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SPECTRAL BOUNDS FOR SINGULAR INDEFINITE STURM-LIOUVILLE OPERATORS WITH L^1–POTENTIALS

JUSSI BEHRNDT, PHILIPP SCHMITZ, AND CARSTEN TRUNK

Abstract. The spectrum of the singular indefinite Sturm-Liouville operator

$$A = \operatorname{sgn}(\cdot)\Big(-\tfrac{d^2}{dx^2} + q\Big)$$

with a real potential $q \in L^1(\mathbb{R})$ covers the whole real line and, in addition, non-real eigenvalues may appear if the potential q assumes negative values. A quantitative analysis of the non-real eigenvalues is a challenging problem, and so far only partial results in this direction were obtained. In this paper the bound

$$|\lambda| \leq \|q\|_{L^1}^2$$

on the absolute values of the non-real eigenvalues λ of A is obtained. Furthermore, separate bounds on the imaginary parts and absolute values of these eigenvalues are proved in terms of the L^1-norm of q and its negative part q_-.

1. Introduction

The aim of this paper is to prove bounds on the absolute values of the non-real eigenvalues of the singular indefinite Sturm-Liouville operator

$$Af = \operatorname{sgn}(\cdot)\big(-f'' + qf\big),$$
$$\operatorname{dom} A = \big\{ f \in L^2(\mathbb{R}) : f, f' \in AC(\mathbb{R}), -f'' + qf \in L^2(\mathbb{R}) \big\},$$

where $AC(\mathbb{R})$ stands for space of all locally absolutely continuous functions. It will always be assumed that the potential q is real-valued and belongs to $L^1(\mathbb{R})$.

The operator A is not symmetric nor self-adjoint in an L^2-Hilbert space due to the sign change of the weight function $\operatorname{sgn}(\cdot)$. However, A can be interpreted as a self-adjoint operator with respect to the Krein space inner product $(\operatorname{sgn}\cdot,\cdot)$ in $L^2(\mathbb{R})$. We summarize the qualitative spectral properties of A in the next theorem, which follows from [4, Theorem 4.2] or [16, Proposition 2.4] and the well-known spectral properties of the definite Sturm-Liouville operator $-\frac{d^2}{dx^2}+q$; cf. [23, 24, 25].

Theorem 1.1. *The essential spectrum of A coincides with $\mathbb{R}$ and the non-real spectrum of A consists of isolated eigenvalues with finite algebraic multiplicity which are symmetric with respect to $\mathbb{R}$.*

Indefinite Sturm-Liouville operators have been studied for more than a century, and have again attracted a lot of attention in the recent past. Early works in this context usually deal with the regular case, that is, the operator A is studied on a finite interval with appropriate boundary conditions at the endpoints; cf. [15, 22] and, e.g., [11, 18, 26]. In this situation the spectrum of A is purely discrete and various estimates on the real and imaginary parts of the non-real eigenvalues were

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obtained in the last few years; cf. [2, 9, 10, 14, 17, 21]. The singular case is much less studied, due to the technical difficulties which, very roughly speaking, are caused by the presence of continuous spectrum.

Explicit bounds on non-real eigenvalues for singular Sturm-Liouville operators with L^∞-potentials were obtained with Krein space perturbation techniques in [5] and under additional assumptions for L^1-potentials in [6, 7], see also [3] for the absence of real eigenvalues and [19] for the accumulation of non-real eigenvalues of a very particular family of potentials. In this paper we substantially improve the earlier bounds in [6, 7] and relax the conditions on the potential. More precisely, here we prove for arbitrary real $q \in L^1(\mathbb{R})$ the following bound.

Theorem 1.2. *Let $q \in L^1(\mathbb{R})$ be real. Every non-real eigenvalue λ of the indefinite Sturm-Liouville operator A satisfies*

$$|\lambda| \leq \|q\|_{L^1}^2. \tag{1.1}$$

Moreover, we prove two bounds in terms of the negative part q_- of q.

