# VARIATIONAL PRINCIPLES FOR TOPOLOGICAL PRESSURE

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## Abstract

Let $X$ be a compact metric space and $T : X \to X$ be continuous. We introduce topological pressure with respect to $T$ for sequences $\Phi = (\varphi_n)_{n\geq 1}$ of arbitrary functions $\varphi_n : X \to [-\infty, \infty]$. We prove an upper variational inequality for the pressure of $\Phi$. We show in addition that if $\varphi_n$ are Borel measurable, then a lower variational inequality holds. This establishes a unifying framework for proving variational principles for the topological pressure of continuous $\mathbb{Z}_+$-actions on compact metric spaces.



## Zusammenfassung

Sei $X$ ein kompakter metrischer Raum und $T : X \to X$ eine stetige Abbildung. Wir führen zunächst den Begriff des topologischen Drucks bezüglich von Folgen $\Phi = (\varphi_n)_{n\geq 1}$ beliebiger Funktionen $\varphi_n : X \to [-\infty, \infty]$ für $T$ ein. Danach beweisen wir eine obere Variationsungleichung für den Druck von $\Phi$. Wir zeigen außerdem, dass eine untere Variationsungleichung gilt, falls alle $\varphi_n$ Borel messbar sind. Beide Ungleichungen ermöglichen einen vereinheitlichenden Rahmen, um Variationsprinzipien für den topologischen Druck stetiger $\mathbb{Z}_+$-Operationen auf kompakten metrischen Räumen zu beweisen.

# Contents

# 1. Introduction

This introduction is to provide an overview of the present thesis, and is divided into three parts: The first part serves as an introduction to thermodynamic formalism for someone with a background in fractal geometry. The second part gives a description of the aim and scope of the present work. The third part states the main results of the thesis.

## §1.1. An introductory example

One of the most celebrated results in the field of fractal geometry was given by Hutchinson [Hut81]:

**Observation** (Hutchinson's formula)**.** *Let $S_1,\dots S_N$ be contractive similarities of $\mathbb{R}^d$ with ratios $r_1,\dots r_N$ satisfying the open set condition. Denote by $X$ the corresponding self-similar set. Then $\dim_H X = s$, if and only if*

$$\sum_{i=1}^{N} r_i^s = 1. \tag{1.1}$$

The above theorem tells us that the problem of finding the Hausdorff dimension of a fractal set (which in general is difficult) has an easy solution in the self-similar setting. Equation (1.1) leads to a probability theoretical point of view: To compute the Hausdorff dimension of $X$, the value $s$ has to be chosen such that $r_1^s,\dots,r_N^s$ give rise to a probability measure on the self-similar set $X$. The construction of a suitable measure (or, more precisely, a mass distribution) is exactly the strategy in the proof of the statement. From a practical point of view, on the other hand, one can readily apply Hutchinson's formula and just calculate the unique zero of the function

$$t \mapsto \log \sum_{i=1}^{N} r_i^t. \tag{1.2}$$

In many popular examples (e.g. middle third Cantor set, Sierpinski triangle), this can even be done by hand, without the help of a computer.

There is another point of view, namely a *thermodynamic interpretation*, which leads eventually into the realm of dynamical systems and ergodic theory. We shall illustrate this here in a simplified and informal way and will closely follow the terminology given in Chapter 1 of [Kel98].

Let us first rephrase Hutchinson's formula. Denote $a_i := r_i^{-1}$ and

$$p(-t\log a) := \log \sum_{i=1}^{N} \exp(-t\log a_i).$$

We then have the following:

**Observation.** *Under the assumptions of Hutchinson's formula, the Hausdorff dimension of the self-similar set $X$ is the unique zero of the function*

$$t \mapsto p(-t\log a).$$

What first looks like a complication, turns out to be a special case of a so-called pressure function. If one has a finite set $\Omega = \{1,\ldots,N\}$ (the so-called configuration space) and a mapping $u : \Omega \to \mathbb{R}$ (which is called energy), the *pressure* of $-\beta u$ for any $\beta \in \mathbb{R}$ is defined to be

$$p(-\beta u) := \log Z(\beta) := \log \sum_{i=1}^{N} \exp(-\beta u_i).$$

Furthermore, if $\mu$ is a probability measure on $\Omega$ (a so-called state of the configuration space $\Omega$), one defines

$$H(\mu) := -\sum_{i=1}^{N} \mu(\{i\}) \log \mu(\{i\})$$

to be the *entropy* of $\mu$. The striking connection between entropy and pressure is interfered by the *variational principle*, which is the main object of study in the present thesis.

**Observation** (Variational principle, finite case)**.** *For each $\beta \in \mathbb{R}$ and $u : \Omega \to \mathbb{R}$ one has*

$$p(-\beta u) = \sup \left\{ H(\mu) - \beta \int_{\Omega} u \, \mathrm{d}\mu \right\}, \tag{1.3}$$

*where the supremum is taken over all probability measures $\mu$ on $\Omega$.*

From a physical point of view, the variational principle is an equivalent reformulation of the principle of minimum free energy:

**Observation** (Principle of minimum free energy)**.** *Fix $\beta \in \mathbb{R}$ and $u : \Omega \to \mathbb{R}$. We call for each probability measure $\mu$ of $\Omega$ the quantity*

$$F(\mu) := \int_{\Omega} u \, \mathrm{d}\mu - T H(\mu)$$

*to be the free energy of state $\mu$, where $T := \frac{1}{\beta}$ is called temperature (with Boltzmann constant $k_B := 1$). Then*

$$-T \log Z(\beta) = \inf_{\mu} F(\mu), \tag{1.4}$$

*where the infimum is taken over all probability measures on $\Omega$. In this context, $F := -T \log Z(\beta)$ is called Helmholtz free energy.*

Each measure which maximizes (1.3) (or minimizes (1.4)) is called equilibrium state. One can show that there is exactly one equilibrium state $\mu_{-\beta u}$. It is called Gibbs measure for $\beta$ (with respect to $u$), and one has

$$\mu_{-\beta u}(\{i\}) = \frac{\exp(-\beta u_i)}{\sum_{i=1}^{N} \exp(-\beta u_i)}$$

for each $i \in \Omega$. Now, if we go back to the self-similar set $X$ with Hausdorff dimension $\dim_H X = s$, it follows by Hutchinson's formula that

$$p(-s \log a) = 0, \quad \text{hence} \quad \sum_{i=1}^{N} \exp(-s \log a_i) = 1.$$

Using $a_i = r_i^{-1}$, this yields

$$\mu_{-s\log a}(\{i\}) = r_i^s$$

for $i \in \Omega$. Thus, the Gibbs measure, which maximises (1.3) for $\beta = s$ and $u = \log a$, is in fact the distribution in (1.1). Moreover, as $p(-s\log a) = 0$, Equation (1.3) immediately implies the following variational principle for the Hausdorff dimension of $X$:

**Observation** (Variational principle for the Hausdorff dimension)**.** *Under the assumptions of Hutchinson's formula, one has*

$$\dim_H X = s = \sup\left\{ \frac{\sum_{i=1}^N p_i \log p_i}{\sum_{i=1}^N p_i \log r_i} : p_i \in [0,1] \text{ for all } i, \text{ and } \sum_{i=1}^N p_i = 1 \right\}.$$

*In addition, the maximum is exactly attained at the probability vector $(r_1^s, \ldots, r_N^s)$, which corresponds to the Gibbs measure $\mu_{-s\log a}$.*

Note that the above statement yields an explicit representation of the Hausdorff dimension of $X$, whereas (1.1) provides only an implicit one. Thus, by applying the variational principle, which resembles a physical law from thermodynamics, we eventually gain a deeper insight about both Hutchinson's formula and fractal sets.

Although only being a brief heuristic, the above exposition aims to shed some light on the thermodynamic background of dimension theory and its usefulness. Let us summarize our observations so far:

1) Hausdorff dimensions can be related to zeros of certain pressure functions.
2) Pressure functions stem from a *thermodynamic formalism*, and one key connection is the variational principle, which follows the physical principle of minimum free energy.
3) Equilibrium states of the variational principle are distinguished measures, which ought to play an important role in their respective interrogations.

As set out in the next subsection, the present thesis is primarily concerned with the second topic of the above list.

### §1.2. Aim and scope

*Historical background.* The topological pressure $P$ for $\mathbb{Z}^d$-actions on a compact metric space $X$ was first introduced by Ruelle [Rue73]. Given a continuous function $\varphi : X \to \mathbb{R}$, Ruelle's pressure $P(\varphi)$ can be seen as a weighted version of the topological entropy $h_{\text{top}}$. Topological entropy itself was introduced by Adler, Konheim, and McAndrew in [AKM65] as an analogue to measure theoretic entropy, which quantifies the complexity of the dynamics from a probabilistic point of view. Dinaburg [Din70] and, more generally, Goodman [Goo71], proved a variational principle, which relates both topological and measure theoretic entropy. Ruelle extended it to the topological pressure for actions satisfying expansiveness and specification. Walters [Wal75] generalized this result to general $\mathbb{Z}_+$-actions: For a continuous mapping $T : X \to X$ and continuous $\varphi : X \to \mathbb{R}$, one has

$$P(\varphi) = \sup\left\{ h_\mu(T) + \int_X \varphi \,\mathrm{d}\mu \right\},$$

where the supremum is taken over all $T$-invariant Borel probability measures $\mu$ on $X$. In the special case $\varphi \equiv 0$, one regains the variational principle for topological entropy, namely $h_{\text{top}}(T) = \sup_\mu h_\mu(T)$.

Since then, topological pressure and the variational principle were intensively studied in the literature. Bowen [Bow73] introduced topological entropy for non-compact sets and his idea was later generalized by Pesin and Pitskel' [PP84]. They introduced topological pressure $P_Z(\varphi)$ on arbitrary subsets $Z \subseteq X$, and proved a variational principle for $T$-invariant, not necessarily compact Borel sets $Z \subseteq X$. Pesin developed this approach into the theory of Carathéodory dimensions (see [Pes97]), which explains that entropy, pressure, and dimension are related by the same construction principle.

A sub-additive, and more generally, a non-additive thermodynamic formalism was introduced by Falconer [Fal88] and Barreira [Bar96], respectively. Here one considers the topological pressure $P_Z(\Phi)$ of a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ of functions $\varphi_n : X \to \mathbb{R}$, which is not necessarily additive (that is, there is not necessarily a function $\psi : X \to \mathbb{R}$ such that $\varphi_n = \sum_{i<n} \psi \circ T^i$). Falconer proved a variational principle for mixing repellers and Barreira a variational principle under strict requirements on $\Phi$, following the more general approach of [PP84]. Later, the sub-additive formalism was refined and further developed by Cao, Feng, and Huang [CFH08, FH10]. Inverse variational principles for sub- and super-additive sequences were given by Cao, Hu, and Zhao [CHZ13]. A recent promising direction is the extension of the sub-additive formalism by Feng and Huang [FH16] to a weighted formalism in factor systems. Their approach plays a key role in the present work.

*Rationales and objectives.* Most of the thermodynamic formalism we reviewed so far is carried out in a continuous, compact setting. This means, there is a continuous mapping $T : X \to X$ on a compact, metric space $X$. The functions $\varphi_n : X \to \mathbb{R}$ are supposed to be continuous, with some additional structure like sub-additivity or tempered variation. Moreover, in many examples studied in literature one additionally restricts to certain characteristics (mixing systems, expansiveness, some type of specification, shift spaces of finite type, etc.), and exploits the underlying traits of the dynamics to treat pressure and prove variational principles. However, there are examples, were the continuous setting is too restrictive. The mapping $T$ might not be continuous (e.g. piecewise interval maps), or the space $X$ might be non-compact (e.g. countable shift spaces). Also, the functions $\varphi_n$ might be discontinuous or unbounded, or carry an opaque structure. This is exemplified by super-additive sequences: Although super-additivity is a strong assumption and the sub-additive formalism is well developed, there is no analog variational principle for super-additive sequences (beside special cases, see e.g. [BCH10]). Also, it is known that some of the methods still work if one assumes $\varphi_n$ to be upper semi-continuous (see e.g. [Kel98] in the setting of $\mathbb{Z}_+^d$-actions). But much less is known, if $\varphi_n$ are lower semi-continuous or only assumed to be Borel measurable (for several special cases see [Mum06, CMP10, Rau17]). On the other hand, if one studies the literature, a pattern in proving variational principles emerges, which ought to be independent of subtle differences such as upper or lower semi-continuity. Indeed, one of the key observations of this thesis is that the existence of a variational principle for sequences $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ follows from the existence of a pointwise ergodic theorem for $\Phi$.

With that said, in the present work we will still assume $X$ to be compact and

$T$ to be continuous but consider arbitrary functions $\varphi_n : X \to [-\infty,\infty]$ instead. Our first aim is to introduce a suitable topological pressure $P_Z(\Phi)$ for $Z \subseteq X$ and $\Phi = (\varphi_n)_{n\in\mathbb{N}}$. By suitable topological pressure we think of a pressure function which

- obeys the rules of monotonicity and countable stability,
- extends other approaches given in the literature (in particular, includes the classical additive formalism),
- and admits variational inequalities and principles as general as possible.

Next, we aim to verify the integrity of the approach by embracing and generalizing many of the variational principles given in the literature, as well as formulating new ones. Lastly, we aim to apply the developed framework to the dimension theory of expanding systems in order to derive new estimates for the Hausdorff dimensions of subsets of their phase spaces.

### §1.3. Main results

Let $(X,d)$ be a compact metric space and $T : X \to X$ be continuous. Fix a sequence $\Phi = (\varphi_n)_{n\geq 1}$ of arbitrary functions $\varphi_n : X \to [-\infty,\infty]$ (we call $\Phi$ to be a potential, see Definition 3.1). For a subset $Z \subseteq X$, denote by $P_Z(\Phi)$ the topological pressure of $\Phi$ on $Z$, as defined in Definition 3.6. By $\mathscr{M}_T(X)$ we denote the set of all $T$-invariant Borel probability measures on $X$, and by $\mathscr{E}_T(X)$ the set of ergodic ones. Given a $\mu \in \mathscr{M}_T(X)$, the quantity $h_\mu(T)$ denotes the measure-theoretic entropy of $\mu$ with respect to $T$. By $h_{\text{top}}(T)$ we denote the topological entropy of the dynamical system $(X,T)$.

*Variational inequalities and principles.* The core results of this thesis are Theorem 1.1 and Theorem 1.2. They provide a framework for establishing variational principles for the topological pressure: Given certain subsets $Z \subseteq X$, both theorems can be used to establish upper or lower bounds for $P_Z(\Phi)$, respectively.

**Theorem 1.1** (Conditional variational inequality). *Let $\lambda : \mathscr{M}_T(X) \to [-\infty,\infty]$ be a mapping and $h_{\text{top}}(T) < \infty$. Then one has for every subset $\mathscr{Y} \subseteq \mathscr{M}_T(X)$*

$$P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) \leq \sup\left\{ h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y} \right\}, \tag{1.5}$$

*where the set $A(\Phi,\lambda,\mathscr{Y}) \subseteq X$ is defined in (4.3). In particular, if one has $Z \subseteq X$ such that $Z \subseteq A(\Phi,\lambda,\mathscr{Y})$, then*

$$P_Z(\Phi) \leq \sup\left\{ h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y} \right\}.$$

Theorem 1.1 basically states that for each choice of $\Phi, \lambda$ and $\mathscr{Y}$ there exists a corresponding (possibly empty) set $A(\Phi,\lambda,\mathscr{Y})$, such that the variational inequality (1.5) holds. The structure of $A(\Phi,\lambda,\mathscr{Y})$ can be precisely described (see (4.3)). To give an example, in the classical setting one considers $\Phi = \left(\sum_{i<n} \varphi \circ T^i\right)_{n\in\mathbb{N}}$ and $\lambda(\mu) = \int_X \varphi \, \mathrm{d}\mu$, where $\varphi : X \to \mathbb{R}$ is continuous. In this particular case one has

$$A(\Phi,\lambda,\mathscr{Y}) = \left\{ x \in X : V_T(x) \cap \mathscr{Y} \neq \varnothing \right\},$$

where $V_T(x)$ denotes the set of all $T$-invariant sublimits of $\left(\frac{1}{n}\sum_{i<n} \delta_{T^i x}\right)_{n\in\mathbb{N}}$ in the weak$^*$-topology. We want to emphasize that our result holds true for arbitrary

choices of $\Phi, \lambda$ and $\mathscr{Y}$. In particular, the functions $\varphi_n : X \to [-\infty, \infty]$ do not need to be continuous or Borel measurable or carry any other additional structure. Special cases of Theorem 1.1 for continuous, additive $\Phi$ were given in [Bow73] and [CP10] (see also Remark 4.5). We prove a slightly stronger formulation of Theorem 1.1, which allows $h_{\text{top}}(T) = \infty$, in Theorem 4.4.

**Theorem 1.2** (Mass distribution principle)**.** *Fix $\mu \in \mathscr{E}_T(X)$, and let $\Phi = (\varphi_n)_{n\geq 1}$ be Borel measurable, that is, each $\varphi_n : X \to [-\infty, \infty]$ is Borel measurable. Suppose there exist a constant $b \in [-\infty, \infty]$ and a Borel set $B \subseteq X$ such that $\mu(B) > 0$ and $\liminf_{n\to\infty} \frac{1}{n}\varphi_n(x) \geq b$ for each $x \in B$. Then, if $h_\mu(T) + b$ is well-defined, one has*

$$P_B(\Phi) \geq h_\mu(T) + b.$$

*In particular, if*

$$h_\mu(T) + \operatorname{ess\,inf}_\mu \left( \liminf_{n\to\infty} \frac{1}{n}\varphi_n \right)$$

*is well-defined (the essential infimum $\operatorname{ess\,inf}_\mu$ is recalled in (4.18)), one has*

$$h_\mu(T) + \operatorname{ess\,inf}_\mu \left( \liminf_{n\to\infty} \frac{1}{n}\varphi_n \right) \leq \inf\left\{ P_Z(\Phi) : \mu(Z) > 0 \right\}.$$

The proof of Theorem 1.2 is given in Theorem 4.8 and Theorem 4.12. Special cases of the above statement, with a similar method of proof, were given in [CHZ13] for sub/super-additive continuous potentials, in [Rau15] for multivariate continuous potentials, in [FH16] for weighted pressure of sub-additive potentials, and in [Rau17] quasi-integrable additive potentials (see also Remark 4.11). We want to emphasize that in our statement the only essential assumption imposed on $\Phi$ is that all $\varphi_n$ are Borel measurable. In particular,

- $\Phi$ does not need to carry any additional structure like sub-additivity;
- $\liminf_{n\to\infty} \frac{1}{n}\varphi_n$ does not need to be constant $\mu$-almost everywhere (see Remark 4.13 for more details).

The combination of Theorem 1.5 and Theorem 1.2 leads to the following general variational principle:

**Theorem 1.3** (General variational principle)**.** *Let $h_{\text{top}}(T) < \infty$ and $\lambda : \mathscr{M}_T(X) \to [-\infty, \infty]$ be a mapping. Fix $\mathscr{Y} \subseteq \mathscr{E}_T(X)$ and suppose that $\Phi$ is Borel measurable and that for each $\mu \in \mathscr{Y}$*

$$\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \lambda(\mu) \tag{1.6}$$

*holds for $\mu$-almost all $x \in X$. Then one has*

$$P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) = \sup\left\{ h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y} \right\}.$$

*In particular, one can choose for each $\mu \in \mathscr{Y}$ a Borel set $B_\mu \subseteq A(\Phi, \lambda, \mathscr{Y})$ such that $\mu(B_\mu) = 1$, and*

$$P_Z(\Phi) = \sup\left\{ h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y} \right\},$$

*where $Z = \bigcup_{\mu\in\mathscr{Y}} B_\mu$ (see (1.7) for a concrete example).*

We prove a slightly stronger formulation of the above theorem in Theorem 5.2, which also allows the case $h_{\text{top}}(T) = \infty$.

Theorem 1.3 can be seen as blueprint for various different variational principles, depending on the choice of $\Phi$ and $\lambda$. Condition (1.6) provides a description of the relationship between both of these: It basically states that, if there exists a pointwise ergodic theorem, then a variational principle holds. Thus, proving a variational principle for a given $\Phi$ reduces to finding a corresponding mapping $\lambda$ which satisfies (1.6). We give a list of possible applications in Section 5. It contains generalizations of known variational principles (e.g. additive, sub-additive, multivariate case) and provides variational principles, which, to our knowledge were not known before (variational principles for ratios, see Proposition 5.35, and for invariant functions, see Theorem 5.38). Here we want to give generalizations of two well-known variational principles, namely variational principles for sub-additive and additive $\Phi$. Denote by $G_\mu$ the generic points of a measure $\mu \in \mathcal{M}_T(X)$, and by $\mathcal{E}_T(X) \subseteq \mathcal{M}_T(X)$ the ergodic measures (see Section 2 for the definitions).

**Theorem 1.4** (Variational principle for sub- or super-additive $\Phi$)**.** *Let $h_{\text{top}}(T) < \infty$ and $\Phi$ be Borel measurable. Suppose, $\Phi$ is sub- or super-additive (see §5.3 for the precise definition). Suppose, furthermore, that the conditions in Kingman's sub-additive ergodic theorem are satisfied for $\Phi$ and each $\mu \in \mathcal{M}_T(X)$ (that is, $\varphi_1^+ \in L^1(X,\mu)$ in the sub-additive case, and $\varphi_1^- \in L^1(X,\mu)$ in the super-additive case). Then one has*

$$P_Z(\Phi) = \sup\left\{ h_\mu(T) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, \mathrm{d}\mu : \mu \in \mathcal{M}_T(X) \right\},$$

*where*

$$Z = \bigcup_{\mu \in \mathcal{E}_T(X)} \left\{ x \in G_\mu : \lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, \mathrm{d}\mu \right\}. \tag{1.7}$$

The above theorem is a consequence of the slightly stronger formulated Theorem 5.30, and it generalizes the results in [CFH08] and [CMP10] to arbitrary Borel measurable sub-additive $\Phi$. Moreover, we want to emphasize that the statement includes Borel measurable super-additive potentials, which, to our knowledge, was not established in literature before (see also Remark 5.31). We want to point out that if $\Phi$ is upper semi-continuous and sub-additive, there is also a variational principle for every $T$-invariant Borel set $Z$ (see Theorem 5.28 and Remark 5.29). In this case, it is in addition possible to express the sub-additive pressure $P_X(\Phi)$ by means of the additive pressure $P_{R(\Phi)}(\varphi)$ of the invariant function $\varphi = \lim_{n\to\infty} \frac{1}{n}\varphi_n$ (see Theorem 5.39).

If $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is both sub- and super-additive, it is called additive and can be represented as $\Phi = \left(\sum_{i<n} \varphi_1 \circ T^i\right)_{n\in\mathbb{N}}$ (see §5.2 for more details). Theorem 1.3 then immediately implies a variational principle for Borel measurable functions $\varphi$:

**Corollary 1.5** (Variational principle for measurable $\varphi$)**.** *Suppose $h_{\text{top}}(T) < \infty$ and $\varphi : X \to \mathbb{R}$ to be Borel measurable and quasi-integrable (see (5.3) for the notion of quasi integrability) with respect to every $\mu \in \mathcal{M}_T(X)$. Then one has*

$$P_Z(\varphi) := P_Z(\Phi) = \sup\left\{ h_\mu(T) + \int_X \varphi \, \mathrm{d}\mu : \mu \in \mathcal{M}_T(X) \right\},$$

*where $\Phi = \left(\sum_{i<n} \varphi \circ T^i\right)_{n\in\mathbb{N}}$, and $Z$ is defined in (1.7).*

We provide a stronger version of the above statement in Theorem 5.16, which permits the cases of $h_{\mathrm{top}}(T) = \infty$ and Borel measurable $\varphi : X \to [-\infty, \infty]$. Corollary 1.5 generalizes the classical variational principle for continuous functions $\varphi$ given in [Wal75] to measurable ones. It also generalizes previous results for certain discontinuous functions $\varphi$, as given in [Mum07] and [Rau17].

Next, we want to mention two results concerning saturated systems $(X, T)$. We call a system $(X, T)$ saturated if for every $\mu \in \mathcal{M}_T(X)$, one has $h_{\mathrm{top}}(G_\mu) = h_\mu(T)$ and $G_\mu \neq \varnothing$ (see §5.8 and Definition 5.46 for further details and examples).

**Theorem 1.6** (Variational principle for saturated systems)**.** *Let $\varphi : X \to \mathbb{R}$ be continuous and $(X, T)$ be saturated. Then, for every subset $\mathscr{Y} \subseteq \mathcal{M}_T(X)$, one has*

$$P_Z(\varphi) = \sup\left\{ h_\mu(T) + \int_X \varphi \, \mathrm{d}\mu : \mu \in \mathscr{Y} \right\},$$

*where $Z = \bigcup_{\mu \in \mathscr{Y}} G_\mu$.*

A proof of the above theorem is given in Theorem 5.48. It answers the following question:

> Given a continuous function $\varphi : X \to \mathbb{R}$ and some subset $\mathscr{Y} \subseteq \mathcal{M}_T(X)$, is there a subset $Z \subseteq X$ such that
>
> $$P_Z(\varphi) = \sup\left\{ h_\mu(T) + \int_X \varphi \, \mathrm{d}\mu : \mu \in \mathscr{Y} \right\} ?$$

Thus, according to Theorem 1.6, in saturated systems one can always relate the variational pressure $\mathscr{Y} \mapsto \sup\{h_\mu(T) + \int_X \varphi \, \mathrm{d}\mu : \mu \in \mathscr{Y}\}$ to the topological pressure $P_Z(\varphi)$ on the collection of generic points $Z = \bigcup_{\mu \in \mathscr{Y}} G_\mu$. A special case of Theorem 1.6 was proven in [CP10], namely for systems with the $g$-almost product property and closed subsets $\mathscr{Y}$. We want to emphasize that in our case the theorem holds true for arbitrary $\mathscr{Y}$, in every saturated system. Also note that for general system $(X, T)$, Theorem 1.3 can answer the above question for general non-additive, discontinuous $\Phi$ (see Question 5.1), as long as one assumes $\mathscr{Y} \subseteq \mathscr{E}_T(X)$.

An immediate application of Theorem 1.5 is the variational principle for level sets in saturated systems.

**Corollary 1.7** (Variational principle for level sets)**.** *Assume $(X, T)$ to be saturated and $\varphi, \psi : X \to \mathbb{R}$ to be continuous. For $\alpha \in \mathbb{R}$, let $K(\varphi, \alpha)$ be the $\alpha$-level set for $\varphi$ as defined in §5.7. Fix an arbitrary subset $U \subseteq \mathbb{R}$ and define $K(\varphi, U) := \bigcup_{\alpha \in U} K(\varphi, \alpha)$. Then one has*

$$P_{K(\varphi,U)}(\psi) = \sup\left\{ h_\mu(T) + \int_X \psi \, \mathrm{d}\mu : \int_X \varphi \, \mathrm{d}\mu \in U \right\}.$$

Special cases of the above statement were proven in [Tho09] for $U = \{\alpha\}$ and, more generally, in [CP10] for closed sets $U$. Statement and proof of Corollary 1.7 are given in Theorem 5.52.

In addition to saturated systems, we are able to derive results for level sets in general systems, which are not necessarily saturated, and for potentials, which are not necessarily continuous. In particular, we have the following result:

**Theorem 1.8** (Pressure on level sets)**.** *Suppose $h_{top}(T) < \infty$. Let $\varphi : X \to \mathbb{R}$ be continuous and $\psi : X \to [-\infty, \infty)$ be upper semi-continuous. Define*

$$E_\varphi := \left\{ \alpha \in \mathbb{R} : \exists \mu \in \mathscr{E}_T(X) \text{ such that } \int_X \varphi \, d\mu = \alpha \right\}.$$

*If $\alpha \in E_\varphi$, then at least one of the following cases holds:*

*(1)*

$$P_{K(\varphi,\alpha)}(\psi) = \sup \left\{ h_\mu(T) + \int_X \psi \, d\mu : \int_X \varphi \, d\mu = \alpha \right\}.$$

*(2) There exists a measure $\nu \in \mathscr{M}_T(X)$ such that $\int_X \varphi \, d\nu = \alpha$ and $P_{K(\varphi,\alpha)}(\psi) = h_\nu(T) + \int_X \psi \, d\nu$.*

The above theorem is proven in Theorem 5.43. Under stronger assumptions, one can show that always case (2) holds (see Corollary 5.44). We want to emphasize that $\psi$ does not need to be continuous, and, to our knowledge, this is the first result concerning the topological pressure of discontinuous potentials on level sets in the literature.

