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SPECTRAL BOUNDS FOR SINGULAR INDEFINITE STURM-LIOUVILLE OPERATORS WITH $L^1$-POTENTIALS

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Abstract. The spectrum of the singular indefinite Sturm-Liouville operator

$$A = \text{sgn}(\cdot) \left(-\frac{d^2}{dx^2} + q\right)$$

with a real potential $q \in L^1(\mathbb{R})$ covers the whole real line and, in addition, non-real eigenvalues may appear if the potential $q$ assumes negative values. A quantitative analysis of the non-real eigenvalues is a challenging problem, and so far only partial results in this direction were obtained. In this paper the bound

$$|\lambda| \leq \|q\|^2_{L^1}$$
on the absolute values of the non-real eigenvalues $\lambda$ of $A$ is obtained. Furthermore, separate bounds on the imaginary parts and absolute values of these eigenvalues are proved in terms of the $L^1$-norm of $q$ and its negative part $q_-$. 

1. Introduction

The aim of this paper is to prove bounds on the absolute values of the non-real eigenvalues of the singular indefinite Sturm-Liouville operator

$$Af = \text{sgn}(\cdot)(-f'' + qf),$$

$${\text{dom}} A = \{f \in L^2(\mathbb{R}) : f, f' \in AC(\mathbb{R}), -f'' + qf \in L^2(\mathbb{R})\},$$

where $AC(\mathbb{R})$ stands for space of all locally absolutely continuous functions. It will always be assumed that the potential $q$ is real-valued and belongs to $L^1(\mathbb{R})$.

The operator $A$ is not symmetric nor self-adjoint in an $L^2$-Hilbert space due to the sign change of the weight function $\text{sgn}(\cdot)$. However, $A$ can be interpreted as a self-adjoint operator with respect to the Krein space inner product $(\text{sgn} \cdot, \cdot)$ in $L^2(\mathbb{R})$. We summarize the qualitative spectral properties of $A$ in the next theorem, which follows from [4, Theorem 4.2] or [16, Proposition 2.4] and the well-known spectral properties of the definite Sturm-Liouville operator $-\frac{d^2}{dx^2} + q$; cf. [23, 24, 25].

Theorem 1.1. The essential spectrum of $A$ coincides with $\mathbb{R}$ and the non-real spectrum of $A$ consists of isolated eigenvalues with finite algebraic multiplicity which are symmetric with respect to $\mathbb{R}$.

Indefinite Sturm-Liouville operators have been studied for more than a century, and have again attracted a lot of attention in the recent past. Early works in this context usually deal with the regular case, that is, the operator $A$ is studied on a finite interval with appropriate boundary conditions at the endpoints; cf. [15, 22] and, e.g., [11, 18, 26]. In this situation the spectrum of $A$ is purely discrete and various estimates on the real and imaginary parts of the non-real eigenvalues were obtained.

Key words and phrases. Non-real eigenvalue, indefinite Sturm-Liouville operator, Krein space, Birman-Schwinger principle.
obtained in the last few years; cf. [2, 9, 10, 14, 17, 21]. The singular case is much less studied, due to the technical difficulties which, very roughly speaking, are caused by the presence of continuous spectrum.

Explicit bounds on non-real eigenvalues for singular Sturm-Liouville operators with $L^\infty$-potentials were obtained with Krein space perturbation techniques in [5] and under additional assumptions for $L^1$-potentials in [6, 7], see also [3] for the absence of real eigenvalues and [19] for the accumulation of non-real eigenvalues of a very particular family of potentials. In this paper we substantially improve the earlier bounds in [6, 7] and relax the conditions on the potential. More precisely, here we prove for arbitrary real $q \in L^1(\mathbb{R})$ the following bound.

**Theorem 1.2.** Let $q \in L^1(\mathbb{R})$ be real. Every non-real eigenvalue $\lambda$ of the indefinite Sturm-Liouville operator $A$ satisfies

\[
|\lambda| \leq \|q\|_{L^1}^2.
\]

Moreover, we prove two bounds in terms of the negative part $q_-$ of $q$.

**Theorem 1.3.** Let $q \in L^1(\mathbb{R})$ be real. Every non-real eigenvalue $\lambda$ of the indefinite Sturm-Liouville operator $A$ satisfies

\[
|\lambda| \leq 24 \cdot \sqrt{3}\|q_-\|_{L^1},
\]

and

\[
|\lambda| \leq 24 \cdot \sqrt{3}\|q_-\|_{L^1} + 6\|q_-\|_{L^1} + \|q\|_{L^1}.
\]

The bound (1.1) is proved in Section 2. It’s proof is based on the Birman-Schwinger principle using similar arguments as in [1, 13], [12, Chapter 14.3]; see also [8]. The bounds (1.2) and (1.3) are obtained in Section 3 by adapting the techniques from the regular case in [2, 9, 21] to the present singular situation.