Theorem 1.3. *Let $q \in L^1(\mathbb{R})$ be real. Every non-real eigenvalue λ of the indefinite Sturm-Liouville operator A satisfies*

$$|\operatorname{Im}\lambda| \leq 24 \cdot \sqrt{3}\|q_-\|_{L^1}^2 \tag{1.2}$$

and

$$|\lambda| \leq 24 \cdot \sqrt{3}\|q_-\|_{L^1}^2 + 6\|q_-\|_{L^1}\big(\|q_-\|_{L^1} + \|q\|_{L^1}\big). \tag{1.3}$$

The bound (1.1) is proved in Section 2. It's proof is based on the Birman-Schwinger principle using similar arguments as in [1, 13], [12, Chapter 14.3]; see also [8]. The bounds (1.2) and (1.3) are obtained in Section 3 by adapting the techniques from the regular case in [2, 9, 21] to the present singular situation.

2. Proof of Theorem 1.2

In this section we prove the bound (1.1) for the non-real eigenvalues of A. We adapt a technique similar to the Birman-Schwinger principle in [12] and apply it to the indefinite operator A. The main ingredient is a bound for the integral kernel of the resolvent of the operator

$$B_0 f = \operatorname{sgn}(\cdot)\big(-f''\big), \quad \operatorname{dom} B_0 = \big\{f \in L^1(\mathbb{R}) : f, f' \in AC(\mathbb{R}), -f'' \in L^1(\mathbb{R})\big\},$$

in $L^1(\mathbb{R})$.

Lemma 2.1. *The operator B_0 is closed in $L^1(\mathbb{R})$ and for all λ in the open upper half-plane $\mathbb{C}^+$ the resolvent of B_0 is an integral operator*

$$\big[(B_0 - \lambda)^{-1} g\big](x) = \int_{\mathbb{R}} K_\lambda(x, y) g(y)\, \mathrm{d}y, \quad g \in L^1(\mathbb{R}),$$

where the kernel $K_\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is bounded by $|K_\lambda(x, y)| \leq |\lambda|^{-\frac{1}{2}}$ for all $x, y \in \mathbb{R}$.

Proof. Here and in the following we define $\sqrt{\lambda}$ for $\lambda \in \mathbb{C}^+$ as the principal value of the square root, which ensures $\operatorname{Im}\sqrt{\lambda} > 0$ and $\operatorname{Re}\sqrt{\lambda} > 0$. For $\lambda \in \mathbb{C}^+$ consider the integral operator

$$(T_\lambda g)(x) = \int_{\mathbb{R}} K_\lambda(x, y) g(y)\, \mathrm{d}y, \quad g \in L^1(\mathbb{R}), \tag{2.1}$$

with the kernel $K_\lambda(x,y) = C_\lambda(x,y) + D_\lambda(x,y)$ of the form

$$C_\lambda(x,y) = \frac{1}{2\alpha\sqrt{\lambda}} \begin{cases} \alpha e^{i\sqrt{\lambda}(x+y)}, & x \geq 0,\ y \geq 0, \\ -e^{\sqrt{\lambda}(ix+y)}, & x \geq 0,\ y < 0, \\ e^{\sqrt{\lambda}(x+iy)}, & x < 0,\ y \geq 0, \\ -\overline{\alpha} e^{\sqrt{\lambda}(x+y)}, & x < 0,\ y < 0, \end{cases}$$

and

$$D_\lambda(x,y) = \frac{1}{2\alpha\sqrt{\lambda}} \begin{cases} \overline{\alpha} e^{i\sqrt{\lambda}|x-y|}, & x \geq 0,\ y \geq 0, \\ 0, & x \geq 0,\ y < 0, \\ 0, & x < 0,\ y \geq 0, \\ -\alpha e^{-\sqrt{\lambda}|x-y|}, & x < 0,\ y < 0, \end{cases}$$

where $\alpha := \frac{1-i}{2}$. Hence,

$$|K_\lambda(x,y)| = |C_\lambda(x,y) + D_\lambda(x,y)| \leq \frac{1}{\sqrt{|\lambda|}}$$

