{\mathsf{q}}\!\perp^a{}_b\,{}^{\mathrm{Q}}\!\perp^b{}_c$. For the index notation convention in the provided notebooks, see appendix A.

3.1 Basic Idea and Objects

The basic idea of the DF approach is to describe a region of spacetime in two different frames. These two frames are called the *lower case frame* and the *upper case frame*. In this work, the lower case frame is a coordinate frame associated with coordinates x^μ. The zeroth component is the usual time coordinate t, which foliates the spacetime. The frame consists of the four vectors ∂^a_μ and the associated co-frame is $\nabla_a x^\mu$ (see also [Wald, 1984]).

For the lower case frame the standard choice of NR is adopted, the Eulerian frame, and standard textbook notation as by [Alcubierre, 2008] is used for the related lower case quantities. The future pointing timelike unit normal vector to slices of constant t is, as usual, denoted by n^a. Tensors orthogonal to n^a are called *lower case spatial* or sometimes just *lower case*. For convenience an additional frame is introduced. It consists of the future pointing timelike unit normal vector n^a plus three linearly independent lower case vectors which are introduced as they are required. For almost all calculations, this frame will be used. Since it has the same timelike unit normal n^a as the lower case coordinate frame, it is also referred to as the lower case frame.

The upper case frame consists of a future pointing timelike unit normal vector N^a, plus any three linearly independent vector fields orthogonal to N^a, which will be chosen for convenience. Tensors orthogonal to N^a are called *upper case spatial* or sometimes just *upper case*. The results in this chapter hold for a general future pointing timelike unit normal vector N^a, but it is mentioned that N^a is identified later with the fluid four-velocity u^a.

The upper case future pointing timelike unit normal vector N^a may be 3+1 decomposed as

$$N^a = W(n^a + v^a)\,, \tag{3.1}$$

with the boost vector v^a, $n_a v^a = 0$, and the Lorentz factor $W = -n_a N^a$. Using the normalization of the normal vectors,

$$N_a N^a = -1\,, \qquad n_a n^a = -1\,, \tag{3.2}$$

one obtains

$$1 = -N_a N^a = -W^2(n_a + v_a)(n^a + v^a) = W^2(1 - v^a v_a) = W^2 - \hat{v}^a \hat{v}_a \tag{3.3}$$

and thereby

$$W = \frac{1}{\sqrt{1 - v^a v_a}} = \sqrt{1 + \hat{v}^a \hat{v}_a}\,. \tag{3.4}$$

This justifies the name Lorentz factor for W. In equation (3.3) the weighted boost vector $\hat{v}^a = W v^a$ is introduced.

In an analogous way the lower case normal vector is 3+1 decomposed in terms of the upper case normal N^a and the upper case boost vector V^a, $N_a V^a = 0$, that is

$$n^a = W(N^a + V^a), \tag{3.5}$$

where the Lorentz factor can be written as $W = (1 - V^a V_a)^{-1/2}$. Comparing this result with the above relation (3.4), the norms of the lower and upper boost vectors must be identical, $V^a V_a = v^a v_a$.

Furthermore, the projection operators related to the normal vectors n^a and N^a are defined by

$$\gamma^b{}_a = g^b{}_a + n^b n_a\,, \qquad {}^{(\mathrm{N})}\gamma^b{}_a = g^b{}_a + N^b N_a\,, \tag{3.6}$$

respectively. These are by construction orthogonal to their associated normal vectors, $\gamma^b{}_a n_b = 0$, ${}^{(\mathrm{N})}\gamma^b{}_a N_b = 0$. The projection operator $\gamma^a{}_b$ becomes the natural induced metric γ_{ij} on slices of constant t in the coordinate frame when both indices are lowered. Hence, γ_{ab} and ${}^{(\mathrm{N})}\gamma_{ab}$ are called the lower and upper case spatial metrics, respectively. Projecting the upper case spatial metric with $\gamma^b{}_a$ on both indices yields

$$\mathrm{g}_{ab} := \gamma^c{}_a \gamma^d{}_b {}^{(\mathrm{N})}\gamma_{cd} = \gamma_{ab} + \hat{v}_a \hat{v}_b \,. \tag{3.7}$$

This can be easily verified by inserting the definition of the upper case projector given in equation (3.6), expressing the upper normal vector in terms of lower case quantities by equation (3.1), and using orthogonality relations. The tensor g_{ab} is called the *boost metric*. Its inverse can be calculated by the Sherman-Morrison formula or by direct computation, and is given by

$$(\mathrm{g}^{-1})^{ab} = \gamma^{ab} - v^a v^b \,, \tag{3.8}$$

which is called the *inverse boost metric*. Please note that in contrast to the four-metric tensor, in general $(\mathrm{g}^{-1})^{ab} \neq \mathrm{g}^{ab}$.

In the same way but by projecting the lower case projector $\gamma^b{}_a$ with ${}^{(\mathrm{N})}\gamma^b{}_a$ on both indices the upper case boost metric and its inverse are defined as

$${}^{(\mathrm{N})}\mathrm{g}_{ab} := {}^{(\mathrm{N})}\gamma^c{}_a {}^{(\mathrm{N})}\gamma^d{}_b \gamma_{cd} = {}^{(\mathrm{N})}\gamma_{ab} + W^2 V_a V_b \,, \qquad {}^{(\mathrm{N})}(\mathrm{g}^{-1})^{ab} = {}^{(\mathrm{N})}\gamma^{ab} - V^a V^b . \tag{3.9}$$

These various definitions are collected in table 3.1.

The vector n^a is by construction hypersurface orthogonal. The lapse function α, shift vector β^a and time vector $t^a \equiv \partial_t^a$ are defined and related via [Alcubierre, 2008]:

$$\begin{aligned} \alpha &= (-\nabla_a t \nabla^a t)^{-\frac{1}{2}} \,, & n^a &= -\alpha \nabla^a t \,, \\ \beta^a &= \gamma^a{}_b t^b = t^a - \alpha n^a \,, & t^a \nabla_a t &= 1 \,. \end{aligned} \tag{3.10}$$

	Upper case frame	Lower case frame
Unit normal vector	$N^a = W(n^a + v^a)$	$n^a = W(N^a + V^a)$
Boost vector	V^a	$v^a = \hat{v}^a / W$
Lorentz factor	$W = (1 - V^a V_a)^{-1/2}$	$W = (1 - v^a v_a)^{-1/2}$
Projector	${}^{(\mathrm{N})}\gamma^a{}_b = g^a{}_b + N^a N_b$	$\gamma^a{}_b = g^a{}_b + n^a n_b$
Boost metric	${}^{(\mathrm{N})}\mathrm{g}_{ab} := {}^{(\mathrm{N})}\gamma_{ab} + W^2 V_a V_b$	$\mathrm{g}_{ab} := \gamma_{ab} + \hat{v}_a \hat{v}_b$
Inverse boost metric	${}^{(\mathrm{N})}(\mathrm{g}^{-1})^{ab} = {}^{(\mathrm{N})}\gamma^{ab} - V^a V^b$	$(\mathrm{g}^{-1})^{ab} = \gamma^{ab} - v^a v^b$

TABLE 3.1: Overview of the relationship between the upper and lower case quantities.

The spacetime metric can be expanded in the lower case frame as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} , \tag{3.11}$$

with inverse,

$$g^{\mu\nu} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix} , \tag{3.12}$$

and the line element takes the form

$$ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j \, . \tag{3.13}$$

The determinants of four-metric $g_{\mu\nu}$ and three-metric γ_{ij} are denoted and related by $\sqrt{-g} = \alpha\sqrt{\gamma}$. The intrinsic covariant derivative operator is denoted by D_a and has connection ${}^{(\mathrm{n})}\Gamma^\mu_{\nu\sigma}$ in the coordinate basis. The Lie derivative (for the definition, see e.g. [Carroll, 2003]) along a vector field z^a is written as $\mathcal{L}_z$. Finally, the extrinsic curvature K_{ab} is defined using the standard NR sign convention [Alcubierre, 2008],

$$K_{ab} = -\gamma^c{}_a \nabla_c n_b \, . \tag{3.14}$$

In the current work there is no need to define any such connection variables associated with the upper case frame, since it will be used exclusively in an algebraic manner to simplify the various matrices that appear in the analysis. The key idea is that by using the DF formalism one may express the equations of motion in a Lagrangian frame that is, for fluid matter, in some sense preferred. This allows to exploit structure in the field equations that is otherwise not obvious. Consequently, the computation necessary to analyze hyperbolicity becomes relatively straightforward.

The various 3+1 quantities have now been defined. In application, lapse, shift, Christoffels, and extrinsic curvature will play only a minor role and are hidden in the derivative operators and/or appear in the source terms. With the 2+1 decomposition introduced next, it is even possible to write down the principal symbol only in terms of matter variables and the 2+1 quantities. By simply adjusting the normalization of the 2+1 quantities, i.e., taking the flat metric insead of g_{ab}, the analyses become applicable to both SR and GR.

2+1 Decomposition

As shown in section 2.2, for the PDE analysis a 2+1 decomposition of the two spatial projectors $\gamma^a{}_b$ and ${}^{(\mathrm{N})}\gamma^a{}_b$ against various arbitrary unit spatial vectors has to be performed. The spatial vectors and associated orthogonal projectors are collected in table 3.2.

	Upper case	Lower case	Lower case
Unit normal vector	N^a	n^a	n^a
Spatial 1-form	S_a	$\mathbb{s}_a$	s_a
Spatial vector	$S^a = {}^{(\mathrm{N})}\gamma^{ab} S_b$	$\hat{\mathbb{s}}^a = (\mathbb{g}^{-1})^{ab}\mathbb{s}_b$	$s^a = \gamma^{ab} s_b$
Norm	$S_a S^a = 1$	$\mathbb{s}_a (\mathbb{g}^{-1})^{ab}\mathbb{s}_b = 1$	$s_a s^a = 1$
Orthogonal projector	${}^{\mathrm{Q}}\!\perp^b{}_a = {}^{(\mathrm{N})}\gamma^b{}_a - S^b S_a$	${}^{\mathbb{q}}\!\perp^b{}_a = \gamma^b{}_a - \hat{\mathbb{s}}^b \mathbb{s}_a$	${}^{\mathrm{q}}\!\perp^b{}_a = \gamma^b{}_a - s^b s_a$
Index notation	${}^{\mathrm{Q}}\!\perp^B{}_A$	${}^{\mathbb{q}}\!\perp^{\mathbb{B}}{}_{\hat{\mathbb{A}}}$	${}^{\mathrm{q}}\!\perp^B{}_A$
Orthogonality	${}^{\mathrm{Q}}\!\perp^b{}_a S_b = 0$	${}^{\mathbb{q}}\!\perp^b{}_a \mathbb{s}_b = 0$	${}^{\mathrm{q}}\!\perp^b{}_a s_b = 0$
	${}^{\mathrm{Q}}\!\perp^b{}_a S^a = 0$	${}^{\mathbb{q}}\!\perp^b{}_a \hat{\mathbb{s}}^a = 0$	${}^{\mathrm{q}}\!\perp^b{}_a s^a = 0$

TABLE 3.2: Summary of the various unit spatial vectors and 1-forms appearing in the 2+1 decomposed equations, plus their associated projection operators.

Please note that $g_{cb}{}^{\mathbb{q}}\!\perp^b{}_a$ is *not* symmetric. To take account of this fact, it is distinguished between the abstract indices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\hat{\mathbb{A}}$, $\hat{\mathbb{B}}$, $\hat{\mathbb{C}}$ of ${}^{\mathbb{q}}\!\perp^{\mathbb{B}}{}_{\hat{\mathbb{A}}}$ when applied to a tensor. Relations between the upper and lower case 2+1 objects will be given later. To write mixed upper and lower case terms in a short way, the summation convention applies also for abstract capital letters of different type but same letter. For example, for upper and lower case vector fields Z^a and z^a, respectively, it is written $z_{\hat{\mathbb{B}}} Z^B \equiv z_a{}^{\mathbb{q}}\!\perp^a{}_{\hat{\mathbb{B}}}{}^{\mathrm{Q}}\!\perp^B{}_c Z^c \equiv z_a{}^{\mathbb{q}}\!\perp^a{}_b{}^{\mathrm{Q}}\!\perp^b{}_c Z^c$.

The first two columns of table 3.2 can be set into relation. The relations are summarized in table 3.3. Please note that there is freedom in the relation between the upper and lower spatial 1-forms S_a and $\mathbb{s}_a$. The freedom is fixed by the choice of both normalizations of the spatial vectors or by choosing one normalization and postulate a projection rule, in the present case $\mathbb{s}_a(\mathbb{g}^{-1})^{ab}\mathbb{s}_b = 1$ and $\mathbb{s}_a = \gamma^b{}_a S_b$.

	Upper case	Lower case
Unit normal vector	N^a	n^a
Boost vector	V^a	v^a
Spatial vector	$S^a = {}^{(\mathrm{N})}\gamma^{ab} S_b$	$\hat{\mathbb{s}}^a = (\mathbb{g}^{-1})^{ab}\mathbb{s}_b$
Spatial 1-form	$S_a = \mathbb{s}_a + v^{\mathbb{s}} n_a$	$\mathbb{s}_a = S_a + W^2 V^S (N_a + V_a)$
	$= {}^{(\mathrm{N})}\gamma_{ab} (\mathbb{g}^{-1})^{bc}\mathbb{s}_c$	$= {}^{(\mathrm{N})}\mathbb{g}_{ab} S^b + W^2 V^S N_a$
	$= {}^{(\mathrm{N})}\gamma_{ab}\hat{\mathbb{s}}^b$	$= \gamma^b{}_a S_b$
Normalization	$S_a{}^{(\mathrm{N})}\gamma^{ab} S_b = 1$	$\mathbb{s}_a (\mathbb{g}^{-1})^{ab}\mathbb{s}_b = 1$

TABLE 3.3: The relation between upper and lower case unit spatial vectors and 1-forms.

PDE Notation and Characteristic Analysis

It is started with a four-dimensional formulation of a quasi-linear first order system of the form

$$\boldsymbol{\mathcal{A}}^a \partial_a \mathbf{U} + \boldsymbol{\mathcal{S}} = 0\,, \tag{3.15}$$

which may be 3+1 decomposed against N^a or n^a by using the identity $\delta^b{}_a = {}^{(\mathrm{N})}\gamma^b{}_a - N^b N_a = \gamma^b{}_a - n^b n_a$. Placing these identities between $\boldsymbol{\mathcal{A}}^a$ and the derivative operator ∂_a, one arrives at the two evolution systems for $\mathbf{U}$ in terms of the timelike normals n^a and N^a:

$$\begin{aligned} \boldsymbol{\mathcal{A}}^n \partial_n \mathbf{U} =& \boldsymbol{\mathcal{A}}^a \gamma^b{}_a \partial_b \mathbf{U} + \boldsymbol{\mathcal{S}}\,, \\ \boldsymbol{\mathcal{A}}^N \partial_N \mathbf{U} =& \boldsymbol{\mathcal{A}}^{a\,(\mathrm{N})}\gamma^b{}_a \partial_b \mathbf{U} + \boldsymbol{\mathcal{S}}\,. \end{aligned} \tag{3.16}$$

To denote clearly the properties of the coefficient matrices, the following definitions are made:

$$\begin{aligned} &\boldsymbol{\mathcal{A}}^n \equiv \mathbf{A}^{\mathrm{n}}, \qquad && \boldsymbol{\mathcal{A}}^a \gamma^b{}_a \equiv \mathbf{A}^b, \qquad && \mathbf{A}^b n_b = 0\,, \\ &\boldsymbol{\mathcal{A}}^N \equiv \mathbf{B}^{\mathrm{N}}, \qquad && \boldsymbol{\mathcal{A}}^{a\,(\mathrm{N})}\gamma^b{}_a \equiv \mathbf{B}^b, \qquad && \mathbf{B}^b N_b = 0\,. \end{aligned} \tag{3.17}$$

Note that by definition $\mathbf{A}^{\mathrm{n}} \neq \mathbf{A}^n = \mathbf{A}^b n_b = 0$ holds[1] and in an analogous way for the upper case matrices $\mathbf{B}^{\mathrm{N}}$ and $\mathbf{B}^N$. Let $\mathfrak{s}_a$ be an arbitrary lower case spatial 1-form against n^a, normalized with respect to the inverse boost metric,

$$\mathfrak{s}_a n^a = 0\,, \qquad \mathfrak{s}_a (\mathrm{g}^{-1})^{ab} \mathfrak{s}_b = 1, \tag{3.18}$$

and let S^a be an arbitrary unit upper case spatial vector against N^a,

$$S^a S_a = 1\,, \qquad S^a N_a = 0. \tag{3.19}$$

The eigenvalue problems of these systems in direction $\mathfrak{s}_a$ and S_a read

$$\begin{aligned} \mathbf{l}^{\mathrm{n}}_{\lambda}((\mathbf{A}^{\mathrm{n}})^{-1}\mathbf{A}^{\mathfrak{s}} - \mathbb{1}\lambda) = 0\,, \\ \mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}((\mathbf{B}^{\mathrm{N}})^{-1}\mathbf{B}^{S} - \mathbb{1}\lambda_{\mathrm{N}}) = 0\,, \end{aligned} \tag{3.20}$$

with principal symbols $(\mathbf{A}^{\mathrm{n}})^{-1}\mathbf{A}^{\mathfrak{s}}$ and $(\mathbf{B}^{\mathrm{N}})^{-1}\mathbf{B}^{S}$, left eigenvectors $\mathbf{l}^{\mathrm{n}}_{\lambda}$ and $\mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}$, and eigenvalues λ and λ_{N} for lower and upper case, respectively. Note that no sub-/superscript n is

[1] Note the difference between roman superscript and the *italic* index.

placed on the lower case eigenvalues. In general, the eigenvalues will depend on the spatial vector chosen for the 2+1 decomposition to obtain the principal symbols. Sometimes the dependence on spatial vectors will be explicitly indicated by use of square brackets. However, square brackets also serve as an alternative to round brackets. With the help of the four-vectors $\boldsymbol{\phi}^a$, $\bar{\boldsymbol{\phi}}^a$ the eigenvalue problems in (3.20) may be written as

$$\begin{aligned} \mathbf{l}^{\mathrm{n}}_{\lambda}\boldsymbol{\mathcal{A}}^a\boldsymbol{\phi}_a &= 0, & \boldsymbol{\phi}_a &= -\lambda n_a + \mathfrak{s}_a\,, \\ \mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}\boldsymbol{\mathcal{A}}^a\bar{\boldsymbol{\phi}}_a &= 0, & \bar{\boldsymbol{\phi}}_a &= -\lambda_{\mathrm{N}} N_a + S_a. \end{aligned} \tag{3.21}$$

This covariant notation is commonly used in the literature. In equation (3.48) the connection between upper and lower case characteristic analysis is given, where $\boldsymbol{\phi}_a$ is expressed in terms of upper case quantities.

This completes the introduction of the main concept and definitions of the DF formalism. The important property of frame independence of strong hyperbolicity is treated in the next section.

3.2 Frame Independence of Strong Hyperbolicity

After defining the lower and upper case frames in the last section, the relation between both frames in the context of PDE systems is studied in the following. In [Hilditch, 2015] it is shown that strong hyperbolicity is unaffected by a switch of coordinates provided that the boost vector is sufficiently small. Following this result it is proven below that strong hyperbolicity is independent of the choice of frame provided that the boost vector satisfies a specific estimate that depends on the maximum absolute eigenvalue of the system.

First, the system of equations for the state vector $\mathbf{U}$ in the upper case frame is considered,

$$\partial_N\mathbf{U} = \mathbf{B}^p\partial_p\mathbf{U} + \boldsymbol{\mathcal{S}}\,, \tag{3.22}$$

and is assumed to be strongly hyperbolic, so that there is a complete set of upper case (left) eigenvectors in all upper case spatial directions. The upper case PDE system (3.22) will now be expressed in terms of lower case quantities. On the left-hand side of equation (3.22), the upper case unit normal is expressed as $N^a = Wn^a + \hat{v}^a$. On the right-hand side, the intrinsically upper case spatial matrix $\mathbf{B}^p$ is split as

$$\mathbf{B}^p = \mathbf{B}^a g^p{}_a = \mathbf{B}^a\gamma^p{}_a - \mathbf{B}^n n^p = \mathbf{B}^a\gamma^p{}_a - W\mathbf{B}^V n^p, \tag{3.23}$$

where in the third step the lower case normal was written in terms of N^a and V^a, and indices n, V were used for contractions with the respective 1-form. Sorting the terms in a lower case sense and adding $0 = \mathbf{B}^a V_a \hat{v}^p - \mathbf{B}^V \hat{v}^p$ on the right-hand side, one arrives at the lower case PDE system

$$W\left(\mathbb{1} + \mathbf{B}^V\right)\partial_n \mathbf{U} = \left[\mathbf{B}^a\left(\gamma^p{}_a + \hat{v}^p V_a\right) - \left(\mathbb{1} + \mathbf{B}^V\right)\hat{v}^p\right]\partial_p \mathbf{U} + \boldsymbol{\mathcal{S}}\,. \tag{3.24}$$

First, the question of invertibility of $\mathbf{A}^{\mathrm{n}} = W\left(\mathbb{1} + \mathbf{B}^V\right)$ needs to be addressed. Let the upper case boost vector be written as $V^a = |V| S_V^a$ with norm $|V| = (V^a V_a)^{1/2}$ and upper case unit spatial vector S_V^a in the direction of V^a. Since $\mathbf{B}^{S_V}$ is diagonalizable with diagonal form $\boldsymbol{\Lambda}^{S_V}$, it has a complete set of right eigenvectors written as columns in the matrix $\mathbf{T}_{S_V}$, and $\mathbf{T}_{S_V}$ is invertible. Performing a similarity transformation one obtains

$$\left(\mathbf{T}_{S_V}\right)^{-1}\left(\mathbb{1} + \mathbf{B}^V\right)\mathbf{T}_{S_V} = \mathbb{1} + |V|\boldsymbol{\Lambda}^{S_V} \tag{3.25}$$

and invertibility of $(\mathbb{1} + \mathbf{B}^V)$ is guaranteed if for each eigenvalue $\lambda_{\mathrm{N}}[S_V^a]$ the inequality

$$1 + |V|\lambda_{\mathrm{N}}[S_V^a] > 0 \tag{3.26}$$

for arbitrary upper case unit spatial S_V^a holds. This condition will be guaranteed by a more restrictive assumption in the proof that follows. Consequently, the PDE system (3.24) is written as

$$\partial_n \mathbf{U} = \frac{1}{W}\left(\mathbb{1} + \mathbf{B}^V\right)^{-1}\left[\mathbf{B}^a\left(\gamma^p{}_a + \hat{v}^p V_a\right) - \left(\mathbb{1} + \mathbf{B}^V\right)\hat{v}^p\right]\partial_p \mathbf{U} + \frac{1}{W}\left(\mathbb{1} + \mathbf{B}^V\right)^{-1}\boldsymbol{\mathcal{S}}\,. \tag{3.27}$$

Let S^a be an arbitrary unit upper case spatial vector. The eigenvalue problem in direction S^a corresponding to the PDE system (3.22) in the upper frame reads

$$\mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}\left[\mathbf{B}^S - \mathbb{1}\lambda_{\mathrm{N}}[S^a]\right] = 0\,, \tag{3.28}$$

where $\mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}$ is the upper case left eigenvector for the principal symbol $\mathbf{B}^S$ with eigenvalue $\lambda_{\mathrm{N}}[S^a]$. The eigenvalue problem for direction $\mathbb{s}_a$ in the lower frame for the PDE system (3.27) is

$$\mathbf{l}^{\mathrm{n}}_{\lambda}\frac{1}{W}\left(\mathbb{1} + \mathbf{B}^V\right)^{-1}\left[\mathbf{B}^a\left(\gamma^p{}_a + \hat{v}^p V_a\right) - \left(\mathbb{1} + \mathbf{B}^V\right)\hat{v}^p\right]\mathbb{s}_p = \lambda[\mathbb{s}_a]\mathbf{l}^{\mathrm{n}}_{\lambda} \tag{3.29}$$

and, using ${}^{(N)}\gamma^b{}_a\,(\gamma^p{}_b + \hat{v}^p V_b)\,\mathfrak{s}_p = S_a$, may be written as

$$\mathbf{l}^{\mathrm{n}}_\lambda(\mathbb{1}+\mathbf{B}^V)^{-1}\left[\mathbf{B}^S - (\mathbb{1}+\mathbf{B}^V)(\hat{v}^{\mathfrak{s}} + W\lambda)\right] = 0\,, \tag{3.30}$$

for lower case left eigenvector $\mathbf{l}^{\mathrm{n}}_\lambda$ with eigenvalue $\lambda = \lambda[\mathfrak{s}_a]$. The associated principal symbol is

$$\mathbf{P}^{\mathfrak{s}} = \frac{1}{W}\left[(\mathbb{1}+\mathbf{B}^V)^{-1}\mathbf{B}^S - \mathbb{1}\hat{v}^{\mathfrak{s}}\right] \tag{3.31}$$

and the lower case spatial 1-form $\mathfrak{s}_a$ is related to the upper case one by $\mathfrak{s}_a = \gamma^b{}_a S_b = S_a + W^2 V^S(N_a + V_a)$, see also table 3.3.

Introducing the modified lower case left eigenvector $\mathbf{L}^{\mathrm{n}}_\lambda = \mathbf{l}^{\mathrm{n}}_\lambda(\mathbb{1}+\mathbf{B}^V)^{-1}$ and collecting terms including $\mathbf{B}$, equation (3.30) is cast into the form

$$\mathbf{L}^{\mathrm{n}}_\lambda\left[\mathbf{B}^{S-V(\hat{v}^{\mathfrak{s}}+W\lambda)} - \mathbb{1}(\hat{v}^{\mathfrak{s}}+W\lambda)\right] = 0\,, \qquad \mathbf{B}^{S-V(\hat{v}^{\mathfrak{s}}+W\lambda)} \equiv \mathbf{B}^a(S_a - V_a(\hat{v}^{\mathfrak{s}}+W\lambda))\,. \tag{3.32}$$

By defining the new upper case unit spatial vector

$$S^a_\lambda[S^b,\lambda] := \frac{1}{N}\left(S^a - V^a(\hat{v}^{\mathfrak{s}}+W\lambda)\right)\,, \tag{3.33}$$

with normalization

$$\begin{aligned} N &= \left[(S^a - V^a(\hat{v}^{\mathfrak{s}}+W\lambda))\,(S_a - V_a(\hat{v}^{\mathfrak{s}}+W\lambda))\right]^{1/2} \\ &= \sqrt{W^2(\lambda+v^{\mathfrak{s}})^2 + 1 + (v^{\mathfrak{s}})^2 - \lambda^2} \\ &= \sqrt{W^2(\lambda - WV^S)^2 + 1 + (V^S)^2W^2 - \lambda^2}\,, \end{aligned} \tag{3.34}$$

the eigenvalue problem in the lower case finally reads

$$\mathbf{L}^{\mathrm{n}}_\lambda\left[\mathbf{B}^{S_\lambda} - \mathbb{1}\frac{1}{N}(\hat{v}^{\mathfrak{s}}+W\lambda)\right] = 0\,, \tag{3.35}$$

for the redefined lower case left eigenvector $\mathbf{L}^{\mathrm{n}}_\lambda$, principal symbol $\mathbf{B}^{S_\lambda} \equiv \mathbf{B}^a S^\lambda_a$, and eigenvalue $(\hat{v}^{\mathfrak{s}}+W\lambda)/N$ in direction of S^a_λ. The equality $WV^S = -v^{\mathfrak{s}}$ follows by using relations given in tables 3.1 and 3.3. The lower case eigenvalue problem (3.35) for fixed λ is the same eigenvalue problem as for the upper case for eigenvalue $(\hat{v}^{\mathfrak{s}}+W\lambda)/N$ in (3.28) where the spatial direction S^a is replaced by S^a_λ. Therefore,

$$\frac{1}{N}(\hat{v}^{\mathfrak{s}} + W\lambda) = \lambda_{\mathrm{N}}[S^a_\lambda] \tag{3.36}$$

must hold for fixed eigenvalue $\lambda = \lambda[\mathfrak{s}_a]$.

Equation (3.36) is a strong result, since it enables to calculate the lower case frame eigenvalues from knowledge of the upper case results. Nevertheless, solving for λ may be difficult, since both N and λ_{N} contain polynomials in λ. The lower case left eigenvector to eigenvalue λ is then simply given by

$$\mathbf{l}^{\mathrm{n}}_\lambda[\mathfrak{s}_b] = \mathbf{l}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}[S^a_\lambda]\left(\mathbb{1} + \mathbf{B}^V\right), \tag{3.37}$$

and the right eigenvectors by

$$\mathbf{r}^{\mathrm{n}}_\lambda[\mathfrak{s}_b] = \mathbf{r}^{\mathrm{N}}_{\lambda_{\mathrm{N}}}[S^a_\lambda]\,. \tag{3.38}$$

Proof of Frame Independence of Strong Hyperbolicity

The proof is as follows: It is known that for arbitrary unit upper case spatial S^a, unit with respect to g_{ab}, the upper case principal symbol $\mathbf{P}^S$ has:

(1) real eigenvalues $\lambda_{\mathrm{N}}[S^a]$,

(2) a complete set of left and right eigenvectors obeying $|\mathbf{T}_S| + |\mathbf{T}_S^{-1}| \le K$, where $\mathbf{T}_S$ is the matrix of right (or left) eigenvectors written as columns (or rows), and K is independent of S^a.

Furthermore it is assumed that:

(3) all upper case eigenvalues satisfy the inequality $1 - |\lambda_{\mathrm{N}}||V| > 0$, for all upper case unit spatial S^a. This assumption automatically guarantees the condition (3.26) for the invertibility of $(\mathbb{1} + \mathbf{B}^V)$.

The lower case eigenvalues are real. First, it is shown that the lower case system is at least weakly hyperbolic. Using equation (3.36) with normalization factor (3.34), one obtains

$$\lambda = \frac{W^3V^S(1-\lambda_{\mathrm{N}}^2) + \lambda_{\mathrm{N}}W\sqrt{1+\lambda_{\mathrm{N}}^2(1/W^2 - 1 + (V^S)^2)}}{W^2(1-\lambda_{\mathrm{N}}^2(1-1/W^2))} \tag{3.39}$$

for given $\lambda_{\rm N}$. One has to beware that the terms inside the square root are negative, but can estimate them from below,

$$1+\lambda_{\rm N}^2\left(1/W^2-1+(V^S)^2\right)\geq 1+\lambda_{\rm N}^2\left(1/W^2-1\right)=1-\lambda_{\rm N}^2|V|^2>0\,, \tag{3.40}$$

where assumptions (1) and (3) were used. Therefore, all lower case eigenvalues are real. Additionally, it is thereby shown that the denominator of (3.39) can never become zero.

The lower case eigenvectors are linearly independent. Take a lower case eigenvalue λ with algebraic multiplicity k. Then, by equation (3.36), the corresponding upper case eigenvalue $\lambda_{\rm N}[S^a_\lambda]$ has also algebraic multiplicity k. Thus, by assumptions (2), which ensures that one can find k linearly independent eigenvectors to the associated eigenvalue problem (3.35), and (3), which guarantees the invertibility of $(\mathbb{1}+\mathbf{B}^V)$, and the use of equation (3.37), it is known that one can find k linearly independent lower case left eigenvectors in the eigenspace of λ. This statement holds also for the right eigenvectors. Therefore, the lower case principal symbol is diagonalizable.

