# On the Dynamics of Marcus type Stochastic Differential Equations

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*"For twenty pages perhaps, he reads slowly, carefully, dutifully, with pauses for self-examination and working out examples. Then, just as it was working up and the pauses should have been more scrupulous than ever, a kind of swoon and ecstasy would fall on him, and he read ravening on, sitting up till dawn to finish the book, as though it were a novel. After that his passion was stayed; the book went back to the Library and he was done with mathematics till the next bout. Not much remained with him after these orgies, but something remained: a sensation in the mind, a worshiping acknowledgment of something isolated and unassailable, or a remembered mental joy at the rightness of thoughts coming together to a conclusion, accurate thoughts, thoughts in just intonation, coming together like unaccompanied voices coming to a close."*

— SYLVIA TOWNSEND WARNER *(1893 - 1978)*

## Abstract

In this work metric dynamical systems (MDS) driven by Lévy processes in positive and negative time are constructed. Ergodicity and invariance for such classes of MDS are shown.
Further a perfection theorem for càdlàg processes and the conjugacy of solution of Marcus type SDEs driven by Lévy processes and solutions of certain RDEs is proven. This result is applied to verify locally conjugacy of solutions of Marcus type SDEs and solutions of linearised Marcus type SDEs (referring to the results of Hartman–Grobman for deterministic ODEs).
Subsequently, stable and unstable manifolds are constructed using the Lyapunov–Perron method. Furthermore, the Lyapunov–Perron method is modified to prove a foliation of the stable manifold. Conclusively, Marcus type stochastic differential delay equations (MSDDEs) are considered. Conditions for existence and uniqueness of solutions are deduced, which implies the semiflow property for solutions of MSDDEs.

## Zusammenfassung


In der vorliegenden Arbeit werden Lévy Prozesse mit positiver und negativer Zeit betrachtet und das zugehörige metrische dynamische System (MDS) hergeleitet. Invarianz und Ergodizität wird für diese MDSe nachgewiesen.

Anschließend wird ein Perfektionierungssatz zur Vervollständigung von groben Kozyklen bewiesen, wenn die zugrundeliegenden Prozesse càdlàg Pfade besitzen. Nachfolgend wird gezeigt, dass die Lösung von stochastischen Differentialgleichungen vom Marcus–Typ (MSDgl) und die Lösung von bestimmten zufälligen Differentialgleichungen (ZDgl) konjugiert sind. Dieses Resultat wird verwendet, um die lokale Konjugation von Lösungen von MSDgl und deren zugehörigen linearisierten MSDgl zu verifizieren.

Im Anschluss werden mit Hifle der Lyapunov–Perron Methode stabile und unstabile Mannigfaltigkeiten konstruiert. Ferner wird mit einer Modifikation der Lyapunov–Perron Methode eine Blätterung der stabilen Mannigfaltigkeit hergeleitet.

Abschließend werden stochastische Differentialgleichungen vom Marcus–Typ mit Gedächtnis (MSDDgl) studiert. Existenz und Eindeutigkeit der Lösung werden bewiesen. Daraus kann schlussendlich gefolgert werden, dass Lösungen von MSDDgl einen stochastischen Kozyklus erzeugen.

## Acknowledgement

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# Contents

# 1 Introduction

*"There was yet another disadvantage attaching to the whole of Newton's physical inquiries,... the want of an appropriate notation for expressing the conditions of a dynamical problem, and the general principles by which its solution must be obtained. By the labours of LaGrange, the motions of a disturbed planet are reduced with all their complication and variety to a purely mathematical question. It then ceases to be a physical problem; the disturbed and disturbing planet are alike vanished: the ideas of time and force are at an end; the very elements of the orbit have disappeared, or only exist as arbitrary characters in a mathematical formula."*

