The Eilenberg-Moore spectral sequence in group cohomology

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Zusammenfassung

Wir untersuchen den Einsatz der Eilenberg-Moore-Spektralsequenz als Hilfsmittel zur Berechnung in der Gruppen-Kohomologie (vgl. [Rus87]). Zuerst leiten wir die Grundlagen der Anwendung der Eilenberg-Moore-Spektralsequenz in der Gruppen-Kohomologie her. Dazu präsentieren wir eine explizite Konstruktion des relevanten Diagramms mittels simplizialer Mengen. Danach konstruieren wir Analoga zu den Steenrod-Operationen auf der E_1-Seite der Eilenberg-Moore-Spektralsequenz, welche hier mithilfe der Koszulauflösung beschrieben wird. Hierfür nutzen wir die gutartigen Eigenschaften der Standardauflösung. Schlussendlich bringen wir die Eilenberg-Moore-Spektralsequenz an zwei Beispielen zur Anwendung, nämlich $32\Gamma_3 f$ und $SU_3(n)$, wobei ersteres bereits in [Oeh16] behandelt wurde, dort allerdings nur unter Verwendung der Lyndon-Hochschild-Serre-Spektralsequenz. Dabei werden wir die zusätzliche Struktur in Form der Kohomologieoperationenanaloga ausnutzen und gleichzeitig die Grenzen des Ansatzes aufzeigen, insbesondere hinsichtlich des Falls, dass die Eilenberg-Moore-Spektralsequenz nicht auf der E_2-Seite kollabiert.

Abstract

We examine the use of the Eilenberg-Moore spectral sequence as a tool in computation of group cohomology as in [Rus87]. We first present an explicit construction of the relevant diagram via simplicial sets to allow the application of the Eilenberg-Moore spectral sequence to group cohomology. Then we construct analogues of the Steenrod squares on the E_1-page of the Eilenberg-Moore spectral sequence given via the Koszul resolution by harnessing the good properties of the bar resolution. Finally we apply the Eilenberg-Moore spectral sequence to two examples ($32\Gamma_3 f$ and $SU_3(n)$, for the first see also [Oeh16]) utilizing the additional structure provided by the faux cohomology operations, highlighting the limitations of this approach, especially when the Eilenberg-Moore spectral sequence does not collapse on the E_2-page.

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Errata

The argument in section 4.2 unfortunately turned out to be flawed. The constructed cohomology operations seem to be well-defined, but are lacking some properties required of Steenrod operations. That is the definition 4.2.2 does not give Steenrod operations. We present to this end an argument by the second reviewer professor Henn below. This argument causes serious doubts whether the goal of section 4.2 is achievable at all. Furthermore this error causes the computations with the Eilenberg-Moore spectral sequence in chapter 5 to be based on a faulty assumption.

Henn's argument

In the case of the principal fibration associated to a central group extension $0 \to \mathbb{Z}/2 \to G \to Q \to 1$ the attempt to define Steenrod operations on the Koszul complex for the polynomial algebra $H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2) \cong \mathbb{F}_2[x_0,x_1,\dots]$ inducing operations on $\mathrm{Tor}^{**}_{H^*(K(\mathbb{Z}/2,2);\mathbb{F}_2)}(H^*(Q;\mathbb{F}_2),\mathbb{F}_2)$ which agree with the Steenrod operations on the E_2-page of the Eilenberg-Moore spectral sequence as proposed in [Rus87] is not tractable.

Recall that the mod-2 cohomology of $K(\mathbb{Z}/2,2)$ is isomorphic to the free unstable algebra $UF(2)$ and that the polynomial generators x_i of degree 2^i+1, $i \geq 0$, are given as $Sq^I x_0$ where I runs through all admissable monomials of excess ≤ 1. The unstable algebra $UF(2)$ contains the free unstable module $F(2)$ which has an additive basis given by all $Sq^I x_0$ with I running through all admissable monomials of excess ≤ 2, with those of excess 2 corresponding to all iterated squares $x_i^{2^k}$, $k \geq 0$, of the polynomial generators x_i, $i \geq 0$.

Lemma: In $UF(2) = \mathbb{F}_2[x_0,x_1,\dots]$ the total Steenrod operation $Sq = Sq^0 + Sq^1 + \cdots$ satisfies

$$Sq(x_i) = \begin{cases} x_i + x_{i-1}^2 + x_{i+1} + x_i^2 & i > 1 \\ x_i + x_{i+1} + x_i^2 & i = 0,1 \end{cases}$$

Proof. It is clear that $Sq^0 x_i = xi$, $Sq^{2^i} x_i = x_{i+1}$ and $Sq^{2^i+1} x_i = x_i^2$ for all $i \geq 0$. For $j \geq 0$ the element $Sq^j x_i$ must be of the form $Sq^I x_0$ for some admissible sequence of excess at most 2. The element $Sq^{2n} x_i$ cannot be a square so by instability it must be zero unless $2n = 0$ or $2n = |x_i| - 1 = 2^i$. Furthermore, because of $Sq^{2n+1} = Sq^1 Sq^{2n}$ we get $Sq^{2n+1} x_i = 0$ unless $2n = 0$ or $2n = 2^i$. If $2n = 2^i$ we obtain $Sq^{2^i+1} x_i = x_i^2$, and if $2n = 0$ and $i > 1$ we obtain $Sq^1 x_i = Sq^1 Sq^{2^{i-1}} x_{i-1} = Sq^{2^{i-1}+1} x_{i-1} = x_{i-1}^2$. This gives the result for $i > 1$.

If $i = 0$ we get $Sq^1 x_0 = x_1$ and if $i = 1$ we get $Sq^1 x_1 = Sq^1 Sq^1 x_0 = 0$. □

Proposition: Suppose M is an unstable module with a $UF(2)$-module structure such that

- the A-module structure and the $UF(2)$-module structure are compatible via the Cartan formula
- as $UF(2)$-module M is generated by elements u_i with $|u_i| = 2^i + 1$
- there is a (necessarily unique) map $d : M \to UF(2)$ which is $UF(2)$-linear, A-linear and which satisfies $d(u_i) = x_i$.

Then there exists $\beta \in \mathbb{F}_2$ such that

$$x_2 u_1 + x_1(u_2 + \beta(x_1 u_0 + x_0 u_1)) = 0,$$

in particular M is not isomorphic to the free $UF(2)$-module generated by the elements u_i for $i \geq 0$.

Proof. Suppose there is such an M. By instability we must have

$$Sq(u_1) = u_1 + \alpha_1 x_0 u_0 + \beta_1 u_2 + \beta_2 x_1 u_0 + \beta_3 x_0 u_1 + \gamma_0 x_1 u_1 + \gamma_2 x_0^2 u_0.$$

for some $\alpha_1, \beta i, \gamma j \in \mathbb{F}_2$. Then the linearity assumptions on d and the lemma imply $\alpha_1 = 0$, $\beta_1 = 1$, $\beta_2 = \beta_3 =: \beta$, $\gamma_0 = 1$ and $\gamma_2 = 0$. In particular we have

$$Sq^1(u_1) = 0, \quad Sq^2(u_1) = u_2 + \beta(x_1 u_0 + x_0 u_1), \quad Sq^3(u_1) = x_1 u_1.$$

Then the Cartan formula shows

$$Sq^2 Sq^3(u_1) = Sq^2(x_1 u_1) = x_2 u_1 + x_1(u_2 + \beta(x_1 u_0 + x_0 u_1))$$

for some $\beta \in \mathbb{F}_2$. On the other hand the Adem relations give $Sq^2 Sq^3 = Sq^5 + Sq^4 Sq^1$ and hence $Sq^2 Sq^3 u_1 = 0$ by the lemma and instability. □

Remark: More precisely the proposition shows that there is relation in degree 8. What happens in higher degrees is not really relevant.

Corollary: The Koszul complex $K^\bullet$ of the polynomial algebra $UF(2)$ does not admit the structure of a DGA for which each K^i is an unstable module with a compatible $UF(2)$-module structure (compatible with the A-module structure with respect to the Cartan formula) for which the Koszul-differentials are both $UF(2)$-linear and A-linear.

1 Introduction

The study of finite groups often tries to reduce its object of investigation to more manageable (that is mostly smaller) parts. One major technique is to find a normal subgroup $N \triangleleft G$ and view the group G via the corresponding extension.

$$1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1 \tag{$*$}$$

This approach is well understood and the classification of finite simple groups gives a list of all possible final constituents. These extensions can be understood in the terms of the group cohomology $H^*(Q; N)$.

However a lot of questions about the structure of p-groups P are still open and seem to be ill-fitted for the above approach. Here one can hope to garner some knowledge about P directly from the group cohomology $H^*(P; \mathbb{F}_p)$.

In toto the cohomology $H^*(G; M)$ of a group G with coefficients M is a useful object, however its computation is itself a venerable task. So all help is appreciated and the tool of spectral sequences offers a way to utilize an extension like $(*)$ to split the problem into computing the cohomology of smaller groups plus some assembly work to get back $H^*(G; M)$.

There are several different flavors of spectral sequences, each giving different advantages or applicability in different situations. For group cohomology probably the most used spectral sequence is the Lyndon-Hochschild-Serre spectral sequence. In contrast the Eilenberg-Moore spectral sequence has had only few applications in group cohomology, with the most notable one probably being [Rus87].

In this work we first look at a central extension

$$1 \to Z \hookrightarrow G \twoheadrightarrow Q \to 1$$

for which we want to utilize the Eilenberg-Moore spectral sequence. The derivation involves the diagram

$$\begin{array}{ccc} BZ & = & BZ \\ \downarrow & & \downarrow \\ E \simeq BG & \longrightarrow & PK(Z,2) \\ \downarrow & & \downarrow \\ BQ & \longrightarrow & K(Z,2) \end{array}$$

where E is the pullback of the lower square and should be homotopy equivalent to the space BG. The author was unable to find a proof for this arguably obvious equivalence in the literature, so we present an argument via simplicial sets proving this assertion in chapter 3. This argument looks a bit over-engineered, but no more appealing alternative was found.

One hindrance to usage of the Eilenberg-Moore spectral sequence was that many arguments containing spectral sequences exploit additional structure, which is quite often provided by cohomology operations in the form of Steenrod squares, which were not readily available to the desired extent for the Eilenberg-Moore spectral sequence. In [Rus87] Steenrod squares were used for the E_1-page of the Eilenberg-Moore spectral sequence without these operations being rigorously introduced, casting some doubt on the argumentation. Whereas the situation is well understood from the E_2-page onward (see for example [Smi70]) it is actually somewhat questionable whether it is possible to equip the E_1-page with its algebraic nature with cohomology operations which have a rather more geometric nature.

In chapter 4 we then give an explicit construction for operations, that behave like Steenrod squares on the E_1-page of the Eilenberg-Moore spectral sequence, recovering the operations used in [Rus87]. For this we construct an embedding of the Koszul complex into the bar complex. On the latter we have more structure allowing us to define well-behaved operations which we then transfer to the Koszul complex.

Finally in chapter 5 we apply the Eilenberg-Moore spectral sequence to the computation of the group cohomology for the groups $32\Gamma_3 f$ and $SU_3(n)$ for $n \in \{4, 8\}$. The first of these computations incorporates material from [Oeh16], where a disagreement in several computations of the cohomology of $32\Gamma_3 f$ was resolved by an explicit approach. However in [Oeh16] the Eilenberg-Moore spectral sequence was not used since the Lyndon-Hochschild-Serre spectral sequence was a more well-known tool, which sufficed for the task at hand. Here we try to perform the task while using the Eilenberg-Moore spectral sequence discovering some issues regarding the determination of the multiplicative structure of the cohomology. Thus we will resort to the Lyndon-Hochschild-Serre spectral sequence to help us out. The second example of the special unitary groups on the other hand will exhibit the difficulties in the case that the Eilenberg-Moore spectral sequence does not collapse on the E_2-page.

2 Preliminaries

2.1 Notations and conventions

Throughout this work G will denote a finite group and most of the time it will be a p-group for a prime $p \in \mathbb{P}$ (which will more often than not be 2). Furthermore k will be a commutative ring, which will specialise most times to $\mathbb{Z}$ or a finite field $\mathbb{F}_q$ for a prime-power q. Then kG is the group ring. Moreover all rings are assumed to include a unit. Most of the occurring symbols are listed with an explanation in the table on page 82.

2.2 Topological background

We will employ some topology to derive the spectral sequences. For this we need fibrations. A thorough introduction to this topic can be found in [Hat02].

Definition 2.2.1: A continuous map $E \xrightarrow{\pi} B$ is a (Serre) fibration, if it has the homotopy lifting property for all simplicial complexes X. That is for the following commutative diagram there always exists a map Φ.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \\ \big\downarrow & \overset{\Phi}{\nearrow} & \big\downarrow \pi \\ X \times [0,1] & \xrightarrow{\varphi} & B \end{array}$$

In the following we will restrict the treatment to Serre fibrations (as opposed to Hurewicz fibrations, which assume the homotopy lifting property for all spaces) and thus omit the qualification and simply call them fibrations. The preimage $F := \pi^{-1}(b_0)$ of a point $b_0 \in B$ is called a fiber and any two choices for b_0 in the same path component of B result in homotopy equivalent fibers.

One important aspect of fibrations is that they admit a long exact sequence in homotopy arising from a fibration.

Proposition 2.2.2 ([Hat02, Theorem 4.41]): Let $F \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} B$ be a fibration with base points $b_0 \in B$ and $x_0 \in F = \pi^{-1}(b_0)$. Then there exists the following long exact sequence of homotopy groups.

$$\ldots \xrightarrow{\pi_*} \pi_{n+1}(B, b_0) \longrightarrow \pi_n(F, x_0) \xrightarrow{\iota_*} \pi_n(E, x_0) \xrightarrow{\pi_*} \pi_n(B, b_0) \longrightarrow \ldots$$

Furthermore we will encounter the Eilenberg-MacLane spaces, which are spaces with very well behaved homotopy.

Definition 2.2.3: Let n be a positive integer and G be an abelian group (or any finite group, if $n = 1$). The Eilenberg-MacLane space $K(G, n)$ is the up to homotopy equivalence unique space such that for $x_0 \in K(G, n)$

$$\pi_k(K(G,n), x_0) = \begin{cases} G & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

In the special case of $n = 1$ these are also denoted $BG := K(G, 1)$ and called classifying spaces.

2.3 Simplicial Sets

Simplicial sets allow a more categorical approach to homology and homotopy; furthermore they offer the means to do combinatorial calculations which are not obviously possible for the more direct approaches like singular homology. We will use [Wei94, § 8] as basis for the following summary of the necessary theory.

Let Δ be the category with the sets $[n] := \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}_0$ as objects and monotonic, non-decreasing maps as morphisms. Then a simplicial object in a category $\mathcal{A}$ is a contravariant functor $\Xi : \Delta^{\mathrm{op}} \to \mathcal{A}$.

The morphisms in Δ are generated by the face maps $\varepsilon_i : [n-1] \to [n]$ and the degeneracy maps $\eta_i : [n+1] \to [n]$ for $i \in \{0, 1, \ldots, n\}$. They are given as follows.

$$\varepsilon_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \qquad \eta_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

Note that every morphism in Δ can be factored into an epi followed by a mono and those can be decomposed into degeneracy and face maps respectively. We will denote the images under Ξ of the objects by $\Xi_n := \Xi([n])$ and the morphisms by $s_i := s_i^{(n)} := \Xi(\eta_i)$ with $\eta_i : [n+1] \to [n]$ and $d_i := d_i^{(n)} := \Xi(\varepsilon_i)$ with $\varepsilon_i : [n-1] \to [n]$. Note that we drop the superscript (n) if it is implicitly given by the context.

We will need more specific knowledge about classifying spaces BG and Eilenberg-MacLane spaces $K(Z, 2)$ understood as simplicial sets. For the former there is a ready answer in [Wei94] and the latter we will construct via the Dold-Kan correspondence.

Fact 2.3.1 ([Wei94]): The classifying space BG for a group G is given as a simplicial set by $BG_n = G^n$. The operations of s_k and d_k on $BG_n = G^n$ are given for $0 \leq k \leq n$ by

$$s_k : BG_n \to BG_{n+1} : (g_0, g_1, \ldots, g_{n-1}) \mapsto (g_0, g_1, \ldots, g_{k-1}, 1, g_k, \ldots, g_{n-1})$$

and for $0 \neq k \neq n$

$$d_k : BG_n \to BG_{n-1} : (g_0, g_1, \ldots, g_{n-1}) \mapsto (g_0, g_1, \ldots, g_{k-2}, g_{k-1}g_k, g_{k+1}, \ldots, g_{n-1})$$

as well as $d_0(g_0, \ldots, g_{n-1}) = (g_1, \ldots, g_{n-1})$ and $d_n(g_0, \ldots, g_{n-1}) = (g_0, \ldots, g_{n-2})$.

Theorem 2.3.2 (Dold-Kan, [Wei94, Theorem 8.4.1])
For an abelian category $\mathcal{A}$ there is an equivalence of categories between the simplicial objects $\mathcal{SA}$ and the bounded below chain complexes $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ mapping simplicial homotopy to homology.

$$\mathcal{SA} \underset{\bar{K}}{\overset{N}{\rightleftarrows}} \mathrm{Ch}_{\geq 0}(\mathcal{A})$$

Here N is the so-called normalized chain complex functor. We will however only need its inverse functor $\bar{K}$. It is often denoted K, but we already use this for the Eilenberg-MacLane spaces, so that we notate it as $\bar{K}$. For a chain complex $C = (C_*, d)$ the functor $\bar{K}$ is given on objects as $\bar{K}(C)_n = \bigoplus_\eta C_p^{(\eta)}$ where η ranges over all epimorphisms $\eta : [n] \to [p]$ for all $p \leq n$ and each $C_p^{(\eta)}$ is a copy of C_p. For any morphism $\theta : [m] \to [n]$ in Δ now $\bar{K}(\theta) : \bar{K}(C)_n \to \bar{K}(C)_m$ is determined componentwise on $C_p^{(\eta)}$ by the epi-mono-factorisation

$$\begin{array}{ccc} [m] & \xrightarrow{\theta} & [n] \\ \downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} \\ [q] & \xhookrightarrow{\varepsilon} & [p] \end{array}$$

whence $\bar{K}(\theta)$ maps $C_p^{(\eta)}$

- by natural identification to $C_p^{(\eta')}$, if $p = q$ (i. e. ε is the identity),
- by d (the differential of the chain complex) to $C_{p-1}^{(\eta')}$, if $p = q + 1$ and $\varepsilon = \varepsilon_p$,
- to zero otherwise.

Now we determine $K(Z, 2)$ for an abelian group Z as image under $\bar{K}$ of the chain complex C_*^Z concentrated at degree 2 and there being isomorphic to Z. For a concise representation of $K(Z, 2)$ (and to a lesser extent BG) we define notation for two specific sorts of morphisms in Δ.

Definition 2.3.3: In the abstract simplex category Δ we set for $n \in \mathbb{N}_0$

$$\theta_i := \theta_i^{(n)} : [n] \to [1] : k \mapsto \begin{cases} 0 & \text{if } k \leq i, \\ 1 & \text{if } i < k, \end{cases}$$

for $0 \leq i < n$ and

$$\theta_{ij} := \theta_{ij}^{(n)} : [n] \to [2] : k \mapsto \begin{cases} 0 & \text{if } k \leq i, \\ 1 & \text{if } i < k \leq j, \\ 2 & \text{if } j < k, \end{cases}$$

for $0 \leq i < j < n$.

The non-trivial components of $K(Z, 2)$ are indexed by $\theta_{ij}^{(n)}$ and isomorphic to Z. We will denoted them by by $Z_{ij} := Z_{ij}^{(n)}$, where we again omit the superscript (n) if the degree is clear from the context.

The components of BG will similarly be indexed by $\theta_i^{(n)}$, all being isomorphic to G. We will denote them by $G_i := G_i^{(n)}$. Since unlike Z above G does not need to be abelian, we have to be a bit more careful. We explicitly define an ordering for the components of BG in the obvious manner $BG_n := G_0^{(n)} \times G_1^{(n)} \times \cdots \times G_{n-1}^{(n)}$. Thus every element in BG_n has a well-defined representation as $(g_0, g_1, \ldots, g_{n-1})$ with $g_i \in G_i$ for $0 \leq i < n$.

Now we have to determine the effects of the basic operations $s_k = \bar{K}(\varepsilon_k)$ and $d_k = \bar{K}(\eta_k)$ on these components to have a good grasp of these simplicial spaces.

Fact 2.3.4: The operations of s_k and d_k on $K(Z, 2)_n = \bar{K}(C_*^Z)_n$ are given componentwise for $0 \leq k \leq n$ by

$$s_k : K(Z, 2)_n \to K(Z, 2)_{n+1} : Z_{ij} \overset{\text{id}}{\longmapsto} Z_{\varepsilon_k(i)\varepsilon_k(j)}$$

and

$$d_k : K(Z, 2)_n \to K(Z, 2)_{n-1} : Z_{ij} \mapsto \begin{cases} 0 & \begin{array}{l}\text{if } k = 0 = i \text{ or } k = i + 1 = j \\ \text{or } k = n = j + 1,\end{array} \\ Z_{\eta_{k-1}(i)\eta_{k-1}(j)} & \begin{array}{l}\text{otherwise} \\ \text{(by natural identification).}\end{array} \end{cases}$$

Proof. We utilize the Dold-Kan correspondence. Since the complex C_*^Z is concentrated in one degree all differentials are trivial and we can only have maps by natural identification or zero maps.

First we check $s_k : K(Z,2)_n \to K(Z,2)_{n+1}$, which corresponds to $\eta_k : [n+1] \to [n]$. We look at how the component Z_{ij} corresponding to θ_{ij} with $0 \le i < j < n+1$ is mapped and hence the relevant square is (2.1).

$$\begin{array}{ccc} [n+1] & \xrightarrow{\eta_k} & [n] \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \theta_{ij}} \\ [q] & \xhookrightarrow{\varepsilon} & [2] \end{array} \tag{2.1}$$

Since $\theta_{ij} \circ \eta_k$ is surjective, we have $q = 2$ and $\varepsilon = \mathrm{id}$. Furthermore $\eta = \theta_{i,j+1}$ exactly if $i < k$ and $j \ge k$. If $j < k$ we get $\eta = \theta_{ij}$ and if $i \ge k$ we get $\eta = \theta_{i+1,j+1}$. Together we retrieve $Z_{ij} \overset{\mathrm{id}}{\longmapsto} Z_{\varepsilon_k(i)\varepsilon_k(j)}$.

We finish with $d_k : K(Z,2)_n \to K(Z,2)_{n-1}$ with square (2.2).

$$\begin{array}{ccc} [n-1] & \xrightarrow{\varepsilon_k} & [n] \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \theta_{ij}} \\ [q] & \xhookrightarrow{\varepsilon} & [2] \end{array} \tag{2.2}$$

Now $\theta_{ij} \circ \varepsilon_k$ is surjective, except for $k = 0 = i$ (which misses 0), $k = i+1 = j$ (which misses 1) and $k = n = j+1$ (which misses 2). In these cases $q = 1$ and d_k vanishes, otherwise $q = 2$ and $\varepsilon = \mathrm{id}$. In the non-exceptional cases we have $\eta = \theta_{i,j-1}$ exactly if $i < k \le j$. For $j < k$ we have $\eta = \theta_{ij}$ and for $i \ge k$ we have $\eta = \theta_{i-1,j-1}$. Together we retrieve $Z_{ij} \mapsto Z_{\eta_{k-1}(i)\eta_{k-1}(j)}$. □

Remark 2.3.5: One can try to improve the notation for BG and make it more similar to that of $K(Z,2)$. However we do not have enough structure to allow making the component-wise description entirely work. More precisely the part $d_k : BG_n \to G^{(n-1)}_{k-1} : (g_0, g_1, \ldots, g_{n-1}) \mapsto g_{k-1}g_k$ cannot be split up with respect to the domain without doing some additional lifting.

If we add the convention that if several components of BG_n are mapped into $G^{(n-1)}_i$ they are multiplied in order, we can retrieve a description, which looks more similar, but sadly is not categorical. It takes the form of

$$s_k : BG_n \to BG_{n+1} : G_i \overset{\mathrm{id}}{\longmapsto} G_{\varepsilon_k(i)}$$

and

$$d_k : BG_n \to BG_{n-1} : G_i \rightsquigarrow \begin{cases} 1 & \text{if } k = 0 = i \text{ or } k = n = i+1, \\ G_{\eta_{k-1}(i)} & \text{otherwise.} \end{cases}$$

This similarity seems to hint, that something like the Dold-Kan correspondence also governs the case of BG. However one the technical level there seems to be quite a large gap (e. g. the prerequisite of an abelian category in theorem 2.3.2).

Remark 2.3.6: We can simplify the description of d_k even more by introducing the convention that any components with invalid indices vanish. That is in the case of BG letting $G_i^{(n)} := 0$ for $i < 0$ or $i > n$ and in the case of $K(Z,2)$ letting $Z_{ij}^{(n)} := 0$ for $i < 0$ or $i \geq j$ or $j > n$. With this the result takes the form of

$$d_k : BG_n \to BG_{n-1} : G_i \rightsquigarrow G_{\eta_{k-1}(i)},$$
$$d_k : K(Z,2)_n \to K(Z,2)_{n-1} : Z_{ij} \mapsto Z_{\eta_{k-1}(i)\eta_{k-1}(j)}.$$

Remark 2.3.7: Sadly this is not symmetric. One would wish to have d_k acting as η_k. Instead it acts as η_{k-1}. This is due to the fact, that θ_i and θ_{ij} are defined by specifying the upper boundaries. If we would create lower boundary variants $\theta^i := \theta_{i-1}$ and $\theta^{i,j} := \theta_{i-1,j-1}$, then d_k would indeed act as η_k, but s_k would no longer act as ε_k. Having both would however not simplify matters. Thus an entirely satisfying notation is eluding us at this point.

We need one further aspect in simplicial sets, namely path-spaces (and the path-loop-fibration). They are constructed via a functor $P : \Delta \to \Delta$ defined by $P[n] := [n+1]$ via adding an element -1 at the beginning of $[n]$ and identifying the resulting thing with $[n+1]$. This induces a functor, i. e. it behaves well on the morphisms of Δ.

Definition 2.3.8: Let Ξ be a simplicial object and P the functor above. The path space $P\Xi$ of Ξ is then given as the composition of P with Ξ.

In effect the maps d_k and s_k in the path space are given by the maps d_{k+1} and s_{k+1} in the initial simplicial object.

We can now move to the path-loop-fibration. The loop space ΩX is in the topological setting given as $\Omega X = [S^1, X]$ and there is a similar definition for the simplicial setting. However we will only use one property, namely that the loop space functor Ω acts nicely on Eilenberg-MacLane spaces in both cases, i. e. we have

$$\Omega K(G, n+1) = K(G, n).$$

Fact 2.3.9 ([Wei94, §8.3.1]): Let Ξ be a simplicial object, then the projection $\pi : P\Xi \to \Xi$ given as $d_0 : \Xi_{n+1} \to \Xi_n$ on $P\Xi_n$ induces the path-loop-fibration

$$\Omega\Xi \overset{\iota}{\hookrightarrow} P\Xi \overset{\pi}{\twoheadrightarrow} \Xi.$$

2.4 Group cohomology

There are two equivalent approaches to group cohomology. On one side there is the algebraic approach and on the other side there is the topological approach. For the first one [Eve91] is a good introductory text, for the second one we refer to [Ben91].

Definition 2.4.1: Let G be a finite group, k a commutative ring and M a kG-module. The cohomology of G is defined as

$$H^*(G; M) := \mathrm{Ext}^*_{kG}(k, M).$$

Definition 2.4.2: Let G be a finite group and k a commutative ring. The cohomology of G is defined as

$$H^*(G; k) := H^*(BG; k).$$

The geometric approach of definition 2.4.2 is amenable to theoretic treatment since one can apply many general principles about the properties of spaces and their functors. The algebraic approach of definition 2.4.1 is more suited to computational treatment where for example explicit calculations with computer algebra systems are desired. Both approaches yield equivalent results where they are applicable (see for example [Ben91, theorem 2.2.3]).

If the commutative ring k has positive characteristic p, then cohomology of a finite group G is essentially given by the cohomology of its Sylow-p-subgroup (see for example [CTV$^+$03, proposition 2.3.8]). In this case it is thus only sensible to restrict the treatment to p-groups.

2.4.1 The cocycle of a central extension

To understand finite groups G one often factors into a normal subgroup $N \triangleleft G$ and a quotient $Q := G/N$ giving an extension

$$1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1.$$

An especially useful case is, if the normal subgroup is central, i. e. $N =: Z \subseteq Z(G)$ because then the action of Z on Q is trivial and we have a central extension

$$0 \to Z \hookrightarrow G \twoheadrightarrow Q \to 1.$$

These central extensions are classified by the second cohomology group $H^2(Q, Z)$. For a thorough introduction to this (and general low dimensional group cohomology) see [Bro82], we only present the bits relevant to us.

An extension is classified by a cohomology element $\alpha \in H^2(Q, Z)$ which is represented by a cocycle α. Since all the viewpoints are equivalent we name all the occurring objects α even though the are only equivalent and not the same. A cohomology class $\alpha \in H^2(Q, Z)$ is equivalent to a homotopy class $\alpha \in [BQ, K(Z, 2)]$ by the general theorem $H^n(Q, Z) \cong [BQ, K(Z, n)]$. Furthermore looking at such a homotopy class through the lens of simplicial complexes, one sees that α can also be understood as a map $\alpha : Q \times Q \to Z$. It is constructed by choosing a section

$s : Q \hookrightarrow G$ (which is just a map of sets) of the projection $G \twoheadrightarrow Q$. Note that we require s to be normalized, that is s maps the neutral element of Q to the neutral element of G. This has the consequence, that α is also normalized, meaning that it vanishes if one of its arguments is a neutral element. More specifically for any $q_1, q_2 \in Q$ the map α is given by the formula

$$\alpha(q_1, q_2) = s(q_1)s(q_2)s(q_1q_2)^{-1}.$$

Now α is a cocycle and hence fulfills the cocycle condition. That is it vanishes under the differential. This takes the explicit form

$$\alpha(q_1, q_2) - \alpha(q_0q_1, q_2) + \alpha(q_0, q_1q_2) - \alpha(q_0, q_1) = 0. \tag{2.3}$$

2.5 Spectral sequences

Spectral sequences are a very useful tool in computing some specific cohomology as well as in theoretic arguments. The reference book is [McC01]. The following describes the cohomology situation, there is a dual homology situation, which works mostly the same – for details see the book. In general we want to compute a graded k-module $\hat{A}$ with a given filtration

$$\hat{A} \cdots \supseteq F^n\hat{A} \supseteq F^{n+1}\hat{A} \supseteq \cdots \{0\}$$

which should be exhaustive (that is the union of all $F^n\hat{A}$ is all of $\hat{A}$) and separated (that is the intersection of all $F^n\hat{A}$ is $\{0\}$), otherwise one cannot expect to be in control of the whole situation. More precisely the spectral sequence computes $E_0^{pq} = F^p\hat{A}^{p+q}/F^{p+1}\hat{A}^{p+q}$, which can be reassembled into $\hat{A}$, but in general only up to extension problems. That is one possibly needs information how the different slices of $\hat{A}$ interact, which is often not readily available.

To facilitate the spectral sequence, we now let $\hat{A}$ be the cohomology of a filtered differential graded k-module A with differential d of degree $+1$, that is $\hat{A}^* = H^*(A, d)$ (where the filtration is compatible with the differential).

A spectral sequence now arises from $E_1^{pq} = H^{p+q}(F^pA/F^{p+1}A)$ with differential $d_1 : E_1^{**} \to E_1^{**}$ of bidegree $(1, 0)$ induced by d. Now we iteratively get $E_{r+1}^{**} = H^{**}(E_r, d_r)$, where each d_r of bidegree $(r, 1-r)$ is induced by d. Under some additional assumptions (such as boundedness of the filtration) this stabilizes and converges to $E_\infty^{pq} \cong E_0^{pq} = F^pH^{p+q}A/F^{p+1}H^{p+q}A$.