Show necessary regularity conditions. The left and right eigenvectors and eigenvalues are now labeled, making duplicates to account for their multiplicity if necessary, with an index[2], writing $\mathbf{l}_{\lambda_{(i)}}$, $\mathbf{r}_{\lambda_{(i)}}$ and $\lambda_{(i)}$, respectively. The matrix of lower case right eigenvectors is denoted by $\mathbf{T}_{\$}$, where the i-th column of $\mathbf{T}_{\$}$ is $\mathbf{r}_{\lambda_{(i)}}$. It is ordered in a way such that the i-th row of $\mathbf{T}_{\$}^{-1}$ is $\mathbf{l}_{\lambda_{(i)}}$. Thus, $\mathbf{l}_{\lambda_{(i)}}\mathbf{r}_{\lambda_{(j)}}=\delta_{(ij)}$ does hold.[3] By equations (3.37) and (3.38), one can express for each i the lower case eigenvectors $\mathbf{l}_{\lambda_{(i)}},\mathbf{r}_{\lambda_{(i)}}$ by $\mathbf{l}^{\rm N}_{\lambda^{\rm N}_{(i)}}[S^a_{\lambda_{(i)}}]\left(\mathbb{1}+\mathbf{B}^V\right)$ and $\mathbf{r}^{\rm N}_{\lambda^{\rm N}_{(i)}}[S^a_{\lambda_{(i)}}]$, respectively. The upper case principal symbol is diagonalizable by assumption (2), so for each i the corresponding left or right eigenvector can be extended by the remaining linearly independent eigenvectors of the upper case principal symbol for spatial vector $S^a_{\lambda_{(i)}}$. The matrices of those completed sets of eigenvectors expanding the chosen $\mathbf{r}^{\rm N}_{\lambda^{\rm N}_{(i)}}[S^a_{\lambda_{(i)}}]$ (and $\mathbf{l}^{\rm N}_{\lambda^{\rm N}_{(i)}}[S^a_{\lambda_{(i)}}]$) written as columns (rows), are denoted by $\mathbf{T}_{S_{\lambda_{(i)}}}$. The chosen i-th right (left) eigenvector is placed in the i-th column (row). By assumption (2) it follows then

$$|\mathbf{T}^{-1}_{S_{\lambda_{(i)}}}|+|\mathbf{T}_{S_{\lambda_{(i)}}}|\leq K_{(i)} \tag{3.41}$$

for each i.

[2] Only in the current proof, indices i,j are not taken to be spatial spacetime indices. This is indicated by placing round brackets around them.

[3] Each left and right eigenvector is defined up to a scalar factor. It is implicitly assumed that these factors are chosen such that the expression is valid.

Next, define the square diagonal quadratic matrices $\mathbf{D}_{(i)}$, which have in the $i-$th entry of their diagonal a 1 and otherwise zeros,

$$\mathbf{D}_{(i)} := \mathrm{diag}(\underbrace{0,\ldots,0}_{i-1 \text{ times}},1,0,\ldots,0)\,, \qquad \sum_i \mathbf{D}_{(i)} = \mathbb{1}.$$

Their norm is $|\mathbf{D}_{(i)}| = \max\limits_{|\mathbf{y}|=1} |\mathbf{y}_{(i)}| = 1$ with the i-th component $\mathbf{y}_{(i)}$ of $\mathbf{y}$. Then, with the above definitions,

$$\begin{aligned} \mathbf{T}_{\$} &= \sum_i \mathbf{T}_{S_{\lambda_{(i)}}} \mathbf{D}_{(i)}\,, \\ \mathbf{T}_{\$}^{-1} &= \sum_i \mathbf{D}_{(i)} \mathbf{T}^{-1}_{S_{\lambda_{(i)}}} \left(\mathbb{1} + \mathbf{B}^V\right), \end{aligned} \tag{3.42}$$

and one can give the estimate:

$$\begin{aligned} |\mathbf{T}_{\$}^{-1}| + |\mathbf{T}_{\$}| &\leq \sum_i \left(|\mathbf{T}^{-1}_{S_{\lambda_{(i)}}}||\mathbb{1} + \mathbf{B}^V| + |\mathbf{T}_{S_{\lambda_{(i)}}}| \right) \\ &\leq \sum_i \left(|\mathbf{T}^{-1}_{S_{\lambda_{(i)}}}| + |\mathbf{T}_{S_{\lambda_{(i)}}}| \right) \max\{1, |\mathbb{1} + \mathbf{B}^V|\} \\ &\leq \sum_i K_{(i)} \max\{1, |\mathbb{1} + \mathbf{B}^V|\} \equiv K. \end{aligned} \tag{3.43}$$

In the first step, the expressions for the matrices (3.42) were inserted and the sub-multiplicity of the norm was used. In the second step, the prefactors were estimated and finally, in the last step, the assumption (2) given by (3.41) was used for each i. Hence, the inequality (2.11) is obtained, which together with the properties above gives strong hyperbolicity in the lower case frame and completes the proof.

Multiplicity and Degeneracies

The definition of strong hyperbolicity does not require a constant multiplicity of the eigenvalues as the spatial direction is varied. In the literature on relativistic fluids, special cases in which the algebraic multiplicity of a particular eigenvalue increases when looking in particular special directions are called *degeneracies* or *degenerate states* of the system. All such possible degeneracies must be taken into account in the demonstration of strong hyperbolicity, since diagonalizability of the principal symbol is required in all directions. The relation between the occurrence of degeneracies in the upper case and lower case systems is, however, not trivial. The key point is that when transforming from the lower case system to the associated upper case eigenvalue problem (3.35), the latter one is considered only for a fixed eigenvalue. Different eigenvalues are naturally assigned to *different* upper case eigenvalue problems. Therefore, it may be the case that, for example, upper case degeneracies always occur in pairs, whilst the same is not true in the lower case frame. Indeed, this is the case for a particular formulation of GRMHD. The relation between the degeneracies plays no role in the foregoing proof of the equivalence of strong hyperbolicity across the two frames.

Validity of Frame Independence in the Context of Relativistic Fluid Models

By construction, all studied relativistic systems possess boost velocities v_a smaller than the speed of light. Similarly, the EOS is confined in a way, at least in the present work, such that the eigenvalues of the principal symbol[4], called the *wave speeds*, are always smaller than or equal to the speed of light, and thus the waves are supposed to be *subluminal*. The confinement of the EOS is reasonable in this work, since only relativistic fluid models are considered. However, theories with gauge freedom such as GR and electromagnetism, do admit hyperbolic formulations with superluminal speeds. For those PDE systems with constraints, addition of multiples of constraints can influence the characteristic structure of the principal symbol, such that the constraint eigenvalues become *superluminal*. In that case when the boost vector becomes too large, upper case strong hyperbolicity will not be sufficient to guarantee strong hyperbolicity in the lower case frame, since the crucial inequality $|\lambda_{\mathrm{N}}||V| < 1$ can be violated. In fact, the systems of GRMHD and RGRMHD are constrained systems and underlie this subtlety. For the considered (constrained) evolution systems in this work, however, the waves speeds are always subluminal.

3.3 Recovering the Eigenvalues and Eigenvectors of the Lower Case Frame

In the current section it is explained how the above results are used in the application. As mentioned before, the upper frame will be chosen as the frame of a comoving observer with the fluid, so from now on,

$$N^a \equiv u^a, \qquad {}^{(\mathrm{u})}\gamma^a{}_b = g^a{}_b + u^a u_b\,, \tag{3.44}$$

[4]For a unit spatial vector. Unit means here that the spatial vector in the 2+1 decomposition is normalized with respect to the four-metric of the spacetime or, using orthogonality relations, against the induced metric. The vectors s^a and S^a are examples, whereas the vector $\hat{\mathfrak{s}}^a$ is a counter example, see table 3.2 for their normalization.

is adopted, with the four-velocity of the fluid u^a. Despite the fact that u^a defines the fluid frame or so-called Lagrangian frame, the boost vectors are never set to zero. As will be seen, the characteristic analysis for GRHD and GRMHD is much easier if performed in the fluid frame, which justifies the approach to take the upper case frame as the 'known' frame.

Since most of the results are obtained using computer algebra, it is appropriate to introduce a basis to obtain scalar quantities as entries in the matrices. The various basis vectors are collected in table 3.4.

	Upper case		Lower case	
Unit normal vector	N^a	N^a	n^a	n^a
Spatial 1-form	S^λ_a	S_a	$\mathbb{s}_a$	s_a
Spatial vector	S^a_λ	S^a	$\hat{\mathbb{s}}^a$	s^a
Orthogonal 1-forms	$Q_1{}^\lambda_a, Q_2{}^\lambda_a$	Q_{1a}, Q_{2a}	$\mathbb{q}_{1a}, \mathbb{q}_{2a}$	q_{1a}, q_{2a}
Orthogonal vectors	$Q_1{}^a_\lambda, Q_2{}^a_\lambda$	$Q_1{}^a, Q_2{}^a$	$\hat{\mathbb{q}}^a_1, \hat{\mathbb{q}}^a_2$	$q_1{}^a, q_2{}^a$
Basis abbreviation	$\mathbf{S}_\lambda = \{S^a_\lambda, Q_1{}^a_\lambda, Q_2{}^a_\lambda\}$	$\mathbf{S} = \{S^a, Q_1{}^a, Q_2{}^a\}$		

TABLE 3.4: Overview of the upper and lower case basis vectors.

The spatial vectors $s^a, q_1{}^a, q_2{}^a$ form a right-handed orthogonal system. Additionally, the spatial vectors $q_1{}^a, q_2{}^a$ are normalized in the same way as their associated s^a vector. The other bases are supposed to behave in the same way. The orthogonal projectors of the 2+1 decomposition can thus be 1+1 decomposed, and the orthogonal projectors may be expressed via the orthogonal system of vectors and 1-forms, e.g., ${}^{\mathbb{q}}\!\perp^a{}_b = \hat{\mathbb{q}}^a_1 \mathbb{q}_{1b} + \hat{\mathbb{q}}^a_2 \mathbb{q}_{2b}$. The relation between the orthogonal lower and upper case spatial 1-forms (and vectors) are the same as for their associated spatial 1-forms (and vectors), namely $\mathbb{s}_a = \gamma^b{}_a S_b$, and vice versa, see table 3.3. Then it is possible to write $(\mathbb{g}^{-1})^{ac}\,{}^{Q}\!\perp_{cd}\,\gamma^d{}_b = (\mathbb{g}^{-1})^{ac}\,(Q_{1c}Q_{1d} + Q_{2c}Q_{2d})\,\gamma^d{}_b = \hat{\mathbb{q}}^a_1 \mathbb{q}_{1b} + \hat{\mathbb{q}}^a_2 \mathbb{q}_{2b} = {}^{\mathbb{q}}\!\perp^a{}_b$. For upper and lower vector fields Z^a, z^a, one also finds:

$$
\begin{aligned}
{}^{Q}\!\perp^{A}{}_{b} Z_A &= Z_{Q_1} Q_{1b} + Z_{Q_2} Q_{2b}\,, & {}^{Q}\!\perp^{b}{}_{A} Z^A &= Z^{Q_1} Q_1{}^b + Z^{Q_2} Q_2{}^b\,,\\
{}^{\mathbb{q}}\!\perp^{\mathbb{A}}{}_{b} z_{\hat{\mathbb{A}}} &= z_{\hat{\mathbb{q}}_1} \mathbb{q}_{1b} + z_{\hat{\mathbb{q}}_2} \mathbb{q}_{2b}\,, & {}^{\mathbb{q}}\!\perp^{b}{}_{\hat{\mathbb{A}}} z^{\mathbb{A}} &= z^{\mathbb{q}_1} \hat{\mathbb{q}}^b_1 + z^{\mathbb{q}_2} \hat{\mathbb{q}}^b_2\,,\\
Z_A z^{\mathbb{A}} &= Z_{Q_1} z^{\mathbb{q}_1} + Z_{Q_2} z^{\mathbb{q}_2}\,, & Z^A z_{\hat{\mathbb{A}}} &= Z^{Q_1} z_{\hat{\mathbb{q}}_1} + Z^{Q_2} z_{\hat{\mathbb{q}}_2}\,.
\end{aligned} \tag{3.45}
$$

Consider a strongly hyperbolic system of PDEs as in (3.22) with $N^a \equiv u^a$, which is 2+1 decomposed by an arbitrary unit upper case spatial vector S^a with known eigenvalues $\lambda_{\mathrm{u}}[S^a]$ and a full set of left eigenvectors $\mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a]$ obtained by (3.28) and right eigenvectors $\mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a]$. Then the lower case eigenvalues are given by equation (3.36) and the lower case left eigenvectors $\mathbf{l}^{\mathrm{n}}_{\lambda}$ for eigenvalue λ are given by equation (3.37). For a particular choice of a basis they can be obtained according to

$$
\begin{aligned}
\mathbf{l}^{\mathrm{n}}_{\lambda}\big|_{\mathbb{s}} &= \mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]\big|_{\mathbf{S}} \left(\mathbb{1} + \mathbf{B}^V\big|_{\mathbf{S}}\right) \\
&= \mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]\big|_{\mathbf{S}_\lambda} \mathbf{T}_\lambda \left(\mathbb{1} + \mathbf{B}^V\big|_{\mathbf{S}}\right) = \mathbf{l}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]\big|_{\mathbf{S}_\lambda} \left(\mathbb{1} + \mathbf{B}^V\big|_{\mathbf{S}_\lambda}\right) \mathbf{T}_\lambda \,.
\end{aligned} \tag{3.46}
$$

The lower case right eigenvectors $\mathbf{r}^{\mathrm{n}}_{\lambda}$ can be calculated via

$$
\begin{aligned}
\mathbf{r}^{\mathrm{n}}_{\lambda}\big|_{\mathbb{s}} &= \mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]\big|_{\mathbf{S}} \\
&= (\mathbf{T}_\lambda)^{-1}\, \mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]\big|_{\mathbf{S}_\lambda} \,,
\end{aligned} \tag{3.47}
$$

for a given upper case right eigenvector $\mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a_\lambda]$. The transformation matrix is denoted by $\mathbf{T}_\lambda$ and transforms between bases associated to S^a and S^a_λ on the level of eigenvectors and matrices.

There exist two ways to obtain the lower eigenvectors: Either the upper case principal symbol $\mathbf{B}^{S_\lambda}\big|_{\mathbf{S}}$ in a basis associated to S^a is taken and for given $\lambda_{\mathrm{u}}[S^a_\lambda]$ the new upper case eigenvectors are calculated or, the upper case eigenvectors to $\mathbf{B}^{S}\big|_{\mathbf{S}}$ in a basis associated to S^a are considered and the replacement $\boldsymbol{S} \to \boldsymbol{S}_\lambda = \{S^a_\lambda, {Q_1}^a_\lambda, {Q_2}^a_\lambda\}$ is made which naturally defines a SO(3)-transformation $\mathbf{R}$. Using the first way, the left and right eigenvectors are given by the formulas in the first line of equations (3.46) and (3.47). However, the principal symbol might lose its easy form which could be especially crucial for a high number of evolved variables. Therefore, the second procedure is chosen in the accompanying notebooks, where the second lines of equations (3.46) and (3.47) are used to obtain the lower eigenvectors.

The recovery is explained in more detail for the system of GRMHD in chapter 5 and is performed in the provided notebook. For the analysis of GRHD the procedure is only given in the corresponding notebook, but not in this thesis. For the notebooks see appendix A.

For the sake of clarity, all the explanations are related to the covariant form of characteristic analysis, using the four-vector $\boldsymbol{\phi}_a$ and the eigenvalue problem as in (3.21). Taking the four-vector of the form $\boldsymbol{\phi}_a = -\lambda n_a + \mathbb{s}_a$ with $\lambda = \lambda[\mathbb{s}_b]$ and writing the lower case vectors in terms of u^a, V^a, and $\mathbb{s}_a = S_a + W^2 V^S (u_a + V_a)$, one obtains

$$
\begin{aligned}
\boldsymbol{\phi}_a &= -\lambda n_a + \mathbb{s}_a \\
&= -\lambda (W u_a + W V_a) + S_a + W^2 V^S (u_a + V_a) \\
&= (W^2 V^S - W\lambda) u_a + S^a + (W^2 V^S - W\lambda) V^a \\
&= N\left(-\lambda_{\mathrm{u}}[S^b_\lambda] u_a + S^\lambda_a\right) \propto -\lambda_{\mathrm{u}}[S^b_\lambda] u_a + S^\lambda_a \,.
\end{aligned} \tag{3.48}
$$

The last step is valid since $\boldsymbol{\phi}_a$ is defined up to an arbitrary scalar factor and here always upper case unit spatial vectors, normalized with respect to the spacetime metric, are considered for the characteristic analysis.

3.4 Variable Independence of Strong Hyperbolicity

Finally, the hyperbolicity properties under a change of variables are investigated. Let $\mathbf{U}$ be a state vector for which the principal symbol $\mathbf{P}^s_{\mathbf{U}}$ is diagonalizable for each unit spatial 1-form s_a. Let $\mathbf{V}$ be another state vector of the same dimension whose components depend smoothly on the components of $\mathbf{U}$. Derivatives of the two state vectors are then related by the invertible Jacobian $\mathbf{J}$,

$$\partial_a \mathbf{V} = \mathbf{J}\partial_a \mathbf{U}\,. \tag{3.49}$$

The principal symbol for $\mathbf{V}$ is then

$$\mathbf{P}^s_{\mathbf{V}} = \mathbf{J}\mathbf{P}^s_{\mathbf{U}}\mathbf{J}^{-1}\,. \tag{3.50}$$

Since this transformation is nothing but a similarity transformation, the eigenvalues remain the same and the (left) right eigenvectors for $\mathbf{V}$ are just modified by a matrix multiplication with the (inverse) Jacobian: $\mathbf{l}_{\mathbf{V}} = \mathbf{l}_{\mathbf{U}}\mathbf{J}^{-1}$; $\mathbf{r}_{\mathbf{V}} = \mathbf{J}\mathbf{r}_{\mathbf{U}}$.

Thus, as is well-known, strong hyperbolicity is independent of the choice of evolved variables as long as the aforementioned assumptions are satisfied. For the hyperbolicity analysis of a set of evolution equations, a suitable choice of variables can simplify practical computations considerably.

All basics and tools necessary to treat PDE systems used to describe neutron stars in NR are now provided. As a first example, the well studied system of GRHD using the DF formalism is considered in the next chapter.

Chapter 4

Hyperbolicity Analysis of Ideal Hydrodynamics

In the following chapters, the hyperbolicity analysis is applied to numerically relevant PDE systems of fluid models with the help of the results of the previous chapters, starting with the investigation of the system of an ideal fluid (also called perfect fluid). The ideal fluid model is well studied in the literature. A full characteristic analysis of the numerically used set of equations has been given in [Font et al., 1994] (see also [Anile, 1990] for an augmented system) and several authors have (re-) done the characteristic analysis for (the same or) a different set of evolution variables (e.g., [Donat et al., 1998]) and numerical applications were given. Other properties of the numerically used set of evolution equations were also studied in the past, e.g., a convexity analysis is given by [Ibáñez et al., 2013]. For a detailed review of the system of GRHD in NR see for example the Living Reviews of [Martí and Müller, 2003] as well as [Font, 2003; Font, 2008].

Due to the variety of analyses in the literature, the following calculations serve first as a sanity check of the DF formalism and second as a proof of principle that the DF approach to the analysis results in an economic treatment. First, the relevant quantities and equations are shown.

4.1 The PDE System of GRHD

The energy-momentum tensor for the model of GRHD has the form

$$T^{ab} = \rho_0 h u^a u^b + p g^{ab} \,, \tag{4.1}$$

with the four-velocity of the fluid elements u^a, the rest mass density ρ_0, the specific enthalpy h, and the pressure p. The specific enthalpy h can be expressed in terms of ρ_0, p and the specific internal energy ε as

$$h = 1 + \varepsilon + \frac{p}{\rho_0}\,. \tag{4.2}$$

The evolution equations of the system are the conservation of energy-momentum,

$$\nabla_a(T^{ab}) = 0\,, \tag{4.3}$$

and the conservation of the number of particles,

$$\nabla_a(\rho_0 u^a) = 0\,. \tag{4.4}$$

The latter one can also be named as the conservation of rest mass or baryonic mass, that is, the constant rest mass per particle times the number of particles. Projecting equation (4.3) along and perpendicular to the fluid four-velocity u^a, the scalar equation

$$\rho_0 h \nabla_a u^a + u^a \nabla_a(\rho_0 + \varepsilon\rho_0) = 0 \tag{4.5}$$

and the vector equation

$$\rho_0 h\, {}^{(\mathrm{u})}\gamma^c{}_b u^a \nabla_a u^b + {}^{(\mathrm{u})}\gamma^{ca} \nabla_a p = 0 \tag{4.6}$$

are obtained, respectively. Additionally, an arbitrary equation of state (EOS) of the form

$$p = p(\rho_0, \varepsilon) \tag{4.7}$$

is chosen and the case of an identically vanishing pressure, $p \equiv 0$, is explicitly excluded (see section 4.5 for this particular case). In numerical simulations in the context of astrophysics, the *ideal fluid* EOS $p = (\Gamma - 1)\,\rho\varepsilon$ with adiabatic index Γ, and the *polytropic* EOS $p = K\rho^\Gamma$ with polytropic constant K are commonly used due to their simplicity [Font, 2008]. However, more sophisticated EOSs have been employed, for example microphysical EOSs, to describe the interiors of compact stars such as neutron stars [Font, 2008]. Nevertheless, the *true* EOS(s) that describe the (different) interior region(s) of neutron stars is (are) still an ongoing part of the current research. For further information in this direction see [Font, 2008; Martí and Müller, 2015] as well as [Faber and Rasio, 2012], and the references mentioned therein. In this work, it is never made a specific choice of the EOS, but the general form of the EOS (4.7) is confined in the aforementioned way. EOSs that lead to superluminal speeds are explicitly excluded.

Equations (4.4) - (4.7) provide six equations for the six unknown quantities $(\rho_0, \varepsilon, p, \hat{v}_a)$. By using equation (4.7), it is sufficient to only evolve the state vector $\mathbf{U} = (p, \hat{v}_a, \varepsilon)^T$. The components of $\mathbf{U}$, expanded in the lower case (Eulerian) tensor basis, may be viewed as a slightly modified version of the *primitive variables* ρ_0, ε, v_i commonly used in the literature (see also section 4.4 where v_a is evolved instead of $\hat{v}_a$). The characteristic analysis in this chapter is performed on the system of evolution equations (4.4) - (4.6) for the components of the state vector $\mathbf{U}$. In particular, the resulting evolution equations are in a non-flux-balance law form[1]. Since there is no gauge freedom in the system, the analysis given below applies unambiguously even after a change of variables, for example to the flux-conservative variables D, S_i, τ defined in terms of the primitive ones as[2],

$$D = \rho_0 W\,, \qquad S_i = \rho_0 h W^2 v_i\,, \qquad \tau = \rho_0 h W^2 - p - D\,, \tag{4.8}$$

satisfying the set of evolution equations [Font et al., 2000]:

$$\begin{aligned}
\partial_t(\sqrt{\gamma}D) &= -\partial_k\left[\sqrt{\gamma}D(\alpha v^k - \beta^k)\right]\,,\\
\partial_t(\sqrt{\gamma}S_i) &= -\partial_k\left[\sqrt{\gamma}\left\{S_i(\alpha v^k - \beta^k) + \alpha p \delta^k{}_i\right\}\right] + \alpha\sqrt{\gamma}\Gamma^{\mu}_{\nu i}T^{\nu}{}_{\mu}\,,\\
\partial_t(\sqrt{\gamma}\tau) &= -\partial_k\left[\sqrt{\gamma}\left\{\tau(\alpha v^k - \beta^k) + \alpha p v^k\right\}\right] + \alpha^2\sqrt{\gamma}\left(T^{0\mu}\partial_\mu \ln\alpha - \Gamma^0_{\mu\nu}T^{\mu\nu}\right)\,.
\end{aligned} \tag{4.9}$$

This is guaranteed by the proof in section 3.4.

4.2 Lower Case Formulation

The equations (4.4) - (4.6) are now split against the lower case unit normal vector n^a and associated orthogonal projector $\gamma^a{}_b$ to get a system of first order PDEs for the variables $(p, \hat{v}_a, \varepsilon)$. After some algebraic manipulations and linear combination of the

[1] In the literature, the "flux-balance law form" is sometimes called the "flux-conservative form".

[2] The flux-conservative set of variables as components of the state vector, was firstly proposed in a different notation by [Martí et al., 1991] working in special relativity (SR).

equations, the system of equations can be rewritten as (see appendix A for the notebook)

$$\mathcal{L}_n p = (c_s^2 - 1) W_{c_s}^2 v^a D_a p - c_s^2 \rho_0 h \frac{W_{c_s}^2}{W} (\mathrm{g}^{-1})^{ab} D_a \hat{v}_b + c_s^2 \rho_0 h W_{c_s}^2 (\mathrm{g}^{-1})^{ab} K_{ab} \,, \tag{4.10}$$

$$\begin{aligned} \gamma^b{}_a \mathcal{L}_n \hat{v}_b = &- \frac{1}{W \rho_0 h} \left(\gamma^c{}_a + c_s^2 W_{c_s}^2 v^c v_a \right) D_c p - v^c D_c \hat{v}_a + c_s^2 W_{c_s}^2 v_a (\mathrm{g}^{-1})^{bc} D_b \hat{v}_c \\ &- c_s^2 W_{c_s}^2 (\mathrm{g}^{-1})^{bc} K_{bc} \hat{v}_a - W D_a \ln \alpha \,, \end{aligned} \tag{4.11}$$

$$\mathcal{L}_n \varepsilon = \frac{p}{\rho_0^2 h} \frac{W_{c_s}^2}{W^2} v^a D_a p - \frac{p}{\rho_0} \frac{W_{c_s}^2}{W} (\mathrm{g}^{-1})^{ab} D_a \hat{v}_b - v^a D_a \varepsilon + \frac{p}{\rho_0} W_{c_s}^2 (\mathrm{g}^{-1})^{ab} K_{ab} \,. \tag{4.12}$$

Here, the local speed of sound c_s is introduced which is defined as

$$c_s^2 = \frac{1}{h} \left(\chi + \frac{p}{\rho_0^2} \kappa \right) , \quad \chi = \left(\frac{\partial p}{\partial \rho_0} \right)_\varepsilon , \; \kappa = \left(\frac{\partial p}{\partial \varepsilon} \right)_{\rho_0} , \tag{4.13}$$

and the abbreviation $W_{c_s} = 1/\sqrt{1 - c_s^2 v^2}$ is used. Unless otherwise stated, only matter or EOSs with speed of sound $0 < c_s \leq 1$ are considered. As one can see, the Lie derivative $\mathcal{L}_n$ along the timelike unit normal vector n^a is used instead of ∂_t and the covariant derivative D_p associated with the three-metric γ_{ab} is written instead of ∂_p. As discussed in the very end of chapter 2, the analysis is unaffected by this since only the source terms differ. The relations to the derivative operator ∂_a in a coordinate basis can be found in [Alcubierre, 2008]. Writing the PDE system (4.10) - (4.12) as a vectorial equation of the form

$$\mathbf{A}^{\mathrm{n}} \mathcal{L}_n \mathbf{U} = \mathbf{A}^p D_p \mathbf{U} + \boldsymbol{\mathcal{S}} \,, \tag{4.14}$$

the coefficient matrices in front of the derivative operators can be identified as

$$\mathbf{A}^{\mathrm{n}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma^b{}_a & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad \mathbf{A}^p = \begin{pmatrix} (c_s^2 - 1) W_{c_s}^2 v^p & -c_s^2 \rho_0 h \frac{W_{c_s}^2}{W} (\mathrm{g}^{-1})^{pc} & 0 \\ -\frac{1}{W \rho_0 h} f^p{}_a & c_s^2 W_{c_s}^2 (\mathrm{g}^{-1})^{pc} v_a - v^p \gamma^c{}_a & 0 \\ \frac{p}{\rho_0^2 h} \frac{W_{c_s}^2}{W^2} v^p & -\frac{p}{\rho_0} \frac{W_{c_s}^2}{W} (\mathrm{g}^{-1})^{pc} & -v^p \end{pmatrix} , \tag{4.15}$$

with shorthand notation $f^p{}_a = \gamma^p{}_a + c_s^2 W_{c_s}^2 v^p v_a$ and the source vector can be written here as

$$\boldsymbol{\mathcal{S}} = \begin{pmatrix} c_s^2 \rho_0 h W_{c_s}^2 (\mathrm{g}^{-1})^{ab} K_{ab} \\ -c_s^2 W_{c_s}^2 (\mathrm{g}^{-1})^{bc} K_{bc} \hat{v}_a - W D_a \ln \alpha \\ \frac{p}{\rho_0} W_{c_s}^2 (\mathrm{g}^{-1})^{ab} K_{ab} \end{pmatrix} . \tag{4.16}$$

As mentioned in section 3.1 on page 19, the principal parts of special and general relativistic hydrodynamics take an almost identical form. All curvature properties that arise when going from SR to GR are included in the derivative operators, the source term and the projector $\gamma^a{}_b$.

Let $\mathbb{s}_a$ be an arbitrary lower case spatial 1-form, normalized with respect to the inverse boost metric, $(\mathrm{g}^{-1})^{ab}\mathbb{s}_a\mathbb{s}_b = 1$, and let ${}^{\mathbb{q}}\!\perp^b{}_a := \gamma^b{}_a - (\mathrm{g}^{-1})^{bc}\mathbb{s}_c\mathbb{s}_a$ be the orthogonal projector (see also table 3.2). Recalling the definition of $\hat{\mathbb{s}}^a = (\mathrm{g}^{-1})^{ab}\mathbb{s}_b$, the lower case metric is written as $\gamma^a{}_b = \hat{\mathbb{s}}^a\mathbb{s}_b + {}^{\mathbb{q}}\!\perp^a{}_b$. The left-hand side of equation (4.14) then becomes

$$
\begin{aligned}
\mathbf{A}^{\mathrm{n}}\mathcal{L}_n\mathbf{U} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma^b{}_a & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{L}_n p \\ \mathcal{L}_n \hat{v}_b \\ \mathcal{L}_n \varepsilon \end{pmatrix} = \begin{pmatrix} \mathcal{L}_n p \\ \gamma^b{}_a \mathcal{L}_n \hat{v}_b \\ \mathcal{L}_n \varepsilon \end{pmatrix} = \begin{pmatrix} \mathcal{L}_n p \\ \hat{\mathbb{s}}^b\mathbb{s}_a \mathcal{L}_n \hat{v}_b + {}^{\mathbb{q}}\!\perp^b{}_a \mathcal{L}_n \hat{v}_b \\ \mathcal{L}_n \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}_n p \\ \mathbb{s}_a (\mathcal{L}_n \hat{v})_{\hat{\mathbb{s}}} + {}^{\mathbb{q}}\!\perp^{\mathbb{A}}{}_a (\mathcal{L}_n \hat{v})_{\hat{\mathbb{A}}} \\ \mathcal{L}_n \varepsilon \end{pmatrix} \equiv \begin{pmatrix} \mathcal{L}_n p \\ (\mathcal{L}_n \hat{v})_{\hat{\mathbb{s}}} \\ (\mathcal{L}_n \hat{v})_{\hat{\mathbb{A}}} \\ \mathcal{L}_n \varepsilon \end{pmatrix} \equiv (\mathcal{L}_n \mathbf{U})_{\hat{\mathbb{s}},\, \hat{\mathbb{A}}} \,.
\end{aligned} \tag{4.17}
$$

As explained earlier in chapter 3, the indices $\mathbb{A}$ and $\hat{\mathbb{A}}$ are introduced here which are abstract but indicate application of the orthogonal projector ${}^{\mathbb{q}}\!\perp^b{}_a$, meaning $z_{\hat{\mathbb{A}}} = {}^{\mathbb{q}}\!\perp^a{}_{\hat{\mathbb{A}}} z_a$ and $z^{\mathbb{A}} = {}^{\mathbb{q}}\!\perp^{\mathbb{A}}{}_b z^b$ for any object z and $(\delta\mathbf{U})_{\hat{\mathbb{s}},\, \hat{\mathbb{A}}} = (\delta p, (\delta\hat{v})_{\hat{\mathbb{s}}}, (\delta\hat{v})_{\hat{\mathbb{A}}}, \delta\varepsilon)^T$ for the state vector (see also table 3.4 and the explanations below it). Furthermore, for any derivative operator δ and vector z^a, the notation $(\delta\hat{v})_z \equiv z^a\delta\hat{v}_a$ is used since no commutation with the derivative operator occurs. Treating the right-hand side of equation (4.14) in the same way as shown in equation (4.17), the expanded equation (4.14) can be cast into the form

$$
(\mathcal{L}_n\mathbf{U})_{\hat{\mathbb{s}},\, \hat{\mathbb{A}}} \simeq \mathbf{P}^{\mathbb{s}} \left(D_{\hat{\mathbb{s}}}\mathbf{U}\right)_{\hat{\mathbb{s}},\, \hat{\mathbb{B}}} \,, \tag{4.18}
$$

with the principal symbol

$$
\mathbf{P}^{\mathbb{s}} = \mathbf{A}^{\mathbb{s}} = \begin{pmatrix}
W_{c_s}^2(c_s^2-1)v^{\mathbb{s}} & -\frac{W_{c_s}^2}{W}c_s^2\rho_0 h & 0^{\mathbb{B}} & 0 \\
-\frac{W^2+c_s^2(v^{\mathbb{s}})^2W_{c_s}^2}{W^3\rho_0 h} & -\frac{W^2-c_s^2W_{c_s}^2}{W^2}v^{\mathbb{s}} & 0^{\mathbb{B}} & 0 \\
-\frac{c_s^2W_{c_s}^2}{W\rho_0 h}v_{\hat{\mathbb{A}}}v^{\mathbb{s}} & c_s^2W_{c_s}^2 v_{\hat{\mathbb{A}}} & -v^{\mathbb{s}}\,{}^{\mathbb{q}}\!\perp^{\mathbb{B}}{}_{\hat{\mathbb{A}}} & 0_{\hat{\mathbb{A}}} \\
\frac{pW_{c_s}^2}{W^2\rho_0^2 h}v^{\mathbb{s}} & -\frac{pW_{c_s}^2}{W\rho_0} & 0^{\mathbb{B}} & -v^{\mathbb{s}}
\end{pmatrix}. \tag{4.19}
$$

The symbol " $\simeq$ " denotes equality up to transverse principal and source terms.