## 2. Proof of Theorem 1.2

In this section we prove the bound (1.1) for the non-real eigenvalues of $A$. We adapt a technique similar to the Birman-Schwinger principle in [12] and apply it to the indefinite operator $A$. The main ingredient is a bound for the integral kernel of the resolvent of the operator

\[
B_0 f = \text{sgn}(\cdot)(-f''), \quad \text{dom } B_0 = \{ f \in L^1(\mathbb{R}) : f, f' \in AC(\mathbb{R}), -f'' \in L^1(\mathbb{R}) \},
\]

in $L^1(\mathbb{R})$.

**Lemma 2.1.** The operator $B_0$ is closed in $L^1(\mathbb{R})$ and for all $\lambda$ in the open upper half-plane $\mathbb{C}^+$ the resolvent of $B_0$ is an integral operator

\[
[(B_0 - \lambda)^{-1}g](x) = \int_{\mathbb{R}} K_\lambda(x, y)g(y) \, dy, \quad g \in L^1(\mathbb{R}),
\]

where the kernel $K_\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is bounded by $|K_\lambda(x, y)| \leq |\lambda|^{-1}$ for all $x, y \in \mathbb{R}$.

**Proof.** Here and in the following we define $\sqrt{\lambda}$ for $\lambda \in \mathbb{C}^+$ as the principal value of the square root, which ensures $\text{Im} \sqrt{\lambda} > 0$ and $\text{Re} \sqrt{\lambda} > 0$. For $\lambda \in \mathbb{C}^+$ consider the integral operator

\[
(T_\lambda g)(x) = \int_{\mathbb{R}} K_\lambda(x, y)g(y) \, dy, \quad g \in L^1(\mathbb{R}),
\]
with the kernel \( K_\lambda(x, y) = C_\lambda(x, y) + D_\lambda(x, y) \) of the form

\[
C_\lambda(x, y) = \frac{1}{2\sqrt{\lambda}} \begin{cases} 
\alpha e^{\text{sgn}(\lambda)L_{x+y}}, & x \geq 0, y \geq 0, \\
-\alpha e^{\text{sgn}(\lambda)L_{x+y}}, & x \geq 0, y < 0, \\
\alpha \sqrt{\lambda}e^{i(x+y)}, & x < 0, y \geq 0, \\
-\alpha \sqrt{\lambda}e^{i(x+y)}, & x < 0, y < 0,
\end{cases}
\]

and

\[
D_\lambda(x, y) = \frac{1}{2\sqrt{\lambda}} \begin{cases} 
\alpha e^{\text{sgn}(\lambda)L_{y-x}}, & x \geq 0, y \geq 0, \\
0, & x \geq 0, y < 0, \\
0, & x < 0, y \geq 0, \\
-\alpha e^{-\text{sgn}(\lambda)L_{y-x}}, & x < 0, y < 0,
\end{cases}
\]

where \( \alpha := \frac{1}{\sqrt{\pi}} \). Hence,

\[
|K_\lambda(x, y)| = |C_\lambda(x, y) + D_\lambda(x, y)| \leq \frac{1}{\sqrt{|\lambda|}}
\]

and the integral in (2.1) converges for every \( g \in L^1(\mathbb{R}) \). We have

\[
\sup_{y \geq 0} \int_\mathbb{R} |C_\lambda(x, y)| \, dx = \frac{1}{2\sqrt{|\lambda|}} \left( \frac{1}{\Im \sqrt{\lambda}} + \frac{\sqrt{2}}{\Re \sqrt{\lambda}} \right)
\]

and

\[
\sup_{y < 0} \int_\mathbb{R} |C_\lambda(x, y)| \, dx = \frac{1}{2\sqrt{|\lambda|}} \left( \frac{\sqrt{2}}{\Im \sqrt{\lambda}} + \frac{1}{\Re \sqrt{\lambda}} \right).
\]