The usefulness for computations now arises from the fact, that quite often there exists an alternative and rather easy way to obtain E_2^{**}, without complete knowledge of A or H^*A. However in this case there is in general no a priori knowledge of the differentials d_r since the action of the differential d on A is unknown as A is unknown. Thus often additional arguments are required to compute E_∞. And even then the aforementioned extension problems may still occur.

2.5.1 Serre spectral sequence

This is one of the first if not the first spectral sequence. We start with a fibration

$$F \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} B.$$

The target is the computation of the cohomology $H^*(E;k)$. The Serre spectral sequence (sometimes called Leray-Serre spectral sequence) allows us to reduce this to a problem on the cohomology of the base space B and the fiber F. More precisely we have the following.

Theorem 2.5.1 (Serre, [McC01, Theorem 5.2])
Let $F \overset{\iota}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} B$ be a fibration where the base space B is path-connected and the fiber F is connected. Additionally assume that $\pi_1(B, b_0)$ with $b_0 \in B$ operates trivially on the fiber. Further let k be a commutative ring. Then there exists a spectral sequence (E_r^{**}, d_r) converging to $H^*(E;k)$ as an algebra with

$$E_2^{pq} \cong H^p(B; H^q(F;k)).$$

Moreover the cup product $\smile$ in $H^*(E;k)$ is related to the product $*$ on the E_2-page via $u * v = (-1)^{p'q} u \smile v$ for $u \in E_2^{pq}$ and $v \in E_2^{p'q'}$.

Remark 2.5.2: If the fundamental group $\pi_1(B, b_0)$ does not operate trivially on the fiber then the spectral sequence still exists, but the coefficients $H^q(F;k)$ have to be replaced by a system of local coefficients.

2.5.2 Eilenberg-Moore spectral sequence

We give a short summary of the necessary terms to define the Eilenberg-Moore spectral sequence for a more complete introduction see [McC01, §7.1].

Differential homological algebra

We start with a differential graded algebra (Γ, d_Γ) over a commutative ring k. From there we can define a differential graded module M over Γ as a differential graded module over k with an action $\varphi : \Gamma \otimes_k M \to M$, that is compatible with all structure. Then the tensor product $M \otimes_\Gamma N$ for a right Γ-module M and a left Γ-module N with actions φ and ψ respectively is given as cokernel in the following exact sequence.

$$M \otimes_k \Gamma \otimes_k N \xrightarrow{\varphi \otimes 1 - 1 \otimes \psi} M \otimes_k N \longrightarrow M \otimes_\Gamma N \longrightarrow 0.$$

Now to describe the derived functor of $\otimes_\Gamma$ we need proper projective resolutions. For a left Γ-module (N, d_N) that is a sequence (P^n, d^n) of projective Γ-modules and a morphism ε such that

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\delta} & (P^{n+1}, d^{n+1}) & \xrightarrow{\delta} & (P^n, d^n) & \xrightarrow{\delta} \cdots \xrightarrow{\delta} & (P^0, d^0) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow \varepsilon & & \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & (N, d_N) & \longrightarrow & 0
\end{array}$$

is a chain equivalence of sequences of graded modules over the graded algebra Γ (forgetting the differentials for now) and we require three subresolutions of projective graded k-modules induced by this.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \ker d^2 & \longrightarrow & \ker d^1 & \longrightarrow & \ker d^0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ker d_N & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \operatorname{im} d^2 & \longrightarrow & \operatorname{im} d^1 & \longrightarrow & \operatorname{im} d^0 & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{im} d_N & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H(P^2, d^2) & \longrightarrow & H(P^1, d^1) & \longrightarrow & H(P^0, d^0) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H(N, d_N) & \longrightarrow & 0
\end{array}$$

From this resolution we construct a total complex which is a differential graded module $(\mathrm{total}(P), D)$ as $\mathrm{total}(P)^k = \bigoplus_{m+n=k}(P^m)^n$ with differential $D = \sum_m(\delta + (-1)^m d^m)$. This inherits a Γ-action from P and thus we can form the Γ-module $(M \otimes_\Gamma \mathrm{total}(P), \partial)$ with $\partial = d_M \otimes 1 + (-1)^q 1 \otimes D$ on $M^q \otimes \mathrm{total}(P)$. With this laid out, we come to the definition.

Definition 2.5.3: Given a differential graded algebra Γ and Γ-modules M, N in the setting above we define

$$\mathrm{Tor}_\Gamma^{**}(M, N) := H^{**}(M \otimes_\Gamma \mathrm{total}(P), \partial).$$

The spectral sequence

We start with a fibration $F \xhookrightarrow{\iota} E \xrightarrow{\pi} B$ and a map $X \xrightarrow{f} B$. From this we construct the pullback E_f as in the following diagram.

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ E_f & \longrightarrow & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & B \end{array} \tag{2.4}$$

Now the Eilenberg-Moore spectral sequence allows the computation of the cohomology $H^*(E_f)$ with knowledge of the other constituents of the pullback square. For this we understand $H^*(X)$ and $H^*(E)$ as differential graded modules over the differential graded algebra $H^*(B)$ via the maps f^* and π^*. For an even more geometric approach we note the paper [Smi70], where the Eilenberg-Moore spectral sequence is presented as a special type of Künneth spectral sequence.

Theorem 2.5.4 ([McC01, Theorem 7.15])
Let k be a commutative ring and $F \xhookrightarrow{\iota} E \xrightarrow{\pi} B$ be a fibration where the base space B is simply-connected and the fiber F is connected. Let $f : X \to B$ be a continuous map and E_f be the pullback of $f : X \to B \leftarrow E : \pi$. Then there exists a spectral sequence (E_r^{**}, d_r) converging to $H^*(E_f; k)$ as an algebra with

$$E_2^{pq} \cong \operatorname{Tor}^{pq}_{H(B;k)}(H(X;k), H(E;k)).$$

2.6 Resolutions

To compute the spectral sequences we are going to use two resolutions: the bar resolution and the Koszul resolution. We will employ the expositions from [McC01, §7] in an abbreviated form.

Let Γ be a differential graded algebra over a commutative ring k and M be a Γ-module. We present the desired two resolutions of M, which will need different assertions about the situation. First only some mild assumptions will be required, but for the Koszul resolutions some rather specific prerequisites are necessary.

2.6.1 Bar resolution

For the bar resolution we assume additionally, that the degree zero part of Γ is isomorphic to k (in other words that Γ is connected) and we set $\bar{\Gamma} := \{\gamma \in$

$\Gamma | \deg\gamma > 0\}$ to be everything else. We define the stages of the resolution

$$B^{-n}(\Gamma, M) := \Gamma \otimes_k \underbrace{\bar{\Gamma} \otimes_k \cdots \otimes_k \bar{\Gamma}}_{n \text{ times}} \otimes_k M$$

where each generator is notated as

$$\gamma[\gamma_1|\dots|\gamma_n]m := \gamma \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes m$$

with $\gamma \in \Gamma$, $\gamma_i \in \bar{\Gamma}$ and $m \in M$; it has bidegree $(-n, I)$, where the internal degree I is $\deg\gamma + \sum_{i=1}^n \deg\gamma_i + \deg m$. Let us denote for this section only $\tilde{x} := (-1)^{1+\deg x}x$. In the later sections we will specialise to characteristic 2 and drop all signs.

Denote by d_Γ and d_M the internal differentials of Γ and M respectively. Then the internal differential of the bar resolution is given on the generators by

$$\delta^{-n}(\gamma[\gamma_1|\dots|\gamma_n]m) := (d_\Gamma\gamma)[\gamma_1|\dots|\gamma_n]m + \sum_{i=1}^{n} \tilde{\gamma}[\tilde{\gamma}_1|\dots|\tilde{\gamma}_{i-1}|d_\Gamma\gamma_i|\gamma_{i+1}|\dots|\gamma_n]m$$
$$+ \tilde{\gamma}[\tilde{\gamma}_1|\dots|\dots|\tilde{\gamma}_n](d_M m).$$

Furthermore the external differential on the generators is given as follows. The signs in this are a bit tricky and the following variant is in agreement with [May67].

$$d_B^{-n}(\gamma[\gamma_1|\dots|\gamma_n]m) := -\tilde{\gamma}\gamma_1[\gamma_2|\dots|\gamma_n]m$$
$$- \sum_{i=1}^{n-1} \tilde{\gamma}[\tilde{\gamma}_1|\dots|\tilde{\gamma}_{i-1}|\tilde{\gamma}_i\gamma_{i+1}|\gamma_{i+2}|\dots|\gamma_n]m$$
$$- \tilde{\gamma}[\tilde{\gamma}_1|\dots|\dots|\tilde{\gamma}_{n-1}](\gamma_n m)$$

Now with ε being the normal action of Γ on M we complete the resolution.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_B} & B^{-2}(\Gamma, M) & \xrightarrow{d_B} & B^{-1}(\Gamma, M) & \xrightarrow{d_B} & B^0(\Gamma, M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\varepsilon} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

With the necessary prerequisites there is a multiplicative structure on the bar complex which is given by the Eilenberg-Zilber map. These will be given in the case of $M = k$ being a field which is the case we will examine. Let $\beta, \gamma \in \Gamma$ and $\beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n \in \bar{\Gamma}$, then we have

$$\beta[\beta_1|\dots|\beta_m] \cdot \gamma[\gamma_1|\dots|\gamma_n] = \sum_{(m,n)\text{-shuffles } \sigma} \beta\gamma[\omega_{\sigma(1)}|\omega_{\sigma(2)}|\dots|\omega_{\sigma(m+n)}]$$

where $\omega_i := \beta_i$ for $i \leq m$ and $\omega_{m+i} := \gamma_i$ otherwise. An $(m,n)-$shuffle σ is a permutation from S_{m+n} such that $\sigma(1) < \sigma(2) < \cdots < \sigma(m)$ and $\sigma(m+1) < \sigma(m+2) < \cdots < \sigma(m+n)$. Note that we have omitted the signs, as this will only be applied to the case of characteristic 2.

2.6.2 Koszul resolution

For the Koszul resolution we need two additional prerequisites. First the module we want to resolve must be the trivial module, that is $M = k$. Second Γ has to be a free graded algebra (that is no non-trivial differentials) on a generating set J. The latter exhibits different structures for different characteristics. If the ground ring has characteristic 2 we get $\Gamma = k[J]$. Whereas in odd characteristic we have to differentiate between generators of odd and even degree and get $\Gamma = \Lambda(J^{\mathrm{odd}}) \otimes_k k[J^{\mathrm{even}}]$, where Λ stands for the exterior algebra. We will now assume that the characteristic is 2 and describe only this case, since the general case includes quite some additional machinery (like divided power algebras). We again use the description from [McC01].

We need one additional ingredient, the desuspension s^{-1} which operates on J by shifting everything down one degree, that is $(s^{-1}J)^n = J^{n+1}$. Now the Koszul complex is given as

$$\mathcal{K}(\Gamma) := \Lambda(s^{-1}J) \otimes_k \Gamma.$$

We often denote the generators by x_0, x_1, x_2, etc. and hence the generating set is $J = \{x_0, x_1, x_2, \ldots, x_m\}$, where the number of generators may possibly be infinite, and the desuspension of x_i by $u_i := s^{-1}(x_1)$. Then the Koszul complex has a bidegree induced by $\deg(1 \otimes x_i) = (0, \deg(x_i))$ and $\deg(u_i \otimes 1) = (-1, \deg(x_i))$. Furthermore there is a differential $d_{\mathcal{K}} : \mathcal{K}(\Gamma)^{n-1,*} \to \mathcal{K}(\Gamma)^{n,*}$ given by

$$\begin{aligned} d_{\mathcal{K}}(1 \otimes x_i) &= 0, \\ d_{\mathcal{K}}(u_i \otimes 1) &= 1 \otimes x_i \end{aligned}$$

continued as a derivation – this would imply signs, if we where not in characteristic 2 – giving

$$d_{\mathcal{K}}(u_{i_1} \cdots u_{i_n} \otimes \gamma) = \sum_{j=1}^{n} u_{i_1} \cdots u_{i_{j-1}} u_{i_{j+1}} \cdots u_{i_n} \otimes \gamma x_{i_j} .$$

Together with the projection $\varepsilon : \Gamma \to k$ we get the Koszul resolution with trivial internal differential.

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{-2,*}(\Gamma) & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{-1,*}(\Gamma) & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{0,*}(\Gamma) = \Gamma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

3 Spectral sequences in group cohomology

We want to determine the group cohomology $H^*(G; M)$ of a finite group G and a finitely generated kG-module M, where k is a commutative ring. If G is not simple we can find a normal subgroup $N \lhd G$ and get an extension

$$1 \to N \hookrightarrow G \twoheadrightarrow Q := G/N \to 1.$$

To stitch together the information about the cohomology of the sub- and quotient-group we will employ spectral sequences.

3.1 Lyndon-Hochschild-Serre spectral sequence

This is an application of the Serre spectral sequence. For a detailed account see [Ben91, section 3.5]. The group extension induces a fibration $BN \hookrightarrow BG \twoheadrightarrow BQ$ and thus the Serre spectral sequence leads to the following (note that special care is required for coefficients in form of a module).

Theorem 3.1.1 ([CTV$^+$03, Theorem 5.4.4])
Let G be a group and N a normal subgroup and M a kG-module for a field k. Set $Q := G/N$. Then there exists a spectral sequence (E_r^{**}, d_r) converging to $H^*(G; M)$ as an algebra with

$$E_2^{pq} \cong H^p(Q; H^q(N; M)).$$

We want to examine the Eilenberg-Moore spectral sequence, but will use the Lyndon-Hochschild-Serre spectral sequence in the course of our examination. It is one of the most common tools in this regard and will be useful here too. Additionally we will in the end do a comparison between these two spectral sequences.

3.2 Eilenberg-Moore spectral sequence in group cohomology

Now we restrict to central extensions, i. e. $Z \subseteq Z(G)$,

$$0 \to Z \hookrightarrow G \twoheadrightarrow Q := G/Z \to 1.$$

This does not hurt us, since the important case of p-groups always has non-trivial center. Note that we will use additive notation for the operation in Z since Z is abelian.

3.2.1 Construction

We construct a situation from the central extension where we can apply the Eilenberg-Moore spectral sequence to compute $H^*(G)$. This is described in [Ben91, §3.7]. The first ingredient is the path-loop-fibration in our case over the space $K(Z, 2)$ as follows

$$\Omega K(Z,2) = BZ \hookrightarrow PK(Z,2) \xrightarrow{\pi_Z} K(Z,2).$$

Where for the path space we choose a base point $z_0 \in K(Z, 2)$ and the map π_Z gives to each path its endpoint. The second ingredient is the extension cocycle $\alpha : BQ \to K(Z, 2)$ as described in section 2.4.1. With both ingredients together we may now create the pullback square.

$$\begin{array}{ccc} E & \longrightarrow & PK(Z,2) \\ \downarrow & & \downarrow{\scriptstyle \pi_Z} \\ BQ & \xrightarrow{\alpha} & K(Z,2) \end{array}$$

Theorem 3.2.1 (Rusin)
In the above situation E is homotopy equivalent to the classifying space BG.

Remark 3.2.2: The author was unable to find a proof of this in the literature. In [Rus87] this is deemed obvious and in [Ben91] there is a more elaborate technical introduction, but the actual assertion is also said to be easy and no further explanation is given. The following is an explicit but rather long proof; there should exist a more elegant argument.

Proof. On the right side we have the path-loop-fibration $BZ \hookrightarrow PK(Z, 2) \twoheadrightarrow K(Z, 2)$. The pullback of a fibration is again a fibration with equal fiber, so we get $BZ \hookrightarrow E \twoheadrightarrow BQ$ on the left side. We can employ the long exact homotopy sequence of proposition 2.2.2 with base points $x_0 \in E$ and $b_0 \in BQ$ and retrieve

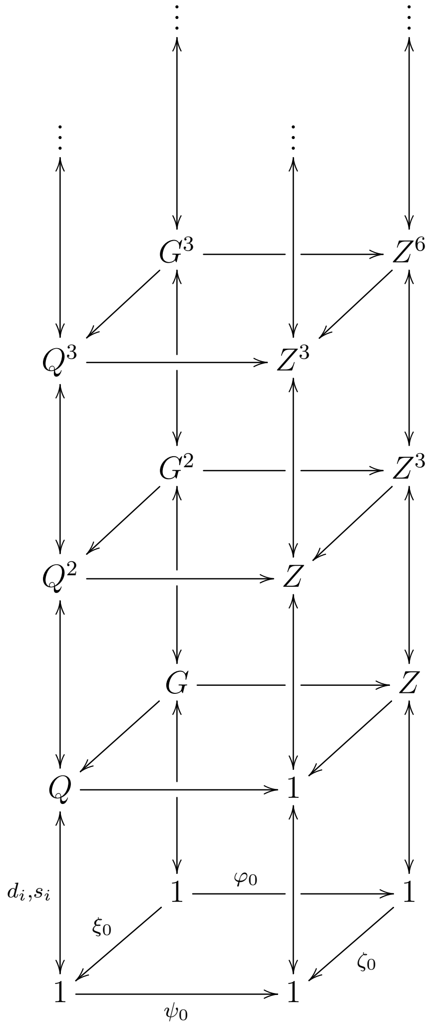
$$\ldots \longrightarrow \pi_2(BQ, b_0) \longrightarrow \pi_1(BZ, x_0) \longrightarrow \pi_1(E, x_0) \longrightarrow$$
$$\longrightarrow \pi_1(BQ, b_0) \longrightarrow \pi_0(BZ, x_0) \longrightarrow \ldots$$

which reduces to

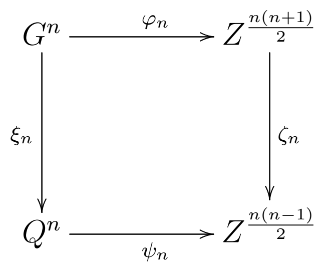
$$\ldots \longrightarrow 0 \longrightarrow Z \longrightarrow \pi_1(E, x_0) \longrightarrow Q \longrightarrow 0 \longrightarrow \ldots$$

showing, that E is a space BX for a suitable group X.

It remains to show that $X \cong G$. We will do this by explicitly exhibiting BG as a possible completion of the square, which will imply by universality that it is the pullback. We do this by using simplicial sets as described in section 2.3. So our situation is as in the following diagram.



The goal now is to construct the maps $\varphi : BG \to PK(Z,2)$, $\xi : BG \to BQ$, $\zeta : PK(Z,2) \to K(Z,2)$ and $\psi : BQ \to K(Z,2)$. In degree n we have the square



and between the degrees we have the simplicial maps d_i and s_i. We denote the components in degree n of the classifying spaces Q_i and G_i with $0 \leq i < n$ and those of $K(Z,2)$ as Z_{ij} with $0 \leq i < j < n$ (and $PK(Z,2)$ appropriately shifted) so that we can use the description of the simplicial maps from fact 2.3.1 and fact 2.3.4.

Now ζ_n is given as the simplicial map $d_0 : K(Z,2)_{n+1} \to K(Z,2)_n$ and $\xi_n : G^n \to Q^n$ is simply the projection. Furthermore $\psi_2 : Q^2 \to Z$ is given as the cocycle α. Now to give the remaining maps we first have to scrutinize the group extension to retrieve a map $\tau : G \to Z$.

The cocycle α is defined via a specific section σ of the projection $G \twoheadrightarrow Q$; more precisely

$$\alpha(q_1, q_2) = \iota^{-1}(\sigma(q_1)\sigma(q_2)\sigma(q_1 q_2)^{-1})$$

and σ maps neutral element to neutral element. Now for every element $g \in G$ we have a unique factorization $g = \sigma(\pi(g))z_g$ for an element z_g from $\iota(Z)$. With this we define $\tau : G \to Z : g \mapsto \iota^{-1}(z_g)$. Then $\tau \circ \iota$ is the identity since σ is normalized. Thus τ is the desired section of ι and furthermore τ is also normalized. Now we can explicitly write down φ_n and ψ_n. We begin by defining ψ_n, we do this componentwise on the component Z_{ij} for $0 \leq i < j < n$ as

$$\psi_n : Q^n \to Z_{ij} : (q_0, \ldots, q_{n-1}) \mapsto \alpha\left(\prod_{k=i}^{j-1} q_k, q_j\right) - \alpha\left(\prod_{k=i+1}^{j-1} q_k, q_j\right)$$

now for φ_n we analogously give the mapping componentwise, first for $i = 0$ and $0 < j < n$ we have

$$\varphi_n : G^n \to Z_{0j} : (g_0, \ldots, g_{n-1}) \mapsto \tau\left(\prod_{k=0}^{j-2} g_k\right) - \tau\left(\prod_{k=0}^{j-1} g_k\right)$$

and on the other hand for $0 < i < j < n$ we have

$$\varphi_n : G^n \to Z_{ij} : (g_0, \ldots, g_{n-1}) \mapsto \alpha\left(\prod_{k=i-1}^{j-2} \pi(g_k), \pi(g_{j-1})\right) - \alpha\left(\prod_{k=i}^{j-2} \pi(g_k), \pi(g_{j-1})\right)$$

which is just like ψ_n on $Z_{i-1,j-1}$.

Now to verify that this is well-defined we take the following steps. During these we will use that α is a cocycle which manifests itself in the cocycle condition

$$\alpha(q_1, q_2) - \alpha(q_0 q_1, q_2) + \alpha(q_0, q_1 q_2) - \alpha(q_0, q_1) = 0$$

for all $q_0, q_1, q_2 \in Q$.

(i) We first need a small observation about τ, namely that

$$\alpha(\pi(g_0), \pi(g_1)) = \tau(g_0 g_1) - \tau(g_0) - \tau(g_1).$$

(ii) We show that the horizontal squares commute.

$$\begin{array}{ccc} G^n & \xrightarrow{\varphi_n} & Z^{\frac{n(n+1)}{2}} \\ {\scriptstyle \xi_n}\downarrow & & \downarrow{\scriptstyle \zeta_n} \\ Q^n & \xrightarrow[\psi_n]{} & Z^{\frac{n(n-1)}{2}} \end{array}$$

(iii) We show that the front squares commute.

$$\begin{array}{ccc} Q^{n+1} & \xrightarrow{\psi_{n+1}} & Z^{\frac{n(n+1)}{2}} \\ {\scriptstyle d_i,s_i}\updownarrow & & \updownarrow{\scriptstyle d_i,s_i} \\ Q^n & \xrightarrow[\psi_n]{} & Z^{\frac{n(n-1)}{2}} \end{array}$$

(iv) We show that the back squares commute.

$$\begin{array}{ccc} G^{n+1} & \xrightarrow{\varphi_{n+1}} & Z^{\frac{(n+1)(n+2)}{2}} \\ {\scriptstyle d_i,s_i}\updownarrow & & \updownarrow{\scriptstyle d_i,s_i} \\ G^n & \xrightarrow[\varphi_n]{} & Z^{\frac{n(n+1)}{2}} \end{array}$$

Now the left and right squares commute by general theory and everything together shows that BG is a possible candidate for the space E.

Now let BX be the actual pullback, then we get by the pullback property the diagram

$$\begin{array}{ccccc} BG & & & & \\ & \searrow^{\cdots} & & \searrow & \\ & & BX & \longrightarrow & PK(Z,2) \\ & \searrow & \downarrow & & \downarrow \\ & & BQ & \longrightarrow & K(Z,2) \end{array}$$

with a map $\lambda : BG \to BX$. Now in degree 1 this is a map $\lambda_1 : G \to X$ which is a set isomorphism, since both G and X are set-isomorphic to $Q \times Z$ and the corresponding diagram commutes. Now in degree 2 we have a map $\lambda_2 : G^2 \to X^2$ which sits in the following commutative square.

$$\begin{array}{ccc} G^2 & \xrightarrow{\lambda_2} & X^2 \\ \downarrow{\scriptstyle d_i} & & \downarrow{\scriptstyle d_i} \\ G & \xrightarrow{\lambda_1} & X \end{array}$$

For y_0, y_1 from either G or X we have $d_0(y_0, y_1) = y_1$ and $d_2(y_0, y_1) = y_0$ and hence $\lambda_2(g_0, g_1) = (\lambda_1(g_0), \lambda_1(g_1))$ for $g_0, g_1 \in G$. Furthermore $d_1(y_0, y_1) = y_0 y_1$ which means, that λ transports the group structure and thus X is isomorphic to G as a group.

To conclude the proof it remains to execute the steps (i) to (iv) above.

Step (i) To treat combinations of α and π we use the decomposition $g = \sigma(\pi(g))\iota(\tau(g))$ and the fact that the images of ι are central as well as the homomorphism property of ι. With these facts we can transform

$$\begin{aligned}
\alpha(\pi(g_0), \pi(g_1)) &= \iota^{-1}\Big(\sigma(\pi(g_0))\sigma(\pi(g_1))\sigma(\pi(g_0)\pi(g_1))^{-1}\Big) \\
&= \tau\Big(g_0\iota(\tau(g_0))^{-1}g_1\iota(\tau(g_1))^{-1}\big(g_0g_1\iota(\tau(g_0g_1))^{-1}\big)^{-1}\Big) \\
&= \tau\Big(g_0\iota(\tau(g_0))^{-1}g_1\iota(\tau(g_1))^{-1}(g_0g_1)^{-1}\iota(\tau(g_0g_1))\Big) \\
&= \tau\Big(g_0g_1(g_0g_1)^{-1}\iota(\tau(g_0))^{-1}\iota(\tau(g_1)^{-1})\iota(\tau(g_0g_1))\Big) \\
&= \tau\Big(\iota(\tau(g_0))^{-1}\iota(\tau(g_1))^{-1}\iota(\tau(g_0g_1))\Big) \\
&= \tau\Big(\iota\big(\tau(g_0g_1)\tau(g_0)^{-1}\tau(g_1)^{-1}\big)\Big) \\
&= \tau(g_0g_1) - \tau(g_0) - \tau(g_1).
\end{aligned}$$

Thus we acquire the desired formula.

Step (ii) The map ζ_n maps all $Z_{ij} \in PK(Z,2)_n = K(Z,2)_{n+1}$ onto 0 if $i = 0$ or $Z_{i-1,j-1}$ otherwise. Since ξ_n is just the projection, a comparison of ψ_n on one hand and φ_n for the components Z_{ij} with $i > 0$ on the other hand directly shows, that the horizontal squares commute.

Step (iii) We have to check commutativity for the maps $s_0, \ldots, s_n$ as well as $d_0, \ldots, d_{n+1}$. We start by checking s_k and begin with $(q_0, q_1, \ldots, q_{n-1}) \in Q^n$ in the lower left corner, which is mapped to $(q_0, q_1, \ldots, q_{k-1}, 1, q_k, \ldots, q_{n-1}) =: (\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_n)$ in the upper left corner. On the right side of the diagram Z_{ij} is mapped to $Z_{\varepsilon_k(i)\varepsilon_k(j)}$. We check five cases of components in the final codomain $Z_{i'j'} \in K(Z,2)_{n+1}$.

- $j' < k$: We get $\psi_{n+1} \circ s_k = s_k \circ \psi_n$ trivially since s_k does not touch the components used by ψ.
- $k = j'$: In this case $Z_{i'j'}$ does not lie in the image of s_k and for ψ_{n+1} we get $\alpha\left(\prod q_\bullet, 1\right) - \alpha\left(\prod q_\bullet, 1\right) = 0$ since $\tilde{q}_k = 1$ and α being normalised.
- $i' < k < j'$: Here $Z_{i'j'}$ is the image of $Z_{i',j'-1}$ and we have to check

$$\alpha\left(\prod_{l=i'}^{j'-2} q_l, q_{j'-1}\right) - \alpha\left(\prod_{l=i'+1}^{j'-2} q_l, q_{j'-1}\right) = \alpha\left(\prod_{l=i'}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right)$$

which is true since one of the $\tilde{q}_l$ is 1.

- $k = i'$: In this case $Z_{i'j'}$ again does not lie in the image of s_k and the image of ψ_{n+1} is $\alpha\left(\prod_{l=i'}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right) = 0$, since $\tilde{q}_{i'} = 1$.

- $k < i'$: Here $Z_{i'j'}$ is the image of $Z_{i'-1,j'-1}$ and we have to check

$$\alpha\left(\prod_{l=i'-1}^{j'-2} q_l, q_{j'-1}\right) - \alpha\left(\prod_{l=i'}^{j'-2} q_l, q_{j'-1}\right) = \alpha\left(\prod_{l=i'}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right)$$

which is just an index shift.

We move on to d_k. We start with $(q_0, q_1, \ldots, q_n) \in Q^{n+1}$ in the upper left corner, which is mapped to $(q_0, q_1, \ldots, q_{k-2}, q_{k-1}q_k, q_{k+1} \ldots, q_n) =: (\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_{n-1})$ in the lower left corner in the general case of $0 \neq k \neq n+1$. On the right side of the diagram Z_{ij} is mapped to $Z_{\eta_{k-1}(i)\eta_{k-1}(j)}$ (remember, that all invalid index combinations vanish by convention). We treat the exceptional cases for k first. If $k = 0$ we see that the diagram is simply an index shift. If $k = n+1$ we see, that everything involving q_n is dropped and otherwise unchanged. For the general case we do again five cases of components in the final codomain $Z_{i'j'} \in K(Z, 2)_n$.

- $j' + 1 < k$: We get $d_k \circ \psi_{n+1} = \psi_n \circ d_k$ trivially since d_k does not touch the components used by ψ.

- $k = j' + 1$: In this case $Z_{i'j'}$ is the target of $Z_{i'j'}$ and $Z_{i',j'+1}$ under ψ. Thus we have to check

$$\alpha\left(\prod_{l=i'}^{j'-1} q_l, q_{j'}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} q_l, q_{j'}\right) + \alpha\left(\prod_{l=i'}^{j'} q_l, q_{j'+1}\right) - \alpha\left(\prod_{l=i'+1}^{j'} q_l, q_{j'+1}\right)$$
$$= \alpha\left(\prod_{l=i'}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right)$$

where $\tilde{q}_{j'} = q_{j'}q_{j'+1}$. Substituting this equality the RHS is equivalent to $\alpha\left(\prod_{l=i'}^{j'-1} q_l, q_{j'}q_{j'+1}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} q_l, q_{j'}q_{j'+1}\right)$ which can be easily transformed into the LHS using the cocycle condition $\alpha(x, yz) = \alpha(xy, z) - \alpha(y, z) + \alpha(x, y)$.

- $i' + 1 < k \leq j'$: Here $Z_{i'j'}$ is the image of $Z_{i',j'+1}$. This means we have to check

$$\alpha\left(\prod_{l=i'}^{j'} q_l, q_{j'+1}\right) - \alpha\left(\prod_{l=i'+1}^{j'} q_l, q_{j'+1}\right) = \alpha\left(\prod_{l=i'}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right)$$

where $\tilde{q}_{k-1} = q_{k-1}q_k$, which is part of all products and thus confirms the equation.

- $k = i' + 1$: In this case $Z_{i'j'}$ is the target of $Z_{i',j'+1}$ and $Z_{i'+1,j'+1}$ under ψ. Thus we have to check

$$\alpha\left(\prod_{l=i'}^{j'} q_l, q_{j'+1}\right) - \alpha\left(\prod_{l=i'+1}^{j'} q_l, q_{j'+1}\right) + \alpha\left(\prod_{l=i'+1}^{j'} q_l, q_{j'+1}\right) - \alpha\left(\prod_{l=i'+2}^{j'} q_l, q_{j'+1}\right)$$
$$= \alpha\left(\prod_{l=i'}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right)$$

where $\tilde{q}_{i'} = q_{i'}q_{i'+1}$ so the RHS is $\alpha\left(\prod_{l=i'}^{j'} q_l, q_{j'+1}\right) - \alpha\left(\prod_{l=i'+2}^{j'} q_l, q_{j'+1}\right)$ which is the LHS after cancelling the middle two terms.

- $k \leq i'$: Here $Z_{i'j'}$ is the image of $Z_{i'+1,j'+1}$. This means we have to check

$$\alpha\left(\prod_{l=i'+1}^{j'} q_l, q_{j'+1}\right) - \alpha\left(\prod_{l=i'+2}^{j'} q_l, q_{j'+1}\right) = \alpha\left(\prod_{l=i'}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right) - \alpha\left(\prod_{l=i'+1}^{j'-1} \tilde{q}_l, \tilde{q}_{j'}\right)$$

which is just an index shift.