*Estimates for Hausdorff dimensions.* If one considers systems $(X, T)$ which are uniformly expanding conformal, one can use the framework developed so far to estimate the Hausdorff dimension $\dim_H Z$ of certain subsets $Z \subseteq X$. A system $(X, T)$ is called uniformly expanding conformal if

$$x \mapsto a(x) := \lim_{y \to x} \frac{d(Tx, Ty)}{d(x, y)}$$

exists, and is continuous and strictly greater than 1 (see §6.1 and Definition 6.1). A general Bowen formula for uniformly expanding conformal systems was proven in [Cli11] (see Theorem 6.3). If we combine this result with Theorem 1.5 and Theorem 1.2, we obtain the following theorem, which is proven in §6.1 (see Theorem 6.6 and Theorem 6.7):

**Theorem 1.9** (Estimates for the Hausdorff dimension)**.** *Assume that $(X, T)$ is uniformly expanding conformal. Fix $\emptyset \neq Z \subseteq X$. Choose for each $x \in Z$ a measure $\mu_x \in V_T(x)$ and define $\mathscr{Y}_Z := \{\mu_x : x \in Z\}$. Then one has the following:*

*1.*

$$\dim_H Z \leq \sup \left\{ \frac{h_\mu(T)}{\int_X \log a \, d\mu} : \mu \in \mathscr{Y}_Z \right\}. \tag{1.8}$$

*In particular, given some $\emptyset \neq \mathscr{Y} \subseteq \mathscr{M}_T(X)$, Inequality (1.8) holds for $Z = \{x \in X : V_T(x) \cap \mathscr{Y} \neq \emptyset\}$ and $\mathscr{Y}_Z = \mathscr{Y}$.*

*2. If $Z$ is in addition a Borel set, one has*

$$\dim_H Z \geq \sup_\mu \frac{h_\mu(T)}{\int_X \log a \, d\mu},$$

*where the supremum is taken over all $\mu \in \mathscr{M}_T(X)$ such that either $\mu(Z) = 1$, or $\mu$ is ergodic and $\mu(Z) > 0$.*

Theorem 1.9 basically tells us that for any non-empty subset $Z \subseteq X$, one can find an upper estimate of $\dim_H Z$ by means of the ratios $\frac{h_{\mu_x}(T)}{\int_X \log a \, d\mu_x}$, where the invariant measures $\mu_x$ correspond to the points $x \in Z$ in the described way. Moreover, if $Z$ is Borel and carries mass for some measure $\mu \in \mathscr{M}_T(X)$, we obtain a lower bound too. Thus, Theorem 1.9 can be seen as a generalization of the well-know result that for a conformal $\mathscr{C}^1$-repeller $f : J \to J$, one has

$$\dim_H J = \sup\left\{ \frac{h_\mu(f)}{\int_X \log \|Df\| \, d\mu} : \mu \in \mathscr{M}_f(J) \right\}$$

(see e.g. [GP97]). The next corollary is a simple application of Theorem 1.9, and basically shows that generic points of measures with zero Hausdorff dimension can only have zero Hausdorff dimension. Its proof is given in Corollary 6.9.

**Corollary 1.10.** *Let $(X,T)$ be uniformly expanding conformal. Define for each $\mu \in \mathscr{M}_T(X)$ the upper Hausdorff dimension of $\mu$ to be*

$$\overline{\dim}_H \mu := \inf\{\dim_H B : \mu(B) = 1\}.$$

*Denote $\mathscr{Y}_0 := \{\mu \in \mathscr{M}_T(X) : \overline{\dim}_H \mu = 0\}$ and $X_0 := \{x \in X : V_T(x) \cap \mathscr{Y}_0 \neq \varnothing\}$. Then, if $X_0 \neq \varnothing$, one has $\dim_H X_0 = 0$.*

We want to add here that, if one drops the conformality assumption in Theorem 1.9, we are still able to derive a dimension estimate with the present framework.

**Theorem 1.11.** *Let $f : J \to J$ be a $\mathscr{C}^1$-repeller (see Definition 6.14). Then, for each Borel set $Z \subseteq J$, one has*

$$\dim_H Z \geq \sup_\mu \frac{h_\mu(f)}{\int_J \log \|Df\| \, d\mu},$$

*where the supremum is taken over all $\mu \in \mathscr{M}_T(J)$ such that either $\mu(Z) = 1$, or $\mu$ is ergodic and $\mu(Z) > 0$.*

The proof of the above theorem is given in Theorem 6.15 and relies on Theorem 3.1 (1) in [CWZ14].

Note that Theorem 1.9 (or Theorem 1.11) cannot give a lower bound if $Z$ has no mass for any $\mu \in \mathscr{M}_T(X)$. A typical example would be the generic points $G_\mu$ for a non-ergodic measure $\mu \in \mathscr{M}_T(X)$. Nevertheless, if $(X,T)$ is saturated and uniformly expanding conformal, one can use Theorem 1.6 instead to calculate the Hausdorff dimension of unions of generic points.

**Theorem 1.12** (Hausdorff dimension of generic points)**.** *Assume $(X,T)$ to be saturated and uniformly expanding conformal. If $\varnothing \neq \mathscr{Y} \subseteq \mathscr{M}_T(X)$, then*

$$\dim_H \bigcup_{\mu \in \mathscr{Y}} G_\mu = \sup_{\mu \in \mathscr{Y}} \dim_H G_\mu = \sup\left\{ \frac{h_\mu(T)}{\int_X \log a \, d\mu} : \mu \in \mathscr{Y} \right\}.$$

A proof of Theorem 1.12 can be found in Theorem 6.10. The above statement was known for closed $\mathscr{Y}$ by the results in [CP10] and [Cli11] and is now generalized to arbitrary non-empty $\mathscr{Y} \subseteq \mathscr{M}_T(X)$. This shows that, under the above assumptions, the Hausdorff dimension behaves stable under arbitrary unions of $G_\mu$. It shows also that the suprema of ratios $\frac{h_\mu(T)}{\int_X \log a \, d\mu}$ always give Hausdorff dimensions of unions of generic points.

Finally, we want to give the following application of our framework:

**Corollary 1.13** (Relative multifractal spectrum of ergodic averages)**.** *Assume $(X,T)$ to be saturated and uniformly expanding conformal. Fix some set $\varnothing \neq I$, let $\varphi_i, \psi_i : X \to \mathbb{R}$ be continuous functions and $\psi_i > 0$ for all $i \in I$. Let $C \subseteq \mathbb{R}^I$ be arbitrary and define*

$$K_C := \left\{ x \in X : \left( \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \varphi_i(T^j x)}{\sum_{j=0}^{n-1} \psi_i(T^j x)} \right)_{i\in I} \in C \right\}.$$

*Then, if $K_C \neq \varnothing$, one has*

$$\dim_H K_C = \sup \left\{ \frac{h_\mu(T)}{\int_X \log a \, d\mu} : \left( \frac{\int_X \varphi_i \, d\mu}{\int_X \psi_i \, d\mu} \right)_{i\in I} \in C \right\}.$$

The proof of Corollary 1.13 can be found in Corollary 6.11. A special case of the above statement was derived for finite sets $I$ and closed, convex $C \subseteq \mathbb{R}^I$ in [Ols03], Theorem 4, for graph directed self-conformal function systems. Corollary 1.13 generalizes this result in three ways: First, we allow $I$ to be an arbitrary, possibly uncountable index set, second, we allow $C \subseteq \mathbb{R}^I$ to be an arbitrary subset, and, third, we obtain an equality. Another result which is related to Corollary 1.13 and [Ols03], was given in Theorem E of [Cao13]. Here it was shown that Corollary 1.13 holds for average conformal $\mathscr{C}^1$-repellers and asymptotically additive sequences $\Phi, \Psi$, where $C \subseteq \mathbb{R}$ is compact. We want to note that with the same method of proof, Corollary 1.13 also holds for this case; moreover, it generalizes to arbitrary families of asymptotically additive sequences $\Phi_i, \Psi_i$ (see also Remark 6.12).

## 2. Preliminaries

In this section we briefly recall and collect some basic notions and notations, which will be used in several places of the present work. Most of the terminology follows [Wal82] and [Bar11]. Terminology, which is specific for a certain section, is introduced there in the beginning.

**Definition 2.1.** Let $(X, \mathfrak{B})$ be a measurable space and $T : X \to X$ be a measurable mapping. A measure $\mu$ on $(X, \mathfrak{B})$ is said to be $T$-invariant, if $\mu(T^{-1}B) = \mu(B)$ for each $B \in \mathfrak{B}$. Likewise, a set $Z \subseteq X$ is called $T$-invariant, if $T^{-1}Z = Z$. A measure $\mu$ on $(X, \mathfrak{B})$ is said to be ergodic, if $\mu(B) \in \{0,1\}$ for each $T$-invariant set $B \in \mathfrak{B}$. The set of all $T$-invariant probability measures on $(X, \mathfrak{B})$ is denoted by $\mathscr{M}_T(X)$, and the set of all $T$-invariant, ergodic ones is denoted by $\mathscr{E}_T(X) \subseteq \mathscr{M}_T(X)$. For each measurable set $Z \subseteq X$, define

$$\mathscr{M}_T(Z) := \left\{ \mu \in \mathscr{M}_T(X) : \mu(Z) = 1 \right\} \quad \text{and} \quad \mathscr{E}_T(Z) := \left\{ \mu \in \mathscr{E}_T(X) : \mu(Z) > 0 \right\}.$$

Recall for $\mu \in \mathscr{M}_T(X)$ the quantity $h_\mu(T)$ to be the measure-theoretic entropy of $\mu$ with respect to $T$.

Now let $(X,d)$ be a compact metric space. The $d$-open ball centered around $x \in X$ with radius $\epsilon > 0$ is defined as

$$B_d(x,\epsilon) := \left\{ y \in X : d(x,y) < \epsilon \right\}.$$

We call $\mathscr{U} \subseteq 2^X$ to be a finite open cover of $X$, if $\operatorname{card}\mathscr{U} < \infty$ and $X \subseteq \bigcup_{U\in\mathscr{U}} U$, where all $U \in \mathscr{U}$ are $d$-open.

Let $T : X \to X$ furthermore be continuous. We call the tuple $(X,T)$ to be a dynamical system. The quantity

$$d_n(x,y) := \max\left\{d(T^i x, T^i y) : 0 \le i < n\right\}$$

gives rise to a metric on $X$ for each $n \ge 1$. Each $d_n$ induces the same topology as $d$, and by $B_n(x,\epsilon) := B_{d_n}(x,\epsilon)$ we denote the $n$-th Bowen ball with center $x$ and radius $\epsilon$. By $h_{\text{top}}(T)$ we denote the topological entropy of $(X,T)$. For a finite open cover $\mathscr{U}$ and $\mathbf{U} = (U_1,\dots,U_n) \in \mathscr{U}^n$, define $m(\mathbf{U}) := n$ and

$$X(\mathbf{U}) := \left\{x \in X : T^{i-1}(x) \in U_i \text{ for } i = 1,\dots,m(\mathbf{U})\right\}.$$

Let $\mu,\mu_1,\mu_2,\mu_3,\dots$ be a sequence of Borel probability measures on $X$. We say that $\lim_{n\to\infty}\mu_n = \mu$ in the weak$^*$-topology, if

$$\lim_{n\to\infty}\int_X \varphi\,\mathrm{d}\mu_n = \int_X \varphi\,\mathrm{d}\mu$$

for every continuous function $\varphi : X \to \mathbb{R}$. Given $x \in X$ and $n \ge 1$, define the probability measures

$$\delta_{x,n} := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^i(x)}, \tag{2.1}$$

where $\delta_y$ is the Dirac measure at point $y \in X$. The measures $\delta_{x,n}$ are called $n$-th empirical measures of $x$. Denote by $V_T(x) \subseteq \mathscr{M}_T(X)$ the set of all $T$-invariant sublimits of $(\delta_{x,n})_{n\in\mathbb{N}}$ in the weak$^*$-topology. For dynamical systems $(X,T)$ one can show that $V_T(x) \neq \varnothing$ for every $x \in X$, and that $\mathscr{M}_T(X)$ is a compact metrizable space. For each $\mu \in \mathscr{M}_T(X)$, the set

$$G_\mu := \left\{x \in X : \lim_{n\to\infty}\delta_{x,n} = \mu\right\}$$

is called set of generic points of $\mu$. Note that $\mu(G_\mu) = 1$, if $\mu$ is ergodic.

Finally, recall the following important facts:

**Theorem 2.2** (Ergodic decomposition theorem)**.** *Let $(X,T)$ be a dynamical system. Fix $\mu \in \mathscr{M}_T(X)$ and denote by $\mu = \int_{\mathscr{E}_T(X)} \nu\,\mathrm{d}m_\mu(\nu)$ the ergodic decomposition of $\mu$. If $\varphi : X \to \mathbb{R}$ is Borel measurable and bounded, then one has*

$$\int_X \varphi\,\mathrm{d}\mu = \int_{\mathscr{E}_T(X)}\left(\int_X \varphi\,\mathrm{d}\nu\right)\mathrm{d}m_\mu(\nu).$$

**Theorem 2.3** (Kingman's sub-additive ergodic theorem)**.** *Let $(X,T)$ be a dynamical system and $\mu \in \mathscr{M}_T(X)$. Suppose $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of Borel measurable functions $\varphi_n : X \to [-\infty,\infty)$ such that*

*(a) $\varphi_1^+ \in L^1(X,\mu)$;*

*(b) for each $m,n \in \mathbb{N}$ one has $\varphi_{n+m} \le \varphi_n + \varphi_m \circ T^n$ $\mu$-almost everywhere.*

*Then there is a Borel measurable function* $\varphi : X \to [-\infty,\infty)$ *such that* $\varphi^+ \in L^1(X,\mu)$, $\varphi \circ T = \varphi$ *$\mu$-almost everywhere,* $\lim_{n\to\infty} \frac{1}{n}\varphi_n = \varphi$ *$\mu$-almost everywhere, and*

$$\lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, \mathrm{d}\mu = \int_X \varphi \, \mathrm{d}\mu.$$

Kingman's ergodic theorem immediately implies Birkhoff's ergodic theorem (we state only the special case for ergodic measures here):

**Corollary 2.4** (Birkhoff's ergodic theorem for ergodic measures)**.** *Let* $(X,T)$ *be a dynamical system and* $\mu \in \mathscr{E}_T(X)$*. Suppose* $\varphi \in L^1(X,\mu)$*. Then one has*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \int_X \varphi \, \mathrm{d}\mu$$

*for $\mu$-almost every* $x \in X$*.*

## 3. Topological pressure

In this section we define topological pressure, prove basic properties, and finally compare it to known definitions given in the literature. Throughout this section, let $(X,T)$ be a dynamical system.

### §3.1. Pressure via $(\delta,N)$-covers

The following definition is designed in a way that the usual properties hold, and that one can apply the known methods of proof for the variational principle. It also extends the classical definition of topological pressure for continuous potentials with tempered variation (see §3.3).

**Definition 3.1.** We call a sequence $\Phi = (\varphi_n)_{n\geq 1}$ to be an potential on $(X,T)$, if $\varphi_n : X \to [-\infty,\infty]$ are functions for all $n \geq 1$. The potential $\Phi$ is said to have a certain property, if $\varphi_n$ have this property for all $n \geq 1$. For instance, $\Phi$ is called Borel measurable, if $\varphi_n : X \to [-\infty,\infty]$ is Borel measurable for each $n \geq 1$.

Given some $\varnothing \neq A \subseteq X$, define for each $n \geq 1$

$$\Phi_n(A) := \sup_{x\in A} \varphi_n(x)$$

and $\Phi_n(\varnothing) := -\infty$. For $\delta > 0$, $N \geq 1$ and $Z \subseteq X$, let $\mathscr{C}_Z(\delta,N)$ be the set of all $\Gamma = \{(n_i,B_i)\}_{i\in I}$, such that

($\alpha$) $I \subseteq \mathbb{N}$ and $n_i \geq N$ for all $i \in I$;

($\beta$) $B_i \subseteq X$ are Borel sets for all $i \in I$;

($\gamma$) For each $i \in I$ there is an $x_i \in X$ such that $B_i \subseteq B_{n_i}(x_i,\delta)$;

($\delta$) $Z \subseteq \bigcup_{i\in I} B_i$.

Each element $\Gamma \in \mathscr{C}_Z(\delta,N)$ is called $(\delta,N)$-cover of $Z$.

The sets $\mathscr{C}_Z(\delta,N)$ are actually non-empty:

**Lemma 3.2.** *Given* $Z \subseteq X$ *one has* $\mathscr{C}_Z(\delta,N) \neq \varnothing$ *for all* $\delta > 0$ *and* $N \geq 1$*.*

*Proof.* As $X$ is compact, $\overline{Z}$ is compact too. Thus, $\overline{Z} \subseteq \bigcup_{z\in\overline{Z}} B_N(z,\delta) = \bigcup_{i=1}^{l} B_N(z_i,\delta)$ for some $z_1,\ldots,z_l \in \overline{Z}$. This means $\Gamma := \{(N, B_N(z_i,\delta))\}_{i=1}^{l} \in \mathscr{C}_Z(\delta,N)$.

□

**Definition 3.3.** Let $\Phi$ be a potential on $(X,T)$. Fix $\varnothing \neq Z \subseteq X$, $\delta > 0$, $\alpha \in \mathbb{R}$ and $N \geq 1$. Set $\exp(-\infty) := 0$, $\exp(\infty) := \infty$ and define

$$\Lambda_Z(\Phi,\delta,\alpha,N) := \inf_{\Gamma \in \mathscr{C}_Z(\delta,N)} \sum_{(n,B)\in\Gamma} \exp\big(-\alpha n + \Phi_n(Z\cap B)\big).$$

As $\mathscr{C}_Z(\delta,N+1) \subseteq \mathscr{C}_Z(\delta,N)$ for all $N \geq 1$, the following limit is well-defined:

$$\Lambda_Z(\Phi,\delta,\alpha) := \lim_{N\to\infty} \Lambda_Z(\Phi,\delta,\alpha,N) = \sup_{N\in\mathbb{N}} \Lambda_Z(\Phi,\delta,\alpha,N).$$

**Remark 3.4.** The above definitions are based on the definition of weighted topological pressure, which was introduced in [FH16]. We will discuss this further in §3.3.

**Lemma 3.5.** *Let $\beta \in \mathbb{R}$ and $Z \subseteq X$. If $\Lambda_Z(\Phi,\delta,\beta) < \infty$, then*

$$\Lambda_Z(\Phi,\delta,\alpha) = \Lambda_Z(\Phi,\delta,\alpha,N) = 0$$

*for all $\alpha > \beta$ and $N \geq 1$.*

*Proof.* The case $Z = \varnothing$ is clear. Choose some $C \in \mathbb{R}$ such that $\Lambda_Z(\Phi,\delta,\beta) < C$. Then $\Lambda_Z(\Phi,\delta,\beta,N) < C$ for all $N \geq 1$. Hence

$$\begin{aligned}
0 &\leq \inf_{\Gamma \in \mathscr{C}_Z(\delta,N)} \sum_{(n,B)\in\Gamma} \exp\big(-\alpha n + \Phi_n(Z\cap B)\big) \\
&= \inf_{\Gamma \in \mathscr{C}_Z(\delta,N)} \sum_{(n,B)\in\Gamma} \exp\big(-\beta n + \Phi_n(Z\cap B)\big)\exp(n(\beta-\alpha)) \\
&\leq C\big(\exp(\beta-\alpha)\big)^N \to 0
\end{aligned}$$

as $N \to \infty$. This shows the statement.

□

The above lemma shows that similarly to the definition of the Hausdorff dimension, there exists a unique jump value for $\delta \mapsto \Lambda_Z(\Phi,\delta,\alpha)$. This value will be used for the definition of pressure.

**Definition 3.6.** Let $\Phi$ be a potential on $(X,T)$. Fix $Z \subseteq X$ and $\delta > 0$. By Lemma 3.5, the following quantity is well-defined:

$$P_Z(\Phi,\delta) := \inf\big\{\alpha \in \mathbb{R} : \Lambda_Z(\Phi,\delta,\alpha) = 0\big\}.$$

Furthermore, one has $\mathscr{C}_Z(\delta',N) \subseteq \mathscr{C}_Z(\delta,N)$ for $0 < \delta' < \delta$ and $N \geq 1$. Hence the following limit is also well-defined:

$$P_Z(\Phi) := \lim_{\delta\to 0} P_Z(\Phi,\delta) = \sup_{\delta>0} P_Z(\Phi,\delta).$$

The quantity $P_Z(\Phi)$ is called topological pressure of $\Phi$ on $Z$ with respect to $(X,T)$. Note that if $\varphi : X \to [-\infty,\infty]$ is a function, we will define for simplicity $P_Z(\varphi) := P_Z(\Phi)$, where $\Phi := \big(\sum_{i<n} \varphi\circ T^i\big)_{n\geq 1}$. See Definition 5.6 and (5.5) for a precise description.

**Remark 3.7.** We want to emphasize that one has $P_{\varnothing}(\Phi) = -\infty$ for each potential $\Phi$ . This follows from $\Lambda_{\varnothing}(\Phi,\delta,\alpha,N) = 0$ for every $\delta > 0$, $\alpha \in \mathbb{R}$ and $N \geq 1$. Thus, $P_Z(\Phi) > -\infty$ implies $Z \neq \varnothing$. For other definitions of pressure or entropy this does not need to be the case (see Remark 3.12 and Remark 3.19).

## §3.2. Basic properties of pressure

We state well-known properties of topological pressure like monotonicity and countable stability. Full proofs are given for the convenience of the reader. Let $\Phi$ be a potential on $(X,T)$.

**Lemma 3.8** (Monotonicity of pressure)**.** *Fix $\delta>0$, $\alpha\in\mathbb{R}$, $N\geq 1$ and $Y\subseteq Z\subseteq X$. Then one has $\Lambda_Y(\Phi,\delta,\alpha,N)\leq\Lambda_Z(\Phi,\delta,\alpha,N)$. In particular, $P_Y(\Phi,\delta)\leq P_Z(\Phi,\delta)$ for all $\delta>0$.*

*Proof.* Note that $\mathscr{C}_Z(\delta,N)\subseteq\mathscr{C}_Y(\delta,N)$ for all $\delta>0$ and $N\geq 1$. Furthermore, $\Phi_n(Y\cap B)\leq\Phi_n(Z\cap B)$ for all $n\geq 1$ and Borel sets $B$. Thus $\Lambda_Y(\Phi,\delta,\alpha,N)\leq\Lambda_Z(\Phi,\delta,\alpha,N)$ for each $\delta>0$, $\alpha\in\mathbb{R}$ and $N\geq 1$, which shows the statement.

□

**Remark 3.9.** The above lemma does not hold in general, if one requires $x_i\in Z$ in condition $(\gamma)$ of Definition 3.1.

**Lemma 3.10** (Countable stability of pressure)**.** *Suppose $Z=\bigcup_{j\in J}Z_j$, where $J\subseteq\mathbb{N}$ and $Z_j\subseteq X$ for all $j\in J$. Then one has $\sup_{j\in J}P_{Z_j}(\Phi,\delta)=P_Z(\Phi,\delta)$ for all $\delta>0$. In addition, one has $P_Z(\Phi)=\sup_{j\in J}P_{Z_j}(\Phi)$.*

*Proof.* Fix $\delta>0$. The estimate $\sup_{j\in J}P_{Z_j}(\Phi,\delta)\leq P_Z(\Phi,\delta)$ follows immediately from Lemma 3.8. Suppose $\sup_{j\in J}P_{Z_j}(\Phi,\delta)<\alpha<\infty$. Then $\Lambda_{Z_j}(\Phi,\delta,\alpha)=0$ for all $j\in J$. Given an $\epsilon>0$, there exists for each $N\geq 1$ and $j\in J$ by Lemma 3.5 a $\Gamma_j^N\in\mathscr{C}_{Z_j}(\delta,N)$ such that

$$\sum_{(n,B)\in\Gamma_j^N}\exp\big(-\alpha n+\Phi_n(Z\cap B)\big)\leq\frac{\epsilon}{2^j}.$$

Set $\Gamma^N:=\bigcup_{j\in J}\Gamma_j^N$. As

$$Z=\bigcup_{j\in J}Z_j\subseteq\bigcup_{j\in J}\bigcup_{(n,B)\in\Gamma_j^N}B=\bigcup_{(n,B)\in\Gamma^N}B,$$

we have that $\Gamma^N\in\mathscr{C}_Z(\delta,N)$. Hence

$$\Lambda_Z(\Phi,\delta,\alpha,N)\leq\sum_{j\in J}\sum_{(n,B)\in\Gamma_j^N}\exp\big(-\alpha n+\Phi_n(Z\cap B)\big)\leq\sum_{j=1}^{\infty}\frac{\epsilon}{2^j}=\epsilon.$$

As $\epsilon>0$ was arbitrarily chosen, we have $\Lambda_Z(\Phi,\delta,\alpha,N)=0$ for all $N\geq 1$. Thus $\Lambda_Z(\Phi,\delta,\alpha)=0$ for each $\alpha>\sup_{j\in J}P_{Z_j}(\Phi,\delta)$. But this means $\sup_{j\in J}P_{Z_j}(\Phi,\delta)\geq P_Z(\Phi,\delta)$. Furthermore, as $\delta>0$ was arbitrarily chosen, and as $\delta\mapsto P_{Z_j}(\Phi,\delta)$ is monotone, we derive

$$P_Z(\Phi)=\lim_{\delta\to 0}P_Z(\Phi,\delta)=\lim_{\delta\to 0}\sup_{j\in J}P_{Z_j}(\Phi,\delta)=\sup_{j\in J}\lim_{\delta\to 0}P_{Z_j}(\Phi,\delta)=\sup_{j\in J}P_{Z_j}(\Phi).$$

□

## §3.3. Comparison to other notions of pressure

In this subsection we compare our definition of topological pressure to other notions of pressure given in the literature.

*Potentials with tempered variation.* We first consider the special case of potentials $\Phi$ which satisfy tempered variation. They were considered in [Bar96] as extension of the classical notions of topological pressure and entropy for non-compact sets, which again were introduced in [Bow73] and [PP84].