— GEORGE BOOLE *(1815 - 1864)*

The theory of *dynamical systems* was widely investigated in the last century. Inspired by time homogenous physical processes such as particle movement due to Newton's laws, an extensive mathematical formalism was developed.
The book by G.D. Birkhoff [Bir27] published in 1927 already covers most of the fundamental results related to deterministic dynamical systems. The opening example in this book deals with a particle in a vacuum falling at the surface of the earth.
Let $h$ be the distance fallen and $v$ be the velocity of the particle. Then the movement of this particle satisfies the following *ordinary differential equation* (ODE):

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}\big(t, h_0, v_0\big) = v\big(t, v_0\big), & h\big(0, h_0, v_0\big) = h_0 \\ \frac{\mathrm{d}v}{\mathrm{d}t}\big(t, v_0\big) = g, & v\big(0, v_0\big) = v_0, \end{cases}$$

where $h_0$ is the initial distance fallen, $v_0$ the initial velocity and $g$ the gravitational acceleration. The unique solution is given by $h\big(t, h_0, v_0\big) = gt^2 + v_0 t + h_0$.
Now let $s > 0$ be a fixed time. Then it can be easily verified, that $h$ satisfies $h\big(t + s, h_0, v_0\big) = h\big(t, h(s, h_0, v_0), v(s, v_0)\big)$. This is the so-called *cocycle property* of dynamical systems. Roughly speaking, this property indicates that it makes no difference whether we consider the position of the particle starting in $h_0$ with initial velocity $v_0$ at time $t + s$ or the position of a particle already falling for time $s$ $\big($i.e. with initial state $h(s, h_0, v_0)$ and initial velocity $v(s, v_0)\big)$ at time $t$.
This structure of dynamical systems can be used to investigate the dynamics of $h$, such as attractors or stable and unstable states.

What happens if there is no vacuum and we face random perturbation? Collisions might slow down the particle or it is accelerated due to air flows. We can handle these new conditions by using a random acceleration $G(\omega)$ instead of the gravitational acceleration $g$. Then we get

$$\begin{cases} \frac{\mathrm{d}h}{\mathrm{d}t}\big(t, h_0, v_0, \omega\big) = v\big(t, v_0, \omega\big), & h\big(0, h_0, v_0, \omega\big) = h_0 \\ \frac{\mathrm{d}v}{\mathrm{d}t}\big(t, v_0, \omega\big) = G(\omega), & v\big(0, v_0, \omega\big) = v_0. \end{cases}$$

If we assume $G(\omega) = G\big(Z_t(\omega)\big)$ with driving noise $Z_t(\omega, t)$ satisfying $Z_t(\omega) := Z(A(\omega), t)$, where $A$ is a finite dimensional random vector, then we can compute the joint probability density function of the solution according to [SC73] (in the framework of so-called random differential equations (RDEs) with finite degrees of randomness).
However, the probability law itself does not allow to compare trajectories (i.e. random paths) of stochastic processes (in the sence of indistinguishability). Hence it is not possible to study the dynamical behavior using this approach.

One of the most investigated class of stochastic processes is given by solutions of stochastic differential equations (SDEs). In the time-continuous setting it is sufficient to consider Brownian driven SDEs, i.e. processes $X$ satisfying

$$X(t) = X(0) + \int_0^t f\big(X(s)\big) \circ \mathrm{d}B(s),$$

where $B$ is a Brownian motion. Under certain regularity conditions on $f$ it can be shown that $X$ generates a random dynamical system (RDS), see [Arn98], [Sch96] or [Led01]. The concept of random dynamical systems is exceptionally suitable to study the dynamical behavior of these stochastic processes. For the continuous case there are various results on random dynamics such as attractors and manifolds, for instance see Scheutzow in [DDR08], Arnold and Chueshov in [AC98], Schmalfuß et al. in [CGASV10] or Mohammed in [MS03].

However, there are processes we can discover in real life, which are not continuous in time. As an example we can look at stock prices or discretised processes to a dense grid. To cover these types of processes, we have to generalise the theory for SDEs with continuous driving noise to processes with jumps. Therefore a new class of SDEs is needed: In 1978 Steven Marcus introduced a generalised version of Stratonovich SDEs for general semimartingales (with jumps), see [Mar78] and [Mar81], so-called *Marcus type stochastic differential equations* (MSDEs). In 1995 Kurtz et all. [KPP95] slightly modified the definition by Marcus and proved several very useful properties for MSDEs. In the following work we study the dynamics generated by MSDEs driven by Lévy processes.