Step (iv) We have to check commutativity for the maps $s_0, \ldots, s_n$ as well as $d_0, \ldots, d_{n+1}$. Before we start with the explicit calculations, we note that we already saw in step (ii), that on the components $Z_{ij} \in PK(Z,2)$ with $i > 0$ the map φ acts just like an index-shifted version of ψ. The same index shift happens for the face and degeneracy maps in $PK(Z,2)$ as part of the construction of the path space, because of which Z_{ij} with $i > 0$ will never be a target component for a Z_{0l} for any $l > 0$ under s_k or d_k. This means, that we can adopt the reasoning of step (iii) for the components Z_{ij} with $i > 0$ and only have to deal explicitly with the case $i = 0$.

We start by checking s_k. We start with $(g_0, g_1, \ldots, g_{n-1}) \in G^n$ in the lower left corner, which is mapped to $(g_0, g_1, \ldots, g_{k-1}, 1, g_k, \ldots, g_{n-1}) =: (\tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_n)$ in the upper left corner. On the right side of the diagram the operation s_k and d_k in the path space are given by the operations s_{k+1} and d_{k+1} in $K(Z,2)$. Thus Z_{ij} is mapped to $Z_{\varepsilon_{k+1}(i)\varepsilon_{k+1}(j)}$. By the above argumentation we can restrict ourselves to the cases where the first index is zero. Thus we check three cases of components in the final codomain $Z_{0j'} \in PK(Z,2)_{n+1}$ where $0 < j' \leq n+1$.

- $j' \leq k$: We get $\varphi_{n+1} \circ s_k = s_k \circ \varphi_n$ trivially since s_k does not touch the components used by φ.

- $k = j' - 1$: In this case $Z_{0j'}$ does not lie in the image of s_k and for φ_{n+1} we get $\tau\left(\prod_{l=0}^{j'-2} \tilde{g}_l\right) - \tau\left(\prod_{l=0}^{j'-1} \tilde{g}_l\right) = 0$ since $\tilde{g}_{j'-1} = 1$.

- $k < j' - 1$: Here $Z_{0j'}$ is the image of $Z_{0,j'-1}$ and we have to check
$$\tau\left(\prod_{l=0}^{j'-3} g_l\right) - \tau\left(\prod_{l=0}^{j'-2} g_l\right) = \tau\left(\prod_{l=0}^{j'-2} \tilde{g}_l\right) - \tau\left(\prod_{l=0}^{j'-1} \tilde{g}_l\right)$$
which is true since one of the $\tilde{g}_l$ is 1.

Coming to d_k, we start with $(g_0, g_1, \ldots, g_n) \in G^{n+1}$ in the upper left corner, which is mapped to $(g_0, g_1, \ldots, g_{k-2}, g_{k-1}g_k, g_{k+1} \ldots, g_n) =: (\tilde{g}_0, \tilde{g}_1, \ldots, \tilde{g}_{n-1})$ in the lower left corner in the general case of $0 \neq k \neq n+1$. On the right side of the diagram Z_{ij} is mapped to $Z_{\eta_k(i)\eta_k(j)}$. Again we can restrict ourselves to the cases where the first index is zero, since all other cases where treated in step (iii). We do the exceptional cases in k first. If $k = 0$ the diagram is again an index shift. If $k = n+1$ analogously the terms involving g_n are dropped, keeping everything. Now we can turn to the remaining four cases of components in the final codomain $Z_{0j'} \in PK(Z,2)_n$.

- $j' < k$: We get $d_k \circ \varphi_{n+1} = \varphi_n \circ d_k$ trivially since d_k does not touch the components used by φ.

- $k = j'$: In this case $Z_{0j'}$ is the target of $Z_{0j'}$ and $Z_{0,j'+1}$ under φ. Thus we have to check
$$\tau\left(\prod_{l=0}^{j'-2} g_l\right) - \tau\left(\prod_{l=0}^{j'-1} g_l\right) + \tau\left(\prod_{l=0}^{j'-1} g_l\right) - \tau\left(\prod_{l=0}^{j'} g_l\right)$$
$$= \tau\left(\prod_{l=0}^{j'-2} \tilde{g}_l\right) - \tau\left(\prod_{l=0}^{j'-1} \tilde{g}_l\right)$$
where $\tilde{g}_{j'-1} = g_{j'-1}g_{j'}$ so after cancelling the middle two terms of the LHS equality is established.

- $0 < k < j'$: Here $Z_{0j'}$ is the image of $Z_{0,j'+1}$. This means we have to check
$$\tau\left(\prod_{l=0}^{j'-1} g_l\right) - \tau\left(\prod_{l=0}^{j'} g_l\right) = \tau\left(\prod_{l=0}^{j'-2} \tilde{g}_l\right) - \tau\left(\prod_{l=0}^{j'-1} \tilde{g}_l\right)$$
which is just an index shift.

- $k = 0$: In this case $Z_{0j'}$ is the target of $Z_{0j'+1}$ and $Z_{1,j'+1}$ under φ. Thus we have to check

$$\begin{aligned}\tau\left(\prod_{l=0}^{j'-1} g_l\right) - \tau\left(\prod_{l=0}^{j'} g_l\right) + \alpha\left(\prod_{l=0}^{j'-1} \pi(g_l), \pi(g_{j'})\right) - \alpha\left(\prod_{l=1}^{j'-1} \pi(g_l), \pi(g_{j'})\right) \\ = \tau\left(\prod_{l=0}^{j'-2} \tilde{g}_l\right) - \tau\left(\prod_{l=0}^{j'-1} \tilde{g}_l\right)\end{aligned}$$

where $\tilde{g}_l = g_{l+1}$ and thus the RHS is $\tau\left(\prod_{l=1}^{j'-1} g_l\right) - \tau\left(\prod_{l=1}^{j'} g_l\right)$. Now we use the identity from step (i) to replace

$$\alpha\left(\prod_{l=0}^{j'-1} \pi(g_l), \pi(g_{j'})\right) - \alpha\left(\prod_{l=1}^{j'-1} \pi(g_l), \pi(g_{j'})\right)$$

with

$$\tau\left(\prod_{l=0}^{j'} g_l\right) - \tau\left(\prod_{l=0}^{j'-1} g_l\right) - \tau(g_{j'}) - \left(\tau\left(\prod_{l=1}^{j'} g_l\right) - \tau\left(\prod_{l=1}^{j'-1} g_l\right) - \tau(g_{j'})\right)$$

giving us the desired equality.

This completes the proof of lemma 3.2.1. $\square$

Now we can apply the Eilenberg-Moore spectral sequence to the following situation.

$$\begin{array}{ccc} BZ & = & \Omega K(Z,2) \\ \downarrow & & \downarrow \\ BG & \longrightarrow & PK(Z,2) \\ \downarrow & & \downarrow{\scriptstyle \pi_Z} \\ BQ & \xrightarrow{\alpha} & K(Z,2) \end{array}$$

Note that the path space is contractible, that is $PK(Z,2) \simeq *$. Hence we have $H^*(PK(Z,2);k) \cong H^*(*;k) \cong k$. Thus we get the following theorem.

Theorem 3.2.3
Let k be a commutative ring, G a group and $Z \hookrightarrow G \twoheadrightarrow Q$ a central extension. Then there exists a spectral sequence (E_r^{**}, d_r) converging to $H^*(G;k)$ as an algebra with

$$E_2^{pq} \cong \mathrm{Tor}^{pq}_{H^*(K(Z,2);k)}(H^*(Q;k), k).$$

3.2.2 Previous work

There are only few works utilizing the Eilenberg-Moore spectral sequence in group cohomology. Rusin [Rus87] did a computation of the cohomology of all metacyclic 2-groups using the Eilenberg-Moore spectral sequence. This is probably the most notable usage. Adem and Milgram [AM97] used it in the computation of several cohomology rings in the vicinity of $H^*(\mathrm{Syl}_2(M_{22}))$. Pakianathan and Yalçın [PY12] determined the effect of Bockstein operations for certain cohomology rings utilizing the Eilenberg-Moore spectral sequence. Finally Benson [Ben09] used it briefly for a theoretical argument about loop spaces. But these are the only instances of the Eilenberg-Moore spectral sequence applied to group cohomology in the literature.

There is a further book by Benson [Ben91] about the theory of the cohomology of groups writing down some details about the general usage of the Eilenberg-Moore spectral sequence in group cohomology. This basically references [Rus87] for examples of application.

4 Steenrod Operations and the Eilenberg-Moore spectral sequence

The tool of spectral sequences allows computation of a certain cohomolgy, but itself needs some information to allow these computation to happen. One important source of information are cohomology operations in the form of Steenrod operations. For Steenrod operations on the Eilenberg-Moore spectral sequence one looks into [Smi70] or [Sin73]. There they are provided on the pages E_r for $r \geq 2$ and are compatible with the Steenrod operations on the target. However in the application like [Rus87] one is interested in Steenrod operations already on the E_1-page, for which the author did not find a reliable source.

Actually it is somewhat questionable whether this is at all possible in this situation and in the full sense of cohomology operations. Since the used Koszul resolution is an algebraic and not a geometric object the construction of cohomology operations seems somewhat inappropriate. We will now derive, that at least in the situation of [Rus87] the basic need can be accommodated and something that works like Steenrod operations can be had. We will specialize to central extensions by a $\mathbb{Z}/2\mathbb{Z}$. This does not limit the applicability since we can break down every extension into such smaller extensions.

In the following we will restrict to the case where the characteristic of the ground field k is 2. Due to the rather larger amount of work involved in sorting out signs (which the author did for the case without internal grading), we will use the property of characteristic 2, that we are able to suppress all signs. The treatment of odd primes is somewhat more complicated. However it should work mostly the same, but is left for another time.

4.1 Koszul and bar resolution

We start by presenting an embedding of the Koszul into the bar resolution via chain homomorphisms. This will later be used to transport our desired structure from the bar resolution onto the Koszul resolution.

We are in the situation of theorem 3.2.3, with a group extension

$$Z = \mathbb{Z}/2\mathbb{Z} \hookrightarrow G \twoheadrightarrow Q$$

with a central subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for a 2-group G. We thus want to resolve k as an $H^*(K(\mathbb{Z}/2\mathbb{Z}, 2); k)$-module. Now it is known (see for example [Hat02]) that

$$\Gamma := H^*(K(\mathbb{Z}/2\mathbb{Z}, 2); k) = k[x_0, x_1, \dots]$$

where $\deg(x_i) = 2^i + 1$ and $x_{i+1} = Sq^{2^i} x_i$ is the critical relation. This is a free graded algebra as required by section 2.6 and the generating set is $J = \{x_0, x_1, \dots\}$. Note that the first generator $x_0 \in H^2(K(\mathbb{Z}/2\mathbb{Z}, 2); k) \simeq [K(\mathbb{Z}/2\mathbb{Z}, 2), K(\mathbb{Z}/2\mathbb{Z}, 2)]$ represents the identity.

We denote the desuspensions as usual by $u_i = s^{-1}(x_i)$ or summarized as $s^{-1}J = \{u_0, u_1, \dots\}$. Thus the bar resolution takes the form

$$B^{-n}(\Gamma, k) = \langle \gamma[\gamma_1|\gamma_2|\dots|\gamma_n] \mid \gamma, \gamma_i \in k[x_0, x_1, \dots], \deg(\gamma_i) > 0 \rangle$$

whereas the Koszul resolution is

$$\mathcal{K}(\Gamma) = \Lambda(u_0, u_1, \dots) \otimes_k k[x_0, x_1, \dots].$$

Note that the last term in the differential of the bar resolution vanishes, because $\bar{\Gamma}$ as the positive degree part of Γ annihilates k which is concentrated in degree zero. Note also, that the internal differential in the bar resolution vanishes (like the internal differential of the Koszul resolution), since Γ has trivial differentials.

4.1.1 Embedding the Koszul resolution in the bar resolution

We now give an embedding of the Koszul resolution into the bar resolution. For this we need maps ζ_n with $n \geq 0$ making the following diagram commutative (using the common unrolled variant of resolutions to suppress another 3D-diagram).

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{-2}(\Gamma) & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{-1}(\Gamma) & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{0}(\Gamma) & \xrightarrow{\varepsilon} & k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \zeta_2} & & \downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle \zeta_0} & & \| & & \\
\cdots & \xrightarrow{d_B} & B^{-2}(\Gamma, k) & \xrightarrow{d_B} & B^{-1}(\Gamma, k) & \xrightarrow{d_B} & B^{0}(\Gamma, k) & \xrightarrow{\varepsilon} & k & \longrightarrow & 0
\end{array}$$

Definition 4.1.1: For $\gamma \in \Gamma$ and $i_1, \dots, i_n \in \mathbb{N}$ pairwise different we set in degree n

$$\zeta_n(u_{i_1} u_{i_2} \cdots u_{i_n} \otimes \gamma) := \sum_{\sigma \in S_n} \gamma[x_{i_{\sigma(1)}} | x_{i_{\sigma(2)}} | \dots | x_{i_{\sigma(n)}}] \, .$$

This induces a map $\zeta_n : \mathcal{K}^{-n}(\Gamma) \to B^{-n}(\Gamma, k)$.

Proposition 4.1.2: The map $\zeta_* : \mathcal{K}^*(\Gamma) \to B^*(\Gamma, k)$ assembled from the ζ_n above is an injective chain map of differential graded chain complexes.

Proof. First we note that ζ respects the internal degree and the trivially vanishing internal differential.

Second we note, that ζ is injective, since $u_i^2 = 0$ in the exterior algebra.

Third we check the rightmost square involving ε. Here we have $\zeta_0(\gamma) = \gamma[]$ and thus an isomorphism.

After the easy bits are out of the way, we check an arbitrary square.

$$\begin{array}{ccc} \mathcal{K}^{-n}(\Gamma) & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{-n+1}(\Gamma) \\ \downarrow{\scriptstyle \zeta_n} & & \downarrow{\scriptstyle \zeta_{n-1}} \\ B^{-n}(\Gamma, k) & \xrightarrow{d_B} & B^{-n+1}(\Gamma, k) \end{array}$$

We start with $u_{i_1} \cdots u_{i_n} \otimes \gamma$ in the upper left corner and let $X := \zeta_{n-1}(d_{\mathcal{K}}(u_{i_1} \cdots u_{i_n} \otimes \gamma))$ and $Y := d_B(\zeta_n(u_{i_1} \cdots u_{i_n} \otimes \gamma))$. We now want to show $X = Y$.

In the following computations a hat denotes omission (like $\hat{u}_{i_j}$ for omitting u_{i_j}). Starting with X we see

$$\begin{aligned} X &= \zeta_n(d_{\mathcal{K}}(u_{i_1} \cdots u_{i_n} \otimes \gamma)) \\ &= \zeta_n \left(\sum_{j=1}^{n} u_{i_1} \cdots \hat{u}_{i_j} \cdots u_{i_n} \otimes \gamma x_{i_j} \right) \\ &= \sum_{j=1}^{n} \sum_{\substack{\sigma' \in S_n \\ \sigma'(j) = j}} (\gamma x_{i_j})[x_{i_{\sigma'(1)}} | \dots | \hat{x}_{i_{\sigma'(j)}} | \dots | x_{i_{\sigma'(n)}}] \,. \end{aligned} \tag{4.1}$$

Now in contrast for Y we get

$$\begin{aligned} Y &= d_B(\zeta_n(u_{i_1} \cdots u_{i_n} \otimes \gamma)) \\ &= d_B \left(\sum_{\sigma \in S_n} \gamma[x_{i_{\sigma(1)}} | \dots | x_{i_{\sigma(n)}}] \right) \\ &= \sum_{\sigma \in S_n} \Bigg((\gamma x_{i_{\sigma(1)}})[x_{i_{\sigma(2)}} | \dots | x_{i_{\sigma(n)}}] \\ &\qquad + \sum_{j=1}^{n-1} \gamma[x_{i_{\sigma(1)}} | \dots | x_{i_{\sigma(j-1)}} | x_{i_{\sigma(j)}} x_{i_{\sigma(j+1)}} | x_{i_{\sigma(j+2)}} | \dots | x_{i_{\sigma(n)}}] \Bigg) \,. \end{aligned}$$

We split this sum into two parts namely Y_1 containing the summands where an $x_{i_{\sigma(1)}}$ got pushed out on the front and Y_2 containing all summands having an entry $x_{i_{\sigma(j)}} x_{i_{\sigma(j+1)}}$ of degree two.

$$Y_1 = \sum_{\sigma \in S_n} (\gamma x_{i_{\sigma(1)}})[x_{i_{\sigma(2)}}|\dots|x_{i_{\sigma(n)}}] \tag{4.2}$$

$$Y_2 = \sum_{\sigma \in S_n} \sum_{j=1}^{n-1} \gamma[x_{i_{\sigma(1)}}|\dots|x_{i_{\sigma(j-1)}}|x_{i_{\sigma(j)}} x_{i_{\sigma(j+1)}}|x_{i_{\sigma(j+2)}}|\dots|x_{i_{\sigma(n)}}] \tag{4.3}$$

We show $Y_1 = X$ as well as $Y_2 = 0$.

For equation (4.3) we observe, that exactly two permutations generate the same generator in the bar resolution, if one of them is ρ, then the other is obtained by first applying the permutation τ transposing j and $j+1$. Thus the two terms associated to these permutations cancel each other out. This shows $Y_2 = 0$.

For equation (4.2) we construct a one-to-one correspondence $\sigma \leftrightarrow (j, \sigma')$ between each summand in (4.2) and (4.1). Given σ we obtain $j := \sigma(1)$ and

$$\sigma'(l) := \begin{cases} \sigma(l+1) & \text{if } l < j, \\ j & \text{if } l = j, \\ \sigma(l) & \text{if } l > j, \end{cases}$$

then $\sigma' = \sigma \circ \tau$ for the permutation τ which consists of the cycle $(1\ 2\ \dots\ j)$. Clearly this is a one-to-one correspondence matching each summand in the two sums. Thus we get $Y_1 = X$.

Thus ζ_* is a chain map of differential graded chain complexes. □

This shows that the Koszul resolution can be understood as a subcomplex of the bar resolution. We further show that ζ is compatible with the multiplicative structure.

Proposition 4.1.3: The map $\zeta_* : \mathcal{K}^*(\Gamma) \to B^*(\Gamma, k)$ respects the multiplicative structure.

Proof. Let $i_1, \dots, i_m \in \mathbb{N}$ as well as $i_{m+1}, \dots, i_{m+n} \in \mathbb{N}$ be pairwise different and $\gamma, \gamma' \in \Gamma$.

If not all i_j are pairwise different then the product in the Koszul complex vanishes. Otherwise we get

$$\begin{aligned} \zeta\big((u_{i_1} \cdots u_{i_m} \otimes \gamma) \cdot (u_{i_{m+1}} \cdots u_{i_{m+n}} \otimes \gamma')\big) &= \zeta(u_{i_1} \cdots u_{i_{m+n}} \otimes \gamma\gamma') \\ &= \sum_{\sigma \in S_{m+n}} \gamma[x_{i_{\sigma(1)}}|x_{i_{\sigma(2)}}|\dots|x_{i_{\sigma(m+n)}}] \,. \end{aligned} \tag{4.4}$$

For the product on the bar complex side we get

$$\begin{aligned}
&\zeta(u_{i_1}\cdots u_{i_m}\otimes\gamma)\cdot\zeta(u_{i_{m+1}}\cdots u_{i_{m+n}}\otimes\gamma') \\
&= \left(\sum_{\sigma_1\in S_m}\gamma[x_{i_{\sigma_1(1)}}|x_{i_{\sigma_1(2)}}|\dots|x_{i_{\sigma_1(m)}}]\right)\cdot\left(\sum_{\sigma_2\in S_n}\gamma'[x_{i_{\sigma_2(m+1)}}|\dots|x_{i_{\sigma_2(m+n)}}]\right) \\
&= \sum_{\substack{(m,n)\text{-shuffle }\tau\\ \sigma_1\in S_m,\sigma_2\in S_n}}\gamma\gamma'[x_{i_{\tau(\sigma_1(1))}}|x_{i_{\tau(\sigma_1(2))}}|\dots|x_{i_{\tau(\sigma_1(m))}}|x_{i_{\tau(\sigma_2(m+1))}}|\dots|x_{i_{\sigma_2(m+n)}}]. \qquad (4.5)
\end{aligned}$$

The combination of a shuffle with permutations on each part of the shuffle gives exactly each element of S_{m+n}. If not all i_j are pairwise different, then each different summand appears an even number of times and hence the image vanishes. Thus the expressions (4.4) and (4.5) are equal and we are done. □

4.1.2 Mapping the bar resolution onto the Koszul resolution

For the reverse direction we give maps ξ_n for $n \geq 0$ from the bar resolution to the Koszul resolution making the following diagram commutative.

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d_B} & B^{-2}(\Gamma,k) & \xrightarrow{d_B} & B^{-1}(\Gamma,k) & \xrightarrow{d_B} & B^{0}(\Gamma,k) & \xrightarrow{\varepsilon} & k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\xi_2} & & \downarrow{\scriptstyle\xi_1} & & \downarrow{\scriptstyle\xi_0} & & \| & & \\
\cdots & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{-2}(\Gamma) & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{-1}(\Gamma) & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{0}(\Gamma) & \xrightarrow{\varepsilon} & k & \longrightarrow & 0
\end{array}$$

Any $\gamma\in\Gamma$ has a unique representation as

$$\gamma = c_0(\gamma) + \sum_{i=0}^{\infty} x_i f_i(\gamma)$$

with $c_0(\gamma)\in k$ and $f_i(\gamma)\in k[J_i]$ for $J_i=\{x_{i'} \mid i'\geq i\}$. Note that this definition relies on a choice of ordering the x_i. The resulting ambiguity will be discussed in §4.3. Furthermore define F_i for $i\geq 0$ by

$$F_i(\gamma) := c_0(\gamma) + \sum_{i'\geq i} x_{i'} f_{i'}(\gamma).$$

Definition 4.1.4: For $\gamma,\gamma_1,\dots,\gamma_n\in\Gamma$ we set in degree n

$$\xi_n(\gamma[\gamma_1|\dots|\gamma_n]) := \sum_{i_1<i_2<\cdots<i_n} u_{i_1}\cdots u_{i_n}\otimes\gamma f_{i_1}(\gamma_1)\cdots f_{i_n}(\gamma_n).$$

where the sum runs over all tuples $(i_1,i_2.\dots,i_n)\in\mathbb{N}^n$ which are in ascending order. This induces a map $\xi_n : B^{-n}(\Gamma,k)\to\mathcal{K}^{-n}(\Gamma)$.

Note that the sum on the right hand side is always finite, since only finitely many f_i are non-zero. Before we start looking closer at ξ we prepare a small fact about f_i and F_i.

Lemma 4.1.5: For $\gamma_1, \gamma_2 \in \Gamma$ we have

$$f_i(\gamma_1\gamma_2) = f_i(\gamma_1)F_i(\gamma_2) + F_i(\gamma_1)f_i(\gamma_2) - x_i f_i(\gamma_1)f_i(\gamma_2)\,.$$

Proof. We have representations $\gamma_l = c_0(\gamma_l) + \sum_{i=0}^{\infty} x_i f_i(\gamma_l)$ for $l \in \{1,2\}$. Hence we get

$$\begin{aligned}\gamma_1\gamma_2 &= \left(c_0(\gamma_1) + \sum_{j=0}^{\infty} x_j f_j(\gamma_1)\right)\left(c_0(\gamma_2) + \sum_{j=0}^{\infty} x_j f_j(\gamma_2)\right)\\ &= c_0(\gamma_1)c_0(\gamma_2) + \sum_{j=0}^{\infty} x_j\big(c_0(\gamma_1)f_j(\gamma_2) + c_0(\gamma_2)f_j(\gamma_1)\big) + \sum_{j,j'=0}^{\infty} x_j x_{j'} f_j(\gamma_1)f_{j'}(\gamma_2)\,.\end{aligned}$$

Thus we retrieve for the terms of the assertion

$$\begin{aligned}f_i(\gamma_1\gamma_2) &= c_0(\gamma_1)f_i(\gamma_2) + c_0(\gamma_2)f_i(\gamma_1)\\ &\qquad + x_i f_i(\gamma_1)f_i(\gamma_2) + \sum_{j>i} x_j\big(f_i(\gamma_1)f_j(\gamma_2) + f_j(\gamma_1)f_i(\gamma_2)\big)\\ &= c_0(\gamma_1)f_i(\gamma_2) + \sum_{j\geq i} x_j f_j(\gamma_1)f_i(\gamma_2) + c_0(\gamma_2)f_i(\gamma_1) + \sum_{j\geq i} f_i(\gamma_1)f_j(\gamma_2)\\ &\qquad - x_i f_i(\gamma_1)f_i(\gamma_2)\\ &= \left(c_0(\gamma_1) + \sum_{j\geq i} x_j f_j(\gamma_1)\right) f_i(\gamma_2) + \left(c_0(\gamma_2) + \sum_{j\geq i} f_j(\gamma_2)\right) f_i(\gamma_1)\\ &\qquad - x_i f_i(\gamma_1)f_i(\gamma_2)\\ &= F_i(\gamma_1)f_i(\gamma_2) + f_i(\gamma_1)F_i(\gamma_2) - x_i f_i(\gamma_1)f_i(\gamma_2)\,.\end{aligned}$$

□

Proposition 4.1.6: The map $\xi_* : B^*(\Gamma, k) \to \mathcal{K}^*(\Gamma)$ assembled from the ξ_n above is a chain map of differential graded chain complexes.

Proof. First we note that ξ respects the internal degree and the trivially vanishing internal differential.

Second we check the rightmost square involving ε. Here we have $\xi_0(1 \otimes \gamma) = \gamma[]$ and everything works out

So it remains to check the general diagram

$$\begin{array}{ccc} B^{-n}(\Gamma,k) & \xrightarrow{d_B} & B^{-n+1}(\Gamma,k)\\ \downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi_{n-1}}\\ \mathcal{K}^{-n}(\Gamma) & \xrightarrow{d_{\mathcal{K}}} & \mathcal{K}^{-n+1}(\Gamma)\end{array}$$

where we start with $\gamma[\gamma_1|\dots|\gamma_n]$ for $\gamma, \gamma_1, \dots, \gamma_n \in \Gamma$ in the upper left hand corner. We let $X := \xi_{n-1}(d_B(\gamma[\gamma_1|\dots|\gamma_n]))$ and $Y := d_{\mathcal{K}}(\xi_n(\gamma[\gamma_1|\dots|\gamma_n]))$. We now show $X = Y$.

For this we start by computing X which is

$$\begin{aligned}
X &= \xi_{n-1}(d_B(\gamma[\gamma_1|\dots|\gamma_n])) \\
&= \xi_{n-1}\left((\gamma\gamma_1)[\gamma_2|\dots|\gamma_n] + \sum_{i=1}^{n-1} \gamma[\gamma_1|\dots|\gamma_{i-1}|\gamma_i\gamma_{i+1}|\gamma_{i+2}|\dots|\gamma_n]\right) \\
&= \sum_{r_2<\dots<r_n} u_{r_2}\cdots u_{r_n} \otimes \gamma\gamma_1 f_{r_2}(\gamma_2)\cdots f_{r_n}(\gamma_n) \\
&\quad + \sum_{i=1}^{n-1} \sum_{r_1<\dots<r_{n-1}} \Big(u_{r_1}\cdots u_{r_{n-1}} \\
&\qquad\qquad \otimes \gamma f_{r_1}(\gamma_1)\cdots f_{r_{i-1}}(\gamma_{i-1}) f_{r_i}(\gamma_i\gamma_{i+1}) f_{r_{i+1}}(\gamma_{i+2})\cdots f_{r_{n-1}}(\gamma_n)\Big)
\end{aligned}$$

and second we retrieve for Y, where again a hat stands for omission

$$\begin{aligned}
Y &= d_{\mathcal{K}}(\xi_n(\gamma[\gamma_1|\dots|\gamma_n])) \\
&= d_{\mathcal{K}}\left(\sum_{r_1<r_2<\dots<r_n} u_{r_1}\cdots u_{r_n} \otimes \gamma f_{r_1}(\gamma_1)\cdots f_{r_n}(\gamma_n)\right) \\
&= \sum_{r_1<r_2<\dots<r_n} \sum_{i=1}^{n} u_{r_1}\cdots \widehat{u_{r_i}} \cdots u_{r_n} \otimes x_{r_i}\gamma f_{r_1}(\gamma_1)\cdots f_{r_n}(\gamma_n)
\end{aligned}$$

We now compare the coefficients (which is in this case the second part of the tensor product) for a specific set $S \subseteq \mathbb{N}$ of size $n-1$. We set $S =: \{s_1, s_2, \dots, s_{n-1}\}$ such that $s_1 < s_2 < \dots < s_{n-1}$. Now we collect the second part of all things of the form $u_{s_1}\cdots u_{s_{n-1}} \otimes \gamma'$.

For X we get the part X_N as

$$\begin{aligned}
X_N &= \gamma\gamma_1 f_{s_1}(\gamma_2)\cdots f_{s_{n-1}}(\gamma_n) \\
&\quad + \sum_{i=1}^{n-1} \gamma f_{s_1}(\gamma_1)\cdots f_{s_{i-1}}(\gamma_{i-1}) f_{s_i}(\gamma_i\gamma_{i+1}) f_{s_{i+1}}(\gamma_{i+2})\cdots f_{s_{n-1}}(\gamma_n)
\end{aligned}$$

and for Y we retrieve Y_N to be

$$Y_N = \sum_{i=1}^{n} \sum_{s_{i-1}<j<s_i} x_j\gamma f_{s_1}(\gamma_1)\cdots f_{s_{i-1}}(\gamma_{i-1}) f_j(\gamma_i) f_{s_i}(\gamma_{i+1})\cdots f_{s_{n-1}}(\gamma_n)$$

where s_0 and s_n are understood to be synthetic lowest and highest elements respectively.

Now we transform X_N into Y_N. First we substitute by lemma 4.1.5

$$f_{s_i}(\gamma_i\gamma_{i+1}) = f_{s_i}(\gamma_i)F_{s_i}(\gamma_{i+1}) + F_{s_i}(\gamma_i)f_{s_i}(\gamma_{i+1}) - x_{s_i}f_{s_i}(\gamma_i)f_{s_i}(\gamma_{i+1}) \tag{4.6}$$

in X_N to find a telescope sum. For an arbitrary $l \in \{1, 2, \ldots, n-1\}$ we take the terms $t_1 := f_{s_l}(\gamma_l)F_{s_l}(\gamma_{l+1})f_{s_{l+1}}(\gamma_{l+2})$ and $t_2 := x_{s_l}f_{s_l}(\gamma_l)f_{s_l}(\gamma_{l+1})f_{s_{l+1}}(\gamma_{l+2})$ from the iteration $i = l$ of (4.6) while adding an $f_{s_{l+1}}(\gamma_{l+2}) = f_{s_{i+1}}(\gamma_{i+2})$ on the right which is taken from X_N and furthermore the term $t_3 := f_{s_l}(\gamma_l)F_{s_{l+1}}(\gamma_{l+1})f_{s_{l+1}}(\gamma_{l+2})$ from the iteration $i = l+1$ of (4.6) while adding an $f_{s_l}(\gamma_l) = f_{s_{i-1}}(\gamma_{i-1})$ on the left again from X_N. Now $F_{s_l}(\gamma_{l+1}) + F_{s_{l+1}}(\gamma_{l+1}) = \sum_{s_l \leq m < s_{l+1}} x_m f_m(\gamma_{l+1})$ since most terms cancel out each other. Hence we see

$$t_1 + t_2 + t_3 = \sum_{s_l < m < s_{l+1}} f_{s_l}(\gamma_l)x_m f_m(\gamma_{l+1})f_{s_{l+1}}(\gamma_{l+2}).$$

The edge cases $l = 1$ and $l = n-1$ are unproblematic since the relevant parts are only the F_{s_i}, however we have to be a tiny bit careful and have to define $f_{s_0} \equiv 1 \equiv f_{s_n}$. Furthermore we need to account for the unused terms at the edges, these are $t_3 = f_{s_0}(\gamma_0)F_{s_1}(\gamma_1)f_{s_1}(\gamma_2) = F_{s_1}(\gamma_1)f_{s_1}(\gamma_2)$ at the beginning in the case $l = 1$ and $t_1 = f_{s_{n-1}}(\gamma_{n-1})F_{s_{n-1}}(\gamma_n)f_{s_n}(\gamma_{n+1}) = f_{s_{n-1}}(\gamma_{n-1})F_{s_{n-1}}(\gamma_n)$ as well as $t_2 = x_{s_{n-1}}f_{s_{n-1}}(\gamma_{n-1})f_{s_{n-1}}(\gamma_n)f_{s_n}(\gamma_{n+1}) = x_{s_{n-1}}f_{s_{n-1}}(\gamma_{n-1})f_{s_{n-1}}(\gamma_n)$ at the end in the case $l = n-1$.