**Definition 3.11.** Let $\Phi$ be a potential on $(X,T)$. For each finite open cover $\mathscr{U}$ of $X$, define

$$\operatorname{var}_n(\Phi,\mathscr{U}) := \sup\left\{ |\varphi_n(x)-\varphi_n(y)| : x,y \in X(\mathbf{U}) \text{ for some } \mathbf{U} \in \mathscr{U}^n \right\}.$$

We say $\Phi$ satisfies tempered variation, if

$$\limsup_{\operatorname{diam}\mathscr{U}\to 0} \limsup_{n\to\infty} \frac{\operatorname{var}_n(\Phi,\mathscr{U})}{n} = 0.$$

Here $\operatorname{diam}\mathscr{U} := \sup_{U\in\mathscr{U}} \operatorname{diam} U$. Denote

$$\varphi(\mathbf{U}) := \begin{cases} \sup_{x\in X(\mathbf{U})} \varphi_{m(\mathbf{U})}(x), & X(\mathbf{U}) \neq \varnothing \\ -\infty, & X(\mathbf{U}) = \varnothing \end{cases}.$$

Given a subset $Z \subseteq X$ and $\alpha \in \mathbb{R}$, define

$$M_Z(\Phi,\mathscr{U},\alpha) := \lim_{N\to\infty} \inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp\big(-\alpha m(\mathbf{U}) + \varphi(\mathbf{U})\big), \tag{3.1}$$

where the infimum is taken over all $\Gamma \subseteq \bigcup_{m\geq N} \mathscr{U}^m$ of $Z$, such that $Z \subseteq \bigcup_{\mathbf{U}\in\Gamma} X(\mathbf{U})$. Similarly to Lemma 3.5 one can show that there is a unique jump value for $\alpha \mapsto M_Z(\Phi,\mathscr{U},\alpha)$, which we call $P_Z^B(\Phi,\mathscr{U})$. The limit

$$P_Z^B(\Phi) := \lim_{\operatorname{diam}\mathscr{U}\to 0} P_Z^B(\Phi,\mathscr{U})$$

is called non-additive topological pressure of $\Phi$ on $Z$. If one replaces in (3.1) the quantity $\varphi(\mathbf{U})$ by 0, one obtains in the same way as above for each $Z \subseteq X$ the quantity $h_{\text{top}}(Z)$, which is called topological entropy of $Z$.

**Remark 3.12.** We want to give some remarks about $P^B$ and $h_{\text{top}}$ in the literature:

(a) In [Bar96], the topological entropy of $Z$ was defined to be $P_Z^B(\mathbf{0})$, where $\mathbf{0} := (0,0,\dots)$. It is easy to see that $P_Z^B(\mathbf{0}) = h_{\text{top}}(Z)$ for all $\varnothing \neq Z \subseteq X$. However if one assumes $\operatorname{diam}\varnothing \leq 0$ (and this is what we will do here), one has $-\infty = P_\varnothing^B(\mathbf{0}) < h_{\text{top}}(\varnothing) = 0$. This is because if $\varnothing \in \mathscr{U}$ for an open cover $\mathscr{U}$, one has $\varnothing \subseteq X(\mathbf{U}) = \varnothing$ for $\mathbf{U} = (\varnothing,\dots,\varnothing) \in \mathscr{U}^m$.

(b) In [PP84], for the definition of pressure the quantity $Z(\mathbf{U}) := X(\mathbf{U}) \cap Z$ instead of $X(\mathbf{U})$ is used. This gives nevertheless the same definition due to the tempered variation property.

(c) In [PP84] it was shown that $h_{\text{top}}(\cdot)$ is equivalent to Bowen's definition of topological entropy for non-compact sets in [Bow73] .

(d) Usually, if $\Phi$ has tempered variation, all $\varphi_n$ are assumed to be continuous. The existence of weak Gibbs measures for possibly discontinuous $\varphi_n$ was for instance examined in [Kes01].

We now have the following relation between $P^B$ and $P$:

**Proposition 3.13.** *If $\Phi$ is a potential on $(X,T)$ which satisfies tempered variation, then $P_Z(\Phi) = P_Z^B(\Phi)$ and $P_Z(\mathbf{0}) = P_Z^B(\mathbf{0}) = h_{\text{top}}(Z)$ for all $\varnothing \neq Z \subseteq X$. In addition, $-\infty = P_\varnothing(\mathbf{0}) = P_\varnothing^B(\mathbf{0}) < h_{\text{top}}(\varnothing) = 0$.*

*Proof.* The proof of the statement works in same way as the proof of Proposition A.2.1 in [Cli10]. The case $Z = \varnothing$ is clear, so assume $Z \neq \varnothing$. Fix a finite open cover $\mathscr{U}$ with $\operatorname{diam}\mathscr{U} < \delta$. Pick some cover $\Gamma \subseteq \bigcup_{m\geq N}\mathscr{U}^m$ in the sense of Definition 3.11. Without loss of generality we may assume $X(\mathbf{U}) \cap Z \neq \varnothing$ for every $\mathbf{U} \in \Gamma$. Therefore one can choose some epicenter $x_{\mathbf{U}} \in X(\mathbf{U}) \cap Z$. As $X(\mathbf{U}) \subseteq B_{m(\mathbf{U})}(x_{\mathbf{U}}, \delta)$, we immediately see that $\Gamma := \{(m(\mathbf{U}), X(\mathbf{U}))\}_{\mathbf{U}\in\Gamma}$ is also a $(\delta, N)$-cover in the sense of Definition 3.1. Note that

$$\varphi(\mathbf{U}) \geq \Phi_{m(\mathbf{U})}(Z \cap X(\mathbf{U})).$$

Thus $M(\Phi,\mathscr{U},\alpha) \geq \Lambda_Z(\Phi,\delta,\alpha)$ and $P_Z^{\mathrm{B}}(\Phi,\mathscr{U}) \geq P_Z(\Phi,\delta)$. Letting $\delta \to 0$ yields $\operatorname{diam}\mathscr{U} \to 0$, which means

$$P_Z^B(\Phi) \geq \lim_{\delta\to 0} P_Z(\Phi,\delta).$$

For the other direction, fix a finite open cover $\mathscr{U}$ of $X$ with corresponding Lebesgue number $l(\mathscr{U})$. Assume without restriction that

$$\limsup_{n\to\infty} \frac{\operatorname{var}_n(\Phi,\mathscr{U})}{n} < \infty.$$

Now pick an $\left(\frac{1}{2}l(\mathscr{U}), N\right)$-cover $\Gamma^*$ of $Z$. By definition of the metric $d_n$ and the Lebesgue number $l(\mathscr{U})$, for each $(n_i, B_i) \in \Gamma$ there exists a vector $\mathbf{U}_i \in \mathscr{U}^{n_i}$ such that $B_i \subseteq B_{n_i}\left(x_i, \frac{1}{2}l(\mathscr{U})\right) \subseteq X(\mathbf{U}_i)$. The collection $\Gamma_N$ of those vectors forms a cover of $Z$ in the sense of Definition 3.11. Choose $N_0 \in \mathbb{N}$ such that

$$\frac{\operatorname{var}_n(\Phi,\mathscr{U})}{n} \leq \limsup_{n\to\infty} \frac{\operatorname{var}_n(\Phi,\mathscr{U})}{n} + l(\mathscr{U})$$

for all $n \geq N_0$. Then

$$\begin{aligned}
\inf_{\Gamma} \sum_{\mathbf{U}\in\Gamma} \exp\big(-\alpha m(\mathbf{U}) + \varphi(\mathbf{U})\big) &\leq \sum_{\mathbf{U}\in\Gamma_N} \exp\left(-\alpha m(\mathbf{U}) + \sup_{x\in X(\mathbf{U})} \varphi_{m(\mathbf{U})}(x)\right) \\
&\leq \sum_{(n_i,B_i)\in\Gamma^*} \exp\big(-\alpha n_i + \Phi_{n_i}(Z\cap B_i) + \operatorname{var}_{n_i}(\Phi,\mathscr{U})\big) \\
&\leq \sum_{(n_i,B_i)\in\Gamma^*} \exp\left(-\left(\alpha - \limsup_{n\to\infty}\frac{1}{n}\operatorname{var}_n(\Phi,\mathscr{U}) - l(\mathscr{U})\right)n_i + \Phi_{n_i}(Z\cap B_i)\right)
\end{aligned}$$

for all $N \geq N_0$, where the infimum is taken over all covers $\Gamma \subseteq \bigcup_{m\geq N}\mathscr{U}^m$ in the sense of Definition 3.11. Hence, as $\Gamma^*$ was chosen arbitrarily, by $N \to \infty$ we obtain

$$M_Z(\Phi,\mathscr{U},\alpha) \leq \Lambda_Z\left(\Phi, \frac{1}{2}l(\mathscr{U}), \alpha - \limsup_{n\to\infty}\frac{1}{n}\operatorname{var}(\Phi,\mathscr{U}) - l(\mathscr{U})\right).$$

Therefore

$$P_Z^B(\Phi,\mathscr{U}) \leq P_Z\left(\Phi, \frac{1}{2}l(\mathscr{U})\right) + \limsup_{n\to\infty}\frac{1}{n}\operatorname{var}(\Phi,\mathscr{U}) + l(\mathscr{U}).$$

Letting $\operatorname{diam}\mathscr{U} \to 0$ results in $l(\mathscr{U}) \to 0$. Using tempered variation of $\Phi$ we end at

$$P_Z^B(\Phi) \leq \lim_{\delta\to 0} P_Z(\Phi,\delta).$$

□

*Weighted pressure.* We want to compare our definition to the pressure introduced in [FH16]. In this work, a finite sequence $(X_i,T_i)_{i=1,\dots,k}$ of dynamical systems is considered, which have the additional property that $(X_{i+1},T_{i+1})$ is a factor of $(X_i,T_i)$. For this sequence, a weight $\mathbf{a}=(a_1,\dots,a_k)$ is considered, where each $a_i$ corresponds to $(X_i,T_i)$. A weighted version of topological pressure $P_Z^{\mathbf{a}}(\Phi)$ for sub-additive sequences $\Phi$ is introduced, although the definition makes also sense for arbitrary $\Phi$. We don't want to give the full definition here, as in our setting we consider only the case $k=1$ and $\mathbf{a}=(1)$. But in this particular case, we have the following:

**Definition 3.14.** Define for $Z\subseteq X$, $\delta>0$, $N\geq 1$ and $\alpha\in\mathbb{R}$

$$\widetilde{\Lambda}_Z(\Phi,\delta,\alpha,N):=\inf_{\Gamma\in\mathscr{C}_Z(\delta,N)}\sum_{(n,B)\in\Gamma}\exp\big(-\alpha n+\Phi_n(B)\big). \tag{3.2}$$

Then, similarly to Lemma 3.5 and Definition 3.6, one defines $\widetilde{P}_Z(\Phi)$. One has then $\widetilde{P}_Z(\Phi)=P_Z^{(1)}(\Phi)$.

The only difference between $\widetilde{P}_Z(\Phi)$ and $P_Z(\Phi)$ is that one considers in (3.2) the quantity $\Phi_n(B)$ instead of $\Phi_n(Z\cap B)$. The reason why we modify the approach of [FH16] is of a technical nature: We need this in the proof of the conditional variational inequality (see (4.11), and compare also to Remark 3.12 (b)). The following is an immediate consequence:

**Proposition 3.15.** *One has $P_Z(\Phi)\leq\widetilde{P}_Z(\Phi)$. If $Z\subseteq X$ is a Borel set, then $P_Z(\Phi)=\widetilde{P}_Z(\Phi)$.*

*Proof.* If $Z$ is a Borel set and $\Gamma=\{(n_i,B_i)\}_{i\in I}$ is a $(\delta,N)$-cover of $Z$, then $\Gamma^*:=\{(n_i,Z\cap B_i)\}_{i\in I}$ is also a $(\delta,N)$-cover of $Z$. Thus $\Lambda_Z(\Phi,\delta,\alpha,N)=\widetilde{\Lambda}_Z(\Phi,\delta,\alpha,N)$ for all $\delta>0$, $\alpha\in\mathbb{R}$ and $N\geq 1$.

□

**Remark 3.16.** Proposition 3.13 also holds if one replaces $P_Z(\Phi)$ by $\widetilde{P}_Z(\Phi)$. This means, if $\Phi$ has tempered variation, then $\widetilde{P}_Z(\Phi)=P_Z^B(\Phi)$ for all $Z\subseteq X$.

*Pressure via Bowen balls.* Another definition of pressure, given in [CHZ13] and [CWZ14] for sub-additive and super-additive potentials, is as follows:

**Definition 3.17.** Define for $Z\subseteq X$, $\delta>0$, $N\geq 1$ and $\alpha\in\mathbb{R}$

$$m_Z(\Phi,\delta,\alpha,N):=\inf_{\Gamma}\sum_{B_n(x,\delta)\in\Gamma}\exp\big(-\alpha n+\Phi_n(B_n(x,\delta))\big),$$

where the infimum is taken over all at most countable $\Gamma=\{B_{n_i}(x_i,\delta)\}_i$ such that $n_i\geq N$, $x_i\in X$ and $Z\subseteq\bigcup_i B_{n_i}(x_i,\delta)$. Then, similarly to Lemma 3.5 and Definition 3.6, one can define $P_Z^C(\Phi):=\liminf_{\delta\to 0}P_Z^C(\Phi,\delta)$ (note that here actually one has to take the limes inferior).

As $\Lambda_Z(\Phi,\delta,\alpha,N)\leq m_Z(\Phi,\delta,\alpha,N)$ for all $\delta>0$, $N\geq 1$ and $\alpha\in\mathbb{R}$, we have the following:

**Proposition 3.18.** *One has $P_Z(\Phi)\leq P_Z^C(\Phi)$ for each $Z\subseteq X$.*

**Remark 3.19.** If $\Phi$ has tempered variation, one can show like in the proof of Proposition 3.13 that $P_Z^C(\Phi)=P_Z^B(\Phi)$ for all $\varnothing\neq Z\subseteq X$. However, one has $P_\varnothing^C(\mathbf{0})=h_{\text{top}}(\varnothing)=0$, which means by Remark 3.12 that $P_Z^C(\mathbf{0})=h_{\text{top}}(Z)$ for all $Z\subseteq X$.

*Pressure via separated sets.* Finally, we want to compare our definition to the topological pressure via separating sets (see [Wal75] and [Wal82]). We follow closely [Rau17].

**Definition 3.20.** Given some $\epsilon > 0$, a subset $\varnothing \neq E \subseteq Z \subseteq X$ is called $(\epsilon, n)$-separated in $Z$, if

$$\inf\{d_n(x,y) : x \neq y \in E\} \geq \epsilon.$$

In addition, $E \subseteq Z$ is called maximal $(\epsilon, n)$-separated in $Z$, if for all $z \in Z$ the set $E \cup \{z\}$ is not $(\epsilon, n)$-separated anymore. One can easily verify then for each $(\epsilon, n)$-separated $E \subseteq Z$ there exists some $E \subseteq E' \subseteq Z$ such that $E'$ is maximal $(\epsilon, n)$-separated. A finite subset $\varnothing \neq F \subseteq Z$ is called $(\epsilon, n)$-spanning in $Z$, if

$$Z \subseteq \bigcup_{z \in F} B_n(z, \epsilon).$$

One can show that every $(\epsilon, n)$-separated set in X is finite, and each maximal $(\epsilon, n)$-separated set is $(\epsilon, n)$-spanning in $Z$. For given potential $\Phi$, $\varnothing \neq Z \subseteq X$, $\epsilon > 0$ and $n \in \mathbb{N}$ define

$$\overline{M}_Z(\Phi, \epsilon, n) := \sup_E \sum_{x \in E} \exp \varphi_n(x),$$

where the supremum is taken over all $(\epsilon, n)$-separated sets $E$ in $Z$. Define

$$\overline{P}_Z(\Phi) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \overline{M}_Z(T, \Phi, \epsilon, n),$$

The above quantity is called upper topological pressure via separating sets of $\Phi$ on $Z$.

Recall that $P_Z(\Phi) = P_Z^B(\Phi)$, if $\Phi$ has tempered variation. The following relation was established in [Bar96] and [CFH08]:

**Proposition 3.21** (Theorem 4.2.7 in [Bar11]). *Let $\Phi$ be a potential with tempered variation. Suppose in addition that*

- *for each $n \geq 1$ one has $\varphi_n : X \to \mathbb{R}$ to be continuous;*
- *for each $n, m \in \mathbb{N}$ one has $\varphi_{n+m} \leq \varphi_n + \varphi_m \circ T^n$.*

*Then, for each $T$-invariant compact subset $Z \subseteq X$, one has $P_Z(\Phi) = P_Z^B(\Phi) = \overline{P}_Z(\Phi)$.*

We can in addition give the following relation:

**Proposition 3.22.** *Suppose $\varphi_n : X \to [-\infty, \infty)$ to be upper semi-continuous for each $n \geq 1$. Then, if $Z \subseteq X$ is compact, one has $P_Z(\Phi) \leq \overline{P}_Z(\Phi)$.*

*Proof.* The proof is a modification of the proof of Theorem 4.6 in [Rau17].

We may assume $P_Z(\Phi) > -\infty$, which implies $Z \neq \varnothing$. Fix some $\delta_0 > 0$ such that $P_Z(\Phi, \delta) > -\infty$ for all $0 < \delta < \delta_0$. Fix furthermore an $-\infty < \alpha < P_Z(\Phi, \delta)$. Then

$$\begin{aligned}\infty = \Lambda_Z(\Phi, \delta, \alpha) &= \lim_{N \to \infty} \inf_{\Gamma \in \mathscr{C}_Z(\delta, N)} \sum_{(n,B) \in \Gamma} \exp\big(-\alpha n + \Phi_n(Z \cap B)\big) \\ &\leq \limsup_{N \to \infty} \inf_{\Gamma \in \overline{\mathscr{C}}_Z(\delta, N)} \sum_{(N,B) \in \Gamma} \exp\big(-\alpha N + \Phi_N(Z \cap B)\big),\end{aligned}$$

where $\overline{\mathscr{C}}_Z(\delta,N) \subseteq \mathscr{C}_Z(\delta,N)$ denotes all $(\delta,N)$-covers such that each covering Bowen ball has exact length $N$. Hence there has to be an $N_0 \in \mathbb{N}$ large enough, such that

$$\exp(\alpha N) \leq \inf_{\Gamma \in \overline{\mathscr{C}}_Z(\delta,N)} \sum_{(N,B)\in\Gamma} \exp \Phi_N(Z \cap B) \tag{3.3}$$

for $N \geq N_0$. Now, as $\varphi_N$ is upper semi-continuous and $Z$ is closed, one can pick an $x_1^N \in Z$ such that $\varphi_N\left(x_1^N\right) = \sup_{x\in Z} \varphi_N(x)$. Similarly, we find $x_j^N \in Z \setminus \bigcup_{i=1}^{j-1} B_N\left(x_i^N,\delta\right)$ such that

$$\varphi_N\left(x_j^N\right) = \sup_{x \in Z \setminus \bigcup_{i=1}^{j-1} B_N(x_i^N,\delta)} \varphi_N(x). \tag{3.4}$$

Those points form a maximal $(\delta,N)$-separated set $E = \left\{x_1^N,\dots,x_{j_N}^N\right\}$ in $Z$. Define Borel sets $B_j := B_N\left(x_j^N,\delta\right) \setminus \bigcup_{i=1}^{j-1} B_N\left(x_i^N,\delta\right)$. As $E$ is $(\delta,N)$-spanning in $Z$, we have

$$Z \subseteq \bigcup_{i=1}^{j_N} B_N\left(x_i^N,\delta\right) = \bigcup_{j=1}^{j_N} B_j.$$

Thus $\Gamma := \{(N,B_j)\}_{j=1}^{j_N} \in \overline{\mathscr{C}}_Z(\delta,N)$, and by the definition of $B_j$ one has

$$Z \cap B_j \subseteq Z \setminus \bigcup_{i=1}^{j-1} B_N(x_i^N,\delta).$$

Hence it follows by (3.4) that $\Phi_n(Z \cap B_j) = \varphi_N\left(x_j^N\right)$. Now applying this to (3.3) we obtain

$$\exp(\alpha N) \leq \sum_{(N,B)\in\Gamma} \exp \Phi_N(Z \cap B) = \sum_{x \in E} \exp \varphi_N(x) \leq \overline{M}_Z(\Phi,\delta,N)$$

for all $N \geq N_0$. Hence there is an $N_1 \geq N_0$ such that

$$\alpha \leq \frac{1}{N} \log \overline{M}_Z(\Phi,\delta,N) \leq \limsup_{n\to\infty} \frac{1}{n} \log \overline{M}_Z(\Phi,\delta,n) + \delta$$

for all $N \geq N_1$. As $\alpha < P_Z(\Phi,\delta)$ was arbitrarily chosen, letting $\alpha \to P_Z(\Phi,\delta)$ yields

$$P_Z(\Phi,\delta) \leq \limsup_{n\to\infty} \frac{1}{n} \log \overline{M}_Z(\Phi,\delta,n) + \delta.$$

Letting $\delta \to 0$ shows the statement.

□

**Remark 3.23.** The statement of Proposition 3.22 holds also for potentials $\Phi$ such that each $\varphi_n$ attains its supremum on each non-empty closed subset of $X$. Finally, we remark that by Proposition 3.15 the statement holds as well, if one replaces $P_Z(\Phi)$ by $\widetilde{P}_Z(\Phi)$.

## 4. Variational inequalities

In this section, we prove variational inequalities, which later will be used as framework for proving variational principles. The methods of the proofs are well-known in literature, but pushed to the boundaries to obtain results even for discontinuous potentials. Throughout this section, let $(X,T)$ be a dynamical system with potential $\Phi$.

### §4.1. The conditional variational inequalities

We first introduce variational pressure and an abstract notion of Lyapunov exponents.

**Definition 4.1.** A mapping $\lambda : \mathcal{M}_T(X) \to [-\infty, \infty]$ is called Lyapunov exponent. The corresponding set

$$\mathscr{A}(\lambda) := \left\{ \mu \in \mathcal{M}_T(X) : h_\mu(T) < \infty \text{ or } \lambda(\mu) > -\infty \right\} \tag{4.1}$$

is called set of all allowed $T$-invariant measures with respect to $\lambda$. This means for measures $\mu \in \mathscr{A}(\lambda)$ the quantity $h_\mu(T) + \lambda(\mu)$ is well-defined. Fix some $\mu \in \mathcal{M}_T(X)$. A point $x \in X$ such that $\mu \in V_T(x)$ is called allowed point with respect to $\lambda$, $\Phi$ and $\mu$, if

$$\limsup_{l \to \infty} \frac{1}{n_l} \varphi_{n_l}(x) \leq \lambda(\mu) \tag{4.2}$$

for all subsequences $(n_l)_{l \geq 1}$ which satisfy $\delta_{x,n_l} \to \mu$ as $l \to \infty$ (recall $\delta_{x,n}$ to be the empirical measures of $x \in X$, see (2.1)). The set of all those points is denoted by $A(\Phi, \lambda, \mu)$. For a subset $\mathscr{Y} \subseteq \mathcal{M}_T(X)$, denote in addition

$$A(\Phi, \lambda, \mathscr{Y}) := \bigcup_{\mu \in \mathscr{Y}} A(\Phi, \lambda, \mu) \tag{4.3}$$

and

$$V(\Phi, \lambda, \mathscr{Y}) := \bigcap_{\mu \in \mathscr{Y}} A(\Phi, \lambda, \mu).$$

Note that $A(\Phi, \lambda, \mu)$ can be empty. This is in particular the case if $\mu \notin V_T(x)$ for each $x \in X$, and in general depends on the choice of $\Phi$ and $\lambda$. To understand the structure of the above sets better, the following proposition is useful:

**Proposition 4.2.** *Let $\varphi : X \to [-\infty, \infty)$ be upper semi-continuous, $\lambda(\mu) := \int_X \varphi \, \mathrm{d}\mu$ and $\varphi_n := \sum_{i<n} \varphi \circ T^i$. Then one has*

$$A(\Phi, \lambda, \mathscr{Y}) = \left\{ x \in X : V_T(x) \cap \mathscr{Y} \neq \varnothing \right\},$$
$$V(\Phi, \lambda, \mathscr{Y}) = \left\{ x \in X : \mathscr{Y} \subseteq V_T(x) \right\},$$

*that is, both sets are independent of $\varphi$.*

*Proof.* Let $x \in X$ and $\mu \in V_T(x) \cap \mathscr{Y}$. If $(n_l)_{l \in \mathbb{N}}$ is any subsequence such that $\lim_{l \to \infty} \delta_{x,n_l} = \mu$, one has

$$\limsup_{l \to \infty} \frac{1}{n_l} \varphi_{n_l}(x) = \limsup_{l \to \infty} \frac{1}{n_l} \sum_{i=0}^{n-1} \varphi(T^i x) = \limsup_{l \to \infty} \int_X \varphi \, \mathrm{d}\delta_{x,n_l} \leq \int_X \varphi \, \mathrm{d}\mu = \lambda(\mu),$$

as $\varphi$ is upper semi-continuous (see Lemma A.2 (d)). Thus $x \in A(\Phi, \lambda, \mu)$ and $\left\{ x \in X : V_T(x) \cap \mathscr{Y} \neq \varnothing \right\} \subseteq A(\Phi, \lambda, \mathscr{Y})$. If $x \in A(\Phi, \lambda, \mathscr{Y})$, then there exists a $\mu \in \mathscr{Y}$ such that $x \in A(\Phi, \lambda, \mu)$. But by definition this means $V_T(x) \cap \mathscr{Y} \neq \varnothing$, and hence $A(\Phi, \lambda, \mathscr{Y}) \subseteq \left\{ x \in X : V_T(x) \cap \mathscr{Y} \neq \varnothing \right\}$. The statement for $V(\Phi, \lambda, \mathscr{Y})$ now follows easily. $\square$

Define $\sup\varnothing := -\infty$. Then for each $\mathscr{Y} \subseteq \mathscr{A}(\lambda)$, the quantities

$$p_{\mathscr{Y}}(\lambda) := \sup\left\{h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y}\right\}$$

and

$$q_{\mathscr{Y}}(\lambda) := \inf\left\{h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y}\right\}$$

are well-defined and called upper and lower variational pressure of $\lambda$ over $\mathscr{Y}$.

**Remark 4.3.** We want to give an interpretation of the above definitions.

(a) Suppose $T$ to be a differentiable mapping and denote $\psi_n := |(T^n)'|$. Then the (upper) Lyapunov exponent $\chi$ at point $x_0 \in X$ can be defined to be the exponential growth rate of $\psi_n$ in $x_0$, namely

$$\chi(x_0) := \limsup_{n\to\infty} \frac{1}{n} \log \psi_n(x_0) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left|T'(T^i x_0)\right|.$$

For given $\nu \in \mathscr{E}_T(X)$, the limit agrees $\nu$-almost everywhere with $\int_X \log|T'|\,\mathrm{d}\nu$. Thus from a measure-theoretic point of view, it is natural to call the mapping $\lambda : \mathscr{M}_T(X) \to \mathbb{R}$, $\lambda(\mu) := \int_X \log|T'|\,\mathrm{d}\mu$, to be a Lyapunov exponent. Now set $\varphi_n := \log \psi_n$ for each $n \geq 1$, and $\Phi := (\varphi_n)_{n\in\mathbb{N}}$. Then, roughly speaking, $A(\Phi, \lambda, \mu)$ consists of points $x \in X$ such that the empirical measures of $x$ converge to $\mu$, and $\chi(x) \leq \lambda(\mu)$.

(b) In this context, we say that $\mathscr{Y}$ is a condition, imposed on the measures, over which is taken the supremum in the variational pressure. This is a notion from multifractal analysis, where entropy or dimension spectra can be often expressed in terms of conditional variational principles (see for instance [Cli13]).

The main result of this subsection is as follows:

**Theorem 4.4** (Conditional variational inequalities)**.** *Let $\lambda$ be a Lyapunov exponent. If $\mathscr{Y} \subseteq \mathscr{A}(\lambda)$, then one has*

$$P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) \leq \sup\left\{h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y}\right\} \tag{4.4}$$

*and*

$$P_{V(\Phi,\lambda,\mathscr{Y})}(\Phi) \leq \inf\left\{h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y}\right\}.$$

**Remark 4.5.** Before we proceed with the proof, we want to give three remarks:

(a) The above statement can be seen as generalization of Theorem 2 in [Bow73], which is as follows: For each $t \in \mathbb{R}$ one has $h_{\mathrm{top}}(B_t) \leq t$, where

$$B_t := \left\{x \in X : \exists \mu \in V_T(x) \text{ such that } h_\mu(T) \leq t\right\}.$$

Indeed, if one sets $\varphi_n := 0$, $\lambda := 0$ and $\mathscr{Y} := \{\mu \in \mathscr{M}_T(X) \text{ such that } h_\mu(T) \leq t\}$, then one has $B_t = A(\Phi, \lambda, \mathscr{Y})$. By (4.4) it follows that

$$h_{\mathrm{top}}(B_t) = P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) \leq p_{\mathscr{Y}}(\lambda) = \sup_{\mu\in\mathscr{Y}} h_\mu(T) \leq t.$$

(b) We emphasize that (4.4) works for arbitrary $\lambda$ and $\Phi$. We only have to assume that $\mathscr{Y} \subseteq \mathscr{A}(\lambda)$, which is automatically satisfied if either $h_{\text{top}}(T) < \infty$ or $\lambda > -\infty$.