Before the characteristic analysis is performed, a comment should be made: By the use of $\hat{v}_a$ in the state vector, the inverse boost metric arose in the principal part (4.15) (compare with section 4.4). By taking $\mathbb{s}_a$ to be normalized by $(\mathrm{g}^{-1})^{ab}$, it is possible to get rid of this complication in the principal symbol, which becomes 'easy' in the sense that it is highly structured.[3] The principal symbol as well as the eigenvalues and eigenvectors for a state vector (p, v_a, ε) can be found in the end of this chapter in section 4.4. Since the spatial 1-form $\mathbb{s}_a$ is normalized with respect to the inverse boost metric, the eigenvalues and eigenvectors take a form that is slightly modified in comparison with the literature, but these differences are purely artificial.

Solving the characteristic polynomial $P_\lambda = \det(\mathbf{P}^{\mathbb{s}} - \lambda \mathbb{1})$ one gets the five real eigenvalues

$$\lambda_{(0,1,2)} = -v^{\mathbb{s}}\,, \qquad \lambda_{(\pm)} = -\frac{1}{1 - c_s^2 v^2}\left((1 - c_s^2)v^{\mathbb{s}} \pm \frac{c_s}{W}\sqrt{1 - c_s^2 v_\perp^2}\right), \tag{4.20}$$

with the shorthand notation $v_\perp^2 := v_{\hat{\mathbb{A}}} v^{\mathbb{A}} \equiv v_a {}^{\mathbb{q}}\!\perp^a{}_b v^b$.

Please note that all eigenvalues in this thesis have the opposite sign in comparison to the literature due to the definition of the principal symbol. In the one-dimensional limit, i.e., $v_\perp^2 = 0$, the eigenvalues $\lambda_{(\pm)}$ reduce to

$$\lambda_{(\pm)} = -\frac{v^{\mathbb{s}} \pm W c_s}{1 \pm \frac{c_s v^{\mathbb{s}}}{W}}\,,$$

which, as noted by [Alcubierre, 2008], is just the special relativistic addition of two velocities, here modified by factors of W. Due to the choice of a three-basis normalized with respect to the inverse boost metric, the eigenvalues are slightly different compared to the results in section 4.4. The left eigenvectors of the principal symbol $\mathbf{P}^{\mathbb{s}}$ with evolved variables $(\delta p, (\delta\hat{v})_{\hat{\mathbb{s}}}, (\delta\hat{v})_{\hat{\mathbb{A}}}, \delta\varepsilon)$ in the order of the respective eigenvalues $\{\lambda_{(0,1,2)}, \lambda_{(\pm)}\}$ are

$$\begin{pmatrix} -\frac{p}{c_s^2\rho_0^2 h} & 0 & 0^{\mathbb{A}} & 1 \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{\rho_0 h}\hat{v}_{\hat{\mathbb{C}}} & 0 & {}^{\mathbb{q}}\!\perp^{\mathbb{A}}{}_{\hat{\mathbb{C}}} & 0 \end{pmatrix}, \qquad \begin{pmatrix} \pm\frac{\sqrt{1 - c_s^2 v_\perp^2}}{c_s \rho_0 h} & 1 & 0^{\mathbb{A}} & 0 \end{pmatrix}, \tag{4.21}$$

respectively. The associated right eigenvectors are

$$\begin{pmatrix} 0 \\ 0 \\ 0_{\hat{\mathbb{B}}} \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ {}^{\mathbb{q}}\!\perp^{\mathbb{C}}{}_{\hat{\mathbb{B}}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{c_s^2\rho_0^2 h}{p} \\ \pm\frac{c_s\rho_0}{p}\sqrt{1 - c_s^2 v_\perp^2} \\ -\frac{c_s^2\rho_0}{p}\hat{v}_{\hat{\mathbb{B}}} \\ 1 \end{pmatrix}, \tag{4.22}$$

respectively. Since there is a complete set of eigenvectors for each $\mathbb{s}_a$ which depend

[3] The attribute 'easy' is used because by this choice the principal symbol has a block triangular form and the 3×3 -block is diagonal.

furthermore continuously on $\mathfrak{s}_a$, the system is strongly hyperbolic. The characteristic variables corresponding to the speeds $\{\lambda_{(0,1,2)}, \lambda_{(\pm)}\}$ are given by

$$\hat{\mathrm{U}}_0 = \delta\varepsilon - \frac{p}{c_s^2\rho_0^2 h}\delta p\,,\ \hat{\mathrm{U}}_{\hat{\mathbb{A}}} = (\delta\hat{v})_{\hat{\mathbb{A}}} + \frac{1}{\rho_0 h}\hat{v}_{\hat{\mathbb{A}}}\delta p\,,\ \hat{\mathrm{U}}_\pm = (\delta\hat{v})_{\hat{\mathfrak{s}}} \pm \frac{\sqrt{1-c_s^2 v_\perp^2}}{c_s\rho_0 h}\delta p\,. \tag{4.23}$$

In the next section, the characteristic analysis in the upper case frame is performed.

4.3 Upper Case Formulation

Once again, the starting point are the equations (4.4), (4.5) and (4.6). They are split against the upper case normal vector u^a and orthogonal projector ${}^{(\mathrm{u})}\gamma^b{}_a$. To compare the results between the lower and the upper case frame, the same state vector as before, $\mathbf{U} = (p, \hat{v}_a, \varepsilon)^T$, is chosen. Using the definition of the local speed of sound (4.13) and after some algebra the following PDEs for the components of the state vector are obtained:

$$\nabla_u p = -c_s^2\rho_0 h\,{}^{(\mathrm{u})}\gamma^b{}_d(\mathrm{g}^{-1})^{dc}\nabla_b\hat{v}_c - c_s^2 W\rho_0 h\,{}^{(\mathrm{u})}\gamma^b{}_d(\mathrm{g}^{-1})^{dc}\nabla_b n_c\,, \tag{4.24}$$

$${}^{(\mathrm{u})}\gamma_{ab}(\mathrm{g}^{-1})^{bc}\nabla_u\hat{v}_c = -\frac{1}{\rho_0 h}{}^{(\mathrm{u})}\gamma^b{}_a\nabla_b p - W\,{}^{(\mathrm{u})}\gamma_{ab}(\mathrm{g}^{-1})^{bc}\nabla_u n_c\,, \tag{4.25}$$

$$\nabla_u\varepsilon = -\frac{p}{\rho_0}{}^{(\mathrm{u})}\gamma^b{}_d(\mathrm{g}^{-1})^{dc}\nabla_b\hat{v}_c - \frac{Wp}{\rho_0}{}^{(\mathrm{u})}\gamma^b{}_d(\mathrm{g}^{-1})^{dc}\nabla_b n_c\,. \tag{4.26}$$

The detailed derivation of this system is given in the accompanying notebook, see appendix A. As an interesting relation, one can find that ${}^{(\mathrm{u})}\gamma_{ab}(\mathrm{g}^{-1})^{bc} = {}^{(\mathrm{u})}(\mathrm{g}^{-1})_{ab}\gamma^{bc}$ holds. Writing the system (4.24) - (4.26) as a vectorial equation for the state vector $\mathbf{U}$ in the form

$$\mathbf{B}^{\mathrm{u}}\nabla_u\mathbf{U} = \mathbf{B}^p\nabla_p\mathbf{U} + \boldsymbol{\mathcal{S}}, \tag{4.27}$$

the coefficient matrices

$$\mathbf{B}^{\mathrm{u}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & {}^{(\mathrm{u})}\gamma_{ab}(\mathrm{g}^{-1})^{bc} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{B}^p = \begin{pmatrix} 0 & -c_s^2\rho_0 h\,{}^{(\mathrm{u})}\gamma^p{}_d(\mathrm{g}^{-1})^{dc} & 0 \\ -\frac{1}{\rho_0 h}{}^{(\mathrm{u})}\gamma^p{}_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0}{}^{(\mathrm{u})}\gamma^p{}_d(\mathrm{g}^{-1})^{dc} & 0 \end{pmatrix} \tag{4.28}$$

can be identified and the source vector is written as

$$\boldsymbol{\mathcal{S}} = \begin{pmatrix} -c_s^2 W\rho_0 h\,{}^{(\mathrm{u})}\gamma^b{}_d(\mathrm{g}^{-1})^{dc}\nabla_b n_c \\ -W\,{}^{(\mathrm{u})}\gamma_{ab}(\mathrm{g}^{-1})^{bc}\nabla_u n_c \\ -\frac{Wp}{\rho_0}{}^{(\mathrm{u})}\gamma^b{}_d(\mathrm{g}^{-1})^{dc}\nabla_b n_c \end{pmatrix}. \tag{4.29}$$

Let S_a be an arbitrary upper case spatial 1-form, $S_a S^a = 1$, and let ${}^{\mathrm{Q}}\!\perp^{b}{}_{a} = {}^{(\mathrm{u})}\gamma^{b}{}_{a} - S^b S_a$ be the orthogonal projector. Decomposing ${}^{(\mathrm{u})}\gamma^{a}{}_{b}$ against S^a and ${}^{\mathrm{Q}}\!\perp^{b}{}_{a}$, and apply relations in table 3.3 to $\mathbb{s}_a$ and $\hat{\mathbb{s}}^a$, e.g., $\hat{\mathbb{s}}^a = (\mathbb{g}^{-1})^{ab} S_b$, the left-hand side of equation (4.27) may be rewritten as

$$\mathbf{B}^{\mathrm{u}}\nabla_u \mathbf{U} = \begin{pmatrix} \nabla_u p \\ S_a(\nabla_u \hat{v})_{\hat{\mathbb{s}}} + {}^{\mathrm{Q}}\!\perp^{\mathbb{A}}{}_a(\nabla_u \hat{v})_{\hat{\mathbb{A}}} \\ \nabla_u \varepsilon \end{pmatrix} \equiv \begin{pmatrix} \nabla_u p \\ (\nabla_u \hat{v})_{\hat{\mathbb{s}}} \\ (\nabla_u \hat{v})_{\hat{\mathbb{A}}} \\ \nabla_u \varepsilon \end{pmatrix} \equiv (\nabla_u \mathbf{U})_{\hat{\mathbb{s}},\,\hat{\mathbb{A}}} \, . \tag{4.30}$$

Since the upper case projector is pushed through the lower case inverse boost metric, one finds $S_a S_b (\mathbb{g}^{-1})^{bc} (\delta\hat{v})_c = S_a \hat{\mathbb{s}}^c (\delta\hat{v})_c = S_a (\delta\hat{v})_{\hat{\mathbb{s}}}$. Analogously, the orthogonal part is ${}^{\mathrm{Q}}\!\perp^{A}{}_{b}\,{}^{(\mathrm{u})}\gamma_{Ac}(\mathbb{g}^{-1})^{cd} (\delta\hat{v})_d = {}^{\mathrm{Q}}\!\perp^{\mathbb{A}}{}_b (\delta\hat{v})_{\hat{\mathbb{A}}}$. With the help of the findings in sections 3.1 and 3.3, the convention ${}^{\mathrm{Q}}\!\perp^{\mathbb{A}}{}_a \equiv {}^{\mathrm{q}}\!\perp^{\mathbb{A}}{}_c(\mathbb{g}^{-1})^{cb}\,{}^{\mathrm{Q}}\!\perp_{ba}$ is used. See also appendix B for the detailed calculation and simplification of the upper case coefficient matrices in equation (4.27).

Treating the right-hand side of equation (4.27) in the same way as in equation (4.30), the expanded equation (4.27) reads

$$(\nabla_u \mathbf{U})_{\hat{\mathbb{s}},\,\hat{\mathbb{A}}} \simeq \mathbf{P}^S \, (\nabla_S \mathbf{U})_{\hat{\mathbb{s}},\,\hat{\mathbb{B}}} \, , \tag{4.31}$$

with principal symbol

$$\mathbf{P}^S = \mathbf{B}^S = \begin{pmatrix} 0 & -c_s^2 \rho_0 h & 0^B & 0 \\ -\frac{1}{\rho_0 h} & 0 & 0^B & 0 \\ 0_A & 0_A & 0^B{}_A & 0_A \\ 0 & -\frac{p}{\rho_0} & 0^B & 0 \end{pmatrix} . \tag{4.32}$$

By employing the upper case frame with the fluid four-velocity as the normal vector, the principal symbol has become much simpler than in the lower case, see (4.19), exhibiting now essentially the same shape as that of a simple wave equation (2.7). For the system of GRHD the extra structure is not required to complete the analysis, because in practice computer algebra tools can already manage the more complicated form. In more sophisticated models, however, additional structure may become essential to successfully proceed with a characteristic analysis. The reason why the form of the principal symbol is much easier than in the lower case is somehow obvious: the energy-momentum tensor (4.1) and hence, the four-dimensional form of the fluid equations of motion contain the fluid four-velocity and its orthogonal projector. Thereby, any frame adapted to that fact naturally annihilates many terms in the principal symbol and thus uncovers the very simple structure of (4.32).

The five eigenvalues of $\mathbf{P}^S$ are

$$\lambda_{(0,1,2)} = 0\,, \qquad \lambda_{(\pm)} = \pm c_s\,, \tag{4.33}$$

with the corresponding left eigenvectors

$$\begin{pmatrix} -\frac{p}{c_s^2\rho_0^2 h} & 0 & 0^A & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & {}^{\mathrm{Q}}\!\perp^A{}_C & 0 \end{pmatrix}, \quad \begin{pmatrix} \mp\frac{1}{c_s\rho_0 h} & 1 & 0^A & 0 \end{pmatrix}, \tag{4.34}$$

right eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ {}^{\mathrm{Q}}\!\perp^C{}_B \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{c_s^2\rho_0^2 h}{p} \\ \mp\frac{c_s\rho_0}{p} \\ 0_B \\ 1 \end{pmatrix}, \tag{4.35}$$

and characteristic variables

$$\hat{\mathrm{U}}_0 = \delta\varepsilon - \frac{p}{c_s^2\rho_0^2 h}\delta p\,,\ \hat{\mathrm{U}}_A = (\delta\hat{v})_{\hat{\mathtt{A}}}\,,\ \hat{\mathrm{U}}_\pm = (\delta\hat{v})_{\hat{\mathtt{s}}} \mp \frac{\delta p}{c_s\rho_0 h}\,. \tag{4.36}$$

In regard to the application of the DF approach, it is straightforward to see that $(\mathbb{1}+\mathbf{B}^V)$ is invertible for all $v_a v^a = V_a V^a < 1$. The various speeds in the upper case system are subluminal, that is $|\lambda| \leq 1$, and there is no gauge freedom in the system. Therefore, by following the argument of section 3.2 the analysis of strong hyperbolicity is equivalent in the upper and lower case frame. Using the recovery procedure described in section 3.3 gives the same results for eigenvalues and eigenvectors as well as characteristic variables as in the lower case analysis in 4.2. Details about the application of the recovery procedure to GRHD can be found in the accompanying notebook, see appendix A.

4.4 GRHD using the Boost Vector

In the following, the lower case PDE system of GRHD is revisited, where the three-velocity v_a is taken to be an evolution variable instead of the earlier used $\hat{v}_a$.

The Lie derivative of $\hat{v}_a$ under a change to v_a behaves according to

$$\gamma^b{}_a\mathcal{L}_n\hat{v}_b = W\mathrm{g}^b{}_a\mathcal{L}_n v_b + \hat{v}_a K_{cd}\hat{v}^c\hat{v}^d\,, \tag{4.37}$$

and expressing $\hat{v}_a = W v_a$ in the covariant derivative one finds

$$\gamma^b{}_a D_c \hat{v}_b = W \mathbf{g}^b{}_a D_c v_b \,. \tag{4.38}$$

Let s_a be an unit spatial 1-form with $s_a s^a = 1,\ s_a n^a = 0$ and denote the orthogonal projector by ${}^{\mathrm{q}}\!\perp^b{}_a := \gamma^b{}_a - s^b s_a$. Using the state vector $\mathbf{U} = (p, v_a, \varepsilon)$ and rewriting the system of equations (4.10) - (4.12) while respecting the aforementioned transformation to v_a, the resulting system of PDEs can be written as a vectorial equation of the form

$$(\mathcal{L}_n \mathbf{U})_{s,\,A} \simeq \mathbf{P}^s (D_s \mathbf{U})_{s,\,B} \,, \tag{4.39}$$

with the principal symbol given by

$$\mathbf{P}^s = \begin{pmatrix} W_{c_s}^2 (c_s^2 - 1) v^s & -W_{c_s}^2 c_s^2 \rho_0 h & 0^B & 0 \\ -\frac{1 + (c_s^2 - 1)(v^s)^2 W_{c_s}^2}{W^2 \rho_0 h} & W_{c_s}^2 (c_s^2 - 1) v^s & 0^B & 0 \\ \frac{(1 - c_s^2) W_{c_s}^2}{W^2 \rho_0 h} v^s v_A & \frac{c_s^2 W_{c_s}^2}{W^2} v_A & -v^s \, {}^{\mathrm{q}}\!\perp^B{}_A & 0_A \\ \frac{p W_{c_s}^2}{W^2 \rho_0^2 h} v^s & -\frac{p W_{c_s}^2}{\rho_0} & 0^B & -v^s \end{pmatrix} . \tag{4.40}$$

The principal symbol (4.40) for state vector $\mathbf{U} = (p, v_a, \varepsilon)$ has by use of an appropriate 2+1 decomposition the same structure as the lower case principal symbol (4.19) for state vector $\mathbf{U} = (p, \hat{v}_a, \varepsilon)$. The analysis is given here in regard to the analysis of RGRMHD in chapter 6.

The eigenvalues of the principal symbol (4.40) for material and acoustic waves are

$$\lambda_{(0,1,2)} = -v^s \,,\ \lambda_{(\pm)} = -\frac{1}{1 - c_s^2 v^2} \left((1 - c_s^2) v^s \pm \frac{c_s}{W} \sqrt{(1 - c_s^2 v^2) - (1 - c_s^2)(v^s)^2} \right) , \tag{4.41}$$

respectively. They coincide with the literature [Ibáñez et al., 2013]. The corresponding left eigenvectors are given by

$$\begin{aligned} &\left(-\tfrac{p}{c_s^2 \rho_0^2 h} \quad 0 \quad 0^A \quad 1 \right), \ \left(\tfrac{1}{W^2 \rho_0 h} v_C \quad v^s v_C \quad (1 - (v^s)^2)\, {}^{\mathrm{q}}\!\perp^A{}_C \quad 0 \right), \\ &\qquad \left(\pm \tfrac{\sqrt{(1 - c_s^2 v^2) - (1 - c_s^2)(v^s)^2}}{c_s \rho_0 h W} \quad 1 \quad 0^A \quad 0 \right) . \end{aligned} \tag{4.42}$$

For the same variables and the same order the right eigenvectors are

$$\begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ {}^{\mathrm{q}}\!\perp^{C}{}_{B} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{c_s^2\rho_0^2 h}{p}(1-(v^s)^2) \\ \pm\frac{c_s\rho_0}{pW}\sqrt{(1-c_s^2v^2)-(1-c_s^2)(v^s)^2}(1-(v^s)^2) \\ -\frac{c_s\rho_0}{pW}\left(\frac{c_s}{W}\pm v^s\sqrt{(1-c_s^2v^2)-(1-c_s^2)(v^s)^2}\right)v_B \\ 1-(v^s)^2 \end{pmatrix}. \tag{4.43}$$

The eigenvectors are in agreement with the ones given in [Ibáñez et al., 2013] up to the chosen set of variables, the normalization of the spatial vector s^a and lower case curvature terms. The characteristic variables corresponding to the speeds $\{\lambda_{(0,1,2)}, \lambda_{(\pm)}\}$ are given by

$$\begin{aligned}
\hat{\mathrm{U}}_0 &= \delta\varepsilon - \frac{p}{c_s^2\rho_0^2 h}\delta p, \\
\hat{\mathrm{U}}_A &= (\delta v)_A + v^s(v_A(\delta v)_s - v_s(\delta v)_A) + \frac{1}{\rho_0 h W^2}\hat{v}_A\delta p, \\
\hat{\mathrm{U}}_\pm &= (\delta v)_s \pm \frac{\sqrt{(1-c_s^2v^2)-(1-c_s^2)(v^s)^2}}{c_s\rho_0 h W}\delta p\,.
\end{aligned} \tag{4.44}$$

4.5 Dust

A special case for the EOS (4.7) is the one of dust, in which the pressure is identically zero everywhere, $p \equiv 0$, and the energy density coincides with the rest mass density. As a consequence, the specific internal energy density is zero, $\varepsilon = 0$, and the specific enthalpy becomes unity. Hence, the energy-momentum tensor (4.1) simplifies to

$$T^{ab} = \rho_0 u^a u^b. \tag{4.45}$$

Taking the divergence and projecting along the streamlines u^a and orthogonal to them, one obtains the equation of conservation of particles and the geodesic equation,

$$\begin{aligned}
\rho_0\nabla_a u^a + u^a\nabla_a\rho_0 &= 0\,, \\
\rho_0\,{}^{(\mathrm{u})}\gamma^c{}_b u^a\nabla_a u^b &= 0\,,
\end{aligned} \tag{4.46}$$

respectively. Thereby, the particles follow timelike geodesics and the conservation of rest mass does not have to be postulated in contrast to the system with non-vanishing EOS.

Using equations (4.46) for the state vector $\mathbf{U} = (\rho_0, \hat{v}_a)$ and 3+1 decomposing the equations for example against n^a and $\gamma^a{}_b$, the PDE system can be written in the lower case as

$$
\begin{aligned}
\mathcal{L}_n \rho_0 &= -\, v^a D_a \rho_0 - \frac{\rho_0}{W} (\mathbb{g}^{-1})^{ab} D_a \hat{v}_b + \rho_0 (\mathbb{g}^{-1})^{ab} K_{ab} \,, \\
\gamma^b{}_a \mathcal{L}_n \hat{v}_b &= -\, v^b D_b \hat{v}_a - W D_a \ln \alpha \,.
\end{aligned} \tag{4.47}
$$

Using again an arbitrary spatial 1-form $\mathbb{s}_a$ as in section 4.2, one immediately arrives at the principal symbol $\mathbf{P}^{\mathbb{s}}$ for $(\delta \mathbf{U})_{\hat{\mathbb{s}},\, \hat{\mathbb{A}}}$,

$$
\mathbf{P}^{\mathbb{s}} = \begin{pmatrix} -v^{\mathbb{s}} & -\frac{\rho_0}{W} & 0^{\mathbb{B}} \\ 0 & -v^{\mathbb{s}} & 0^{\mathbb{B}} \\ 0_{\hat{\mathbb{A}}} & 0_{\hat{\mathbb{A}}} & -v^{\mathbb{s}}\, {}^{\mathbb{q}}\!\perp^{\mathbb{B}}{}_{\hat{\mathbb{A}}} \end{pmatrix} , \tag{4.48}
$$

which contains a Jordan block. The principal symbol is thus missing an eigenvector. The system is only weakly hyperbolic and the IVP ill-posed.

Chapter 5

Hyperbolicity Analysis of Ideal Magnetohydrodynamics

After presenting the DF formalism and its usability for GRHD, the main part of this thesis is arrived where the investigation of the hyperbolicity structure of PDE systems to GRMHD is performed. The evolution equations are derived for a set of eight variables in regard to the numerically evolved set. The first characteristic analysis for RMHD was performed by [Anile and Pennisi, 1987]. They work covariantly and consider an augmented system of *ten* evolved variables, assuming implicitly a 'free-evolution' style [Hilditch, 2013] to treat the two additional algebraic constraints, $u^a u_a = -1, u^a b_a = 0$, as well as the Maxwell constraint for the magnetic field. The analysis was reviewed and expanded in [Anile, 1990]. Another augmented system for RMHD using ten variables was later derived in [van Putten, 1991]. On the basis of [Anile and Pennisi, 1987; Anile, 1990], several authors, e.g., [Komissarov, 1999; Antón et al., 2010], reinvestigated the hyperbolicity structure especially in relation to occurring degeneracies and how they can be classified. In particular, a very detailed discussion is given in [Antón et al., 2010]. It is found that this augmented formulation of RMHD is strongly hyperbolic.

By the occurrence of shocks, a flux-balance law form [Godunov, 1959] of the set of evolution equations is taken in the numerical implementation. This form is needed to treat them with sophisticated methods such as HRSC schemes [Hawke et al., 2005]. The PDE system of GRMHD using HRSC schemes in slightly different forms is for example considered by [Komissarov, 1999; Balsara, 2001; Gammie et al., 2003; Antón et al., 2006; Giacomazzo and Rezzolla, 2007; Antón et al., 2010], where a total of *eight* variables including the magnetic field is evolved, which is here referred to as the flux-balance law form. Sometimes the system is augmented by an auxiliary scalar field to drive the magnetic field constraint [Zanotti et al., 2015]. This technique is called "divergence cleaning" [Liebling et al., 2010]. Over the past years, also schemes have been developed and modified where the magnetic four-potential is evolved instead of the magnetic field [Giacomazzo et al., 2011; Etienne et al., 2015]. The related PDE system is not considered here. In the following, only the system of equations for eight variables, i.e., five hydrodynamical and three magnetic field evolution PDEs, is investigated.

It is important to stress that the analysis of [Anile, 1990] does not necessarily apply to the system used in applications. Changing the number of variables can influence the hyperbolicity properties of the system under consideration. In general, it is not enough to know that there is *some* convenient form of the system being treated but it is rather required that the *particular* formulation being employed should itself be at least strongly hyperbolic.

The careful reconsideration of the hyperbolicity analysis of GRMHD is motivated by two interesting observations. First, when numerical schemes are constructed to treat GRMHD in flux-balance law form one sometimes sees that the longitudinal component of the magnetic field is ignored in evaluating the fluxes by striking the corresponding row and column in the principal symbol. From the numerical point of view, this is a legitimate approach because the approximation works by repeated application of a one-dimensional scheme. However, from the mathematical point of view, namely investigating the hyperbolicity of the PDE system, it is not permitted to discard any of the variables, for example in a particular direction. To show strong hyperbolicity, a complete set of eigenvectors of the principal symbol must be found, including those associated with the Gauss constraint for the magnetic field. It must therefore be distinguished between 'computational tricks' and an actual change of the system of equations itself. Second, the system of GRMHD is constrained (see the end of section 2.2) and due care is needed when considering the evolution equations. The way how the constraint is present in the evolution equations influences the mathematical structure of the system and leads to *different* formulations.

Neither of these subtleties have been completely taken care of in the earlier analyses. Indeed, a first indication can be found that the flux-balance law system of GRMHD used in numerics, e.g., by [Antón et al., 2006], differs from that used in the analysis of [Anile and Pennisi, 1987]. The eigenvalues associated with the Gauss constraint differ between the two systems. In [Anile and Pennisi, 1987] the 'entropy eigenvalue' is found with multiplicity two, where one of these corresponds to the Maxwell constraint. In [Ibáñez et al., 2015], for the system of eight variables, the 'entropy eigenvalue' has only multiplicity one, and the 'constraint eigenvalue' is zero.

In the following sections, a reconsideration of the PDE system of GRMHD using eight evolution variables is performed. The ultimate aim is to analyze the original numerically used flux-balance law formulation of GRMHD as in [Antón et al., 2006] where the magnetic field is evolved. The investigation benefits significantly from the DF formalism by performing the characteristic analysis at first in the upper case. An earlier approach of a direct characteristic analysis in the lower case failed by the fact that the principal symbol became a complicated matrix whose structure is difficult to spot.[1] To support this statement, by the naive lower case approach, for example, the magnetosonic eigenvalues arrived with more then 10^4 terms.

The further course of this chapter is as follows: In section 5.1, the basic definitions and equations for GRMHD following [Anile, 1990; Antón et al., 2010] are recapitulated. Afterwards an upper case 3+1 decomposition of the PDEs is performed. After the derivation of the evolution equations, in each of them multiples of the Gauss constraint are manually added to take different formulations into account (see section 5.2). Subsequently, in section 5.3, the particular choice where all constraint addition coefficients are set to zero is adopted, which leads to the first formulation under consideration. By this, one obtains a PDE system that is in some sense analogous to the set of equations in [Anile and Pennisi, 1987], but with their algebraic constraints explicitly imposed. The characteristic analysis of the corresponding principal symbol is performed in the upper case including a full degeneracy analysis (section 5.4) which shows that the system is strongly hyperbolic. The lower case characteristic quantities derived by the recovery procedure in section 3.3 are given in section 5.5 including the treatment of the degeneracies. Finally, in section 5.6, a different choice of constraint addition coefficients is adopted to obtain a set of equations equal to the flux-balance law system, comparing explicitly with [Ibáñez et al., 2015], and it is shown that this particular formulation of GRMHD used in NR is only weakly hyperbolic.

5.1 Basics of GRMHD

In this section, the basic definitions and equations of GRMHD are repeated, following the works of [Anile, 1990; Antón et al., 2010]. However, since the aim is an investigation of hyperbolicity structure of the system of equations, the presentation is primarily done in a mathematical fashion, where some important physical insights and statements are suppressed. Throughout Lorentz-Heaviside units for electro-magnetic quantities with $\varepsilon_0 = \mu_0 = 1$ are used, where ε_0 is the vacuum permittivity (or electric constant)

[1] In the author's opinion, the fact that the aforementioned points have not been carefully unrevealed is due to the highly complicated structure of the lower case GRMHD principal symbol. Especially the lower case right constraint eigenvector is quite lengthy. Nevertheless, [Ibáñez et al., 2015] were able to derive the other seven right eigenvectors of the system for the usage in a convexity analysis.

and μ_0 is the vacuum permeability (or magnetic constant).

The Maxwell Equations and Ohm's Law

It is initially started with the introduction of the *Faraday electromagnetic tensor field* or *field strength tensor* F^{ab}. For a generic observer with four-velocity $\mathtt{N}^a$ the field strength tensor can be expressed via the electric and magnetic fields, $\mathtt{E}^a$, $\mathtt{B}^a$, as

$$F^{ab} = \mathtt{N}^a\mathtt{E}^b - \mathtt{N}^b\mathtt{E}^a + \epsilon^{abcd}\mathtt{N}_c\mathtt{B}_d\,, \tag{5.1}$$

with the Levi-Cività tensor,

$$\epsilon^{abcd} = -\frac{1}{\sqrt{-g}}\,[abcd]\ , \tag{5.2}$$

where g is the determinant of the spacetime metric g_{ab} and $[abcd]$ is the completely antisymmetric Levi-Cività symbol with $[0123] = 1$. Both the electric and magnetic field are spatial against $\mathtt{N}^a$ and thus satisfy the orthogonality relations $\mathtt{E}^a\mathtt{N}_a = \mathtt{B}^a\mathtt{N}_a = 0$. The dual of the field strength tensor is defined as

$$^*F^{ab} = -\frac{1}{2}\epsilon^{abcd}F_{cd}\,, \tag{5.3}$$

or expressed in terms of the electric and magnetic fields,

$$^*F^{ab} = \mathtt{N}^a\mathtt{B}^b - \mathtt{N}^b\mathtt{B}^a - \epsilon^{abcd}\mathtt{N}_c\mathtt{E}_d\,. \tag{5.4}$$

Within the scope of this thesis, the sign convention of [Alcubierre et al., 2009] is applied.