Second we substitute $\gamma_1 = \sum_{i=0}^{\infty} x_i f_i(\gamma_1)$ and the previous equality to find

$$\begin{aligned} X_N = \gamma &\left(\sum_{i=0}^{\infty} x_i f_i(\gamma_1)\right) f_{s_1}(\gamma_2)\cdots f_{s_{n-1}}(\gamma_n) \\ &+ \sum_{i=1}^{n-2} \gamma f_{s_1}(\gamma_1)\cdots \left(\sum_{s_i < m < s_{i+1}} f_{s_i}(\gamma_i)x_m f_m(\gamma_{i+1})f_{s_{i+1}}(\gamma_{i+2})\right)\cdots f_{s_{n-1}}(\gamma_n) \\ &+ \gamma F_{s_1}(\gamma_1)f_{s_1}(\gamma_2)\cdots f_{s_{n-1}}(\gamma_n) + \gamma f_{s_1}(\gamma_1)\cdots f_{s_{n-1}}(\gamma_{n-1})F_{s_{n-1}}(\gamma_n) \\ &+ x_{s_{n-1}}\gamma f_{s_1}(\gamma_1)\cdots f_{s_{n-1}}(\gamma_{n-1})f_{s_{n-1}}(\gamma_n) \end{aligned}$$

Third we replace $F_{s_l}(\gamma) = c_0(\gamma) + \sum_{i' \geq s_l} x_{i'} f_{i'}(\gamma)$ for $l = 1$ and $l = n-1$. Note that $c_0(\gamma) = 0$ by the definition of $B^*(\Gamma, k)$. Now the term with $F_{s_1}(\gamma_1)$ matches against the term with $\sum_{i=0}^{\infty} x_i f_i(\gamma_1)$ and the last two terms fit at the end of the big sum to give us

$$X_N = \sum_{i=0}^{n-1} \sum_{s_i < m < s_{i+1}} x_m \gamma f_{s_1}(\gamma_1)\cdots f_{s_i}(\gamma_i)f_m(\gamma_{i+1})f_{s_{i+1}}(\gamma_{i+2})\cdots f_{s_{n-1}}(\gamma_n) = Y_N$$

Thus the ξ_* form a chain map. □

Again we show that ξ also works with respect to the multiplicative structure.

Proposition 4.1.7: The map $\xi_* : B^*(\Gamma, k) \to \mathcal{K}^*(\Gamma)$ is compatible with the multiplicative structure.

Proof. Let $\beta, \gamma \in \Gamma$ and $\beta_1, \dots, \beta_m, \gamma_1, \dots, \gamma_n \in \bar{\Gamma}$. Furthermore let $\omega_i := \beta_i$ for $i \leq m$ and $\omega_{m+i} := \gamma_i$ otherwise. Now we can compute the first direction.

$$\begin{aligned}
&\xi(\beta[\beta_1|\dots|\beta_m] \cdot \gamma[\gamma_1|\dots|\gamma_n]) \\
&= \xi\left(\sum_{(m,n)\text{-shuffle } \sigma} \beta\gamma[\omega_{\sigma(1)}|\dots|\omega_{\sigma(m+n)}]\right) \\
&= \sum_{\substack{(m,n)\text{-shuffle } \sigma \\ i_1<i_2<\dots<i_{m+n}}} u_{i_1}\cdots u_{i_{m+n}} \otimes \beta\gamma f_{i_1}(\omega_{\sigma(1)})\cdots f_{i_{m+n}}(\omega_{\sigma(m+n)}) \qquad (4.7)
\end{aligned}$$

And the reverse direction is

$$\begin{aligned}
&\xi(\beta[\beta_1|\dots|\beta_m]) \cdot \xi(\gamma[\gamma_1|\dots|\gamma_n]) \\
&= \left(\sum_{i_1<i_2<\dots<i_m} u_{i_1}\cdots u_{i_m} \otimes \beta f_{i_1}(\beta_1)\cdots f_{i_m}(\beta_m)\right) \\
&\qquad \cdot \left(\sum_{i_{m+1}<i_{m+2}<\dots<i_{m+n}} u_{i_{m+1}}\cdots u_{i_{m+n}} \otimes \gamma f_{i_{m+1}}(\gamma_1)\cdots f_{i_{m+n}}(\gamma_n)\right) \\
&= \sum_{\substack{i_1<i_2<\dots<i_m \\ i_{m+1}<i_{m+2}<\dots<i_{m+n}}} u_{i_1}\cdots u_{i_{m+n}} \otimes \beta\gamma f_{i_1}(\beta_1)\cdots f_{i_m}(\beta_m) f_{i_{m+1}}(\gamma_1)\cdots f_{i_{m+n}}(\gamma_n)\,. \qquad (4.8)
\end{aligned}$$

We note that in (4.8) any pair of equal indices causes the corresponding summand to vanish. Thus the shuffle in (4.7) causes exactly the same summation to take place. Therefore we have established equality. □

4.2 Steenrod Operations on the E_1-page

Proposition 4.2.1: The maps ζ_* and ξ_* provide an embedding of the Koszul resolution $\mathcal{K}^*(\Gamma)$ into the bar resolution $B^*(\Gamma, k)$.

Proof. It remains to show that $\xi \circ \zeta = \text{id}$, which we will verify shortly. We then have a concrete description of the in general only abstract equivalence between the two resolutions. Furthermore in one of the two directions the composition actually yields identity, thus giving us an embedding. This allows us to find the structure of the Koszul resolution inside the bar resolution, leading to the definitions following this proof.

We take $u_{i_1}\cdots u_{i_n}\otimes\gamma\in\mathcal{K}^{-n}(\Gamma)$ and compute its image

$$\begin{aligned}&\xi(\zeta(u_{i_1}\cdots u_{i_n}\otimes\gamma))\\&\quad=\xi\left(\sum_{\sigma\in S_n}\gamma[x_{i_{\sigma(1)}}|x_{i_{\sigma(2)}}|\dots|x_{i_{\sigma(n)}}]\right)\\&\quad=\sum_{\sigma\in S_n}\sum_{j_1<j_2<\dots<j_n}u_{j_1}\cdots u_{j_n}\otimes\gamma f_{j_1}(x_{i_{\sigma(1)}})\cdots f_{j_n}(x_{i_{\sigma(n)}})\\&\quad=u_{i_1}\cdots u_{i_n}\otimes\gamma\end{aligned}$$

where the last equality stems from the facts that $f_j(x_i)=\delta_{ij}$ (with the Kronecker δ function) and $j_1<j_2<\dots<j_n$. □

To begin with we have the Steenrod operations Sq^i by general theory on $\Gamma=k[J]=H^*(K(\mathbb{Z}/2\mathbb{Z},2);k)$. Now we can easily get similar operations Sq^i_{bar} on the bar resolution $B^{-n}(\Gamma,k)=\Gamma\otimes\bar{\Gamma}^n$ by letting

$$Sq^i_{bar}(\gamma[\gamma_1|\dots|\gamma_n]):=\sum_{j+j_1+\cdots j_n=i}Sq^j(\gamma)[Sq^{j_1}(\gamma_1)|\dots|Sq^{j_n}(\gamma_n)].$$

This is a natural definition and inherits the properties of the Steenrod operations, since the tensor product behaves well with respect to cohomology operations.

With this we can now transfer these to the Koszul resolution via ζ and ξ. We define

$$Sq^i_{Koszul}:=\xi\circ Sq^i_{bar}\circ\zeta\,.$$

This inherits the properties of Sq^i_{bar} since ζ and ξ transport the structure between the different resolutions.

Now remember that the E_1-page of the Eilenberg-Moore spectral sequence is given by

$$E_1^{pq}=\bigoplus_{i+j=q}H^i(Q)\otimes_\Gamma\mathcal{K}^{pj}(\Gamma)\,.$$

We can now define our target operation.

Definition 4.2.2: Let $p,q,i,j\in\mathbb{Z}$ such that $i+j=q$. Furthermore let $x\in H^i(Q)$ and $y\in\mathcal{K}^{pj}(\Gamma)$ which means $x\otimes y\in E_1^{pq}$. For $l\in\mathbb{N}$ we set

$$Sq^l_{EM}(x\otimes y):=\sum_{l_1+l_2=l}Sq^{l_1}(x)\otimes Sq^{l_2}_{Koszul}(y)\,.$$

This recovers the operation given in Rusin's paper [Rus87].

Remark 4.2.3: During review it turned out, that the above doesn't actully do what it set out to do. In short it doesn't work and it is questionable whether it could work at all. For a detailed explanation see page ix.

4.3 Ambiguity

Most of this chapter generalises to more abstract situations. However one point of complication is that for a general Koszul complex one has to choose an ordering of the generators. In our case, there was a natural ordering so that we could suppress this issue. But in the general case ξ will depend on the chosen ordering. On one hand this is not a big issue, since all possible ξ are lifts of the same map and thus chain homotopic. On the other hand this is irritating as all the definitions on top of ξ are inheriting this ambiguity.

If however in Γ the image of each generator under the Steenrod operations is a power of some generator (different or equal) there is no ambiguity. Since in that case there are no mixed monomials nothing can be shuffled by a different ordering.

Especially in our case of $\Gamma = H^*(K(\mathbb{Z}/2\mathbb{Z}, 2); k)$ we have that each generator x_i is mapped under Steenrod actions to either another generator x_j or the square of the same generator x_i^2. On the Koszul resolution, we now have the generators $u_i := s^{-1}(x_i) \otimes 1$ and their images under Steenrod operations are thus either $u_j \otimes 1$ or $u_i \otimes x_i$.

5 Applications

We will now present two applications of the Eilenberg-Moore spectral sequence and evaluate its versatility for actual computations of group cohomology. We especially want to compare it to the Lyndon-Hochschild-Serre spectral sequence which is another common tool for the computation of group cohomology.

5.1 $32\Gamma_3 f$

We start with the computation of the cohomology $H^*(32\Gamma_3 f;\mathbb{F}_2)$. We denote by $32\Gamma_3 f$ the group of size 32 with name $\Gamma_3 f$ in the sense of Hall–Senior [HS64]. Alternatively it is the group number 15 of size 32 in the numbering of the Small Groups library [BEO02].

We examine the versatility and limitations of the Eilenberg-Moore spectral sequence. For this we need the work of § 3 and § 4 as prerequisites. These were not available before, so that a previous treatment of this matter by the author in [Oeh16] utilized only the Lyndon-Hochschild-Serre spectral sequence, as this was a well proven technique that sufficed for the purpose. There a disagreement between the four works [Rus87; Hue89; CTV⁺03; GK11] was resolved by explicitly computing the cohomology of $32\Gamma_3 f$ (there exists now a revised manuscript [Hue16] by Huebschmann rectifying the issue in his approach).

However we will encounter shortcomings of the Eilenberg-Moore spectral sequence approach, due to which we will need to use the Lyndon-Hochschild-Serre approach of [Oeh16] to fill some gaps.

5.1.1 Presentation of $32\Gamma_3 f$

A common presentation for the group $32\Gamma_3 f$ uses two generators, that is

$$32\Gamma_3 f = \left\langle h_1, h_2 \middle| h_2^8 = 1, h_1^4 = h_2^4, {}^{h_1}h_2 = h_2^3 \right\rangle .$$

For the approach with spectral sequences a power-conjugate presentation (see for example [HEO05, § 9.4.1]) is more amenable, as it makes the occurring central extensions more directly visible. We substitute $f_1 := h_1$, $f_2 := h_2$, $f_3 := h_2^2$, $f_4 := h_1^2$

and $f_5 := h_1^4 = h_2^4$ for the presentation

$$32\Gamma_3 f = \left\langle f_1, f_2, f_3, f_4, f_5 \,\middle|\, \begin{matrix} f_1^2 = f_4, f_2^2 = f_3, f_3^2 = f_4^2 = f_5, f_5^2 = 1, \\ f_2^{f_1} = f_2 f_3, f_3^{f_1} = f_3 f_5, f_i^{f_j} = f_i \text{ for all other } j < i \end{matrix} \right\rangle$$
$$\overset{\text{as set}}{=} \left\{ f_1^a f_2^b f_3^c f_4^d f_5^e \mid a, b, c, d, e \in \mathbb{F}_2 \right\} .$$

Additionally this presentation behaves very well in conjunction with the bar resolution used, since all generators have relative order two making the description of the cochains very concise.

In the following we will for $i \in \mathbb{N}$ use variables $g_i \in 32\Gamma_3 f$. Each such variable shall correspond to the presentation

$$g_i = f_1^{a_i} f_2^{b_i} f_3^{c_i} f_4^{d_i} f_5^{e_i}$$

so that g_i implicitly defines a_i to e_i.

The group operation can now be formulated in terms of the second presentation. For any $g_1, g_2 \in 32\Gamma_3 f$ we retrieve

$$g_1 g_2 = (f_1^{a_1} f_2^{b_1} f_3^{c_1} f_4^{d_1} f_5^{e_1})(f_1^{a_2} f_2^{b_2} f_3^{c_2} f_4^{d_2} f_5^{e_2}) = f_1^{a_1+a_2} f_2^{b_1+b_1} f_3^{c_1+c_2+\tilde{c}} f_4^{d_1+d_2+\tilde{d}} f_5^{e_1+e_2+\tilde{e}}$$

with the deviations from the elementary abelian case being

$$\begin{aligned} \tilde{c} &= b_1 a_2 + b_1 b_2, \\ \tilde{d} &= a_1 a_2, \\ \tilde{e} &= d_1 d_2 + a_1 a_2 d_2 + a_1 d_1 a_2 \\ &\quad + c_1 c_2 + b_1 b_2 c_2 + b_1 c_1 b_2 + b_1 a_2 c_2 + b_1 c_1 a_2 + c_1 a_2 + b_1 a_2 b_2. \end{aligned}$$

To leverage the spectral sequence we want to describe $32\Gamma_3 f$ as a central extension. For this we first observe from the above listing, that the center of $32\Gamma_3 f$ is generated by f_5. We thus use the central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \cong \langle f_5 \rangle \hookrightarrow 32\Gamma_3 f \twoheadrightarrow 16\Gamma_2 c_2 \to 1 \tag{5.1}$$

where the quotient is the group $\Gamma_2 c_2$ of size 16 according to Hall–Senior or the group number 4 of size 16 according to the Small Groups Library. We present $16\Gamma_2 c_2$ by abuse of notation as

$$16\Gamma_2 c_2 = \langle f_1, f_2, f_3, f_4, f_5 | f_5 = 1 \rangle$$

where the f_i otherwise behave as in the case of $32\Gamma_3 f$. Furthermore by omitting the line for $\tilde{e}$ above we see that the center of $16\Gamma_2 c_2$ is generated by f_3 and f_4. We can thus describe $16\Gamma_2 c_2$ as a central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \cong \langle f_4 \rangle \hookrightarrow 16\Gamma_2 c_2 \twoheadrightarrow D_8 \to 1 \tag{5.2}$$

where the quotient
$$D_8 = \langle f_1, f_2, f_3, f_4, f_5 | f_4 = f_5 = 1\rangle$$
is easily identifiable as the dihedral group of order 8.

We denote by $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$, the set maps $32\Gamma_3 f \to \mathbb{F}_2$ mapping $f_1^a f_2^b f_3^c f_4^d f_5^e$ to the respective exponent. We will utilize the bar construction to describe the cohomology of the group G ($32\Gamma_3 f$ or a quotient thereof), so that in degree n the cochains are given by the vector space of maps $G^n \to \mathbb{F}_2$ as given above, but with indices to indicate the relevant copy of G. Thus for example we have $\bar{b}_1\bar{a}_2\bar{c}_2 : G^2 \to \mathbb{F}_2 : (g_1, g_2) \mapsto b_1 a_2 c_2$.

5.1.2 Cohomology of $16\Gamma_2 c_2$

The cohomology of the D_8 is (see for example [AM04, Theorem 2.7], where the group generators are different so that our u is the sum of their degree one cohomology generators)
$$H^*(D_8) = \mathbb{F}_2[u_{\underline{1}}, v_{\underline{1}}, w_{\underline{2}}]/(v^2 + uv)$$
where the classes of degree one are given by $u = [\bar{a}_1]$, $v = [\bar{b}_1]$ with square brackets denoting the appropriate equivalence classes and the underlined subscripts denoting the degree (these underlined subscripts will mostly appear upon introduction of a variable). The class of degree two is approximately $w \approx [\bar{c}_1\bar{c}_2]$ as $\langle f_2, f_3\rangle$ is a $\mathbb{Z}/4\mathbb{Z}$. We find (with some computer assistance) the candidate
$$\Xi := \bar{c}_1\bar{c}_2 + \bar{b}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{b}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{b}_2 + \bar{c}_1\bar{a}_2 + \bar{b}_1\bar{a}_2\bar{b}_2$$
which we confirm by the following calculation.
$$\begin{aligned}
&d(\Xi)(g_1, g_2, g_3)\\
&\quad = \Xi(g_2, g_3) - \Xi(g_1 g_2, g_3) + \Xi(g_1, g_2 g_3) - \Xi(g_1, g_2)\\
&\quad = (c_2c_3 + b_2a_3c_3 + b_2c_2a_3 + b_2b_3c_3 + b_2c_2b_3 + c_2a_3 + b_2a_3b_3)\\
&\qquad - \big((c_1 + c_2 + b_1a_2 + b_1b_2)c_3 + (b_1 + b_2)a_3c_3\\
&\qquad\quad + (b_1 + b_2)(c_1 + c_2 + b_1a_2 + b_1b_2)a_3 + (b_1 + b_2)b_3c_3\\
&\qquad\quad + (b_1 + b_2)(c_1 + c_2 + b_1a_2 + b_1b_2)b_3 + (c_1 + c_2 + b_1a_2 + b_1b_2)a_3\\
&\qquad\quad + (b_1 + b_2)a_3b_3\big)\\
&\qquad + \big(c_1(c_2 + c_3 + b_2a_3 + b_2b_3) + b_1(a_2 + a_3)(c_2 + c_3 + b_2a_3 + b_2b_3)\\
&\qquad\quad + b_1c_1(a_2 + a_3) + b_1(b_2 + b_3)(c_2 + c_3 + b_2a_3 + b_2b_3) + b_1c_1(b_2 + b_3)\\
&\qquad\quad + c_1(a_2 + a_3) + b_1(a_2 + a_3)(b_2 + b_3)\big)\\
&\qquad - (c_1c_2 + b_1a_2c_2 + b_1c_1a_2 + b_1b_2c_2 + b_1c_1b_2 + c_1a_2 + b_1a_2b_2)\\
&\quad = 0
\end{aligned}$$
Hence we set $w = [\bar{c}_1\bar{c}_2 + \bar{b}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{b}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{b}_2 + \bar{c}_1\bar{a}_2 + \bar{b}_1\bar{a}_2\bar{b}_2]$.

Furthermore the only non-trivial Steenrod operation is $Sq^1(w) = uw$ (see again [AM04, Theorem 2.7]).

Now we can determine the cocycle of the extension (5.2). We select according to § 2.4.1 a splitting $\sigma : D_8 \to 16\Gamma_2 c_2$ on the set level and compute $\alpha : D_8^2 \to \langle f_4 \rangle : (g_1, g_2) \mapsto \sigma(g_1)\sigma(g_2)\sigma(g_1 g_2)^{-1}$, which is a representative for the cocycle in $H^2(D_8, \langle f_4 \rangle \cong \mathbb{F}_2)$. We choose the canonical splitting which maps $f_1^a f_2^b f_3^c$ to $f_1^a f_2^b f_3^c f_4^0$. Thus we get

$$\begin{aligned}
\alpha(g_1, g_2) &= \sigma(f_1^{a_1} f_2^{b_1} f_3^{c_1})\sigma(f_1^{a_2} f_2^{b_2} f_3^{c_2})\sigma(f_1^{a_1+a_2} f_2^{b_1+b_1} f_3^{c_1+c_2+b_1a_2+b_1b_2})^{-1} \\
&= f_1^{a_1} f_2^{b_1} f_3^{c_1} f_1^{a_2} f_2^{b_2} f_3^{c_2} (f_1^{a_1+a_2} f_2^{b_1+b_1} f_3^{c_1+c_2+b_1a_2+b_1b_2})^{-1} \\
&= f_1^{a_1+a_2} f_2^{b_1+b_1} f_3^{c_1+c_2+b_1a_2+b_1b_2} f_4^{a_1a_2} (f_1^{a_1+a_2} f_2^{b_1+b_1} f_3^{c_1+c_2+b_1a_2+b_1b_2})^{-1} \\
&= f_1^{a_1+a_2} f_2^{b_1+b_1} f_3^{c_1+c_2+b_1a_2+b_1b_2} (f_1^{a_1+a_2} f_2^{b_1+b_1} f_3^{c_1+c_2+b_1a_2+b_1b_2})^{-1} f_4^{a_1a_2} \\
&= f_4^{a_1a_2}
\end{aligned}$$

and the cocycle α is $[\bar{a}_1\bar{a}_2] = u^2$.

We can now set in motion the Eilenberg-Moore spectral sequence as described in § 3.2 in the general form of

$$E_2^{**} \cong \mathrm{Tor}^{**}_{H^*(K(Z,2);\mathbb{F}_2)}(H^*(Q;\mathbb{F}_2), \mathbb{F}_2) \implies H^*(G)$$

for the central extension $Z \hookrightarrow G \twoheadrightarrow Q$. Now in this case the E_1-page is given as

$$E_1^{pq} = \bigoplus_{i+j=q} H^i(D_8) \otimes_\Gamma \mathcal{K}^{pj}(\Gamma)$$

where $\Gamma := H^*(K(\mathbb{Z}/2\mathbb{Z}, 2); \mathbb{F}_2) = \mathbb{F}_2[x_0, x_1, \dots]$. The Koszul resolution is given as $\Lambda(u_0, u_1, \dots)$ as described in § 2.6.2 with u_i corresponding to x_i. Note that $u_1 \in \mathcal{K}^{pj}$ and $u = u_{\underline{1}} \in H^i(D_8)$ are different, the underlined subscript just indicates the degree. Together we retrieve the E_1-page as seen in table 5.1. Note that for the negative degrees only the generators from the Koszul resolution are present and all combinations of them with $H^*(D_8)$ are suppressed.

Now $d_1(u_0) \in E_1^{0,2}$ is by definition $d_1(1 \otimes_\Gamma (u_0 \otimes_{\mathbb{F}_2} 1)) = 1 \otimes_\Gamma (1 \otimes_{\mathbb{F}_2} x_0)$ and the latter is understood as a pullback along the extension cocycle α (compare diagram (2.4)) where x_0 represents the homotopy class of the identity, resulting in the extension cocycle $\alpha \otimes_\Gamma (1 \otimes_{\mathbb{F}_2} 1)$ itself (see [Rus87] for another explanation of this). According to § 4 we have a Steenrod structure on the Koszul complex and thus get $d_1(u_i) = d_1((Sq_1)^i(u_0)) = (Sq_1)^i(d_1(u_0)) = (Sq_1)^i(\alpha)$. But we have $Sq^1(u^2) = 0$ and hence $d_1(u_i) = 0$ for $i > 0$. The algebra structure of the spectral sequence now determines d_1 completely and we get $E_2 \cong (H^*(D_8)/(u^2)) \otimes \Lambda(u_i \mid i > 0)$.

Now $d_j(u_i) = 0$ for $j \geq 2$ and by the algebra structure we see a collapse on the E_2-page which is displayed in table 5.2. Now in bidegree $(0, n)$ for $n > 0$ we have

$\ddots$			
u_1u_2			8
u_0u_2			7
			6
u_0u_1	u_2		5
		$\vdots$	4
	u_1	u^3, u^2v, uw, vw	3
	u_0	u^2, uv, w	2
		u, v	1
		1	0
-2	-1	0	

Table 5.1: E_1-page of the Eilenberg-Moore spectral sequence for $H^*(16\Gamma_2 c_2)$

$\ddots$		$\vdots$	
u_1u_2		w^4, uvw^3	8
		uw^3, vw^3	7
		w^3, uvw^2	6
	u_2	uw^2, vw^2	5
		w^2, uvw	4
	u_1	uw, vw	3
		uv, w	2
		u, v	1
		1	0
-2	-1	0	

Table 5.2: E_2-page of the Eilenberg-Moore spectral sequence for $H^*(16\Gamma_2 c_2)$

dimension 2 and for each total degree $2m$ with $m > 0$ we have exactly one element from the Koszul complex. Thus we retrieve the Poincaré series $1, 2, 3, 4, 5, \ldots \triangleq 1/_{1-t}$ for $H^*(16\Gamma_2 c_2)$.

At this point we would like to determine the multiplicative structure of the cohomology ring $H^*(16\Gamma_2 c_2)$, but the Koszul resolution is not amenable to ungrading and we need some additional argument to resolve the multiplicative structure. For this we turn to the Lyndon-Hochschild-Serre spectral sequence.

There we have $E_2^{pq} = H^p(D_8) \otimes H^q(\langle f_4 \rangle) \implies H^*(16\Gamma_2 c_2)$, where $H^*(\langle f_4 \rangle) = H^*(\mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[t_1]$ (and $t = [\bar{d}_1]$). Again the cocycle $u^2 \in H^2(D_8)$ has to be killed and hence $d_2(t) = u^2$. Now by the algebra structure we can compute d_2 everywhere and get $E_3 \cong (H^*(D_8)/(u^2)) \otimes \mathbb{F}_2[t^2]$. This now has the dimensions required by the Poincaré series, (the rationale is totally analogous to the argument when determining the Poincaré series). Thus the spectral sequence must collapse. We see

$$H^*(16\Gamma_2c_2) \cong \mathbb{F}_2[u_{\underline{1}}, v_{\underline{1}}, w_{\underline{2}}, x_{\underline{2}}]/(v^2+uv, u^2)\,.$$

Here u, v and w are simply inflated from $H^*(D_8)$.

Now x is roughly $[\bar{d}_1\bar{d}_2]$ and we verify

$$\begin{aligned}
&d(\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2)(g_1, g_2, g_3)\\
&\quad= (\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2)(g_2, g_3) - (\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2)(g_1g_2, g_3)\\
&\qquad\quad + (\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2)(g_1, g_2g_3) - (\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2)(g_1, g_2)\\
&\quad= (d_2d_3 + a_2a_3d_3 + a_2d_2a_3)\\
&\qquad\quad - ((d_1 + d_2 + a_1a_2)d_3 + (a_1 + a_2)a_3d_3 + (a_1 + a_2)(d_1 + d_2 + a_1a_2)a_3)\\
&\qquad\quad + (d_1(d_2 + d_3 + a_2a_3) + a_1(a_2 + a_3)(d_2 + d_3 + a_2a_3) + a_1d_1(a_2 + a_3))\\
&\qquad\quad - (d_1d_2 + a_1a_2d_2 + a_1d_1a_2)\\
&\quad= 0.
\end{aligned}$$

Thus $x = [\bar{d}_1\bar{d}_2+\bar{a}_1\bar{a}_2\bar{d}_2+\bar{a}_1\bar{d}_1\bar{a}_2]$. Furthermore the non-trivial Steenrod-Operations are $Sq^1(w) = uw$ (still valid) and $Sq^1(x) = 0$. For the latter we have to do a bit of work, the result lies in degree three and hence is a linear combination of uw, vw, ux and vx. Now we restrict to the two possible $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ subgroups. For $\langle f_1, f_3, f_4\rangle$ with cohomology $\mathbb{F}_2[p_{\underline{1}}, q_{\underline{1}}, r_{\underline{2}}]/(q^2)$, where $p = [\bar{c}_1]$, $q = [\bar{a}_1]$ and $r = [\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2]$, we get the restrictions

$$u \mapsto q, \qquad v \mapsto 0, \qquad w \mapsto p^2 + pq, \qquad x \mapsto r$$

and for $\langle f_2, f_3, f_4\rangle$ with cohomology $\mathbb{F}_2[p'_{\underline{1}}, q'_{\underline{1}}, r'_{\underline{2}}]/(q'^2)$, where $p' = [\bar{d}_1]$, $q' = [\bar{b}_1]$ and $r' = [\bar{c}_1\bar{c}_2 + \bar{b}_1\bar{b}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{b}_2]$, we get the restrictions

$$u \mapsto 0, \qquad v \mapsto q', \qquad w \mapsto r', \qquad x \mapsto p'^2.$$

Now $Sq^1(p'^2) = 0 = Sq^1(r)$ (for the second one see e. g. [CTV⁺03, § 7.4]). Hence $Sq^1(x)$ cannot contain any of the four listed constituents and thus must vanish.

5.1.3 Cohomology of $32\Gamma_3 f$

We determine the extension cocycle of (5.1) again according to § 2.4.1. We analogously choose the canonical splitting σ mapping $f_1^a f_2^b f_3^c f_4^d$ to $f_1^a f_2^b f_3^c f_4^d f_5^0$. The calculation now gives us

$$\begin{aligned}
\alpha(g_1, g_2) &= \sigma(f_1^{a_1} f_2^{b_1} f_3^{c_1} f_4^{d_1})\sigma(f_1^{a_2} f_2^{b_2} f_3^{c_2} f_4^{d_2})\\
&\qquad\qquad \sigma(f_1^{a_1+a_2} f_2^{b_1+b_1} f_3^{c_1+c_2+b_1a_2+b_1b_2} f_4^{d_1+d_2+a_1a_2})^{-1}\\
&= f_5^{d_1d_2+a_1a_2d_2+a_1d_1a_2+c_1c_2+b_1b_2c_2+b_1c_1b_2+b_1a_2c_2+b_1c_1a_2+c_1a_2+b_1a_2b_2}\,.
\end{aligned}$$

Thus we receive for the cocycle α the value $w + x$.

Again we use the Eilenberg-Moore spectral sequence this time with

$$E_2 = \operatorname{Tor}_\Gamma(H^*(16\Gamma_2 c_2), H^*(*) \cong \mathbb{F}_2) \implies H^*(32\Gamma_3 f)$$

where $\Gamma := H^*(K(\langle f_5\rangle \cong \mathbb{Z}/2\mathbb{Z}, 2); \mathbb{F}_2) \cong \mathbb{F}_2[x_0, x_1, \dots]$.

The E_1-page is given as $E_1^{**} = H^*(16\Gamma_2 c_2) \otimes_\Gamma \mathcal{K}^{**}(\Gamma)$. It is shown in table 5.3.