(c) In the case $\lambda(\mu) := \int_X \varphi \, \mathrm{d}\mu$ and $\varphi_n := \sum_{i<n} \varphi \circ T^i$ for continuous $\varphi : X \to \mathbb{R}$, Theorem 4.4 was already proven in [CP10] for closed subsets $\mathscr{Y}$.

*Proof of Theorem 4.4.* In case the $p_{\mathscr{Y}}(\Phi) = \infty$ we are done, therefore assume $p_{\mathscr{Y}}(\Phi) < \infty$. This implies $0 \le h_\mu(T) < \infty$ and $\lambda(\mu) < \infty$ for each $\mu \in \mathscr{Y}$. Thus we can divide $\mathscr{Y}$ into two parts $\mathscr{Y}_{-\infty} := \left\{ \mu \in \mathscr{Y} : \lambda(\mu) = -\infty \right\}$ and $\mathscr{Y}' := \left\{ \mu \in \mathscr{Y} : \lambda(\mu) > -\infty \right\}$. As a result we obtain by Lemma 3.10 and $A(\Phi, \lambda, \mathscr{Y}) = A(\Phi, \lambda, \mathscr{Y}_{-\infty}) \cup A(\Phi, \lambda, \mathscr{Y}')$

$$P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) = \max\left\{ P_{A(\Phi,\lambda,\mathscr{Y}_{-\infty})}(\Phi), P_{A(\Phi,\lambda,\mathscr{Y}')}(\Phi) \right\}. \tag{4.5}$$

Now suppose we have already shown (4.4) for each subset $\mathscr{T} \subseteq \mathscr{A}(\lambda)$ such that $-\infty < \lambda(\mu) < \infty$ for all $\mu \in \mathscr{T}$. Define a sequence of Lyapunov exponents $\lambda_N(\mu) := -h_\mu(T) - N$ for $N \in \mathbb{N}$ and $\mu \in \mathscr{M}_T(X)$. As the entropies $h_\mu(T)$ are finite for all $\mu \in \mathscr{Y}_{-\infty}$, one has $-\infty < \lambda_N(\mu) < \infty$ for all $\mu \in \mathscr{Y}_{-\infty}$ and $N \in \mathbb{N}$. In addition

$$A(\Phi, \lambda, \mathscr{Y}_{-\infty}) \subseteq A(\Phi, \lambda_N, \mathscr{Y}_{-\infty})$$

holds for each $N \in \mathbb{N}$. Thus by using Lemma 3.8 and (4.4) we obtain

$$\begin{aligned} P_{A(\Phi,\lambda,\mathscr{Y}_{-\infty})}(\Phi) &\le P_{A(\Phi,\lambda_N,\mathscr{Y}_{-\infty})}(\Phi) \le p_{\mathscr{Y}_{-\infty}}(\lambda_N) \\ &= \sup\left\{ h_\mu(T) - h_\mu(T) - N : \mu \in \mathscr{Y}_{-\infty} \right\} \le -N. \end{aligned}$$

Note that if $\mathscr{Y}_{-\infty} = \varnothing$, we already have $P_{A(\Phi,\lambda,\mathscr{Y}_{-\infty})}(\Phi) = p_{\mathscr{Y}_{-\infty}}(\lambda_N) = -\infty$. Otherwise letting $N \to \infty$ yields $P_{A(\Phi,\lambda,\mathscr{Y}_{-\infty})}(\Phi) = -\infty$, and by (4.5) we end at

$$P_{A(\Phi,\lambda,\mathscr{Y})} = P_{A(\Phi,\lambda,\mathscr{Y}')} \le p_{\mathscr{Y}'}(\lambda) \le p_{\mathscr{Y}}(\lambda).$$

Hence it remains to show (4.4) for all subsets $\mathscr{Y} \subseteq \mathscr{A}(\lambda)$ satisfying $h_\mu(T) < \infty$ and $-\infty < \lambda(\mu) < \infty$ for all $\mu \in \mathscr{Y}$. Now pick such a set $\mathscr{Y}$ and suppose $p_{\mathscr{Y}}(\lambda) = -\infty$. This implies $\mathscr{Y} = \varnothing$; otherwise a measure $\mu \in \mathscr{Y}$ exists such that $p_{\mathscr{Y}}(\lambda) \ge h_\mu(T) + \lambda(\mu) > -\infty$, which is a contradiction. This means $A(\Phi, \lambda, \mathscr{Y}) = \varnothing$, thus $P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) = p_{\mathscr{Y}}(\lambda) = -\infty$. Therefore without restriction we may assume

$$-\infty < p_{\mathscr{Y}}(\lambda) < \infty. \tag{4.6}$$

To proceed we need some technical lemmas.

**Lemma 4.6** (Lemma 2.16, [Bow75]). *Let $E$ be a finite set and $q \in \mathbb{N}$. Fix $a = (a_1, \dots, a_q) \in E^q$, define the probability measure $\nu_a$ as*

$$\nu_a(e) := \frac{1}{q} \operatorname{card}\left\{ j \in \mathbb{N} : a_j = e \right\}$$

*for every $e \in E$, and set the entropy of $a$ to*

$$H(a) := -\sum_{e \in E} \nu_a(e) \log \nu_a(e).$$

*Then for $h \ge 0$ one has*

$$\limsup_{q \to \infty} \frac{1}{q} \log \operatorname{card}\left\{ a \in E^q : H(a) \le h \right\} \le h.$$

**Lemma 4.7.** *Let $x \in X$ and $\mu \in V_T(x)$ such that $x$ is allowed with respect to $\lambda$, $\Phi$ and $\mu$. Fix $\delta > 0$ and a finite open cover $\mathscr{U}$ of $X$. Then there exists a number $m \geq 1$ such that for any $j \geq 1$ one can find an $N \geq j$ and a $\mathbf{U} \in \mathscr{U}^N$ satisfying:*

1. *$x \in X(\mathbf{U})$;*
2. *$\mathbf{U}$ contains a subvector $\mathbf{V}$ of length $qm$, $N \geq qm \geq N - m$ for a $q \in \mathbb{N}$, that seen as an element of $(\mathscr{U}^m)^q$ satisfies*
$$\frac{1}{m} H(\mathbf{V}) \leq h_\mu(T) + \delta;$$
3. *$\varphi_N(x) \leq N(\lambda(\mu) + \delta)$.*

*Proof.* The first and second condition can be proven like Lemma 2.15 in [Bow75]. In the proof one constructs an increasing natural sequence $(N'_j)_{j\in\mathbb{N}}$ and corresponding vectors $(\mathbf{U}_j)_{j\in\mathbb{N}}$ such that conditions *(1)* and *(2)* are satisfied, and $\delta_{x,N'_j} \to \mu$ as $j \to \infty$. Then by (4.2) one has

$$\limsup_{j\to\infty} \frac{1}{N'_j} \varphi_{N'_j}(x) \leq \lambda(\mu).$$

Thus there is an $j_0 \geq 1$ such that $\varphi_{N'_j}(x) \leq N'_j(\lambda(\mu)+\delta)$ for all $j \geq j_0$. Hence for each $j \geq 1$ the number $N_j := N'_{j+j_0}$ together with $\mathbf{U}_{j+j_0}$ satisfies all three conditions. □

The rest of the proof works in the same way as the proof of the upper estimate for the non-additive variational principle in [Bar96]. The first goal is to cover $A(\Phi,\lambda,\mathscr{Y})$ with countable many suitable subsets. To do so we fix $\delta > 0$ and a finite open cover $\mathscr{U}$ of $X$ satisfying $\operatorname{diam}\mathscr{U} < \delta$. In addition fix for each $x \in A(\Phi,\lambda,\mathscr{Y})$ a measure $\mu_x \in \mathscr{Y}$ such that $x \in A(\Phi,\lambda,\mu_x)$. Choose some $u_1, u_2, u_3, \ldots \in \mathbb{R}$ such that for every $z \in \mathbb{R}$ there exists a $u_i$ satisfying $|u_i - z| < \delta$. Now denote for $m, i \geq 1$ by $Z_{m,i}$ the set of points $x \in A(\Phi,\lambda,\mathscr{Y})$, which meet the following criteria:

- The measure $\mu_x$ fulfills $\lambda(\mu_x) \in [u_i - \delta, u_i + \delta]$.
- All three properties in Lemma 4.7 are satisfied by $\mu_x$, $\delta$, $\mathscr{U}$ and $m$.

As $\{u_i\}_{i\in\mathbb{N}}$ is $\delta$-dense in $X$ and by (4.6) one has $\lambda(\mu_x) \in \mathbb{R}$, Lemma 4.7 ensures for every $x \in A(\Phi,\lambda,\mathscr{Y})$ the existence of some corresponding $m, i \in \mathbb{N}$. Hence we obtain

$$A(\Phi,\lambda,\mathscr{Y}) = \bigcup_{m\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} Z_{m,i}. \tag{4.7}$$

For simplicity we may assume that all $Z_{m,i}$ are nonempty, else they can be cancelled out of the union.

Now fix $Z_{m,i} \neq \emptyset$ and denote for each $q \geq 1$

$$R_q := \left\{ \mathbf{V} \in (\mathscr{U}^m)^q : H(\mathbf{V}) \leq m(p_{\mathscr{Y}}(\lambda) - u_i + 2\delta) \right\}. \tag{4.8}$$

Pick some $x \in Z_{m,i}$, then by Lemma 4.7 one finds arbitrarily large $N \geq 1$ and corresponding $q \geq \frac{N}{m}$, $\mathbf{U} \in \mathscr{U}^N$, $\mathbf{V} \in (\mathscr{U}^m)^q$ satisfying

$$0 \leq \frac{1}{m} H(\mathbf{V}) \leq h_{\mu_x}(T) + \delta \leq h_{\mu_x}(T) + \lambda(\mu_x) - u_i + 2\delta \leq p_{\mathscr{Y}}(\lambda) - u_i + 2\delta.$$

This means $\mathbf{V} \in R_q$ and especially

$$0 \le m\big(p_{\mathscr{Y}}(\lambda) - u_i + 2\delta\big). \tag{4.9}$$

Applying Lemma 4.6 to (4.8) and (4.9), there exists a $q_0 \in \mathbb{N}$ such that

$$\frac{1}{q}\log \operatorname{card} R_q \le m\big(p_{\mathscr{Y}}(\lambda) - u_i + 3\delta\big)$$

for all $q \ge q_0$. Fix $N \ge N_0 := q_0 m$, count all vectors $\mathbf{U}$ which can appear in the above situation for any $x \in Z_{m,i}$, and denote that number by $b_N$. Hence, as $q \ge q_0$,

$$b_N \le (\operatorname{card}\mathscr{U})^m \operatorname{card} R_q \le (\operatorname{card}\mathscr{U})^m \exp\big(qm(p_{\mathscr{Y}}(\lambda) - u_i + 3\delta)\big).$$

This means, as $N = qm + r$ for some corresponding $0 \le r \le m$,

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N}\log b_N &\le \limsup_{N\to\infty}\frac{1}{N}\big(m\log\operatorname{card}\mathscr{U} + qm(p_{\mathscr{Y}}(\lambda) - u_i + 3\delta)\big)\\ &\le p_{\mathscr{Y}}(\lambda) - u_i + 3\delta.\end{aligned}$$

As a result there exists some $N_1 \ge N_0$ such that

$$b_N \le \exp\big(N(p_{\mathscr{Y}}(\lambda) - u_i + 4\delta)\big) \tag{4.10}$$

for all $N \ge N_1$.

Continuing with the proof, for each $l \ge N_1$ define the collection $\Gamma_l$ containing all $\mathbf{U} \in \bigcup_{N\ge l}\mathscr{U}^N$ which satisfy the properties of Lemma 4.7 for any $x \in Z_{m,i}$. Therefore $Z_{m,i} \subseteq \bigcup_{\mathbf{U}\in\Gamma_l} X(\mathbf{U})$ and $X(\mathbf{U}) \cap Z_{m,i} \ne \varnothing$ for each $\mathbf{U} \in \Gamma_l$. Hence one can choose some epicenter $x_{\mathbf{U}} \in X(\mathbf{U}) \cap Z_{m,i}$. As $\operatorname{diam} U < \delta$ for each $U \in \mathscr{U}$, we have $X(\mathbf{U}) \subseteq B_{m(\mathbf{U})}(x_{\mathbf{U}}, \delta)$. This means $\{(m(\mathbf{U}), X(\mathbf{U}))\}_{\mathbf{U}\in\Gamma_l}$ is a $(\delta, l)$-cover of $Z_{m,i}$. Note that by Lemma 4.7 for each $\mathbf{U} \in \Gamma_l$ one has

$$\Phi_{m(\mathbf{U})}(Z_{m,i} \cap X(\mathbf{U})) = \sup_{x\in Z_{m,i}\cap X(\mathbf{U})} \varphi_{m(\mathbf{U})}(x) \le m(\mathbf{U})(u_i + 2\delta). \tag{4.11}$$

Hence we can estimate for $\alpha \in \mathbb{R}$ and $l \ge N_1$

$$\begin{aligned}\Lambda_{Z_{m,i}}(\Phi,\delta,\alpha,l) &\le \sum_{\mathbf{U}\in\Gamma_l} \exp\big(-\alpha m(\mathbf{U}) + m(\mathbf{U})(u_i + 2\delta)\big)\\ &\le \sum_{N=l}^{\infty} b_N \exp\big(-\alpha N + N(u_i + 2\delta)\big)\\ &\le \sum_{N=l}^{\infty}\big(\exp(-\alpha + p_{\mathscr{Y}}(\lambda) + 6\delta)\big)^N.\end{aligned}$$

Here in the last step we used the estimate (4.10). Now for every $\alpha > p_{\mathscr{Y}}(\lambda) + 6\delta$ we obtain

$$\beta := \exp\big(-\alpha + p_{\mathscr{Y}}(\lambda) + 6\delta\big) < 1,$$

and hence

$$\Lambda_{Z_{m,i}}(\Phi,\delta,\alpha) \le \limsup_{l\to\infty}\sum_{N=l}^{\infty}\beta^N = 0.$$

This means $P_{Z_{m,i}}(\Phi,\delta) \le p_{\mathscr{Y}}(\lambda) + 6\delta$ for fixed $Z_{m,i}$. To finish the proof, we take the supremum over all $m, i$ and apply (4.7) together with Theorem 3.10:

$$P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi,\delta) = \sup_{m,i} P_{Z_{m,i}}(\Phi,\delta) \le p_{\mathscr{Y}}(\lambda) + 6\delta.$$

Finally, letting $\delta \to 0$ results in $P_{A(\Phi,\lambda,\mathcal{Y})}(\Phi) \le p_{\mathcal{Y}}(\lambda)$.

For the second statement, fix $\mu \in \mathcal{Y}$. As $V(\Phi,\lambda,\mathcal{Y}) \subseteq A(\Phi,\lambda,\mu)$, one has by Lemma 3.8 and (4.4)

$$P_{V(\Phi,\lambda,\mathcal{Y})}(\Phi) \le P_{A(\Phi,\lambda,\mu)}(\Phi) \le h_\mu(T) + \lambda(\mu).$$

Taking the infimum over all $\mu \in \mathcal{Y}$ yields the result.

□

### §4.2. The mass distribution principle

The mass distribution principle basically tells us that if one has a Borel set $B$ such that $\mu(B) > 0$ for some $\mu \in \mathcal{E}_T(X)$, then the pressure on $B$ can be estimated from below by lower bounds of $\liminf_{n\to\infty} \frac{1}{n}\varphi_n$ on $B$.

**Theorem 4.8** (Ergodic mass distribution principle)**.** *Fix $\mu \in \mathcal{E}_T(X)$ and let $\Phi$ be a Borel measurable potential on $(X,T)$. Suppose there exist a constant $b \in [-\infty,\infty]$ and a Borel set $B \subseteq X$ satisfying $\mu(B) > 0$, such that*

$$\liminf_{n\to\infty} \frac{1}{n}\varphi_n(x) \ge b \tag{4.12}$$

*for each $x \in B$. Then, if $h_\mu(T) + b$ is well-defined, one has*

$$P_B(\Phi) \ge h_\mu(T) + b.$$

**Remark 4.9.** The term "mass distribution principle" is borrowed from fractal geometry. Here a mass distribution usually refers to a certain probability measure supported on a fractal set $F$, which can be used to get a lower bound of the Hausdorff dimension of $F$ (see for instance [Fal14]). If one considers pressure to be some type of dimension (which was rigorously treated in [Pes97]), the ergodic measures $\mu$ with property (4.12) can be seen as mass distributions of the topological pressure for the whole space $X$. Actually, as we will see in Theorem 6.7, those measures can give rise to mass distributions in the sense of fractal geometry. See also Remark 6.8.

*Proof of Theorem 4.8.* We use a modified version of the proof of Proposition 4.2 in [FH16]. Before we start the proof, recall the Brin-Katok theorem for pointwise entropy:

**Lemma 4.10** ([BK83])**.** *Fix $\mu \in \mathcal{E}_T(X)$ and define $h_\mu(x,\epsilon,n) := -\frac{1}{n}\log\mu(B_n(x,\epsilon))$. Then one has*

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty} h_\mu(x,\epsilon,n) = \lim_{\epsilon\to 0}\limsup_{n\to\infty} h_\mu(x,\epsilon,n) = h_\mu(T) \tag{4.13}$$

*for $\mu$-almost all $x \in X$.*

Let $G \subseteq B$ such that (4.13) holds for each $x \in G$. Note that $\mu(G) > 0$. Assume first that $h_\mu(T) + b$ is finite. Let $\epsilon > \epsilon' > 0$ and $\delta > 0$. Define the Borel sets

$$G^{\delta,\epsilon} := \left\{ x \in G : \liminf_{n\to\infty} h_\mu(x,\epsilon,n) > h_\mu(T) - \delta \right\}.$$

Then $G^{\delta,\epsilon} \subseteq G^{\delta,\epsilon'}$ and $G = \bigcup_{\epsilon>0} G^{\delta,\epsilon}$, hence

$$0 < \mu(G) = \lim_{m\to\infty} \mu\big(G^{\delta,\frac{1}{m}}\big).$$

This shows that there is an $\epsilon_\delta > 0$ such that $0 < \mu(G^{\delta,\epsilon_\delta}) \le \mu(G^{\delta,\epsilon})$ for all $0 < \epsilon \le \epsilon_\delta$. For each $x \in G^{\delta,\epsilon_\delta}$ there exists a minimal $N(\delta,x) \in \mathbb{N}$ such that

$$\exp\big(-n(h_\mu(T)-\delta)\big) \ge \mu(B_n(x,\epsilon_\delta)), \tag{4.14}$$

$$\frac{1}{n}\varphi_n(x) \ge b-\delta, \tag{4.15}$$

for all $n \ge N(\delta,x)$. Define for each $N \ge 1$ the Borel sets

$$G^{\delta,\epsilon_\delta,N} := \Big\{x \in G^{\delta,\epsilon_\delta} : x \text{ satisfies (4.14) and (4.15) for all } n \ge N\Big\}.$$

One can see like above that there exists an $M(\delta) \in \mathbb{N}$ such that $0 < \mu\big(G^{\delta,\epsilon_\delta,M(\delta)}\big) \le \mu\big(G^{\delta,\epsilon_\delta,N}\big)$ for all $N \ge M(\delta)$. Now define

$$A_\delta := G^{\delta,\epsilon_\delta,M(\delta)}.$$

If $\Gamma = \{(n_l,B_l)\}_{l\in L}$ is an $(\epsilon,N)$-cover of $A_\delta$, then $\Gamma^* := \{(n_l,B_l\cap A_\delta)\}_{l\in L'}$ is also an $(\epsilon,N)$-cover of $A_\delta$, where $L' := \big\{l \in L : B_l \cap A_\delta \neq \varnothing\big\}$. By the definition of $\Gamma$ there is for each $l \in L'$ an $x_l \in X$ such that $B_l \cap A_\delta \subseteq B_l \subseteq B_{n_l}(x_l,\epsilon)$. Fix $0 < \epsilon < \frac{1}{2}\epsilon_\delta$ and $N \ge M(\delta)$. Fix $y_l \in B_l \cap A_\delta$ and let $x \in B_l \cap A_\delta$. Then $d(T^j y_l, T^j x) \le d(T^j y_l, T^j x_l) + d(T^j x_l, T^j x) \le 2\epsilon < \epsilon_\delta$ for all $0 \le j < n_l$. Thus $B_l \cap A_\delta \subseteq B_{n_l}(y_l,\epsilon_\delta)$ for all $l \in L'$. Hence, as $y_l \in A_\delta$ for all $l \in L'$,

$$\exp\big(-n_l(h_\mu(T)-\delta)\big) \ge \mu\big(B_{n_l}(y_l,\epsilon_\delta)\big) \ge \mu(B_l \cap A_\delta). \tag{4.16}$$

In addition, one has by (4.15)

$$\Phi_n(B_l \cap A_\delta) = \sup_{x\in B_l\cap A_\delta} \varphi_{n_l}(x) \ge n_l(b-\delta). \tag{4.17}$$

Hence, setting $\alpha_\delta := h_\mu(T) + b - 2\delta$, one has using (4.16) and (4.17)

$$\exp\big(-n_l\alpha_\delta + \Phi_n(B_l\cap A_\delta)\big) \ge \mu(B_l \cap A_\delta)$$

for all $l \in L'$. Thus, for each $(\epsilon,N)$-cover $\Gamma = \{(n_l,B_l)\}_{l\in L} \in \mathscr{C}_{A_\delta}(\epsilon,N)$, where $0 < \epsilon < \frac{1}{2}\epsilon_\delta$ and $N \ge M(\delta)$, there is the estimate

$$\begin{aligned}\sum_{(n,B)\in\Gamma} \exp\big(-\alpha n + \Phi_n(B\cap A_\delta)\big) &\ge \sum_{(n,B)\in\Gamma^*} \exp\big(-\alpha n + \Phi_n(B\cap A_\delta)\big)\\ &\ge \sum_{l\in L'} \mu(B_l\cap A_\delta) \ge \mu\left(\bigcup_{l\in L'} B_l \cap A_\delta\right) = \mu(A_\delta).\end{aligned}$$

This shows $\Lambda_{A_\delta}(\Phi,\epsilon,\alpha_\delta) \ge \mu(A_\delta) > 0$, and hence by Lemma 3.5 and Lemma 3.8

$$P_B(\Phi,\epsilon) \ge P_{A_\delta}(\Phi,\epsilon) \ge \alpha_\delta = h_\mu(T) + b - 2\delta$$

for all $0 < \epsilon < \frac{1}{2}\epsilon_\delta$. Now letting $\epsilon \to 0$ and $\delta \to 0$ shows that $P_B(\Phi) \ge h_\mu(T) + b$, if $h_\mu(T) + b$ is finite.

If $b = \infty$, replace (4.15) by $\frac{1}{n}\varphi_n(x) \ge \frac{1}{\delta}$ and if $h_\mu(T) = \infty$, replace (4.14) by $\exp\big(\frac{-n}{\delta}\big) \ge \mu(B_n(x,\epsilon))$ and set $G^{\delta,\epsilon} := \big\{x \in G : \liminf_{n\to\infty} h_\mu(x,\epsilon,n) > \frac{1}{\delta}\big\}$. Then the proof works in the same way.

□

**Remark 4.11.** Related statements to Theorem 4.8 with a similar method of proof were obtained in [CHZ13] (sub/super-additive continuous potentials), [Rau15] (multivariate continuous potentials), [FH16] (weighted pressure of sub-additive potentials) and [Rau17] (quasi-integrable additive potentials). They all have in common that the Brin-Katok theorem and the convergence of $\frac{1}{n}\varphi_n$ is used to establish a lower estimate for pressure. However, by refining the method, we are able to derive a stronger result, which also includes sets with positive $\mu$-measure.

The next statement shows that a global lower bound for the all pressures $P_Z(\Phi)$ of sets $Z$, which satisfy $\mu(Z) > 0$, exists.

**Corollary 4.12** (Ergodic inverse variational inequality)**.** *Let $\mu \in \mathscr{E}_T(X)$ and $\Phi$ be a Borel measurable potential on $(X,T)$. Define $\varphi := \liminf_{n\to\infty} \frac{1}{n}\varphi_n$ and*

$$\operatorname{ess\,inf}_\mu \varphi := \sup\Big\{ b \in \mathbb{R} : \mu\big(\big\{ x \in X : \varphi(x) < b \big\}\big) = 0 \Big\}. \tag{4.18}$$

*Then, if $h_\mu(T) + \operatorname{ess\,inf}_\mu \varphi$ is well-defined, one has*

$$h_\mu(T) + \operatorname{ess\,inf}_\mu \varphi \le \inf\{P_Z(\Phi) : \mu(Z) > 0\}.$$

*Proof.* We may assume that $-\infty < \operatorname{ess\,inf}_\mu \varphi$. Pick a number $-\infty < b < \operatorname{ess\,inf}_\mu \varphi$. If $Z \subseteq X$ is a Borel set such that $\mu(Z) > 0$, one has $\mu(\{x \in Z : \varphi(x) < b\}) = 0$. This means there exists a Borel set $Z' \subseteq Z$ satisfying $\mu(Z') = \mu(Z)$ and

$$\liminf_{n\to\infty} \frac{1}{n}\varphi_n(x) \ge b$$

for each $x \in Z'$. Hence it follows from Theorem 4.8 and Lemma 3.8 that $h_\mu(T) + b \le P_{Z'}(\Phi) \le P_Z(\Phi)$. As $Z$ can be chosen arbitrarily, taking the infimum on the right yields

$$h_\mu(T) + b \le \inf\{P_Z(\Phi) : \mu(Z) > 0\}.$$

Now letting $b \to \operatorname{ess\,inf}_\mu \varphi$ proves the statement.

□

**Remark 4.13.** If there is a constant $C$ such that $\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \varphi(x) = C$ for $\mu$-almost all $x \in X$, then $\operatorname{ess\,inf}_\mu \varphi = C$. Consider for instance an $\psi \in L^1(X,\mu)$. Then by Birkhoff's ergodic theorem, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ T^i = \int_X \psi \, d\mu = \operatorname{ess\,inf}_\mu \psi$$

$\mu$-almost everywhere. Thus it follows from Corollary 4.12 that

$$h_\mu(T) + \int_X \psi \, d\mu \le \inf\{P_Z(\Phi) : \mu(Z) > 0\}.$$

One the other hand, Corollary 4.12 applies also to situations, where the limit $\lim_{n\to\infty} \frac{1}{n}\varphi_n$ exists, but might be not constant almost everywhere. For instance, if $\psi \in L^2(X,\mu)$ and $p$ is a polynomial with integer coefficients, it was shown in [Bou88] that

$$A_n\psi(x) := \frac{1}{n} \sum_{i=1}^{n} \psi\big(T^{p(i)}x\big)$$

converges $\mu$-almost everywhere. Hence Corollary 4.12 applies to $\Psi := (nA_n\psi)_{n\ge 1}$, that is, one has

$$h_\mu(T) + \operatorname{ess\,inf}_\mu \Big(\lim_{n\to\infty} A_n\psi\Big) \le \inf\{P_Z(\Psi) : \mu(Z) > 0\}.$$

In the formulation of Theorem 4.8 and Corollary 4.12 we assume $\mu$ to be ergodic. This assumption can be dropped, if $\lambda$ is an ergodic decomposable Lyapunov exponent.