First we introduce random dynamical systems generated by Lévy driven MSDEs. Therefore we generalise the canonical construction of Brownian noise in Chapter 3. Then we prove, that the solutions generate a random dynamical system in Chapter 4 by using a perfection theorem specifically modified for càdlàg processes. Moreover we prove, that the solutions of MSDEs are conjugated with respect to solutions of certain RDEs. This can be used to prove a modified version of the random Hartman–Grobman theorem in Section 5.1. Subsequently we apply the Lyapunov–Perron method to construct stable and unstable manifolds in Section 5.2. After that we adapt the Lyapunov–Perron method to obtain a foliation of the stable manifold. In the final Chapter 6 we consider equations with memory, i.e. the position $X(t)$ possibly depends on the whole history $\{X(s)\colon s \in [t-\alpha, t]\}$ for some fixed time horizon $\alpha > 0$. We prove existence and uniqueness for solutions of Marcus type delay differential equations (MSDDEs) by modifying the methods from [MS03]. Moreover, these results imply the semiflow property for solutions of MSDDEs.

# 2 Preliminaries

*“It requires a very unusual mind to undertake the analysis of the obvious."*
— ALFRED NORTH WHITEHEAD *(1861 - 1947)*

## 2.1 Definitions

In the following section we provide basic tools which are needed to formulate the main results of this work.

### Function spaces

To formulate the main results of this work, we need two different classes of function spaces. The driving noise will be given by a càdlàg process, and the solution of the equations we study will be differentiable (with respect to the initial condition.
First we define the spaces $\mathscr{C}^k$ of continuously differentiale functions the space $\mathscr{C}$ of continuous functions included) and then the space $\mathscr{D}$ of càdlàg functions. More precisely, we stick to the following definitions:

Let $n, m \in \mathbb{N}$ and $k \in \mathbb{N}_0$. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is called *k-times continuously differentiable*, if

$$D^\alpha f := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \dots \partial^{\alpha_n} x_n} f\colon \mathbb{R}^n \to \mathbb{R}^m$$

is a continuous function for all $\alpha \in \mathbb{N}_0^n$, such that $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n \leqslant k$. The space of $k$-times continuously differentiable functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is denoted by $\mathscr{C}^k\big(\mathbb{R}^n, \mathbb{R}^m\big)$.
Especially, we define the space of continuous functions $\mathscr{C}\big(\mathbb{R}^n, \mathbb{R}^m\big) := \mathscr{C}^0\big(\mathbb{R}^n, \mathbb{R}^m\big)$.

**Definition 2.1.** A function $f\colon \mathbb{R} \to \mathbb{R}^m$ is called *càdlàg* (continue à droite, limite à gauche), if for each $t \in \mathbb{R}$

$$\begin{aligned} f(t) &= \lim_{s \to t+} f(s) \quad \text{and} \\ f(t-) &:= \lim_{s \to t-} f(s) \quad \text{exists.} \end{aligned}$$

The space of càdlàg functions $f\colon \mathbb{R} \to \mathbb{R}^m$ is denoted by $\mathscr{D}\big(\mathbb{R}, \mathbb{R}^m\big)$.
Moreover, if we focus on functions $f\colon \mathbb{R} \to \mathbb{R}^m$ satisfying $f(0) = 0$, then we write $\mathscr{C}_0^k$ instead of $\mathscr{C}^k$ or $\mathscr{D}_0$ instead of $\mathscr{D}$ respectively.

Besides $\mathscr{C}$ and $\mathscr{D}$ we also need measurable spaces and spaces of integrable functions. Therefore we consider a measurable space $(\Omega, \mathscr{F})$ consisting of a non-empty underlying set $\Omega$ endowed with a $\sigma$-field $\mathscr{F}$ over $\Omega$ and a measure $\mu$ defined on $(\Omega, \mathscr{F})$. If $\mu(\Omega) = 1$ then $\mu$ is called *probability measure*.