Taking a comoving observer with $\mathtt{N}^a = u^a$, the Faraday tensor (5.1) and its dual (5.4) in terms of electric and magnetic fields e^a and b^a, respectively, are

$$F^{ab} = u^a e^b - u^b e^a + \epsilon^{abcd}u_c b_d\,, \tag{5.5}$$

$$^*F^{ab} = u^a b^b - u^b b^a - \epsilon^{abcd}u_c e_d\,. \tag{5.6}$$

By definition, the (dual) field strength tensor is antisymmetric if both indices are lowered or raised. Using the field strength tensor (5.1) and its dual (5.4), Maxwell's equations read

$$\nabla^*_b F^{ab} = 0\,, \qquad \nabla_b F^{ab} = \mathcal{J}^a\,, \tag{5.7}$$

called the homogeneous and inhomogeneous Maxwell equations, respectively. The source term $\mathcal{J}^a$ of the latter one is the so-called electric four-current. An explicit expression is provided by generalized Ohm's law, which determines the physical model one would like to study. A simple version of the four-current is given by

$$\mathcal{J}^a = \rho_{\mathrm{el}} u^a + I^a \,. \tag{5.8}$$

The first part of $\mathcal{J}^a$ is an advection term with the proper charge density ρ_{el} measured by the comoving observer with u^a, where the second is the conductive (upper case three-) current I^a. Under several physical assumptions and conditions (see [Dionysopoulou et al., 2015] and references therein) a good approximation of the conductive current in terms of the scalar electric conductivity σ and the electric field in the fluid frame e^a is

$$I^a = \sigma e^a = \sigma F^{ab} u_b. \tag{5.9}$$

The four-current is then given by

$$\mathcal{J}^a = \rho_{\mathrm{el}} u^a + \sigma F^{ab} u_b \,, \tag{5.10}$$

which is commonly used in NR (see also chapter 6).

Ideal MHD Condition

In the limit of infinite conductivity σ but finite current $\mathcal{J}^a$, the electric field measured by the comoving observer has to vanish,

$$e^a = F^{ab} u_b \equiv 0 \,. \tag{5.11}$$

The first equality holds by using the definition of the field strength tensor (5.5). By equation (5.11), the Eulerian electric field vector E^a, $E^a n_a = 0$ can be calculated to

$$0 = F^{ab} u_b = \left(n^a E^b - n^b E^a + \epsilon^{abcd} n_c B_d\right) u_b \iff E^a = -\epsilon^{abcd} v_b n_c B_d \,, \tag{5.12}$$

where B^a is the Eulerian magnetic field vector. Therefore, only the Maxwell evolution equation for the magnetic field as well as the Gauss constraint have to be taken into account.

Energy-Momentum Tensor of GRMHD

The total energy-momentum tensor of GRMHD is expressed as the sum of the ideal fluid part as in chapter 4,

$$T^{ab}_{\text{fluid}} = \rho_0 h u^a u^b + g^{ab} p \,, \tag{5.13}$$

plus the standard electromagnetic energy-momentum tensor,

$$T^{ab}_{\text{em}} = F^{ac} F^{b}{}_{c} - \frac{1}{4} g^{ab} F_{cd} F^{cd} \,. \tag{5.14}$$

Using the ideal MHD condition (5.11) and expressing the field strength tensor via (5.5), the electromagnetic energy-momentum tensor in terms of the magnetic field is

$$T^{ab}_{\text{em}} = \left(u^a u^b + \frac{1}{2} g^{ab} \right) b^2 - b^a b^b \,, \tag{5.15}$$

and the total energy-momentum tensor is thus given by

$$T^{ab} = \rho_0 h^* u^a u^b + p^* g^{ab} - b^a b^b \,, \tag{5.16}$$

with $h^* = h + b^2/\rho_0$ and $p^* = p + b^2/2$. In equation (5.15), the abbreviation $b^2 = b^a b_a$ is introduced. Also, an EOS of the form $p = p(\rho_0, \varepsilon)$ with the same properties and restrictions as in chapter 4 is employed.

Covariant PDE System of GRMHD

The covariant set of equations of GRMHD are the conservation of the number of particles,

$$\nabla_a (\rho_0 u^a) = 0 \,, \tag{5.17}$$

the conservation of energy-momentum,

$$\nabla_b T^{ab} = 0 \,, \tag{5.18}$$

and the relevant Maxwell equations,

$$\nabla_b {}^* F^{ab} = 0 \,. \tag{5.19}$$

5.2 3+1 Decomposition of the PDE System

Due to the form of the energy-momentum tensor (5.16), a 3+1 decomposition against the fluid four-vector u^a and its orthogonal projector ${}^{(\mathrm{u})}\gamma^b{}_a$, i.e., working in the upper case frame, should lead to a simpler principal symbol than its lower case version, as is the case for GRHD. The PDE system of GRMHD considered in this work is a set of eight evolution equations together with the Maxwell constraint for the magnetic field. The latter is sometimes just called Maxwell constraint or Gauss constraint for short. In contrast to the system of GRHD in chapter 4, more care is needed when splitting the equations. As mentioned in section 2.2, those constrained systems have the property that adding multiples of a constraint to an equation does not change the physics, but leads to different mathematical systems on the level of evolution equations.

The covariant PDE system of GRMHD is first split in the upper case. Then, to take the presence of the Maxwell constraint into account, some parametrized combination of the Maxwell constraint is added to each evolution equation. A concrete choice of the constraint addition parameters results in a set of evolution equations which is called a *formulation of GRMHD*.

Primarily, the focus is here on two specific formulations. The first one of these is called *prototype algebraic constraint free formulation*. This formulation is essentially that of [Anile and Pennisi, 1987], but without the artificial expansion of variables through the definition of the algebraic constraints $u^a u_a = -1$ and $u^a b_a = 0$. The algebraic constraints are instead satisfied a priori by reducing the number of evolution equations and variables. The second formulation, called *flux-balance law formulation*, corresponds to the flux-balance law system used in numerics for example by [Antón et al., 2006; Antón et al., 2010; Ibáñez et al., 2015]. The desired upper case form of evolution equations to the flux-balance law formulation is obtained by matching the values of the formulation parameters with the literature. This was performed using Mathematica (see appendix A for the accompanying notebook).

Splitting the covariant PDE system of GRMHD according to the upper case, the eight equations determining the time evolution of the GRMHD system are

$$\begin{aligned} \nabla_a(\rho_0 u^a) = 0\,, \quad {}^{(\mathrm{u})}\gamma_{ab}\nabla_c T^{bc} = 0\,, \\ u_b \nabla_c T^{bc} = 0\,, \quad {}^{(\mathrm{u})}\gamma_{ab}\nabla_c {}^*F^{bc} = 0\,, \end{aligned} \tag{5.20}$$

together with an EOS $p = p(\rho_0, \varepsilon)$ and the Gauss constraint

$$0 = u_c \nabla_b {}^*F^{bc} = {}^{(\mathrm{u})}\gamma^{bc}\nabla_b b_c\,. \tag{5.21}$$

The upper case magnetic field vector b^a can be split in the lower case as

$$b^a = (\hat{b}^c v_c) n^a + \hat{b}^a \,, \qquad n_a b^a = -(v_a \hat{b}^a) \,, \qquad \gamma^a{}_b \, b^b = \hat{b}^a \,, \tag{5.22}$$

with $n_a \hat{b}^a = 0$. Furthermore, the Eulerian magnetic field vector B^a is connected to the upper case magnetic field via

$$\hat{b}_a = \frac{1}{W} \mathrm{g}_{ab} B^b = \frac{1}{W} B_a + (B^b \hat{v}_b) v_a \,, \qquad B^a = W (\mathrm{g}^{-1})^{ab} \hat{b}_b = W \hat{b}^a - (\hat{b}^c \hat{v}_c) v^a \,, \tag{5.23}$$

where the lower case Gauss constraint reads

$$\gamma^{ab} \nabla_a B_b = 0 \,. \tag{5.24}$$

Taking equations (5.20), a straightforward calculation similar to that for GRHD in section 4.3 provides evolution equations for the pressure,

$$\nabla_u p = - \, c_s^2 \rho_0 h \, {}^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d \hat{v}_e + S^{(p)} + \omega^{(p)} \left({}^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d \bot b_e + S^{(\mathrm{c})} \right) \,, \tag{5.25}$$

the weighted boost vector,

$$\begin{aligned} {}^{(\mathrm{u})}\gamma_{ab} (\mathrm{g}^{-1})^{bc} \nabla_u \hat{v}_c = & - \left(\frac{b^d b_a}{\rho_0^2 h h^*} + \frac{{}^{(\mathrm{u})}\gamma^d{}_a}{\rho_0 h^*} \right) \nabla_d p + \frac{2}{\rho_0 h^*} {}^{(\mathrm{u})}\gamma^{[b}{}_a b^{d]} \, {}^{(\mathrm{u})}\gamma_{bc} (\mathrm{g}^{-1})^{ce} \nabla_d \bot b_e \\ & + S_a^{(\hat{\mathbf{v}})} + \omega_a^{(\hat{v})} \left({}^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d \bot b_e + S^{(\mathrm{c})} \right) \,, \end{aligned} \tag{5.26}$$

the (auxiliary) magnetic field,

$$\begin{aligned} {}^{(\mathrm{u})}\gamma_{ab} (\mathrm{g}^{-1})^{bc} \nabla_u \bot b_c = & \, 2 \, {}^{(\mathrm{u})}\gamma_{ab} \, {}^{(\mathrm{u})}\gamma^{[b}{}_c b^{d]} (\mathrm{g}^{-1})^{ce} \nabla_d \hat{v}_e + S_a^{(\bot \mathbf{b})} \\ & + \omega_a^{(\bot b)} \left({}^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d \bot b_e + S^{(\mathrm{c})} \right) \,, \end{aligned} \tag{5.27}$$

and finally, the specific internal energy,

$$\nabla_u \varepsilon = - \frac{p}{\rho_0} {}^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d \hat{v}_e + S^{(\varepsilon)} + \omega^{(\varepsilon)} \left({}^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d \bot b_e + S^{(\mathrm{c})} \right) \,. \tag{5.28}$$

By equation (5.21), the Gauss constraint becomes

$$ {}^{(\mathrm{u})}\gamma^{ac} \nabla_a b_c = {}^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d \bot b_e + S^{(\mathrm{c})}. \tag{5.29}$$

The sources are given by

$$
\begin{aligned}
S^{(p)} &= -c_s^2 W \rho_0 h^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d n_e \,, \\
S_a^{(\hat{\mathbf{v}})} &= -W^{(\mathrm{u})}\gamma_{ab} (\mathrm{g}^{-1})^{be} \nabla_u n_e + \frac{2W}{\rho_0 h^*}{}^{(\mathrm{u})}\gamma^{[b}{}_a b^{e]} V_b b^d \nabla_d n_e \,, \\
S_a^{(\perp\mathbf{b})} &= 2W^{(\mathrm{u})}\gamma_{ab}{}^{(\mathrm{u})}\gamma^{[b}{}_c b^{d]} (\mathrm{g}^{-1})^{ce} \nabla_d n_e + 2W^{(\mathrm{u})}\gamma^e{}_{[a} V_{b]} b^b \nabla_u n_e \,, \\
S^{(\varepsilon)} &= -\frac{Wp}{\rho_0}{}^{(\mathrm{u})}\gamma^d{}_c (\mathrm{g}^{-1})^{ce} \nabla_d n_e \,, \\
S^{(\mathrm{c})} &= \left(W V^d b^e - W (b^c V_c)^{(\mathrm{u})}\gamma^{de} \right) \nabla_d n_e \,.
\end{aligned} \tag{5.30}
$$

For convenience and to further simplify the resulting principal symbol, the *auxiliary magnetic field* $\perp b^c$ is introduced. It is defined by the relation

$$
{}^{(\mathrm{u})}\gamma_{ac} (\mathrm{g}^{-1})^{cd} \nabla_b \perp b_d := {}^{(\mathrm{u})}\gamma_{ac} (\mathrm{g}^{-1})^{cd} \nabla_b \hat{b}_d + V_a b_d (\mathrm{g}^{-1})^{de} \nabla_b \hat{v}_e \tag{5.31}
$$

and only exists as an abbreviation in the sense of this relation, despite '$\perp b_a$' is written as a component in the state vector. As usual, square brackets around indices denote anti-symmetrization, so that $2\hat{v}^{[a} b^{b]} = \hat{v}^a b^b - \hat{v}^b b^a$ holds. In the system (5.25) - (5.28) multiples of the Maxwell constraint (5.29) connected to constraint coefficients $\omega^{(p)}, \omega_a^{(\hat{v})}, \omega_a^{(\perp b)}$, and $\omega^{(\varepsilon)}$ were already added. Using an upper case unit spatial 1-form S_a and the related lower case unit spatial 1-form $\mathfrak{s}_a = \gamma^b{}_a S_b$, the auxiliary magnetic field may be written as

$$
S_c (\mathrm{g}^{-1})^{cd} \nabla_b \perp b_d = (\nabla_b \perp b)_{\hat{\mathfrak{s}}} \simeq (\nabla_b B)_{\mathfrak{s}} + (B^a \hat{v}_a)(\nabla_b v)_{\mathfrak{s}} - B^{\mathfrak{s}} \hat{v}^c (\nabla_b v_c) \,, \tag{5.32}
$$

where source terms are neglected, which is indicated by '$\simeq$'.

5.3 Prototype Algebraic Constraint Free Formulation

In the next two sections, the characteristic analysis for the prototype algebraic constraint free formulation of GRMHD is performed. First, in section 5.4 the equations (5.25) - (5.28) with constraint parameters $\omega^{(p)} = 0$, $\omega_a^{(\hat{v})} = 0$, $\omega_a^{(\perp b)} = 0$, and $\omega^{(\varepsilon)} = 0$ are used and analyzed in the upper case. The resulting system is connected to the augmented system of equations of [Anile and Pennisi, 1987] as follows: Take the equations of [Anile and Pennisi, 1987], project the momentum equation and the evolution equation for the magnetic field with ${}^{(\mathrm{u})}\gamma^a{}_b$ orthogonal to the four-velocity of the fluid, change the evolved variables to $(p, \hat{v}_a, \perp b_a, \varepsilon)$, and replace the derivative of the pressure p in the evolution equation for the magnetic field using the evolution equation for p. After doing so and taking an upper case unit spatial vector S^a one would arrive with the principal symbol given below. As remarked in section 3.1 on page 19, the fact that [Anile and Pennisi, 1987] work exclusively in RMHD is of no consequence, since the principal symbols of RMHD and GRMHD are essentially the same.

With the results in the upper case, the procedure given in section 3.3 is used to obtain all lower case characteristic quantities, such as eigenvalues, eigenvectors and characteristic variables. This is done in section 5.5. As previously mentioned, a direct computation of the lower case characteristic quantities is quite lengthy and was not successful. For both frames, a detailed discussion of degenerate states is provided in the respective section.

The analysis of the flux-balance law formulation of GRMHD is given afterwards in section 5.6.

5.4 Upper Case Formulation

Writing the equations (5.25) - (5.28) with $\omega^{(p)} = 0$, $\omega_a^{(\hat{v})} = 0$, $\omega_a^{(\perp b)} = 0$, and $\omega^{(\varepsilon)} = 0$ in a vectorial form with state vector $\mathbf{U} = (p, \hat{v}_a, \perp b_a, \varepsilon)^T$,

$$\mathbf{B}^{\mathrm{u}} \nabla_u \mathbf{U} = \mathbf{B}^p \nabla_p \mathbf{U} + \boldsymbol{\mathcal{S}}\,, \tag{5.33}$$

one can identify the coefficient matrix of the timelike derivative as

$$\mathbf{B}^{\mathrm{u}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & {}^{(\mathrm{u})}\gamma_{ab}(\mathrm{g}^{-1})^{bc} & 0 & 0 \\ 0 & 0 & {}^{(\mathrm{u})}\gamma_{ab}(\mathrm{g}^{-1})^{bc} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.34}$$

and the upper case spatial coefficient matrix reads

$$\mathbf{B}^p = \begin{pmatrix} 0 & -c_s^2 \rho_0 h\, {}^{(\mathrm{u})}\gamma^p{}_c (\mathrm{g}^{-1})^{ce} & 0 & 0 \\ f^p{}_a & 0 & l^{pe}{}_a & 0 \\ 0 & 2\,{}^{(\mathrm{u})}\gamma_{ab}\,{}^{(\mathrm{u})}\gamma^{[b}{}_c b^{p]} (\mathrm{g}^{-1})^{ce} & 0 & 0 \\ 0 & -\frac{p}{\rho_0}\,{}^{(\mathrm{u})}\gamma^p{}_c (\mathrm{g}^{-1})^{ce} & 0 & 0 \end{pmatrix}, \tag{5.35}$$

with abbreviations

$$l^{pe}{}_a = \frac{2}{\rho_0 h^*}\,{}^{(\mathrm{u})}\gamma^{[b}{}_a b^{p]}\,{}^{(\mathrm{u})}\gamma_{bc} (\mathrm{g}^{-1})^{ce}\,, \qquad f^p{}_a = -\left(\frac{b^p b_a}{\rho_0^2 h h^*} + \frac{{}^{(\mathrm{u})}\gamma^p{}_a}{\rho_0 h^*} \right), \tag{5.36}$$

and source vector $\boldsymbol{\mathcal{S}} = (S^{(p)}, S_a^{(\hat{\mathbf{v}})}, S_a^{(\perp \mathbf{b})}, S^{(\varepsilon)})^T$. A straightforward calculation shows that $(\mathbb{1} + \mathbf{B}^V)$ is invertible for all $v^a v_a = V^a V_a < 1$.

Only in this section, the sub-/superscript ‘u’ to indicate the upper case eigenvalues is omitted for simplicity.

5.4.1 2+1 Decomposition

The 2+1 decomposition and notation is similar to the one of GRHD performed in chapter 4. Therefore, some steps are skipped. Let S_a be an arbitrary unit spatial 1-form and let ${}^{\mathrm{Q}}\!\perp^b{}_a$ be the associated orthogonal projector. Let s_a and ${}^{\mathrm{q}}\!\perp^b{}_a$ be their lower case projected versions (see tables 3.2 and 3.3 for definitions and relations). Decomposing ${}^{(\mathrm{u})}\gamma^b{}_a$ and $\gamma^b{}_a$ against S_a and s_a, respectively, equation (5.33) can be written in the form

$$(\nabla_u \mathbf{U})_{\hat{\mathsf{s}},\,\hat{\mathbb{A}}} \simeq \mathbf{P}^S\, (\nabla_S \mathbf{U})_{\hat{\mathsf{s}},\,\hat{\mathbb{B}}}\,, \tag{5.37}$$

with $(\delta\mathbf{U})_{\hat{\mathsf{s}},\hat{\mathbb{A}}} = (\delta p, (\delta\hat{v})_{\hat{\mathsf{s}}}, (\delta\hat{v})_{\hat{\mathbb{A}}}, (\delta\!\perp\! b)_{\hat{\mathsf{s}}}, (\delta\!\perp\! b)_{\hat{\mathbb{A}}}, \delta\varepsilon)^T$ and principal symbol

$$\mathbf{P}^S = \mathbf{B}^S = \begin{pmatrix} 0 & -c_s^2\rho_0 h & 0^B & 0 & 0^B & 0 \\ -\frac{\left(b^S\right)^2+\rho_0 h}{\rho_0^2 h h^*} & 0 & 0^B & 0 & -\frac{b^B}{\rho_0 h^*} & 0 \\ -\frac{b^S b_A}{\rho_0^2 h h^*} & 0_A & 0^B{}_A & 0_A & \frac{b^S}{\rho_0 h^*}{}^{\mathrm{Q}}\!\perp^B{}_A & 0_A \\ 0 & 0 & 0^B & 0 & 0^B & 0 \\ 0_A & -b_A & b^S\,{}^{\mathrm{Q}}\!\perp^B{}_A & 0_A & 0^B{}_A & 0_A \\ 0 & -\frac{p}{\rho_0} & 0^B & 0 & 0^B & 0 \end{pmatrix}. \tag{5.38}$$

The characteristic polynomial P_λ for the principal symbol (5.38) can be written as

$$P_\lambda = \frac{\lambda^2}{(\rho_0 h^*)^2} P_{\text{Alfvén}} P_{\text{mgs}}\,, \tag{5.39}$$

with the quadratic polynomial for Alfvén waves

$$P_{\text{Alfvén}} = -\left(b^S\right)^2 + \lambda^2 \rho_0 h^*, \tag{5.40}$$

and the quartic polynomial for magnetosonic waves

$$P_{\text{mgs}} = \left(\lambda^2 - 1\right)\left(\lambda^2 b^2 - \left(b^S\right)^2 c_s^2\right) + \lambda^2\left(\lambda^2 - c_s^2\right)\rho_0 h\,. \tag{5.41}$$

By solving (5.39) different kinds of speeds of waves propagating along the S^a-direction are provided. All speeds are real and the system is strongly hyperbolic, as will be seen later. The entropy waves have speed

$$\lambda_{(\mathrm{e})} = 0\,. \tag{5.42}$$

The constraint waves have the same speed, given by

$$\lambda_{(\mathrm{c})} = 0\,. \tag{5.43}$$

The Alfvén waves are given by solving $P_{\text{Alfvén}} = 0$, which results in the two different speeds

$$\lambda_{(\mathrm{a}\pm)} = \pm \frac{b^S}{\sqrt{\rho_0 h^*}}\,, \tag{5.44}$$

where subscripts '±' refer to the '±' on the right-hand side of equation (5.44). Solving the quartic equation $P_{\mathrm{mgs}} = 0$, four different speeds of the magnetosonic waves are obtained. The two slow magnetosonic waves are

$$\lambda_{(\mathrm{s}\pm)} = \pm\sqrt{\zeta_{\mathrm{S}} - \sqrt{\zeta_{\mathrm{S}}^2 - \xi_{\mathrm{S}}}}\,, \tag{5.45}$$

and the two fast magnetosonic waves read

$$\lambda_{(\mathrm{f}\pm)} = \pm\sqrt{\zeta_{\mathrm{S}} + \sqrt{\zeta_{\mathrm{S}}^2 - \xi_{\mathrm{S}}}}\,, \tag{5.46}$$

where the abbreviations

$$\zeta_{\mathrm{S}} = \frac{\left(b^2 + c_s^2\left[\left(b^S\right)^2 + \rho_0 h\right]\right)}{2\rho_0 h^*}\,, \quad \xi_{\mathrm{S}} = \frac{\left(b^S\right)^2 c_s^2}{\rho_0 h^*}\,, \tag{5.47}$$

are used. Please note that the subscript 'S' in ζ_{S} and ξ_{S} is not a contraction with a vector, but rather a reminder[2] that the vector S^a is used for the 2+1 decomposition. Again, subscripts '±' refer to the '±' on the right-hand side of the equations (5.45) and (5.46). Since $(b^S)^2 \leq b^2$ and $c_s^2 \leq 1$, all eigenvalues have an absolute value smaller than or equal to one. Thereby, the relation $|\lambda_{\mathrm{u}}||V| < 1$ holds for all upper case eigenvalues λ_{u} and for all boost velocities with an absolute value smaller than one, which is required for straightforward application of the formalism of section 3.3. Thus, the recovering procedure can be applied.

The eight upper case left eigenvectors corresponding to $\lambda_{(\mathrm{e})}, \lambda_{(\mathrm{c})}, \lambda_{(\mathrm{a}\pm)}$ and $\lambda_{(\mathrm{m}\pm)}$

[2] This will become relevant when recovering the lower case quantities.

with $\mathrm{m} = \mathrm{s}, \mathrm{f}$ are:

$$\begin{aligned}
\text{Entropy:} \quad & \begin{pmatrix} -\frac{p}{c_s^2 \rho_0^2 h} & 0 & 0^A & 0 & 0^A & 1 \end{pmatrix}, && (5.48)\\
\text{Constraint:} \quad & \begin{pmatrix} 0 & 0 & 0^A & 1 & 0^A & 0 \end{pmatrix}, && (5.49)\\
\text{Alfvén:} \quad & \begin{pmatrix} 0 & 0 & \mp{}^{(\mathrm{S})}\epsilon^{AC} b_C \sqrt{\rho_0 h^*} & 0 & -{}^{(\mathrm{S})}\epsilon^{AC} b_C & 0 \end{pmatrix}, && (5.50)\\
\text{Magnetosonic:} \quad & \begin{pmatrix} \frac{\rho_0 h^* \left(\lambda_{(\mathrm{m}\pm)}\right)^2 - b^2}{c_s^2 \rho_0 h} & \frac{\left(b^S\right)^2 - \rho_0 h^* \left(\lambda_{(\mathrm{m}\pm)}\right)^2}{\lambda_{(\mathrm{m}\pm)}} & \frac{b^S b^A}{\lambda_{(\mathrm{m}\pm)}} & 0 & b^A & 0 \end{pmatrix}, && (5.51)
\end{aligned}$$

respectively. Here, the antisymmetric upper case Levi-Cività two- and three-tensors are defined as ${}^{(\mathrm{S})}\epsilon^{AB} := S_d {}^{(\mathrm{u})}\epsilon^{dAB} := S_d u_c {}^{\mathrm{Q}}\perp^A{}_a {}^{\mathrm{Q}}\perp^B{}_b \epsilon^{cdab}$. The right eigenvectors can be obtained by inverting the matrix of left eigenvectors or by solving the eigenvalue problem directly. Explicitly, in the order to their corresponding eigenvalues $\lambda_{(\mathrm{e})}, \lambda_{(\mathrm{c})}, \lambda_{(\mathrm{a}\pm)}$ and $\lambda_{(\mathrm{m}\pm)}$ they can be expressed as

$$\begin{pmatrix} 0 \\ 0 \\ 0_B \\ 0 \\ 0_B \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \\ 0_B \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ \mp \frac{{}^{(\mathrm{S})}\epsilon_{BC}}{\sqrt{\rho_0 h^*}} b^C \\ 0 \\ -{}^{(\mathrm{S})}\epsilon_{BC} b^C \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{c_s^2 \rho_0^2 h}{p} \\ -\frac{\rho_0 \lambda_{(\mathrm{m}\pm)}}{p} \\ \frac{\rho_0 \lambda_{(\mathrm{m}\pm)}}{p b^S b_\perp^2} \left[\left(b^S\right)^2 + \rho_0 h^* \left((\lambda_{(\mathrm{m}\pm)})^2 - 2\zeta_{\mathrm{S}} \right) \right] b_B \\ 0 \\ \frac{\rho_0}{b_\perp^2 p} \left[b^2 + \rho_0 h^* \left((\lambda_{(\mathrm{m}\pm)})^2 - 2\zeta_{\mathrm{S}} \right) \right] b_B \\ 1 \end{pmatrix}, \tag{5.52}$$

with $\mathrm{m} = \mathrm{s}, \mathrm{f}$ as before.

In the magnetosonic eigenvectors the perpendicular magnetic field vector $b_\perp^a \equiv {}^{\mathrm{Q}}\perp^a{}_b\, b^b$ with $b_\perp^2 = b_\perp^a b_a^\perp = b^A b_A$ is introduced. By definition, it is orthogonal to the upper case spatial vector S^a. To avoid confusion: this vector field should not be mixed up with the auxiliary magnetic field defined by equation (5.31). The redundancy of writing $b_\perp^A$ in this section is in anticipation of the lower case.

At this stage of the characteristic analysis, a complete set of eigenvectors for real eigenvalues exists. However, it must be checked, whether any of the eigenvalues may change their multiplicity (for particular directions), and if so, whether or not a complete set of eigenvectors is still available. The situation where a priori distinct eigenvalues coincide and their multiplicity changes, is called degenerate state or for short degeneracy. To show strong hyperbolicity of the system, it has to be shown that for each possible degenerate state a complete set of eigenvectors still exists. For the augmented system of RMHD, this was already described in [Anile and Pennisi, 1987; Anile, 1990; Komissarov, 1999; Balsara, 2001]. A full account was furthermore given by [Antón et al., 2010].

It should be also mentioned, that in the appendix of [Komissarov, 1999], the eigenvalues and right eigenvectors in the fluid rest frame are given for seven variables in a one-dimensional analysis of RMHD. They are obtained by explicitly setting (locally) the spatial entries of the four-velocity to zero. This approach is ultimately quite similar to the DF approach used in this work.

5.4.2 Degeneracy Analysis of the Upper Case

The prototype algebraic constraint free formulation of GRMHD has the same degeneracies as they exist in the augmented system of [Anile and Pennisi, 1987]. Two different types of degeneracies can occur. For type I degeneracy, the magnetic field along the spatial vector, b^S, is equal to zero. For degeneracy of type II, the magnetic field is parallel to S^a, so that $b^a_\perp = {}^{\mathrm{Q}}\!\perp^a{}_b\, b^b = 0$ holds. To describe the different situations properly, the upper case magnetic field vector is cast into the form

$$b^a = b^S S^a + b^a_\perp\,, \qquad b^2 = (b^S)^2 + b^2_\perp\,. \tag{5.53}$$

In this section, $b^S S^a$ and b^S are called the *parallel magnetic field*, and $b^a_\perp$ and $|b_\perp|$ are referred to as the *perpendicular magnetic field* with respect to S^a.

First, the characteristic polynomial is considered. The Alfvén polynomial (5.40) and the magnetosonic polynomial (5.41) have solutions

$$\left.\frac{b^S}{\lambda}\right|_{(\mathrm{a}\pm)} = \pm\sqrt{\rho_0 h^*}, \tag{5.54}$$

$$\begin{aligned}\left.\frac{b^S}{\lambda}\right|_{(\mathrm{m}\pm)} &= \pm\sqrt{\left(\rho_0 h + \frac{b^2}{c_s^2}\right) + \rho_0 h\left(1 - \frac{1}{c_s^2}\right)\frac{\lambda^2_{(\mathrm{m}\pm)}}{1-\lambda^2_{(\mathrm{m}\pm)}}} \\ &= \pm\sqrt{(b^S)^2 + \left(\rho_0 h + \frac{b^2}{c_s^2}\right) - \rho_0 h^* \frac{\lambda^2_{(\mathrm{m}\pm)}}{c_s^2}}\,,\end{aligned} \tag{5.55}$$

respectively. These expressions are well defined even for degenerate states.