**Definition 4.14.** Let $\mu \in \mathcal{M}_T(X)$ and $\mu = \int_{\mathcal{E}_T(X)} \nu \, \mathrm{dm}_\mu(\nu)$ its ergodic decomposition. We call a Lyapunov exponent $\lambda$ ergodic decomposable with respect to $\mu$, if

$$\lambda(\mu) = \int_{\mathcal{E}_T(X)} \lambda(\nu) \, \mathrm{dm}_\mu(\nu).$$

The above definition requires that $\lambda|_{\mathcal{E}_T(X)} : \mathcal{E}_T(X) \to [-\infty, \infty]$ is Borel measurable, and quasi-integrable with respect to $\mathrm{m}_\mu$.

**Remark 4.15.** We will later see practical examples, where $\lambda$ actually is ergodic decomposable (Proposition 5.11, Proposition 5.25).

**Theorem 4.16** (Non-ergodic mass distribution principle)**.** *Fix $\mu \in \mathcal{M}_T(X)$, let $\Phi$ be a Borel measurable potential on $(X, T)$ and $\lambda$ be a Lyapunov exponent which is ergodic decomposable with respect to $\mu$. Denote by $\mathrm{m}_\mu$ the ergodic decomposition of $\mu$ and suppose that $\mathcal{E}_T(X) \cup \{\mu\} \subseteq \mathcal{A}(\lambda)$. Then, if $B \subseteq X$ is a subset, such that for $\mathrm{m}_\mu$-almost every $\nu \in \mathcal{E}_T(X)$ one has $P_B(\Phi) \geq h_\nu(T) + \lambda(\nu)$, it follows that*

$$P_B(\Phi) \geq h_\mu(T) + \lambda(\mu).$$

*Proof.* This is immediate, as

$$P_B(\Phi) \geq \int_{\mathcal{E}_T(X)} \big(h_\nu(T) + \lambda(\nu)\big) \, \mathrm{dm}_\mu(\nu) = h_\mu(T) + \lambda(\mu).$$

□

**Theorem 4.17** (Non-ergodic inverse variational inequality)**.** *Fix $\mu \in \mathcal{M}_T(X)$, let $\Phi$ be a Borel measurable potential on $(X, T)$ and $\lambda$ be a Lyapunov exponent which is ergodic decomposable with respect to $\mu$. Suppose that for each $\nu \in \mathcal{E}_T(X)$*

$$\liminf_{n\to\infty} \frac{1}{n} \varphi_n(x) \geq \lambda(\nu)$$

*for $\nu$-almost all $x \in X$. Suppose furthermore that $\mathcal{E}_T(X) \cup \{\mu\} \subseteq \mathcal{A}(\lambda)$. Then one has*

$$h_\mu(T) + \lambda(\mu) \leq \inf\big\{P_Z(\Phi) : \mu(Z) = 1\big\}.$$

*Proof.* Assume $Z \subseteq X$ to be a Borel set such that $\mu(Z) = 1$. As $1 = \mu(Z) = \int_{\mathcal{E}_T(X)} \nu(Z) \, \mathrm{dm}_\mu(\nu)$, we have $\nu(Z) = 1$ for $\mathrm{m}_\mu$-almost all $\nu \in \mathcal{E}_T(X)$. By Theorem 4.8 it follows that $h_\nu(T) + \lambda(\nu) \leq P_Z(\Phi)$ for $\mathrm{m}_\mu$-almost every $\nu \in \mathcal{E}_T(X)$. Thus

$$h_\mu(T) + \lambda(\mu) = \int_{\mathcal{E}_T(X)} \big(h_\nu(T) + \lambda(\nu)\big) \, \mathrm{dm}_\mu(\nu) \leq P_Z(\Phi).$$

□

**Remark 4.18.** A special case of the above statements were first proven in [PP84]. They furthermore generalize Lemma 2 in [Bar96] to Borel measurable, possibly non-additive potentials.

## 5. Variational principles

In this section we shall use Theorem 4.4 and Theorem 4.8 to state and prove variational principles for various types of potentials $\Phi$. In the following, let $(X,T)$ be a dynamical system with Borel measurable potential $\Phi$. If $\lambda$ is a Lyapunov exponent, recall for each $\mathscr{Y} \subseteq \mathscr{M}_T(X)$ the notions $\mathscr{A}(\lambda)$, $A(\Phi,\lambda,\mathscr{Y})$ and $p_{\mathscr{Y}}(\lambda)$ (see Definition 4.1). Define in addition for each $Z \subseteq X$ the subset

$$\mathscr{L}(Z) := \left\{ x \in Z : V_T(x) \cap \mathscr{M}_T(Z) \neq \varnothing \right\}.$$

Note that if $Z$ is $T$-invariant and compact, one has $Z = \mathscr{L}(Z)$. Also recall $\delta_{x,n} = \frac{1}{n}\sum_{i<n} \delta_{T^i x}$ to be the empirical measures of $x \in X$ (see (2.1)).

### §5.1. The general case

In view of Theorem 4.4, it is natural to ask the following question:

Given some subset $\mathscr{Y} \subseteq \mathscr{M}_T(X)$, is there a subset $G \subseteq X$ such that

$$P_G(\Phi) = \sup\left\{ h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y} \right\} \ ? \tag{5.1}$$

The above question has a partial answer, if one restricts to ergodic measures, for which $\frac{1}{n}\varphi_n$ converges almost surely to a constant.

**Definition 5.1.** Let $\lambda$ be a Lyapunov exponent. Define

$$\mathscr{G}(\Phi,\lambda) := \left\{ \mu \in \mathscr{A}(\lambda) \cap \mathscr{E}_T(X) : \lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \lambda(\mu) \text{ for } \mu-\text{almost all } x \in X \right\}. \tag{5.2}$$

The set $\mathscr{G}(\Phi,\lambda)$ is called set of ergodic measures, which satisfy a variational principle for $\Phi$ and $\lambda$.

Indeed, one has the following:

**Theorem 5.2** (Variational principle for $\mathscr{G}(\Phi,\lambda)$)**.** *For each subset $\mathscr{Y} \subseteq \mathscr{G}(\Phi,\lambda)$ one has*

$$P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) = \sup\left\{ h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y} \right\}.$$

*In particular, one can choose for each $\mu \in \mathscr{Y}$ a Borel set $B_\mu \subseteq A(\Phi,\lambda,\mathscr{Y})$ such that $\mu(B_\mu) = 1$, and*

$$P_B(\Phi) = \sup\left\{ h_\mu(T) + \lambda(\mu) : \mu \in \mathscr{Y} \right\},$$

*where $B = \bigcup_{\mu\in\mathscr{Y}} B_\mu$.*

*Proof.* By Theorem 4.4 one has $P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) \leq p_{\mathscr{Y}}(\lambda)$. For each $\mu \in \mathscr{Y}$ there is a Borel set $B_\mu \subseteq G_\mu$ such that $\mu(B_\mu) = 1$ and $\lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \lambda(\mu)$ for all $x \in B_\mu$. Hence $B_\mu \subseteq A(\Phi,\lambda,\mu) \subseteq A(\Phi,\lambda,\mathscr{Y})$, and this shows by Theorem 4.8 and Lemma 3.8 that

$$p_{\{\mu\}}(\lambda) \leq P_{B_\mu}(\Phi) \leq P_B(\Phi) \leq P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi).$$

Taking the supremum on the left side yields the result.

□

**Remark 5.3.** The above theorem is the most general formulation of a variational principle in this work. It basically states that if $\Phi$ and $\lambda$ are connected via a pointwise ergodic theorem, then a variational principle holds on a certain set. This connection is precisely described by (5.2) and (4.2) . Note that if $h_{\mathrm{top}}(T) < \infty$, one always has $\mathscr{A}(\lambda) = \mathscr{M}_T(X)$, and in many cases one can show that $\mathscr{G}(\Phi, \lambda) = \mathscr{E}_T(X)$.

An immediate consequence is the so-called inverse variational principle:

**Corollary 5.4** (Inverse variational principle)**.** *Let $\mu \in \mathscr{G}(\Phi, \lambda)$. Then one has*

$$h_\mu(T) + \lambda(\mu) = \inf\left\{P_Z(\Phi) : \mu(Z) = 1\right\} = \inf\left\{P_Z(\Phi) : \mu(Z) > 0\right\}.$$

*Moreover, one has $h_\mu(T) + \lambda(\mu) = P_B(\Phi)$ for each Borel set $B \subseteq A(\Phi, \lambda, \mu)$ such that $\mu(B) > 0$.*

*Proof.* By Theorem 5.2 and Corollary 4.12 one has

$$P_{A(\Phi,\lambda,\{\mu\})}(\Phi) = h_\mu(T) + \lambda(\mu) \le \inf\left\{P_Z(\Phi) : \mu(Z) > 0\right\} \le \inf\left\{P_Z(\Phi) : \mu(Z) = 1\right\}$$

Now pick a Borel set $B \subseteq A(\Phi, \lambda, \mu)$ such that $\mu(B) = 1$; note that there exists at least one $B$ with that property. Then $h_\mu(T) + \lambda(\mu) \le P_B(\Phi) \le P_{A(\Phi,\lambda,\mu)}(\Phi)$, which shows both statements.

□

**Remark 5.5.** The term "inverse variational principle" was first used in [Pes97] for the special case of additive, continuous potentials. Here it was proven that for $\varphi : X \to \mathbb{R}$ continuous one has

$$h_\mu(T) + \int_X \varphi \, \mathrm{d}\mu = \inf\left\{P_Z(\varphi) : \mu(Z) = 1\right\}.$$

The above equation was later generalized to continuous sub- and super-additive potentials, as well as multivariate potentials (see also Remark 4.11). However, we want to emphasize that in our statement the infimum can also be taken over Borel sets $Z$ such that $\mu(Z) > 0$.

### §5.2. The additive case

In this subsection we want to study variational principles for additive potentials. Continuous additive potentials were first studied in [Rue73] and [Wal75]. First recall that given a measure $\mu$ on $X$, a Borel measurable function $\varphi : X \to [-\infty, \infty]$ is said to be quasi-integrable with respect to $\mu$, if either $\int_X \varphi^+ \, \mathrm{d}\mu < \infty$ or $\int_X \varphi^- \, \mathrm{d}\mu < \infty$. In this case, one defines

$$\int_X \varphi \, \mathrm{d}\mu := \int_X \varphi^+ \, \mathrm{d}\mu - \int_X \varphi^- \, \mathrm{d}\mu. \tag{5.3}$$

**Definition 5.6.** Let $\Phi = (\varphi_n)_{n\ge1}$ be a not necessarily measurable potential on $(X, T)$. We call $\Phi$ to be additive, if $\varphi_{n+m} = \varphi_n + \varphi_m \circ T^n$ for all $m, n \ge 1$. In this case, by induction one sees that $\varphi_n = \sum_{i=0}^{n-1} \varphi_1 \circ T^i$ for each $n \ge 1$. For simplicity we denote $P_Z(\varphi_1) := P_Z(\Phi)$ (and we use this also for the other notions of pressure considered in §3.3). Now suppose $\varphi_1$ is Borel measurable. Then $\Phi$ is also Borel measurable. In case $\varphi_1$ is quasi-integrable with respect to $\mu \in \mathcal{M}_T(X)$, and if not stated otherwise, we slightly abuse the notation and define

$$\lambda(\mu) := \int_X \varphi_1 \,\mathrm{d}\mu.$$

Also, for simplicity, denote

$$\mathcal{G}(\varphi_1) := \mathcal{G}(\Phi,\lambda), \quad \mathcal{A}(\varphi) := \mathcal{A}(\lambda), \quad A(\varphi,\mathcal{Y}) := A(\Phi,\lambda,\mathcal{Y}), \tag{5.4}$$

where $\mathcal{G}(\Phi,\lambda)$ is defined in (5.2), and $\mathcal{A}(\lambda)$ as well as $A(\Phi,\lambda,\mathcal{Y})$ are defined in Definition 4.1.

Now let $I \in \{(-\infty,\infty],[-\infty,\infty)\}$. If $\varphi : X \to I$, then $\Phi := \left(\sum_{i<n} \varphi \circ T^n\right)_{n\geq 1}$ defines an additive potential. On the other hand, if there is an $x \in X$ and $n > i, j \geq 0$ such that $\varphi(T^i x) = \infty$ and $\varphi(T^j x) = -\infty$, the value $\sum_{k=0}^{n-1} \varphi(T^k x)$ is not well-defined. Therefore, throughout this work, we will always assume that

$$\sum_{i=0}^{n-1} \varphi(T^i x) \text{ is well-defined for all } x \in X, n \geq 1. \tag{5.5}$$

**Remark 5.7.** Suppose $\varnothing \neq Z \subseteq X$ to be $T$-invariant. If $\varphi|_Z = C$ for some $C \in [-\infty,\infty]$, then it easy to see that $P_Z(\varphi) = h_{\mathrm{top}}(Z) + C$ in the case of $h_{\mathrm{top}}(Z) < \infty$. In the case $C = -\infty$ and $h_{\mathrm{top}}(Z) = \infty$, one has $P_Z(\varphi) = -\infty$. Moreover, if $C = \infty$, one has by Lemma 3.8 that $P_X(\varphi) = \infty$. An illustration for such a system is given in Figure 1, where $P_{\{0\}}(-s\log|T'|) = -\infty$ and $P_{\{1\}}(-s\log|T'|) = P_X(-s\log|T'|) = \infty$ for each $s > 0$. Observe furthermore that for each convex combination $\mu = \alpha\delta_0 + (1-\alpha)\delta_1$ the function $\log|T'|$ is not quasi-integrable with respect to $\mu$.

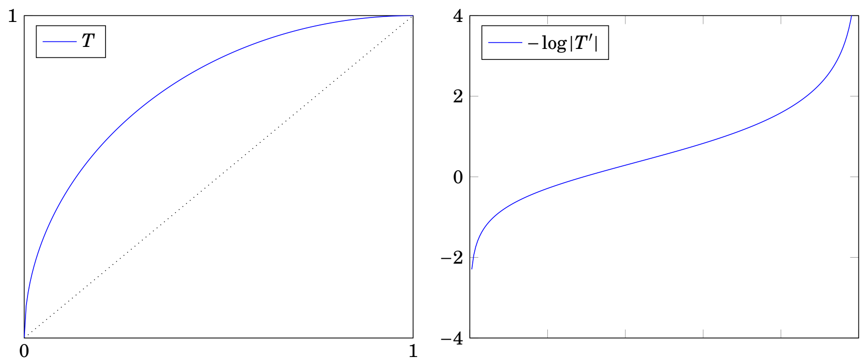


Figure 1: A system with infinite pressure function $s \mapsto P_Z(-s\log|T'|)$.

To motivate our framework further, we first want to restate and reprove the classical variational principle for continuous $\varphi$.

**Theorem 5.8** (Variational principle for continuous $\varphi$)**.** *If $\varphi : X \to \mathbb{R}$ is continuous, then*

$$P_{\mathcal{L}(Z)}(\varphi) = \sup\left\{ h_\mu(T) + \int_X \varphi \,\mathrm{d}\mu : \mu \in \mathcal{M}_T(Z) \right\}$$

*for each $T$-invariant Borel set $Z \subseteq X$.*

*Proof.* We assume $\mathscr{L}(Z) \neq \varnothing$, else nothing is to be shown. First note that $\varphi$ is bounded on $X$. Thus $\int_X \varphi \, d\mu$ is well-defined and finite on $\mathscr{M}_T(X)$. Hence $\mathscr{A}(\varphi) = \mathscr{M}_T(X)$, and using Birkhoff's ergodic theorem, $\mathscr{G}(\varphi) = \mathscr{E}_T(X)$. On the other hand, by Proposition 4.2 one has

$$\mathscr{L}(Z) \subseteq \{x \in X : V_T(x) \cap \mathscr{M}_T(Z) \neq \varnothing\} = A(\varphi, \mathscr{M}_T(Z)).$$

Now applying Lemma 3.8 and Theorem 4.4, we obtain

$$P_{\mathscr{L}(Z)}(\varphi) \leq P_{A(\varphi,\mathscr{M}_T(Z))}(\varphi) \leq p_{\mathscr{M}_T(Z)}(\varphi).$$

Fix $\mu \in \mathscr{M}_T(Z)$ and denote by $\mathrm{m}_\mu$ its ergodic decomposition. As $Z$ is $T$-invariant and $\mu(Z) = 1$, we have that $\nu(Z \cap G_\nu) = 1$ for $\mathrm{m}_\mu$-almost every $\nu \in \mathscr{E}_T(X)$. Hence by Theorem 4.8 and Lemma 3.8

$$h_\nu(T) + \int_X \varphi \, d\nu \leq P_{Z \cap G_\nu}(\varphi) \leq P_{\mathscr{L}(Z)}(\varphi)$$

for $\mathrm{m}_\mu$-almost every $\nu \in \mathscr{E}_T(X)$. By Theorem 4.16 this yields

$$h_\mu(T) + \int_X \varphi \, d\mu \leq P_{\mathscr{L}(Z)}(\varphi),$$

and as $\mu$ was chosen arbitrarily, taking the supremum on the left side finishes the proof. □

**Remark 5.9.**

(a) The above theorem was first proven in [Wal75] for compact $T$-invariant $Z$. More precisely it was proven that $\overline{P}_Z(\varphi) = p_{\mathscr{M}_T(Z)}(\varphi)$. Thus by Proposition 3.21 the estimate $P_Z(\varphi) \leq p_{\mathscr{M}_T(Z)}(\varphi)$ immediately follows, without involving Theorem 5.8.

(b) In [PP84] it was proven that $P^B_{\mathscr{L}(Z)}(\varphi) = p_{\mathscr{M}_T(Z)}(\varphi)$ for arbitrary $T$-invariant Borel sets $Z$. One then can use Proposition 3.13 to see that $P_{\mathscr{L}(Z)}(\varphi) = p_{\mathscr{M}_T(Z)}(\varphi)$, without involving Theorem 5.8.

(c) Observe that Theorem 5.8 does not hold in general, if one drops the $T$-invariance assumption: Suppose $\mu \in \mathscr{E}_T(X)$ and $Z \subseteq G_\mu$ such that $0 < \mu(Z) < 1$. Then $\mathscr{E}_T(Z) = \{\mu\}$ and

$$-\infty = P_{\mathscr{L}(Z)}(\varphi) = p_{\mathscr{M}_T(Z)}(\varphi) < p_{\{\mu\}}(\varphi) = P_{A(\varphi,\mu)}(\varphi) = h_\mu(T) + \int_X \varphi \, d\mu.$$

Note that in this case, we also have $Z \subseteq A(\varphi, \mu)$, thus $P_Z(\varphi) = h_\mu(T) + \int_X \varphi \, d\mu$. This generalizes Theorem 3 in [PP84] to sets with positive $\mu$-measure, and is a special case of Theorem 5.2.

The proof of Theorem 5.8 uses Birkhoff's ergodic theorem, the continuity of $\mu \mapsto \int_X \varphi \, d\mu$, and the ergodic decomposition of $\int_X \varphi \, d\mu$. These tools can also be applied to functions $\varphi$, which are not continuous. To do so, we first state the ergodic theorem and ergodic decomposition theorem for quasi-integrable functions.

**Proposition 5.10** (Birkhoff's ergodic theorem for quasi-integrable functions)**.** *Let $\mu \in \mathscr{E}_T(X)$ and $\varphi : X \to [-\infty,\infty]$ be quasi-integrable with respect to $\mu$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \int_X \varphi \, \mathrm{d}\mu$$

*for $\mu$-almost every $x \in X$.*

*Sketch of the proof.* As $\varphi$ is quasi-integrable, either $\varphi(x) < \infty$ or $-\infty < \varphi(x)$ for $\mu$-almost all $x \in X$. The proof is then straight forward using Kingman's sub-additive ergodic theorem (see §A.2).

□

**Proposition 5.11** (Ergodic decomposition for quasi-integrable functions)**.** *Fix some $\mu \in \mathscr{M}_T(X)$ and denote by $\mathrm{m}_\mu$ the ergodic decomposition $\mu = \int_{\mathscr{E}_T(X)} \nu \, \mathrm{d}\mathrm{m}_\mu(\nu)$ of $\mu$. If $\varphi : X \to [-\infty,\infty]$ is quasi-integrable with respect to $\mu$, then one has*

$$\int_X \varphi \, \mathrm{d}\mu = \int_{\mathscr{E}_T(X)} \left( \int_X \varphi \, \mathrm{d}\nu \right) \mathrm{d}\mathrm{m}_\mu(\nu).$$

*In particular, $\mathscr{E}_T(X) \ni \nu \mapsto \int_X \varphi \, \mathrm{d}\nu$ is quasi-integrable with respect to $\mathrm{m}_\mu$.*

*Sketch of the proof.* For Borel measurable bounded $\varphi$, the above statement follows from Theorem 2.2. If $\varphi$ is bounded from below or above, the statement follows by applying the monotone convergence theorem. Thus, as $\varphi^+$ and $-\varphi^-$ are bounded from below and above, the statement follows also for quasi-integrable, possibly unbounded $\varphi$ (see §A.2 for more details).

□

Now we can generalize Theorem 5.8 to quasi-integrable functions $\varphi$, which satisfy an additional upper semi-continuity property.

**Theorem 5.12.** *Let $\varphi : X \to [-\infty,\infty]$ be Borel measurable (satisfying (5.5)) and fix a $T$-invariant Borel set $Z \subseteq X$. Suppose $h_{\mathrm{top}}(T) < \infty$ and assume that the following holds:*

*(1) For each $\mu \in \mathscr{M}_T(Z)$ one has that $\varphi$ is quasi-integrable with respect to $\mu$.*

*(2) For each $x \in A(\varphi, \mathscr{M}_T(Z))$ and $\mu \in V_T(x) \cap \mathscr{M}_T(Z)$ one has*

$$\limsup_{l\to\infty} \int_X \varphi \, \mathrm{d}\delta_{x,n_l} \le \int_X \varphi \, \mathrm{d}\mu \tag{5.6}$$

*for all subsequences $(n_l)_{l\in\mathbb{N}}$ such that $\lim_{l\to\infty} \delta_{x,n_l} = \mu$.*

*Then $P_{\mathscr{L}(Z)}(\varphi) = p_{\mathscr{M}_T(Z)}(\varphi)$. In particular, one has $P_{G_\mu}(\varphi) = h_\mu(T) + \int_X \varphi \, \mathrm{d}\mu$ for each $\mu \in \mathscr{E}_T(X)$.*

*Proof.* As $h_{\mathrm{top}}(T) < \infty$ and by (1) we have $\mathscr{M}_T(Z) \subseteq \mathscr{A}(\varphi)$. By Proposition 5.10 and (1) it also follows that $\mathscr{E}_T(Z) \subseteq \mathscr{G}(\varphi)$. Using (2) and Proposition 5.11, the rest of the proof works in exactly the same way as for continuous $\varphi$.

□

**Corollary 5.13** (Variational principle for upper semi-continuous $\varphi$). *If $h_{\mathrm{top}}(T) < \infty$ and $\varphi : X \to [-\infty, \infty)$ is upper semi-continuous, then for each $T$-invariant Borel set $Z \subseteq X$ one has*

$$P_{\mathcal{L}(Z)}(\varphi) = \sup\left\{ h_\mu(T) + \int_X \varphi \,\mathrm{d}\mu : \mu \in \mathcal{M}_T(Z) \right\}.$$

*Proof.* This follows from Theorem 5.12, together with the fact, that $\varphi$ is bounded from above and $\mu \mapsto \int_X \varphi \,\mathrm{d}\mu$ is upper semi-continuous on $\mathcal{M}_T(X)$ (see Lemma A.2). □

**Remark 5.14.** The condition $h_{\mathrm{top}}(T) < \infty$ in Theorem 5.12 and Corollary 5.13 can be dropped, if one assumes $\int_X \varphi \,\mathrm{d}\mu > -\infty$ for each $\mu \in \mathcal{M}_T(Z)$.

**Remark 5.15.** Theorem 5.12 and Corollary 5.13 generalize known results in the literature:

(a) In [Rau17] a statement similar to Theorem 5.12 under a stronger assumption, called upper semi-continuity with respect to $T$, was proven for the special case $Z = X$ (and using the topological pressure $\overline{P}$).

(b) In [BF12] together with [CFH08], and independently [Kel98], the statement of Corollary 5.13 was proven for the special case of $T$-invariant, compact $Z \subseteq X$, again using $\overline{P}$. Thus, $\overline{P}_Z(\varphi) = P_Z(\varphi) = p_{\mathcal{M}_T(Z)}(\varphi)$ for all $T$-invariant, compact $Z \subseteq X$. The result in the present work on the other hand includes $T$-invariant, non-compact $Z \subseteq X$ as well. Thus Corollary 5.13 is a generalization of the result in [PP84] to functions $\varphi$ which satisfy conditions (1) and (2) in Theorem 5.12.

If one allows arbitrary measurable $\varphi : X \to [-\infty, \infty]$, then one can still prove the following:

**Theorem 5.16** (Variational principle for measurable $\varphi$). *If $\varphi : X \to [-\infty, \infty]$ is Borel measurable (and satisfies (5.5)), then one has $P_{A(\varphi,\mathcal{Y})}(\varphi) = p_{\mathcal{Y}}(\varphi)$ for each $\mathcal{Y} \subseteq \mathcal{G}(\varphi)$. In particular, if $\mathcal{A}(\varphi) = \mathcal{M}_T(X)$, then*

$$P_{A(\varphi,\mathcal{E}_T(X))}(\varphi) = \sup\left\{ h_\mu(T) + \int_X \varphi \,\mathrm{d}\mu : \mu \in \mathcal{M}_T(X) \right\}.$$

*Proof.* The first statement is a direct consequence of Theorem 5.2. The condition $\mathcal{A}(\varphi) = \mathcal{M}_T(X)$ implies that $\varphi$ is quasi-integrable with respect to each $\mu \in \mathcal{M}_T(X)$, and that $p_{\mathcal{M}_T(X)}(\varphi)$ is well-defined. By Proposition 5.10 it follows that $\mathcal{G}(\varphi) = \mathcal{E}_T(X)$, thus the second statement follows by using Proposition 5.11. □

**Corollary 5.17.** *Let $Z \subseteq X$ be a $T$-invariant Borel set and $\varphi : X \to [-\infty, \infty]$ be Borel measurable. Define $\mathcal{Y} := \{\mu \in \mathcal{E}_T(Z) : \varphi \in L^1(Z, \mu)\}$. Then one has $P_{A(\varphi,\mathcal{Y})}(\varphi) = p_{\mathcal{Y}}(\varphi)$.*

*Proof.* As $\mathcal{Y} \subseteq \mathcal{G}(\varphi)$, the statement follows from Theorem 5.16. □

**Remark 5.18.** A special case of Corollary 5.17 was proven in [Mum07] for measurable functions $\varphi : X \to \mathbb{R}$. Here an additional assumption was used, namely that there exists an increasing sequence $Z_l \subseteq Z_{l+1}$ such that $Z = \bigcup_{l\geq 1} Z_l$ and $\varphi|_{\overline{Z}_l}$ is continuous for each $l \geq 1$. More precisely it was basically shown there that

$$\sup_{l\geq 1} P_{A(\varphi,\mathscr{Y})\cap Z_l}(\varphi) = p_{\mathscr{Y}}(\varphi).$$

Thus, under this additional assumption, Corollary 5.17 can also be proven by combining Lemma 3.10 and the statement in [Mum07]. Our statements on the other hand are much stronger, as they include functions, which are nowhere continuous, which can attain $\pm\infty$, or which are not integrable at all. An example for a function $\varphi$ which is not upper semi-continuous and does not satisfy the condition in [Mum07], but satisfies conditions (1) and (2) in Theorem 5.12, was given in [Rau17].

**Remark 5.19.** We want to give a final remark concerning the nature of sets, for which the variational principle calculates the pressure.