Let $(E, \|\cdot\|_E)$ be a separable Banach space. If $E = \mathbb{R}^n$ we write $|\cdot|$ instead of $\|\cdot\|_{\mathbb{R}^n}$ to indicate the Euclidean norm. The space of measurable functions $f\colon \Omega \to E$, satisfying

$$\|f\|_{L^p}^p := \int_\Omega \|f(\omega)\|_E^p \, \mu(\mathrm{d}\omega) < \infty$$

is a semi-normed vector space, since $\|f\|_{L^p} = 0$ implies $f(\omega) = 0$ for almost each $\omega \in \Omega$ (not each and every $\omega$ though), i.e. $\mu(\{\omega\colon f(\omega) \neq 0\}) = 0$, while probably $f(\omega) \neq 0$ for some $\omega \in \Omega$. Therefore we consider the space of equivalence classes $[f] = f/\sim$, where $f \sim g$ if and only if $\mu(\{\omega\colon f(\omega) \neq g(\omega)\}) = 0$.
We denote the space of equivalence classes $[f]$ satisfying $\|f\|_{L^p} < \infty$ by $L^p$. Then $(L^p, \|\cdot\|_{L^p})$ is a normed vector space. As usual we write $f$ instead of $[f]$ and treat $f$ like an ordinary function, while keep in mind that $f$ is an equivalence class.

### Stochastic Processes and Measurability

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathrm{P})$ be a filtered probability space, where $\mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}}$ is an increasing sequence of $\sigma$-fields such that $\mathscr{F}_s \subset \mathscr{F}_t \subset \mathscr{F}$ for each $s < t$.

**Definition 2.2.** Let $\mathbb{T} \subset \mathbb{R}$ be a non-empty time space. A family $X = (X_s)_{s\in\mathbb{T}}$ of random variables (r.v.) is called *stochastic process* on $\mathbb{T}$.
The stochastic process $X$ is $\mathbb{F}$-*adapted*, if $X_t$ is $\mathscr{F}_t$-measurable, for each $t \in \mathbb{T}$. The process $X$ is called jointly measurable, if the mapping $\Omega \times \mathbb{T} \ni (\omega, t) \mapsto X_t(\omega)$ is $\mathscr{F} \otimes \mathscr{B}(\mathbb{T})$-measurable.

## 2.2 Stochastic Integration

In this section we briefly introduce and motivate the theory of stochastic integration. For details see the book by Jacod and Shiryaev [JS02, Section 4].

First of all we highlight two important classes of stochastic processes – martingales and processes with finite variation – since they are crucial for the definition of stochastic integration:
To define the variation of $X$ (over $[0, T]$) we fix a partition $\{0 = t_0 < t_1 < \cdots < t_n = T\}$ and consider $\sum_{k=1}^n |X_{t_i} - X_{t_{i-1}}|$. Then the *variation of* $X$ (over $[0, T]$) is obtained by taking the supremum over all partitions $\mathscr{P}$ of $[0, T]$. That way we can somehow measure the speed $X$ can fluctuate with, e.g. constant processes have zero variation and the variation of an increasing process $X$ is given by $X(T) - X(0)$.
If $X$ is a process of finite variation, then we can define an integral with respect to $X$ in the sense

of Riemann–Stieltjes.

However, there are processes of infinite variation (such as the Brownian motion), which appear in nature. Hence we are interested in a theory to define integrals with respect to certain processes of infinite variation. This can be done in the framework of martingales:
Given an adapted process $X$ on a time set $\mathbb{T}$ (mainly $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{T} = [0, T]$), such that $X_t \in L^1(\mathrm{P})$ for each $t \in \mathbb{T}$. Then $X$ is called *martingale*, if $\mathbb{E}\left[X_t | \mathscr{F}_s\right] = X_s$ a.s. for each $s < t$, where $\mathbb{E}\left[X_t | \mathscr{F}_s\right]$ denotes the conditional expectation of $X_t$ with respect to $\mathscr{F}_s$.
However, there are processes $Y$ that might not satisfy the martingale property completely. Instead there is a sequence of stopping times $\left(\tau_n\right)_{n=1}^{\infty}$ tending to infinity a.s., such that $\mathbb{1}_{\tau_n > 0} Y^{\tau_n}$ is a martingale for each $n \in \mathbb{N}$. Then $Y$ is called *local martingale*.