To describe the degenerate states properly, the pairs of Alfvén $\lambda^\pm_{(\mathrm{a})}$ as well as slow $\lambda^\pm_{(\mathrm{s})}$ and fast $\lambda^\pm_{(\mathrm{f})}$ magnetosonic eigenvalues are divided into two classes denoted by a superscript '+' or '−'. The superscripts '+' and '−' refer to the higher or lower value of each pair, respectively. In the upper case, the pairs of eigenvalues are symmetrically distributed with the entropy eigenvalue $\lambda_{(\mathrm{e})} = 0$ in the center. By doing estimates, one can show that the upper case eigenvalues can be ordered as

$$\lambda^-_{\text{(f)}} \leq \lambda^-_{\text{(a)}} \leq \lambda^-_{\text{(s)}} \leq \lambda_{\text{(e)}} \leq \lambda^+_{\text{(s)}} \leq \lambda^+_{\text{(a)}} \leq \lambda^+_{\text{(f)}}\,. \tag{5.56}$$

The order is the same as for lower case GRMHD and the Newtonian case [Antón et al., 2010]. In the prototype algebraic constraint free formulation of GRMHD the constraint and entropy eigenvalues coincide. Due to the simple expressions of the upper case eigenvalues, it is easily found that $\lambda^\pm_{\text{(s)}} = \lambda_{\text{(s}\pm)}$ and $\lambda^\pm_{\text{(f)}} = \lambda_{\text{(f}\pm)}$, where the subscripts '+' or '−' indicate the respective sign of the right-hand side of the slow (5.45) and fast (5.46) magnetosonic eigenvalues. For Alfvén eigenvalues (5.44), where again the subscripts '+' or '−' indicate the respective sign of the right-hand side, the classification depends on the sign of the magnetic field b^a in the direction of S^a:

$$\begin{aligned} b^S \geq 0: &\qquad \lambda^\pm_{\text{(a)}} = \lambda_{\text{(a}\pm)}\,;\\ b^S < 0: &\qquad \lambda^\pm_{\text{(a)}} = \lambda_{\text{(a}\mp)}\,. \end{aligned} \tag{5.57}$$

Type I degeneracy. For type I degeneracy in the upper case where $b^S = 0$ and $b^2 = b^2_\perp$, the waves of entropy and constraint as well as both Alfvén waves and both slow magnetosonic waves propagate at the same speed:

$$\lambda_{\text{(e)}} = \lambda_{\text{(c)}} = \lambda^\pm_{\text{(a)}} = \lambda^\pm_{\text{(s)}} = 0,\ \lambda^\pm_{\text{(f)}} = \pm\frac{\sqrt{b^2 + c_s^2\rho_0 h}}{\sqrt{\rho_0 h^*}}\,. \tag{5.58}$$

The multiplicity of the entropy eigenvalue increases under such a degeneracy from two to six. The well defined solutions of the magnetosonic polynomial then become

$$\left.\frac{b^S}{\lambda}\right|_{\text{(s}\pm)} = \pm\sqrt{\rho_0 h + \frac{b^2}{c_s^2}}, \qquad \left.\frac{b^S}{\lambda}\right|_{\text{(f}\pm)} = 0\,. \tag{5.59}$$

Type II degeneracy. For type II degeneracy, namely when $b^a_\perp = 0$ and $b^2 = (b^S)^2$, the pair of slow or of fast magnetosonic waves propagate with the same speed as their respective Alfvén wave, depending on the numerical value of the speed of sound:

$$\begin{aligned} \lambda^\pm_{\text{(s)}} = \lambda^\pm_{\text{(a)}} = \pm\frac{|b^S|}{\sqrt{\rho_0 h^*}},\ \lambda^\pm_{\text{(f)}} = \pm c_s,\ \text{ if }\ c_s^2 > \frac{(b^S)^2}{\rho_0 h^*}\,;\\ \lambda^\pm_{\text{(f)}} = \lambda^\pm_{\text{(a)}} = \pm\frac{|b^S|}{\sqrt{\rho_0 h^*}},\ \lambda^\pm_{\text{(s)}} = \pm c_s,\ \text{ if }\ c_s^2 < \frac{(b^S)^2}{\rho_0 h^*}\,. \end{aligned} \tag{5.60}$$

Then the multiplicities of eigenvalues corresponding to both Alfvén waves change from one to two. The well defined solutions of the magnetosonic polynomial are now

$$\left.\frac{b^S}{\lambda}\right|_{(\mathrm{m}\pm\neq\mathrm{a}\pm)} = \pm\frac{b^S}{c_s}\,, \qquad \left.\frac{b^S}{\lambda}\right|_{(\mathrm{m}\pm=\mathrm{a}\pm)} = \pm\sqrt{\rho_0 h^*}\,.$$

Type II′ degeneracy. The type II′ degeneracy is essentially a type II degeneracy, but leads to a further increase of the multiplicity of the Alfvén waves. Both Alfvén waves have multiplicity three. This occurs if the numerical value of the speed of sound reaches a special value determined by the relation $(b^S)^2 = c_s^2 \rho_0 h^*$. In such a case, the fast and slow magnetosonic as well as the Alfvén waves of the same class travel at the same speed:

$$\lambda^{\pm}_{(\mathrm{f})} = \lambda^{\pm}_{(\mathrm{s})} = \lambda^{\pm}_{(\mathrm{a})} = \pm\frac{|b^S|}{\sqrt{\rho_0 h^*}}\,. \tag{5.61}$$

The case of a vanishing magnetic field. The case of a vanishing magnetic field vector, $b^a = 0$, can be regarded as the occurrence of a type I and type II degeneracy at the same time. The eigenvalues of the entropy, the constraint, both Alfvén, and both slow magnetosonic waves become zero. The fast magnetosonic waves become the acoustic waves of the GRHD case, namely equal to the speed of sound with the respective sign. Since the local speed of sound is assumed to have always a positive nonzero value, it is not possible for type I and type II′ degeneracies to occur simultaneously.

As one can easily show, the left and right eigenvectors given in the last subsection will not form a set of eight linearly independent eigenvectors if degeneracies occur. However, it is possible to give a complete set of rescaled eigenvectors based on the ones above, which forms in all degenerate states a linearly independent set of eigenvectors. In the following two subsections sets of renormalized upper case left and right eigenvectors are obtained.

5.4.3 Renormalized Upper Case Left Eigenvectors

Entropy and constraint eigenvectors. The upper case eigenvectors are rescaled in an analogous way as to [Antón et al., 2010]. The procedure can also be found in the accompanying notebook, see appendix A. The entropy (5.48) and constraint (5.49) left eigenvectors remain the same and are displayed here again for the sake of consistency:

$$\text{Entropy:} \qquad \begin{pmatrix} -\frac{p}{c_s^2 \rho_0^2 h} & 0 & 0^A & 0 & 0^A & 1 \end{pmatrix}, \tag{5.62}$$

$$\text{Constraint:} \qquad \begin{pmatrix} 0 & 0 & 0^A & 1 & 0^A & 0 \end{pmatrix}. \tag{5.63}$$

Alfvén eigenvectors. The upper case Alfvén eigenvectors (5.50) are well defined under a type I degeneracy. For type II degeneracy, they become zero since the perpendicular magnetic field $b_\perp^a$ vanishes. Hence, the terms $b^C = b_\perp^C$ have to be divided by the norm of the perpendicular magnetic field, $|b_\perp|$. The rescaled Alfvén eigenvectors to eigenvalues $\lambda_{(\mathrm{a}\pm)}$ are then:

$$\text{Alfvén:} \qquad \begin{pmatrix} 0 & 0 & \pm^{(\mathrm{S})}\epsilon^{AC}\sqrt{\rho_0 h^*}\frac{b_C^\perp}{|b_\perp|} & 0 & ^{(\mathrm{S})}\epsilon^{AC}\frac{b_C^\perp}{|b_\perp|} & 0 \end{pmatrix}. \tag{5.64}$$

The rescaled perpendicular magnetic field is just a unit vector in the respective direction.

Magnetosonic eigenvectors. The rescaling of the four magnetosonic left eigenvectors (5.51),

$$\begin{pmatrix} \frac{\rho_0 h^* \left(\lambda_{(\mathrm{m}\pm)}\right)^2 - b^2}{c_s^2 \rho_0 h} & \frac{\left(b^S\right)^2 - \rho_0 h^* \left(\lambda_{(\mathrm{m}\pm)}\right)^2}{\lambda_{(\mathrm{m}\pm)}} & \frac{b^S b^A}{\lambda_{(\mathrm{m}\pm)}} & 0 & b^A & 0 \end{pmatrix}, \tag{5.65}$$

is far more involved. First, it is noted that they are already well defined under type I degeneracies. They become neither zero nor have singular entries. The terms $b^S/\lambda_{(\mathrm{s}\pm)}$ have an appropriate limiting value given above. For fast magnetosonic waves, $b^S/\lambda_{(\mathrm{f}\pm)}$ is just zero.

In regard to the renormalization, the four eigenvectors need to be separated into two groups, depending on which eigenvalues under a type II degeneracy coincide with the Alfvén eigenvalues. First, the eigenvectors with eigenvalues closer to the Alfvén waves are rescaled as follows. Considering the characteristic polynomial for magnetosonic waves (5.41), and using the expression $b^2 = (b^S)^2 + b_\perp^2$, one obtains

$$P_{\mathrm{mgs}} = \left(\lambda^2 - 1\right)\left(\lambda^2\left((b^S)^2 + b_\perp^2\right) - \left(b^S\right)^2 c_s^2\right) + \lambda^2\left(\lambda^2 - c_s^2\right)\rho_0 h\,, \tag{5.66}$$

and, for the magnetosonic eigenvalues where $P_{\mathrm{mgs}} = 0$, the squared parallel magnetic field can be expressed as

$$(b^S)^2 = \frac{\lambda_{(\mathrm{m}\pm)}^2\left(b_\perp^2(\lambda_{(\mathrm{m}\pm)}^2 - 1) + (\lambda_{(\mathrm{m}\pm)}^2 - c_s^2)\rho_0 h\right)}{\left(c_s^2 - \lambda_{(\mathrm{m}\pm)}^2\right)\left(\lambda_{(\mathrm{m}\pm)}^2 - 1\right)}. \tag{5.67}$$

Using expression (5.67) to replace all $(b^S)^2$ terms in the first two entries of the magnetosonic eigenvectors (also the hidden ones in h^*), they become

$$\begin{pmatrix} \frac{b_\perp^2\left(\lambda_{(\mathrm{m}\pm)}^2-1\right)}{\left(c_s^2-\lambda_{(\mathrm{m}\pm)}^2\right)\rho_0 h} & \frac{(c_s^2-1)b_\perp^2\lambda_{(\mathrm{m}\pm)}}{\lambda_{(\mathrm{m}\pm)}^2-c_s^2} & \frac{b^S b^A}{\lambda_{(\mathrm{m}\pm)}} & 0 & b^A & 0 \end{pmatrix}. \tag{5.68}$$

Expressing the magnetosonic eigenvectors in this form, it is obvious that they will become zero under type II degeneracy. Hence, they are divided by $|b_\perp|$ to rescale them. By use of the abbreviation $\mathcal{H} = |b_\perp|/(c_s^2 - \lambda_{(\mathrm{m}\pm)}^2)$, finally, the rescaled upper case magnetosonic left eigenvectors which have eigenvalues closer to the Alfvén eigenvalues are written as:

$$\text{Magnetosonic closer to Alfvén:} \quad \begin{pmatrix} \frac{\mathcal{H}(\lambda^2-1)}{\rho_0 h} & (1-c_s^2)\mathcal{H}\lambda & \left(\frac{b^S}{\lambda}\right)\frac{b_\perp^A}{|b_\perp|} & 0 & \frac{b_\perp^A}{|b_\perp|} & 0 \end{pmatrix}_{(\mathrm{m}_1\pm)}. \tag{5.69}$$

These eigenvectors are well defined under type I and type II degeneracies. However, in a type II′ degenerate state the numerator as well as the denominator of $\mathcal{H}$ vanish. In such a case $\mathcal{H} = 0$ is taken, although the limit value of $\mathcal{H}$ may not be zero. Under all circumstances, $\mathcal{H}$ will not diverge when approaching a type II′ degeneracy, as can be explicitly shown.

To obtain a complete set of eight linear independent eigenvectors, the remaining two upper case magnetosonic eigenvectors need a different rescaling, which is done as follows. Considering once more the characteristic polynomial for magnetosonic waves (5.41) and solving it for $(b^S)^2$ directly, one finds

$$(b^S)^2 = \frac{\lambda_{(\mathrm{m}\pm)}^2\left(b^2(\lambda_{(\mathrm{m}\pm)}^2-1)+(\lambda_{(\mathrm{m}\pm)}^2-c_s^2)\rho_0 h\right)}{c_s^2\left(\lambda_{(\mathrm{m}\pm)}^2-1\right)}. \tag{5.70}$$

Replacing $(b^S)^2$ in the second entry of the magnetosonic eigenvectors, dividing the eigenvectors by $\rho_0 h\lambda_{(\mathrm{m}\pm)}^2 - b^2$, respectively, and taking the abbreviation $\mathcal{F}^A = b_\perp^A/(\rho_0 h^* \lambda_{(\mathrm{m}\pm)}^2 - b^2)$, the other two rescaled magnetosonic left eigenvectors are finally given by:

$$\text{Remaining magnetosonic:} \quad \begin{pmatrix} \frac{1}{c_s^2\rho_0 h} & \frac{(1-c_s^2)\lambda}{c_s^2(\lambda^2-1)} & \left(\frac{b^S}{\lambda}\right)\mathcal{F}^A & 0 & \mathcal{F}^A & 0 \end{pmatrix}_{(\mathrm{m}_2\pm)}. \tag{5.71}$$

These eigenvectors are also well defined for type I and type II degeneracies. Similar to $\mathcal{H}$, the numerator as well as the denominator of $\mathcal{F}^A$ become zero under type II′ degeneracies. Nevertheless, $\mathcal{F}^A = 0$ is taken. By taking the appropriate type II′ limit of $\mathcal{F}^A$ one can show that $\mathcal{F}^A$ will not diverge in any case.

The eight rescaled upper case left eigenvectors (5.62), (5.63), (5.64), (5.69), and (5.71) with abbreviations,

$$\mathcal{H} = \frac{|b_\perp|}{c_s^2 - \lambda_{(\mathrm{m}\pm)}^2}\,, \qquad \mathcal{F}^A = \frac{b_\perp^A}{(\rho_0 h^* \lambda_{(\mathrm{m}\pm)}^2 - b^2)}\,, \tag{5.72}$$

form a linearly independent set for all possible states. For type II degeneracy, the abbreviations $\mathcal{H}$ and $\mathcal{F}^A$ vanish automatically and it is arranged to take

$$\frac{b_C^\perp}{|b_\perp|} = \frac{1}{\sqrt{2}}(Q_{1C} + Q_{2C})\,. \tag{5.73}$$

For type II′ degeneracy, in agreement with the type II degenerate case, it is arranged to take

$$\frac{b_C^\perp}{|b_\perp|} = \frac{1}{\sqrt{2}}(Q_{1C} + Q_{2C})\,, \qquad \mathcal{H} = 0\,, \qquad \mathcal{F}^A = 0^A\,. \tag{5.74}$$

The equations (5.73) - (5.74) are just a canonical choice to represent the complete set of upper case eigenvectors under a type II or type II′ degenerate limit. For type II′ degeneracies, depending on how the limit is taken, their values may not vanish but the form (5.69) and (5.71) for the upper case magnetosonic left eigenvectors with $\mathcal{H} = 0$, $\mathcal{F}^A = 0$ can nevertheless be obtained by taking appropriate linear combinations of the resulting eigenvectors of the respective subspace.

5.4.4 Renormalized Upper Case Right Eigenvectors

The rescaled upper case right eigenvectors are obtained from the old ones (5.52) in a similar way and with the same abbreviations. The rescaled entropy and constraint upper case right eigenvectors remain the same, and the rescaled Alfvén eigenvectors are just the old ones divided by the norm of the perpendicular magnetic field. The rescaled upper case right eigenvectors are:

$$\text{Entropy:} \qquad \begin{pmatrix} 0 & 0 & 0_B & 0 & 0_B & 1 \end{pmatrix}^T, \tag{5.75}$$

$$\text{Constraint:} \qquad \begin{pmatrix} 0 & 0 & 0_B & 1 & 0_B & 0 \end{pmatrix}^T, \tag{5.76}$$

$$\text{Alfvén:} \qquad \begin{pmatrix} 0 & 0 & \pm {}^{(\mathrm{S})}\epsilon_{BC}\frac{b^C}{|b_\perp|} & 0 & {}^{(\mathrm{S})}\epsilon_{BC}\sqrt{\rho_0 h^*}\frac{b^C}{|b_\perp|} & 0 \end{pmatrix}^T. \tag{5.77}$$

The rescaled upper case magnetosonic right eigenvectors with eigenvalues closer to the Alfvén waves are the previous ones (5.52), multiplied by $\mathcal{H}p/\rho_0$, where the third and fifth component are simplified with the help of identities following from $P_{\mathrm{mgs}}(\lambda_{(\mathrm{m}\pm)}) = 0$.

They can be explicitly written as:

$$\text{Closer to Alfvén:} \quad \left(c_s^2\rho_0 h\mathcal{H} \quad -\mathcal{H}\lambda \quad -\left(\tfrac{b^S}{\lambda}\right)\tfrac{b_B^\perp}{|b_\perp|} \quad 0 \quad \tfrac{\rho_0 h}{(\lambda^2-1)}\tfrac{b_B^\perp}{|b_\perp|} \quad \tfrac{p}{\rho_0}\mathcal{H} \right)^T_{(\mathrm{m}_1\pm)} . \tag{5.78}$$

The remaining pair of upper case magnetosonic eigenvectors does not need a rescaling. They are just multiplied for convenience by the factor p/ρ_0, and the third and fifth entry are reexpressed in a suitable form to treat the occurring degeneracies with the help of P_{mgs}. The rescaled pair of remaining upper case magnetosonic eigenvectors is then given by:

$$\text{Remaining:} \quad \left(c_s^2\rho_0 h \quad -\lambda \quad c_s^2(1-\lambda^2)\left(\tfrac{b^S}{\lambda}\right)\mathcal{F}_B \quad 0 \quad c_s^2\rho_0 h\mathcal{F}_B \quad \tfrac{p}{\rho_0} \right)^T_{(\mathrm{m}_2\pm)} . \tag{5.79}$$

These eight vectors form a linear independent set of upper case right eigenvectors of the principal symbol in all possible states.

5.4.5 Upper Case Characteristic Variables

The eight upper case characteristic variables, as defined in section 2.2, valid for all degeneracies are

$$\begin{aligned}
\hat{\mathrm{U}}_{\mathrm{e}} &= \delta\varepsilon - \frac{p}{c_s^2\rho_0^2 h}\delta p\,, \qquad\qquad \hat{\mathrm{U}}_{\mathrm{c}} = (\delta\perp b)_{\hat{\mathsf{S}}}\,,\\
\hat{\mathrm{U}}_{\mathrm{a}\pm} &= \pm\, {}^{(\mathrm{S})}\epsilon^{AC}\sqrt{\rho_0 h^*}\frac{b_C^\perp}{|b_\perp|}(\delta\hat{v})_{\hat{\mathsf{A}}} + {}^{(\mathrm{S})}\epsilon^{AC}\frac{b_C^\perp}{|b_\perp|}(\delta\perp b)_{\hat{\mathsf{A}}}\,,\\
\hat{\mathrm{U}}_{\mathrm{m}_1\pm} &= \frac{\mathcal{H}(\lambda^2-1)}{\rho_0 h}\delta p + (1-c_s^2)\mathcal{H}\lambda(\delta\hat{v})_{\hat{\mathsf{S}}} + \left(\frac{b^S}{\lambda}\right)\frac{b_\perp^A}{|b_\perp|}(\delta\hat{v})_{\hat{\mathsf{A}}} + \frac{b_\perp^A}{|b_\perp|}(\delta\perp b)_{\hat{\mathsf{A}}}\,,\\
\hat{\mathrm{U}}_{\mathrm{m}_2\pm} &= \frac{1}{c_s^2\rho_0 h}\delta p + \frac{(1-c_s^2)\lambda}{c_s^2(\lambda^2-1)}(\delta\hat{v})_{\hat{\mathsf{S}}} + \left(\frac{b^S}{\lambda}\right)\mathcal{F}^A(\delta\hat{v})_{\hat{\mathsf{A}}} + \mathcal{F}^A(\delta\perp b)_{\hat{\mathsf{A}}}\,,
\end{aligned} \tag{5.80}$$

with $\{\mathrm{m}_1, \mathrm{m}_2\}$ for magnetosonic waves closer to the Alfvén waves (m_1) and the remaining magnetosonic waves (m_2), equal to $\{\mathrm{s}, \mathrm{f}\}$ or $\{\mathrm{f}, \mathrm{s}\}$, depending on the numerical value of the speed of sound c_s. On the right-hand side of $\hat{\mathrm{U}}_{\mathrm{m}_1\pm}$ and $\hat{\mathrm{U}}_{\mathrm{m}_2\pm}$ the respective eigenvalue is taken. Since the resulting similarity transformation matrix $\mathbf{T}_S$ and its inverse $\mathbf{T}_S^{-1}$ always exist and have bounded components, the regularity condition (2.11) is satisfied. This shows that the prototype algebraic constraint free system is in the upper case strongly hyperbolic. Since all the eigenvalues have absolute values smaller than or equal to one, the system must also be strongly hyperbolic in the lower case frame. Also, the procedure to recover the lower case characteristic quantities can be applied.

5.5 Lower Case Formulation

By the results of the last section it is clear that the prototype algebraic constraint free formulation is strongly hyperbolic, independent of the frame. Nevertheless, the lower case eigenvalues and eigenvectors would be important to employ the system numerically and therefore, their derivation is performed in the next subsections. Since the transformations from upper to lower case depend on several quantities, the unscaled upper case eigenvectors are used for the recovery. Afterwards, degeneracies in the lower case are studied and the rescaled eigenvectors are derived. From now on all upper case characteristic quantities are explicitly indicated by a script ‘u’.

5.5.1 Recovering the Lower Case Quantities

To obtain the lower case eigenvalues and eigenvectors as well as the characteristic variables the procedure described in section 3.3 is used. The recovery is performed in several steps, which are explained in the following paragraphs.

Step one. First of all, the calculated upper case eigenvalues (5.42) - (5.46) are taken and the upper case unit spatial vector S^a is replaced by $S^a_\lambda = (S^a - W(\lambda - WV^S))/N$ (with $\mathfrak{s}_a = \gamma^b{}_a S_b$), whereby the new upper case eigenvalues

$$\lambda^{\mathrm{u}}_{(\mathrm{e})} = 0\,, \tag{5.81}$$

$$\lambda^{\mathrm{u}}_{(\mathrm{c})} = 0\,, \tag{5.82}$$

$$\lambda^{\mathrm{u}}_{(\mathrm{a}\pm)} = \pm\frac{b^{S_\lambda}}{\sqrt{\rho_0 h^*}}\,, \tag{5.83}$$

$$\lambda^{\mathrm{u}}_{(\mathrm{s}\pm)} = \pm\sqrt{\zeta_{\mathrm{S}_\lambda} - \sqrt{\zeta^2_{\mathrm{S}_\lambda} - \xi_{\mathrm{S}_\lambda}}}\,, \tag{5.84}$$

$$\lambda^{\mathrm{u}}_{(\mathrm{f}\pm)} = \pm\sqrt{\zeta_{\mathrm{S}_\lambda} + \sqrt{\zeta^2_{\mathrm{S}_\lambda} - \xi_{\mathrm{S}_\lambda}}}\,, \tag{5.85}$$

are obtained. Please note that the right-hand sides of the eigenvalues contain the respective lower case eigenvalues in all S_λ-terms. The abbreviations $\zeta_{\mathrm{S}_\lambda}$ and ξ_{S_λ} are explicitly written as

$$\zeta_{\mathrm{S}_\lambda} = \frac{\left(b^2 + c_s^2\left[\left(b^{S_\lambda}\right)^2 + \rho_0 h\right]\right)}{2\rho_0 h^*}\,, \quad \xi_{\mathrm{S}_\lambda} = \frac{\left(b^{S_\lambda}\right)^2 c_s^2}{\rho_0 h^*}\,, \tag{5.86}$$

and depend now on the considered lower case eigenvalue itself. The magnetic field vector in the new lower case eigenvalue dependent direction S^a_λ becomes

$$b^{S_\lambda} = b^a S^\lambda_a = \frac{1}{N}\left(b^S - W(b^a V_a)(\lambda - WV^S)\right) , \tag{5.87}$$

$$N = \sqrt{(W\lambda - W^2 V^S)^2 + 1 + (V^S)^2 W^2 - \lambda^2} \,. \tag{5.88}$$

Step two. With the results of step one, the lower case eigenvalues are now calculated by using equation (3.36), that is

$$\frac{1}{N} W(\lambda - WV^S) = \lambda_\mathrm{u}[S^a_\lambda] \,. \tag{5.89}$$

For example, taking the upper case entropy eigenvalue $\lambda_\mathrm{u}[S^a_\lambda] = \lambda^\mathrm{u}_{(\mathrm{e})} = 0$, one arrives at the lower case entropy wave speed $\lambda_{(\mathrm{e})} = WV^S$. In this case, the normalization factor N becomes unity, and S^a and S^a_λ are identical.

Step three. By the last step, the lower case eigenvalues are already known. Hence, the eigenvectors can be treated, beginning with the left ones. The transformation of the upper case left eigenvectors for S^a_λ into the lower case left eigenvectors is performed for the state vector

$$(\delta \mathbf{U})_{\hat{\mathbb{S}}, \hat{\mathbb{A}}} = (\delta p, (\delta \hat{v})_{\hat{\mathbb{S}}}, (\delta \hat{v})_{\hat{\mathbb{A}}}, (\delta {\perp} b)_{\hat{\mathbb{S}}}, (\delta {\perp} b)_{\hat{\mathbb{A}}}, \delta \varepsilon)^T ,$$

which is the same as used in the upper case analysis. The transformation is λ-dependent and therefore has to be performed in each eigenspace independently. The lower case left eigenvectors are obtained by equation (3.46), that is

$$\mathbf{l}^\mathrm{n}_\lambda\big|_{\mathbb{S}} = \mathbf{l}^\mathrm{u}_{\lambda_\mathrm{u}}[S^a_\lambda]\Big|_{\mathbf{S}_\lambda} \left(\mathbb{1} + \mathbf{B}^V\big|_{\mathbf{S}_\lambda}\right) \mathbf{T}_\lambda \,, \tag{5.90}$$

where the eigenvectors $\mathbf{l}^\mathrm{u}_{\lambda_\mathrm{u}}[S^a]\big|_{\mathbf{S}}$ explicitly written in (5.48) - (5.51) are used. The second strategy (see the explanation in section 3.3) is followed and all basis vectors are replaced by the ones associated with S^a_λ. The replacement causes automatically a rotation of the basis in which the eigenvector is expanded. Therefore, on the right-hand side of equation (5.90) the transformation matrix $\mathbf{T}_\lambda$ is in use. The matrices $\left(\mathbb{1} + \mathbf{B}^V\big|_{\mathbf{S}}\right)$ and $\left(\mathbb{1} + \mathbf{B}^V\big|_{\mathbf{S}_\lambda}\right)$ for bases $\mathbf{S} = \{S^a, {Q_1}^a, {Q_2}^a\}$ and $\mathbf{S}_\lambda = \{S^a_\lambda, {Q_1}^a_\lambda, {Q_2}^a_\lambda\}$ can be found in the accompanying notebook, see appendix A.

To obtain the basis transformation $\mathbf{T}_\lambda$ more details are needed. Writing S^a_λ in the basis $\mathbf{S}$, one gets

$$\begin{aligned} S_\lambda{}^a &= c_S S^a + c_1 Q_1{}^a + c_2 Q_2{}^a\,, & c_S &= \frac{1+(W^2V^S - W\lambda)V^S}{N}\,, \\ c_1 &= \frac{(W^2V^S - W\lambda)V^{Q_1}}{N}\,, & c_2 &= \frac{(W^2V^S - W\lambda)V^{Q_2}}{N}\,. \end{aligned} \tag{5.91}$$

This relation defines a rotation of the basis, depending on the coefficients c_S, c_1 and c_2 and thus on the respective lower case eigenvalue. Thus, a transformation matrix which is an element of SO(3) can be build. By denoting $Q_1{}_\lambda^a$ and $Q_2{}_\lambda^a$ as rotated basis vectors $Q_1{}^a$ and $Q_2{}^a$ (see also table 3.4 and the associated explanations in the text below it), respectively, the rotation matrix is given by

$$\mathbf{R} = \begin{pmatrix} c_S & c_1 & c_2 \\ -c_1 & \frac{c_S c_1^2 + c_2^2}{c_1^2+c_2^2} & \frac{(c_S-1)c_1c_2}{c_1^2+c_2^2} \\ -c_2 & \frac{(c_S-1)c_1c_2}{c_1^2+c_2^2} & \frac{c_1^2+c_S c_2^2}{c_1^2+c_2^2} \end{pmatrix} \tag{5.92}$$

such that

$$\begin{pmatrix} S_\lambda^a \\ Q_1{}_\lambda^a \\ Q_2{}_\lambda^a \end{pmatrix} = \mathbf{R} \begin{pmatrix} S^a \\ Q_1{}^a \\ Q_2{}^a \end{pmatrix}. \tag{5.93}$$

Since $\mathbf{R} \in \mathrm{SO}(3)$, its transpose is equal to its inverse, $\mathbf{R}^T = \mathbf{R}^{-1}$. The associated lower case bases obey the same transformation rule, as can be easily seen by multiplying equation (5.93) with $\gamma^b{}_a$. The transformation matrix is taken to be $\mathbf{T}_\lambda = \mathrm{diag}(1, \mathbf{R}, \mathbf{R}, 1)$, adjusted to the order of vector and scalar variables in the state vector. The derivative of the state vector transforms according to

$$(\delta\mathbf{U})_\mathbf{S} = \mathbf{T}_\lambda^T (\delta\mathbf{U})_{\mathbf{S}_\lambda}. \tag{5.94}$$

By definition, $\mathbf{T}_\lambda$ shares the properties of $\mathbf{R}$, for example, its transpose is equal to the inverse, $\mathbf{T}_\lambda^T = \mathbf{T}_\lambda^{-1}$. For an upper case eigenvalue equal to zero, the transformation matrix becomes just the identity matrix and the bases $\mathbf{S}$ and $\mathbf{S}_\lambda$ coincide.

Step four. As a last step, the right eigenvectors have to be calculated by equation (3.47), that is

$$\mathbf{r}_\lambda^{\mathrm{n}}\big|_{\mathbf{s}} = \mathbf{T}_\lambda^T\, \mathbf{r}_{\lambda_\mathrm{u}}^{\mathrm{u}}[S_\lambda^a]\big|_{\mathbf{S}_\lambda}. \tag{5.95}$$

For this the right eigenvectors $\mathbf{r}^{\mathrm{u}}_{\lambda_{\mathrm{u}}}[S^a]\big|_{\mathbf{S}}$ given in (5.52) are taken with replaced basis vectors.

5.5.2 Definitions and Formulas

In order to write the lower case eigenvectors in a short form, but also in a suitable way to both compare them to the literature [Antón et al., 2010] and take into account the long expressions of the lower case magnetosonic eigenvalues, some new abbreviations are defined in this subsection. Another purpose is the easy 'translation' of degeneracies, which are explained in the next subsection.

First, the quantity a is introduced, which is just the upper case eigenvalue (depending on S_λ) times N, where N is the normalization factor (3.34) to guarantee $S^a_\lambda S^\lambda_a = 1$. The newly introduced quantity a can be expressed as

$$a := N\lambda_{\mathrm{u}} = W\lambda - W^2 V^S = W\lambda + \hat{v}^{\mathfrak{s}}\,, \tag{5.96}$$

where the relation between the eigenvalues (5.89) and the equality $WV^S = -v^{\mathfrak{s}}$ is used.

As the second quantity the variable $\mathcal{B}$ is introduced. It is just the upper case magnetic field in direction S^a_λ times N and can be seen as a weighted version of the magnetic field in direction of S^a_λ. Explicitly, it is given by

$$\mathcal{B} := N b^{S_\lambda} = N b^a S^\lambda_a = b^S - (b^a V_a) a = b^S + (b^a V_a) W (V^S W - \lambda)\,, \tag{5.97}$$

or in terms of lower case spatial quantities

$$\mathcal{B} = \hat{b}^{\mathfrak{s}} + (\hat{b}^a v_a)\lambda = \frac{1}{W}(B^{\mathfrak{s}} + (B^a \hat{v}_a)\hat{v}^{\mathfrak{s}}) + (B^a \hat{v}_a)\lambda\,. \tag{5.98}$$

Last, the quantity $\mathcal{G}$ is introduced as

$$\mathcal{G} := 1 + (V^S)^2 W^2 - \lambda^2 \tag{5.99}$$

such that

$$N^2 = a^2 + \mathcal{G} \tag{5.100}$$

holds.

The introduced quantities a, $\mathcal{B}$ and $\mathcal{G}$ are motivated by the ones made in [Anile and Pennisi, 1987; Antón et al., 2010] in regard to the covariant approach of the characteristic analysis shown by equation (3.48). They slightly differ from their respective definition in the literature. This is due to the fact that the unit spatial lower case 1-form $\mathfrak{s}_a$ is normalized with respect to the inverse boost metric (instead of the inverse three-metric γ^{ab}).

