An estimate on the non-real spectrum of a singular indefinite Sturm-Liouville operator

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August 2017

URN: urn:nbn:de:gbv:ilm1-2017200464
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Abstract

It will be shown with the help of the Birman-Schwinger principle that the non-real spectrum of the singular indefinite Sturm-Liouville operator $\text{sgn}(\cdot)(-d^2/dx^2 + q)$ with a real potential $q \in L^1 \cap L^2$ is contained in a circle around the origin with radius $\|q\|_{L^1}^2$.

Keywords: indefinite Sturm-Liouville, Birman-Schwinger, singular, non-real spectrum, eigenvalues

1 Introduction and main result

Consider the operators

$$A_0 f = \text{sgn}(\cdot)(-f'')$$

and

$$Af := A_0 f + \text{sgn}(\cdot)q f = \text{sgn}(\cdot)(-f'' + q f), \quad f \in H^2(\mathbb{R}),$$

in $L^2(\mathbb{R})$, where $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a real function with $\lim_{x \to \pm \infty} q(x) = 0$.

Note that $q$ is a relatively compact perturbation of $A_0$ (cf. Theorem 11.2.11 in [10]). The operator $A$ (and $A_0$) is neither symmetric nor self-adjoint with respect to the usual scalar product in $L^2(\mathbb{R})$, but symmetric and self-adjoint with respect to the indefinite inner product

$$[f, g] := \int_{\mathbb{R}} \text{sgn}(x)f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(\mathbb{R}),$$

and the essential spectrum is given by $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_0) = \sigma(A_0) = \mathbb{R}$; cf. [9] and Corollary 4.4 in [2]. It is well known that the operator $A$ may have non-real spectrum, see e.g. [5]. The main objective of this note is to prove the following theorem.

**Theorem 1.1.** Let $q \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\lim_{x \to \pm \infty} q(x) = 0$. Then the non-real spectrum of $A$ consists only of isolated eigenvalues and every non-real eigenvalue $\lambda$ of $A$ satisfies $|\lambda| \leq \|q\|_{L^1}^2$. 
This result improves the bounds in [6] for certain potentials and is based on the techniques in [1]. For further bounds on the non-real spectrum of indefinite Sturm-Liouville operators we refer to [4] for the case of a bounded potential \( q \) and [3, 7, 8, 11–13] for the regular case.

2 Proof of Theorem 1.1

Lemma 2.1. For every \( \lambda \in \mathbb{C}^+ \) the resolvent of \( A_0 \) is an integral operator of the form

\[
[(A_0 - \lambda)^{-1}g](x) = \int_{\mathbb{R}} K_\lambda(x, y) g(y) \, dy, \quad g \in L^2(\mathbb{R}),
\]

with a kernel function \( K_\lambda \) which is bounded by \( |K_\lambda(x, y)| \leq |\lambda|^{-\frac{1}{2}} \).

Proof. For \( \lambda \in \mathbb{C}^+ \) consider the solutions \( u, v \) of the differential equation \( -\sgn(\cdot)f'' = \lambda f \) defined by

\[
\begin{align*}
  u(x) &= \begin{cases} 
    e^{i \sqrt{\lambda} x}, & x \geq 0, \\
    \alpha e^{i \sqrt{\lambda} x} + e^{-i \sqrt{\lambda} x}, & x < 0,
  \end{cases} \\
  v(x) &= \begin{cases} 
    \alpha e^{i \sqrt{\lambda} x} + e^{-i \sqrt{\lambda} x}, & x \geq 0, \\
    e^{i \sqrt{\lambda} x}, & x < 0,
  \end{cases}
\end{align*}
\]

where \( \alpha = \frac{1+i}{2} \). For a non-real \( \lambda \) we define \( \sqrt{\lambda} \) as the principle value of the square root, so that, \( \Re \sqrt{\lambda} > 0 \) and \( \Im \sqrt{\lambda} > 0 \) for \( \lambda \in \mathbb{C}^+ \). As the Wronskian determinant equals \( 2\alpha \sqrt{\lambda} \) these two solutions are linearly independent. Moreover, for all \( x \in \mathbb{R} \) the restrictions \( u|_{(x, \infty)} \) and \( v|_{(-\infty, x)} \) are square integrable functions. One verifies that for \( g \in L^2(\mathbb{R}) \)

\[
(T_\lambda g)(x) := \frac{1}{2\alpha \sqrt{\lambda}} \left( u(x) \int_{-\infty}^{x} v(y) \sgn(y) g(y) \, dy + v(x) \int_{x}^{\infty} u(y) \sgn(y) g(y) \, dy \right)
\]