**Definition 2.3.** An adapted stochastic process $X = \left(X_t\right)_{t \geqslant 0}$ is called *semimartingale*, if $X$ can be decomposed according to $X_t = X_0 + M_t + A_t$, where $X_0$ is $\mathscr{F}_0$-measurable and finite-valued, $M$ is a local martingale and $A$ is an adapted process with almost surely local finite variation.

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathrm{P})$ be a filtered probability space, where $\mathbb{F}$ is now right-continuous (not necessarily complete though), which means that $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$. Further let $H$ be a *simple previsible* process, i.e.

$$H_t = H_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_{i-1} \mathbb{1}_{(t_{i-1}, t_i]}(t), \tag{2.1}$$

where $0 = t_0 \leqslant \ldots \leqslant t_n \leqslant T < +\infty$ is a finite sequence of deterministic times, $H_i \in \mathscr{F}_{t_i}$ and $|H_i| < +\infty$ almost surely, $i = 0, \ldots, n$. Moreover let $X$ be a semimartingale.
For each $t > 0$ we define the stochastic integral of $H$ with respect to $X$ as

$$\int_0^t H_s \, \mathrm{d}X_s := H_0 X_0 + \sum_{i=1}^{n} H_{i-1} \Big( X_{t_i \wedge t} - X_{t_{i-1} \wedge t} \Big). \tag{2.2}$$

Then we can extend the definition above to predictable processes $H$ which are locally bounded, such that

(i) the stochastic integral has a càdlàg modification;

(ii) the mapping $H \mapsto \int H_s \, \mathrm{d}X_s$ is linear up to indistinguishability, i.e. the trajectories of $\int \big(\alpha H_s + K_s\big) \, \mathrm{d}X$ and $\alpha \int H_s \, \mathrm{d}X + \int K_s \, \mathrm{d}X$ are the same for almost every $\omega \in \Omega$, where $H, K$ are predictable and locally bounded processes and $\alpha \in \mathbb{R}$;

(iii) if $\left(H^n\right)_{n \in \mathbb{N}}$ is a sequence of predictable processes which is uniformly bounded by a previsible and locally bounded process, such that $H^n$ converge pointwise to a process $H$, then $\int_0^t H_s^n \, \mathrm{d}X \to \int_0^t H_s \, \mathrm{d}X$ in probability as $n \to \infty$ for each fixed $t \in \mathbb{R}^+$.

The extension is unique up to indistinguishability and the stochastic integral considered as stochastic process is a semimartingale again, see [JS02, Theorem 4.31, p. 46 et seqq.].

*Remark.* The richest class of integrators for which we can well-define a stochastic integral as we did in (2.2), coincides with the linear space of semimartingales, see [Pro04, p.52 & Theorem 43, p. 144].

There are several ways to define stochastic integration properly. The definition above is based on [JS02]. One small detail in this definition is the lack of completion of the underlying filtration. Usually there is no downside in completing the probability space by standard techniques, see [Pro04] or [App04] for instance. If so, we can start with simple processes using $\mathbb{1}_{(\sigma,\tau]}$ instead of $\mathbb{1}_{(s,t]}$, where $\sigma$ and $\tau$ are stopping times, and define the integral accordingly. Then we have to assume completeness of the underlying filtration, to ensure that the first hitting times of an optional event is a stopping time. Nevertheless, for the sake of completeness we use the more general definition which is given with respect to non-completed probability spaces (while right-continuity is still essential) and according to [JS02, Proposition 4.44, p. 51] there is no restriction compared to the definition with respect to a completed filtration. In general, the underlying filtration is not right-continuous, though. Then we can use the completed natural filtration of the underlying stochastic process which is right-continuous, see [KS91, Proposition 7.7, p. 90].