In analogy to the upper case degeneracy analysis, the upper case magnetic field vector is cast into the form

$$b^a = b^{S_\lambda} S^a_\lambda + b^a_\perp \,, \qquad b^2 = (b^{S_\lambda})^2 + b^2_\perp \,, \tag{5.101}$$

with

$$|b_\perp|^2 = b^2 - (b^{S_\lambda})^2 = b^a_\perp b^\perp_a \,, \tag{5.102}$$

where again the adjectives *parallel* and *perpendicular* are used to denote the parts of the magnetic field in direction of S^a_λ and orthogonal to it, respectively. Please note that nevertheless capital letters are still taken for contraction with ${}^{\mathrm{Q}}\!\perp$, e.g., $b^A_\perp = {}^{\mathrm{Q}}\!\perp^A{}_a b^a_\perp$. In general, $b^S_\perp \neq 0$ is not vanishing and therefore $b^A_\perp \neq b^A$, in contrast to the definition of the perpendicular magnetic field in the last section. Although the parallel and perpendicular parts of the magnetic field are introduced in a similar way to the upper case analysis, one has to keep in mind that these names now highly depend on the wave under consideration. Hence, this notation is always meant with respect to the wavefront determined by the eigenvalue, despite being not explicitly mentioned anymore. Definition (5.101) and the explained notation of its parts is taken for all lower case characteristic quantities. Since $b^a_\perp$ is orthogonal to S^a_λ, the relation $b^S_\perp = a(b^a_\perp V_a)$ is used several times according to which respective expression is suitable.

5.5.3 Degeneracy Analysis of the Lower Case

Before the lower case eigenvalues and eigenvectors are recovered, the lower case degeneracies are explained. The degeneracy analysis for RMHD is already given in the literature, see [Komissarov, 1999; Antón et al., 2010]. The lower case frame degeneracy analysis is essentially the same as in the upper case setting and becomes much easier by having knowledge about the upper case system (i.e., the discussion in the rest frame of the fluid). One only has to replace the vector S^a by S^a_λ defined in equation (3.33),

$$S^a \longrightarrow S^a_\lambda = \frac{1}{N}\left(S^a - V^a(\hat{v}^{\mathfrak{s}} + W\lambda)\right) \,, \qquad S^a_\lambda S^\lambda_a = 1 \,,\; S^a S_a = 1 \,,\; \mathfrak{s}_a = \gamma^b{}_a S_b \,, \tag{5.103}$$

and the corresponding orthogonal basis vectors as well. The lower case eigenvalues are again ordered as

$$\lambda^-_{(\mathrm{f})} \leq \lambda^-_{(\mathrm{a})} \leq \lambda^-_{(\mathrm{s})} \leq \lambda_{(\mathrm{e})} \leq \lambda^+_{(\mathrm{s})} \leq \lambda^+_{(\mathrm{a})} \leq \lambda^+_{(\mathrm{f})}\,, \tag{5.104}$$

where the lower case entropy eigenvalue $\lambda_{(\mathrm{e})} = -v^{\mathfrak{s}}$ separates each pair of eigenvalues into two classes, denoted by a superscript '+' or '−', referring to the higher or lower value of each pair, respectively. Subscripts will always refer to the respective plus or minus sign on the right-hand side of their definition, where the eigenvalues are casually referred to as the 'positive' and 'negative' eigenvalue.

Type I degeneracy. For type I degeneracy in the upper case, the parallel magnetic field (for both Alfvén waves) vanishes, leading to a total multiplicity of six for the entropy subspace. However, in the lower case, the unit spatial vector is now λ-dependent and a priori has different directions for the two Alfvén eigenvalues, $S^a_{\lambda_{(\mathrm{a}+)}} \neq S^a_{\lambda_{(\mathrm{a}-)}}$. Hence, consider first $b^a S^\lambda_a = 0$ for $\lambda_{(\mathrm{a}+)}$ (and thereby also $a_{(\mathrm{a}+)} = 0$ and $\mathcal{B}_{(\mathrm{a}+)} = 0$ by equations (5.96) and (5.97)). Then, by definition, consulting equation (5.83), the Alfvén eigenvalue $\lambda^{\mathrm{u}}_{(\mathrm{a}+)}[S^a_\lambda]$ is equal to zero and the lower case Alfvén eigenvalue coincides with the entropy eigenvalue, $\lambda_{(\mathrm{a}+)} = \lambda_{(\mathrm{e})} = WV^S$. Consequently, considering again equation (5.97) (or (5.87)), one finds that the magnetic field in direction of S^a vanishes,

$$\text{Lower case type I degeneracy:} \qquad b^S = 0\,. \tag{5.105}$$

By taking the lower case eigenvalue $\lambda = WV^S$ one also finds $S^a_\lambda = S^a$ for the positive Alfvén wave. The result (5.105) serves as a λ-independent classification of the lower case type I degeneracy leading to exactly the same multiplicities as in the upper case system. The entropy wave, the constraint wave, the two Alfvén waves, and the two slow magnetosonic waves propagate at the same speed:

$$\lambda_{(\mathrm{e})} = \lambda_{(\mathrm{c})} = \lambda^\pm_{(\mathrm{a})} = \lambda^\pm_{(\mathrm{s})} = -v^{\mathfrak{s}}. \tag{5.106}$$

Taking these eigenvalues, their respective S^a_λ coincide in the type I degeneracy with S^a, $S^a_\lambda = S^a$. One also finds $a = 0$ and $N = \mathcal{G} = 1$.

Type II and II′ degeneracy. For type II degeneracy the perpendicular magnetic field vector, $b^a_\perp = {}^{Q_\lambda}\!\perp^a{}_b\, b^b$, ${}^{Q_\lambda}\!\perp^a{}_b = {}^{(u)}\gamma^a{}_b - S^a_\lambda S^\lambda_b$, vanishes for one of the Alfvén waves:

$$\text{Lower case type II degeneracy:} \qquad b^a_\perp = 0\,, \qquad \text{for } \lambda = \lambda_{(a+)} \text{ or } \lambda = \lambda_{(a-)}\,. \tag{5.107}$$

In this case, one of the Alfvén waves and one of the magnetosonic waves of the same class have the same speed:

$$\lambda^+_{(a)} = \lambda^+_{(s)} \quad \text{or} \quad \lambda^-_{(a)} = \lambda^-_{(s)} \quad \text{or} \quad \lambda^+_{(a)} = \lambda^+_{(f)} \quad \text{or} \quad \lambda^-_{(a)} = \lambda^-_{(f)}\,. \tag{5.108}$$

In the type II′ degeneracy one of the Alfvén waves and the slow and fast magnetosonic wave of the same class travel at the same speed:

$$\lambda^+_{(s)} = \lambda^+_{(a)} = \lambda^+_{(f)} \qquad \text{or} \qquad \lambda^-_{(s)} = \lambda^-_{(a)} = \lambda^-_{(f)}\,. \tag{5.109}$$

In the upper case for type II and type II′ degeneracy both Alfvén speeds are degenerate at the same time. Replacing S^a by S^a_λ leads to different SO(3) transformations for different values of λ. The resulting two spatial vectors are in general not parallel:

$$S^a_\lambda|_{\lambda_{(a+)}} \nparallel S^a_\lambda|_{\lambda_{(a-)}} \tag{5.110}$$

Therefore, in the lower case this cannot be satisfied in general. However, if the upper case velocity V^a is parallel to S^a, one finds

$$V^a \parallel S^a: \quad S^a_\lambda = \frac{1}{N}\left(1 - V^S(\hat{v}^{\mathfrak{s}} + W\lambda)\right) S^a = \frac{\left(1 - V^S(\hat{v}^{\mathfrak{s}} + W\lambda)\right)}{\sqrt{\left(1 - V^S(\hat{v}^{\mathfrak{s}} + W\lambda)\right)^2}} S^a = \pm S^a, \tag{5.111}$$

and both Alfvén waves can become degenerate at the same time. Hence, the fact that for $V^a \nparallel S^a$ only one Alfvén wave exhibits a type II or type II′ degeneracy at the same time, is a consequence of aberration.[3]

As a last comment, one should note that

$$V^a \parallel S^a \Longleftrightarrow V^a \parallel S^a_\lambda \tag{5.112}$$

holds independently of λ, as can be obtained by considering the definition of S^a_λ. In both cases one finds $S^a = \pm S^a_\lambda$ and vice versa, independently of λ. For a vanishing fluid

[3] This was already mentioned in [Antón et al., 2010].

velocity with $V^a = 0$, one finds $S^a = \pm S^a_\lambda$ independently of λ.

5.5.4 Lower Case Eigenvalues and Eigenvectors

The lower case eigenvalues and eigenvectors are given below. For simplicity, the already rescaled lower case left and right eigenvectors are partially given directly. They are obtained by first transforming a given unscaled upper case eigenvector to the respective lower case one, and then performing a rescaling procedure in the same way as in the upper case analysis in section 5.4. The rescaling can be found in the provided notebook, see appendix A.

Entropy wave. Taking $\lambda_\mathrm{u} = 0$ as in (5.81), the corresponding lower case entropy eigenvalue is calculated to

$$\lambda_{(\mathrm{e})} = WV^S = -v^{\$}\,, \tag{5.113}$$

in agreement with the results of [Antón et al., 2010]. Inserting this eigenvalue in the defining equation (3.34) for N, one finds $N = 1$ and $S^a_\lambda = S^a$. Thus, the transformation matrix $\mathbf{T}_\lambda$ becomes the identity matrix and the upper (5.48) and lower case left eigenvector for entropy waves coincide, given by

$$\begin{pmatrix} -\frac{p}{c_s^2\rho_0^2 h} & 0 & 0^A & 0 & 0^A & 1 \end{pmatrix}. \tag{5.114}$$

The same holds for the upper (5.52) and corresponding lower case right eigenvector for entropy waves, given by

$$\begin{pmatrix} 0 & 0 & 0_B & 0 & 0_B & 1 \end{pmatrix}^T. \tag{5.115}$$

Both lower case entropy eigenvectors are well defined under degeneracies and are valid for all states.

Constraint wave. Taking again $\lambda_\mathrm{u} = 0$ as in (5.82), the lower case constraint eigenvalue is given by

$$\lambda_{(\mathrm{c})} = WV^S = -v^{\$}\,. \tag{5.116}$$

The lower case constraint eigenvalue agrees with the one given in [Antón et al., 2010]. Similar to the entropy eigenvalue one finds $N = 1$ and $S^a_\lambda = S^a$ and the lower case left and right eigenvectors for constraint waves are equal to their corresponding upper case versions in (5.49) and (5.52), given by

$$\begin{pmatrix} 0 & (b^C V_C) & -b^S V^A & 1 & 0^A & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & 0_B & 1 & 0_B & 0 \end{pmatrix}^T, \tag{5.117}$$

respectively. Both eigenvectors are well defined under degeneracies and are valid for all states.

Alfvén waves. The Alfvén waves are obtained by taking the upper case Alfvén eigenvalues (5.83) and solving for λ. Thus, the lower case Alfvén eigenvalues become

$$\lambda_{(\mathrm{a}\pm)} = \frac{b^S + V^S W^2 \left[(b^a V_a) \pm \sqrt{\rho_0 h^*}\right]}{W \left[(b^a V_a) \pm \sqrt{\rho_0 h^*}\right]} . \tag{5.118}$$

In terms of lower case quantities they can be expressed as

$$\lambda_{(\mathrm{a}\pm)} = -\frac{\hat{b}^{\$} \mp \hat{v}^{\$} \sqrt{\rho_0 h^*}}{(\hat{b}^a v_a) \mp W \sqrt{\rho_0 h^*}} = -\frac{(B^{\$} + (B^a \hat{v}_a)\hat{v}^{\$}) \mp \hat{v}^{\$} W \sqrt{\rho_0 h^*}}{W \left((B^a \hat{v}_a) \mp W \sqrt{\rho_0 h^*}\right)} . \tag{5.119}$$

They coincide up to a minus sign and factor W (due to the choice of the spatial vector) with the derived ones in [Antón et al., 2010]. Taking the unscaled upper case left (5.50) and right (5.52) eigenvectors, transforming them in the respective way and dividing both by $|b_\perp|$, the rescaled lower case left and right Alfvén eigenvectors to $\lambda_{(\mathrm{a}\pm)}$ read

$$\begin{pmatrix} \pm \frac{{}^{(\mathrm{S})}\epsilon_{BC}}{\sqrt{\rho_0 h^*}} \frac{V^B b_\perp^C}{|b_\perp|} \\ b^S \, {}^{(\mathrm{S})}\epsilon_{BC} \frac{b_\perp^B V^C}{|b_\perp|} \\ \left((b^a V_a) \pm \sqrt{\rho_0 h^*}\right) {}^{(\mathrm{u})}\epsilon^A{}_{bc} \frac{N S^b_{\lambda_{(\mathrm{a}\pm)}} b_\perp^c}{|b_\perp|} \\ 0 \\ -{}^{(\mathrm{S})}\epsilon^A{}_B \left(\frac{b_\perp^B}{|b_\perp|} \pm \frac{|b_\perp| V^B}{\sqrt{\rho_0 h^*}} \right) \\ 0 \end{pmatrix}^T , \quad \begin{pmatrix} 0 \\ \frac{b^S}{\sqrt{\rho_0 h^*}} {}^{(\mathrm{S})}\epsilon_{AC} \frac{b_\perp^A V^C}{|b_\perp|} \\ \frac{(b^b V_b) \pm \sqrt{\rho_0 h^*}}{\sqrt{\rho_0 h^*}} {}^{(\mathrm{u})}\epsilon_{Bac} \frac{N S^a_{\lambda_{(\mathrm{a}\pm)}} b_\perp^c}{|b_\perp|} \\ \pm b^S \, {}^{(\mathrm{S})}\epsilon_{AC} \frac{b_\perp^A V^C}{|b_\perp|} \\ \left(\sqrt{\rho_0 h^*} \pm (b^b V_b)\right) {}^{(\mathrm{u})}\epsilon_{Bac} \frac{N S^a_{\lambda_{(\mathrm{a}\pm)}} b_\perp^c}{|b_\perp|} \\ 0 \end{pmatrix} , \tag{5.120}$$

respectively. The transformation procedure as well as the rescaling can be found in the accompanying notebook, see appendix A. The transformation matrices are too lengthy and therefore not given here.

Magnetosonic waves. The upper case slow and fast magnetosonic eigenvalues are defined in (5.84) and (5.85), respectively. Inserting one of these eigenvalues into equation (5.89), one can show after some manipulations (given in the notebook) that the lower case magnetosonic eigenvalues are solutions of the quartic equation

$$\mathcal{N}_4 = \rho_0 h \left(\frac{1}{c_s^2} - 1 \right) a^4 - \left(\rho_0 h + \frac{b^2}{c_s^2} \right) a^2 \mathcal{G} + \mathcal{B}^2 \mathcal{G} = 0 \,, \tag{5.121}$$

where $\mathcal{N}_4$ is the same polynomial as obtained by [Anile and Pennisi, 1987], but with slightly differently defined abbreviations a, $\mathcal{B}$ and $\mathcal{G}$ in subsection 5.5.2. It is always possible to find analytic expressions for the zeros of a polynomial with order less than or equal to four. Therefore, in the case under consideration, the magnetosonic eigenvalues can be given explicitly. However, the expressions for them are quite long and are (partially) only analytically amenable under special circumstances. Therefore, in regard to the numerical usage, the numerical computation relying on the characteristic information by using some root-finder should be more convenient.

By the recovery procedure, it is nevertheless easily possible to obtain the magnetosonic eigenvectors. The long expressions for the magnetosonic eigenvalues are hidden in various quantities. A brute force computation with Mathematica performed in advance, considering the eigenvalue problem for the lower case principal symbol, was unsuccessful.

The rescaled lower case magnetosonic eigenvectors are obtained as follows. First, the upper case magnetosonic left (5.51) and right (5.52) eigenvectors are considered, the basis vectors replaced by their λ-depend versions, and multiplied with the transformation matrices in the respective way. As a result, one arrives at a first set of lower case left and right eigenvectors, valid in non-degenerate states. After a simplification, they can be rewritten in a short form using the abbreviations a, $\mathcal{B}$ and $\mathcal{G}$ defined in subsection 5.5.2. The unscaled lower case magnetosonic left and right eigenvectors are

$$\begin{pmatrix} 1 - aV^S - \frac{\mathcal{B} b^S}{a^2} \frac{\mathcal{G}}{\rho_0 h} \\ \frac{\mathcal{B}^2}{a} + (a^2 + \mathcal{G}) \frac{(b^S)^2}{a} + \frac{\mathcal{B}}{a}(aV^S - 1) b^S - a\rho_0 h^* \\ \frac{b^S}{a} \left((a^2 + \mathcal{G}) b^A + a\mathcal{B} V^A \right) \\ 0 \\ a b^S V^A + b^A (1 - aV^S) \\ 0 \end{pmatrix}^T , \begin{pmatrix} c_s^2 \rho_0 h \left(\rho_0 h \left[a^2 + \mathcal{G} - \frac{\mathcal{B}^2}{a^2} \mathcal{G} \right] \right) \\ \frac{\mathcal{B}}{a} \mathcal{G} b^S + \rho_0 h a (aV^S - 1) \\ \frac{\mathcal{B}}{a} \mathcal{G} b_B + \rho_0 h a^2 V_B \\ \rho_0 h \left(\left[a^2 + \mathcal{G} \right] b^S + \mathcal{B}(aV^S - 1) \right) \\ \rho_0 h \left(\left[a^2 + \mathcal{G} \right] b_B + a\mathcal{B} V_B \right) \\ \frac{p}{\rho_0} \left(\rho_0 h \left[a^2 + \mathcal{G} - \frac{\mathcal{B}^2}{a^2} \mathcal{G} \right] \right) \end{pmatrix} , \tag{5.122}$$

respectively. With this result, the study of the different degeneracies and the rescaling can be performed. To clarify the behavior of the eigenvectors, the upper case magnetic field vector is split into the parallel and perpendicular part as in equation (5.101). The eigenvectors are already well defined under a type I degeneracy, where $a = \mathcal{B} = 0$. By various relations linked to the magnetosonic polynomial (5.121), one can show that all entries of all left and right eigenvectors (5.122) are proportional to $|b_\perp|$. Hence, they become zero under type II degeneracies and both need to be divided by $|b_\perp|$. As in the upper case, to guarantee a complete set of linearly independent eigenvectors, only the eigenvectors with eigenvalues closer to the Alfvén waves are rescaled in this way. The remaining two left and two right eigenvectors are divided by $(a^2\rho_0 h^* - \mathcal{G}b^2)$ and simplified by use of the magnetosonic polynomial (5.121).

The rescaled lower case left and right magnetosonic eigenvectors with eigenvalues closer to the Alfvén speeds can be expressed as

$$\begin{pmatrix} -\frac{\mathcal{G}}{\rho_0 h}\frac{(b_\perp^a V_a)}{|b_\perp|}\left(\frac{\mathcal{B}}{a}\right) - \frac{(1-aV^S)\mathcal{G}}{\rho_0 h}\mathcal{F} \\ a(a^2+\mathcal{G})\left((1-c_s^2)\mathcal{F} + \left[\left(\frac{\mathcal{B}}{a}\right)+(b^a V_a)\right]\frac{(b_\perp^a V_a)}{|b_\perp|}\right) \\ (a^2+\mathcal{G})\left[\left(\frac{\mathcal{B}}{a}\right)+(b^a V_a)\right]\frac{b_\perp^A}{|b_\perp|} \\ 0 \\ a\frac{b_\perp^S V^A}{|b_\perp|} + (1-aV^S)\frac{b_\perp^A}{|b_\perp|} \\ 0 \end{pmatrix}^T_{(\mathrm{m}_1\pm)}, \quad \begin{pmatrix} -c_s^2\rho_0 h\mathcal{G}(a^2+\mathcal{G})\mathcal{F} \\ \mathcal{G}\left(\frac{\mathcal{B}}{a}\right)\frac{b_\perp^S}{|b_\perp|} + a(1-aV^S)\mathcal{G}\mathcal{F} \\ \mathcal{G}\left(\frac{\mathcal{B}}{a}\right)\frac{b_B^\perp}{|b_\perp|} - a^2\mathcal{G}\mathcal{F}V_B \\ (a^2+\mathcal{G})\frac{\rho_0 h}{|b_\perp|}b_\perp^S \\ (a^2+\mathcal{G})\frac{\rho_0 h}{|b_\perp|}b_B^\perp \\ -\frac{p}{\rho_0}(a^2+\mathcal{G})\mathcal{G}\mathcal{F} \end{pmatrix}_{(\mathrm{m}_1\pm)}, \tag{5.123}$$

respectively. The remaining two lower case left and right magnetosonic eigenvectors are given by

$$\begin{pmatrix} -\frac{\mathcal{G}}{\rho_0 h}\left(\frac{\mathcal{B}}{a}\right)(\mathcal{C}^a V_a) + \frac{(1-aV^S)}{c_s^2\rho_0 h(a^2+\mathcal{G})} \\ \left(1-\frac{1}{c_s^2}\right)\frac{a}{\mathcal{G}} + a(a^2+\mathcal{G})\left[\left(\frac{\mathcal{B}}{a}\right)+(b^a V_a)\right](\mathcal{C}^b V_b) \\ (a^2+\mathcal{G})\left[\left(\frac{\mathcal{B}}{a}\right)+(b^a V_a)\right]\mathcal{C}^A \\ 0 \\ a\mathcal{C}^S V^A + (1-aV^S)\mathcal{C}^A \\ 0 \end{pmatrix}^T_{(\mathrm{m}_2\pm)}, \quad \begin{pmatrix} \rho_0 h \\ \left(\frac{\mathcal{B}}{a}\right)\mathcal{G}\mathcal{C}^S - \frac{a(1-aV^S)}{c_s^2(a^2+\mathcal{G})} \\ \left(\frac{\mathcal{B}}{a}\right)\mathcal{G}\mathcal{C}_B + \frac{a^2}{c_s^2(a^2+\mathcal{G})}V_B \\ (a^2+\mathcal{G})\rho_0 h\mathcal{C}^S \\ (a^2+\mathcal{G})\rho_0 h\mathcal{C}_B \\ \frac{p}{c_s^2\rho_0} \end{pmatrix}_{(\mathrm{m}_2\pm)}, \tag{5.124}$$

respectively. Here the definitions

$$\mathcal{C}^a = \frac{b_\perp^a}{a^2\rho_0 h - \mathcal{G}b^2}\,, \qquad \mathcal{F} = \frac{|b_\perp|}{c_s^2(a^2+\mathcal{G}) - a^2} \tag{5.125}$$

are introduced. Under a type II degeneracy, $\mathcal{F}$ and $\mathcal{C}^a$ vanish, and accordingly under a type II$'$ degeneracy they are also taken to be

$$\mathcal{C}^a = 0\,, \qquad \mathcal{F} = 0\,, \tag{5.126}$$

and the rescaled perpendicular magnetic field is taken to be under type II and type II′ degeneracies:

$$\frac{b^{\perp}_{C}}{|b_{\perp}|} = \frac{1}{\sqrt{2}}(Q_{1}{}^{\lambda}_{C} + Q_{2}{}^{\lambda}_{C})\,. \tag{5.127}$$

Taking these values is again just a canonical choice for a representation of linearly independent eigenvectors of an eigenspace under type II and type II′ degeneracies, similar to the upper case analysis.

5.5.5 Lower Case Characteristic Variables

The lower case characteristic variables valid for all degeneracies are

$$\begin{aligned}
\hat{\mathrm{U}}_{0} =& \delta\varepsilon - \frac{p}{c_s^2\rho_0^2 h}\delta p\,, \qquad \hat{\mathrm{U}}_{\mathrm{c}} = (\delta\perp b)_{\hat{\mathbb{s}}} + (b^A V_A)(\delta\hat{v})_{\hat{\mathbb{s}}} - b^S V^A(\delta\hat{v})_{\hat{\mathbb{A}}}\,,\\
\hat{\mathrm{U}}_{\mathrm{a}\pm} =& \pm\frac{{}^{(\mathrm{S})}\epsilon_{BC}}{\sqrt{\rho_0 h^*}}\frac{V^B b_\perp^C}{|b_\perp|}\delta p + b^{S\,(\mathrm{S})}\epsilon_{BC}\frac{b_\perp^B V^C}{|b_\perp|}(\delta\hat{v})_{\hat{\mathbb{s}}} + \left((b^a V_a) \pm \sqrt{\rho_0 h^*}\right)\frac{N S^b_{\lambda_{(\mathrm{a}\pm)}} b_\perp^c}{|b_\perp|}{}^{(\mathrm{u})}\epsilon^A{}_{bc}(\delta\hat{v})_{\hat{\mathbb{A}}}\\
&-\left(\frac{b_\perp^B}{|b_\perp|} \pm \frac{|b_\perp| V^B}{\sqrt{\rho_0 h^*}}\right){}^{(\mathrm{S})}\epsilon^A{}_B(\delta\perp b)_{\hat{\mathbb{A}}}\,,
\end{aligned} \tag{5.128}$$

for entropy, constraint and Alfvén waves, and

$$\begin{aligned}
\hat{\mathrm{U}}_{\mathrm{m}_1\pm} =& -\left(\frac{\mathcal{G}}{\rho_0 h}\frac{(b_\perp^a V_a)}{|b_\perp|}\left(\frac{\mathcal{B}}{a}\right) + \frac{(1-aV^S)\mathcal{G}}{\rho_0 h}\mathcal{F}\right)\delta p\\
&+ a(a^2+\mathcal{G})(1-c_s^2)\mathcal{F}(\delta\hat{v})_{\hat{\mathbb{s}}} + N^2\left[\left(\frac{\mathcal{B}}{a}\right) + (b^a V_a)\right]\left(\frac{b_\perp^S}{|b_\perp|}(\delta\hat{v})_{\hat{\mathbb{s}}} + \frac{b_\perp^A}{|b_\perp|}(\delta\hat{v})_{\hat{\mathbb{A}}}\right)\\
&+\left(a\frac{b_\perp^S V^A}{|b_\perp|} + (1-aV^S)\frac{b_\perp^A}{|b_\perp|}\right)(\delta\perp b)_{\hat{\mathbb{A}}}\,,\\
\hat{\mathrm{U}}_{\mathrm{m}_2\pm} =& \left(\frac{(1-aV^S)}{c_s^2\rho_0 h(a^2+\mathcal{G})} - \frac{\mathcal{G}}{\rho_0 h}\left(\frac{\mathcal{B}}{a}\right)(\mathcal{C}^a V_a)\right)\delta p\\
&+\left(1-\frac{1}{c_s^2}\right)\frac{a}{\mathcal{G}}(\delta\hat{v})_{\hat{\mathbb{s}}} + N^2\left[\left(\frac{\mathcal{B}}{a}\right) + (b^a V_a)\right]\left(C^S(\delta\hat{v})_{\hat{\mathbb{s}}} + C^A(\delta\hat{v})_{\hat{\mathbb{A}}}\right)\\
&+\left(aC^S V^A + (1-aV^S)\mathcal{C}^A\right)(\delta\perp b)_{\hat{\mathbb{A}}}\,,
\end{aligned} \tag{5.129}$$

for magnetosonic waves closer to the Alfvén waves (m_1) and the remaining magnetosonic waves (m_2), with $\{\mathrm{m}_1, \mathrm{m}_2\}$ equal to $\{\mathrm{s}, \mathrm{f}\}$ or $\{\mathrm{f}, \mathrm{s}\}$. The functions on the right-hand side of $\hat{\mathrm{U}}_{\mathrm{m}\pm}$ are evaluated at the corresponding eigenvalue.

5.6 Flux-Balance Law Formulation

In the last three sections, the prototype algebraic constraint free formulation of GRMHD was studied. It turned out that this formulation is indeed strongly hyperbolic and thus complete sets of linearly independent rescaled eigenvectors can be found in the upper and lower case. This particular formulation where all constraint parameters are set to zero has the property that a priori the constraint and entropy eigenvalue coincide. Therefore, the degeneracies are the same as for the augmented system derived by [Anile and Pennisi, 1987].

5.6.1 Analysis of the Flux-Balance Law Formulation of GRMHD

In this subsection, a different set of values for the constraint parameters $\omega^{(p)}$, $\omega_a^{(\hat{v})}$, $\omega_a^{(\perp b)}$, and $\omega^{(\varepsilon)}$ is considered in regard to analyze whether or not the numerically used flux-balance law formulation of GRMHD is strongly hyperbolic. The analysis is performed in the upper case frame. The values of the constraint parameters are found by taking linear combinations of the equations (5.25) - (5.28) and comparing them with the system in the flux-balance law form given in [Antón et al., 2006; Cerda-Duran et al., 2008; Antón et al., 2010; Ibáñez et al., 2015], called the "Valencian" form of (G)RMHD equations, using the same set of evolved variables. The Valencian flux-balance law form of GRMHD as considered in [Antón et al., 2006] is

$$\partial_0(\sqrt{\gamma}\mathbf{F}^0) + \partial_i(\sqrt{-g}\mathbf{F}^i) = \sqrt{-g}\mathbf{S}\,, \tag{5.130}$$

with state vector $\mathbf{F}^0$, fluxes $\mathbf{F}^i$ and source term $\mathbf{S}$ given by

$$\mathbf{F}^0 = \begin{pmatrix} D \\ S_j \\ \tau \\ B^k \end{pmatrix} = \begin{pmatrix} \rho_0 W \\ \rho_0 h^* W^2 v_j - \alpha b^0 b_j \\ \rho_0 h^* W^2 - p^* - \alpha^2 (b^0)^2 - D \\ B^k \end{pmatrix},$$
$$\mathbf{F}^i = \begin{pmatrix} D\tilde{v}^i \\ S_j \tilde{v}^i + p^* \delta^i_j - b_j B^i/W \\ \tau \tilde{v}^i + p^* v^i - \alpha b^0 B^i / W \\ \tilde{v}^i B^k - \tilde{v}^k B^i \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 \\ T^{\mu\nu}\left(\partial_\mu g_{\nu j} - \Gamma^\delta_{\nu\mu} g_{\delta j}\right) \\ \alpha\left(T^{\mu 0}\partial_\mu \ln\alpha - T^{\mu\nu}\Gamma^0_{\nu\mu}\right) \\ 0^k \end{pmatrix}, \tag{5.131}$$

with $\tilde{v}^i = v^i - \beta^i/\alpha$ and the determinants γ and g of the three- and four-metric, respectively.

It is stressed again, that the discussion that follows only applies to those flux-balance law forms with eight evolved variables including the magnetic field. In fact, several of these flux-balance law forms of (G)RMHD exist. Remarkably, for the variables taken in the prototype algebraic constraint free formulation, they differ only by a linear combination of the conservation of number of particles equation. As far as the author knows, all flux-balance law forms of (G)RMHD in the literature use an Eulerian coordinate frame associated with coordinates x^μ. Also, all of the flux-balance law forms in the literature have the common feature that they initially start with the covariant PDE system of (G)RMHD as in section 5.1 using an energy-momentum tensor and dual field strength tensor as given in section 5.1 (sometimes in terms of other variables), and split them in a lower case sense. This is done either by projecting the covariant PDE system of GRMHD with n^a and its orthogonal projector, as e.g., by [Antón et al., 2006], or by taking directly the components (e.g., [Komissarov, 1999; Gammie et al., 2003]) such that the eight evolution equations become

$$\begin{aligned} &\nabla_\mu(\rho_0 u^\mu) = 0\,, && \nabla_\mu({}^*F^{i\mu}) = 0\,, \\ &\nabla_\mu(T^{0\mu}) = 0\,, && \nabla_\mu(T^{i\mu}) = 0\,, \end{aligned} \tag{5.132}$$

and the constraint is $\nabla_\mu({}^*F^{0\mu}) = 0$. These two ways are essentially the same and differ only by geometric terms in the state and source vectors.

The most important common feature of the various flux-balance law forms is, that the Maxwell constraint for the magnetic field is neither added to the evolution equations (5.132) (or to the respective projected versions of these) nor 'used' by setting the divergence of the magnetic field to zero in the evolution equations (5.132). By this fact, a classification is made as follows: All flux-balance law forms who inherit the Gauss constraint for the magnetic field in the evolution equations in the same way as in [Antón et al., 2006] and only

- differ by linear combinations of the evolution equations,
- use a different state vector in the sense of section 3.4,
- use a given background (i.e., work in the Cowling approximation),
- use Minkowski spacetime (i.e., work in RMHD),
- differ in the source terms,

with respect to the flux-balance law form in [Antón et al., 2006], share the same hyperbolicity properties. They can be seen as representatives of the flux-balance law formulation of GRMHD investigated here.

These are for example [Komissarov, 1999; Balsara, 2001; Gammie et al., 2003; Komissarov, 2004; Anninos et al., 2005; Duez et al., 2005; Shibata and Sekiguchi, 2005; Giacomazzo and Rezzolla, 2007; Del Zanna et al., 2007].