For \( y \geq 0 \) we estimate

\[
\int_0^\infty |D_\lambda(x, y)| \, dx = \frac{1}{2\sqrt{|\lambda|}} \left( \int_0^\infty e^{-\Im \sqrt{\lambda}L_{x-y}} \, dx \right) \frac{2 - e^{-\Im \sqrt{\lambda}y}}{2\sqrt{|\lambda|} \Im \sqrt{\lambda}} \leq \frac{1}{\sqrt{|\lambda|} \Im \sqrt{\lambda}}
\]

and analogously for \( y < 0 \)

\[
\int_{-\infty}^0 |D_\lambda(x, y)| \, dx = \frac{1}{2\sqrt{|\lambda|}} \left( \int_{-\infty}^0 e^{-\Re \sqrt{\lambda}L_{x-y}} \, dx \right) \frac{2 - e^{\Re \sqrt{\lambda}y}}{2\sqrt{|\lambda|} \Re \sqrt{\lambda}} \leq \frac{1}{\sqrt{|\lambda|} \Re \sqrt{\lambda}}
\]

Hence,

\[
e := \sup_{y \in \mathbb{R}} \int_\mathbb{R} |K_\lambda(x, y)| \, dx < \infty
\]

and Fubini’s theorem yields

\[
\|T_\lambda g\|_{L^1} \leq \int_\mathbb{R} |g(y)| \int_\mathbb{R} |K_\lambda(x, y)| \, dx \, dy \leq e \|g\|_{L^1}.
\]

Therefore \( T_\lambda \) in (2.1) is an everywhere defined bounded operator in \( L^1(\mathbb{R}) \).

We claim that \( T_\lambda \) is the inverse of \( B_0 - \lambda \). In fact, consider the functions \( u, v \) given by

\[
u(x) = \begin{cases} 
\alpha e^{\sqrt{\lambda}x}, & x \geq 0, \\
\alpha e^{-\sqrt{\lambda}x}, & x < 0,
\end{cases}
\]

\[
u(x) = \begin{cases} 
\alpha e^{-\sqrt{\lambda}x}, & x \geq 0, \\
\alpha e^{\sqrt{\lambda}x}, & x < 0,
\end{cases}
\]

which solve the differential equation \( \text{sgn}(\cdot)(-f''(\cdot)) = \lambda f \), that is, \( u \) and \( v \), and their derivatives, belong to \( AC(\mathbb{R}) \) and satisfy the differential equation almost everywhere. Since the Wronskian equals \( 2\alpha \sqrt{\lambda} \), these solutions are linearly independent.
Note that $u, v \notin L^1(\mathbb{R})$ and one concludes that $B_0 - \lambda$ is injective. A simple calculation shows the identity

$$K_{\lambda}(x, y) = C_{\lambda}(x, y) + D_{\lambda}(x, y) = \frac{1}{2\alpha \sqrt{\lambda}} \begin{cases} u(x)v(y) \text{sgn}(y), & y < x, \\ v(x)u(y) \text{sgn}(y), & x < y, \end{cases}$$

and hence we have

$$(T_{\lambda}g)(x) = \frac{1}{2\alpha \sqrt{\lambda}} \left( u(x) \int_{-\infty}^{x} v(y) \text{sgn}(y)g(y) \, dy + v(x) \int_{x}^{\infty} u(y) \text{sgn}(y)g(y) \, dy \right).$$

One verifies $T_{\lambda}g, (T_{\lambda}g)' \in AC(\mathbb{R})$ and $T_{\lambda}g$ is a solution of $\text{sgn}(\cdot)(-f'') - \lambda f = g$. This implies $(T_{\lambda}g)'' \in L^1(\mathbb{R})$ and hence $T_{\lambda}g \in \text{dom} \, B_0$ satisfies

$$(B_0 - \lambda)T_{\lambda}g = g \quad \text{for all } g \in L^1(\mathbb{R}).$$

Therefore, $B_0 - \lambda$ is surjective and we have $T_{\lambda} = (B_0 - \lambda)^{-1}$. It follows that $B_0$ is a closed operator in $L^1(\mathbb{R})$ and that $\lambda$ belongs to the resolvent set of $B_0$.

**Proof of Theorem 1.2.** Since the non-real point spectrum of $A$ is symmetric with respect to the real line (see Theorem 1.1) it suffices to consider eigenvalues in the upper half plane. Let $\lambda \in \mathbb{C}^+$ be an eigenvalue of $A$ with a corresponding eigenfunction $f \in \text{dom} \, A$. Since $q \in L^1(\mathbb{R})$ and $-\frac{d^2}{dx^2} + q$ is in the limit point case at $\pm \infty$ (see, e.g., [23, Lemma 9.37]) the function $f$ is unique up to a constant multiple. As $-f'' + qf = \lambda f$ on $\mathbb{R}^+$ and $f'' - qf = \lambda f$ on $\mathbb{R}^-$ with $q$ integrable one has the well-known asymptotical behaviour

$$f(x) = \alpha_+ \left( 1 + o(1) \right) e^{i\sqrt{\lambda}x}, \quad x \to +\infty,$$