Assume $\varphi : X \to \mathbb{R}$ to be continuous. Choose for each $\mu \in \mathscr{E}_T(X)$ a Borel set $B_\mu \subseteq G_\mu$ such that $\mu(B_\mu) > 0$, and define $B := \bigcup_{\mu\in\mathscr{E}_T(X)} B_\mu$. Then it follows that $P_B(\varphi) = P_X(\varphi)$. Thus the topological pressure on $X$ is determined by the topological pressure on a possibly smaller subset $B$, which consists of points $x \in X$, such that $V_T(x)\cap\mathscr{E}_T(X) \neq \varnothing$. In particular, one has the possibly uncountable stability

$$P_B(\Phi) = \sup_{\mu\in\mathscr{E}_T(X)} P_{B_\mu}(\Phi).$$

Similarly, by ergodic decomposition the variational pressure is determined by supremum over the ergodic measures. Hence, from the view point of the variational principle, only ergodic measures and the topological pressures on their associated points seem to matter. However, in so-called saturated systems, this is not true anymore (see §5.8).

### §5.3. The sub- and super-additive cases

The goal of this subsection is to extend the sub-additive thermodynamic formalism, which was developed in [CFH08] and [FH10].

**Definition 5.20.** Let $\Phi = (\varphi_n)_{n\geq 1}$ be a potential on $(X,T)$. We call $\Phi$ to be sub-additive, if $\varphi_n : X \to [-\infty,\infty)$ and $\varphi_{n+m} \leq \varphi_n + \varphi_m \circ T^n$ for all $m,n \geq 1$. It is called super-additive, if $-\Phi := (-\varphi_n)_{n\geq 1}$ is sub-additive, that is, if $\varphi_{n+m} \geq \varphi_n + \varphi_m \circ T^n$ for all $m,n \geq 1$. Define also for $\mu \in \mathscr{M}_T(X)$

$$\lambda(\mu) := \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \,\mathrm{d}\mu, \tag{5.7}$$

whenever $\Phi$ is Borel measurable, and the right side is well-defined.

Our first goal is to establish a relation similarly to (5.6).

**Proposition 5.21.** *Let $\Phi$ be an upper semi-continuous, sub-additive potential and $x \in X$. If $\mu \in V_T(x)$ and $(n_l)_{l\in\mathbb{N}}$ is a subsequence, such that $\lim_{l\to\infty} \delta_{x,n_l} = \mu$, then*

$$\limsup_{l\to\infty} \frac{1}{n_l}\varphi_{n_l}(x) \leq \lim_{k\to\infty} \frac{1}{k} \int_X \varphi_k \,\mathrm{d}\mu.$$

*Proof.* The proof is based on Lemma 2.2 and Lemma 2.3 in [CFH08]. First observe that for all continuous $\psi : X \to \mathbb{R}$ and $k \in \mathbb{N}$ one has

$$\left| \int_X \psi \, \mathrm{d}\delta_{x,n_l} - \frac{(n_l - k + 1)}{n_l} \int_X \psi \, \mathrm{d}\delta_{x,n_l-k+1} \right| \le \frac{(k-1)}{n_l} \|\psi\|_\infty \to 0$$

as $l \to \infty$. This shows

$$\lim_{l\to\infty} \delta_{x,n_l-k+1} = \mu.$$

To proceed we need the following lemma:

**Lemma 5.22** (Lemma 2.2 in [CFH08])**.** *Let $\Phi$ be a sub-additive potential and $C := \max\left(0, \sup_{x\in X} \varphi_1(x)\right)$. Then for any $x \in X$ and $n \ge 2k$ one has*

$$\varphi_n(x) \le 2kC + \frac{1}{k} \sum_{i=0}^{n-k} \varphi_k(T^i x).$$

Using the above lemma, one has

$$\varphi_{n_l}(x) \le 2kC + \frac{1}{k} \sum_{i=0}^{n_l-k} \varphi_k(T^i x) = 2kC + \frac{(n_l - k + 1)}{k} \int_X \varphi_k \, \mathrm{d}\delta_{x,n_l-k+1}$$

for $n_l \ge 2k$. Note that, as $\varphi_k$ are upper semi-continuous, we have $C < \infty$ and the upper semi-continuity of the mapping $\mu \mapsto \int_X \varphi_k \, \mathrm{d}\mu$ (see Lemma A.2). Hence

$$\limsup_{l\to\infty} \frac{1}{n_l} \varphi_{n_l}(x) \le \frac{1}{k} \limsup_{l\to\infty} \int_X \varphi_k \, \mathrm{d}\delta_{x,n_l-k+1} \le \frac{1}{k} \int_X \varphi_k \, \mathrm{d}\mu$$

for each $k \ge 1$. Letting $k \to \infty$ and applying Kingman's sub-additive ergodic theorem one the right side yields the result.

□

We have for upper semi-continuous, sub-additive potentials $\Phi$ a simple description of the sets $A(\Phi, \lambda, \mathscr{Y})$, similarly to the additive case:

**Corollary 5.23.** *If $\Phi$ is an upper semi-continuous, sub-additive potential, and $\lambda$ defined as in (5.7), one has*

$$A(\Phi, \lambda, \mathscr{Y}) = \left\{ x \in X : V_T(x) \cap \mathscr{Y} \neq \varnothing \right\},$$
$$V(\Phi, \lambda, \mathscr{Y}) = \left\{ x \in X : \mathscr{Y} \subseteq V_T(x) \right\},$$

*for each $\mathscr{Y} \subseteq \mathscr{M}_T(X)$.*

*Proof.* Using Proposition 5.21, this can be proven like Proposition 4.2. □

**Remark 5.24.** As far as we know, there is no analogue to Proposition 5.21 for super-additive $\Phi$, even if $\Phi$ is continuous. This is because Lemma 5.22 is established for sub-additive potentials only. Generally speaking, upper semi-continuous or sub-additive potentials are behaving well with respect to condition (4.2). But it is unknown whether this is also the case for lower semi-continuous or super-additive potentials.

We also need an ergodic decomposition theorem for sub-additive potentials. As we did not find a proof in the literature, we will provide one in here.

**Proposition 5.25.** *Fix $\mu \in \mathcal{M}_T(X)$ and its ergodic decomposition $\mathrm{m}_\mu$. Assume $\Phi$ to be a sub-additive potential such that $\varphi_n : X \to [-\infty,\infty)$ is Borel measurable for each $n \geq 1$ and $\varphi_1^+ \in L^1(X,\mu)$. Then*

$$\lim_{n\to\infty} \frac{1}{n}\int_X \varphi_n \,\mathrm{d}\mu = \int_{\mathscr{E}_T(X)} \left(\lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n \,\mathrm{d}\nu\right)\mathrm{dm}_\mu(\nu).$$

*Proof.* The idea of the proof is to decompose the integral of the invariant function $\lim_{n\to\infty}\frac{1}{n}\varphi_n$.

First, as $\varphi_1^+ \in L^1(X,\mu)$, we have by Proposition 5.11

$$\int_X \varphi_1^+ \,\mathrm{d}\mu = \int_{\mathscr{E}_T(X)} \left(\int_X \varphi_1^+ \,\mathrm{d}\nu\right)\mathrm{dm}_\mu(\nu).$$

This means that $\varphi_1^+ \in L^1(X,\nu)$ for $\mathrm{m}_\mu$-almost every $\nu \in \mathscr{E}_T(X)$. Hence applying Kingman's sub-additive ergodic theorem, there exists for $\mathrm{m}_\mu$-almost every $\nu \in \mathscr{E}_T(X)$ a function $\psi_\nu : X \to [-\infty,\infty)$ such that $f_\nu^+ \in L^1(X,\nu)$, $\psi_\nu(Tx) = \psi_\nu(x)$ and

$$\lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n \,\mathrm{d}\nu = \int_X \psi_\nu \,\mathrm{d}\nu = \lim_{n\to\infty}\frac{1}{n}\varphi_n(x) = \psi_\nu(x) \tag{5.8}$$

for $\nu$-almost every $x \in X$. Also by Kingman's sub-additive ergodic theorem, there is a function $\psi_\mu : X \to [-\infty,\infty)$ such that $f_\mu^+ \in L^1(X,\mu)$, $\psi_\mu(Tx) = \psi_\mu(x) = \lim_{n\to\infty}\frac{1}{n}\varphi_n(x)$ for $\mu$-almost every $x \in X$ and

$$\lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n \,\mathrm{d}\mu = \int_X \psi_\mu \,\mathrm{d}\mu.$$

Define

$$B := \left\{x \in X : \lim_{n\to\infty}\frac{1}{n}\varphi_n(x) = \psi_\mu(x)\right\},$$

then we have $\mu(B) = 1$. Thus, by ergodic decomposition, $\nu(B) = 1$ for $\mathrm{m}_\mu$-almost every $\nu \in \mathscr{E}_T(X)$. This means by (5.8) that for $\mathrm{m}_\mu$-almost every $\nu \in \mathscr{E}_T(X)$ we have

$$\psi_\mu = \psi_\nu = \lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n \,\mathrm{d}\nu$$

$\nu$-almost everywhere. Now applying again Proposition 5.11 yields

$$\int_X \psi_\mu \,\mathrm{d}\mu = \int_{\mathscr{E}_T(X)} \left(\int_X \psi_\mu \,\mathrm{d}\nu\right)\mathrm{dm}_\mu(\nu) = \int_{\mathscr{E}_T(X)} \left(\lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n \,\mathrm{d}\nu\right)\mathrm{dm}_\mu(\nu),$$

which shows the statement.

□

**Corollary 5.26.** *Fix $\mu \in \mathcal{M}_T(X)$ and its ergodic decomposition $\mathrm{m}_\mu$. Assume $\Phi$ to be a super-additive potential such that $\varphi_n : X \to (-\infty,\infty]$ is Borel measurable for each $n \geq 1$ and $\varphi_1^- \in L^1(X,\mu)$. Then*

$$\lim_{n\to\infty} \frac{1}{n}\int_X \varphi_n \,\mathrm{d}\mu = \int_{\mathscr{E}_T(X)} \left(\lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n \,\mathrm{d}\nu\right)\mathrm{dm}_\mu(\nu).$$

*Proof.* As $-\Phi$ is sub-additive, the statement follows immediately from Proposition 5.25.

□

**Remark 5.27.** Proposition 5.25 was proven in [FH10] for continuous, sub-additive $\Phi$.

Kingman's sub-additive ergodic theorem, Proposition 5.21 and Proposition 5.25 are the foundations of the next theorem, which is a generalization of Theorem 5.13 to sub-additive potentials.

**Theorem 5.28.** *Let $\Phi$ be an upper semi-continuous, sub-additive potential. If $h_{\text{top}}(T) < \infty$, then for each $T$-invariant Borel set $Z \subseteq X$ one has*

$$P_{\mathscr{L}(Z)}(\Phi) = \sup\left\{ h_\mu(T) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \,\mathrm{d}\mu : \mu \in \mathscr{M}_T(Z) \right\}.$$

*In particular, one has*

$$P_{G_\mu}(\Phi) = h_\mu(T) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \,\mathrm{d}\mu$$

*for each $\mu \in \mathscr{E}_T(X)$.*

*Proof.* The proof works in the same way as the proof of Theorem 5.13

□

**Remark 5.29.** In [CFH08], the statement of Theorem 5.28 was proven for the special case of $T$-invariant, compact $Z \subseteq X$, using $\overline{P}$. Thus, $\overline{P}_Z(\varphi) = P_Z(\varphi) = p_{\mathscr{M}_T(Z)}(\varphi)$ for all $T$-invariant, compact $Z \subseteq X$ (note that the results there were formulated for continuous, sub-additive potentials, but the given proof includes also the upper semi-continuous case). As our result also holds for $T$-invariant, non-compact $Z \subseteq X$, Theorem 5.28 is moreover a generalization of the result in [PP84] to upper semi-continuous, sub-additive potentials.

If one allows arbitrary sub- or super-additive potentials, one has the following:

**Theorem 5.30.** *Let $\Phi$ be a Borel measurable and sub- or super-additive. Then one has*

$$P_{A(\Phi,\lambda,\mathscr{Y})}(\Phi) = \sup\left\{ h_\mu(T) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \,\mathrm{d}\mu : \mu \in \mathscr{Y} \right\}$$

*for each $\mathscr{Y} \subseteq \mathscr{G}(\Phi,\lambda)$. In particular, if $\mathscr{A}(\lambda) = \mathscr{M}_T(X)$ (that is Kingman's sub-additive ergodic theorem holds for $\Phi$ and each $\mu \in \mathscr{M}_T(X)$), then*

$$P_{A(\Phi,\lambda,\mathscr{E}_T(X))}(\Phi) = \sup\left\{ h_\mu(T) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \,\mathrm{d}\mu : \mu \in \mathscr{M}_T(X) \right\}.$$

*Proof.* The statement follows from Theorem 5.2 and Proposition 5.25.

□

**Remark 5.31.** We want to provide some remarks concerning the literature:

(a) The above theorem was basically proven in [CHZ13] for the special case of continuous sub- or super-additive potentials and $\mathscr{Y} := \{\mu\}$, $\mu \in \mathscr{E}_T(X)$.

(b) As Theorem 5.30 allows potentials, which are not upper semi-continuous, it is a generalization of the result given in [CFH08]. A result for discontinuous sub-additive potentials with tempered variation was proven in [CMP10], following the approach of [Mum07] (see also Remark 5.18).

(c) As far as we know, Theorem 5.30 is the first general variational for super-additive potentials $\Phi$. A special case of a super-additive variational principle for average conformal repeller was proven in [BCH10].

### §5.4. The multivariate case

The topological pressure of multivariate potentials was first considered in [Rau15].

**Definition 5.32.** Let $d \geq 1$ and $\varphi : X^d \to \mathbb{R}$ be a function. Define for $x \in X$ and $n \geq 1$

$$\varphi_n(x) := \frac{1}{n^{d-1}} \sum_{i=0}^{n-1} \varphi(T^{i_1}x, \dots, T^{i_d}x).$$

Thus $\Phi := (\varphi_n)_{n\geq 1}$ is a potential on $(X,T)$. We call $P_Z(\Phi)$ multivariate pressure of $\varphi$ on $Z \subseteq X$. Note that in the case $d > 1$ the potential $\Phi$ in general is neither sub- nor super-additive.

In [Rau15] it was shown that if $\varphi$ is continuous on $X^d$, then $\Phi$ has tempered variation, and that furthermore one has for each $T$-invariant Borel set $Z \subseteq X$

$$\begin{aligned} P^B_{\mathscr{F}(Z)}(\Phi) &= \sup\left\{ h_\mu(T) + \int_{X^d} \varphi \, \mathrm{d}\mu^d : \mu \in \mathscr{E}_T(Z) \right\} \\ &= \sup\left\{ h_\mu(T) + \int_{X^d} \lim_{n\to\infty} \frac{1}{n} \varphi_n \, \mathrm{d}\mu : \mu \in \mathscr{M}_T(Z) \right\}. \end{aligned}$$

Here $\mathscr{F}(Z) := \{x \in Z : V_T(x) \cap \mathscr{E}_T(Z) \neq \varnothing\}$. Recall that by Proposition 3.13 we have $P^B_{\mathscr{F}(Z)}(\Phi) = P_{\mathscr{F}(Z)}(\Phi)$. We want to generalize the above variational principle to a special class of measurable $\varphi : X^d \to \mathbb{R}$:

**Theorem 5.33.** *Let $B(X) := \{f : X \to \mathbb{R}$ Borel measurable and bounded$\}$ and pick $\varphi_{ij} \in B(X)$ for $1 \leq l$, $1 \leq j \leq d$. Define $\varphi(x_1,\dots,x_d) := \sum_{i\leq l} \prod_{j\leq d} \varphi_{ij}(x_i)$ and $\Phi$ according to Definition 5.32. Then for each $T$-invariant Borel set $Z \subseteq X$ one has*

$$P_G(\Phi) = \sup\left\{ h_\mu(T) + \int_{X^d} \varphi \, \mathrm{d}\mu^d : \mu \in \mathscr{E}_T(Z) \right\},$$

*where*

$$G := \bigcup_{\mu\in\mathscr{E}_T(Z)} \bigcap_{i\leq l} \bigcap_{j\leq d} \left\{ x \in G_\mu : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi_{ij}(x) = \int_X \varphi_{ij} \, \mathrm{d}\mu \right\}.$$

*Proof.* As $\varphi_{ij} \in L^1(X,\mu)$ for all $i,j$ and each $\mu \in \mathscr{E}_T(Z)$, the statement is a direct consequence of Birkhoff's ergodic theorem and Theorem 5.2.

□

**Remark 5.34.**

(a) The core of the mentioned result in [Rau15] is the proof of Theorem 5.33 for continuous $\varphi_{ij}$. Then one can approximate the multivariate pressure of arbitrary continuous $\varphi : X^d \to \mathbb{R}$ with the multivariate pressures of $\sum_{i\leq l} \prod_{j\leq d} \varphi_{ij}$.

(b) A possible direction to generalize the above theorem is to combine Theorem 5.2 with the multivariate ergodic theorem for von Mises statistics, which was developed in [DG14].

### §5.5. The ratio case

We want to prove a new variational principle for Birkhoff ratios, which is inspired by Hopf's ratio ergodic theorem (see [Zwe04]). Those ratios give rise to examples for potentials, which are neither sub- nor super-additive.

**Theorem 5.35.** *Let $\mu \in \mathscr{E}_T(X)$ and $f, g \in L^1(X,\mu)$. Suppose that $g > 0$ and define*

$$\varphi_n := n \frac{\sum_{i<n} f \circ T^i}{\sum_{i<n} g \circ T^i}$$

*for each $n \geq 1$. Then*

$$P_{B_\mu(f,g)}(\Phi) = h_\mu(T) + \frac{\int_X f \,\mathrm{d}\mu}{\int_X g \,\mathrm{d}\mu},$$

*where*

$$B_\mu(f,g) := \left\{ x \in G_\mu : \lim_{n\to\infty} \frac{\sum_{i<n} f(T^i x)}{\sum_{i<n} g(T^i x)} = \frac{\int_X f \,\mathrm{d}\mu}{\int_X g \,\mathrm{d}\mu} \right\}.$$

*In particular, if $f$ and $g$ are continuous, the for each $\mathscr{Y} \subseteq \mathscr{E}_T(X)$ one has*

$$P_G(\Phi) = \sup\left\{ h_\mu(T) + \frac{\int_X f \,\mathrm{d}\mu}{\int_X g \,\mathrm{d}\mu} : \mu \in \mathscr{Y} \right\}, \tag{5.9}$$

*where $G := \bigcup_{\mu\in\mathscr{Y}} G_\mu$.*

*Proof.* By Birkhoff's ergodic theorem one has $\mu(B_\mu(f,g)) = 1$. Thus the first statement follows from Theorem 5.2. The second statement works in the same way, together with the fact, that $\mu \mapsto \int_X \varphi \,\mathrm{d}\mu$ is continuous on $\mathscr{M}_T(X)$ for each continuous $\varphi : X \to \mathbb{R}$.

□

**Remark 5.36.** As far as we know, the above proposition does not follow from any known method in literature. An even stronger statement is true, namely equation (5.9) also holds, if one assumes

$$f, g \in \bigcap_{\mu\in\mathscr{Y}} L^1(X,\mu),$$

and replaces $G$ by $\bigcup_{\mu\in\mathscr{Y}} B_\mu(f,g)$.

### §5.6. The invariant case

In this subsection we want to establish variational principles for invariant functions. These functions naturally appear as limits in the pointwise ergodic theorems.

**Definition 5.37.** A Borel measurable function $\varphi : X \to [-\infty,\infty]$ is called $T$-invariant, if $\varphi \circ T = \varphi$.

If $\varphi$ is $T$-invariant and $\mu \in \mathscr{E}_T(X)$, then there exists a constant $\varphi_\mu \in [-\infty,\infty]$ such that $\varphi(x) = \varphi_\mu$ for $\mu$-almost every $x \in X$ (see Lemma A.3). One has then the following variational principle:

**Theorem 5.38** (Variational principle for invariant $\varphi$)**.** *Let $h_{\mathrm{top}}(T) < \infty$. For each $T$-invariant, Borel measurable function $\varphi : X \to [-\infty,\infty]$ one has*

$$P_G(\varphi) = \sup\left\{ h_\mu(T) + \varphi_\mu : \mu \in \mathscr{E}_T(X) \right\},$$

*where $G := \bigcup_{\mu \in \mathscr{E}_T(X)} \{ x \in G_\mu : \varphi(x) = \varphi_\mu \}$.*

*Proof.* This is a direct consequence of Theorem 5.2.

□

If one has a sub-additive potential $\Phi$, then $\varphi := \lim_{n\to\infty} \frac{1}{n}\varphi_n$ gives rise to an invariant function almost everywhere, for each invariant measure. Hence we can use the additive pressure of $\varphi$ to express the sub-additive pressure of $\Phi$.

**Theorem 5.39** (Reduction to the additive case)**.** *Let $h_{\mathrm{top}}(T) < \infty$ and $\Phi$ be an upper semi-continuous, sub-additive potential. Define*

$$\begin{aligned} \varphi &:= \lim_{n\to\infty} \frac{1}{n}\varphi_n, \\ R(\Phi) &:= \left\{ x \in X : \varphi(x) \text{ exists} \right\}. \end{aligned}$$

*Then one has the equalities*

$$\begin{aligned} P_{R(\Phi)}(\varphi) &= \sup\left\{ h_\mu(T) + \int_{R(\Phi)} \varphi \,\mathrm{d}\mu : \mu \in \mathscr{M}_T(X) \right\} \\ &= \sup\left\{ h_\mu(T) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \,\mathrm{d}\mu : \mu \in \mathscr{M}_T(X) \right\} \\ &= P_{R(\Phi)}(\Phi) = P_X(\Phi). \end{aligned}$$

*Proof.* First note that $\varphi(x)$ might not be well-defined for every $x \in X$. However, it can be extended to a Borel measurable function $\tilde{\varphi} : X \to [-\infty,+\infty]$, for instance by

$$\tilde{\varphi}(x) := \begin{cases} \varphi(x), & x \in R(\Phi) \\ C, & x \in X \setminus R(\Phi) \end{cases} \quad \text{for some constant } C \in \mathbb{R}.$$

Secondly observe that $P_{R(\Phi)}(\tilde{\varphi}) = P_{R(\Phi)}(\varphi')$ for two arbitrary extensions $\tilde{\varphi}, \varphi'$. This follows directly from the definition of topological pressure. Thus it is justified to write $P_{R(\Phi)}(\varphi)$ instead of $P_{R(\Phi)}(\tilde{\varphi})$. Next, if $\mu \in \mathscr{M}_T(X)$, by Kingman's sub-additive ergodic theorem one has $\mu(R(\Phi)) = 1$ and

$$\int_{R(\Phi)} \varphi \,\mathrm{d}\mu = \int_X \tilde{\varphi} \,\mathrm{d}\mu = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \,\mathrm{d}\mu.$$

Hence, if $x \in R(\Phi)$ and $\mu \in V_T(x)$ such that $\lim_{l\to\infty} \delta_{x,n_l} = \mu$, we obtain

$$\varphi(x) = \lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = \limsup_{l\to\infty} \frac{1}{n_l}\varphi_{n_l}(x) \le \int_{R(\Phi)} \varphi \,\mathrm{d}\mu.$$

The statements then follow by applying both Theorem 4.4 and Theorem 4.17 to $P_{R(\Phi)}(\varphi)$ and $P_{R(\Phi)}(\Phi)$, respectively.

□

**Remark 5.40.**

(a) The set $R(\Phi)$ is also called regular set of $\Phi$. We will study it further in §5.7.

(b) A statement similar to Theorem 5.39 could also be formulated for continuous multivariate potentials or Birkhoff ratios. Furthermore, if one considers continuous, additive potentials $\Phi$, one can drop the $h_{\text{top}}(T) < \infty$ assumption in Theorem 5.39.

## §5.7. The case of level sets

Let $\varphi : X \to [-\infty, \infty]$ be a function. One object of study in multifractal analysis is the so-called multifractal decomposition

$$X = K(\varphi) \cup \bigcup_{\alpha \in [-\infty,\infty]} K(\varphi, \alpha),$$

where $K(\varphi,\alpha) := \{x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i<n} \varphi \circ T^i(x) = \alpha\}$ for each $\alpha \in [-\infty, \infty]$, and

$$K(\varphi) := \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) \text{ does not exist} \right\}.$$

The set $K(\varphi)$ is called irregular set (also called historic set), the sets $K(\varphi,\alpha)$ are called level sets, and the set

$$R(\varphi) := \bigcup_{\alpha \in [-\infty,\infty]} K(\varphi,\alpha) = \left\{ x \in X : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) \text{ does exist} \right\}$$

is called regular set with respect to $\varphi$ (we refer to [Bar08] for an introduction).

We are first concerned with $R(\varphi)$. As $R(\varphi)$ can be "seen" by invariant measures, we are able to provide an estimate for the topological pressure on $R(\varphi)$ with the methods developed so far. Recall that by Theorem 5.39 one has

$$P_{R(\varphi)}(\varphi) = P_X(\varphi) = \sup \left\{ h_\mu(T) + \int_X \varphi \, \mathrm{d}\mu : \mu \in \mathcal{M}_T(X) \right\},$$

if $h_{\text{top}}(T) < \infty$ and $\varphi : X \to [-\infty, +\infty)$ is upper semi-continuous. The above observation can be generalized in the following way:

**Theorem 5.41** (Pressure of regular sets)**.** *Suppose $h_{\text{top}}(T) < \infty$. Fix $\mu \in \mathcal{M}_T(X)$, let $\varphi \in L^1(X,\mu)$ and $\psi : X \to [-\infty, +\infty)$ be upper semi-continuous. Then one has $h_\mu(T) + \int_X \psi \, \mathrm{d}\mu \le P_{R(\varphi)}(\psi)$, and at least one of the following cases holds:*

*(1) $P_{R(\varphi)}(\psi) = P_X(\psi)$.*

*(2) There exists a measure $\nu \in \mathcal{M}_T(X)$ such that $P_{R(\varphi)}(\psi) = h_\nu(T) + \int_X \psi \, \mathrm{d}\nu$.*

*In particular, if $\varphi$ is Borel measurable and bounded, then (1) always holds. If $\mu \mapsto h_\mu(T)$ is upper semi-continuous on $\mathcal{M}_T(X)$, then (2) always holds.*

*Proof.* By Birkhoff's ergodic theorem we have $\mu(R(\varphi)) = 1$. Hence using Proposition 5.10, Proposition 5.11 and Theorem 4.17 one has

$$h_\mu(T) + \int_X \psi \, \mathrm{d}\mu \le P_{R(\varphi)}(\psi). \tag{5.10}$$

Now assume (1) does not hold. By Corollary 5.13 there exists a measure $\sigma \in \mathcal{M}_T(X)$ such that $P_{R(\varphi)}(\psi) < h_\sigma(T) + \int_X \psi \, d\sigma$. Hence by (5.10) we find a $\beta \in [0,1]$ such that

$$\beta\left(h_\mu(T) + \int_X \psi \, d\mu\right) + (1-\beta)\left(h_\sigma(T) + \int_X \psi \, d\sigma\right) = P_{R(\varphi)}(\psi). \tag{5.11}$$

Setting $\nu := \beta\mu + (1-\beta)\sigma$ shows (2).

In addition, if $\varphi$ is bounded, then by Birkhoff's ergodic theorem one has $\nu(R(\varphi)) = 1$ for all $\nu \in \mathcal{M}_T(X)$. Case (1) follows then by Theorem 4.8 and Corollary 5.13. If the entropy mapping $\mu \mapsto h_\mu(T)$ is upper semi-continuous, then $\mu \mapsto h_\mu(T) + \int_X \psi \, d\mu$ is upper semi-continuous too. Hence there exists an equilibrium state $\sigma \in \mathcal{M}_T(X)$ for $\psi$, that is,

$$h_\sigma(T) + \int_X \psi \, d\sigma = P_X(\psi).$$

This yields (2) by applying (5.11) to $\int_X \psi \, d\mu$ and $\int_X \psi \, d\sigma$.