$\ddots$			
u_1u_2			8
u_0u_2			7
			6
u_0u_1	u_2	$\vdots$	5
		uvw, uvx, w^2, wx, x^2	4
	u_1	uw, vw, ux, vx	3
	u_0	uv, w, x	2
		u, v	1
		1	0
-2	-1	0	

Table 5.3: E_1-page of the Eilenberg-Moore spectral sequence for $H^*(32\Gamma_3 f)$

Now $d_1(u_0) = \alpha = w + x$ is the cocycle. Furthermore by the Steenrod structure $d_1(u_1) = Sq^1(\alpha) = uw$ and thus $d_1(u_i) = (Sq_1)^i(\alpha) = uw^{2^{i-1}}$ for all $i > 0$. We make a substitution

$$\hat{u}_i := \begin{cases} u_i & i \in \{0,1\} \\ u_i - w^{2^{i-2}} u_{i-1} & i > 1 \end{cases}$$

to get rid of nearly all differentials, since $d(\hat{u}_i) = d(u_i) - d(w^{2^{i-1}} u_{i-1}) = uw^{2^{i-1}} - w^{2^{i-2}} uw^{2^{i-2}} = 0$ for $i > 1$. Since $\hat{u}_i^2 = 0$ these new generators actually generate the Koszul complex, i. e. $\Lambda(\hat{u}_i \mid i \geq 0) = \mathcal{K}(\Gamma)$. Furthermore for $i > 1$ we have $Sq_1(\hat{u}_i) = Sq^{2^i}(u_i - w^{2^{i-2}} u_{i-1}) = u_{i+1} - w^{2^{i-1}} u_i = \hat{u}_{i+1}$ giving us the same Steenrod structure as before. The algebra structure of the spectral sequence now determines d_1 completely and we get $E_2 \cong (H^*(\Gamma_2 c_2)/(w + x, uw)) \otimes \Lambda(\hat{u}_i \mid i > 1)$ as displayed in table 5.4.

Now $d_j(\hat{u}_i) = 0$ for $j \geq 2$ and by the algebra structure we see a collapse on the E_2-page. Now in bidegree $(0, n)$ for $n > 0$ we have dimension 2 and for each total degree $4m$ with $m > 0$ we have exactly one element from the Koszul complex. Thus we retrieve the Poincaré series $1, 2, 2, 2, 3, 4, 4, 4, 5, 6, 6, 6, \dots \mathrel{\hat{=}} 1/(1-t)(1-t+t^2)$ for $H^*(32\Gamma_3 f)$.

As before the Eilenberg-Moore spectral sequence is not amenable to ungrading so we use the Lyndon-Hochschild-Serre spectral sequence to determine the multiplicative structure. We have $E_2^{pq} = H^p(16\Gamma_2 c_2) \otimes H^q(\langle f_5\rangle) \implies H^*(32\Gamma_3 f)$, where

$\ddots$		
$\hat{u}_3$		9
		8
		7
		6
$\hat{u}_2$	$\vdots$	5
	$uvw = uvx,\ w^2 = wx = x^2$	4
	$uw = ux,\ vw = vx$	3
	$uv,\ w = x$	2
	$u,\ v$	1
	1	0
-1	0	

Table 5.4: E_2-page of the Eilenberg-Moore spectral sequence for $H^*(32\Gamma_3 f)$

$H^*(\langle f_5 \rangle) = H^*(\mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[t_{\underline{1}}]$ (and $t = [\bar{e}_1]$). Now the cocycle $w + x \in H^2(\Gamma_2 c_2)$ has to be killed and hence $d_2(t) = w + x$. Now by the algebra structure we can compute d_2 everywhere and get $E_3 \cong (H^*(\Gamma_2 c_2)/(w + x)) \otimes \mathbb{F}_2[t^2]$. Furthermore uw has to be killed and can only be hit by $d_3(t^2) = uw$ and hence we can determine d_3 everywhere. Note that $d_3(ut^2) = u^2 w = 0$ gives us a new generator, which we will name $T_{\underline{3}}$. The relations for T are $T^2 = 0 = uT$. This gives us $E_4 \cong (H^*(\Gamma_2 c_2) \otimes \mathbb{F}_2[T, t^4])/(w + x, uw, T^2, uT)$ which is listed in table 5.5.

	$\vdots$				$\iddots$	
5						
4	t^4	$ut^4,\ vt^4$	$uvt^4,\ wt^4 = xt^4$	$vwt^4 = vxt^4$	$\dots$	
3						
2		$T(= ut^2)$	$vT(= uvt^2)$	$wT(= uwt^2 = uxt^2)$	$\dots$	
1						
0	1	$u,\ v$	$uv,\ w = x$	$vw = vx$	$w^2 = wx = x^2$	$\cdots$
	0	1	2	3	4	5

Table 5.5: E_4-page of the Lyndon-Hochschild-Serre spectral sequence for $H^*(32\Gamma_3 f)$

We take a look at the remaining parts and list the dimensions by bidegree. In bidegree $(n, 4m)$ we have dimension 1 for $n \neq 1, 2$ and in the exceptional cases we have dimension 2. In bidegree $(n, 4m + 2)$ we have dimension 1 for $n > 0$ and nothing for $n = 0$. This now gives exactly the dimensions required by the Poincaré series and the spectral sequence must collapse.

Now we have the generators $u_{\underline{1}}$, $v_{\underline{1}}$, $W_{\underline{2}}$ ($w_{\underline{2}}$ equalling $x_{\underline{2}}$), $y_{\underline{3}}$ given by T (equalling

ut^2) and $z_{\underline{4}}$ given by t^4. We also get the relations for u, v and W and furthermore z has no relations. However the relations involving y still need to be determined.

Now y corresponds roughly to $\bar{a}\bar{e}^2$. With $x_{12} := \bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2$ and $w_{12} := \bar{c}_1\bar{c}_2 + \bar{b}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{b}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{b}_2 + \bar{c}_1\bar{a}_2 + \bar{b}_1\bar{a}_2\bar{b}_2$ we can define

$$\begin{aligned}\Theta := x_{12}\bar{d}_3 + \Big(&x_{12}w_{12} + (\bar{e}_1 + \bar{e}_2)(x_{12} + w_{12}) \\ &+ \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{c}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{d}_2 + \bar{c}_1\bar{a}_2\bar{c}_2 + \bar{e}_1\bar{a}_2 + \bar{e}_1\bar{e}_2\Big)\bar{a}_3\end{aligned} \tag{5.3}$$

We can use this term as representative for y according to the following fact, which will be proven in § 5.1.4.

Fact 5.1.1: The differential of Θ vanishes, making it a cocycle.

To determine the relations of y we use the subgroup $K := \langle f_2, f_3, f_4, f_5\rangle \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, where the generators are f_2 and f_3f_4. It has cohomology $H^*(K) \cong \mathbb{F}_2[\xi_{\underline{1}}, \varphi_{\underline{1}}, \chi_{\underline{2}}]/(\varphi^2)$, where ξ belongs to the C_2.

We first compute the restrictions on the chain level. The inclusion is $K \hookrightarrow G : f_2^i f_3^j f_5^k (f_3f_4)^l \mapsto f_2^i f_3^{j+l} f_4^l f_5^k$. Hence we have restrictions

$$\bar{a} \mapsto 0, \quad \bar{b} \mapsto \bar{i}, \quad \bar{c} \mapsto \bar{j} + \bar{l}, \quad \bar{d} \mapsto \bar{l}, \quad \bar{e} \mapsto \bar{k}$$

Now the relevant cohomology classes are $\xi = [\bar{l}_1]$ and $\varphi = [\bar{i}_1]$. For y we note that Θ has the form $\bar{d}_1\bar{d}_2\bar{d}_3 + \Theta'$, where every summand of Θ' contains at least one $\bar{a}$. Thus we get the following restrictions.

$$\begin{aligned} u &= [\bar{a}_1] \mapsto 0 \\ v &= [\bar{b}_1] \mapsto [\bar{i}_1] = \varphi \\ W &= [\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2] \mapsto [\bar{l}_1\bar{l}_2] = \xi^2 \\ y &= [\bar{d}_1\bar{d}_2\bar{d}_3 + \Theta'] \mapsto [\bar{l}_1\bar{l}_2\bar{l}_3] = \xi^3 \end{aligned}$$

We have our uncertainties $y^2 = ?W^3 + ?vWy$ and $uy = ?W^2$ which upon restriction are seen to be $y^2 = W^3$ and $uy = 0$. Thus the cohomology ring is

$$H^*(\Gamma_3 f) \cong \mathbb{F}_2[u_{\underline{1}}, v_{\underline{1}}, W_{\underline{2}}, y_{\underline{3}}, z_{\underline{4}}]/(v^2 + uv, u^2, uW, y^2 + W^3, uy)\,.$$

5.1.4 Proof of vanishing differential

We will now proof fact 5.1.1, which asserts that the cohomology class Θ defined in (5.3) actually is a cocycle. First we expand Θ to retrieve the terms that comprise it.

$$
\begin{aligned}
\Theta = {} & (\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2)\bar{d}_3 \\
& + \Big((\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2)(\bar{c}_1\bar{c}_2 + \bar{b}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{b}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{b}_2 \\
& \qquad + \bar{c}_1\bar{a}_2 + \bar{b}_1\bar{a}_2\bar{b}_2) \\
& \quad + (\bar{e}_1 + \bar{e}_2)(\bar{d}_1\bar{d}_2 + \bar{a}_1\bar{a}_2\bar{d}_2 + \bar{a}_1\bar{d}_1\bar{a}_2 \\
& \qquad + \bar{c}_1\bar{c}_2 + \bar{b}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{b}_2\bar{c}_2 + \bar{b}_1\bar{c}_1\bar{b}_2 + \bar{c}_1\bar{a}_2 + \bar{b}_1\bar{a}_2\bar{b}_2) \\
& \quad + \bar{b}_1\bar{c}_1\bar{a}_2 + \bar{b}_1\bar{c}_1\bar{a}_2\bar{c}_2 + \bar{b}_1\bar{d}_2 + \bar{c}_1\bar{a}_2\bar{c}_2 + \bar{e}_1\bar{a}_2 + \bar{e}_1\bar{e}_2\Big)\bar{a}_3 \\
= {} & \bar{d}_1\bar{d}_2\bar{d}_3 + \bar{a}_1\bar{a}_2\bar{d}_2\bar{d}_3 + \bar{a}_1\bar{d}_1\bar{a}_2\bar{d}_3 \\
& + \bar{c}_1\bar{d}_1\bar{c}_2\bar{d}_2\bar{a}_3 + \bar{b}_1\bar{d}_1\bar{a}_2\bar{c}_2\bar{d}_2\bar{a}_3 + \bar{b}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{d}_2\bar{a}_3 + \bar{b}_1\bar{d}_1\bar{b}_2\bar{c}_2\bar{d}_2\bar{a}_3 + \bar{b}_1\bar{c}_1\bar{d}_1\bar{b}_2\bar{d}_2\bar{a}_3 \\
& \quad + \bar{c}_1\bar{d}_1\bar{a}_2\bar{d}_2\bar{a}_3 + \bar{b}_1\bar{d}_1\bar{a}_2\bar{b}_2\bar{d}_2\bar{a}_3 \\
& + \bar{a}_1\bar{c}_1\bar{a}_2\bar{c}_2\bar{d}_2\bar{a}_3 + \bar{a}_1\bar{b}_1\bar{a}_2\bar{c}_2\bar{d}_2\bar{a}_3 + \bar{a}_1\bar{b}_1\bar{c}_1\bar{a}_2\bar{d}_2\bar{a}_3 + \bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2\bar{c}_2\bar{d}_2\bar{a}_3 \\
& \quad + \bar{a}_1\bar{b}_1\bar{c}_1\bar{a}_2\bar{b}_2\bar{d}_2\bar{a}_3 + \bar{a}_1\bar{c}_1\bar{a}_2\bar{d}_2\bar{a}_3 + \bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2\bar{d}_2\bar{a}_3 \\
& + \bar{a}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{c}_2\bar{a}_3 + \bar{a}_1\bar{b}_1\bar{d}_1\bar{a}_2\bar{c}_2\bar{a}_3 + \bar{a}_1\bar{b}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{a}_3 + \bar{a}_1\bar{b}_1\bar{d}_1\bar{a}_2\bar{b}_2\bar{c}_2\bar{a}_3 \\
& \quad + \bar{a}_1\bar{b}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{b}_2\bar{a}_3 + \bar{a}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{a}_3 + \bar{a}_1\bar{b}_1\bar{d}_1\bar{a}_2\bar{b}_2\bar{a}_3 \\
& + \bar{d}_1\bar{e}_1\bar{d}_2\bar{a}_3 + \bar{a}_1\bar{e}_1\bar{a}_2\bar{d}_2\bar{a}_3 + \bar{a}_1\bar{d}_1\bar{e}_1\bar{a}_2\bar{a}_3 + \bar{c}_1\bar{e}_1\bar{c}_2\bar{a}_3 + \bar{b}_1\bar{e}_1\bar{a}_2\bar{c}_2\bar{a}_3 + \bar{b}_1\bar{c}_1\bar{e}_1\bar{a}_2\bar{a}_3 \\
& \quad + \bar{b}_1\bar{e}_1\bar{b}_2\bar{c}_2\bar{a}_3 + \bar{b}_1\bar{c}_1\bar{e}_1\bar{b}_2\bar{a}_3 + \bar{c}_1\bar{e}_1\bar{a}_2\bar{a}_3 + \bar{b}_1\bar{e}_1\bar{a}_2\bar{b}_2\bar{a}_3 \\
& + \bar{d}_1\bar{d}_2\bar{e}_2\bar{a}_3 + \bar{a}_1\bar{a}_2\bar{d}_2\bar{e}_2\bar{a}_3 + \bar{a}_1\bar{d}_1\bar{a}_2\bar{e}_2\bar{a}_3 + \bar{c}_1\bar{c}_2\bar{e}_2\bar{a}_3 + \bar{b}_1\bar{a}_2\bar{c}_2\bar{e}_2\bar{a}_3 + \bar{b}_1\bar{c}_1\bar{a}_2\bar{e}_2\bar{a}_3 \\
& \quad + \bar{b}_1\bar{b}_2\bar{c}_2\bar{e}_2\bar{a}_3 + \bar{b}_1\bar{c}_1\bar{b}_2\bar{e}_2\bar{a}_3 + \bar{c}_1\bar{a}_2\bar{e}_2\bar{a}_3 + \bar{b}_1\bar{a}_2\bar{b}_2\bar{e}_2\bar{a}_3 \\
& + \bar{b}_1\bar{c}_1\bar{a}_2\bar{a}_3 + \bar{b}_1\bar{c}_1\bar{a}_2\bar{c}_2\bar{a}_3 + \bar{b}_1\bar{d}_2\bar{a}_3 + \bar{c}_1\bar{a}_2\bar{c}_2\bar{a}_3 + \bar{e}_1\bar{a}_2\bar{a}_3 + \bar{e}_1\bar{e}_2\bar{a}_3
\end{aligned}
$$

Now we look at its differential.

$$
\begin{aligned}
& d(\Theta)(g_1, g_2, g_3, g_4) \\
& \quad = \Theta(g_2, g_3, g_4) + \Theta(g_1g_2, g_3, g_4) + \Theta(g_1, g_2g_3, g_4) + \Theta(g_1, g_2, g_3g_4) + \Theta(g_1, g_2, g_3)
\end{aligned}
$$

We will examine this term by term. In the following table each term is listed with its differential. Each summand of the differential has a superscript i/j which denotes the matching partner against which this summand cancels. Specifically it is the j-th summand of the differential of θ_i. The summands pair perfectly so that none are left and hence the differential of Θ vanishes.

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_1 =$ $\bar{d}_1\bar{d}_2\bar{d}_3$	$a_1a_2d_3d_4{}^{2/2} + d_1a_2a_3d_4{}^{3/3} + d_1d_2a_3a_4{}^{49/10}$
$\theta_2 =$ $\bar{a}_1\bar{a}_2\bar{d}_2\bar{d}_3$	$a_1a_2d_2a_3a_4{}^{49/2} + a_1a_2d_3d_4{}^{1/1} + a_1d_2a_3d_4{}^{3/2}$
$\theta_3 =$ $\bar{a}_1\bar{d}_1\bar{a}_2\bar{d}_3$	$a_1d_1a_2a_3a_4{}^{49/1} + a_1d_2a_3d_4{}^{2/3} + d_1a_2a_3d_4{}^{1/2}$
$\theta_4 =$ $\bar{c}_1\bar{d}_1\bar{c}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2b_2c_3d_3a_4{}^{14/7} + a_1b_1a_2c_3d_3a_4{}^{12/7} + a_1c_1a_2c_3d_3a_4{}^{11/9} + a_1a_2c_2c_3d_3a_4{}^{36/14} + b_1d_1a_2c_3d_3a_4{}^{5/9}$ $+ b_1d_1b_2c_3d_3a_4{}^{7/15} + b_1a_2d_2c_3d_3a_4{}^{39/15} + b_1b_2d_2c_3d_3a_4{}^{41/19} + c_1d_1a_2b_2a_3b_3a_4{}^{22/22} + c_1d_1a_2b_2a_3a_4{}^{20/20}$ $+ c_1d_1a_2c_2a_3a_4{}^{27/28} + c_1d_1a_2a_3c_3a_4{}^{18/18} + c_1d_1b_2d_2a_3a_4{}^{30/26} + c_1d_1b_2d_2b_3a_4{}^{32/26} + c_1d_1b_2a_3d_3a_4{}^{6/16}$ $+ c_1d_1b_2b_3d_3a_4{}^{8/18} + c_1d_1c_2d_3a_4{}^{25/22} + c_1d_1d_2c_3a_4{}^{28/19} + c_1d_2c_3d_3a_4{}^{38/25} + d_1c_2c_3d_3a_4{}^{35/25}$
$\theta_5 =$ $\bar{b}_1\bar{d}_1\bar{a}_2\bar{c}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2a_3c_3d_3a_4{}^{11/1} + a_1a_2b_2a_3c_3d_3a_4{}^{36/9} + b_1d_1a_2b_2d_2a_3a_4{}^{30/17} + b_1d_1a_2b_2d_2b_3a_4{}^{32/17} + b_1d_1a_2b_2a_3d_3a_4{}^{6/10}$ $+ b_1d_1a_2b_2b_3d_3a_4{}^{8/12} + b_1d_1a_2c_2d_3a_4{}^{25/16} + b_1d_1a_2d_2c_3a_4{}^{28/12} + b_1d_1a_2c_3d_3a_4{}^{4/5} + b_1d_1b_2d_2a_3b_3a_4{}^{7/10}$ $+ b_1d_1b_2d_2a_3a_4{}^{7/11} + b_1d_1b_2a_3b_3d_3a_4{}^{7/13} + b_1d_1b_2a_3d_3a_4{}^{7/14} + b_1d_1c_2d_2a_3a_4{}^{30/19} + b_1d_1c_2a_3d_3a_4{}^{6/12}$ $+ b_1d_1d_2a_3c_3a_4{}^{29/7} + b_1d_2a_3c_3d_3a_4{}^{39/33} + d_1b_2a_3c_3d_3a_4{}^{35/20}$
$\theta_6 =$ $\bar{b}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{d}_2\bar{a}_3$	$a_1b_1c_1a_2a_3d_3a_4{}^{26/1} + a_1b_1a_2b_2a_3d_3a_4{}^{9/1} + a_1b_1a_2c_2a_3d_3a_4{}^{26/4} + a_1b_1a_2a_3d_3a_4{}^{9/2} + a_1c_1a_2b_2a_3d_3a_4{}^{11/5}$ $+ a_1a_2b_2c_2a_3d_3a_4{}^{36/3} + b_1c_1d_1a_2d_3a_4{}^{25/11} + b_1c_1d_1d_2a_3a_4{}^{30/13} + b_1c_1d_2a_3d_3a_4{}^{40/17} + b_1d_1a_2b_2a_3d_3a_4{}^{5/5}$ $+ b_1d_1a_2a_3d_3a_4{}^{9/5} + b_1d_1c_2a_3d_3a_4{}^{5/15} + b_1a_2b_2d_2a_3d_3a_4{}^{39/4} + b_1a_2d_2a_3d_3a_4{}^{9/7} + b_1c_2d_2a_3d_3a_4{}^{39/28}$ $+ c_1d_1b_2a_3d_3a_4{}^{4/15} + c_1b_2d_2a_3d_3a_4{}^{38/14} + d_1b_2c_2a_3d_3a_4{}^{35/14}$
$\theta_7 =$ $\bar{b}_1\bar{d}_1\bar{b}_2\bar{c}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2b_3c_3d_3a_4{}^{14/11} + a_1a_2b_2b_3c_3d_3a_4{}^{36/10} + b_1d_1a_2b_2c_2a_3a_4{}^{27/22} + b_1d_1a_2b_2a_3b_3a_4{}^{22/16} + b_1d_1a_2b_2a_3c_3a_4{}^{18/14}$ $+ b_1d_1a_2b_2a_3a_4{}^{20/14} + b_1d_1a_2c_2a_3b_3a_4{}^{22/17} + b_1d_1a_2a_3b_3c_3a_4{}^{21/14} + b_1d_1b_2c_2d_3a_4{}^{25/17} + b_1d_1b_2d_2a_3b_3a_4{}^{5/10}$ $+ b_1d_1b_2d_2a_3a_4{}^{5/11} + b_1d_1b_2d_2c_3a_4{}^{28/13} + b_1d_1b_2a_3b_3d_3a_4{}^{5/12} + b_1d_1b_2a_3d_3a_4{}^{5/13} + b_1d_1b_2c_3d_3a_4{}^{4/6}$ $+ b_1d_1c_2d_2b_3a_4{}^{32/19} + b_1d_1c_2b_3d_3a_4{}^{8/14} + b_1d_1d_2b_3c_3a_4{}^{31/7} + b_1d_2b_3c_3d_3a_4{}^{41/31} + d_1b_2b_3c_3d_3a_4{}^{35/21}$
$\theta_8 =$ $\bar{b}_1\bar{c}_1\bar{d}_1\bar{b}_2\bar{d}_2\bar{a}_3$	$a_1b_1c_1a_2b_3d_3a_4{}^{15/3} + a_1b_1a_2b_2b_3d_3a_4{}^{12/4} + a_1b_1a_2c_2b_3d_3a_4{}^{14/9} + a_1b_1a_2b_3d_3a_4{}^{17/3} + a_1c_1a_2b_2b_3d_3a_4{}^{11/6}$ $+ a_1a_2b_2c_2b_3d_3a_4{}^{36/4} + b_1c_1d_1a_2b_2a_3a_4{}^{27/18} + b_1c_1d_1a_2a_3b_3a_4{}^{22/14} + b_1c_1d_1b_2d_3a_4{}^{25/12} + b_1c_1d_1d_2b_3a_4{}^{32/13}$ $+ b_1c_1d_2b_3d_3a_4{}^{42/13} + b_1d_1a_2b_2b_3d_3a_4{}^{5/6} + b_1d_1a_2b_3d_3a_4{}^{10/5} + b_1d_1c_2b_3d_3a_4{}^{7/17} + b_1a_2b_2d_2b_3d_3a_4{}^{39/5}$ $+ b_1a_2d_2b_3d_3a_4{}^{44/9} + b_1c_2d_2b_3d_3a_4{}^{41/26} + c_1d_1b_2b_3d_3a_4{}^{4/16} + c_1b_2d_2b_3d_3a_4{}^{38/15} + d_1b_2c_2b_3d_3a_4{}^{35/15}$

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_9 =$ $\bar{c}_1\bar{d}_1\bar{a}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2b_2a_3d_3a_4{}^{6/2} + a_1b_1a_2a_3d_3a_4{}^{6/4} + a_1c_1a_2a_3d_3a_4{}^{26/6} + a_1a_2c_2a_3d_3a_4{}^{36/13} + b_1d_1a_2a_3d_3a_4{}^{6/11} + b_1d_1b_2a_3d_3a_4{}^{10/7} + b_1a_2d_2a_3d_3a_4{}^{6/14} + b_1b_2d_2a_3d_3a_4{}^{39/22} + c_1d_1a_2d_3a_4{}^{25/21} + c_1d_1d_2a_3a_4{}^{33/19} + c_1d_2a_3d_3a_4{}^{43/19} + d_1c_2a_3d_3a_4{}^{35/24}$
$\theta_{10} =$ $\bar{b}_1\bar{d}_1\bar{a}_2\bar{b}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2a_3b_3d_3a_4{}^{15/8} + a_1a_2b_2a_3b_3d_3a_4{}^{36/8} + b_1d_1a_2b_2d_3a_4{}^{25/15} + b_1d_1a_2d_2b_3a_4{}^{32/18} + b_1d_1a_2b_3d_3a_4{}^{8/13} + b_1d_1b_2d_2a_3a_4{}^{33/13} + b_1d_1b_2a_3d_3a_4{}^{9/6} + b_1d_1d_2a_3b_3a_4{}^{34/7} + b_1d_2a_3b_3d_3a_4{}^{44/21} + d_1b_2a_3b_3d_3a_4{}^{35/19}$
$\theta_{11} =$ $\bar{a}_1\bar{c}_1\bar{a}_2\bar{c}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2a_3c_3d_3a_4{}^{5/1} + a_1b_1b_2a_3c_3d_3a_4{}^{14/18} + a_1c_1a_2b_2d_2a_3a_4{}^{30/10} + a_1c_1a_2b_2d_2b_3a_4{}^{32/10} + a_1c_1a_2b_2a_3d_3a_4{}^{6/5} + a_1c_1a_2b_2b_3d_3a_4{}^{8/5} + a_1c_1a_2c_2d_3a_4{}^{25/6} + a_1c_1a_2d_2c_3a_4{}^{28/6} + a_1c_1a_2c_3d_3a_4{}^{4/3} + a_1c_1b_2d_2a_3b_3a_4{}^{22/11} + a_1c_1b_2d_2a_3a_4{}^{20/9} + a_1c_1b_2a_3b_3d_3a_4{}^{15/10} + a_1c_1b_2a_3d_3a_4{}^{13/6} + a_1c_1c_2d_2a_3a_4{}^{27/14} + a_1c_1c_2a_3d_3a_4{}^{26/7} + a_1c_1d_2a_3c_3a_4{}^{18/11} + a_1c_2a_3c_3d_3a_4{}^{36/30} + b_1a_2b_2a_3c_3d_3a_4{}^{41/7} + b_1a_2a_3c_3d_3a_4{}^{12/16} + c_1a_2a_3c_3d_3a_4{}^{38/10}$
$\theta_{12} =$ $\bar{a}_1\bar{b}_1\bar{a}_2\bar{c}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2b_2d_2a_3a_4{}^{14/3} + a_1b_1a_2b_2d_2b_3a_4{}^{32/6} + a_1b_1a_2b_2a_3d_3a_4{}^{13/3} + a_1b_1a_2b_2b_3d_3a_4{}^{8/2} + a_1b_1a_2c_2d_3a_4{}^{25/5} + a_1b_1a_2d_2c_3a_4{}^{28/4} + a_1b_1a_2c_3d_3a_4{}^{4/2} + a_1b_1b_2d_2a_3b_3a_4{}^{14/14} + a_1b_1b_2d_2a_3a_4{}^{14/16} + a_1b_1b_2a_3b_3d_3a_4{}^{14/17} + a_1b_1b_2a_3d_3a_4{}^{14/19} + a_1b_1c_2d_2a_3a_4{}^{20/7} + a_1b_1c_2a_3d_3a_4{}^{13/5} + a_1b_1d_2a_3c_3a_4{}^{19/7} + a_1b_2a_3c_3d_3a_4{}^{36/27} + b_1a_2a_3c_3d_3a_4{}^{11/19}$
$\theta_{13} =$ $\bar{a}_1\bar{b}_1\bar{c}_1\bar{a}_2\bar{d}_2\bar{a}_3$	$a_1b_1c_1a_2d_3a_4{}^{25/2} + a_1b_1c_1d_2a_3a_4{}^{20/1} + a_1b_1a_2b_2a_3d_3a_4{}^{12/3} + a_1b_1a_2a_3d_3a_4{}^{16/1} + a_1b_1c_2a_3d_3a_4{}^{12/13} + a_1c_1b_2a_3d_3a_4{}^{11/13} + a_1b_2c_2a_3d_3a_4{}^{36/21} + b_1c_1a_2a_3d_3a_4{}^{26/13} + b_1a_2b_2a_3d_3a_4{}^{16/6} + b_1a_2c_2a_3d_3a_4{}^{26/16} + b_1a_2a_3d_3a_4{}^{16/7} + c_1a_2b_2a_3d_3a_4{}^{38/6}$
$\theta_{14} =$ $\bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2\bar{c}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2b_2c_2d_3a_4{}^{25/3} + a_1b_1a_2b_2d_2a_3b_3a_4{}^{22/7} + a_1b_1a_2b_2d_2a_3a_4{}^{12/1} + a_1b_1a_2b_2d_2c_3a_4{}^{28/3} + a_1b_1a_2b_2a_3b_3d_3a_4{}^{15/7} + a_1b_1a_2b_2a_3d_3a_4{}^{26/3} + a_1b_1a_2b_2c_3d_3a_4{}^{4/1} + a_1b_1a_2c_2d_2b_3a_4{}^{32/7} + a_1b_1a_2c_2b_3d_3a_4{}^{8/3} + a_1b_1a_2d_2b_3c_3a_4{}^{31/2} + a_1b_1a_2b_3c_3d_3a_4{}^{7/1} + a_1b_1b_2c_2d_2a_3a_4{}^{27/10} + a_1b_1b_2c_2a_3d_3a_4{}^{26/5} + a_1b_1b_2d_2a_3b_3a_4{}^{12/8} + a_1b_1b_2d_2a_3c_3a_4{}^{18/4} + a_1b_1b_2d_2a_3a_4{}^{12/9} + a_1b_1b_2a_3b_3d_3a_4{}^{12/10} + a_1b_1b_2a_3c_3d_3a_4{}^{11/2} + a_1b_1b_2a_3d_3a_4{}^{12/11} + a_1b_1c_2d_2a_3b_3a_4{}^{22/9} + a_1b_1c_2a_3b_3d_3a_4{}^{15/9} + a_1b_1d_2a_3b_3c_3a_4{}^{21/11} + a_1b_2a_3b_3c_3d_3a_4{}^{36/25} + b_1a_2a_3b_3c_3d_3a_4{}^{41/12}$
$\theta_{15} =$ $\bar{a}_1\bar{b}_1\bar{c}_1\bar{a}_2\bar{b}_2\bar{d}_2\bar{a}_3$	$a_1b_1c_1a_2b_2d_3a_4{}^{25/1} + a_1b_1c_1a_2d_2b_3a_4{}^{32/2} + a_1b_1c_1a_2b_3d_3a_4{}^{8/1} + a_1b_1c_1b_2d_2a_3a_4{}^{27/4} + a_1b_1c_1b_2a_3d_3a_4{}^{26/2} + a_1b_1c_1d_2a_3b_3a_4{}^{22/3} + a_1b_1a_2b_2a_3b_3d_3a_4{}^{14/5} + a_1b_1a_2a_3b_3d_3a_4{}^{10/1} + a_1b_1c_2a_3b_3d_3a_4{}^{14/21} + a_1c_1b_2a_3b_3d_3a_4{}^{11/12} + a_1b_2c_2a_3b_3d_3a_4{}^{36/20} + b_1c_1a_2a_3b_3d_3a_4{}^{42/4} + b_1a_2b_2a_3b_3d_3a_4{}^{41/6} + b_1a_2c_2a_3b_3d_3a_4{}^{41/10} + b_1a_2a_3b_3d_3a_4{}^{17/8} + c_1a_2b_2a_3b_3d_3a_4{}^{38/5}$