and the integral in (2.1) converges for every $g \in L^1(\mathbb{R})$. We have

$$\sup_{y\geq 0} \int_{\mathbb{R}} |C_\lambda(x,y)|\,\mathrm{d}x = \frac{1}{2\sqrt{|\lambda|}} \left(\frac{1}{\operatorname{Im}\sqrt{\lambda}} + \frac{\sqrt{2}}{\operatorname{Re}\sqrt{\lambda}} \right)$$

and

$$\sup_{y<0} \int_{\mathbb{R}} |C_\lambda(x,y)|\,\mathrm{d}x = \frac{1}{2\sqrt{|\lambda|}} \left(\frac{\sqrt{2}}{\operatorname{Im}\sqrt{\lambda}} + \frac{1}{\operatorname{Re}\sqrt{\lambda}} \right).$$

For $y \geq 0$ we estimate

$$\int_0^\infty |D_\lambda(x,y)|\,\mathrm{d}x = \frac{1}{2\sqrt{|\lambda|}} \int_0^\infty e^{-\operatorname{Im}\sqrt{\lambda}|x-y|}\,\mathrm{d}x = \frac{2 - e^{-\operatorname{Im}\sqrt{\lambda}y}}{2\sqrt{|\lambda|}\operatorname{Im}\sqrt{\lambda}} \leq \frac{1}{\sqrt{|\lambda|}\operatorname{Im}\sqrt{\lambda}},$$

and analogously for $y < 0$

$$\int_{-\infty}^0 |D_\lambda(x,y)|\,\mathrm{d}x = \frac{1}{2\sqrt{|\lambda|}} \int_{-\infty}^0 e^{-\operatorname{Re}\sqrt{\lambda}|x-y|}\,\mathrm{d}x = \frac{2 - e^{\operatorname{Re}\sqrt{\lambda}y}}{2\sqrt{|\lambda|}\operatorname{Re}\sqrt{\lambda}} \leq \frac{1}{\sqrt{|\lambda|}\operatorname{Re}\sqrt{\lambda}}.$$

Hence,

$$c := \sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |K_\lambda(x,y)|\,\mathrm{d}x < \infty$$

and Fubini's theorem yields

$$\|T_\lambda g\|_{L^1} \leq \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |K_\lambda(x,y)|\,\mathrm{d}x\,\mathrm{d}y \leq c\|g\|_{L^1}.$$

Therefore T_λ in (2.1) is an everywhere defined bounded operator in $L^1(\mathbb{R})$.

We claim that T_λ is the inverse of $B_0 - \lambda$. In fact, consider the functions u, v given by

$$u(x) = \begin{cases} e^{i\sqrt{\lambda}x}, & x \geq 0, \\ \overline{\alpha} e^{\sqrt{\lambda}x} + \alpha e^{-\sqrt{\lambda}x}, & x < 0, \end{cases} \quad \text{and} \quad v(x) = \begin{cases} \alpha e^{i\sqrt{\lambda}x} + \overline{\alpha} e^{-i\sqrt{\lambda}x}, & x \geq 0, \\ e^{\sqrt{\lambda}x}, & x < 0, \end{cases}$$

which solve the differential equation $\operatorname{sgn}(\cdot)(-f'') = \lambda f$, that is, u and v, and their derivatives, belong to $AC(\mathbb{R})$ and satisfy the differential equation almost everywhere. Since the Wronskian equals $2\alpha\sqrt{\lambda}$, these solutions are linearly independent.

Note that $u,v \notin L^1(\mathbb{R})$ and one concludes that $B_0-\lambda$ is injective. A simple calculation shows the identity

$$K_\lambda(x,y) = C_\lambda(x,y) + D_\lambda(x,y) = \frac{1}{2\alpha\sqrt{\lambda}} \begin{cases} u(x)v(y)\operatorname{sgn}(y), & y<x, \\ v(x)u(y)\operatorname{sgn}(y), & x<y, \end{cases}$$

and hence we have

$$(T_\lambda g)(x) = \frac{1}{2\alpha\sqrt{\lambda}}\left(u(x)\int_{-\infty}^x v(y)\operatorname{sgn}(y)g(y)\,\mathrm{d}y + v(x)\int_x^\infty u(y)\operatorname{sgn}(y)g(y)\,\mathrm{d}y \right).$$