### Itô's Calculus and Stochastic Differential Equations

In the following section we briefly introduce stochastic differential equations (SDEs). For the sake of simplicity we consider the one-dimensional case first. Subsequently we consider the $n$-dimensional case as well.
Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous mapping, $Z$ be a semimartingale and $\xi \in \mathbb{R}$. Then we can ask for an adapted process $X$, such that for each $t > 0$ we have

$$X_t = \xi + \int_0^t f\big(X_s\big)\,\mathrm{d}Z_s \quad \text{a.s..} \tag{2.3}$$

In this case, the process $X = \big(X_t\big)_{t\in\mathbb{R}^+}$ is called *solution* of the *Itô type stochastic differential equation* (SDE)

$$\begin{cases} \mathrm{d}X_t \;=\; f\big(X_t\big)\,\mathrm{d}Z_t, \\ \;\;X_0 \;=\; \xi \end{cases} \tag{2.4}$$

generated by $f$, $Z$ and $\xi$.

**Definition 2.4.** Let $Z$ be a stochastic process. The filtration $\mathbb{F} = \big(\mathscr{F}_t\big)_{t\in\mathbb{R}^+}$ given by

$$\mathscr{F}_t := \sigma\big(Z(s)\colon s \leqslant t\big)$$

is called *natural filtration* generated by $Z$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be globally Lipschitz-continuous, $Z$ be a real-valued semimartingale and $\xi$ an integrable random variable. As stated in [Pro04, Theorem 7, p. 253 et seqq.] there exits a unique solution of (2.3) which is a semimartingale again. Moreover, this remains true in finite dimensional spaces and if $\xi = \xi_t$ is an integrable adapted càdlàg process. We do not explain 'integrable' in detail right now. We specify our assumptions on the functions $f$ and stochastic processes $Z$ later on.

Let $X$ be the solution of the Itô type SDE generated by $f\colon \mathbb{R} \to \mathbb{R}$, $Z$ and $\xi$. Moreover, let $f \in \mathscr{C}^2\big(\mathbb{R}, \mathbb{R}\big)$. Then $f(X) = \big(f(X_t)\big)_{t\in\mathbb{R}^+}$ is a semimartingale and furthermore, $f(X)$ is the solution of a stochastic differential equation as well. All of this is a conclusion of Itô's lemma. Before we can state the results of Itô's lemma – mainly the so-called Itô formula – we fix some definitions first:
Let $X$ and $Y$ be semimartingales. The left-continuous version $X_-$ of $X$ is given by to $X_{s-} := \lim_{r\to s-} X_r$. Then the quadratic covariation process $[X, Y]$ of $X$ and $Y$ is defined by

$$[X, Y]_t := X_tY_t - X_0Y_0 - \int_0^t X_s \,\mathrm{d}Y_s - \int_0^t Y_s \,\mathrm{d}X_s,$$

for each $t \geqslant 0$. Moreover, $X$ can be decomposed according to $X_s = \big(X_s - \Delta X_s\big) + \Delta X_s$ into the purely discontinuous part $\Delta X$ given by $\Delta X_s := X_s - X_{s-}$ and the purely continuous part $X^c$ given by $X^c_s := X_s - \Delta X_s$.

**Theorem 2.5** (Itô's Formula – [Pro04], Theorem 32, p. 78 et seqq.). *Let $f \in \mathscr{C}^2\big(\mathbb{R}, \mathbb{R}\big)$ and $X$ be a real-valued semimartingale. Then we have*

$$\begin{aligned} f(X_t) - f(X_0) = &\int_0^t f'\big(X_s\big)\,\mathrm{d}X_s + \frac{1}{2}\int_0^t f''\big(X_s\big)\,\mathrm{d}[X, X]^c_s \\ &+ \sum_{0<s\leqslant t} \Big\{ f\big(X_s\big) - f\big(X_{s-}\big) - f'\big(X_{s-}\big)\Delta X_s \Big\}. \end{aligned}$$

Itô's formula is the stochastic analogue to the change of variables formula for deterministic differential equations.