$$f'(x) = \alpha_+ i\sqrt{\lambda}(1 + o(1)) e^{i\sqrt{\lambda}x}, \quad x \to +\infty,$$

and

$$f(x) = \alpha_- \left( 1 + o(1) \right) e^{\sqrt{\lambda}x}, \quad x \to -\infty,$$

$$f'(x) = \alpha_- \sqrt{\lambda}(1 + o(1)) e^{\sqrt{\lambda}x}, \quad x \to -\infty,$$

for some $\alpha_+, \alpha_- \in \mathbb{C}$; see, e.g., [20, § 24.2, Example a] or [23, Lemma 9.37]. These asymptotics yield $f, qf \in L^1(\mathbb{R})$ and $-f'' = \lambda \text{sgn}(\cdot)f - qf \in L^1(\mathbb{R})$, and therefore $f \in \text{dom} \, B_0$. Thus, $f$ satisfies

$$0 = (A - \lambda)f = \text{sgn}(\cdot)(-f'') - \lambda f + \text{sgn}(\cdot)qf = (B_0 - \lambda)f + \text{sgn}(\cdot)qf$$

and since $\lambda$ is in the resolvent set of $B_0$ we obtain

$$-qf = q(B_0 - \lambda)^{-1} \text{sgn}(\cdot)qf.$$

Note that $\|qf\|_{L^1} \neq 0$ as otherwise $\lambda$ would be an eigenvalue of $B_0$. With the help of Lemma 2.1 we then conclude

$$0 < \|qf\|_{L^1} \leq \int_{\mathbb{R}} |q(x)| \int_{\mathbb{R}} |K_{\lambda}(x, y)||q(y)f(y)| \, dy \, dx \leq \frac{1}{\sqrt{\lambda}} \|qf\|_{L^1} \|q\|_{L^1},$$

and this yields the desired bound (1.1).
3. Proof of Theorem 1.3

In this section we prove the bounds in (1.2) and (1.3) for the non-real eigenvalues of \(A\) in Theorem 1.3, which essentially depend on the negative part \(q_-(x) = \max\{0, -q(x)\}, \ x \in \mathbb{R}\), of the potential. The following lemma will be useful.

**Lemma 3.1.** Let \(\lambda \in \mathbb{C}^+\) be an eigenvalue of \(A\) and let \(f\) be a corresponding eigenfunction. Define
\[
U(x) := \int_x^\infty \text{sgn}(t)|f(t)|^2 \, dt \quad \text{and} \quad V(x) := \int_x^\infty |f'(t)|^2 + q(t)|f(t)|^2 \, dt.
\]
for \(x \in \mathbb{R}\). Then the following assertions hold:

(a) \(\lambda U(x) = f'(x)\overline{f(x)} + V(x)\);
(b) \(\lim_{x \to -\infty} U(x) = 0\) and \(\lim_{x \to -\infty} V(x) = 0\);
(c) \(||f'||_{L^2} \leq 2||q_-||_{L^1}||f||_{L^2}||
(d) \(||f||_{L^\infty} \leq 2\sqrt{||q_-||_{L^1}} ||f||_{L^2}||

**Proof.** Note that \(f\) satisfies the asymptotics (2.2)-(2.3) and hence \(f\) and \(f'\) vanish at \(\pm \infty\) and \(f' \in L^2(\mathbb{R})\). In particular, \(V(x)\) is well defined. We multiply the identity \(\lambda f(t) = \text{sgn}(t)(-f'''(t) + q(t)f(t))\) by \(\text{sgn}(t)\overline{f(t)}\) and integration by parts yields
\[
\lambda U(x) = \int_x^\infty -f'''(t)\overline{f(t)} + q(t)|f(t)|^2 \, dt = f'(x)\overline{f(x)} + V(x)
\]
for all \(x \in \mathbb{R}\). This shows (a). Moreover, we have
\[
\lambda \int_{\mathbb{R}} \text{sgn}(t)|f(t)|^2 \, dt = \lim_{x \to -\infty} \lambda U(x) = \lim_{x \to -\infty} V(x) = \int_{\mathbb{R}} |f'(t)|^2 + q(t)|f(t)|^2 \, dt.
\]
Taking the imaginary part shows \(\lim_{x \to -\infty} U(x) = 0\) and, hence, \(\lim_{x \to -\infty} V(x) = 0\). This proves (b).