One verifies $T_\lambda g, (T_\lambda g)' \in AC(\mathbb{R})$ and $T_\lambda g$ is a solution of $\operatorname{sgn}(\cdot)(-f'')-\lambda f = g$. This implies $(T_\lambda g)'' \in L^1(\mathbb{R})$ and hence $T_\lambda g \in \operatorname{dom} B_0$ satisfies

$$(B_0-\lambda)T_\lambda g = g \quad \text{for all } g\in L^1(\mathbb{R}).$$

Therefore, $B_0-\lambda$ is surjective and we have $T_\lambda = (B_0-\lambda)^{-1}$. It follows that B_0 is a closed operator in $L^1(\mathbb{R})$ and that λ belongs to the resolvent set of B_0. □

Proof of Theorem 1.2. Since the non-real point spectrum of A is symmetric with respect to the real line (see Theorem 1.1) it suffices to consider eigenvalues in the upper half plane. Let $\lambda\in\mathbb{C}^+$ be an eigenvalue of A with a corresponding eigenfunction $f\in\operatorname{dom} A$. Since $q\in L^1(\mathbb{R})$ and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+q$ is in the limit point case at $\pm\infty$ (see, e.g. [23, Lemma 9.37]) the function f is unique up to a constant multiple. As $-f''+qf=\lambda f$ on $\mathbb{R}^+$ and $f''-qf=\lambda f$ on $\mathbb{R}^-$ with q integrable one has the well-known asymptotical behaviour

$$\begin{aligned} f(x) &= \alpha_+\big(1+o(1)\big)e^{i\sqrt{\lambda}x}, \quad x\to+\infty, \\ f'(x) &= \alpha_+ i\sqrt{\lambda}\big(1+o(1)\big)e^{i\sqrt{\lambda}x}, \quad x\to+\infty, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} f(x) &= \alpha_-\big(1+o(1)\big)e^{\sqrt{\lambda}x}, \quad x\to-\infty, \\ f'(x) &= \alpha_-\sqrt{\lambda}\big(1+o(1)\big)e^{\sqrt{\lambda}x}, \quad x\to-\infty, \end{aligned} \tag{2.3}$$

for some $\alpha_+,\alpha_-\in\mathbb{C}$; see, e.g. [20, § 24.2, Example a] or [23, Lemma 9.37]. These asymptotics yield $f, qf\in L^1(\mathbb{R})$ and $-f''=\lambda\operatorname{sgn}(\cdot)f - qf\in L^1(\mathbb{R})$, and therefore $f\in\operatorname{dom} B_0$. Thus, f satisfies

$$0 = (A-\lambda)f = \operatorname{sgn}(\cdot)(-f'') - \lambda f + \operatorname{sgn}(\cdot)qf = (B_0-\lambda)f + \operatorname{sgn}(\cdot)qf$$

and since λ is in the resolvent set of B_0 we obtain

$$-qf = q(B_0-\lambda)^{-1}\operatorname{sgn}(\cdot)qf.$$

Note that $\|qf\|_{L^1}\neq 0$ as otherwise λ would be an eigenvalue of B_0. With the help of Lemma 2.1 we then conclude

$$0<\|qf\|_{L^1} \le \int_{\mathbb{R}}|q(x)|\int_{\mathbb{R}}|K_\lambda(x,y)||q(y)f(y)|\,\mathrm{d}y\,\mathrm{d}x \le \frac{1}{\sqrt{|\lambda|}}\|qf\|_{L^1}\|q\|_{L^1}$$

and this yields the desired bound (1.1). □

3. Proof of Theorem 1.3

In this section we prove the bounds in (1.2) and (1.3) for the non-real eigenvalues of A in Theorem 1.3, which essentially depend on the negative part $q_-(x) = \max\{0, -q(x)\}$, $x \in \mathbb{R}$, of the potential. The following lemma will be useful.