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_{16} = \bar{a}_1\bar{c}_1\bar{a}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2a_3d_3{a_4}^{13/4} + a_1b_1b_2a_3d_3{a_4}^{17/5} + a_1c_1a_2d_3{a_4}^{25/7} + a_1c_1d_2a_3{a_4}^{23/5} + a_1c_2a_3d_3{a_4}^{36/31} + b_1a_2b_2a_3d_3{a_4}^{13/9} + b_1a_2a_3d_3{a_4}^{13/11} + c_1a_2a_3d_3{a_4}^{26/18}$
$\theta_{17} = \bar{a}_1\bar{b}_1\bar{a}_2\bar{b}_2\bar{d}_2\bar{a}_3$	$a_1b_1a_2b_2d_3{a_4}^{25/4} + a_1b_1a_2d_2b_3{a_4}^{32/8} + a_1b_1a_2b_3d_3{a_4}^{8/4} + a_1b_1b_2d_2a_3{a_4}^{23/4} + a_1b_1b_2a_3d_3{a_4}^{16/2} + a_1b_1d_2a_3b_3{a_4}^{24/3} + a_1b_2a_3b_3d_3{a_4}^{36/26} + b_1a_2a_3b_3d_3{a_4}^{15/15}$
$\theta_{18} = \bar{a}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{c}_2\bar{a}_3$	$a_1b_1d_1a_2a_3c_3{a_4}^{29/1} + a_1b_1d_1b_2a_3c_3{a_4}^{21/8} + a_1b_1a_2d_2a_3c_3{a_4}^{29/2} + a_1b_1b_2d_2a_3c_3{a_4}^{14/15} + a_1c_1d_1a_2b_2a_3{a_4}^{30/9} + a_1c_1d_1a_2b_2b_3{a_4}^{32/9} + a_1c_1d_1a_2c_3{a_4}^{28/5} + a_1c_1d_1b_2a_3b_3{a_4}^{22/10} + a_1c_1d_1b_2a_3{a_4}^{20/8} + a_1c_1d_1c_2a_3{a_4}^{27/12} + a_1c_1d_2a_3c_3{a_4}^{11/16} + a_1d_1c_2a_3c_3{a_4}^{37/15} + a_1c_2d_2a_3c_3{a_4}^{36/28} + b_1d_1a_2b_2a_3c_3{a_4}^{7/5} + b_1d_1a_2a_3c_3{a_4}^{19/10} + b_1a_2b_2d_2a_3c_3{a_4}^{41/4} + b_1a_2d_2a_3c_3{a_4}^{19/11} + c_1d_1a_2a_3c_3{a_4}^{4/12} + c_1a_2d_2a_3c_3{a_4}^{38/9} + d_1a_2c_2a_3c_3{a_4}^{35/7}$
$\theta_{19} = \bar{a}_1\bar{b}_1\bar{d}_1\bar{a}_2\bar{c}_2\bar{a}_3$	$a_1b_1d_1a_2b_2a_3{a_4}^{20/2} + a_1b_1d_1a_2b_2b_3{a_4}^{32/3} + a_1b_1d_1a_2c_3{a_4}^{28/2} + a_1b_1d_1b_2a_3b_3{a_4}^{21/7} + a_1b_1d_1b_2a_3{a_4}^{21/9} + a_1b_1d_1c_2a_3{a_4}^{20/4} + a_1b_1d_2a_3c_3{a_4}^{12/14} + a_1d_1b_2a_3c_3{a_4}^{37/14} + a_1b_2d_2a_3c_3{a_4}^{36/24} + b_1d_1a_2a_3c_3{a_4}^{18/15} + b_1a_2d_2a_3c_3{a_4}^{18/17} + d_1a_2b_2a_3c_3{a_4}^{35/6}$
$\theta_{20} = \bar{a}_1\bar{b}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{a}_3$	$a_1b_1c_1d_2a_3{a_4}^{13/2} + a_1b_1d_1a_2b_2a_3{a_4}^{19/1} + a_1b_1d_1a_2a_3{a_4}^{23/1} + a_1b_1d_1c_2a_3{a_4}^{19/6} + a_1b_1a_2b_2d_2a_3{a_4}^{27/8} + a_1b_1a_2d_2a_3{a_4}^{23/3} + a_1b_1c_2d_2a_3{a_4}^{12/12} + a_1c_1d_1b_2a_3{a_4}^{18/9} + a_1c_1b_2d_2a_3{a_4}^{11/11} + a_1d_1b_2c_2a_3{a_4}^{37/11} + a_1b_2c_2d_2a_3{a_4}^{36/19} + b_1c_1d_1a_2a_3{a_4}^{27/19} + b_1c_1a_2d_2a_3{a_4}^{27/21} + b_1d_1a_2b_2a_3{a_4}^{7/6} + b_1d_1a_2c_2a_3{a_4}^{27/24} + b_1d_1a_2a_3{a_4}^{23/9} + b_1a_2b_2d_2a_3{a_4}^{23/10} + b_1a_2c_2d_2a_3{a_4}^{27/27} + b_1a_2d_2a_3{a_4}^{23/11} + c_1d_1a_2b_2a_3{a_4}^{4/10} + c_1a_2b_2d_2a_3{a_4}^{38/4} + d_1a_2b_2c_2a_3{a_4}^{35/3}$
$\theta_{21} = \bar{a}_1\bar{b}_1\bar{d}_1\bar{a}_2\bar{b}_2\bar{c}_2\bar{a}_3$	$a_1b_1d_1a_2b_2a_3b_3{a_4}^{22/4} + a_1b_1d_1a_2b_2a_3{a_4}^{27/5} + a_1b_1d_1a_2b_2c_3{a_4}^{28/1} + a_1b_1d_1a_2c_2b_3{a_4}^{32/4} + a_1b_1d_1a_2b_3c_3{a_4}^{31/1} + a_1b_1d_1b_2c_2a_3{a_4}^{27/7} + a_1b_1d_1b_2a_3b_3{a_4}^{19/4} + a_1b_1d_1b_2a_3c_3{a_4}^{18/2} + a_1b_1d_1b_2a_3{a_4}^{19/5} + a_1b_1d_1c_2a_3b_3{a_4}^{22/6} + a_1b_1d_2a_3b_3c_3{a_4}^{14/22} + a_1d_1b_2a_3b_3c_3{a_4}^{37/12} + a_1b_2d_2a_3b_3c_3{a_4}^{36/22} + b_1d_1a_2a_3b_3c_3{a_4}^{7/8} + b_1a_2d_2a_3b_3c_3{a_4}^{41/11} + d_1a_2b_2a_3b_3c_3{a_4}^{35/4}$
$\theta_{22} = \bar{a}_1\bar{b}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{b}_2\bar{a}_3$	$a_1b_1c_1d_1a_2b_3{a_4}^{32/1} + a_1b_1c_1d_1b_2a_3{a_4}^{27/2} + a_1b_1c_1d_2a_3b_3{a_4}^{15/6} + a_1b_1d_1a_2b_2a_3b_3{a_4}^{21/1} + a_1b_1d_1a_2a_3b_3{a_4}^{34/1} + a_1b_1d_1c_2a_3b_3{a_4}^{21/10} + a_1b_1a_2b_2d_2a_3b_3{a_4}^{14/2} + a_1b_1a_2d_2a_3b_3{a_4}^{34/2} + a_1b_1c_2d_2a_3b_3{a_4}^{14/20} + a_1c_1d_1b_2a_3b_3{a_4}^{18/8} + a_1c_1b_2d_2a_3b_3{a_4}^{11/10} + a_1d_1b_2c_2a_3b_3{a_4}^{37/10} + a_1b_2c_2d_2a_3b_3{a_4}^{36/18} + b_1c_1d_1a_2a_3b_3{a_4}^{8/8} + b_1c_1a_2d_2a_3b_3{a_4}^{42/3} + b_1d_1a_2b_2a_3b_3{a_4}^{7/4} + b_1d_1a_2c_2a_3b_3{a_4}^{7/7} + b_1d_1a_2a_3b_3{a_4}^{24/6} + b_1a_2b_2d_2a_3b_3{a_4}^{41/3} + b_1a_2c_2d_2a_3b_3{a_4}^{41/9} + b_1a_2d_2a_3b_3{a_4}^{24/7} + c_1d_1a_2b_2a_3b_3{a_4}^{4/9} + c_1a_2b_2d_2a_3b_3{a_4}^{38/3} + d_1a_2b_2c_2a_3b_3{a_4}^{35/2}$

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_{23} =$ $\bar{a}_1\bar{c}_1\bar{d}_1\bar{a}_2\bar{a}_3$	$a_1b_1d_1a_2a_3{a_4}^{20/3} + a_1b_1d_1b_2a_3{a_4}^{24/2} + a_1b_1a_2d_2a_3{a_4}^{20/6} + a_1b_1b_2d_2a_3{a_4}^{17/4} + a_1c_1d_2a_3{a_4}^{16/4} + a_1d_1c_2a_3{a_4}^{37/16} + a_1c_2d_2a_3{a_4}^{36/29} + b_1d_1a_2b_2a_3{a_4}^{27/23} + b_1d_1a_2a_3{a_4}^{20/16} + b_1a_2b_2d_2a_3{a_4}^{20/17} + b_1a_2d_2a_3{a_4}^{20/19} + c_1d_1a_2a_3{a_4}^{27/29} + c_1a_2d_2a_3{a_4}^{27/31} + d_1a_2c_2a_3{a_4}^{35/8}$
$\theta_{24} =$ $\bar{a}_1\bar{b}_1\bar{d}_1\bar{a}_2\bar{b}_2\bar{a}_3$	$a_1b_1d_1a_2b_3{a_4}^{32/5} + a_1b_1d_1b_2a_3{a_4}^{23/2} + a_1b_1d_2a_3b_3{a_4}^{17/6} + a_1d_1b_2a_3b_3{a_4}^{37/13} + a_1b_2d_2a_3b_3{a_4}^{36/23} + b_1d_1a_2a_3b_3{a_4}^{22/18} + b_1a_2d_2a_3b_3{a_4}^{22/21} + d_1a_2b_2a_3b_3{a_4}^{35/5}$
$\theta_{25} =$ $\bar{d}_1\bar{e}_1\bar{d}_2\bar{a}_3$	$a_1b_1c_1a_2b_2d_3{a_4}^{15/1} + a_1b_1c_1a_2d_3{a_4}^{13/1} + a_1b_1a_2b_2c_2d_3{a_4}^{14/1} + a_1b_1a_2b_2d_3{a_4}^{17/1} + a_1b_1a_2c_2d_3{a_4}^{12/5} + a_1c_1a_2c_2d_3{a_4}^{11/7} + a_1c_1a_2d_3{a_4}^{16/3} + a_1d_1a_2d_2d_3{a_4}^{37/8} + a_1e_1a_2d_3{a_4}^{26/9} + a_1a_2e_2d_3{a_4}^{36/16} + b_1c_1d_1a_2d_3{a_4}^{6/7} + b_1c_1d_1b_2d_3{a_4}^{8/9} + b_1c_1a_2d_2d_3{a_4}^{40/8} + b_1c_1b_2d_2d_3{a_4}^{42/7} + b_1d_1a_2b_2d_3{a_4}^{10/3} + b_1d_1a_2c_2d_3{a_4}^{5/7} + b_1d_1b_2c_2d_3{a_4}^{7/9} + b_1a_2b_2d_2d_3{a_4}^{44/3} + b_1a_2c_2d_2d_3{a_4}^{39/11} + b_1b_2c_2d_2d_3{a_4}^{41/13} + c_1d_1a_2d_3{a_4}^{9/9} + c_1d_1c_2d_3{a_4}^{4/17} + c_1a_2d_2d_3{a_4}^{43/10} + c_1c_2d_2d_3{a_4}^{38/21} + d_1e_1a_2a_3{a_4}^{27/32} + d_1e_2d_3{a_4}^{35/27} + e_1d_2d_3{a_4}^{50/20}$
$\theta_{26} =$ $\bar{a}_1\bar{e}_1\bar{a}_2\bar{d}_2\bar{a}_3$	$a_1b_1c_1a_2a_3d_3{a_4}^{6/1} + a_1b_1c_1b_2a_3d_3{a_4}^{15/5} + a_1b_1a_2b_2a_3d_3{a_4}^{14/6} + a_1b_1a_2c_2a_3d_3{a_4}^{6/3} + a_1b_1b_2c_2a_3d_3{a_4}^{14/13} + a_1c_1a_2a_3d_3{a_4}^{9/3} + a_1c_1c_2a_3d_3{a_4}^{11/15} + a_1d_1d_2a_3d_3{a_4}^{37/17} + a_1e_1a_2d_3{a_4}^{25/9} + a_1e_1d_2a_3{a_4}^{27/16} + a_1e_2a_3d_3{a_4}^{36/34} + b_1c_1a_2b_2a_3d_3{a_4}^{42/2} + b_1c_1a_2a_3d_3{a_4}^{13/8} + b_1a_2b_2c_2a_3d_3{a_4}^{41/2} + b_1a_2b_2a_3d_3{a_4}^{41/8} + b_1a_2c_2a_3d_3{a_4}^{13/10} + c_1a_2c_2a_3d_3{a_4}^{38/8} + c_1a_2a_3d_3{a_4}^{16/8} + d_1a_2d_2a_3d_3{a_4}^{35/9} + e_1a_2a_3d_3{a_4}^{50/12}$
$\theta_{27} =$ $\bar{a}_1\bar{d}_1\bar{e}_1\bar{a}_2\bar{a}_3$	$a_1b_1c_1d_1a_2a_3{a_4}^{30/1} + a_1b_1c_1d_1b_2a_3{a_4}^{22/2} + a_1b_1c_1a_2d_2a_3{a_4}^{30/2} + a_1b_1c_1b_2d_2a_3{a_4}^{15/4} + a_1b_1d_1a_2b_2a_3{a_4}^{21/2} + a_1b_1d_1a_2c_2a_3{a_4}^{30/4} + a_1b_1d_1b_2c_2a_3{a_4}^{21/6} + a_1b_1a_2b_2d_2a_3{a_4}^{20/5} + a_1b_1a_2c_2d_2a_3{a_4}^{30/7} + a_1b_1b_2c_2d_2a_3{a_4}^{14/12} + a_1c_1d_1a_2a_3{a_4}^{33/5} + a_1c_1d_1c_2a_3{a_4}^{18/10} + a_1c_1a_2d_2a_3{a_4}^{33/6} + a_1c_1c_2d_2a_3{a_4}^{11/14} + a_1d_1e_2a_3{a_4}^{37/18} + a_1e_1d_2a_3{a_4}^{26/10} + a_1d_2e_2a_3{a_4}^{36/32} + b_1c_1d_1a_2b_2a_3{a_4}^{8/7} + b_1c_1d_1a_2a_3{a_4}^{20/12} + b_1c_1a_2b_2d_2a_3{a_4}^{42/1} + b_1c_1a_2d_2a_3{a_4}^{20/13} + b_1d_1a_2b_2c_2a_3{a_4}^{7/3} + b_1d_1a_2b_2a_3{a_4}^{23/8} + b_1d_1a_2c_2a_3{a_4}^{20/15} + b_1a_2b_2c_2d_2a_3{a_4}^{41/1} + b_1a_2b_2d_2a_3{a_4}^{41/5} + b_1a_2c_2d_2a_3{a_4}^{20/18} + c_1d_1a_2c_2a_3{a_4}^{4/11} + c_1d_1a_2a_3{a_4}^{23/12} + c_1a_2c_2d_2a_3{a_4}^{38/7} + c_1a_2d_2a_3{a_4}^{23/13} + d_1e_1a_2a_3{a_4}^{25/25} + d_1a_2e_2a_3{a_4}^{35/10} + e_1a_2d_2a_3{a_4}^{50/11}$

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_{28} =$ $\bar{c}_1\bar{e}_1\bar{c}_2\bar{a}_3$	$a_1b_1d_1a_2b_2c_3{a_4}^{21/3} + a_1b_1d_1a_2c_3{a_4}^{19/3} + a_1b_1a_2b_2d_2c_3{a_4}^{14/4} + a_1b_1a_2d_2c_3{a_4}^{12/6} + a_1c_1d_1a_2c_3{a_4}^{18/7}$ $+ a_1c_1a_2d_2c_3{a_4}^{11/8} + a_1d_1a_2c_2c_3{a_4}^{37/7} + a_1a_2c_2d_2c_3{a_4}^{36/12} + b_1c_1a_2c_2c_3{a_4}^{40/7} + b_1c_1a_2c_3{a_4}^{46/3}$ $+ b_1c_1b_2c_2c_3{a_4}^{42/6} + b_1d_1a_2d_2c_3{a_4}^{5/8} + b_1d_1b_2d_2c_3{a_4}^{7/12} + b_1e_1a_2c_3{a_4}^{29/10} + b_1e_1b_2c_3{a_4}^{31/10}$ $+ b_1a_2b_2c_2c_3{a_4}^{44/2} + b_1a_2e_2c_3{a_4}^{39/16} + b_1b_2e_2c_3{a_4}^{41/22} + c_1d_1d_2c_3{a_4}^{4/18} + c_1e_1b_2a_3{a_4}^{30/27}$ $+ c_1e_1b_2b_3{a_4}^{32/27} + c_1a_2c_2c_3{a_4}^{43/9} + c_1a_2c_3{a_4}^{48/5} + c_1e_2c_3{a_4}^{38/26} + d_1c_2d_2c_3{a_4}^{35/23}$ $+ e_1c_2c_3{a_4}^{50/19}$
$\theta_{29} =$ $\bar{b}_1\bar{e}_1\bar{a}_2\bar{c}_2\bar{a}_3$	$a_1b_1d_1a_2a_3c_3{a_4}^{18/1} + a_1b_1a_2d_2a_3c_3{a_4}^{18/3} + a_1d_1a_2b_2a_3c_3{a_4}^{37/4} + a_1a_2b_2d_2a_3c_3{a_4}^{36/6} + b_1c_1a_2b_2a_3c_3{a_4}^{40/4}$ $+ b_1c_1c_2a_3c_3{a_4}^{40/15} + b_1d_1d_2a_3c_3{a_4}^{5/16} + b_1e_1a_2b_2a_3{a_4}^{30/20} + b_1e_1a_2b_2b_3{a_4}^{32/20} + b_1e_1a_2c_2c_3{a_4}^{28/14}$ $+ b_1e_1b_2a_3b_3{a_4}^{31/8} + b_1e_1b_2a_3{a_4}^{31/9} + b_1e_1c_2a_3{a_4}^{30/22} + b_1a_2b_2c_2a_3c_3{a_4}^{39/1} + b_1a_2c_2a_3c_3{a_4}^{39/12}$ $+ b_1e_2a_3c_3{a_4}^{39/34} + c_1a_2b_2a_3c_3{a_4}^{43/6} + c_1b_2c_2a_3c_3{a_4}^{38/11} + d_1b_2d_2a_3c_3{a_4}^{35/17} + e_1b_2a_3c_3{a_4}^{50/16}$
$\theta_{30} =$ $\bar{b}_1\bar{c}_1\bar{e}_1\bar{a}_2\bar{a}_3$	$a_1b_1c_1d_1a_2a_3{a_4}^{27/1} + a_1b_1c_1a_2d_2a_3{a_4}^{27/3} + a_1b_1d_1a_2b_2a_3{a_4}^{33/1} + a_1b_1d_1a_2c_2a_3{a_4}^{27/6} + a_1b_1d_1a_2a_3{a_4}^{33/2}$ $+ a_1b_1a_2b_2d_2a_3{a_4}^{33/3} + a_1b_1a_2c_2d_2a_3{a_4}^{27/9} + a_1b_1a_2d_2a_3{a_4}^{33/4} + a_1c_1d_1a_2b_2a_3{a_4}^{18/5} + a_1c_1a_2b_2d_2a_3{a_4}^{11/3}$ $+ a_1d_1a_2b_2c_2a_3{a_4}^{37/1} + a_1a_2b_2c_2d_2a_3{a_4}^{36/1} + b_1c_1d_1d_2a_3{a_4}^{6/8} + b_1c_1a_2b_2c_2a_3{a_4}^{40/1} + b_1c_1a_2b_2a_3{a_4}^{46/1}$ $+ b_1c_1e_2a_3{a_4}^{40/18} + b_1d_1a_2b_2d_2a_3{a_4}^{5/3} + b_1d_1a_2d_2a_3{a_4}^{33/12} + b_1d_1c_2d_2a_3{a_4}^{5/14} + b_1e_1a_2b_2a_3{a_4}^{29/8}$ $+ b_1e_1a_2a_3{a_4}^{33/14} + b_1e_1c_2a_3{a_4}^{29/13} + b_1a_2b_2e_2a_3{a_4}^{39/6} + b_1a_2e_2a_3{a_4}^{33/17} + b_1c_2e_2a_3{a_4}^{39/29}$ $+ c_1d_1b_2d_2a_3{a_4}^{4/13} + c_1e_1b_2a_3{a_4}^{28/20} + c_1a_2b_2c_2a_3{a_4}^{43/3} + c_1a_2b_2a_3{a_4}^{48/3} + c_1b_2e_2a_3{a_4}^{38/16}$ $+ d_1b_2c_2d_2a_3{a_4}^{35/12} + e_1b_2c_2a_3{a_4}^{50/13}$
$\theta_{31} =$ $\bar{b}_1\bar{e}_1\bar{b}_2\bar{c}_2\bar{a}_3$	$a_1b_1d_1a_2b_3c_3{a_4}^{21/5} + a_1b_1a_2d_2b_3c_3{a_4}^{14/10} + a_1d_1a_2b_2b_3c_3{a_4}^{37/5} + a_1a_2b_2d_2b_3c_3{a_4}^{36/7} + b_1c_1a_2b_2b_3c_3{a_4}^{40/5}$ $+ b_1c_1c_2b_3c_3{a_4}^{42/12} + b_1d_1d_2b_3c_3{a_4}^{7/18} + b_1e_1b_2a_3b_3{a_4}^{29/11} + b_1e_1b_2a_3{a_4}^{29/12} + b_1e_1b_2c_3{a_4}^{28/15}$ $+ b_1e_1c_2b_3{a_4}^{32/22} + b_1a_2b_2c_2b_3c_3{a_4}^{39/3} + b_1a_2c_2b_3c_3{a_4}^{44/8} + b_1e_2b_3c_3{a_4}^{41/32} + c_1a_2b_2b_3c_3{a_4}^{43/7}$ $+ c_1b_2c_2b_3c_3{a_4}^{38/13} + d_1b_2d_2b_3c_3{a_4}^{35/18} + e_1b_2b_3c_3{a_4}^{50/17}$

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_{32} =$ $\bar{b}_1\bar{c}_1\bar{e}_1\bar{b}_2\bar{a}_3$	$a_1b_1c_1d_1a_2b_3a_4{}^{22/1} + a_1b_1c_1a_2d_2b_3a_4{}^{15/2} + a_1b_1d_1a_2b_2b_3a_4{}^{19/2} + a_1b_1d_1a_2c_2b_3a_4{}^{21/4} + a_1b_1d_1a_2b_3a_4{}^{24/1}$ $+ a_1b_1a_2b_2d_2b_3a_4{}^{12/2} + a_1b_1a_2c_2d_2b_3a_4{}^{14/8} + a_1b_1a_2d_2b_3a_4{}^{17/2} + a_1c_1d_1a_2b_2b_3a_4{}^{18/6} + a_1c_1a_2b_2d_2b_3a_4{}^{11/4}$ $+ a_1d_1a_2b_2c_2b_3a_4{}^{37/2} + a_1a_2b_2c_2d_2b_3a_4{}^{36/2} + b_1c_1d_1d_2b_3a_4{}^{8/10} + b_1c_1a_2b_2c_2b_3a_4{}^{40/2} + b_1c_1a_2b_2b_3a_4{}^{46/2}$ $+ b_1c_1e_2b_3a_4{}^{42/14} + b_1d_1a_2b_2d_2b_3a_4{}^{5/4} + b_1d_1a_2d_2b_3a_4{}^{10/4} + b_1d_1c_2d_2b_3a_4{}^{7/16} + b_1e_1a_2b_2b_3a_4{}^{29/9}$ $+ b_1e_1a_2b_3a_4{}^{34/8} + b_1e_1c_2b_3a_4{}^{31/11} + b_1a_2b_2e_2b_3a_4{}^{39/7} + b_1a_2e_2b_3a_4{}^{44/10} + b_1c_2e_2b_3a_4{}^{41/27}$ $+ c_1d_1b_2d_2b_3a_4{}^{4/14} + c_1e_1b_2b_3a_4{}^{28/21} + c_1a_2b_2c_2b_3a_4{}^{43/4} + c_1a_2b_2b_3a_4{}^{48/4} + c_1b_2e_2b_3a_4{}^{38/17}$ $+ d_1b_2c_2d_2b_3a_4{}^{35/13} + e_1b_2c_2b_3a_4{}^{50/14}$
$\theta_{33} =$ $\bar{c}_1\bar{e}_1\bar{a}_2\bar{a}_3$	$a_1b_1d_1a_2b_2a_3a_4{}^{30/3} + a_1b_1d_1a_2a_3a_4{}^{30/5} + a_1b_1a_2b_2d_2a_3a_4{}^{30/6} + a_1b_1a_2d_2a_3a_4{}^{30/8} + a_1c_1d_1a_2a_3a_4{}^{27/11}$ $+ a_1c_1a_2d_2a_3a_4{}^{27/13} + a_1d_1a_2c_2a_3a_4{}^{37/6} + a_1a_2c_2d_2a_3a_4{}^{36/11} + b_1c_1a_2c_2a_3a_4{}^{40/6} + b_1c_1a_2a_3a_4{}^{49/3}$ $+ b_1c_1b_2c_2a_3a_4{}^{40/11} + b_1d_1a_2d_2a_3a_4{}^{30/18} + b_1d_1b_2d_2a_3a_4{}^{10/6} + b_1e_1a_2a_3a_4{}^{30/21} + b_1e_1b_2a_3a_4{}^{34/9}$ $+ b_1a_2b_2c_2a_3a_4{}^{39/2} + b_1a_2e_2a_3a_4{}^{30/24} + b_1b_2e_2a_3a_4{}^{39/24} + c_1d_1d_2a_3a_4{}^{9/10} + c_1a_2c_2a_3a_4{}^{43/8}$ $+ c_1a_2a_3a_4{}^{49/8} + c_1e_2a_3a_4{}^{43/20} + d_1c_2d_2a_3a_4{}^{35/22} + e_1c_2a_3a_4{}^{50/18}$
$\theta_{34} =$ $\bar{b}_1\bar{e}_1\bar{a}_2\bar{b}_2\bar{a}_3$	$a_1b_1d_1a_2a_3b_3a_4{}^{22/5} + a_1b_1a_2d_2a_3b_3a_4{}^{22/8} + a_1d_1a_2b_2a_3b_3a_4{}^{37/3} + a_1a_2b_2d_2a_3b_3a_4{}^{36/5} + b_1c_1a_2b_2a_3b_3a_4{}^{40/3}$ $+ b_1c_1c_2a_3b_3a_4{}^{42/11} + b_1d_1d_2a_3b_3a_4{}^{10/8} + b_1e_1a_2b_3a_4{}^{32/21} + b_1e_1b_2a_3a_4{}^{33/15} + b_1a_2b_2c_2a_3b_3a_4{}^{44/1}$ $+ b_1a_2c_2a_3b_3a_4{}^{44/7} + b_1e_2a_3b_3a_4{}^{44/22} + c_1a_2b_2a_3b_3a_4{}^{43/5} + c_1b_2c_2a_3b_3a_4{}^{43/12} + d_1b_2d_2a_3b_3a_4{}^{35/16}$ $+ e_1b_2a_3b_3a_4{}^{50/15}$
$\theta_{35} =$ $\bar{d}_1\bar{d}_2\bar{e}_2\bar{a}_3$	$a_1a_2d_3e_3a_4{}^{36/17} + d_1a_2b_2c_2a_3b_3a_4{}^{22/24} + d_1a_2b_2c_2a_3a_4{}^{20/22} + d_1a_2b_2a_3b_3c_3a_4{}^{21/16} + d_1a_2b_2a_3b_3a_4{}^{24/8}$ $+ d_1a_2b_2a_3c_3a_4{}^{19/12} + d_1a_2c_2a_3c_3a_4{}^{18/20} + d_1a_2c_2a_3a_4{}^{23/14} + d_1a_2d_2a_3d_3a_4{}^{26/19} + d_1a_2e_2a_3a_4{}^{27/33}$ $+ d_1a_2a_3e_3a_4{}^{37/20} + d_1b_2c_2d_2a_3a_4{}^{30/31} + d_1b_2c_2d_2b_3a_4{}^{32/31} + d_1b_2c_2a_3d_3a_4{}^{6/18} + d_1b_2c_2b_3d_3a_4{}^{8/20}$ $+ d_1b_2d_2a_3b_3a_4{}^{34/15} + d_1b_2d_2a_3c_3a_4{}^{29/19} + d_1b_2d_2b_3c_3a_4{}^{31/17} + d_1b_2a_3b_3d_3a_4{}^{10/10} + d_1b_2a_3c_3d_3a_4{}^{5/18}$ $+ d_1b_2b_3c_3d_3a_4{}^{7/20} + d_1c_2d_2a_3a_4{}^{33/23} + d_1c_2d_2c_3a_4{}^{28/25} + d_1c_2a_3d_3a_4{}^{9/12} + d_1c_2c_3d_3a_4{}^{4/20}$ $+ d_1d_2e_3a_4{}^{50/10} + d_1e_2d_3a_4{}^{25/26}$