## Stratonovich integrals and Stratonovich type SDEs

For the usual Riemann integral of a continuous function $f\colon [a, b] \to \mathbb{R}$ we know, that

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{n\to\infty} \sum_{i=1}^{N_n} f(\tau_i^n)\big(t_i^n - t_{i-1}^n\big),$$

where $a = t_0^n < t_1^n < \cdots < t_{N_n}^n = b$ and $\sup_i |t_{i+1}^n - t_i^n| \to 0$ as $n \to \infty$. Moreover, $\tau_i^n \in [t_i^n, t_{i+1}^n]$ can be choosen freely.

For Itô type integrals we have a similar construction (with the convergence in probability), in which we always have to choose the left interval boundary $\tau_i^n = t_{i-1}^n$, i.e. we get

$$\int_a^b X_s \, \mathrm{d}Z_s = (\mathrm{P}) \lim_{n\to\infty} \sum_{i=1}^{N_n} X_{t_{i-1}^n} \big( Z_{t_i^n} - Z_{t_{i-1}^n} \big)$$

for semimartingales $X$ and $Z$, where $\sup_i \left| t_i^n - t_{i-1}^n \right| \to 0$ as $n \to \infty$.

Since the quadratic (co)variation of semimartingales does not vanish in general, we get different objects, depending on the choice of $\tau_i^n$. For instance we can use $\frac{1}{2}\big(X_{t_{i-1}^n} + X_{t_i^n}\big)$ instead of $X_{t_i^n}$. This leads us to the Stratonovich type integral of $X$ with respect to $Z$:

$$\begin{aligned} \int_0^t X_s \circ \mathrm{d}Z_s &= \lim_{n\to\infty} \sum_{i=1}^{n} \left\{ \frac{X_{t_{i-1}^n} + X_{t_i^n}}{2} \big( Z_{t_i^n} - Z_{t_{i-1}^n} \big) \right\} \\ &= \int_0^t X_s \, \mathrm{d}Z_s + \frac{1}{2}[X, Z]_t. \end{aligned}$$

Similarly to Itô type SDEs we can define solutions of Stratonovich type SDEs

$$X_t = X_0 + \int_0^t f\big(X_s\big) \circ \mathrm{d}Z_s.$$

Let $f \in \mathscr{C}^2$. Then Itô's formula implies

$$\begin{aligned} \big[f(X), Z\big]_t &= \left[ f\big(X_0\big) + \int_0^{\cdot} f'\big(X_s\big) \, \mathrm{d}X_s + \frac{1}{2} \int_0^t f''\big(X_s\big) \, \mathrm{d}[X, X]_s, Z \right]_t \\ &= \int_0^t f'\big(X_s\big) \, \mathrm{d}[X, Z]_s + \frac{1}{2} \int_0^t f''\big(X_s\big) \, \mathrm{d}\big[[X, X], Z\big]_s \\ &= \int_0^t f'\big(X_s\big) f\big(X_s\big) \, \mathrm{d}[Z, Z]_s, \end{aligned}$$

which gives us

$$\begin{aligned} X_t &= X_0 + \int_0^t f\big(X_s\big) \circ \mathrm{d}Z_s \\ &= X_0 + \int_0^t f\big(X_s\big) \, \mathrm{d}Y_s + \frac{1}{2} \big[f(X), Z\big]_t \\ &= X_0 + \int_0^t f\big(X_s\big) \, \mathrm{d}Z_s + \frac{1}{2} \int_0^t \big(f' f\big)\big(X_s\big) \, \mathrm{d}\big[Z, Z\big]_s. \end{aligned}$$

Now let $X$ be a purely continuous semimartingale. Then using Stratonovich integrals makes the Itô formula coincide with the usual change of variables rule:

$$f(X_t) - f(X_0) = \int_0^t f'\big(X_s\big) \circ \mathrm{d}X_s.$$

## 2.3 Marcus type Stochastic Differential Equations

Other than Itô and Stratonovich type SDEs we study the dynamic for a new class of equations. These equations are called *Marcus type SDEs* (MSDEs).
To obtain Stratonovich type SDEs we modify the diffusion part of Itô type SDEs. Similar to this we obtain Marcus type SDEs by modifying the jump part of Stratonovich SDEs.