Lemma 3.1. *Let $\lambda \in \mathbb{C}^+$ be an eigenvalue of A and let f be a corresponding eigenfunction. Define*

$$U(x) := \int_x^\infty \operatorname{sgn}(t)|f(t)|^2 \,\mathrm{d}t \quad \text{and} \quad V(x) := \int_x^\infty |f'(t)|^2 + q(t)|f(t)|^2 \,\mathrm{d}t.$$

for $x \in \mathbb{R}$. Then the following assertions hold:

(a) $\lambda U(x) = f'(x)\overline{f(x)} + V(x)$*;*
(b) $\lim_{x\to-\infty} U(x) = 0$ *and* $\lim_{x\to-\infty} V(x) = 0$*;*
(c) $\|f'\|_{L^2} \leq 2\|q_-\|_{L^1}\|f\|_{L^2}$*;*
(d) $\|f\|_\infty \leq 2\sqrt{\|q_-\|_{L^1}}\|f\|_{L^2}$*.*

Proof. Note that f satisfies the asymptotics (2.2)–(2.3) and hence f and f' vanish at $\pm\infty$ and $f' \in L^2(\mathbb{R})$. In particular, $V(x)$ is well defined. We multiply the identity $\lambda f(t) = \operatorname{sgn}(t)(-f''(t) + q(t)f(t))$ by $\operatorname{sgn}(t)\overline{f(t)}$ and integration by parts yields

$$\lambda U(x) = \int_x^\infty -f''(t)\overline{f(t)} + q(t)|f(t)|^2 \,\mathrm{d}t = f'(x)\overline{f(x)} + V(x)$$

for all $x \in \mathbb{R}$. This shows (a). Moreover, we have

$$\lambda \int_{\mathbb{R}} \operatorname{sgn}(t)|f(t)|^2 \,\mathrm{d}t = \lim_{x\to-\infty} \lambda U(x) = \lim_{x\to-\infty} V(x) = \int_{\mathbb{R}} |f'(t)|^2 + q(t)|f(t)|^2 \,\mathrm{d}t.$$

Taking the imaginary part shows $\lim_{x\to-\infty} U(x) = 0$ and, hence, $\lim_{x\to-\infty} V(x) = 0$. This proves (b).

As f is continuous and vanishes at $\pm\infty$ we have $\|f\|_\infty < \infty$. Making use of $\lim_{x\to-\infty} V(x) = 0$ we find

$$\|f'\|_{L^2}^2 = \int_{\mathbb{R}} |f'(t)|^2 \,\mathrm{d}t = -\int_{\mathbb{R}} q(t)|f(t)|^2 \,\mathrm{d}t \leq \int_{\mathbb{R}} q_-(t)|f(t)|^2 \,\mathrm{d}t \leq \|q_-\|_{L^1}\|f\|_\infty^2$$

and this shows (c). In order to verify (d) let $x, y \in \mathbb{R}$ with $x > y$. Then

$$|f(x)|^2 - |f(y)|^2 \leq \int_y^x \big(|f|^2\big)'(t) \,\mathrm{d}t \leq 2\int_y^x |f(t)f'(t)| \,\mathrm{d}t \leq 2\|f\|_{L^2}\|f'\|_{L^2}$$

together with $f(y) \to 0$, $y \to -\infty$, leads to $\|f\|_\infty^2 \leq 2\|f\|_{L^2}\|f'\|_{L^2}$. Hence the estimate in (d) follows by applying (c). □

Proof of (1.2) and (1.3). Let $\lambda \in \mathbb{C}^+$ be a eigenvalue of A and let $f \in \operatorname{dom} A$ be a corresponding eigenfunction. We can assume $\|q_-\|_{L^1} > 0$ as otherwise $f = 0$ by Lemma 3.1 (d). Let U and V be as in Lemma 3.1, let $\delta := (24\|q_-\|_{L^1})^{-1}$ and define the function g on $\mathbb{R}$ by

$$g(x) = \begin{cases} \operatorname{sgn}(x), & |x| > \delta, \\ \frac{x}{\delta}, & |x| \leq \delta. \end{cases}$$