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_{36} =$ $\bar{a}_1\bar{a}_2\bar{d}_2\bar{e}_2\bar{a}_3$	$a_1a_2b_2c_2d_2a_3{a_4}^{30/12} + a_1a_2b_2c_2d_2b_3{a_4}^{32/12} + a_1a_2b_2c_2a_3d_3{a_4}^{6/6} + a_1a_2b_2c_2b_3d_3{a_4}^{8/6} + a_1a_2b_2d_2a_3b_3{a_4}^{34/4}$ $+ a_1a_2b_2d_2a_3c_3{a_4}^{29/4} + a_1a_2b_2d_2b_3c_3{a_4}^{31/4} + a_1a_2b_2a_3b_3d_3{a_4}^{10/2} + a_1a_2b_2a_3c_3d_3{a_4}^{5/2} + a_1a_2b_2b_3c_3d_3{a_4}^{7/2}$ $+ a_1a_2c_2d_2a_3{a_4}^{33/8} + a_1a_2c_2d_2c_3{a_4}^{28/8} + a_1a_2c_2a_3d_3{a_4}^{9/4} + a_1a_2c_2c_3d_3{a_4}^{4/4} + a_1a_2d_2e_3{a_4}^{50/2}$ $+ a_1a_2e_2d_3{a_4}^{25/10} + a_1a_2d_3e_3{a_4}^{35/1} + a_1b_2c_2d_2a_3b_3{a_4}^{22/13} + a_1b_2c_2d_2a_3{a_4}^{20/11} + a_1b_2c_2a_3b_3d_3{a_4}^{15/11}$ $+ a_1b_2c_2a_3d_3{a_4}^{13/7} + a_1b_2d_2a_3b_3c_3{a_4}^{21/13} + a_1b_2d_2a_3b_3{a_4}^{24/5} + a_1b_2d_2a_3c_3{a_4}^{19/9} + a_1b_2a_3b_3c_3d_3{a_4}^{14/23}$ $+ a_1b_2a_3b_3d_3{a_4}^{17/7} + a_1b_2a_3c_3d_3{a_4}^{12/15} + a_1c_2d_2a_3c_3{a_4}^{18/13} + a_1c_2d_2a_3{a_4}^{23/7} + a_1c_2a_3c_3d_3{a_4}^{11/17}$ $+ a_1c_2a_3d_3{a_4}^{16/5} + a_1d_2e_2a_3{a_4}^{27/17} + a_1d_2a_3e_3{a_4}^{37/19} + a_1e_2a_3d_3{a_4}^{26/11}$
$\theta_{37} =$ $\bar{a}_1\bar{d}_1\bar{a}_2\bar{e}_2\bar{a}_3$	$a_1d_1a_2b_2c_2a_3{a_4}^{30/11} + a_1d_1a_2b_2c_2b_3{a_4}^{32/11} + a_1d_1a_2b_2a_3b_3{a_4}^{34/3} + a_1d_1a_2b_2a_3c_3{a_4}^{29/3} + a_1d_1a_2b_2b_3c_3{a_4}^{31/3}$ $+ a_1d_1a_2c_2a_3{a_4}^{33/7} + a_1d_1a_2c_2c_3{a_4}^{28/7} + a_1d_1a_2d_2d_3{a_4}^{25/8} + a_1d_1a_2e_3{a_4}^{50/1} + a_1d_1b_2c_2a_3b_3{a_4}^{22/12}$ $+ a_1d_1b_2c_2a_3{a_4}^{20/10} + a_1d_1b_2a_3b_3c_3{a_4}^{21/12} + a_1d_1b_2a_3b_3{a_4}^{24/4} + a_1d_1b_2a_3c_3{a_4}^{19/8} + a_1d_1c_2a_3c_3{a_4}^{18/12}$ $+ a_1d_1c_2a_3{a_4}^{23/6} + a_1d_1d_2a_3d_3{a_4}^{26/8} + a_1d_1e_2a_3{a_4}^{27/15} + a_1d_2a_3e_3{a_4}^{36/33} + d_1a_2a_3e_3{a_4}^{35/11}$
$\theta_{38} =$ $\bar{c}_1\bar{c}_2\bar{e}_2\bar{a}_3$	$b_1a_2c_3e_3{a_4}^{39/17} + b_1b_2c_3e_3{a_4}^{41/25} + c_1a_2b_2d_2a_3b_3{a_4}^{22/23} + c_1a_2b_2d_2a_3{a_4}^{20/21} + c_1a_2b_2a_3b_3d_3{a_4}^{15/16}$ $+ c_1a_2b_2a_3d_3{a_4}^{13/12} + c_1a_2c_2d_2a_3{a_4}^{27/30} + c_1a_2c_2a_3d_3{a_4}^{26/17} + c_1a_2d_2a_3c_3{a_4}^{18/19} + c_1a_2a_3c_3d_3{a_4}^{11/20}$ $+ c_1b_2c_2a_3c_3{a_4}^{29/18} + c_1b_2c_2a_3{a_4}^{43/13} + c_1b_2c_2b_3c_3{a_4}^{31/16} + c_1b_2d_2a_3d_3{a_4}^{6/17} + c_1b_2d_2b_3d_3{a_4}^{8/19}$ $+ c_1b_2e_2a_3{a_4}^{30/30} + c_1b_2e_2b_3{a_4}^{32/30} + c_1b_2a_3b_3c_3{a_4}^{43/14} + c_1b_2a_3e_3{a_4}^{40/22} + c_1b_2b_3e_3{a_4}^{42/18}$ $+ c_1c_2d_2d_3{a_4}^{25/24} + c_1c_2a_3c_3{a_4}^{43/17} + c_1c_2a_3{a_4}^{43/18} + c_1c_2e_3{a_4}^{50/9} + c_1d_2c_3d_3{a_4}^{4/19}$ $+ c_1e_2c_3{a_4}^{28/24}$
$\theta_{39} =$ $\bar{b}_1\bar{a}_2\bar{c}_2\bar{e}_2\bar{a}_3$	$b_1a_2b_2c_2a_3c_3{a_4}^{29/14} + b_1a_2b_2c_2a_3{a_4}^{33/16} + b_1a_2b_2c_2b_3c_3{a_4}^{31/12} + b_1a_2b_2d_2a_3d_3{a_4}^{6/13} + b_1a_2b_2d_2b_3d_3{a_4}^{8/15}$ $+ b_1a_2b_2e_2a_3{a_4}^{30/23} + b_1a_2b_2e_2b_3{a_4}^{32/23} + b_1a_2b_2a_3b_3c_3{a_4}^{44/4} + b_1a_2b_2a_3e_3{a_4}^{40/19} + b_1a_2b_2b_3e_3{a_4}^{42/15}$ $+ b_1a_2c_2d_2d_3{a_4}^{25/19} + b_1a_2c_2a_3c_3{a_4}^{29/15} + b_1a_2c_2a_3{a_4}^{49/6} + b_1a_2c_2e_3{a_4}^{50/6} + b_1a_2d_2c_3d_3{a_4}^{4/7}$ $+ b_1a_2e_2c_3{a_4}^{28/17} + b_1a_2c_3e_3{a_4}^{38/1} + b_1b_2c_2a_3b_3c_3{a_4}^{41/14} + b_1b_2c_2a_3c_3{a_4}^{44/13} + b_1b_2c_2a_3{a_4}^{49/7}$ $+ b_1b_2d_2a_3b_3d_3{a_4}^{41/17} + b_1b_2d_2a_3d_3{a_4}^{9/8} + b_1b_2e_2a_3b_3{a_4}^{41/20} + b_1b_2e_2a_3{a_4}^{33/18} + b_1b_2a_3b_3c_3{a_4}^{44/16}$ $+ b_1b_2a_3b_3e_3{a_4}^{41/23} + b_1b_2a_3e_3{a_4}^{41/24} + b_1c_2d_2a_3d_3{a_4}^{6/15} + b_1c_2e_2a_3{a_4}^{30/25} + b_1c_2a_3c_3{a_4}^{46/9}$ $+ b_1c_2a_3e_3{a_4}^{40/21} + b_1c_2a_3{a_4}^{45/3} + b_1d_2a_3c_3d_3{a_4}^{5/17} + b_1e_2a_3c_3{a_4}^{29/16}$

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_{40} =$ $\bar{b}_1\bar{c}_1\bar{a}_2\bar{e}_2\bar{a}_3$	$b_1c_1a_2b_2c_2a_3{a_4}^{30/14} + b_1c_1a_2b_2c_2b_3{a_4}^{32/14} + b_1c_1a_2b_2a_3b_3{a_4}^{34/5} + b_1c_1a_2b_2a_3c_3{a_4}^{29/5} + b_1c_1a_2b_2b_3c_3{a_4}^{31/5}$ $+ b_1c_1a_2c_2a_3{a_4}^{33/9} + b_1c_1a_2c_2c_3{a_4}^{28/9} + b_1c_1a_2d_2d_3{a_4}^{25/13} + b_1c_1a_2e_3{a_4}^{50/3} + b_1c_1b_2c_2a_3b_3{a_4}^{42/5}$ $+ b_1c_1b_2c_2a_3{a_4}^{33/11} + b_1c_1b_2a_3b_3c_3{a_4}^{42/8} + b_1c_1b_2a_3b_3{a_4}^{46/4} + b_1c_1b_2a_3c_3{a_4}^{42/9} + b_1c_1c_2a_3c_3{a_4}^{29/6}$ $+ b_1c_1c_2a_3{a_4}^{46/6} + b_1c_1d_2a_3d_3{a_4}^{6/9} + b_1c_1e_2a_3{a_4}^{30/16} + b_1a_2b_2a_3e_3{a_4}^{39/9} + b_1a_2a_3e_3{a_4}^{43/1}$ $+ b_1c_2a_3e_3{a_4}^{39/31} + c_1b_2a_3e_3{a_4}^{38/19}$
$\theta_{41} =$ $\bar{b}_1\bar{b}_2\bar{c}_2\bar{e}_2\bar{a}_3$	$b_1a_2b_2c_2d_2a_3{a_4}^{27/25} + b_1a_2b_2c_2a_3d_3{a_4}^{26/14} + b_1a_2b_2d_2a_3b_3{a_4}^{22/19} + b_1a_2b_2d_2a_3c_3{a_4}^{18/16} + b_1a_2b_2d_2a_3{a_4}^{27/26}$ $+ b_1a_2b_2a_3b_3d_3{a_4}^{15/13} + b_1a_2b_2a_3c_3d_3{a_4}^{11/18} + b_1a_2b_2a_3d_3{a_4}^{26/15} + b_1a_2c_2d_2a_3b_3{a_4}^{22/20} + b_1a_2c_2a_3b_3d_3{a_4}^{15/14}$ $+ b_1a_2d_2a_3b_3c_3{a_4}^{21/15} + b_1a_2a_3b_3c_3d_3{a_4}^{14/24} + b_1b_2c_2d_2d_3{a_4}^{25/20} + b_1b_2c_2a_3b_3c_3{a_4}^{39/18} + b_1b_2c_2a_3b_3{a_4}^{44/12}$ $+ b_1b_2c_2e_3{a_4}^{50/7} + b_1b_2d_2a_3b_3d_3{a_4}^{39/21} + b_1b_2d_2a_3d_3{a_4}^{44/14} + b_1b_2d_2c_3d_3{a_4}^{4/8} + b_1b_2e_2a_3b_3{a_4}^{39/23}$ $+ b_1b_2e_2a_3{a_4}^{44/15} + b_1b_2e_2c_3{a_4}^{28/18} + b_1b_2a_3b_3e_3{a_4}^{39/26} + b_1b_2a_3e_3{a_4}^{39/27} + b_1b_2c_3e_3{a_4}^{38/2}$ $+ b_1c_2d_2b_3d_3{a_4}^{8/17} + b_1c_2e_2b_3{a_4}^{32/25} + b_1c_2a_3b_3c_3{a_4}^{44/19} + b_1c_2a_3b_3{a_4}^{44/20} + b_1c_2b_3e_3{a_4}^{42/17}$ $+ b_1d_2b_3c_3d_3{a_4}^{7/19} + b_1e_2b_3c_3{a_4}^{31/14}$
$\theta_{42} =$ $\bar{b}_1\bar{c}_1\bar{b}_2\bar{e}_2\bar{a}_3$	$b_1c_1a_2b_2d_2a_3{a_4}^{27/20} + b_1c_1a_2b_2a_3d_3{a_4}^{26/12} + b_1c_1a_2d_2a_3b_3{a_4}^{22/15} + b_1c_1a_2a_3b_3d_3{a_4}^{15/12} + b_1c_1b_2c_2a_3b_3{a_4}^{40/10}$ $+ b_1c_1b_2c_2c_3{a_4}^{28/11} + b_1c_1b_2d_2d_3{a_4}^{25/14} + b_1c_1b_2a_3b_3c_3{a_4}^{40/12} + b_1c_1b_2a_3c_3{a_4}^{40/14} + b_1c_1b_2e_3{a_4}^{50/4}$ $+ b_1c_1c_2a_3b_3{a_4}^{34/6} + b_1c_1c_2b_3c_3{a_4}^{31/6} + b_1c_1d_2b_3d_3{a_4}^{8/11} + b_1c_1e_2b_3{a_4}^{32/16} + b_1a_2b_2b_3e_3{a_4}^{39/10}$ $+ b_1a_2b_3e_3{a_4}^{44/11} + b_1c_2b_3e_3{a_4}^{41/30} + c_1b_2b_3e_3{a_4}^{38/20}$
$\theta_{43} =$ $\bar{c}_1\bar{a}_2\bar{e}_2\bar{a}_3$	$b_1a_2a_3e_3{a_4}^{40/20} + b_1b_2a_3e_3{a_4}^{44/18} + c_1a_2b_2c_2a_3{a_4}^{30/28} + c_1a_2b_2c_2b_3{a_4}^{32/28} + c_1a_2b_2a_3b_3{a_4}^{34/13}$ $+ c_1a_2b_2a_3c_3{a_4}^{29/17} + c_1a_2b_2b_3c_3{a_4}^{31/15} + c_1a_2c_2a_3{a_4}^{33/20} + c_1a_2c_2c_3{a_4}^{28/22} + c_1a_2d_2d_3{a_4}^{25/23}$ $+ c_1a_2e_3{a_4}^{50/8} + c_1b_2c_2a_3b_3{a_4}^{34/14} + c_1b_2c_2a_3{a_4}^{38/12} + c_1b_2a_3b_3c_3{a_4}^{38/18} + c_1b_2a_3b_3{a_4}^{48/6}$ $+ c_1b_2a_3c_3{a_4}^{46/10} + c_1c_2a_3c_3{a_4}^{38/22} + c_1c_2a_3{a_4}^{38/23} + c_1d_2a_3d_3{a_4}^{9/11} + c_1e_2a_3{a_4}^{33/22}$
$\theta_{44} =$ $\bar{b}_1\bar{a}_2\bar{b}_2\bar{e}_2\bar{a}_3$	$b_1a_2b_2c_2a_3b_3{a_4}^{34/10} + b_1a_2b_2c_2c_3{a_4}^{28/16} + b_1a_2b_2d_2d_3{a_4}^{25/18} + b_1a_2b_2a_3b_3c_3{a_4}^{39/8} + b_1a_2b_2a_3c_3{a_4}^{46/7}$ $+ b_1a_2b_2e_3{a_4}^{50/5} + b_1a_2c_2a_3b_3{a_4}^{34/11} + b_1a_2c_2b_3c_3{a_4}^{31/13} + b_1a_2d_2b_3d_3{a_4}^{8/16} + b_1a_2e_2b_3{a_4}^{32/24}$ $+ b_1a_2b_3e_3{a_4}^{42/16} + b_1b_2c_2a_3b_3{a_4}^{41/15} + b_1b_2c_2a_3c_3{a_4}^{39/19} + b_1b_2d_2a_3d_3{a_4}^{41/18} + b_1b_2e_2a_3{a_4}^{41/21}$ $+ b_1b_2a_3b_3c_3{a_4}^{39/25} + b_1b_2a_3c_3{a_4}^{48/2} + b_1b_2a_3e_3{a_4}^{43/2} + b_1c_2a_3b_3c_3{a_4}^{41/28} + b_1c_2a_3b_3{a_4}^{41/29}$ $+ b_1d_2a_3b_3d_3{a_4}^{10/9} + b_1e_2a_3b_3{a_4}^{34/12}$

θ_i	$d(\theta_i)(g_1, g_2, g_3, g_4)$
$\theta_{45} =$ $\bar{b}_1\bar{c}_1\bar{a}_2\bar{a}_3$	$b_1a_2b_2a_3{a_4}^{49/5} + b_1a_2a_3{a_4}^{47/1} + b_1c_2a_3{a_4}^{39/32} + c_1b_2a_3{a_4}^{48/7}$
$\theta_{46} =$ $\bar{b}_1\bar{c}_1\bar{a}_2\bar{c}_2\bar{a}_3$	$b_1c_1a_2b_2a_3{a_4}^{30/15} + b_1c_1a_2b_2b_3{a_4}^{32/15} + b_1c_1a_2c_3{a_4}^{28/10} + b_1c_1b_2a_3b_3{a_4}^{40/13} + b_1c_1b_2a_3{a_4}^{49/4}$ $+ b_1c_1c_2a_3{a_4}^{40/16} + b_1a_2b_2a_3c_3{a_4}^{44/5} + b_1a_2a_3c_3{a_4}^{48/1} + b_1c_2a_3c_3{a_4}^{39/30} + c_1b_2a_3c_3{a_4}^{43/16}$
$\theta_{47} =$ $\bar{b}_1\bar{d}_2\bar{a}_3$	$b_1a_2a_3{a_4}^{45/2}$
$\theta_{48} =$ $\bar{c}_1\bar{a}_2\bar{c}_2\bar{a}_3$	$b_1a_2a_3c_3{a_4}^{46/8} + b_1b_2a_3c_3{a_4}^{44/17} + c_1a_2b_2a_3{a_4}^{30/29} + c_1a_2b_2b_3{a_4}^{32/29} + c_1a_2c_3{a_4}^{28/23}$ $+ c_1b_2a_3b_3{a_4}^{43/15} + c_1b_2a_3{a_4}^{45/4} + c_1c_2a_3{a_4}^{49/9}$
$\theta_{49} =$ $\bar{e}_1\bar{a}_2\bar{a}_3$	$a_1d_1a_2a_3{a_4}^{3/1} + a_1a_2d_2a_3{a_4}^{2/1} + b_1c_1a_2a_3{a_4}^{33/10} + b_1c_1b_2a_3{a_4}^{46/5} + b_1a_2b_2a_3{a_4}^{45/1}$ $+ b_1a_2c_2a_3{a_4}^{39/13} + b_1b_2c_2a_3{a_4}^{39/20} + c_1a_2a_3{a_4}^{33/21} + c_1c_2a_3{a_4}^{48/8} + d_1d_2a_3{a_4}^{1/3}$
$\theta_{50} =$ $\bar{e}_1\bar{e}_2\bar{a}_3$	$a_1d_1a_2e_3{a_4}^{37/9} + a_1a_2d_2e_3{a_4}^{36/15} + b_1c_1a_2e_3{a_4}^{40/9} + b_1c_1b_2e_3{a_4}^{42/10} + b_1a_2b_2e_3{a_4}^{44/6}$ $+ b_1a_2c_2e_3{a_4}^{39/14} + b_1b_2c_2e_3{a_4}^{41/16} + c_1a_2e_3{a_4}^{43/11} + c_1c_2e_3{a_4}^{38/24} + d_1d_2e_3{a_4}^{35/26}$ $+ e_1a_2d_2a_3{a_4}^{27/34} + e_1a_2a_3d_3{a_4}^{26/20} + e_1b_2c_2a_3{a_4}^{30/32} + e_1b_2c_2b_3{a_4}^{32/32} + e_1b_2a_3b_3{a_4}^{34/16}$ $+ e_1b_2a_3c_3{a_4}^{29/20} + e_1b_2b_3c_3{a_4}^{31/18} + e_1c_2a_3{a_4}^{33/24} + e_1c_2c_3{a_4}^{28/26} + e_1d_2d_3{a_4}^{25/27}$

5.2 $SU_3(2^n)$

We want to compute the dimensions of the cohomology $H^*(Syl_2(SU_3(q)), \mathbb{F}_2)$, where $q = 2^n$. The group $SU_3(4)$ was investigated in [Cla94], but as noted in [Gre04] there are some errors in Clark's article. We will use the tool of the Eilenberg-Moore spectral sequence to verify the results of [Gre04] and explore the possibilities of the tool. We will work over $\mathbb{F}_{q^2}$ as in [Cla94] and use the approach presented there as this allows an elegant description of the situation.

Let V be an m-dimensional $\mathbb{F}_{q^2}$-vector-space and β be any hermitian form on V. Then the group $U_m(q)$ is given as the $\mathbb{F}_{q^2}$-linear automorphisms of V preserving β. All choices for β result in isomorphic groups and in the usual applications β is taken to be induced by the identity matrix. If however we take the matrix $J = \left(\delta_{i,m-(j-1)}\right)_{i,j=1}^{m}$ then $U_m(q)$ is given as a subset of the upper triangular matrices. We now specialise to $m = 3$. A 2-Sylow subgroup G of $SU_3(q)$ is given by

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & \bar{a} \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b \in \mathbb{F}_{q^2}$ and $b + \bar{b} = a\bar{a}$, where $\bar{x} := x^q$ is a kind of conjugation since $x\bar{x} \in \mathbb{F}_q$ by $(x\bar{x})^{q-1} = (x^{q+1})^{q-1} = x^{q^2-1} = 1$. It has size q^3 and the center $Z(G)$ has size q and is given by those matrices with $a = 0$. The quotient $G/Z(G)$ is elementary abelian of size q^2. Thus we have the central extension

$$1 \longrightarrow Z := Z(G) \longrightarrow G \longrightarrow Q := G/Z(G) \longrightarrow 1.$$

The normalizer of G in $SU_3(q)$ is the cyclic group of diagonal matrices given by $T := \langle \operatorname{diag}(\zeta, \bar{\zeta}\zeta^{-1}, \bar{\zeta}^{-1}) \rangle = \langle \operatorname{diag}(\zeta, \zeta^{q-1}, \zeta^{-q}) \rangle$, where ζ is a primitive root in $\mathbb{F}_{q^2}$.

We step back for a short abstract interlude. Let W be $\mathbb{F}_q$ seen as an $\mathbb{F}_2$ vector space of dimension n and α be a primitive element for the extension $\mathbb{F}_q/\mathbb{F}_2$. Now 1, α, α^2, $\ldots$, α^{n-1} is a basis for W. Let L be the vector map given by $L : W \to W : x \mapsto \alpha x$. The matrix representing L is in rational canonical form and the minimal and characteristic polynomial of L is thus the minimal polynomial m_α of α. Further it is the minimal polynomial of $L^* : W^* \to W^*$ and thus of $L^* : H^1(W, \mathbb{F}_2) \to H^1(W, \mathbb{F}_2)$. Now we extend scalars and see, that $L^* : H^1(W, \mathbb{F}_q) \to H^1(W, \mathbb{F}_q)$ is diagonalizable with eigenvalues the roots α, $F(\alpha)$, $\ldots$, $F^{n-1}(\alpha)$ of m_α, where $F(x) = x^2$ is the Frobenius automorphism of $\mathbb{F}_q/\mathbb{F}_2$. The Frobenius automorphism also acts on $H^*(W; \mathbb{F}_q) = \mathbb{F}_q \otimes H^*(W; \mathbb{F}_2)$ and keeps everything in $H^*(W; \mathbb{F}_2)$ fixed. Hence it commutes with L^* which keeps $\mathbb{F}_q$ fixed. Now we choose an eigenvector x of L^* with eigenvalue α, then $F(x)$ too is an eigenvector with eigenvalue $F(\alpha)$. Therefore x, $F(x)$, $\ldots$, $F^{n-1}(x)$ is a basis of eigenvectors diagonalizing L^*.

The normalizer T acts by conjugation on Z as multiplication with ζ^{q+1} which is a primitive root for $\mathbb{F}_q/\mathbb{F}_2$ and on Q as multiplication with ζ^{2-q} which is a primitive root for $\mathbb{F}_{q^2}/\mathbb{F}_2$. Thus the actions of T on $H^1(Q, \mathbb{F}_{q^2})$ is diagonalizable with eigenvalues ζ^{2-q}, $\zeta^{2(2-q)}$, …, $\zeta^{2^{2n-1}(2-q)}$ corresponding to eigenvectors a_1, a_2, …, a_{2n}. Similarly we get for $H^1(Z, \mathbb{F}_q)$ eigenvectors $v^{(1)}$, $v^{(2)}$, …, $v^{(n)}$ with eigenvalues ζ^{q+1}, $\zeta^{2(q+1)}$, …, $\zeta^{2^{n-1}(q+1)}$ which by extension of scalars work in $H^1(Z, \mathbb{F}_{q^2})$ too.

Next take a look at Steenrod operations which are classically defined for the cohomlogy ring $H^*(W; \mathbb{F}_2)$ and which now must be extended to $H^*(W; \mathbb{F}_q) = \mathbb{F}_q \otimes H^*(W; \mathbb{F}_2)$. Because of the structure of the tensor product we see that F and Sq commute and $Sq^n(F(x)) = x^2$ for $x \in H^n(W, \mathbb{F}_q)$. Thus we have $Sq^1(a_i) = a_{i-1}^2$. Note that we view the index of $a_\bullet$ modulo $2n$ throughout the following section, where the values taken are from 1 to $2n$ (instead of 0 to $2n-1$ as usual).

We now specialise to the case $n > 1$, since the case $n = 1$ would have to be handled separately while yielding no additional insights. Finally we need to determine the extension cocycle. For this we will take a look at the Lyndon-Hochschild-Serre spectral sequence

$$H^*(Q; H^*(Z; \mathbb{F}_{q^2})) \implies H^*(G; \mathbb{F}_{q^2}).$$

This spectral sequence has a differential $d^2 : H^1(Z, \mathbb{F}_{q^2}) \to H^2(Q, \mathbb{F}_{q^2})$ which maps the $v^{(i)}$ to $d^2(v^{(i)})$ inside $H^2(Q, \mathbb{F}_{q^2})$. Since these are the only coboundaries in $H^2(Q, \mathbb{F}_{q^2})$ they will correspond to the extension cocycle in $H^2(Q, Z)$ which always has to be killed.

The spectral sequence is natural w. r. t. the action of T and hence the differentials preserve the eigenvalues. Thus $d^2(v^{(i)})$ has eigenvalue $\zeta^{2^{i-1}(q+1)}$ and lies inside $H^2(Q, \mathbb{F}_{q^2}) = \langle a_r a_s : 1 \leq r \leq s \leq 2n \rangle$, where the generators have eigenvalues $\zeta^{(2^{r-1}+2^{s-1})(2-q)}$. We determine $d^2(v^{(1)})$ by solving

$$q + 1 \equiv (2^{r-1} + 2^{s-1})(2 - q) \mod q^2 - 1\,. \tag{5.4}$$

First we show that $r = 1$ and $s = n + 1$ is a solution. In this case $2^{r-1} = 1$ and $2^{s-1} = q$ and we compute the following.

$$q + 1 - (1 + q)(2 - q) = q + 1 - (2 + q - q^2) = q^2 - 1 \equiv 0 \mod q^2 - 1$$

This solution is unique if $\gcd(q^2 - 1, q - 2) = 1$, since then we can divide by $2 - q$ and the only ambiguity that could occur is $r = s = 2n$, which is in fact not an ambiguity. However using the Euclidean algorithm we calculate

$$\gcd(q^2 - 1, q - 2) = \begin{cases} 1 & \text{if } n \equiv 0 \mod 2, \\ 3 & \text{if } n \equiv 1 \mod 2. \end{cases}$$

We now have to examine the bad case $n = 2m+1$ for an $m > 1$. First we note that $2-q$ is inverted up to the gcd of 3 by $2+q$ since

$$(2-q)(2+q) = 4 - q^2 \equiv 4 - 1 = 3 \mod q^2 - 1\,.$$

So multiplying (5.4) by $2+q$ we get

$$(2+q)(1+q) = 2 + 3q + q^2 \equiv 3(q+1) \overset{!}{\equiv} 3(2^{r-1} + 2^{s-1}) \mod q^2 - 1$$

Now if $3(2^{r-1} + 2^{s-1}) \leq q^2 - 1$ this has only the solution presented above, so we have to check what happens for big r and s. More specifically if $s = 2n$ we always see an overflow and for $s = 2n-1$ we see an overflow for $r \in \{2n-2, 2n-1\}$. For $s = 2n$ and $s = t = 2n-1$ we compute

$$3 \cdot 2^{2n-1} = q^2 + \frac{q^2}{2} \equiv \frac{q^2}{2} + 1 \mod q^2 - 1$$

which does not allow a solution. For the last case $s = 2n-1$, $r = 2n-2$ we get

$$3(2^{2n-3} + 2^{2n-2}) = q^2 + \frac{q^2}{8} \not\equiv 3(q+1) \mod q^2 - 1$$

showing, that indeed our solution is unique. Thus we have $d^1(v^{(1)}) = a_1 a_{n+1}$.

All other parts of the extension cocycle can be easily obtained by $d^1(v^{(j)}) = d^1(F^{j-1}(v^{(1)})) = F^{j-1}(d^1(v^{(1)})) = F^{j-1}(a_1 a_{n+1}) = a_j a_{n+j}$.

Now we use the Eilenberg-Moore spectral sequence as detailed in § 3 giving us

$$E_2^{**} \cong \mathrm{Tor}_\Gamma^{**}(H^*(Q;\mathbb{F}_{q^2}), \mathbb{F}_{q^2}) \implies H^*(G;\mathbb{F}_{q^2})\,.$$

where $\Gamma = H^*(K(Z,2);\mathbb{F}_{q^2})$. We know that $H^*(K(\mathbb{Z}/2\mathbb{Z},2);\mathbb{F}_{q^2}) \cong \mathbb{F}_{q^2}[x_0, x_1, \dots]$ with $\deg(x_i) = 2^i + 1$ and $Sq_1(x_i) = x_{i+1}$. For $Z = (\mathbb{Z}/2\mathbb{Z})^n$ we get a product and thus n copies of the generators: $x_i^{(j)}$ with $1 \leq j \leq n$ and hence $\Gamma \cong \mathbb{F}_{q^2}[x_i^{(j)}]_{i=0,j=1}^{\infty,n}$. The E_1-page is now given as

$$E_1^{pq} = \bigoplus_{i+j=q} H^i(Q;\mathbb{F}_{q^2}) \otimes_\Gamma \mathcal{K}^{pj}(\Gamma)$$

with the Koszul resolution given by $\Lambda(u_0^{(1)}, \dots, u_0^{(n)}, u_1^{(1)}, \dots, u_1^{(n)}, \dots)$ as described in § 2.6.2 with $u_i^{(j)}$ corresponding to $x_i^{(j)}$.

The differential d^1 of degree $(1,0)$ can now be determined. We have (as in [Rus87]) the relations $d(u_i^{(j)}) = d(Sq_1^i(u_0^{(j)})) = Sq_1^i(d(u_0^{(j)}))$, where $d(u_0^{(j)})$ must be the corresponding extension cocycle. Furthermore the Frobenius map F commutes with d^1 and hence we can extend F onto the Koszul complex via $F(u_i^{(j)}) = u_i^{(j+1 \mod n)}$.

We let u_0 be the tuple $(u_0^{(1)}, \dots, u_0^{(n)})$, so that $d^1(u_0)$ is the complete extension cocycle (as this is the only way to kill it). We will only look at the first component $u_i^{(1)}$ since all other components follow from $u_i^{(j)} = F^{j-1}(u_i^{(1)})$.

$$\begin{aligned}
d^1(u_0^{(1)}) &= a_1 a_{n+1} \\
d^1(u_1^{(1)}) &= Sq^1(a_1 a_{n+1}) = a_{2n}^2 a_{n+1} + a_1 a_n^2 \\
d^1(u_2^{(1)}) &= Sq^2(a_{2n}^2 a_{n+1} + a_1 a_n^2) = a_{2n-1}^4 a_{n+1} + a_1 a_{n-1}^4 \\
&\vdots \\
d^1(u_k^{(1)}) &= Sq^{2^{k-1}}(a_{1-(k-1)}^{2^{k-1}} a_{n+1} + a_1 a_{n+1-(k-1)}^{2^{k-1}}) = a_{1-k}^{2^k} a_{n+1} + a_1 a_{n+1-k}^{2^k} \\
&\vdots \\
d^1(u_n^{(1)}) &= Sq^{2^{n-1}}(a_{n+2}^{2^{n-1}} a_{n+1} + a_1 a_2^{2^{n-1}}) = a_1^{2^n+1} + a_{n+1}^{2^n+1} \\
d^1(u_{n+1}^{(1)}) &= Sq^{2^n}(a_1^{2^n+1} + a_{n+1}^{2^n+1}) = a_1 a_{2n}^{2^{n+1}} + a_{n+1} a_n^{2^{n+1}}
\end{aligned}$$

Now we replace $u_{n+1}^{(1)}$ by

$$\tilde{u}_{n+1}^{(1)} := u_{n+1}^{(1)} + (a_n^{2^n-1} a_{n+1} + a_{2n}^{2^n-1} a_1) u_n^{(n)} + (a_n^{2^n-2} a_{n+1} a_{2n}^{2^n} + a_{2n}^{2^n-2} a_1 a_n^{2^n}) u_0^{(n)}$$

and $u_{n+1}^{(j)}$ analogously for $2 \le j \le n$. This lies in the kernel of d^1 as

$$\begin{aligned}
d^1\Big(u_{n+1}^{(1)} &+ (a_n^{2^n-1} a_{n+1} + a_{2n}^{2^n-1} a_1) u_n^{(n)} + (a_n^{2^n-2} a_{n+1} a_{2n}^{2^n} + a_{2n}^{2^n-2} a_1 a_n^{2^n}) u_0^{(n)}\Big) \\
&= a_1 a_{2n}^{2^{n+1}} + a_{n+1} a_n^{2^{n+1}} + (a_n^{2^n-1} a_{n+1} + a_{2n}^{2^n-1} a_1)(a_n^{2^n+1} + a_{2n}^{2^n+1}) \\
&\quad + (a_n^{2^n-2} a_{n+1} a_{2n}^{2^n} + a_{2n}^{2^n-2} a_1 a_n^{2^n}) a_n a_{2n} \\
&= 0
\end{aligned}$$

Furthermore we replace $u_{n+2}^{(j)}$ by $Sq^{2^{n+1}}(\tilde{u}_{n+1}^{(j)})$, $u_{n+3}^{(j)}$ by $Sq^{2^{n+2}}(Sq^{2^{n+1}}(\tilde{u}_{n+1}))^{(j)}$ and so on. The new u_i can now replace the old ones as they are just like them. First they generate the Koszul complex, since all $\tilde{u}_i^{(j)}$ square to zero and are by construction equal to $u_i^{(j)}$ plus terms in $u_{i'}^{(j')}$ with $i' < i$. Second they transform similar under the Frobenius map F, that is $F(\tilde{u}_i^{(j)}) = \tilde{u}_i^{(j+1)}$. Third they transform similar under Steenrod operation by construction. Hence we will drop the tilde from now on.

Now we have all prerequisites ready and can use the Sage code from the appendix to calculate the dimensions of the modules E_2^{**}.

5.2.1 $SU_3(4)$

For $n = 2$ we find the dimensions listed in table 5.7. We see, that the differential d^2 of bidegree $(2, -1)$ has no other option than vanishing. Thus the spectral sequence

1	.	.	16
4	.	.	15
8	.	.	14
10	.	.	13
8	.	.	12
6	4	.	11
.	7	.	10
.	16	.	9
.	20	.	8
.	16	.	7
.	7	.	6
.	4	6	5
.	.	8	4
.	.	10	3
.	.	8	2
.	.	4	1
.	.	1	0
-2	-1	0	

Table 5.7: Dimensions of the E_2-page for $SU_3(4)$

collapses on the E_2-page as desired. Thus we retrieve (by extension of scalars) the following Poincaré series for $H^*(Syl_2(U_3(4)), \mathbb{F}_2)$.

$$1, 4, 8, 10, 12, 13, 16, 20, 16, 13, 12, 10, 8, 4, 1$$

This however gives us only the additive structure, for the multiplicative structure the necessary ungrading requires some additional information which is not readily available from this approach.