In the work of [Zanotti et al., 2015], the eight flux-balance RMHD equations are extended by the evolution of a scalar field ϕ to control the Maxwell constraint for B^i, see also chapter 6. This may affect the hyperbolicity property of the system and such an augmented system is not considered here. In the so-called "internal energy formulation" in [Anninos et al., 2005], which uses GRMHD equations similar to [Villiers and Hawley, 2003], the splitting of the covariant equations is done in an upper case sense, and thus the hyperbolicty analysis here does not apply to the arising system of evolution equations. However, the system is not written in a flux-balance law form in the first place, and is therefore not considered here. As mentioned in the beginning of this chapter, PDE systems of GRMHD where the magnetic four-potential is evolved, e.g., [Etienne et al., 2015], are also not considered here. For an overview and explanations concerning the differences of the existing flux-balance law forms of (G)RMHD, see the Living review of [Font, 2008]. It should be noted that numerical codes are constantly being adjusted and improved and the references above may be outdated. For example, the analysis applies to [Giacomazzo and Rezzolla, 2007], but later the evolution of the magnetic four-potential is taken [Giacomazzo et al., 2011] instead of the magnetic field.

To reproduce the flux-balance law formulation given in [Ibáñez et al., 2015], computer algebra (see appendix A) is used and the linear combination of the upper case system of equations (5.25) - (5.28) that reproduces the flux-balance law set of equations is found.

This was done ignoring all derivatives of the normal vector n^a. In the analyses throughout the thesis, all derivatives of the normal vector may be ignored anyway since they only contribute to the source vector and do not affect the analysis, as mentioned before. The constraint coefficients determining the flux-balance law formulation are found to be

$$\begin{aligned} \omega^{(p)} &= \frac{\kappa}{\rho_0}(b^c V_c)\,, & \omega_a^{(\hat{v})} &= \frac{1}{\rho_0 h} b_a\,, \\ \omega_a^{(\perp b)} &= -V_a\,, & \omega^{(\varepsilon)} &= \frac{1}{\rho_0}(b^c V_c)\,. \end{aligned} \tag{5.133}$$

Proceeding in the same way as for the previous formulation, the upper case principal symbol becomes

$$\mathbf{P}^S = \begin{pmatrix} 0 & -c_s^2\rho_0 h & 0^B & \frac{\kappa}{\rho_0}(b^c V_c) & 0^B & 0 \\ -\frac{\left(b^S\right)^2+\rho_0 h}{\rho_0^2 h h^*} & 0 & 0^B & \frac{b^S}{\rho_0 h} & -\frac{b^B}{\rho_0 h^*} & 0 \\ -\frac{b^S b_A}{\rho_0^2 h h^*} & 0_A & 0^B{}_A & \frac{b_A}{\rho_0 h} & \frac{b^S}{\rho_0 h^*}{}^{\mathrm{Q}}\!\perp^B{}_A & 0_A \\ 0 & 0 & 0^B & -V^S & 0^B & 0 \\ 0_A & -b_A & b^S\,{}^{\mathrm{Q}}\!\perp^B{}_A & -V_A & 0^B{}_A & 0_A \\ 0 & -\frac{p}{\rho_0} & 0^B & \frac{1}{\rho_0}(b^c V_c) & 0^B & 0 \end{pmatrix} \tag{5.134}$$

and the characteristic polynomial is of the form

$$P_\lambda = \frac{1}{(\rho_0 h^*)^2}\lambda(\lambda + V^S) P_{\text{Alfvén}} P_{\text{mgs}}\,, \tag{5.135}$$

where $P_{\text{Alfvén}}$ and P_{mgs} coincide with the polynomials given earlier in equations (5.40) and (5.41). As expected, the upper case eigenvalue associated with the constraint has changed from zero, in the previous formulation, to $-V^S$. Therefore, additional degeneracies have to be considered. One of those new degeneracies occurs when the constraint and entropy speeds become identical, namely if $V^S = 0$. In this particular degeneracy one finds that the principal symbol is not diagonalizable. Hence, the system is only weakly hyperbolic and has an ill-posed IVP. To get an intuitive idea of what precisely goes wrong, one may consider the upper case left eigenvectors associated with the entropy and constraint waves in generic directions, and then takes a limiting direction with $V^S \to 0$. These are,

$$\text{Entropy:} \qquad \begin{pmatrix} -\frac{p\rho_0}{c_s^2\rho_0^2 h - \kappa p}\frac{V^S}{(b^c V_c)} & 0 & 0^A & 1 & 0^A & \frac{c_s^2\rho_0^3 h}{c_s^2\rho_0^2 h - \kappa p}\frac{V^S}{(b^c V_c)} \end{pmatrix}\,, \tag{5.136}$$

and

$$\text{Constraint:} \qquad \begin{pmatrix} 0 & 0 & 0^A & 1 & 0^A & 0 \end{pmatrix}\,, \tag{5.137}$$

with upper case eigenvalues $\lambda_{(\mathrm{e})} = 0$ and $\lambda_{(\mathrm{c})} = -V^S$, respectively. Both upper case right eigenvectors can be found in the accompanying notebook, see appendix A, but are suppressed here because the upper case constraint right eigenvector is quite lengthy. Taking the limit $V^S \to 0$ one immediately arrives at the conclusion that the geometric multiplicity is only one as the two vectors become identical. The eigenvector cannot be rescaled as for the earlier degeneracies since only some entries in the left entropy eigenvector become zero; the limit of the principal symbol is truly problematic. This degeneracy was unfortunately overlooked in [Ibáñez et al., 2015], although there the focus is rather on convexity of the system than on hyperbolicity.

To support the upper case result of weak hyperbolicity, it is explicitly checked in the accompanying notebook, see appendix A, that upon taking the lower case matrices from [Ibáñez et al., 2015] and deriving the lower case left eigenvectors of the entropy and constraint waves, the very same problem is present. However, deriving the right constraint eigenvector in the lower case frame is much more involved than in the upper case. This is the reason why only the left ones are evaluated. It is stressed that using the matrices of [Ibáñez et al., 2015] is a completely independent calculation and stresses the weak hyperbolicity of the system.

5.6.2 Analysis of the Flux-Balance Law Formulation of MHD

With the result of the last subsection, one might wonder about the hyperbolicity of classical magnetohydrodynamics (MHD) when formulated in a flux-balance law form as for example in [Brio and Wu, 1988]. The set of evolution equations as given in [Jeffrey and Taniuti, 1964, p. 170–171], can be written in index notation as

$$\begin{aligned} \partial_t \rho_0 + &\partial_i \rho_0 v^i = 0\,, \\ \partial_t (\rho_0 v^j) + &\partial_i \left[\rho_0 v^j v^i + \left(p + \frac{B^2}{2} \right) \delta^{ji} - B^j B^i \right] = 0\,, \\ \partial_t \left(\frac{\rho_0 v^2 + B^2}{2} + \rho_0 \epsilon \right) + &\partial_i \left[\rho_0 v^i \left(\frac{v^2}{2} + \epsilon + \frac{p}{\rho_0} \right) + v^i B^2 - B^i (B^j v^j) \right] = 0\,, \\ \partial_t B^j + &\partial_i \left(B^j v^i - B^i v^j \right) = 0, \end{aligned} \tag{5.138}$$

where for simplicity $4\pi\mu = 1$ is adopted. The Maxwell constraint reads $\partial_i B^i = 0$. The designation of the classical variables takes place in regard to their respective relativistic versions. This set of evolution equations is just the Newtonian limit of the flux-balance law formulation (5.130) in, for example, [Antón et al., 2006], suffering from the same degeneracy and also being only weakly hyperbolic. Hence, the IVP is ill-posed. See also the accompanying notebook, appendix A.

5.6.3 Comments on the Numerical Consequence of Weak Hyperbolicity

To show the weak hyperbolicity of the flux-balance law formulation of GRMHD, one could consider a convergence test as done in [Cao and Hilditch, 2012], see figure 13. The used GRMHD code should of course take the flux-balance law formulation as above to evolve the hydrodynamical variables as well as the magnetic field. Based on the upper case principal symbol (5.134) of the flux-balance law formulation of GRMHD, the following requirements are recommended for a numerical convergence test:

i) The boost vector, the magnetic field, and their scalar product should not vanish:

$$v^a \neq 0\,, \qquad B^a \neq 0\,, \qquad B^a v_a = -b^a V_a \neq 0. \tag{5.139}$$

ii) A 2+1 dimensional spacetime is minimally needed. One-dimensional schemes are strongly hyperbolic (see [Komissarov, 1999] for a one-dimensional explanation).

iii) An EOS of the form $p(\rho_0, \epsilon) \neq 0$, with $0 < c_s \leq 1$ can be chosen.

iv) Tests including shocks should be avoided, since the convergence would suffer.

As far as the author knows, almost all of the standard test cases used for GRMHD in NR (see [Komissarov, 1999] for various tests) do not satisfy all the aforementioned requirements simultaneously. This partially explains why no one became aware of the weak hyperbolicity of the formulation (and the ill-posedness of the IVP), yet.

Chapter 6

Hyperbolicity Analysis of Resistive Magnetohydrodynamics

In this chapter, the evolution equations used in the literature for the numerical implementation of resistive magnetohydrodynamics are investigated. The analysis holds for the system of RRMHD used in [Komissarov, 2007; Dumbser and Zanotti, 2009; Palenzuela et al., 2009; Mizuno, 2013] and also for the general relativistic settings describing RGRMHD in [Palenzuela, 2013; Bucciantini and Del Zanna, 2013; Qian et al., 2017; Dionysopoulou et al., 2013; Dionysopoulou et al., 2015]. The reason is the same as for the systems of GRHD and GRMHD: all curvature quantities are absorbed in the derivative operators and the normalization of the spatial vectors for 2+1 decomposition or contribute only to the source term which does not affect the analysis. In the aforementioned literature, two different approaches to deal with the charge density exist. In both cases, the evolution equations form a weakly hyperbolic system as is shown separately in the sections 6.2 and 6.3. Again, the analysis below is only valid for the considered formulations and sets of variables in the sense of section 3.4.

In this chapter, the lower case frame is used exclusively. There are two reasons why the lower case is sufficient and adequate. First, the characteristic quantities needed to show weak hyperbolicity of the system of PDEs can be handled with the lower case. Second, the use of the appropriate frame depends on the form of the energy-momentum tensor. By using the Eulerian electric and magnetic fields in the field strength tensor, the energy-momentum tensor contains terms parallel and orthogonal to the Eulerian normal vector and therefore splitting against the fluid velocity is probably not the best choice anymore. As in chapter 5, Lorentz-Heaviside units are used throughout, where vacuum permittivity and vacuum permeability are equal to one. To start the analysis, first of all, the equations of motion for the state vector $\mathbf{U}$ are derived.

6.1 Equations of RGRMHD

Since the primary focus of this thesis is the derivation of the evolution equations and their mathematical structure some interesting physical facts, particularly those related to Ohm's law, are mostly not further considered.

6.1.1 Augmented Maxwell Equations

As in the beginning of the last chapter about GRMHD, the following definition of the field strength tensor is taken for a generic Eulerian observer with four-velocity n^a,

$$F^{ab} = n^a E^b - n^b E^a + \epsilon^{abcd} n_c B_d \,, \tag{6.1}$$

$$^*F^{ab} = n^a B^b - n^b B^a - \epsilon^{abcd} n_c E_d \,, \tag{6.2}$$

with the Levi-Civitá tensor,

$$\epsilon^{abcd} = -\frac{1}{\sqrt{-g}} \left[abcd\right] \,, \tag{6.3}$$

the Levi-Civitá symbol $[abcd]$, $[0123] = 1$ and

$$\epsilon^{abcd} n_a = \epsilon^{bcd} = \frac{1}{\sqrt{\gamma}} \left[bcd\right] \,, \tag{6.4}$$

where the definition and convention by [Alcubierre et al., 2009] is followed. Using this convention, the dual of the field strength tensor can be expressed as

$$^*F^{ab} = -\frac{1}{2} \epsilon^{abcd} F_{cd} \,. \tag{6.5}$$

In numerical applications to RGRMHD divergence cleaning is used. Thus, the augmented scalar fields ψ and ϕ are introduced, e.g., see [Komissarov, 2007; Palenzuela et al., 2009; Dionysopoulou et al., 2015]. Hence, the Maxwell equations become

$$\nabla_b \left(F^{ab} - g^{ab}\psi\right) = \mathcal{J}^a - \frac{1}{\tau} n^a \psi \,, \tag{6.6}$$

$$\nabla_b \left(^*F^{ab} - g^{ab}\phi\right) = -\frac{1}{\tau} n^a \phi \,. \tag{6.7}$$

Note that in the literature the notation $\kappa = \tau^{-1}$ is usually employed. The electric four-current $\mathcal{J}^a$ is split against n^a and $\gamma^b{}_a$, yielding the decomposition

$$\mathcal{J}^a = qn^a + J^a\,, \qquad n_a J^a = 0\,. \tag{6.8}$$

Proceeding with a 3+1 decomposition of the Maxwell equations (6.6) and (6.7) and making use of (6.8) the electromagnetic evolution equations may be written as

$$\gamma^a{}_b \mathcal{L}_n E^b = \epsilon^{abc} D_b B_c - \gamma^{ab} D_b \psi + S^a_{(\mathbf{E})}\,, \tag{6.9}$$

$$\gamma^a{}_b \mathcal{L}_n B^b = -\epsilon^{abc} D_b E_c - \gamma^{ab} D_b \phi + S^a_{(\mathbf{B})}\,, \tag{6.10}$$

$$\mathcal{L}_n \psi = -D_a E^a - \frac{1}{\tau}\psi + q\,, \tag{6.11}$$

$$\mathcal{L}_n \phi = -D_a B^a - \frac{1}{\tau}\phi\,, \tag{6.12}$$

with sources,

$$\begin{aligned} S^a_{(\mathbf{E})} &= \frac{1}{\alpha} B_c \epsilon^{abc} D_b \alpha + K E^a - J^a\,, \\ S^a_{(\mathbf{B})} &= -\frac{1}{\alpha} E_c \epsilon^{abc} D_b \alpha + K B^a\,. \end{aligned}$$

The constant τ is the time scale for the exponential driving of equations (6.11) and (6.12) towards the constraints

$$D_a E^a = q\,, \tag{6.13}$$

$$D_a B^a = 0\,, \tag{6.14}$$

respectively. The three-current J^a is given by generalized Ohm's law, see subsection 6.1.3. It is important to stress that although J^a appears in the source term, it could contain derivatives of the evolved variables. Such terms would then of course contribute to the principal part and must be taken into account.

As a consequence of the antisymmetry of the field strength tensor, additionally a conservation law $\nabla_a \mathcal{J}^a = 0$ for the electric charge can be considered, which reads in the 3+1 language

$$\mathcal{L}_n q = -\gamma^{ab} D_a J_b - \frac{1}{\alpha} J^b D_b \alpha + K\,q\,. \tag{6.15}$$

6.1.2 The Energy-Momentum Tensor

The energy-momentum tensor T^{ab} of RGRMHD contains an ideal fluid part,

$$T^{ab}_{\text{mat}} = \rho_0 h u^a u^b + p g^{ab} \,, \tag{6.16}$$

plus the standard electromagnetic energy-momentum tensor,

$$T^{ab}_{\text{em}} = F^{ac} F^b{}_c - \frac{1}{4} g^{ab} F_{cd} F^{cd} \,, \tag{6.17}$$

with a field strength tensor defined in (6.1). Writing F^{ab} in terms of E^a and B^a, the electromagnetic part becomes

$$T^{ab}_{\text{em}} = \frac{1}{2} \left(B_c B^c + E_c E^c \right) \left(\gamma^{ab} + n^a n^b \right) - B^a B^b - E^a E^b + \left(n^a \epsilon^{bcd} + n^b \epsilon^{acd} \right) E_c B_d \,. \tag{6.18}$$

As mentioned in the very beginning of this chapter, the energy-momentum tensor is partially split against the lower case normal vector and orthogonal terms. Thus, the upper case is not a good choice. However, when writing the field strength tensor in terms of u^a and the corresponding electric and magnetic fields, the upper case frame becomes again the preferred choice.

6.1.3 Generalized Ohm's Law

Generalized Ohm's law provides an expression for the spatial current J^a. Explanations about the physical validity and form of J^a can be found in the literature, see for example [Meier, 2004; Dionysopoulou et al., 2015]. In this thesis, the form of J^a is restricted to an equation of the form

$$J^a = q v^a + \tilde{J}^a \,, \qquad \tilde{J}^a = \tilde{J}^a(p, v_b, \varepsilon, E_c, B_d) \,, \tag{6.19}$$

where $\tilde{J}^a$ contains neither derivatives of the matter and electromagnetic variables nor second order or higher derivatives of the metric tensor.[1] This is a fairly general choice and includes the particular form used in the literature mentioned above, that is

$$J^a = q v^a + W \sigma \left(E^a + \epsilon^{abc} v_b B_c - (v_b E^b) v^a \right) , \tag{6.20}$$

where σ is the conductivity of the fluid and is permitted to be an arbitrary function of the evolved variables except the charge density q, while respecting the limitation

[1] In general, the electric current itself obeys a differential equation, see [Meier, 2004].

of $\tilde{J}^a$. However, the choice of J^a does not contain the so-called force free limit, since the associated current contains derivatives of the electric and magnetic fields which contribute to the principal symbol. For an hyperbolicity analysis of the force free limit see [Pfeiffer and MacFadyen, 2013].

As a last point of this subsection, the connection to the upper case frame is given. Splitting the four-current (6.8) against the fluid four-velocity, it can be expressed as

$$\mathcal{J}^a = \rho_{\rm el} u^a + I^a \,, \qquad u_a I^a = 0\,. \tag{6.21}$$

Here, $\rho_{\rm el}$ is the rest charge density measured by a comoving observer with the fluid. Taking the upper case three-current to be proportional to the upper case electric field, $I^a = \sigma e^a$ with $e^a = F^{ab}u_b$, then the lower case charge density q and current J^a can be related to the upper case ones according to

$$q = W\rho_{\rm el} + \sigma W(E^a v_b)\,, \tag{6.22}$$

$$J^a = W\rho_{\rm el} v^a + \sigma W\left(E^a + \epsilon^{abc} v_b B_c\right)\,. \tag{6.23}$$

The results are obtained by expressing the field strength tensor as in equation (6.1) and writing the four-velocity in terms of n^a and v^a. Using equation (6.22) to replace the rest charge density $\rho_{\rm el}$ in the three-current (6.23), the three-current used in the literature (6.20) is recovered.

6.1.4 Hydrodynamical Equations

The evolution equations for p, v_a and ε are obtained by considering the conservation of the number of particles and the conservation of energy-momentum

$$\nabla_a(\rho_0 u^a) = 0\,, \tag{6.24}$$

$$\nabla_a(T^{ab}) = 0\,, \tag{6.25}$$

and then proceed with the 3+1 split. After combining the equations, using Maxwell evolution equations and introducing the speed of sound, one arrives at the evolution equations for the pressure,

$$\begin{aligned}\mathcal{L}_n p =& (c_s^2 - 1)v^p W_{c_s}^2 D_p p - c_s^2 \rho_0 h W_{c_s}^2 \gamma^{pc} D_p v_c - c_2(E^b v_b)\gamma^{pc} D_p E_c - c_2(B^b v_b)\gamma^{pc} D_p B_c \\ &+ \left(c_1 E^p - c_2 \epsilon^{bdp} B_b v_d\right) D_p \psi + \left(c_1 B^p + c_2 \epsilon^{bdp} E_b v_d\right) D_p \phi + S^{(p)}\,,\end{aligned} \tag{6.26}$$

for the fluid velocity,

$$
\begin{aligned}
\gamma^b{}_a \mathcal{L}_n v_b = & -\frac{1}{W^2 \rho_0 h}\left(\gamma^p{}_a + (c_s^2 - 1) W_{c_s}^2 v^p v_a\right) D_p p + \left(\frac{c_s^2 W_{c_s}^2}{W^2} v_a \gamma^{pc} - v^p \gamma^c{}_a\right) D_p v_c \\
& + \frac{1}{W^2 \rho_0 h}\left(E_a + c_2 (E^b v_b) v_a\right) \gamma^{pc} D_p E_c + \frac{1}{W^2 \rho_0 h}\left(B_a + c_2 (B^b v_b) v_a\right) \gamma^{pc} D_p B_c \\
& + \frac{1}{W^2 \rho_0 h}\left(\gamma_{ad} + c_2 v_a v_d\right) \epsilon^{bdp} B_b D_p \psi - c_5 v_a E^p D_p \psi \\
& - \frac{1}{W^2 \rho_0 h}\left(\gamma_{ad} + c_2 v_a v_d\right) \epsilon^{bdp} E_b D_p \phi - c_5 v_a B^p D_p \phi + S_a^{(\mathbf{v})} \,, \qquad (6.27)
\end{aligned}
$$

and for the internal specific energy,

$$
\begin{aligned}
\mathcal{L}_n \varepsilon = & \frac{p W_{c_s}^2}{W^2 \rho_0^2 h} v^p D_p p - \frac{p W_{c_s}^2}{\rho_0} \gamma^{pc} D_p v_c - v^p D_p \varepsilon - c_4 (E^b v_b) \gamma^{pc} D_p E_c - c_4 (B^b v_b) \gamma^{pc} D_p B_c \\
& + \left(c_3 E^p - c_4 \epsilon^{bdp} B_b v_d\right) D_p \psi + \left(c_3 B^p + c_4 \epsilon^{bdp} E_b v_d\right) D_p \phi + S^{(\varepsilon)} \,. \qquad (6.28)
\end{aligned}
$$

The source terms are given by

$$
\begin{aligned}
S^{(p)} = & c_1 (E^b J_b) + c_2 \epsilon^{bcd} B_b J_c v_d + W_{c_s}^2 c_s^2 \rho_0 h (\mathrm{g}^{-1})^{bc} K_{bc} \,, \\
S_a^{(\mathbf{v})} = & \frac{1}{W^2 \rho_0 h}\left(\gamma_{ad} + c_2 v_a v_d\right) \epsilon^{bde} B_b J_e - c_5 (E^d J_d) v_a - \frac{1}{\alpha} (\mathrm{g}^{-1})^c{}_a D_c \alpha \\
& - c_s^2 \frac{W_{c_s}^2}{W^2} (\mathrm{g}^{-1})^{bc} K_{bc} v_a - K_{bc} v^b v^c v_a \,, \\
S^{(\varepsilon)} = & c_3 (E^b J_b) + c_4 \epsilon^{bcd} B_b J_c v_d + \frac{W_{c_s}^2 p}{\rho_0} (\mathrm{g}^{-1})^{bc} K_{bc} \,, \qquad (6.29)
\end{aligned}
$$

and the used abbreviations are defined as

$$
\begin{aligned}
c_1 = & \frac{W_{c_s}^2}{W^2 \rho_0}\left(\kappa W^2 + c_s^2 (W^2 - 1) \rho_0\right) \,, \qquad & c_2 = & W_{c_s}^2 \left(\frac{\kappa}{\rho_0} + c_s^2\right) \,, \\
c_3 = & \frac{W_{c_s}^2}{W^2 \rho_0^2 h}\left(p (W^2 - 1) + (\chi - \chi W^2 + h W^2) \rho_0\right) \,, \\
c_4 = & \frac{W_{c_s}^2}{W^2 \rho_0^2 h}\left(p W^2 + (\chi - \chi W^2 + h W^2) \rho_0\right) \,, \qquad & c_5 = & \frac{W_{c_s}^2}{W^2 \rho_0^2 h}\left(\kappa + \rho_0\right) \,. \qquad (6.30)
\end{aligned}
$$

The full set of evolution equations determining the time evolution of the components of the state vector $\mathbf{U} = (p, v_a, \varepsilon, q, E_a, B_a, \psi, \phi)^T$ are the hydrodynamical equations (6.26) - (6.28), together with the equations coming from the electromagnetic sector, namely the equation for the evolution of the charge density (6.15) and the augmented Maxwell equations (6.9) - (6.12). The obtained system of equations is identical to the system of evolution equations given in [Dionysopoulou et al., 2015] up to irrelevant source terms.

This was explicitly checked using a Mathematica notebook, where source terms have been ignored according to the earlier discussion. An EOS $p = p(\rho_0, \varepsilon)$ with restrictions as in chapter 4 is assumed.

6.2 Analysis with Evolution of the Charge Density

In this section, the first common set of evolution equations determining the time evolution of the state vector $\mathbf{U} = (p, v_a, \varepsilon, q, E_a, B_a, \psi, \phi)^T$ is considered. In this set, the charge density is evolved by equation (6.15). The analysis of the characteristic structure fits the set of numerically used equations considered in [Komissarov, 2007; Dumbser and Zanotti, 2009; Palenzuela et al., 2009; Palenzuela, 2013; Mizuno, 2013]. As always, first the 2+1 decomposition of the equations is performed, this time using an arbitrary unit spatial 1-form $s_a, s_a s^a = 1, s_a n^a = 0$, and denoting the orthogonal projector by ${}^{\mathrm{q}}\!\perp^b{}_a = \gamma^b{}_a - s^b s_a$. The state vector $\mathbf{U} = (p, v_a, \varepsilon, q, E_a, B_a, \psi, \phi)^T$ and the associated evolution equations (6.26), (6.27), (6.28), (6.15), (6.9), (6.10), (6.11), and (6.12) to the 14 components of $\mathbf{U}$ can be written in matrix form,

$$\mathbf{A}^{\mathrm{n}} \mathcal{L}_n \mathbf{U} = \mathbf{A}^p D_p \mathbf{U} + \boldsymbol{\mathcal{S}} \,. \tag{6.31}$$

The form of the matrices is easily obtained from the system of equations and is not explicitly given here. A simple 2+1 decomposition of this equation yields

$$(\mathcal{L}_n \mathbf{U})_{s,\,A} \simeq \mathbf{P}^s (D_s \mathbf{U})_{s,\,B}, \tag{6.32}$$

with $(\delta \mathbf{U})_{s,\,A} = (\delta p, (\delta v)_s, (\delta v)_A, \delta\varepsilon, \delta q, (\delta E)_s, (\delta E)_A, (\delta B)_s, (\delta B)_A, \delta\psi, \delta\phi)^T$. The principal symbol is of the form

$$\mathbf{P}^s = \mathbf{A}^s = \begin{pmatrix} \mathbf{A}_{6\times6} & \mathbf{B}_{6\times8} \\ \mathbf{0}_{8\times6} & \mathbf{C}_{8\times8} \end{pmatrix} , \tag{6.33}$$

where $\mathbf{B}_{6\times8}$ contains the coefficients of spatial derivatives with respect to the variables (E_a,B_a,ψ,ϕ) in the time evolution of (p,v_a,ε,q). The matrix $\mathbf{C}_{8\times8}$ is the sub-matrix of the electromagnetic variables (E_a, B_a, ψ, ϕ). The matrix $\mathbf{A}_{6\times6}$ can be written as,

$$\mathbf{A}_{6\times6} = \begin{pmatrix} \mathbf{A}_{5\times5} & \mathbf{0}_{5\times1} \\ \mathbf{A}_{1\times5} & -v^s \end{pmatrix} , \tag{6.34}$$

with $\mathbf{A}_{5\times5} = \mathbf{P}^s_{\mathrm{HD}}$ being the principal symbol of the pure hydrodynamical sector (see section 4.4), explicitly given by equation (4.40), and

$$\mathbf{A}_{1\times5} = \begin{pmatrix} -\frac{\partial J^s}{\partial p} & -s_c\frac{\partial J^s}{\partial v_c} & -{}^{\mathrm{q}}\!\perp^B{}_A\frac{\partial J^s}{\partial v_A} & -\frac{\partial J^s}{\partial \varepsilon}\end{pmatrix} . \tag{6.35}$$

This vector contains the coefficients of spatial derivatives of the hydrodynamical variables in the time evolution of q. Since the principal symbol (6.33) is block triangular, the 14 eigenvalues are given by those of $\mathbf{A}_{6\times6}$ and $\mathbf{C}_{8\times8}$, which read:

$$\begin{aligned} \mathbf{A}_{6\times6}: \quad & \lambda = -v^s\,, \ \text{(multiplicity 4)}\,, \\ & \lambda = \lambda_{(\pm)}\,, \quad \text{(see (4.41))}\,; \end{aligned} \tag{6.36}$$

$$\mathbf{C}_{8\times8}: \quad \lambda = \pm1\,, \quad \text{(multiplicity 4)}\,. \tag{6.37}$$

Due to the evolution equation of q, the matter eigenvalue $\lambda = -v^s$ has multiplicity four in contrast to multiplicity three for the pure system of GRHD. Therefore, due care is needed since the eigenspace changed comparing to GRHD.

Continuing the characteristic analysis, it can be shown that in general only 13 eigenvectors exist. The eigenspace of the eigenvalue $\lambda = -v^s$, with algebraic multiplicity four, has only geometric multiplicity three. For example, the linearly independent right eigenvectors can be chosen as:

$$\begin{pmatrix} 0 \\ 0 \\ 0_B \\ 1 \\ 0 \\ \mathbf{0}_{8\times1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0_B \\ 0 \\ 1 \\ \mathbf{0}_{8\times1} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ {}^{(\mathrm{s})}\epsilon_{BC}\,{}^{\mathrm{q}}\!\perp^C{}_A\frac{\partial J^s}{\partial v_A} \\ 0 \\ 0 \\ \mathbf{0}_{8\times1} \end{pmatrix} . \tag{6.38}$$

Here, the antisymmetric lower case Levi-Cività two-tensor for s_a is introduced according to

$$ {}^{(\mathrm{s})}\epsilon^{AB} := s_d n_c\, {}^{\mathrm{q}}\!\perp^A{}_a\, {}^{\mathrm{q}}\!\perp^B{}_b\, \epsilon^{cdab}. \tag{6.39}$$

This result is contrary to an earlier analysis presented in [Cordero-Carrión et al., 2012]. The earlier analysis is erroneous since the three vectors called $r_{\lambda_{H0}}$ corresponding to $\lambda = -v^s$ are not eigenvectors.[2] To substantiate the result of a missing eigenvector, a Jordan

[2]The explicit error is that the 9th component of these vectors may not be zero, since they produce cross terms with the A_{qH} part (corresponding to the $\mathbf{A}_{1\times5}$ part of the principal symbol derived above).

decomposition of the principal symbol (6.33) is performed. The Jordan normal form $\mathbf{J}[\mathbf{P}^s]$ of the principal symbol (6.33) can be written as a block diagonal matrix,

$$\mathbf{J}[\mathbf{P}^s] = \mathrm{diag}(\lambda_{(+)}, \lambda_{(-)}, \mathbf{J}_{v^s}, -\mathbb{1}_2 v^s, -\mathbb{1}_4, \mathbb{1}_4)\,, \tag{6.40}$$

containing the Jordan block

$$\mathbf{J}_{v^s} = \begin{pmatrix} -v^s & 1 \\ 0 & -v^s \end{pmatrix}\,. \tag{6.41}$$

The presence of the Jordon block confirms that $\mathbf{P}^s$ is in general not diagonalizable. Therefore, the system of equations is weakly hyperbolic and has an ill-posed IVP. In particular the numerically used current (6.20) leads to an ill-posed IVP.

To clarify under which circumstances the system is weakly hyperbolic, the terms in the principal symbol that lead to the weak hyperbolicity are considered. These terms can be identified as the coefficients in the spatial derivative of the boost velocity in the time derivative of the charge density,

$$-{}^{q}\!\perp^{B}{}_{A}\frac{\partial J^s}{\partial v_A} = -{}^{q}\!\perp^{B}{}_{a}\frac{\partial \tilde{J}^s}{\partial v_a}. \tag{6.42}$$

If $\frac{\partial \tilde{J}^s}{\partial v_A}$ vanishes identically for arbitrary unit spatial s_a (see equation (6.19) for the introduction of $\tilde{J}^a$), the system is strongly hyperbolic. This result serves as a restriction for currents usable in numerical codes that work with the system of evolution equations considered in this section.

6.3 Analysis without Evolution of the Charge Density

Now the second common set of evolution equations is considered. In this system, the evolution equation for the charge density q is suppressed. Additionally, the scalar field ψ is not evolved and set to zero everywhere. Thus, the augmented Maxwell equation (6.11) becomes the standard Maxwell constraint $D_a E^a = q$ for the electric field E^a. This equation is not a constraint in the PDE sense, it is now rather the definition used to obtain q.