□

**Remark 5.42.** We want to give some remarks:

(a) The above statement remains valid, if one considers the regular set $R(\Phi)$ of a sub- or super-additive potential, for which Kingman's sub-additive ergodic theorem holds. Also the same holds, if one considers the intersection $\bigcap_{n\geq 1} R(\varphi_n)$ for countably many $\varphi_n \in L^1(X,\mu)$.

(b) By (5.10) it follows immediately that

$$\sup\left\{h_\mu(T) + \int_X \psi \, d\mu : \mu \in \mathcal{Y}\right\} \leq P_{R(\varphi)}(\psi),$$

if $\varphi \in \bigcap_{\mu \in \mathcal{Y}} L^1(X,\mu)$ for a subset $\mathcal{Y} \subseteq \mathcal{M}_T(X)$.

(c) Theorem 5.41 implies that $h_{\text{top}}(R(\varphi)) = h_{\text{top}}(X)$, if $\varphi \in \bigcap_{\mu \in \mathcal{E}_T(X)} L^1(X,\mu)$. However, a surprising result in the field of multifractal analysis is that $K(\varphi)$ can have the topological entropy and Hausdorff dimension of the whole space $X$ (see for instance [BS00]).

One can obtain a similar statement for individual level sets $K(\varphi,\alpha)$, provided $\varphi$ is continuous and $\alpha$ chosen in a specific way:

**Theorem 5.43** (Pressure on level sets)**.** *Suppose $h_{\text{top}}(T) < \infty$. Let $\varphi : X \to \mathbb{R}$ be continuous and $\psi : X \to [-\infty,\infty]$ be upper semi-continuous. Define*

$$E_\varphi := \left\{\alpha \in \mathbb{R} : \exists \mu \in \mathcal{E}_T(X) \text{ such that } \int_X \varphi \, d\mu = \alpha\right\}.$$

*If $\alpha \in E_\varphi$, then at least one of the following cases holds:*

*(1)*

$$P_{K(\varphi,\alpha)}(\psi) = \sup\left\{h_\mu(T) + \int_X \psi \, d\mu : \int_X \varphi \, d\mu = \alpha\right\}.$$

*(2) There exists a measure $\nu \in \mathcal{M}_T(X)$ such that $\int_X \varphi \, d\nu = \alpha$ and $P_{K(\varphi,\alpha)}(\psi) = h_\nu(T) + \int_X \psi \, d\nu$.*

*Proof.* Let $x \in K(\varphi,\alpha)$. Then for each $\mu \in V_T(x)$ one has $\int_X \varphi \,\mathrm{d}\mu = \alpha$: Indeed, if $\lim_{l\to\infty} \delta_{x,n_l} = \mu$, then

$$\alpha = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \lim_{l\to\infty} \frac{1}{n_l} \sum_{i=0}^{n_l-1} \varphi(T^i x) = \lim_{l\to\infty} \int_X \varphi \,\mathrm{d}\delta_{x,n_l} = \int_X \varphi \,\mathrm{d}\mu.$$

But this means $K(\varphi,\alpha) \subseteq A(\psi,\mathscr{Y})$ for $\mathscr{Y} := \{\mu \in \mathscr{M}_T(X) : \int_X \varphi \,\mathrm{d}\mu = \alpha\}$, as $\mu \mapsto \int_X \psi \,\mathrm{d}\mu$ is upper semi-continuous. By picking a $\sigma \in \mathscr{E}_T(X)$ such that $\sigma(K(\varphi,\alpha)) = 1$, one has

$$h_\sigma(T) + \int_X \psi \,\mathrm{d}\sigma \le P_{K(\varphi,\alpha)}(\psi) \le P_{A(\psi,\mathscr{Y})}(\psi) \le \sup\left\{ h_\mu(T) + \int_X \psi \,\mathrm{d}\mu : \int_X \varphi \,\mathrm{d}\mu = \alpha \right\}.$$

The rest of the proof works like the proof of Theorem 5.41, and each measure $\nu$ constructed according to (5.11) satisfies $\int_X \varphi \,\mathrm{d}\nu = \alpha$.

□

If one restricts further to upper semi-continuous systems with continuous $\varphi$ possessing unique equilibrium states, the following holds:

**Corollary 5.44.** *Let $(X,T)$ such that $\mu \mapsto h_\mu(T)$ is upper semi-continuous and $h_{\mathrm{top}}(T) < \infty$. Suppose $\varphi : X \to \mathbb{R}$ to be continuous, such that each $\xi \in \mathrm{span}\{\varphi\}$ has a unique equilibrium state. Define*

$$I_\varphi := \left\{ \alpha \in \mathbb{R} : \exists \mu \in \mathscr{M}_T(X) \text{ such that } \int_X \varphi \,\mathrm{d}\mu = \alpha \right\}.$$

*If $\alpha \in \mathrm{int}(I_\varphi)$, then Theorem 5.43 (2) holds for and any upper semi-continuous $\psi : X \to [-\infty,\infty]$, that is, there exists a measure $\nu \in \mathscr{M}_T(X)$ such that $\int_X \varphi \,\mathrm{d}\nu = \alpha$ and*

$$P_{K(\varphi,\alpha)}(\psi) = h_\nu(T) + \int_X \psi \,\mathrm{d}\nu.$$

*Proof.* If $\alpha \in \mathrm{int}(I_\varphi)$, then by [BS01] (or more generally, [BD09] Theorem 1) there exists a $\mu \in \mathscr{E}_T(X)$ such that $\mu(K(\varphi,\alpha)) = 1$. Hence for $x \in G_\mu \cap K(\varphi,\alpha)$ one has

$$\int_X \varphi \,\mathrm{d}\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x) = \alpha.$$

Thus $\alpha \in E_\varphi$. Furthermore, as $\nu \mapsto h_\nu(T) + \int_X \psi \,\mathrm{d}\nu$ is upper semi-continuous, we have by Theorem 5.43

$$h_\mu(T) + \int_X \psi \,\mathrm{d}\mu \le P_{K(\varphi,\alpha)}(\psi) \le \max\left\{ h_\mu(T) + \int_X \psi \,\mathrm{d}\mu : \int_X \varphi \,\mathrm{d}\mu = \alpha \right\}.$$

Now proceeding like in (5.11) shows the statement.

□

**Remark 5.45.**

(a) Note that Theorem 5.43 holds without additional assumptions to be satisfied by $(X,T)$. Furthermore, Theorem 5.43 and Corollary 5.44 are to our knowledge the first results concerning the pressure of an upper semi-continuous function on a level set.

(b) If $(X,T)$ is a system with the specification property (or more generally $g$-almost product property), it was shown in [Tho09] ([CP10]) that case (1) in Theorem 5.43 holds for each $\alpha \in I_\varphi$ and continuous $\psi : X \to \mathbb{R}$. On the other hand we want the remark that the $g$-almost product property does not imply upper semi-continuity of $\mu \mapsto h_\mu(T)$ (an example was given for instance in [PS07]).

(c) The existence of an ergodic measure $\mu$ such that $\mu(K(\varphi,\alpha)) = 1$ in Corollary 5.44 is a result of the sophisticated multifractal analysis done in [BS01], and more generally, [BD09].

(d) A much stronger result can be proven, if one assumes $(X,T)$ to be saturated (see §5.8 and Theorem 5.52).

### §5.8. The saturated case

It is well-known that for every ergodic measure $\mu \in \mathscr{E}_T(X)$ one has $h_{\text{top}}(G_\mu) = h_\mu(T)$ and $\mu(G_\mu) = 1$. The generic points $G_\mu$ are therefore large from a measure-theoretic point of view, and carry the same entropy as $\mu$. For a non-ergodic measure $\mu \in \mathscr{M}_T(X) \setminus \mathscr{E}_T(X)$, this is not necessarily true: In this case, one has $h_{\text{top}}(G_\mu) \le h_\mu(T)$, but it can happen that $h_{\text{top}}(G_\mu) < h_\mu(T)$. Moreover, $\nu(G_\mu) = 0$ for each $\nu \in \mathscr{M}_T(X)$. Hence $G_\mu$ is negligible from a measure-theoretic point of view. It is even possible that $G_\mu = \varnothing$, that is, no generic point for $\mu$ exists at all.

However, there are examples of systems $(X,T)$, which admit non-ergodic measures $\mu \in \mathscr{M}_T(X) \setminus \mathscr{E}_T(X)$ such that $h_{\text{top}}(G_\mu) = h_\mu(T)$. Before we give a brief overview of the literature, we want to give a definition first. It is based on a definition given in [Cao13].

**Definition 5.46.** We call a measure $\mu \in \mathscr{M}_T(X)$ to be saturated, if $h_{\text{top}}(G_\mu) = h_\mu(T)$ and $G_\mu \neq \varnothing$. The set of all saturated, $T$-invariant measures is denoted by $\mathscr{S}_T(X) := \{\mu \in \mathscr{M}_T(X) : \mu \text{ is saturated}\}$. Note that $\mathscr{E}_T(X) \subseteq \mathscr{S}_T(X)$. The dynamical system $(X,T)$ itself is called saturated, if $\mathscr{S}_T(X) = \mathscr{M}_T(X)$.

We have the following equivalent characterization:

**Proposition 5.47.** *A measure $\mu \in \mathscr{M}_T(X)$ is saturated if and only if*

$$P_{G_\mu}(\varphi) = h_\mu(T) + \int_X \varphi \, d\mu$$

*for every continuous $\varphi : X \to \mathbb{R}$.*

*Proof.* Let $\mu \in \mathscr{M}_T(X)$ be saturated. In [Cao13], Lemma 4.2, it was proven that $P^B_{G_\mu}(\varphi) = h_\mu(T) + \int_X \varphi \, d\mu$. By Proposition 3.13 we have $P_{G_\mu}(\varphi) = P^B_{G_\mu}(\varphi)$, which shows one direction.

For the other direction, note that $P_{G_\mu}(0) = h_\mu(T) \ge 0$ implies $G_\mu \neq \varnothing$. Thus $P_{G_\mu}(0) = h_{\text{top}}(G_\mu)$ follows (see also Remark 3.12 (a)).

□

There are several examples for saturated measures and systems described in the literature. In [PS07] it was proven that every dynamical system $(X,T)$, which satisfies the $g$-almost product property, is saturated. This implies in particular that systems with the specification property are saturated (see [PS07], Proposition

2.1). Recently, shift-spaces with non-uniform structure were studied in [CTY15]. It was proven in [CZ16] that those systems are saturated too. In [LLST17], $\mathscr{C}^{1+\alpha}$-diffeomorphisms $T: M \to M$ were considered. It was shown that $\mathscr{M}_T(\tilde{\Lambda}) \subseteq \mathscr{S}_T(M)$ for certain $T$-invariant hyperbolic "cells" $\tilde{\Lambda} \subseteq M$. A multifractal analysis for saturated systems was carried out for instance in [Cao13] and [BCW13].

The main result of this subsection is that for saturated measures and continuous functions, the conditional variational inequality (Theorem 4.4) is actually an equality. Recall the definition of $A(\varphi, \mathscr{Y})$, given in (5.4).

**Theorem 5.48.** *Let $\varphi: X \to \mathbb{R}$ be continuous. Then for any subset $\mathscr{Y} \subseteq \mathscr{S}_T(X)$ one has*

$$P_G(\varphi) = P_{A(\varphi,\mathscr{Y})}(\varphi) = \sup\left\{ h_\mu(T) + \int_X \varphi \,\mathrm{d}\mu : \mu \in \mathscr{Y} \right\}, \tag{5.12}$$

*where $G := \bigcup_{\mu \in \mathscr{Y}} G_\mu$. In particular, if $(X,T)$ is saturated, then (5.12) holds for arbitrary subsets $\mathscr{Y} \subseteq \mathscr{M}_T(X)$.*

*Proof.* For each $\mu \in \mathscr{Y}$ we have $G_\mu \subseteq G \subseteq A(\varphi, \mathscr{Y})$. Using Proposition 5.47 as well as Theorem 4.4 we obtain the estimate

$$h_\mu(T) + \int_X \varphi \,\mathrm{d}\mu = P_{G_\mu}(\varphi) \le P_G(\varphi) \le P_{A(\varphi,\mathscr{Y})}(\varphi) \le \sup\left\{ h_\mu(T) + \int_X \varphi \,\mathrm{d}\mu : \mu \in \mathscr{Y} \right\}.$$

Taking the supremum on the left side yields the result.

□

We want to give two quick applications of the above result. First we show that the thermodynamic pressure, defined in [Tho11], can be expressed as topological pressure we defined in the present work, provided $(X,T)$ is saturated.

**Definition 5.49** (Thermodynamic pressure, see [Tho11])**.** Let $Z \subseteq X$ be a non-empty set and $\varphi: X \to \mathbb{R}$ continuous. The quantity

$$P_Z^*(\varphi) := \sup\left\{ h_\mu(T) + \int_X \varphi \,\mathrm{d}\mu : \mu \in V_T(x) \text{ for some } x \in Z \right\}$$

is called thermodynamic pressure of $\varphi$ on $Z$. One defines in addition $P_\varnothing^*(\varphi) := \inf_{x \in X} \varphi(x)$.

We now have the following:

**Proposition 5.50.** *Assume $(X,T)$ is saturated. Let $Z \subseteq X$ be a non-empty set and $\varphi: X \to \mathbb{R}$ be continuous. Define*

$$\mathscr{K}(Z) := \bigcup_{x \in Z} \bigcup_{\mu \in V_T(x)} G_\mu.$$

*Then one has $P_Z^*(\varphi) = P_{\mathscr{K}(Z)}(\varphi)$.*

*Proof.* Define $\mathscr{Y} := \bigcup_{x \in Z} V_T(x)$. Then the statement follows immediately from Theorem 5.48.

□

**Remark 5.51.**

(a) Note that for the empty set one has $-\infty = P_{\varnothing}(\varphi) < P^*_{\varnothing}(\varphi)$.

(b) It is open, whether Proposition 5.50 still holds if one replaces $\varphi$ by a discontinuous function. However, Theorem 5.48 and Proposition 5.50 both hold, if one replaces $\varphi$ by a continuous asymptotically additive potential (for the proof, one can use Lemma 4.3 in [Cao13]).

Secondly, we want to state a stronger version of Theorem 5.43 for saturated systems.

**Theorem 5.52** (Variational principle for level sets $K(\varphi,U)$)**.** *Assume $(X,T)$ to be saturated and $\varphi,\psi : X \to \mathbb{R}$ to be continuous. For $\alpha \in \mathbb{R}$, let $K(\varphi,\alpha)$ be the $\alpha$-level set for $\varphi$ as defined in §5.7. Fix an arbitrary subset $U \subseteq \mathbb{R}$ and define $K(\varphi,U) := \bigcup_{\alpha \in U} K(\varphi,\alpha)$. Then one has*

$$P_{K(\varphi,U)}(\psi) = \sup\left\{ h_\mu(T) + \int_X \psi \, d\mu : \int_X \varphi \, d\mu \in U \right\}.$$

*Proof.* In case $K(\varphi,U) = \varnothing$, we are done. Now let $K(\varphi,U) \neq \varnothing$. Similar to the proof of Theorem 5.43 one can show that for each $x \in K(\varphi,U)$ and $\mu \in V_T(x)$ one has $\int_X \varphi \, d\mu \in U$. Thus $K(\varphi,U) \subseteq A(\psi,\mathscr{Y})$, and by Theorem 4.4 we derive

$$P_{K(\varphi,U)}(\psi) \le P_{A(\psi,\mathscr{Y})}(\psi) \le \sup\left\{ h_\mu(T) + \int_X \psi \, d\mu : \int_X \varphi \, d\mu \in U \right\}.$$

If $\mu \in \mathscr{M}_T(X)$ such that $\int_X \varphi \, d\mu \in U$, then $G_\mu \subseteq K(\varphi,U)$. Hence using Theorem 5.48 we obtain

$$h_\mu(T) + \int_X \psi \, d\mu = P_{G_\mu}(\psi) \le P_{K(\varphi,U)}(\psi).$$

Taking the supremum on the left side yields the result.

□

**Remark 5.53.** The above theorem was implicitly proven for closed $U$ in [CP10] for systems $(X,T)$ which satisfy the g-almost product property: It follows from the fact that if $U$ is closed, then $\mathscr{Y} := \left\{ \mu \in \mathscr{M}_T(X) : \int_X \varphi \, d\mu \in U \right\}$ is closed too. Then one can apply the results in [CP10]. A result for open set $U$ was recently proven in [CYZ16] in the setting of non-uniformly hyperbolic systems. The present result shows that a variational principle holds for level sets of arbitrary $U \subseteq \mathbb{R}$, in arbitrary saturated systems $(X,T)$.

## 6. Applications to dimension theory

In this section we want to apply the framework developed so far to the dimension theory of dynamical systems. Given a dynamical system $(X,T)$, we denote by $\dim_H Z$ the Hausdorff dimension of a subset $Z \subseteq X$. Recall the lower and upper Hausdorff dimension of a measure $\mu \in \mathscr{M}_T(X)$ to be

$$\underline{\dim}_H \mu := \inf\left\{ \dim_H B : \mu(B) > 0 \right\}$$
$$\overline{\dim}_H \mu := \inf\left\{ \dim_H B : \mu(B) = 1 \right\}.$$

For a continuous function $\varphi : X \to \mathbb{R}$, recall $A(\varphi,\mathscr{Y})$ (see (5.4)). Given $\mathscr{Y} \subseteq \mathscr{M}_T(X)$, we have by Proposition 4.2

$$A(\varphi,\mathscr{Y}) = \left\{ x \in X : V_T(x) \cap \mathscr{Y} \neq \varnothing \right\}, \tag{6.1}$$

which is independent of $\varphi$. Hence for simplicity we define $A(\mathscr{Y}) := A(\varphi,\mathscr{Y})$

### §6.1. Uniformly expanding conformal systems

We first want to introduce a class of systems $(X,T)$, for which the dimension theory is well developed. The next definition follows [Cli11].

**Definition 6.1.** We call $(X,T)$ to be uniformly expanding conformal, if for each $x\in X$

$$a(x) := \lim_{y\to x} \frac{d(Tx,Ty)}{d(x,y)} \in (1,\infty)$$

exists, and $x\mapsto a(x)$ is continuous. The function $x\mapsto \log(a(x))$ is called geometric potential.

**Remark 6.2.** Systems with the above property are called uniformly expanding, because by compactness of $X$ and continuity of $a$ there has to be a $c>1$ such that $a(x)\geq c$ for all $x\in X$. It is called conformal because of the existence of the limits $a(x)$.

For uniformly expanding conformal systems, a strong variant of the so-called Bowen formula holds. It is the key ingredient for this subsection and was proven in [Cli11].

**Theorem 6.3** (Bowen's formula)**.** *Let $(X,T)$ be uniformly expanding conformal. Then for each $\varnothing\neq Z\subseteq X$ the pressure function $s\mapsto P_Z(-s\log a)$ is continuous and strictly decreasing in s, and has a unique zero, which is $\dim_H Z$.*

*Proof.* This follows from Proposition 3.13 and Theorem 2.4 in [Cli11]. □

First we want to provide a quick example, how one can use the conditional variational inequality (Theorem 4.4) to derive an upper bound for the Hausdorff dimension of a set. Define $\mathscr{Y} := \{\mu\in\mathscr{M}_T(X) : h_\mu(T)=0\}$ and $\tilde{X} := A(\mathscr{Y}) = \{x\in X : V_T(x)\cap\mathscr{Y}\neq\varnothing\}$. It is well-known that $h_{\text{top}}(\tilde{X})=0$ (see Remark 4.5 (a)). Here we want to use the conditional variational inequality to compute the Hausdorff dimension of $\tilde{X}$.

**Proposition 6.4.** *If $(X,T)$ is uniformly expanding conformal and $\tilde{X}\neq\varnothing$, then* $\dim_H\tilde{X}=0$.

*Proof.* By Theorem 4.4 and (6.1)

$$\begin{aligned} P_{\tilde{X}}(-s\log a) &= P_{A(-s\log a,\mathscr{Y})}(-s\log a) \\ &\leq \sup\left\{ h_\mu(T) - s\int_X \log a \,\mathrm{d}\mu : \mu\in\mathscr{Y}\right\} \\ &= -s\inf_{\mu\in\mathscr{Y}}\int_X \log a\,\mathrm{d}\mu \leq -s\min_{x\in X}\log(a(x)) < 0 \end{aligned}$$

for all $s>0$. Therefore by Theorem 6.3 one has $\dim_H\tilde{X}\leq 0$. As $\tilde{X}\neq\varnothing$, the statement follows. □

**Remark 6.5.** Another approach for proving the above theorem would be to show that $\tilde{X}$ is at most countable. This is not true in general: Assume that $(X,T)$ is topological conjugated to a full shift. Then one can show that $G_\mu\neq\varnothing$ for all $\mu\in\mathscr{M}_T(X)$ (see [PS07]). As there are uncountable many $\mu$ such that $h_\mu(T)=0$, and $G_\mu\subseteq\tilde{X}$ for each of it, it follows that $\tilde{X}$ is uncountable.

Proposition 6.4 is actually a corollary from the following observation:

**Theorem 6.6.** *Let* $(X,T)$ *be uniformly expanding conformal and* $\varnothing \neq Z \subseteq X$. *Choose for each* $x \in Z$ *a measure* $\mu_x \in V_T(x)$ *and define* $\mathscr{Y}_Z := \{\mu_x : x \in Z\}$. *Then*

$$\dim_H Z \le \sup\left\{\frac{h_\mu(T)}{\int_X \log a \,\mathrm{d}\mu} : \mu \in \mathscr{Y}_Z\right\}. \tag{6.2}$$

*Inequality (6.2) holds true in particular for* $Z = A(\mathscr{Y})$ *and* $\mathscr{Y}_Z = \mathscr{Y}$, *where* $\varnothing \neq \mathscr{Y} \subseteq \mathscr{M}_T(X)$ *is arbitrary.*

*Proof.* By definition of $\mathscr{Y}_Z$ one has $Z \subseteq A(\mathscr{Y}_Z)$. Now denote $D := \dim_H Z$. In case $D = 0$, there is nothing to show. If $D > 0$, we have by Theorem 6.3, Theorem 4.4 and (6.1)

$$\begin{aligned} 0 = P_Z(-D\log a) &\le P_{A(\mathscr{Y}_Z)}(-D\log a) \\ &= P_{A(-D\log a,\mathscr{Y}_Z)}(-D\log a) \\ &\le \sup\left\{h_\mu(T) - D\int_X \log a \,\mathrm{d}\mu : \mu \in \mathscr{Y}_Z\right\}. \end{aligned}$$

Hence for each $\epsilon > 0$ there is a $\mu_\epsilon \in \mathscr{Y}_Z$ such that

$$\frac{-\epsilon}{\int_X \log a \,\mathrm{d}\mu_\epsilon} + D \le \frac{h_{\mu_\epsilon}(T)}{\int_X \log a \,\mathrm{d}\mu_\epsilon} \le \sup\left\{\frac{h_\mu(T)}{\int_X \log a \,\mathrm{d}\mu} : \mu \in \mathscr{Y}_Z\right\}.$$

As $\int_X \log a \,\mathrm{d}\mu$ is uniformly bounded away from 0 and $\infty$ for all $\mu \in \mathscr{M}_T(X)$, letting $\epsilon \to 0$ yields (6.2) for $\dim_H Z$. The statement for $\dim_H A(\mathscr{Y})$ follows in the same way.

□

By using Theorem 4.17, we can get a lower bound for the Hausdorff dimension of arbitrary Borel sets $Z \subseteq X$:

**Theorem 6.7.** *If* $(X,T)$ *is uniformly expanding conformal and* $Z \subseteq X$ *is a Borel set, then*

$$\dim_H Z \ge \sup\left\{\frac{h_\mu(T)}{\int_X \log a \,\mathrm{d}\mu} : \mu \in \mathscr{M}_T(Z) \cup \mathscr{E}_T(Z)\right\}. \tag{6.3}$$

*In addition, one has*

$$\overline{\dim}_H \mu \ge \frac{h_\mu(T)}{\int_X \log a \,\mathrm{d}\mu}$$

*for each* $\mu \in \mathscr{M}_T(X)$, *and*

$$\underline{\dim}_H \nu = \overline{\dim}_H \nu = \dim_H G_\nu = \frac{h_\nu(T)}{\int_X \log a \,\mathrm{d}\nu}$$

*for each* $\nu \in \mathscr{E}_T(X)$.

*Proof.* In case $\mathscr{M}_T(Z) \cup \mathscr{E}_T(Z) = \varnothing$, we have $-\infty$ on the right side of (6.3) and are done. If $\mu \in \mathscr{M}_T(Z) \cup \mathscr{E}_T(Z)$, then $Z \neq \varnothing$, and by Theorem 4.17 and Theorem 6.3 one has for $s_0 := \dim_H Z$

$$0 = P_Z(-s_0 \log a) \ge h_\mu(T) - s_0 \int_X \log a \,\mathrm{d}\mu.$$

Hence

$$\dim_H Z \geq \frac{h_\mu(T)}{\int_X \log a \, d\mu}.$$

Taking the supremum on the right and infimum on the left side yields the first and second statement, respectively. The third statement follows immediately by $P_{G_\nu}(-s\log a) = h_\nu(T) - s\int_X \log a \, d\nu$ for each $s \geq 0$, and by $\nu(G_\nu) = 1$.

□

**Remark 6.8.**

(a) If $Z$ is $T$-invariant, the above statements follow from Theorem 1 in [PP84]. In this case, $\mu(Z) = 1$ for all $\mu \in \mathscr{E}_T(Z)$, and hence $\mathscr{E}_T(Z) \subseteq \mathscr{M}_T(Z)$. Note that our result holds for arbitrary Borel sets $\varnothing \neq Z \subseteq X$. In addition we are able to consider in the supremum of (6.3) all ergodic measures $\mu$ satisfying $\mu(Z) > 0$. If $Z$ is a subset such that there is any $\mu \in \mathscr{M}_T(X)$ satisfying $\mu(Z) > 0$, then immediately by ergodic decomposition we have that $\mathscr{E}_T(Z) \neq \varnothing$. Thus, for those $Z$, we always obtain a lower bound for $\dim_H Z$, which is a strong result (see also Remark 4.9).

(b) Note that for non-ergodic $\mu \in \mathscr{M}_T(X) \setminus \mathscr{E}_T(X)$ it can happen that $\underline{\dim}_H \mu < \overline{\dim}_H \mu$. For instance, assume $x \in X$ to be a fixed point, and $\mu \in \mathscr{M}_T(X)$ such that $\underline{\dim}_H \mu > 0$. Define $\nu := \alpha\mu + (1-\alpha)\delta_x$ for an $\alpha \in (0,1)$. Then $\overline{\dim}_H \nu \geq \underline{\dim}_H \mu > 0 = \underline{\dim}_H \nu$.

If one combines Theorem 6.4 and Theorem 6.7, one can derive the following:

**Corollary 6.9.** *Define $X_0 := \{x \in X : \exists \mu \in V_T(x) \text{ such that } \overline{\dim}_H \mu = 0\}$ and assume $X_0 \neq \varnothing$. Then under the assumptions of Theorem 6.4 one has* $\dim_H X_0 = 0$.

*Proof.* Suppose $\overline{\dim}_H \mu = 0$, then by Theorem 6.7 one has

$$0 = \overline{\dim}_H \mu = \frac{h_\mu(T)}{\int_X \log a \, d\mu}.$$

This implies $h_\mu(T) = 0$, thus $X_0 \subseteq \tilde{X}$. By Theorem 6.4 the statement follows.

□

If $(X,T)$ is uniformly expanding conformal and in addition saturated (see Definition 5.46), one can combine Theorem 6.3 and Theorem 5.48 to calculate the Hausdorff dimensions of the sets of generic points $G_\mu$.