This reproduces the results from [Gre04].

5.2.2 $SU_3(8)$

For $n = 3$ we find the dimensions listed in table 5.8. Here we can not conclude that the spectral sequence collapses on the E_2-page and indeed explicit computations of the lower degrees (which are the only feasible ones) of the cohomology of $H^*(Syl_2(U_3(8)), \mathbb{F}_2)$ show that d^2 does not vanish.

To retrieve d^2 we again need some additional information, so that the computation stalls at this point.

1	.	.	.	.	.	.	51
6	.	.	.	.	.	.	50
18	.	.	.	.	.	.	49
35	.	.	.	.	.	.	48
48	.	.	.	.	.	.	47
51	6	.	.	.	.	.	46
48	31	.	.	.	.	.	45
48	84	.	.	.	.	.	44
48	147	.	.	.	.	.	43
45	198	7	.	.	.	.	42
30	207	18	.	.	.	.	41
18	225	66	.	.	.	.	40
6	255	144	.	.	.	.	39
.	288	231	.	.	.	.	38
.	279	291	6	.	.	.	37
.	231	347	24	.	.	.	36
.	156	420	57	.	.	.	35
.	120	561	108	.	.	.	34
.	99	681	167	.	.	.	33
.	78	687	213	.	.	.	32
.	48	594	285	6	.	.	31
.	24	501	425	18	.	.	30
.	6	465	624	30	.	.	29
.	.	429	765	42	.	.	28
.	.	348	798	71	.	.	27
.	.	222	756	129	.	.	26
.	.	129	756	222	.	.	25
.	.	71	798	348	.	.	24
.	.	42	765	429	.	.	23
.	.	30	624	465	6	.	22
.	.	18	425	501	24	.	21
.	.	6	285	594	48	.	20
.	.	.	213	687	78	.	19
.	.	.	167	681	99	.	18
.	.	.	108	561	120	.	17
.	.	.	57	420	156	.	16
.	.	.	24	347	231	.	15
.	.	.	6	291	279	.	14
.	.	.	.	231	288	.	13
.	.	.	.	144	255	6	12
.	.	.	.	66	225	18	11
.	.	.	.	18	207	30	10
.	.	.	.	7	198	45	9
.	.	.	.	.	147	48	8
.	.	.	.	.	84	48	7
.	.	.	.	.	31	48	6
.	.	.	.	.	6	51	5
.	.	.	.	.	.	48	4
.	.	.	.	.	.	35	3
.	.	.	.	.	.	18	2
.	.	.	.	.	.	6	1
.	.	.	.	.	.	1	0
-6	-5	-4	-3	-2	-1	0	

Table 5.8: Dimensions of the E_2-page for $SU_3(8)$

5.3 Evaluation of versatility

We have seen two examples of applications of the Eilenberg-Moore spectral sequence for computing group cohomology. The main advantage is its fast collapse. If it does not collapse on the E_2-page however, there has to be some additional structure which can be exploited to retrieve the higher differentials. This does not compare unfavorably to other spectral sequences like the Lyndon-Hochschild-Serre spectral sequence where the data from the higher differentials we deduced where already included in the E_2-page in case of the Eilenberg-Moore spectral sequence. For example in the case of $SU_3(8)$ above an approach with another spectral sequence seems just as intractable as the presented approach with the Eilenberg-Moore spectral sequence.

One area where the Eilenberg-Moore spectral sequence is inferior to e. g. the Lyndon-Hochschild-Serre is the ungrading needed to retrieve the multiplicative structure. Due to the used Koszul-resolutions there are often products that vanish in the spectral sequence but do not vanish in the group cohomology.

However as already noted in [McC01, § 7.3] the Eilenberg-Moore spectral sequence is not universally faster than other spectral sequences. In the referenced section an example is given, where the Leray-Serre spectral sequences collapses faster than the Eilenberg-Moore spectral sequence.

In summary one can say that the Eilenberg-Moore spectral sequence can be a very useful tool, but most of the time must be supplemented by other techniques.

Appendix – Code

```
#!/usr/bin/env sage

"""Compute the dimensions of the E2-page for the Eilenberg-Moore
spectral sequence of the group U_3(n).

The important variables are as follows:

* q_generators: List of generators of H^*(Q).

* all_k_generators: List of products of generators of the Koszul
  resolution. This is finite since the square of each generator
  vanishes.

* e2_dims: This records the dimensions of the E2-page -- target of the
  computation. It is a list of lists of integers. The entry
  e2_dims[x][y] is the dimension of E_2^{-y,x}.

* significant_relations: List of generators for the E2-page. Initially
  the relations are the generators of the kernel of d^1. From those we
  remove all coboundaries and then reduce them to a minimal generating
  set yielding the significant relations.
"""

import argparse
import datetime
from itertools import combinations, combinations_with_replacement
import os
import random
import re
import string
import subprocess
import sys
import tempfile

##
## command line interface
##
parser = argparse.ArgumentParser(
    description='Compute dimensions of E2 page.')
parser.add_argument('-v', '--verbose', action='store_true',
                    help='verbose output')
parser.add_argument('-t', '--tex', action='store_true',
                    help='create tex output')
parser.add_argument('-l', '--load', action='store_true',
                    help='load intermediate results from files')
parser.add_argument('-s', '--save', action='store_true',
                    help='save intermediate results to files')
parser.add_argument('-L', '--lateload', action='store_true',
                    help='load intermediate results from files stage 2')
parser.add_argument('-S', '--latesave', action='store_true',
                    help='save intermediate results to files stage 2')
parser.add_argument('-p', '--pickleloadfile', metavar='pickle load file',
                    dest="pload", type=str,
                    help="File to use for loading pickled information from.")
parser.add_argument('-P', '--picklesavefile', metavar='pickle save file',
                    dest="psave", type=str,
                    help="File to use for storing pickled information to.")
parser.add_argument('n', help='parameter n to use', type=int)
args = parser.parse_args()
```

```python
if not args.pload:
    if args.lateload:
        args.pload = "{}-stage2.pickle".format(args.n)
    elif args.load:
        args.pload = "{}-stage1.pickle".format(args.n)
if not args.psave:
    if args.latesave:
        args.psave = "{}-stage2.pickle".format(args.n)
    elif args.save:
        args.psave = "{}-stage1.pickle".format(args.n)

##
## preliminaries
##
now = datetime.datetime.now
START = now()
print("Computing dimensions for U_3(2^{})".format(args.n))

def singular_execute(prog):
    """Helper to call singular."""
    if args.verbose:
        print(">>>")
        print(prog)
        print("<<<")
    command_fd, command_path = tempfile.mkstemp()
    command_file = os.fdopen(command_fd, "w")
    command_file.write(prog.strip() + '\n')
    command_file.close()
    print("{}: Executing {} chars of Singular code.".format(now(), len(prog)))
    ret = subprocess.check_output(["Singular", "-q", "-b", command_path])
    print("{}: Finished execution.".format(now()))
    # clean up
    os.remove(command_path)
    ret = ret.strip()
    if args.verbose:
        print(">>>")
        print(ret)
        print("<<<")
    sys.stdout.flush()
    return ret

##
## initialize variables
##

## Generators of H^1(Q, F_{q^2})
q_generators = ["a{}".format(i) for i in range(1, 2*args.n + 1)]

class KoszulGenerator:
    """Generators of the Koszul complex u_i^{(j)} with images under the
       differential d^1."""
    def __init__(self, name, image, homological_degree, internal_degree):
        self.name = name
        self.image = image
        self.homological_degree = homological_degree
        self.internal_degree = internal_degree
        self.singular_mapping = None
        self.singular_image = None
    def __repr__(self):
        return "{}".format(self.name)
    def __str__(self):
```

```
        return "{} -> {}".format(self.name, self.image)

class Relation:
    """Representations of elements in the kernel of the differentials."""
    def __init__(self, raw_singular, homological_degree, name):
        self.raw_singular = raw_singular
        self.homological_degree = homological_degree
        self.name = name
        self.singular_vector = "vector {} = {};".format(name, raw_singular)
        self.raw_latex = None
        self.transformed = None
        self.internal_degree = None
    def __repr__(self):
        return self.raw_singular
    def __str__(self):
        return self.raw_singular

## Instatiate Koszul generators from u_0^{(1)} to u_n^{n}
k_generators = [
    KoszulGenerator("u0_{}".format(j), "a{}*a{}".format(j, args.n + j), -1, 2)
    for j in range(1, args.n + 1)]
k_generators += [
    KoszulGenerator(
        "u{}_{}".format(i, j),
        "a{}^{}*a{}+a{}*a{}^{}".format(
            (j - i) % (2*args.n) or (2*args.n), 2**i, args.n + j, j,
            args.n - i + j, 2**i),
        -1, 2**i + 1)
    for i in range(1, args.n) for j in range(1, args.n + 1)]
k_generators += [
    KoszulGenerator(
        "u{}_{}".format(args.n, j), "a{}^{}+a{}^{}".format(
            j, 2**args.n + 1, args.n + j, 2**args.n + 1), -1, 2**args.n + 1)
    for j in range(1, args.n + 1)]
print("Differentials are {}.".format(", ".join(str(g) for g in k_generators)))

##
## compute the dimensions of E_2^{0*}
##
## For this we quotient out the ideal of all images of d^1 in degree 0
## and directly retrieve the dimesions with the function hilb().
##
tmpl = string.Template("""
ring R = 2, (${QGENS}), dp;
ideal i = ${IMAGES};
i = groebner(i);
hilb(i);
""")
prog = tmpl.substitute(QGENS=", ".join(q_generators),
                       IMAGES=", ".join(g.image for g in k_generators))
lines = singular_execute(prog).split('\n')
lines = lines[lines.index('') + 1:-2]
regex = re.compile(r"^//\s+(\d+) t\^\d+$")
dims = []
for line in lines:
    mo = regex.match(line)
    dims.append(int(mo.groups(0)[0]))
print("Dimensions of E_2^{{0*}} are {}".format(dims))

dims.append(0)
e2_dims = [dims]
```

```
##
## compute the dimensions of the rest of E_2^{**}
##
## Zeroth step get list of all generators. These are combinations like
## u_0^{(1)}*u_2^{(2)}.
##

## Make generators
all_k_generators = [KoszulGenerator("e", "0", 0, 0)]
for deg in range(1, len(k_generators) + 1):
    for combo in combinations(k_generators, deg):
        image_parts = [
            "({})*{}".format(
                combo[pos].image,
                "".join(combo[i].name for i in range(deg) if i != pos) or "e")
            for pos in range(deg)]
        all_k_generators.append(
            KoszulGenerator("".join(g.name for g in combo),
                            "+".join(image_parts),
                            -deg, sum(g.internal_degree for g in combo)))

## Add singular code describing differential of generator.
image_gens = ["0"]
lookup = {g.name: i for i, g in enumerate(all_k_generators)}
for deg in range(1, len(k_generators) + 1):
    for combo in combinations(k_generators, deg):
        im_vec = []
        for pos in range(deg):
            gen = "".join(combo[i].name for i in range(deg) if i != pos) or "e"
            new_pos = lookup[gen]
            im_vec.append("({})*gen({})".format(combo[pos].image, new_pos + 1))
        image_gens.append("+".join(im_vec))
assert(len(image_gens) == len(all_k_generators))
for i in range(len(all_k_generators)):
    all_k_generators[i].singular_image = \
        "vector d{} = {};".format(all_k_generators[i].name, image_gens[i])

## Add singular code for first step below.
vecs = ["gen(1)"]
index = 1
for deg in range(1, len(k_generators) + 1):
    for combo in combinations(k_generators, deg):
        vec = ["gen({})".format(index + 1)]
        for pos in range(deg):
            ## Here we introduce new generators which are the same as
            ## the old generators, but are separated for computational
            ## purposes. See below for why this is done. The
            ## correspondence is by a fixed offset of
            ## len(all_k_generators).
            gen = "".join(combo[i].name for i in range(deg) if i != pos) or "e"
            new_pos = lookup[gen]
            vec.append("({})*gen({})".format(
                combo[pos].image, len(all_k_generators) + new_pos + 1))
        vecs.append("+".join(vec))
        index += 1
assert(len(vecs) == len(all_k_generators))
for g, v in zip(all_k_generators, vecs):
    g.singular_mapping = "vector {} = {};".format(g.name, v)

print("Instantiated {} generators".format(len(all_k_generators)))
sample = random.randint(0, len(all_k_generators) - 1)
print("The generator number {} is {}".format(sample, all_k_generators[sample]))
```

```
print("Its singular representation is {}.".format(
    all_k_generators[sample].singular_mapping))

##
## First step get all things in the kernel of d^1
##
## To do this we gather all generators of the modules in question
## (e.g. all u_i^{(j)} for homological degree -1) together with new
## generators which represent the images of d^1 (in the example these
## would be d^1(u_i^{(j)}). Now we compute a groebner basis of this
## module such that we retrieve all combinations where the image part
## vanishes, hence the kernel of d^1. This is done by choosing a
## monomial order, corresponding to the numbering and eliminating the
## higher monomials.
##

tmpl = string.Template("""
ring R = 2, (${QGENS}), (C, dp);
${VECTORS}
module m = ${ALLKGENS};
module g = groebner(groebner(m), "fgml");
g;""")
relations = []
min_hom_deg = min(g.homological_degree for g in all_k_generators)
regex = re.compile(r"gen\((\d+)\)")
if not args.load:
    for i in range(-1, min_hom_deg - 1, -1):
        relevant = [g for g in all_k_generators if g.homological_degree == i]
        prog = tmpl.substitute(
            QGENS=", ".join(q_generators),
            VECTORS="\n".join(g.singular_mapping for g in relevant),
            ALLKGENS=", ".join(g.name for g in relevant))
        previous = len(relations)
        s_result = singular_execute(prog)
        for line in s_result.split('\n'):
            rel = line.split('=')[1]
            mo = regex.search(rel)
            if int(mo.groups(0)[0]) <= len(all_k_generators):
                relations.append(Relation(
                    rel, i, "rel{}".format(len(relations))))
        print(("Found {} candidate(s) in degree {} for relations "
               "in the kernel.").format(len(relations) - previous, i))
    print(("Found a total of {} candidates for relations "
           "in the kernel.").format(len(relations)))
relations.sort(key=lambda s: len(s.raw_singular))

##
## Second step purge redundant stuff
##
## We computed the kernel, but it of course contains
## coboundaries. Hence we filter out everything that comes from an
## image.
##
## Additionally what we retrieved is not a minimal generating set for
## the kernel. We eliminate further until it is minimal.
##

## Step 2a eliminate coboundaries

nocoboundaries = []
if not args.load:
    for i in range(-1, min_hom_deg, -1):
```

```
        tmpl = string.Template("""
        ring R = 2, (${QGENS}), (C, dp);
        ${IMVECTORS}
        module coboundarymod = ${IMGENS};
        coboundarymod = groebner(coboundarymod);
        ${THERELATIONS}
        list candidates = (${RELATIONNAMES});
        list nocoboundary;
        for(int i = 1; i <= size(candidates); i = i + 1) {
            if(NF(candidates[i], coboundarymod) != 0) {
                nocoboundary = insert(nocoboundary, candidates[i]);
            }
        }
        nocoboundary;""")

        relevant_gens = [g for g in all_k_generators
                         if g.homological_degree == i - 1]
        relevant_rels = [r for r in relations if r.homological_degree == i]
        if not relevant_rels:
            print("No relations in degree {}.".format(i))
            continue

        prog = tmpl.substitute(
            QGENS=", ".join(q_generators),
            IMVECTORS="\n".join(g.singular_image for g in relevant_gens),
            IMGENS=", ".join("d{}".format(g.name) for g in relevant_gens),
            THERELATIONS="\n".join(r.singular_vector for r in relevant_rels),
            RELATIONNAMES=", ".join(r.name for r in relevant_rels))
        previous = len(nocoboundaries)
        lines = reversed(singular_execute(prog).split('\n'))
        for line in lines:
            if 'gen(' in line:
                nocoboundaries.append(
                    Relation(line.strip(), i, "noco{}".format(
                        len(nocoboundaries))))
        print("Found {} non-coboundary relations in degree {}.".format(
            len(nocoboundaries) - previous, i))
    print("Found a total of {} non-coboundary relations.".format(
        len(nocoboundaries)))

if args.load and not args.lateload:
    with open(args.pload) as f:
        for line in f:
            i, raw = line.split(',')
            nocoboundaries.append(
                Relation(raw.strip(), int(i), "noco{}".format(
                    len(nocoboundaries))))
    print("Loaded {} non-coboundary relations.".format(len(nocoboundaries)))

if args.save and not args.latesave:
    with open(args.psave, 'w') as f:
        for rel in nocoboundaries:
            f.write("{},{}\n".format(rel.homological_degree, rel.raw_singular))
    print("Saved {} non-coboundary relations.".format(len(nocoboundaries)))

## Step 2b remove redundancies

redundancy_check_singular_template = string.Template("""
    ring R = 2, (${QGENS}), (C, dp);
    ${IMVECTORS}
    module coboundarymod = ${IMGENS};
    coboundarymod = groebner(coboundarymod);
```

```
    ${THERELATIONS}
    list candidates = (${RELATIONNAMES});
    list survivors;
    module irredmod;
    int i;

    // First one scan forwards
    irredmod = coboundarymod;
    for(i = 1; i <= size(candidates); i = i + 1) {
        if(NF(candidates[i], irredmod) != 0) {
            survivors = insert(survivors, candidates[i]);
            irredmod = irredmod + candidates[i];
            irredmod = groebner(irredmod);
        }
    }
    candidates = survivors;
    survivors = list();

    // Second one scan backwards
    irredmod = coboundarymod;
    for(i = 1; i <= size(candidates); i = i + 1) {
        if(NF(candidates[i], irredmod) != 0) {
            survivors = insert(survivors, candidates[i]);
            irredmod = irredmod + candidates[i];
            irredmod = groebner(irredmod);
        }
    }

    // Third thoroughly checking for dependencies
    int iterations = size(survivors);
    while (iterations > 0) {
        iterations = iterations - 1;
        irredmod = coboundarymod;
        for(i = 2; i <= size(survivors); i = i + 1) {
            irredmod = irredmod + survivors[i];
        }
        irredmod = groebner(irredmod);
        if(NF(survivors[1], irredmod) == 0) {
            survivors = delete(survivors, 1);
        } else {
            survivors = insert(survivors, survivors[1], size(survivors));
            survivors = delete(survivors, 1);
        }
    }
    survivors;""")

significant_relations = []
if not args.lateload:
    for i in range(-1, min_hom_deg - 1, -1):
        relevant_gens = [g for g in all_k_generators
                         if g.homological_degree == i - 1]
        relevant_rels = [r for r in nocoboundaries if r.homological_degree == i]
        if not relevant_rels:
            print("No relations in degree {}.".format(i))
            continue

        prog = redundancy_check_singular_template.substitute(
            QGENS=", ".join(q_generators),
            IMVECTORS="\n".join(g.singular_image for g in relevant_gens),
            IMGENS=", ".join("d{}".format(g.name) for g in relevant_gens),
            THERELATIONS="\n".join(r.singular_vector for r in relevant_rels),
            RELATIONNAMES=", ".join(r.name for r in relevant_rels))
```

```
        lines = reversed(singular_execute(prog).split('\n'))
        previous = len(significant_relations)
        for line in lines:
            if 'gen(' in line:
                significant_relations.append(
                    Relation(line.strip(), i, "rel{}".format(
                        len(significant_relations))))
        print("Found {} significant relations in degree {}.".format(
            len(significant_relations) - previous, i))
    print("Found a total of {} significant relations.".format(
        len(significant_relations)))

if args.lateload:
    with open(args.pload) as f:
        for line in f:
            i, raw = line.split(',')
            significant_relations.append(
                Relation(raw.strip(), int(i), "rel{}".format(
                    len(significant_relations))))
    print("Loaded {} significant relations.".format(
        len(significant_relations)))

if not significant_relations:
    print("Terminating since nothing to be done anymore.")
    sys.exit()

if args.latesave:
    with open(args.psave, 'w') as f:
        for rel in significant_relations:
            f.write("{},{}\n".format(rel.homological_degree, rel.raw_singular))
    print("Saved {} significant relations.".format(len(significant_relations)))


##
## Third step dissect survivors
##
## Do some prettyfication of the significant relations.
##

def texify(singular_string):
    regex = re.compile(r"u(\d+)_(\d+)")
    ret = singular_string
    for i in range(len(all_k_generators)):
        replacement = all_k_generators[i].name
        replacement = regex.sub(r"u_\1^{(\2)}", replacement)
        ret = ret.replace("gen({})".format(i + 1), replacement)
    ret = ret.replace('a', 'a_')
    ret = ret.replace('*', '')
    return ret

if args.tex:
    print("LaTeX for relations")
    for rel in significant_relations:
        rel.raw_latex = texify(rel.raw_singular)
        print(rel.raw_latex)
    print("End of LaTeX")

for rel in significant_relations:
    s = rel.raw_singular
    for i in range(len(all_k_generators)):
        s = s.replace("gen({})".format(i + 1), all_k_generators[i].name)
    s = s.replace('u', '*u')
```

```
    s = s.replace('**', '*')
    rel.transformed = s

aregex = re.compile(r"^a\d+\^?(\d*)$")
uregex = re.compile(r"^u(\d+)_\d+$")
for rel in significant_relations:
    homological_degree = 0
    internal_degree = 0
    terms = rel.transformed.split('+')[0].split('*')
    for term in terms:
        if 'a' in term:
            mo = aregex.match(term)
            if mo.groups(0)[0]:
                internal_degree += int(mo.groups(0)[0])
            else:
                internal_degree += 1
        elif 'u' in term:
            homological_degree -= 1
            mo = uregex.match(term)
            internal_degree += 2**int(mo.groups(0)[0]) + 1
        else:
            raise RuntimeError("what is this: '{}'?".format(term))
    assert(rel.homological_degree == homological_degree)
    rel.internal_degree = internal_degree

degree_output = {}
for rel in significant_relations:
    deg = (rel.homological_degree, rel.internal_degree)
    degree_output[deg] = degree_output.setdefault(deg, 0) + 1
print("Degrees are {}.".format(degree_output))

##
## Fourth step explicit computations
##
## We use the significant relations to explicitly write down a
## generating set, for each bidegree. Then we remove all redundant
## entries and retrieve a basis allowing us to count the dimension.
##
## However sometimes we can skip the work of explicit computation,
## which happens if we know all but one dimension in a row since we
## then can use a simple arithmetic argument.
##

e2_generator_cache = {}

def compute_e2_dimension(homdeg, intdeg, make_list=False):
    """Compute one specific dimension in the E2-page.

    For this we take everything in the bidegree below (i.e. internal
    degree one less) and multiply it by all generators of H^*(Q)
    (remember that they all have degree one); finally we add the
    significant relations that are new in this bidegree. Then all
    redundancies are removed and the dimension is counted out.

    To get the bidegree below we keep a cache of a basis for each
    bidegree we ever computed. We then compute upwards from the
    closest cache entry we found until we are at the target degree. If
    nothing is found in the cache we start with the minimal degree
    that has any significant relations, meaning that nothing is below.
    """
    relevant_gens = [g for g in all_k_generators
                     if g.homological_degree == homdeg - 1]
```

```
    if homdeg == 0:
        min_int_deg = 0
    else:
        min_int_deg = min(rel.internal_degree for rel in significant_relations
                          if rel.homological_degree == homdeg)
    if homdeg == 0:
        myrel = Relation("gen(1);", 0, "rel0")
        myrel.internal_degree = 0
        eventually_relevant = [myrel]
    else:
        eventually_relevant = [
            rel for rel in significant_relations
            if (rel.homological_degree == homdeg
                and rel.internal_degree <= intdeg)]

    intermediate_result = []
    cache_found = -1
    for current_deg in range(intdeg - 1, min_int_deg, -1):
        if (homdeg, current_deg) in e2_generator_cache:
            print("Use cached {}".format((homdeg, current_deg)))
            intermediate_result = e2_generator_cache[(homdeg, current_deg)]
            cache_found = current_deg
            break
    for current_deg in range(max(min_int_deg, cache_found + 1), intdeg + 1):
        relevant_rels = []
        for one in intermediate_result:
            for qgen in q_generators:
                relevant_rels.append(
                    Relation("{}*({})".format(qgen, one), homdeg,
                             "rel{}".format(len(relevant_rels))))
        for rel in eventually_relevant:
            if rel.internal_degree == current_deg:
                relevant_rels.append(
                    Relation(rel.raw_singular, homdeg,
                             "rel{}".format(len(relevant_rels))))

        prog = redundancy_check_singular_template.substitute(
            QGENS=", ".join(q_generators),
            IMVECTORS="\n".join(g.singular_image for g in relevant_gens),
            IMGENS=", ".join("d{}".format(g.name) for g in relevant_gens),
            THERELATIONS="\n".join(r.singular_vector for r in relevant_rels),
            RELATIONNAMES=", ".join(r.name for r in relevant_rels))
        intermediate_result = []
        lines = singular_execute(prog).split('\n')
        for line in lines:
            if 'gen(' in line:
                intermediate_result.append(line.strip())

    e2_generator_cache[(homdeg, intdeg)] = intermediate_result

    if make_list:
        return intermediate_result
    else:
        return len(intermediate_result)

min_hom_deg = min(rel.homological_degree for rel in significant_relations)

def compute_e1_dimension(homdeg, intdeg):
    """Compute the dimension on the E1-page.

    This is pretty straight forward since the structure on the E1-page
    is not involved.
```

```
    """
    relevant = [g for g in all_k_generators if g.homological_degree == homdeg]
    relevant = [g for g in relevant if g.internal_degree <= intdeg]
    return sum(binomial(2*args.n + intdeg - g.internal_degree - 1, 2*args.n - 1)
               for g in relevant)

## Prepare the e2_dims array with zeros where we know the E2-page to be empty.
e2_dims.extend([[]] * (-min_hom_deg))
for h in range(-1, min_hom_deg - 1, -1):
    min_int_deg = min(rel.internal_degree
                      for rel in significant_relations
                      if rel.homological_degree == h)
    e2_dims[-h] = [0] * min_int_deg
    e2_dims[-h].append(sum(
        1 for rel in significant_relations
        if (rel.homological_degree == h
            and rel.internal_degree == min_int_deg)))

## The actual calculation.
##
## We compute row-wise. We break off once a row is zero (we later
## check, that this does not miss anything)
known = min(len(x) for x in e2_dims)
while(True):
    for l in e2_dims:
        if len(l) == known and l[known - 1] == 0:
            l.append(0)
    unknown = sum(1 for l in e2_dims if len(l) <= known)
    if unknown == 0:
        if sum(l[known] for l in e2_dims) == 0:
            print("Calculated all dimensions.")
            break
        print("Finished degree {} is ({}).".format(
            known, ", ".join(str(l[known]) for l in reversed(e2_dims))))
        known += 1
        continue
    elif unknown > 1:
        gaps = []
        for i in range(len(e2_dims)):
            if len(e2_dims[i]) <= known:
                gaps.append((i, e2_dims[i][-1]))
        todo = min(gaps, key=lambda x: x[1])
        print("Explicitly compute {}.".format((-todo[0], known)))
        e2_dims[todo[0]].append(compute_e2_dimension(-todo[0], known))
        continue
    elif unknown == 1:
        for i, l in enumerate(e2_dims):
            if len(l) <= known:
                index = i
        print("Infer {}.".format((-index, known)))
        e1_dims = [compute_e1_dimension(0, known)]
        while(True):
            d = compute_e1_dimension(-len(e1_dims), known)
            if d == 0:
                break
            else:
                e1_dims.append(d)
        e2_dims[index].append(0)
        for i in range(len(e1_dims)):
            if i < len(e2_dims):
                e1_dims[i] -= e2_dims[i][known]
            e1_dims[i] = (-1)**(abs(index - i)) * e1_dims[i]
```

```
        e2_dims[index][known] = sum(e1_dims)
    else:
        raise RutimeError("What are we doing?")

## Verify that vanishing was used correctly, i.e. we did not miss any
## generator because everything else vanished below it.
for rel in significant_relations:
    if known > rel.internal_degree:
        assert(e2_dims[abs(rel.homological_degree)][rel.internal_degree] > 0)

##
## Output
##
def get(l, i):
    try:
        return l[i]
    except IndexError:
        return '?'

print("Known dimensions for E_2^{{**}} are as follows")
longest = max(len(x) for x in e2_dims)
lines = [tuple(get(l, j) for l in reversed(e2_dims))
          for j in range(longest)]
print("\n".join(str(x) for x in reversed(lines)))

if args.tex:
    lines = [
        "{} & {} \\\\\\hline".format(
            " & ".join(str(x) for x in line), str(i))
        for i, line in enumerate(lines)]
    print("\\begin{{tabular}}{{{}}}".format(
        "|".join(["c"] * (len(e2_dims) + 1))))
    print("\n".join(x for x in reversed(lines)))
    print("{} &  \\\\\\hline".format(
        " & ".join(str(-i) for i in reversed(range(len(e2_dims))))))
    print("\\end{tabular}")

END = now()
print("Terminating after {}.".format(END - START))
```

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Table of symbols

Symbol	Explanation
S_n	symmetric group on n letters
$\mathrm{diag}(a_1, a_2, \ldots, a_n)$	diagonal matrix with entries a_i on the diagonal
$X \hookrightarrow Y$	injective map
$X \twoheadrightarrow Y$	surjective map
BG	classifying space of a group G
$K(G, n)$	Eilenberg-MacLane space of a group G
$\pi_n(X)$	n-th homotopy group of a space X
$[n]$	equivalence class of n *or* for $n \in \mathbb{N}$ the set $\{0, 1, \ldots, n\}$
ε_i	basic injective simplicial map (see §2.3)
η_i	basic surjective simplicial map (see §2.3)
ΩX	Loop space of a space X
PX	Path space of a space X
θ_i, θ_{ij}	Special simplicial maps (see def. 2.3.3)
Δ	Category of the prototypical simplicial simplex
s_k, d_k	images of ε_k and η_k under Dold-Kan
C_*^Z	chain complex concentrated in degree 2
$\bar{K}$	functor in the Dold-Kan correspondence
$C_p^{(\eta)}$	constituent of the Dold-Kan correspondence
$[X, Y]$	set of homotopy classes of maps between two spaces
H^*	cohomology functor
$Z(G)$	center of a group G
E_r^{pq}	(p, q)-cell of the r-th page of a spectral sequence
$B^*(\Gamma, M)$	bar resolution of the Γ-module M
$[a\|b\|c]$	element of the bar resolution
$\tilde{x}$	x up to signs (see §2.6.1)
$\mathcal{K}(\Gamma)$	Koszul resolution over Γ
Sq^i	Steenrod cohomology operation of degree i
LHS	left hand side
RHS	right hand side

Ehrenwörtliche Erklärung

Hiermit erkläre ich,

- dass mir die Promotionsordnung der Fakultät bekannt ist,
- dass ich die Dissertation selbst angefertigt und alle von mir benutzten Hilfsmittel, persönliche Mitteilungen sowie Quellen in meiner Arbeit angegeben habe,
- dass ich die Hilfe eines Promotionsberaters nicht in Anspruch genommen habe und dass Dritte weder unmittelbar noch mittelbar geldwerte Leistungen von mir für Arbeiten erhalten haben, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen,
- dass ich die Dissertation noch nicht als Prüfungsarbeit für eine staatliche oder andere wissenschaftliche Prüfung eingereicht habe.

Bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts haben mich folgende Personen unterstützt:

Prof. Dr. David J. Green, Jena.

Ich habe die gleiche, in wesentlichen Teilen ähnliche bzw. eine andere Abhandlung noch bei keiner anderen Hochschule als Dissertation eingereicht.

Jena, den 3. Dezember 2017