(2.1)

is a solution of \( -\sgn(\cdot)f'' - \lambda f = g \). It remains to show that \( T_\lambda \) is a bounded operator in \( L^2(\mathbb{R}) \). Rearranging the terms in (2.1) one sees that

\[
(T_\lambda g)(x) = (2\alpha \sqrt{\lambda})^{-1} \int_{\mathbb{R}} (k_1(x, y) + k_2(x, y)) g(y) \, dy \quad \text{for } g \in L^2(\mathbb{R})
\]

with

\[
k_1(x, y) := \begin{cases} 
  \alpha e^{i \sqrt{\lambda}(x+y)}, & x > 0, \quad y > 0, \\
  -e^{i \sqrt{\lambda}(x+y)}, & x > 0, \quad y < 0, \\
  e^{i \sqrt{\lambda}(x+y)}, & x < 0, \quad y > 0, \\
  -\alpha e^{i \sqrt{\lambda}(x+y)}, & x < 0, \quad y < 0,
\end{cases}
\]

\[
k_2(x, y) := \begin{cases} 
  e^{i \sqrt{\lambda}(x+y)}, & x > 0, \quad y > 0, \\
  -e^{i \sqrt{\lambda}(x+y)}, & x > 0, \quad y < 0, \\
  \alpha e^{i \sqrt{\lambda}(x+y)}, & x < 0, \quad y > 0, \\
  -\alpha e^{i \sqrt{\lambda}(x+y)}, & x < 0, \quad y < 0.
\end{cases}
\]
and

\[ k_2(x, y) := \begin{cases} \mp e^{\pm \sqrt{|x-y|}}, & x > 0, y > 0, \\ 0, & x > 0, y < 0, \\ 0, & x < 0, y > 0, \\ -\alpha e^{-\sqrt{|x-y|}}, & x < 0, y < 0. \end{cases} \]

We have \( k_1 \in L^2(\mathbb{R}^2) \). Calculating the resolvents of the self-adjoint operator \(-d^2/dx^2\) at the points \( \pm \lambda \) (cf. Satz 11.26 in [14]) yields for \( g \in L^2(\mathbb{R}) \)

\[ \int_{\mathbb{R}} k_2(x, y)g(y) \, dy = \pm 2\alpha \sqrt{\lambda} \left[ \left( -\frac{d^2}{dx^2} \mp \lambda \right)^{-1} (1_{\mathbb{R}^+} + g) \right](x), \quad x \in \mathbb{R}^\pm, \]

where \( 1_{\mathbb{R}^+} \) and \( 1_{\mathbb{R}^-} \) denote the characteristic functions of the positive and negative half-lines, respectively. Hence, \( T_\lambda \) is a bounded operator in \( L^2(\mathbb{R}) \) and \( (A_0 - \lambda)^{-1} = T_\lambda \). It is easy to see that the sum \( k_1 + k_2 \) is bounded by \( 2|\alpha| = \sqrt{2} \). Defining \( K_\lambda(x, y) := \frac{1}{2\alpha \sqrt{\lambda}} (k_1(x, y) + k_2(x, y)) \) completes the proof.

**Proof of Theorem 1.1.** We assume \( \|q\|_{L^1} \neq 0 \) as otherwise there are no non-real eigenvalues of \( A \). Since the operator \( A \) is a self-adjoint operator with respect to \([\cdot, \cdot]\) the point spectrum of \( A \) is symmetric with respect to the real line and hence it suffices to consider an eigenvalue \( \lambda \in \mathbb{C}^+ \) with corresponding eigenfunction \( f \in \text{dom}(A) = H^2(\mathbb{R}) \). Note, that \( f \) is bounded, since \( f \in H^2(\mathbb{R}) \). As \( Af = \lambda f \) we have in terms of the unperturbed operator \( A_0 \)

\[ (A_0 - \lambda)f = -\text{sgn}(\cdot)qf \in L^2(\mathbb{R}). \tag{2.2} \]

Setting \( q^{\frac{1}{2}}(x) := \text{sgn}(q(x))|q(x)|^{\frac{1}{2}} \) we have \( |q|^{\frac{1}{2}}q^{\frac{1}{2}} = q \), and hence (2.2) and \( \lambda \in \rho(A_0) \) yield

\[ g := q^{\frac{1}{2}}f = -q^{\frac{1}{2}}(A_0 - \lambda)^{-1} \left( \text{sgn}(\cdot)|q|^{\frac{1}{2}}q^{\frac{1}{2}}f \right) = -q^{\frac{1}{2}}(A_0 - \lambda)^{-1} \left( \text{sgn}(\cdot)|q|^{\frac{1}{2}}g \right). \]

Here the boundedness of \( f \) implies \( g \in L^2(\mathbb{R}) \). Now with Lemma 2.1 we estimate

\[ \|g\|_{L^2}^2 = \int_{\mathbb{R}} |g(x)| \cdot \left| -\frac{1}{2}(A_0 - \lambda)^{-1} \left( \text{sgn}(\cdot)|q|^{\frac{1}{2}}g \right) \right| (x) \, dx \]

\[ \leq \int_{\mathbb{R}} \left| q^{\frac{1}{2}}(x)g(x) \right| \int_{\mathbb{R}} |K_\lambda(x, y)| \left| q^{\frac{1}{2}}(y)g(y) \right| \, dy \, dx \]

\[ \leq |\lambda|^{-\frac{1}{2}} \left( \int_{\mathbb{R}} \left| q^{\frac{1}{2}}(x)g(x) \right| \, dx \right)^2 \]

\[ \leq |\lambda|^{-\frac{1}{2}} \|g\|_{L^2}^2 \int_{\mathbb{R}} \left| q^{\frac{1}{2}}(x) \right|^2 \, dx = |\lambda|^{-\frac{1}{2}} \|g\|_{L^2}^2 \|g\|_{L^1}. \]

Since \( g \) is non-trivial the estimate \( |\lambda| \leq \|q\|_{L^1}^2 \) follows.
References


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