**Theorem 6.10** (Hausdorff dimension of generic points)**.** *Assume $(X,T)$ to be uniformly expanding conformal and saturated. If $\varnothing \neq \mathscr{Y} \subseteq \mathscr{M}_T(X)$, then*

$$\dim_H A(\mathscr{Y}) = \dim_H \bigcup_{\mu \in \mathscr{Y}} G_\mu = \sup_{\mu \in \mathscr{Y}} \dim_H G_\mu = \sup\left\{ \frac{h_\mu(T)}{\int_X \log a \, d\mu} : \mu \in \mathscr{Y} \right\}.$$

*Proof.* Denote $G := \bigcup_{\mu \in \mathscr{Y}} G_\mu$. As $(X,T)$ is saturated, by Theorem 6.3 and Theorem 5.48 it follows that for each $\mu \in \mathscr{Y}$ one has

$$0 = P_{G_\mu}(-\dim_H G_\mu \log a) = h_\mu(T) - \dim_H G_\mu \int_X \log a \, d\mu.$$

This shows

$$\dim_H G_\mu = \frac{h_\mu(T)}{\int_X \log a \, d\mu} \le \sup_{\mu \in \mathscr{Y}} \dim_H G_\mu = \sup \left\{ \frac{h_\mu(T)}{\int_X \log a \, d\mu} : \mu \in \mathscr{Y} \right\} \le \dim_H G$$

for each $\mu \in \mathscr{Y}$. Now suppose that there exists a $t \in \mathbb{R}$ such that $\sup_{\mu \in \mathscr{Y}} \dim_H G_\mu < t < \dim_H G$. Then, as the pressure functions $s \mapsto P_{G_\mu}(-s \log a)$ are strictly decreasing in $s$, one has

$$h_\mu(T) - t \int_X \log a \, d\mu = P_{G_\mu}(-t \log a) < 0$$

for each $\mu \in \mathscr{Y}$. On the other hand, as $t < \dim_H G$, one also has by Theorem 6.3 and Theorem 5.48

$$0 < P_G(-t \log a) = \sup \left\{ h_\mu(T) - t \int_X \log a \, d\mu : \mu \in \mathscr{Y} \right\},$$

which is a contradiction. Hence $\sup_{\mu \in \mathscr{Y}} \dim_H G_\mu = \dim_H G$. Also, it is easy to see that $G \subseteq A(\mathscr{Y})$. Therefore Theorem 6.6 implies

$$\dim_H A(\mathscr{Y}) = \dim_H G.$$

□

The above theorem can for example be used, to calculate the dimension of level sets. The following corollary generalizes Theorem 4 in [Ols03], which was proven in the context of graph directed self-conformal iterated function systems. Note that our statement admits arbitrary index sets $I$ and arbitrary "conditions" $C$.

**Corollary 6.11** (Relative multifractal spectrum of ergodic averages)**.** *Assume $(X,T)$ to be uniformly expanding conformal and saturated. Fix some set $\emptyset \neq I$, let $\varphi_i, \psi_i : X \to \mathbb{R}$ be continuous functions and $\psi_i > 0$ for all $i \in I$. Let $C \subseteq \mathbb{R}^I$ be arbitrary and define*

$$K_C := \left\{ x \in X : \left( \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \varphi_i(T^j x)}{\sum_{j=0}^{n-1} \psi_i(T^j x)} \right)_{i \in I} \in C \right\}.$$

*Then, if $K_C \neq \emptyset$, one has*

$$\dim_H K_C = \sup \left\{ \frac{h_\mu(T)}{\int_X \log a \, d\mu} : \left( \frac{\int_X \varphi_i \, d\mu}{\int_X \psi_i \, d\mu} \right)_{i \in I} \in C \right\}.$$

*Proof.* Define

$$\mathscr{Y} := \left\{ \mu \in \mathscr{M}_T(X) : \left( \frac{\int_X \varphi_i \, d\mu}{\int_X \psi_i \, d\mu} \right)_{i \in I} \in C \right\}.$$

First note that $\bigcup_{\mu \in \mathscr{Y}} G_\mu \subseteq K_C$. Furthermore, for each $x \in K_C$ and each $\mu \in V_T(x)$ such that there is a subsequence $(n_l)_{l \ge 1}$ satisfying $\lim_{l\to\infty} \delta_{x,n_l} = \mu$, one has

$$C \ni \left( \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \varphi_i(T^j x)}{\sum_{j=0}^{n-1} \psi_i(T^j x)} \right)_{i \in I} = \left( \lim_{l\to\infty} \frac{\sum_{j=0}^{n_l-1} \varphi_i(T^j x)}{\sum_{j=0}^{n_l-1} \psi_i(T^j x)} \right)_{i \in I} = \left( \frac{\int_X \varphi_i \, d\mu}{\int_X \psi_i \, d\mu} \right)_{i \in I}.$$

This shows $\mu \in \mathscr{Y}$, and hence $K_C \subseteq A(\mathscr{Y})$. Therefore

$$\dim_H \bigcup_{\mu \in \mathscr{Y}} G_\mu \leq \dim_H K_C \leq \dim_H A(\mathscr{Y}),$$

and the statement follows from Theorem 6.10.

□

**Remark 6.12.**

(a) The above result can be easily generalized, for instance if one considers asymptotically additive sequences instead of continuous $\varphi_i, \psi_i$. A statement for that case was already given in [Cao13], Theorem E, for compact subsets $C \subseteq \mathbb{R}$.

(b) Note that the upper estimate of $\dim_H K_C$ also holds, if one drops the assumption of $(X, T)$ being saturated. Also, if $K_C$ is Borel measurable, by Theorem 6.7 one has

$$\dim_H K_C \geq \sup\left\{ \frac{h_\mu(T)}{\int_X \log a \, \mathrm{d}\mu} : \mu \in \mathscr{E}_T(K_C) \right\}.$$

**Remark 6.13.** Most results in §6.1 also apply to so-called average conformal $\mathscr{C}^1$-repellers. That is because for those systems, a result similar to Theorem 6.3 holds true (see Theorem B in [Cao13]). Arbitrary $\mathscr{C}^1$-repellers are considered in §6.2.

## §6.2. Repellers on manifolds

Theorem 6.7 can be generalized, if one drops the conformality assumption and considers $\mathscr{C}^1$-repellers. The next definition follows [CWZ14].

**Definition 6.14.** Let $M$ be a smooth $d$-dimensional Riemannian manifold and $f : U \to M$ be a $\mathscr{C}^1$-mapping on an open subset $U \subseteq M$. A compact, $f$-invariant subset $J \subseteq U$ is called $\mathscr{C}^1$-repeller of $f$, if there exist constants $\kappa > 1$ and $K > 0$ such that

$$\| \mathrm{D} f^n(x) u \| \geq K \kappa^n \| u \| \tag{6.4}$$

for all $x \in J$, $u \in T_x M$ and $n \geq 1$.

**Theorem 6.15.** *Let $(J, f|_J)$ be a $\mathscr{C}^1$-repeller. Then for each Borel set $Z \subseteq J$ one has*

$$\dim_H Z \geq \sup\left\{ \frac{h_\mu(f|_J)}{\int_J \log \| \mathrm{D} f \| \, \mathrm{d}\mu} : \mu \in \mathscr{M}_{f|_J}(Z) \cup \mathscr{E}_{f|_J}(Z) \right\}.$$

*Proof.* Assume $\varnothing \neq Z$. By [CWZ14] Theorem 3.1 (1), one has $\dim_H Z \geq s_1$, where $s_1$ is the unique zero of the pressure function

$$s \mapsto P_Z^C(-s \log \| \mathrm{D} f \|).$$

By Proposition 3.18 we know that $P_Z^C(-s \log \| \mathrm{D} f \|) \geq P_Z(-s \log \| \mathrm{D} f \|)$. Thus, if $s_2$ denotes a zero of $P_Z(-s \log \| \mathrm{D} f \|)$, we have $s_1 \geq s_2$. Now if $\mu \in \mathscr{M}_{f|_J}(Z) \cup \mathscr{E}_{f|_J}(Z)$, in the same way as in the proof of Theorem 6.7 one shows that

$$s_2 \geq \frac{h_\mu(f|_J)}{\int_J \log \| \mathrm{D} f \| \, \mathrm{d}\mu},$$

which proves the result.

□

We will finally demonstrate that the zero of pressure of the super-additive potential $(-\log\|\mathrm{D}f^n\|)_{n\geq 1}$ can be used as lower estimate for the Hausdorff dimension of $J$. This was remarked to be unknown in [CWZ14].

**Theorem 6.16.** *Let $(J, f|_J)$ be a $\mathscr{C}^1$-repeller. Define $\Psi^s := (s\log\|\mathrm{D}f^n\|)_{n\geq 1}$ for each $s\in\mathbb{R}$, and*

$$\lambda_1(\mu) := \lim_{n\to\infty}\frac{1}{n}\int_J \log\|\mathrm{D}f^n(x)\|\,\mathrm{d}\mu(x)$$

*for each $\mu\in\mathscr{M}_{f|_J}(J)$. Define also*

$$G_\mu^s := \left\{x\in G_\mu : \lim_{n\to\infty}\frac{1}{n}s\log\|\mathrm{D}f^n(x)\| = s\lambda_1(\mu)\right\},\quad G^s := \bigcup_{\mu\in\mathscr{E}_{f|_J}(J)} G_\mu^s.$$

*Then one has*

$$D := \sup\left\{s\in\mathbb{R} : P_{G^1}(\Psi^{-s})\geq 0\right\} = \sup\left\{\frac{h_\mu(f|_J)}{\lambda_1(\mu)} : \mu\in\mathscr{M}_{f|_J}(J)\right\} \leq \dim_H J.$$

*Proof.* First note that $\Psi^1$ is a continuous, sub-additive potential. By Kingman's sub-additive ergodic theorem and (6.4) it follows that $0<\log\kappa\leq\lambda_1(\mu)<\infty$ for every $\mu\in\mathscr{M}_{f|_J}(J)$. Also note that $D\geq 0$, as $P_{G^1}(\Psi^0) = h_{\mathrm{top}}(f)$. It is easy to see that for each $s\neq 0$ and $\mu\in\mathscr{E}_{f|_J}(J)$ one has $G_\mu^s = G_\mu^1$. As $\dim_H J<\infty$, we know by Theorem 6.15 that $h_{\mathrm{top}}(f)<\infty$. Thus, using Theorem 5.2 and Theorem 5.30,

$$s\mapsto P_{G^1}(\Psi^{-s}) = \sup\left\{h_\mu(f|_J) - s\lambda_1(\mu) : \mu\in\mathscr{E}_{f|_J}(J)\right\}$$

is monotonically decreasing in $s$. Again using Theorem 5.30, we derive that $s\mapsto P_{G_\mu^1}(\Psi^{-s}) = h_\mu(f|_J) - s\lambda_1(\mu)$ is strictly monotonically decreasing in $s$, and

$$P_{G_\mu^1}(\Psi^{-s}) = 0 \quad\text{if and only if}\quad s = s_\mu := \frac{h_\mu(f|_J)}{\lambda_1(\mu)} \tag{6.5}$$

for each $\mu\in\mathscr{E}_{f|_J}(J)$. Now set $\Delta := \sup\left\{s_\mu : \mu\in\mathscr{E}_{f|_J}(J)\right\}$ and note that $0\leq\Delta<\infty$. By (6.5) we have $P_{G_\mu^1}(\Psi^{-\Delta})\leq 0$ for each $\mu\in\mathscr{E}_{f|_J}(J)$. Thus

$$\sup_{\mu\in\mathscr{E}_{f|_J}(J)} P_{G_\mu^1}(\Psi^{-\Delta}) = \sup\left\{h_\mu(f|_J) - \Delta\lambda_1(\mu) : \mu\in\mathscr{E}_{f|_J}(J)\right\} = P_{G^1}(\Psi^{-\Delta})\leq 0.$$

This shows $\Delta\geq D$. Now if $\Delta>D$, there exists a $\mu\in\mathscr{E}_{f|_J}(J)$ such that $\Delta\geq s_\mu>D$. But then by the definition of $D$ and $G^1$ one obtains

$$0 > P_{G^1}(\Psi^{-s_\mu}) \geq P_{G_\mu^1}(\Psi^{-s_\mu}) = 0,$$

which is a contradiction. This shows $\Delta = D$.

To finish the proof, observe that $\lambda_1$ is ergodic decomposable (see Proposition 5.25). Hence

$$P_{G^1}(\Psi^{-s}) = \sup\left\{h_\mu(f|_J) - s\lambda_1(\mu) : \mu\in\mathscr{M}_{f|_J}(J)\right\} \tag{6.6}$$

for each $s \in \mathbb{R}$. By definition one has

$$\Delta = D \le \sup\left\{\frac{h_\mu(f|_J)}{\lambda_1(\mu)} : \mu \in \mathscr{M}_{f|_J}(J)\right\} =: \Lambda < \infty.$$

If $D < \Lambda$, there is a $\mu \in \mathscr{M}_{f|_J}(J) \setminus \mathscr{E}_{f|_J}(J)$ such that

$$D < \frac{h_\mu(f|_J)}{\lambda_1(\mu)} =: s_\mu.$$

By (6.6) that would mean

$$P_{G^1}(\Psi^{-s_\mu}) \ge h_\mu(f|_J) - s_\mu \lambda_1(\mu) = 0,$$

and hence $D \ge s_\mu > D$, which is a contradiction. This shows $D = \Delta = \Lambda$.

By [CWZ14], Theorem 3.3, it follows that $\Delta \le \dim_H J$, which shows the statement.

□


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## A. Appendix

### §A.1. Semi-continuous functions

**Definition A.1.** Let $X$ be a topological space. A function $\varphi : X \to [-\infty,\infty)$ is called upper semi-continuous, if $\{x \in X : \varphi(x) < c\}$ is an open set for each $c \in \mathbb{R}$. It is called lower semi-continuous, if $-\varphi$ is upper semi-continuous.

**Lemma A.2.** *If $X$ is a compact metric space, and $\varphi : X \to [-\infty,\infty)$ is upper semi-continuous, then the following holds:*

*(a) One has $\sup_{x\in X} \varphi(x) < \infty$, hence $\varphi$ is quasi-integrable with respect to every Borel probability measure $\mu$ on $X$.*

*(b) There exists a decreasing sequence of continuous functions $f_n : X \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$.*

*(c) If $x, x_1, x_2, x_3, \dots \in X$ such that $\lim_{n\to\infty} x_n = x$, it follows that*

$$\limsup_{n\to\infty} \varphi(x_n) \le \varphi(x).$$

*(d) If $\mu, \mu_1, \mu_2, \mu_3, \dots$ are Borel probability measures on $X$ such that $\lim_{n\to\infty} \mu_n = \mu$, one has*

$$\limsup_{n\to\infty} \int_X \varphi \, \mathrm{d}\mu_n \le \int_X \varphi \, \mathrm{d}\mu.$$

*Proof.* For (a)-(c), see e.g. [Kel98], Lemma 4.1.5. Statement (d) follows from (a), (b) and the monotone convergence theorem. □

## §A.2. Ergodic theory for quasi-integrable functions

**Lemma A.3.** *Let $(X,T)$ be a dynamical system, $\varphi : X \to [-\infty,+\infty]$ be Borel measurable and $\mu \in \mathscr{E}_T(X)$. Assume $\varphi \circ T = \varphi$ $\mu$-almost everywhere. Then $\varphi$ is constant $\mu$-almost everywhere.*

*Proof.* Define $X_\alpha := f^{-1}(\{\alpha\})$ for $\alpha \in [-\infty,\infty]$. Then one has

$$\begin{aligned}\mu\big(T^{-1}(X_\alpha) \triangle X_\alpha\big) &= \mu\Big(\big(T^{-1}(X_\alpha) \setminus X_\alpha\big) \cup \big(X_\alpha \setminus T^{-1}(X_\alpha)\big)\Big) \\ &\le \mu\big(\{x \in X : f(T(x)) \ne f(x)\}\big) = 0.\end{aligned}$$

Therefore by [Wal82], Theorem 1.5, we obtain $\mu(X_{-\infty}) \in \{0,1\}$ and similarly $\mu(X_{+\infty}) \in \{0,1\}$. Now suppose $\mu(X_{-\infty}) = 1$, then $f(x) = -\infty$ for $\mu$-almost every $x \in X$, and we are done (similarly for $\mu(X_{+\infty}) = 1$). Hence we may assume $-\infty < f < \infty$ $\mu$-almost everywhere, but this case can be proven like Theorem 1.6 (iii) [Wal82]. □

**Lemma A.4.** *Let $(X,\mathfrak{B},\mu)$ be a probability space and $\varphi : \Omega \to [-\infty,+\infty]$ be a quasi-integrable function with respect to $\mu$. Then one has the following:*

*(a)* $-\varphi$ *is quasi-integrable with respect to $\mu$ satisfying* $\int_\Omega -\varphi \, d\mu = -\int_\Omega \varphi \, d\mu$.

*(b) If $(Y,\mathfrak{A},\nu)$ is another probability space such that $T : Y \to X$ is Borel measurable and $\mu = \nu \circ T^{-1}$, one has*

$$\int_X \varphi \, d\mu = \int_Y \varphi \circ T \, d\nu.$$

*(c) If $\psi : X \to [-\infty,+\infty]$ is quasi-integrable with respect to $\mu$ such that $\varphi \le \psi$ $\mu$-almost everywhere, one has*

$$\int_X \varphi \, d\mu \le \int_X \psi \, d\mu.$$

*Proof.* The proofs follow easily from the definition of quasi-integrability. □

**Lemma A.5.** *Let $(X,\mathfrak{B},\mu)$ be a probability space and $\varphi,\psi : X \to [-\infty,+\infty]$ be quasi-integrable functions with respect to $\mu$, such that $\int_X \varphi \, d\mu + \int_X \psi \, d\mu$ is well-defined. Then $\varphi+\psi$ is well-defined $\mu$-almost everywhere and quasi-integrable with respect to $\mu$, and furthermore*

$$\int_X (\varphi+\psi) \, d\mu = \int_X \varphi \, d\mu + \int_X \psi \, d\mu.$$

*Proof.* First let $a,b \in [-\infty,+\infty]$ be arbitrary numbers such that $a+b$ is well-defined. It is then easy to see that

$$0 \le \max(a+b,0) \le \max(a,0) + \max(b,0). \tag{A.1}$$

Now suppose $\int_X \varphi^+ \, d\mu, \int_X \psi^+ \, d\mu < \infty$. Then $\varphi^+,\psi^+ : X \to [0,\infty]$ are integrable, and therefore $\varphi^+ + \psi^+$ is integrable too. In addition $\mu(\{\varphi = \infty\}) = \mu(\{\psi = \infty\}) = 0$, which means that $\varphi+\psi$ is well-defined $\mu$-almost everywhere. By (A.1)

$$0 \le \int_X (\varphi+\psi)^+ \, d\mu \le \int_X (\varphi^+ + \psi^+) \, d\mu = \int_X \varphi^+ \, d\mu + \int_X \psi^+ \, d\mu < \infty$$

follows, which shows that $(\varphi+\psi)^+$ is integrable. Hence $\varphi+\psi$ is quasi-integrable. Define $\varphi_N := \max(\varphi,-N)$ and $\psi_N := \max(\psi,-N)$ for every $N \geq 1$. Then $\varphi_N \geq \varphi_{N+1} \geq \varphi$ and $\lim_{N\to\infty}\varphi_N = \varphi$. Furthermore $\varphi_N^+ = \varphi^+$ and $0 \leq \varphi_N^- \leq N$ for all $N \geq 1$. As $\mu$ is a probability measure, all $\varphi_N^-$ are integrable. Thus

$$\int_X |\varphi_N| \,\mathrm{d}\mu = \int_X \varphi^+ \,\mathrm{d}\mu + \int_X \varphi_N^- \,\mathrm{d}\mu < \infty,$$

which shows that $\varphi_N$ is integrable for every $N \geq 1$. The above properties hold also for all $\psi_N$, $N \geq 1$. By monotone convergence

$$\lim_{N\to\infty}\int_X \varphi_N \,\mathrm{d}\mu = \int_X \varphi \,\mathrm{d}\mu \quad \text{and} \quad \lim_{N\to\infty}\int_X \psi_N \,\mathrm{d}\mu = \int_X \psi \,\mathrm{d}\mu$$

follows. Therefore by Lemma A.4

$$\int_X (\varphi+\psi)\,\mathrm{d}\mu \leq \int_X (\varphi_N+\psi_N)\,\mathrm{d}\mu = \int_X \varphi_N \,\mathrm{d}\mu + \int_X \psi_N \,\mathrm{d}\mu \to \int_X \varphi \,\mathrm{d}\mu + \int_X \psi \,\mathrm{d}\mu$$

as $N \to \infty$. This shows $\int_X(\varphi+\psi)\,\mathrm{d}\mu = -\infty$ in case $\int_X \varphi\,\mathrm{d}\mu = -\infty$ or $\int_X \psi\,\mathrm{d}\mu = -\infty$. Thus

$$\int_X (\varphi+\psi)\,\mathrm{d}\mu = \int_X \varphi \,\mathrm{d}\mu + \int_X \psi \,\mathrm{d}\mu.$$

The case $\int_X \varphi^- \,\mathrm{d}\mu, \int_X \psi^- \,\mathrm{d}\mu < \infty$ follows by using Lemma A.4 (a). □

*Proof of Proposition 5.10.* Let $\int_X \varphi^+ \,\mathrm{d}\mu < \infty$. Define $\varphi_n(x) := \sum_{i=0}^{n-1}\varphi(T^i x)$ for all $x \in X$ and $n \geq 1$. Then $(\varphi_n)_{n=1}^\infty$ is a sub-additive sequence such that $\int_X \varphi_1^+ \,\mathrm{d}\mu = \int_X \varphi^+ \,\mathrm{d}\mu < \infty$. By Kingman's ergodic theorem there exists some quasi-integrable $\psi : X \to [-\infty,\infty)$ satisfying $\psi \circ T = \psi$ and $\lim_{n\to\infty}\frac{1}{n}\varphi_n = \psi$ $\mu$-almost everywhere. The theorem also tells us that $\int_X \varphi_n^+ \,\mathrm{d}\mu < \infty$ for all $n \geq 1$ and

$$\int_X \psi \,\mathrm{d}\mu = \lim_{n\to\infty}\frac{1}{n}\int_X \varphi_n \,\mathrm{d}\mu.$$

We also have $\int_X (\varphi\circ T^i)^+ \,\mathrm{d}\mu = \int_X \varphi^+ \circ T^i \,\mathrm{d}\mu = \int_X \varphi^+ \,\mathrm{d}\mu < \infty$ for each $i \geq 0$. Thus by Lemma A.4 and Lemma A.5

$$\frac{1}{n}\int_X \varphi_n \,\mathrm{d}\mu = \frac{1}{n}\sum_{i=0}^{n-1}\int_X \varphi\circ T^i \,\mathrm{d}\mu = \int_X \varphi \,\mathrm{d}\mu.$$

As $\mu$ is ergodic, this shows together with Lemma A.3

$$\psi = \int_X \psi \,\mathrm{d}\mu = \int_X \varphi \,\mathrm{d}\mu$$

$\mu$-almost everywhere.

The case $\int_X \varphi^- \,\mathrm{d}\mu < \infty$ works in the same way. □

*Proof of Proposition 5.11.* If $\varphi$ is bounded from above and below, the statement follows from Theorem 2.2. Now let $\varphi \leq C$ for some $0 \leq C < \infty$. Define $\varphi_N := \max(\varphi,-N)$ for $N \in \mathbb{N}$, then $(\varphi_N)_{N\in\mathbb{N}}$ is a decreasing and $\lim_{N\to\infty}\varphi_N = \varphi$. Furthermore $|\varphi_N|$ is bounded and integrable for all $N \geq 1$. Using the ergodic decomposition for bounded functions as well as monotone convergence, we obtain

$$\int_X \varphi \,\mathrm{d}\mu = \lim_{N\to\infty}\int_X \varphi_N \,\mathrm{d}\mu = \lim_{N\to\infty}\int_{\mathscr{E}_T(X)}\left(\int_X \varphi_N \,\mathrm{d}\nu\right)\mathrm{d}m_\mu(\nu).$$

Next define $F_N : \mathscr{E}_T(X) \to [-N, C]$, $\nu \mapsto \int_X \varphi_N \, \mathrm{d}\nu$. Then $(F_N)_{N\in\mathbb{N}}$ is a decreasing sequence of $\mathrm{m}_\mu$-integrable functions. Applying monotone convergence twice yields

$$\lim_{N\to\infty} \int_{\mathscr{E}_T(X)} \left( \int_X \varphi_N \, \mathrm{d}\nu \right) \mathrm{dm}_\mu(\nu) = \int_{\mathscr{E}_T(X)} \left( \int_X \lim_{N\to\infty} \varphi_N \, \mathrm{d}\nu \right) \mathrm{dm}_\mu(\nu).$$

But this means

$$\int_X \varphi \, \mathrm{d}\mu = \int_{\mathscr{E}_T(X)} \left( \int_X \varphi \, \mathrm{d}\nu \right) \mathrm{dm}_\mu(\nu).$$

In case $C \le \varphi$ for some $-\infty < C \le 0$, we can obtain the above statement by applying Lemma A.4 (a). Now suppose $\varphi$ is quasi-integrable with respect to $\mu$. As $\varphi^+$ and $\varphi^-$ are bounded from below, we have

$$\begin{aligned} \int_X \varphi \, \mathrm{d}\mu &= \int_X \varphi^+ \, \mathrm{d}\mu - \int_X \varphi^- \, \mathrm{d}\mu \\ &= \int_{\mathscr{E}_T(X)} \left( \int_X \varphi^+ \, \mathrm{d}\nu \right) \mathrm{dm}_\mu(\nu) - \int_{\mathscr{E}_T(X)} \left( \int_X \varphi^- \, \mathrm{d}\nu \right) \mathrm{dm}_\mu(\nu). \end{aligned}$$

Now let $F^\pm : \mathscr{E}_T(X) \to [0, \infty]$, $F^\pm(\nu) := \int_X \varphi^\pm \, \mathrm{d}\nu$. This means by Lemma A.4 (a)

$$\int_X \varphi \, \mathrm{d}\mu = \int_{\mathscr{E}_T(X)} F^+ \, \mathrm{dm}_\mu - \int_{\mathscr{E}_T(X)} F^- \, \mathrm{dm}_\mu = \int_{\mathscr{E}_T(X)} F^+ \, \mathrm{dm}_\mu + \int_{\mathscr{E}_T(X)} -F^- \, \mathrm{dm}_\mu.$$

By Lemma A.5 $F^+ + (-F^-)$ is well-defined $\mathrm{m}_\mu$-almost everywhere and quasi-integrable with respect to $\mathrm{m}_\mu$, satisfying

$$\begin{aligned} &\int_{\mathscr{E}_T(X)} F^+ \, \mathrm{dm}_\mu + \int_{\mathscr{E}_T(X)} -F^- \, \mathrm{dm}_\mu = \int_{\mathscr{E}_T(X)} \left(F^+ + (-F^-)\right) \mathrm{dm}_\mu \\ &= \int_{\mathscr{E}_T(X)} \left( \int_X \varphi^+ \, \mathrm{d}\nu - \int_X \varphi^- \, \mathrm{d}\nu \right) \mathrm{dm}_\mu(\nu) = \int_{\mathscr{E}_T(X)} \left( \int_X \varphi \, \mathrm{d}\nu \right) \mathrm{dm}_\mu(\nu) \end{aligned}$$

Thus

$$\int_X \varphi \, \mathrm{d}\mu = \int_{\mathscr{E}_T(X)} \left( \int_X \varphi \, \mathrm{d}\nu \right) \mathrm{dm}_\mu(\nu).$$

□

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# Index

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- dass mir die Promotionsordnung der Fakultät bekannt ist,
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Bei der Auswahl und Auswertung des Materials sowie der Herstellung des Manuskripts haben mich folgende Personen unterstützt: Prof. Dr. Martina Zähle, FSU Jena.

Ich habe die gleiche, in wesentlichen Teilen ähnliche bzw. eine andere Abhandlung noch bei keiner anderen Hochschule als Dissertation eingereicht.

Jena, den 03.10.2017 Marc Rauch