From Lemma 3.1 (a) we have

$$\lambda \int_{\mathbb{R}} g'(x)U(x) \,\mathrm{d}x = \int_{\mathbb{R}} g'(x)\big(f'(x)\overline{f(x)} + V(x)\big) \,\mathrm{d}x. \tag{3.1}$$

Since g is bounded and $U(x)$ vanishes for $x \to \pm\infty$, integration by parts leads to the estimate

$$\begin{aligned}\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)\,\mathrm{sgn}(x)|f(x)|^2\,\mathrm{d}x \geq \int_{\mathbb{R}\setminus[-\delta,\delta]} |f(x)|^2\,\mathrm{d}x \\ = \|f\|_{L^2}^2 - \int_{-\delta}^{\delta} |f(x)|^2\,\mathrm{d}x \geq \|f\|_{L^2}^2 - 2\delta\|f\|_\infty^2 \\ \geq \|f\|_{L^2}^2 - 8\delta\|q_-\|_{L^1}\|f\|_{L^2}^2 = \frac{2}{3}\|f\|_{L^2}^2;\end{aligned} \tag{3.2}$$

here we have used Lemma 3.1 (d) in the last line of (3.2). Further we see with Lemma 3.1 (c)–(d)

$$\begin{aligned}\left|\int_{\mathbb{R}} g'(x)f'(x)\overline{f(x)}\,\mathrm{d}x\right| \leq \|f\|_\infty\|f'\|_{L^2}\|g'\|_{L^2} \leq 4\|q_-\|_{L^1}^{\frac{3}{2}}\|f\|_{L^2}^2\sqrt{\frac{2}{\delta}} \\ \leq 16\cdot\sqrt{3}\|q_-\|_{L^1}^2\|f\|_{L^2}^2.\end{aligned} \tag{3.3}$$

Since $\|g\|_\infty = 1$ and $V(x)$ vanishes for $x \to \pm\infty$ integration by parts together with Lemma 3.1 (c)–(d) yield

$$\begin{aligned}\left|\int_{\mathbb{R}} g'(x)V(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} g(x)\left(|f'(x)|^2 + q(x)|f(x)|^2\right)\,\mathrm{d}x\right| \\ \leq \|g\|_\infty\|f'\|_{L^2}^2 + \|f\|_\infty^2\|q\|_{L^1} \\ \leq 4\|q_-\|_{L^1}\left(\|q_-\|_{L^1} + \|q\|_{L^1}\right)\|f\|_{L^2}^2.\end{aligned} \tag{3.4}$$

Comparing the imaginary parts in (3.1) we have with (3.2) and (3.3)

$$\begin{aligned}\frac{2}{3}|\operatorname{Im}\lambda|\|f\|_{L^2}^2 \leq |\operatorname{Im}\lambda|\left|\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x\right| \leq \left|\int_{\mathbb{R}} g'(x)f'(x)\overline{f(x)}\,\mathrm{d}x\right| \\ \leq 16\cdot\sqrt{3}\|q_-\|_{L^1}^2\|f\|_{L^2}^2.\end{aligned}$$

In the same way we obtain from (3.2), (3.1) and (3.3)–(3.4) that

$$\begin{aligned}\frac{2}{3}|\lambda|\|f\|_{L^2}^2 \leq \left|\lambda\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} g'(x)\big(f'(x)\overline{f(x)} + V(x)\big)\,\mathrm{d}x\right| \\ \leq 16\cdot\sqrt{3}\|q_-\|_{L^1}^2\|f\|_{L^2}^2 + 4\|q_-\|_{L^1}\left(\|q_-\|_{L^1} + \|q\|_{L^1}\right)\|f\|_{L^2}^2.\end{aligned}$$

This shows the bounds (1.3) and (1.2). □

References

[1] A. A. Abramov, A. Aslanyan, E. B. Davies, *Bounds on complex eigenvalues and resonances*, J. Phys. A: Math. Gen. **34**, 57–72 (2001).

[2] J. Behrndt, S. Chen, F. Philipp, J. Qi, *Estimates on the non-real eigenvalues of regular indefinite Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh Sect. A **144**, 1113–1126 (2014).