The analysis in this section applies to the system of equations used in [Bucciantini and Del Zanna, 2013; Qian et al., 2017; Dionysopoulou et al., 2013; Dionysopoulou et al., 2015]. The set of equations is reduced to 12 evolution equations (6.26), (6.27), (6.28), (6.9), (6.10), and (6.12) for the components of the state vector $\mathbf{U} = (p, v_a, \varepsilon, E_a, B_a, \phi)^T$. Since now q is not evolved by the conservation of charge equation (6.15), all q's must be replaced by $D_a E^a$. Therefore, in equations (6.26), (6.27), (6.28), and (6.9) the three-current J^a is replaced by use of equation (6.19) with

$$J^a = v^a \gamma^{pc} D_p E_c + \tilde{J}^a \,, \tag{6.43}$$

where the first term will contribute to the principal symbol. Writing the system of equations in matrix form,

$$\mathbf{A}^{\mathrm{n}} \mathcal{L}_n \mathbf{U} = \mathbf{A}^p D_p \mathbf{U} + \boldsymbol{\mathcal{S}} \,, \tag{6.44}$$

and decomposing against s_a, $s_a s^a = 1$ and ${}^{q}\!\perp^b{}_a$,

$$(\mathcal{L}_n \mathbf{U})_{s,\,A} \simeq \mathbf{P}^s (D_s \mathbf{U})_{s,\,B} \,, \tag{6.45}$$

the principal symbol can be identified as

$$\mathbf{P}^s = \mathbf{A}^s = \begin{pmatrix} \mathbf{A}_{5\times5} & \mathbf{B}_{5\times7} \\ \mathbf{0}_{7\times5} & \mathbf{C}_{7\times7} \end{pmatrix} . \tag{6.46}$$

Again, $\mathbf{B}_{5\times7}$ contains the coefficients of spatial derivatives with respect to the variables (E_a, B_a, ϕ) in the time evolution of (p, v_a, ε) and $\mathbf{A}_{5\times5} = \mathbf{P}^s_{\mathrm{HD}}$ is the principal symbol of the pure hydrodynamical sector, explicitly given in (4.40). The matrix $\mathbf{C}_{7\times7}$ is the submatrix of the electromagnetic variables (E_a,B_a,ϕ) and reads

$$\mathbf{C}_{7\times7} = \begin{pmatrix} -v^s & 0 & 0 & 0 & 0 & 0 & 0 \\ -v^{q_1} & 0 & 0 & 0 & 0 & -1 & 0 \\ -v^{q_2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} . \tag{6.47}$$

The twelve eigenvalues of the principal symbol (6.46) are given by the ones of $\mathbf{A}_{5\times5}$

and $\mathbf{C}_{7\times 7}$, these are:

$$\begin{aligned}
\mathbf{A}_{5\times 5}: \quad & \lambda = -v^s\,, \ \text{(multiplicity 3)}\,, \\
& \lambda = \lambda_{(\pm)}\,, \quad \text{(see (4.41))}\,, \qquad (6.48)\\
\mathbf{C}_{7\times 7}: \quad & \lambda = \pm 1\,, \ \text{(multiplicity 3)}\,, \\
& \lambda = -v^s\,, \ \text{(multiplicity 1)}\,. \qquad (6.49)
\end{aligned}$$

As in the previous case where the charge density is evolved, the eigenspace of the matter eigenvalue $\lambda = -v^s$ has algebraic multiplicity four. As can be shown, the geometric multiplicity is only three. A set of right eigenvectors is for example given by:

$$\begin{pmatrix} \mathbf{0}_{2\times 1} \\ 1 \\ \mathbf{0}_{9\times 1} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0}_{3\times 1} \\ 1 \\ \mathbf{0}_{8\times 1} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0}_{4\times 1} \\ 1 \\ \mathbf{0}_{7\times 1} \end{pmatrix}. \tag{6.50}$$

Calculating the Jordan normal form $\mathbf{J}[\mathbf{P}^s]$ of the principal symbol (6.46), one finds the block diagonal matrix,

$$\mathbf{J}[\mathbf{P}^s] = \mathrm{diag}(\lambda_{(+)}, \lambda_{(-)}, -\mathbb{1}_2 v^s, \mathbf{J}_{v^s}, -\mathbb{1}_3, \mathbb{1}_3)\,, \tag{6.51}$$

with the Jordan block

$$\mathbf{J}_{v^s} = \begin{pmatrix} -v^s & 1 \\ 0 & -v^s \end{pmatrix}. \tag{6.52}$$

Therefore, the system of equations is also only weakly hyperbolic when the charge density variable q is not evolved. The result also holds if the scalar field ϕ is set to zero as well. In this case equation (6.12) reduces to the usual constraint $D_a B^a = 0$, and just the eleven variables $(p, v_a, \varepsilon, E_a, B_a)$ are evolved[3]. Analyzing the eigenvalue structure of the resulting principal symbol, one finds that a pair of eigenvalues $\lambda = \pm 1$ changes to the single eigenvalue $\lambda = 0$. It should be mentioned that adding multiples of the magnetic field constraint to the remaining eleven evolution equations might change the characteristic structure of the principal symbol. However, this will lead in general to a set of evolution equations which cannot be formulated in a flux-balance law form since the constraint coefficients in the above evolution equations are solution dependent.

For the special subcase $\tilde{J}^a \equiv 0$ the system is strongly hyperbolic. This can be seen as

[3] The new evolution equations are obtained from the previous ones by setting $\phi = 0$. No constraint is manually added.

follows: Starting again with the four-current $\mathcal{J}^a$ and 3+1 decomposing it in lower (6.8) and upper (6.21) case frames,

$$\mathcal{J}^a = qn^a + J^a\,, \qquad n_a J^a = 0\,; \tag{6.53}$$

$$\mathcal{J}^a = \rho_{\rm el} u^a + I^a\,, \qquad u_a I^a = 0\,, \tag{6.54}$$

one finds by projecting as usual

$$q = -n_a \mathcal{J}^a = W\rho_{\rm el} - n_a I^a\,, \tag{6.55}$$

$$\begin{aligned} J^a &= \gamma^a{}_b \mathcal{J}^a = W\rho_{\rm el} v^a + \gamma^a{}_b I^b \\ &= qv^a + (n_b I^b) v^a + \gamma^a{}_b I^b. \end{aligned} \tag{6.56}$$

Comparing the last line with the general choice $J^a = qv^a + \tilde{J}^a$ in (6.19), one finds

$$0 = \tilde{J}^a = (n^b I_b) v^a + \gamma^{ab} I_b = (\mathrm{g}^{-1})^{ab} I_b\,, \tag{6.57}$$

where the last equality holds by expressing n^a as $n^a = u^a/W - v^a$, using the orthogonality between the upper case three-current, and the four-velocity of the fluid and applying the definition of the inverse boost metric. Since the inverse boost metric is invertible, one has $\gamma^a{}_b I^b = 0$ and with the orthogonality condition $u_b I^b = 0$ one finally obtains $I^b = 0$. Thus, the conditions $\tilde{J}^a \equiv 0$ and $I^a \equiv 0$ are equivalent. In such a case, by use of equation (6.55), the charge densities of the lower case frame and the fluids rest frame are related according to

$$q = W\rho_{\rm el} \tag{6.58}$$

and the conservation of the electric charge, $\nabla_a \mathcal{J}^a$, has the simple form

$$\nabla_a(\rho_{\rm el} u^a)\,. \tag{6.59}$$

This is nothing else then the equation of conservation of number of particles multiplied with the constant specific charge of the particles. Instead of using the Gauss constraint for E^a one can now compute the charge density by the rest mass density and the Lorentz factor. Hence, q can be seen as a source term. Then the algebraic multiplicity of the matter eigenvalue $\lambda = -v^s$ changes to three, and a complete set of eigenvectors can be found.

The last two sections 6.2 and 6.3 show that the numerically used RGRMHD-systems of equations (coming from a flux-balance law form) have ill-posed IVPs for a fairly general choice of the three-current. Other formulations[4] of RGRMHD are not considered here. It is possible that these systems can be cured by a carefully chosen constraint addition. These possible formulations need a fresh characteristic analysis.

6.4 Charged Dust

As a last section in this chapter, the charged dust system of equations is analyzed. That is, the system of dust in section 4.5 coupled with the electric and magnetic fields. In this system, the conductivity is set to zero, $\sigma = 0$, as well as the pressure, $p = 0$, and the internal specific energy, $\varepsilon = 0$. Due to the vanishing conductivity, the rest mass density and the rest charge density are proportional to each other, with the specific charge as the constant of proportionality. The system of equations for the charged dust variables (ρ_0, v_i, E_i, B_i) decouples[5] into two parts. First, the evolution equations for (ρ_0, v_i), which were already found to be weakly hyperbolic in section 4.5, and second, the electromagnetic equations which can be given in a symmetric hyperbolic form, see [Alcubierre et al., 2009]. Hence, the whole system is only weakly hyperbolic and thus has an ill-posed IVP.

Interestingly, in [Perlick and Carr, 2010] it is shown that a different set of PDEs of charged dust using (v_i, E_i, B_i) as variables is strongly hyperbolic, at least in the Cowling approximation. In their system, ρ_0 is obtained by the Gauss constraint equation. Thus, the charge density and the rest mass density are related to the divergence of the electric field. In this case, the evolution of ρ_0 may be suppressed, but then the densities have to be expressed in terms of the divergence of the electric field. Under this treatment, however, the minimal coupling condition with the gravitational field equations, see equation (2.20), breaks. Hence, in full GR, the whole coupled system including the Einstein field equations must be considered, which is not done here.

[4] They can be obtained, for example, by adding multiples of constraints to the equations.

[5] Decoupling means here that the principal symbol has a triangular block structure with the dust and the electro-magnetic principal symbols as blocks on the diagonal, where the two blocks on the diagonal do not share the same eigenvalues. This includes the special case of a vanishing minor diagonal block (the block depends only on the addition of constraints to the evolution equation of dust, which has no influence on the hyperbolicity properties here).

Chapter 7

Conclusion

Summary

In the present work, the hyperbolicity of systems of first order partial differential equations (PDEs) of relativistic fluids in full general relativity (GR) was investigated. First of all, in chapter 2 the basic concepts were explained and especially for the case of a linear first order constant coefficient PDE system it was shown that strong hyperbolicity is an indispensable property to guarantee well-posed initial value problems. More precisely, in the constant coefficient case, strong hyperbolicity is also the sufficient property. In the case of a quasi-linear first order system such as the fluid systems, well-posedness of the initial value problem (IVP) is obtained if the PDE system is pointwise strongly hyperbolic and additional smoothness conditions hold, i.e., the symmetrizer $\mathbf{H}$ and the state vector $\mathbf{U}$ must be smooth in all arguments. With regard to the numerical treatment of the equations, this means that only variables of a well-posed IVP should be evolved, since otherwise no statements about the convergence or uniqueness of the solution can be made. The immediate consequence is that only strongly hyperbolic PDE systems may be used. Exactly this aspect was examined in the present work, where the physical systems of ideal hydrodynamics (GRHD), ideal magnetohydrodynamics (GRMHD) and resistive magnetohydrodynamics (RGRMHD) in their numerically considered formulation were analyzed. Showing strong hyperbolicity of a system of PDEs implies that the principal symbol $\mathbf{P}^s$ of the system is diagonalizable. Moreover, the matrix of eigenvectors is invertible and the sum of the matrix of eigenvectors and its inverse is bounded from above by a constant independent of the unit spatial 1-form s_a in the principal symbol.

To study the hyperbolicity structure of the fluid equations in an efficient manner, the dual frame (DF) formalism based on [Hilditch, 2015; Hilditch et al., 2018] was introduced in chapter 3. The key point of the DF formalism is that two different frames can be brought into relation. The frames are referred to as upper case and lower case frame and were identified in the later application with the rest frame of the fluid (also called Lagrangian frame) and an Eulerian (coordinate) frame, respectively. The main result of the DF formalism exploited here is the fact that strong hyperbolicity is independent of the frame, provided that the absolute value of the boost velocity times the highest absolute eigenvalue of the principal symbol is smaller than one. The identification of the upper case frame with the rest frame of the fluid is naturally preferred by the form of the energy-momentum tensors of relativistic fluids. In the Lagrangian frame the principal symbol is expected to become highly structured and thus, the calculation of the upper case characteristic quantities such as eigenvalues and eigenvectors is somehow straight forward. By knowledge of the upper case characteristic quantities and by using the DF formalism, the lower case eigenvalues and eigenvectors can be recovered, which are for example used in high resolution shock capturing (HRSC) schemes. By taking certain derivative operators, a particular set of variables, and a 2+1 decomposition as above, it was possible to study systems in special relativity (SR) and GR at the same time, since source terms do not affect the hyperbolicity properties of the system.

As a first application, the system of GRHD in chapter 4 was examined with the findings of the previous chapters. The advantage of the PDE system of GRHD is that it has been extensively studied in the literature and that the characteristic analysis in upper and lower case can be performed independently. It turned out that the upper case principal symbol actually takes a very simple form and the recovery procedure gave the same results as the direct lower case computation. The numerically used system of equations turned out to be strongly hyperbolic, at least for the form of the equation of state (EOS) considered in this work. In the dust case where the pressure of the fluid vanishes, the set of four evolution equations forms a weakly hyperbolic system and thus has an ill-posed IVP.

Second, the PDE system of GRMHD was examined with respect to strong hyperbolicity in chapter 5. The first characteristic analysis of RMHD was done by [Anile and Pennisi, 1987]. They work covariantly and considered an augmented system of ten evolution variables. However, their analysis does not apply to the numerically used set of evolution equations as in [Antón et al., 2006], where only eight evolution equations in flux-balance law form supplemented by the magnetic field constraint are considered. In this thesis, the aim was to analyze the latter set of equations. This was performed by taking the eight evolution equations related to the numerically used ones, i.e., five for hydrodynamical variables and three for the magnetic field, and adding some parametrized combinations of the Maxwell constraint for the magnetic field to each equation. A particular choice of the constraint addition parameters is called a formulation of GRMHD.

Two formulations of GRMHD were finally considered in this work.

The first one, called prototype algebraic constraint free formulation, is closely related to the augmented system of [Anile and Pennisi, 1987] and turned out to be strongly hyperbolic. All characteristic quantities such as eigenvalues and rescaled eigenvectors, valid for all degeneracies, were obtained in the upper case and by use of the aforementioned recovery procedure also in the lower case. It turned out, that with the help of the DF approach the degeneracy analysis of the upper case can be easily translated into the lower case and the quantities as well as particular limits can be well understood.

Second, the so-called flux-balance law formulation of GRMHD was investigated, were in total eight variables including the magnetic field are evolved. By fitting the constraint coefficients to the literature, however, it was shown that due to a new degeneracy the principal symbol lacks an eigenvector. Thus, the numerically used flux-balance law formulation of GRMHD is only weakly hyperbolic and has an ill-posed IVP. This fundamental problem cannot be cured by any numerical method and therefore the numerically used set of equations has to be altered. In fact, all flux-balance law forms which are in some sense analogous to [Antón et al., 2006] suffer from the same pathological behavior if the degeneracy occurs. The weak hyperbolicity was furthermore checked for matrices given in the literature, which confirms the obtained results completely independently. As a side result one also finds that the flux-balance law form of classical MHD is only weakly hyperbolic. Suggestions for a test setup to show the ill-posedness of the flux-balance law formulation of GRMHD were given in the very end of chapter 5.

Finally, in chapter 6 the system of RGRMHD was considered. In both cases, with and without the evolution of the (Eulerian) charge density, the systems of equations used in NR were only weakly hyperbolic. Weak hyperbolicity is found for a very general choice of the three-current, but the analysis still has to be reperformed if different three-currents than the one taken in this work are considered. In the very end of chapter 6, the system of charged dust was discussed. In the minimally coupled form, the system of equations was found to be only weakly hyperbolic. However, at least in the Cowling approximation, [Perlick and Carr, 2010] found that a strongly hyperbolic form of the PDE system can be obtained. The form of equations used by [Perlick and Carr, 2010] breaks the minimal coupling, however, and in full GR the whole coupled system of equations for matter and metric variables must be considered.

All obtained results are only valid for the particular analyzed set of evolution PDEs in the aforementioned way. Deviations from these might change the hyperbolicity properties of the system under consideration and the analysis must be done from scratch.

Future prospects

As seen for the systems of GRHD and especially GRMHD, the DF formalism [Hilditch, 2015; Hilditch et al., 2018] is a powerful tool to reveal structure in the principal symbol. It is expected that far more sophisticated fluid models, such as multi fluid models, will highly simplify in the upper case whereby the DF formalism provides an easy and clearly structured process to obtain the characteristic quantities in the lower case frame. Taken together, the hyperbolicity analysis of (fluid) systems is just one application of DF – whenever frames or foliations have to be related, or computations are much easier in one frame than another, which is quite often the case in the context of GR and NR, the DF formalism provides the translation between them and brings light into darkness.

For the system of GRMHD, a numerical verification of the ill-posedness of the investigated flux-balance law formulation should be the next goal. Due to the advanced time, it was unfortunately not possible to create a test case as part of this work. The prototype algebraic constraint free formulation is a first example for a strongly hyperbolic system of evolution PDEs to GRMHD. However, it does not obey a flux-balance law form. By use of the DF approach, it is maybe possible to find an appropriate frame where a strongly hyperbolic formulation in flux-balance law form can be obtained, but this thought has not been pursued, yet. Moreover, the PDE systems of GRMHD with divergence cleaning for the magnetic field constraint as well as the ones with evolution of the magnetic four-potential should be analyzed concerning their hyperbolicity properties, since the analysis above does not apply to them.

For the numerically used formulation of the system of RGRMHD the adjustment of the three-currents could immediately heal the system and strong hyperbolicity be obtained. The question is then whether this adjustment is a good choice to describe the physical system one likes to study. Perhaps some other formulations of RGRMHD also form a strongly hyperbolic PDE system.

In this work, for the first time, a complete characteristic analysis of an evolution system of GRMHD using eight evolution variables including the magnetic field is given. Also, it is shown for the first time that the flux-balance law formulation of GRMHD as well as two numerically used formulations of RGRMHD are only weakly hyperbolic. Thus, this work has great impact on codes in NR that use these PDE systems. Additionally, by the DF formalism, a powerful new tool is provided to perform the hyperbolicity analysis of more sophisticated numerically relevant PDE systems to fluid models. Altogether, this thesis can be seen as another small step towards more robust codes in NR.

Appendix A

Notes on the Accompanying Notebooks

Most of the results presented in this work were obtained by using xTensor for Mathematica [Martín-García, 2017]. All notebooks can be downloaded from `http://www.tpi.uni-jena.de/~hild/Hydro_DF.tgz`. The index notation convention in the provided notebooks differs from that used in the thesis. Explanations can be found below.

General Notes

* all notebooks use xTensor and have been tested in different subversions of Mathematica 10
* at the beginning it is recommended to take DFstyle_HD_xTensor_V2_Pub.nb - it contains more explanations than the other notebooks
* sometimes the notation of variables and constants slightly differs from the thesis notation, but identification is possible by consideration of their definition or relation to other quantities
* all indices are abstract and a 1+1+1+1 decomposition is used
* the notebooks are divided in chapters
* in general, the chapters contain additional definitions and rules adjusted to their needs
* all notebooks are ready to run with the "Evaluate Notebook" button
* however, the evaluation of for example Simplify and FullSimplify sometimes differs between versions of Mathematica, this can cause errors
* the Definition-files slightly differ and should be left in their respective folder

The chapters in this thesis are assigned to the folders as follows:

Chapter 4 - Folder GRHD

- contains the two notebooks
  (1) DFstyle_HD_xTensor_V2_Pub.nb (sections 4.2, 4.3), and
  (2) DFstyle_HD_xTensor_V2_for_v_Pub.nb (section 4.4)

- both use Tensor_Definitions_DF_Pub.m to get some of the definitions and rules they need

(1) DFstyle_HD_xTensor_V2_Pub.nb:
First the characteristic analysis in the upper case is performed, then in the lower case, and in the end the recovery using the results of section 3.3 is done

(2) DFstyle_HD_xTensor_V2_for_v_Pub.nb:
It contains just the calculations to obtain the lower case characteristic quantities using the boost vector v^a.

Chapter 5 - Folder GRMHD

- contains the two notebooks
  (3) DFstyle_MHD_xTensor_V2_Pub.nb (sections 5.3 - 5.5), and
  (4) FluxGRMHD_xTensor_WeakHyp_Pub.nb (section 5.6)
- both use MHD_Definitions_DF_Pub.m and Tensor_Definitions_DF_Pub.m to get some of the definitions and rules they need

(3) DFstyle_MHD_xTensor_V2_Pub.nb

  - first the characteristic analysis in the upper case is performed, then the lower case is obtained using the results of section 3.3
  - afterwards the degeneracy analysis is performed for upper and lower case
  - in the very end the transformations to primitive set of variables is given explicitly and the magnetosonic eigenvectors are calculated for these variables
  - note: the lower Alfvén eigenvectors calculation could last several hours, therefore it is excluded and the old results are directly in use (the formulas to calculate them are commented out)

(4) FluxGRMHD_xTensor_WeakHyp_Pub.nb

   - contains two separate parts:
   - first part: obtain upper case equations with particular choice of constraint-coefficients; then show whether upper case has missing eigenvectors; afterwards show the equivalence to flux-balance equations
   - second part: the last chapter is independent and uses the matrices given in the literature

Chapter 5 - Folder Classic MHD

- stand-alone version, the flux-balance formulas of Newtonian MHD are examined
- have to introduce a Manifold and use the covariant derivative (which can be identified as the usual partial derivative operator)
- contractions with normal vector are time derivatives, contractions with induced metric tensor ${\gamma^a}_b$ are spatial derivatives

Chapter 6 - Folder RGRMHD

- contains the two notebooks
  (5) RGRMHD_xTensor_V2_Augmented_System_Pub.nb (section 6.2), and
  (6) RGRMHD_xTensor_V2_without_E-Constraint_Pub.nb (section 6.3)
- both use RGRMHD_Definitions_Pub.m and Tensor_Definitions_Pub.m to get some of the definitions and rules they need

Appendix B

Detailed Calculation of the Coefficient Matrices of Upper Case GRHD

The detailed calculation of equation (4.30) is given by

$$\begin{aligned}
\mathbf{B}^{\mathrm{u}}\nabla_u \mathbf{U} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & {}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \nabla_u p \\ \nabla_u \hat{v}_c \\ \nabla_u \varepsilon \end{pmatrix} = \begin{pmatrix} \nabla_u p \\ {}^{(\mathrm{u})}\gamma_{ab}(\mathbb{g}^{-1})^{bc}\nabla_u \hat{v}_c \\ \nabla_u \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} \nabla_u p \\ S_a S_b (\mathbb{g}^{-1})^{bc}\nabla_u \hat{v}_c + {}^{\mathrm{Q}}\!\perp_{ab}\,(\mathbb{g}^{-1})^{bc}\nabla_u \hat{v}_c \\ \nabla_u \varepsilon \end{pmatrix} = \begin{pmatrix} \nabla_u p \\ S_a (\nabla_u \hat{v})_{\hat{\mathbb{s}}} + {}^{\mathrm{Q}}\!\perp_{ab}\,(\mathbb{g}^{-1})^{bc}\nabla_u \hat{v}_c \\ \nabla_u \varepsilon \end{pmatrix} \\
&\equiv \begin{pmatrix} \nabla_u p \\ S_a (\nabla_u \hat{v})_{\hat{\mathbb{s}}} + {}^{\mathrm{Q}}\!\perp^{\mathbb{A}}{}_a (\nabla_u \hat{v})_{\hat{\mathbb{A}}} \\ \nabla_u \varepsilon \end{pmatrix} \equiv \begin{pmatrix} \nabla_u p \\ (\nabla_u \hat{v})_{\hat{\mathbb{s}}} \\ (\nabla_u \hat{v})_{\hat{\mathbb{A}}} \\ \nabla_u \varepsilon \end{pmatrix} \equiv (\nabla_u \mathbf{U})_{\hat{\mathbb{s}},\hat{\mathbb{A}}} \,.
\end{aligned} \tag{B.1}$$

Here the convention ${}^{\mathrm{Q}}\!\perp^{\mathbb{A}}{}_a \equiv {}^{\mathrm{q}}\!\perp^{\mathbb{A}}{}_c (\mathbb{g}^{-1})^{cb}\,{}^{\mathrm{Q}}\!\perp_{ba}$ is used by taking the relation

$${}^{\mathrm{Q}}\!\perp_{ab}\,(\mathbb{g}^{-1})^{bc} = {}^{\mathrm{Q}}\!\perp_{ab}\,(\mathbb{g}^{-1})^{bd}\gamma^c{}_d = {}^{\mathrm{Q}}\!\perp_{ab}\,(\mathbb{g}^{-1})^{bd}\,{}^{\mathrm{q}}\!\perp^c{}_d = {}^{\mathrm{Q}}\!\perp_{ab}\,(\mathbb{g}^{-1})^{b\mathbb{B}}\,{}^{\mathrm{q}}\!\perp^c{}_{\hat{\mathbb{B}}} \,. \tag{B.2}$$

The right-hand side of equation (4.27) is rewritten as follows:

$$
\begin{aligned}
\mathbf{B}^p \nabla_p \mathbf{U} &= \begin{pmatrix} 0 & -c_s^2 \rho_0 h^{(\mathrm{u})}\gamma^p{}_d (\mathbb{g}^{-1})^{dc} & 0 \\ -\frac{1}{\rho_0 h}{}^{(\mathrm{u})}\gamma^p{}_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0}{}^{(\mathrm{u})}\gamma^p{}_d (\mathbb{g}^{-1})^{dc} & 0 \end{pmatrix} \begin{pmatrix} \nabla_p p \\ \nabla_p \hat{v}_c \\ \nabla_p \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c_s^2 \rho_0 h (S^p S_d + {}^{\mathrm{Q}}\!\perp^p{}_d)(\mathbb{g}^{-1})^{dc} & 0 \\ -\frac{1}{\rho_0 h}(S^p S_a + {}^{\mathrm{Q}}\!\perp^p{}_a) & 0 & 0 \\ 0 & -\frac{p}{\rho_0}(S^p S_d + {}^{\mathrm{Q}}\!\perp^p{}_d)(\mathbb{g}^{-1})^{dc} & 0 \end{pmatrix} \begin{pmatrix} \nabla_p p \\ \nabla_p \hat{v}_c \\ \nabla_p \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c_s^2 \rho_0 h S_d (\mathbb{g}^{-1})^{dc} & 0 \\ -\frac{1}{\rho_0 h} S_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0} S_d (\mathbb{g}^{-1})^{dc} & 0 \end{pmatrix} \begin{pmatrix} \nabla_S p \\ \nabla_S \hat{v}_c \\ \nabla_S \varepsilon \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & -c_s^2 \rho_0 h {}^{\mathrm{Q}}\!\perp^A{}_d (\mathbb{g}^{-1})^{dc} & 0 \\ -\frac{1}{\rho_0 h} {}^{\mathrm{Q}}\!\perp^A{}_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0} {}^{\mathrm{Q}}\!\perp^A{}_d (\mathbb{g}^{-1})^{dc} & 0 \end{pmatrix} \begin{pmatrix} \nabla_A p \\ \nabla_A \hat{v}_c \\ \nabla_A \varepsilon \end{pmatrix} \\
&\simeq \begin{pmatrix} 0 & -c_s^2 \rho_0 h S_d (\mathbb{g}^{-1})^{dc} & 0 \\ -\frac{1}{\rho_0 h} S_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0} S_d (\mathbb{g}^{-1})^{dc} & 0 \end{pmatrix} \begin{pmatrix} \nabla_S p \\ \nabla_S \hat{v}_c \\ \nabla_S \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c_s^2 \rho_0 h (S_d S^d S_e (\mathbb{g}^{-1})^{ec} + S^{d\mathrm{Q}}\!\perp_{ed}\,(\mathbb{g}^{-1})^{ec}) & 0 \\ -\frac{1}{\rho_0 h} S_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0}(S_d S^d S_e (\mathbb{g}^{-1})^{ec} + S^{d\mathrm{Q}}\!\perp_{ed}\,(\mathbb{g}^{-1})^{ec}) & 0 \end{pmatrix} \begin{pmatrix} \nabla_S p \\ \nabla_S \hat{v}_c \\ \nabla_S \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c_s^2 \rho_0 h S^d & 0 \\ -\frac{1}{\rho_0 h} S_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0} S^d & 0 \end{pmatrix} \begin{pmatrix} \nabla_S p \\ (S_d S_e (\mathbb{g}^{-1})^{ec} + {}^{\mathrm{Q}}\!\perp_{ed}\,(\mathbb{g}^{-1})^{ec})(\nabla_S \hat{v}_c) \\ \nabla_S \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c_s^2 \rho_0 h \left((S^d S_d) S^b + (S^{d\mathrm{Q}}\!\perp^A{}_d){}^{\mathrm{Q}}\!\perp^b{}_A\right) & 0 \\ -\frac{1}{\rho_0 h} S_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0}\left((S^d S_d) S^b + (S^{d\mathrm{Q}}\!\perp^A{}_d){}^{\mathrm{Q}}\!\perp^b{}_A\right) & 0 \end{pmatrix} \begin{pmatrix} \nabla_S p \\ (S_b \hat{\mathbb{s}}^c + {}^{\mathrm{Q}}\!\perp_{eb}\,(\mathbb{g}^{-1})^{ec})(\nabla_S \hat{v}_c) \\ \nabla_S \varepsilon \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c_s^2 \rho_0 h \left((S^d S_d) S^b + (S^{d\mathrm{Q}}\!\perp^A{}_d){}^{\mathrm{Q}}\!\perp^b{}_A\right) & 0 \\ -\frac{1}{\rho_0 h} S_a & 0 & 0 \\ 0 & -\frac{p}{\rho_0}\left((S^d S_d) S^b + (S^{d\mathrm{Q}}\!\perp^A{}_d){}^{\mathrm{Q}}\!\perp^b{}_A\right) & 0 \end{pmatrix} \begin{pmatrix} \nabla_S p \\ S_b (\nabla_S \hat{v})_{\hat{\mathbb{s}}} + {}^{\mathrm{Q}}\!\perp^{\mathbb{A}}{}_b (\nabla_S \hat{v})_{\hat{\mathbb{A}}} \\ \nabla_S \varepsilon \end{pmatrix} \\
&\equiv \begin{pmatrix} 0 & -c_s^2 \rho_0 h (S^d S_d) & -c_s^2 \rho_0 h (S^{d\mathrm{Q}}\!\perp^A{}_d) & 0 \\ -\frac{1}{\rho_0 h} S_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{p}{\rho_0}(S^d S_d) & -\frac{p}{\rho_0}(S^{d\mathrm{Q}}\!\perp^A{}_d) & 0 \end{pmatrix} \begin{pmatrix} \nabla_S p \\ (\nabla_S \hat{v})_{\hat{\mathbb{s}}} \\ (\nabla_S \hat{v})_{\hat{\mathbb{A}}} \\ \nabla_S \varepsilon \end{pmatrix}. \qquad \text{(B.3)}
\end{aligned}
$$

If the orthogonality of S^a and ${}^{\mathrm{Q}}\!\perp^A{}_d$ is used, then equation (4.31) is obtained.

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[GEO] GEO600, http://www.geo600.uni-hannover.de/.

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[KAG] KAGRA - Kamioka Gravitational Wave Detector, http://gwcenter.icrr.u-tokyo.ac.jp/en/.

[LIGO] LIGO, http://www.ligo.caltech.edu/.

[LISA] LISA - Laser Interferometer Space Antenna, https://lisa.nasa.gov/

[VIRGO] VIRGO, http://www.virgo.infn.it/.

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Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

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Weitere Personen waren an der inhaltlich-materiellen Erstellung der vorliegenden Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeldliche Hilfe von Vermittlungs- bzw. Beratungsdiensten (Promotionsberater oder andere Personen) in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen.

Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

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Ich versichere ehrenwörtlich, dass ich nach besten Wissen die reine Wahrheit gesagt und nichts verschwiegen habe.

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Andreas Uwe Sven Schoepe