As \(f\) is continuous and vanishes at \(\pm \infty\) we have \(||f||_{L^\infty} < \infty\). Making use of \(\lim_{x \to -\infty} V(x) = 0\) we find
\[
||f'||_{L^2}^2 = \int_{\mathbb{R}} |f'(t)|^2 \, dt = -\int_{\mathbb{R}} q(t)|f(t)|^2 \, dt \leq \int_{\mathbb{R}} q_-(t)|f(t)|^2 \, dt \leq ||q_-||_{L^1} ||f||_{L^\infty}^2
\]
and this shows (c). In order to verify (d) let \(x, y \in \mathbb{R}\) with \(x > y\). Then
\[
|f(x)|^2 - |f(y)|^2 \leq \int_y^x (|f(x)|^2)' \, dt \leq 2 \int_y^x |f'(t)|^2 \, dt \leq 2 ||f||_{L^2} ||f'||_{L^2}
\]
together with \(f(y) \to 0, y \to -\infty\), leads to \(||f||_{L^\infty}^2 \leq 2 ||f||_{L^2} ||f'||_{L^2}||. Hence the estimate in (d) follows by applying (c).

**Proof of (1.2) and (1.3).** Let \(\lambda \in \mathbb{C}^+\) be an eigenvalue of \(A\) and let \(f \in \text{dom} A\) be a corresponding eigenfunction. We can assume \(||q_-||_{L^1} > 0\) as otherwise \(f = 0\) by Lemma 3.1 (d). Let \(U\) and \(V\) be as in Lemma 3.1, let \(\delta := (2||q_-||_{L^1})^{-1}\) and define the function \(g\) on \(\mathbb{R}\) by
\[
g(x) = \begin{cases} \text{sgn}(x), & |x| > \delta, \\ \frac{\delta}{x}, & |x| \leq \delta. \end{cases}
\]
From Lemma 3.1 (a) we have
\[
\lambda \int_{\mathbb{R}} g'(x)U(x) \, dx = \int_{\mathbb{R}} g'(x)(f'(x)\overline{f(x)} + V(x)) \, dx.
\]
Since $g$ is bounded and $U(x)$ vanishes for $x \to \pm \infty$, integration by parts leads to the estimate
\[
\int \frac{g'(x)}{U(x)} \, dx = \int g(x) \, dx \cdot \frac{f(x)}{U(x)} \geq \int_{\mathbb{R} \setminus [-\delta, \delta]} \left| \frac{f(x)}{U(x)} \right| \, dx \geq \|f\|_{L^2}^2 - \|g\|_{L^2}^2.
\]  
(3.2)

Further we see with Lemma 3.1 (d) in the last line of (3.2). Further we see with Lemma 3.1 (c)–(d)
\[
\left| \int g'(x) f(x) \, dx \right| \leq \|f\|_{L^2} \|f\|_{L^2} \|g\|_{L^2} \leq 4 \|q_-\|_{L^2}^2 \|f\|_{L^2}^2 \sqrt{2} \delta^{-1}
\]
(3.3)

Since $\|g\|_{L^\infty} = 1$ and $V(x)$ vanishes for $x \to \pm \infty$ integration by parts together with Lemma 3.1 (c)–(d) yield
\[
\left| \int g'(x) V(x) \, dx \right| = \left| \int g(x) \left( |f'(x)|^2 + q(x) |f(x)|^2 \right) \, dx \right| 
\leq \|g\|_{L^\infty} \|f'\|_{L^2}^2 + \|g\|_{L^2} \|q\|_{L^1} \|f\|_{L^2}^2 
\leq 4 \|q_-\|_{L^1} \left( \|q_-\|_{L^1} + \|q\|_{L^1} \right) \|f\|_{L^2}^2.
\]  
(3.4)

Comparing the imaginary parts in (3.1) we have with (3.2) and (3.3)
\[
\frac{2}{3} \left| \Im \lambda \|f\|_{L^2}^2 \right| \leq \left| \Im \lambda \right| \left| \int g'(x) U(x) \, dx \right| \leq \left| \int g'(x) f'(x) \, dx \right| 
\leq 16 \cdot \sqrt{3} \|q_-\|_{L^1} \|f\|_{L^2}^2.
\]

In the same way we obtain from (3.2), (3.1) and (3.3)–(3.4) that
\[
\frac{2}{3} \left| \Re \|f\|_{L^2}^2 \right| \leq \left| \Re \lambda \right| \left| \int g'(x) U(x) \, dx \right| = \left| \int g'(x) \left( f'(x) + V(x) \right) \, dx \right| 
\leq 16 \cdot \sqrt{3} \|q_-\|_{L^1} \|f\|_{L^2}^2 + 4 \|q_-\|_{L^1} \left( \|q_-\|_{L^1} + \|q\|_{L^1} \right) \|f\|_{L^2}^2.
\]

This shows the bounds (1.3) and (1.2).

References


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