[3] J. Behrndt, Q. Katatbeh, C. Trunk, *Non-real eigenvalues of singular indefinite Sturm-Liouville operators*, Proc. Amer. Math. Soc. **137**, 3797–3806 (2009).

[4] J. Behrndt, F. Philipp, *Spectral analysis of ordinary differential operators with indefinite weigths*, J. Differential Equations **248**, 2015–2037 (2010).

[5] J. Behrndt, F. Philipp, C. Trunk, *Bounds on the non-real spectrum of differential operators with indefinite weights*, Math. Ann. **357**, 185–213 (2013).

[6] J. Behrndt, P. Schmitz, C. Trunk, *Bounds on the non-real spectrum of a singular indefinite Sturm-Liouville operator on* $\mathbb{R}$, Proc. Appl. Math. Mech. **16**, 881–882 (2016).

[7] J. Behrndt, P. Schmitz, C. Trunk, *Estimates for the non-real spectrum of a singular indefinite Sturm-Liouville operator on* $\mathbb{R}$, to appear in Proc. Appl. Math. Mech. **17**.

[8] B. M. Brown, M. S. P. Eastham, *Analytic continuation and resonance-free regions for Sturm-Liouville potentials with power decay*, J. Comput. Appl. Math. **148**, 49–63 (2002).
[9] S. Chen, J. Qi, *A priori bounds and existence of non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Spectr. Theory **4**, 53–63 (2014).
[10] S. Chen, J. Qi, B. Xie, *The upper and lower bounds on the non-real eigenvalus of indefinite Sturm-Liouville problems*, Proc. Amer. Math. Soc. **144**, 547–559 (2016).
[11] B. Ćurgus, H. Langer, *A Krein space approach to symmetric ordinary differential operators with an indefinite weight function*, J. Differential Equations **79**, 31–61 (1989).
[12] E. B. Davies, *Linear Operators and their Spectra*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2007.
[13] E. B. Davies, J. Nath, *Schrödinger operators with slowly decaying potentials*, J. Comput. Appl. Math. **148**, 1–18 (2002).
[14] X. Guo, H. Sun, B. Xie, *Non-real eigenvalues of symmetric Sturm–Liouville problems with indefinite weight functions*, Electron. J. Qual. Theory Differ. Equ. **2017**, 1–14 (2107).
[15] O. Haupt, *Über eine Methode zum Beweise von Oszillationstheoremen*, Math. Ann. **76**, 67–104 (1914).
[16] I. Karabash, C. Trunk, *Spectral properties of singular Sturm-Liouville operators*, Proc. Roy. Soc. Edinburgh Sect. A **139**, 483–503 (2009).
[17] M. Kikonko, A. B. Mingarelli, *Bounds on real and imaginary parts of non-real eigenvalues of a non-definite Sturm-Liouville problem*, J. Differential Equations **261**, 6221–6232 (2016).
[18] A. B. Mingarelli, *A survey of the regular weighted Sturm–Liouville problem–the non-definite case*, in: Proceedings of the Workshop on Applications of Differential Equations, 1986, pp. 109–137.
[19] M. Levitin, M. Seri, *Accumulation of complex eigenvalues of an indefinite Sturm-Liouville operator with a shifted Coulomb potential*, Operators and Matrices **10**, 223–245 (2016).
[20] M. A. Naimark, *Linear Differential Operators. Part II: Linear Differential Operators in Hilbert Space*, Frederick Ungar Publishing Co., New York, 1968.
[21] J. Qi, B. Xie, *Non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Differential Equations **255**, 2291–2301 (2013).
[22] R. Richardson, *Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order*, Amer. J. Math. **40**, 283–316 (1918).
[23] G. Teschl, *Mathematical Methods in Quantum Mechanics; With Applications to Schrödinger Operators*, Amer. Math. Soc., Providence, Rhode Island, 2009.
[24] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Math. 1258, Springer, 1987.
[25] J. Weidmann, *Lineare Operatoren in Hilberträumen Teil II*, Teubner, 2003.
[26] A. Zettl, *Sturm-Liouville Theory*, Mathematical Surveys and Monographs **121**, AMS, Providence, RI, 2005.

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