### Heuristic observation

Let $B$ be a Brownian motion and let $(B^n)_{n\in\mathbb{N}}$ be a sequence of uniformly bounded and continuous processes with bounded variation given by

$$B_t^n := n \int_{t-1/n}^{t} B_s \,\mathrm{d}s.$$

Then $X_t^n := x \cdot \mathrm{e}^{B_t^n - B_0^n}$ satisfies the integral equation

$$X_t^n = x + \int_0^t X_s^n \,\mathrm{d}B_s^n.$$

Indeed, set $f(x,y) := x \cdot \mathrm{e}^y$, such that $X^n = f\big(x \cdot \mathrm{e}^{-B_0^n}, B_t^n\big)$. Then Itô's formula implies

$$\begin{aligned} X_t^n &= f\big(x \cdot \mathrm{e}^{-B_0^n}, B_t^n\big) \\ &= x \cdot \mathrm{e}^{-B_0^n}\mathrm{e}^{B_0^n} + \int_0^t f\big(x\mathrm{e}^{-B_0^n}, B_s^n\big)\,\mathrm{d}B_s^n + \frac{1}{2}\int_0^t f\big(x\mathrm{e}^{-B_0^n}, B_s^n\big)\,\mathrm{d}\underbrace{[B^n, B^n]_s}_{=0} \\ &= x + \int_0^t X_s^n \,\mathrm{d}B_s^n. \end{aligned}$$

Moreover we have $X_t^n \to X_t$ (a.s.) as $n \to \infty$, where $X_t = x \cdot \mathrm{e}^{B_t}$, and $X_t$ satisfies

$$X_t = x + \int_0^t X_s \circ \mathrm{d}B_s,$$

which follows similar to $X^n$ by using Itô's formula. This is a special case of the well-known Wong–Zakai approximation, see [WZ65].

Now we do the same calculations for a purely discontinuous noise. Let $Z$ be a one-dimensional compound Poisson process with finitely many different jump sizes $\Delta Z_t \in \{\alpha_1, \dots, \alpha_K\}$ for a deterministic $K \in \mathbb{N}$ and each $t \in [0,T]$, i.e.

$$Z_t = \sum_{i=1}^{K} \alpha_i N_t^i,$$

where $N^i$ are independent Poisson processes counting the number of jumps with size $\alpha_i$, for each $i = 1, \dots, K$.

Moreover let $Z_t^n$ be given by

$$Z_t^n := n \int_{t-1/n}^{t} Z_s \, \mathrm{d}s.$$

The process $Z_t^n$ is a piecewise linear process with bounded variation.

Then $X_t^n := x \cdot \mathrm{e}^{Z_t^n - Z_0^n}$ satisfies

$$X_t^n = x + \int_0^t X_s^n \, \mathrm{d}Z_s^n$$

similar to the Browian case and $X_t^n \to X_t$ (a.s.) as $n \to \infty$, where $X_t = x \cdot \mathrm{e}^{Z_{t-}}$.

Now $X_t$ does neither satisfy

$$\begin{aligned} X_t &= x + \int_0^t X_s \mathrm{d}Z_s \\ &= x + \sum_{0<s\leqslant t} \sum_{i=1}^{K} \alpha_i X_{s-} \Delta N_s^i = X_{t-} + X_{t-} \Delta Z_t. \end{aligned} \tag{2.5}$$

nor

$$X_t^\circ = x + \int_0^t X_s^\circ \circ \mathrm{d}Z_s. \tag{2.6}$$

Indeed, the process $Y_t := \prod_{0<s\leqslant t}(1 + \Delta Z_s)$ is the unique process, satisfying equation (2.5). To solve (2.6), we are looking for a process $Y^\circ$, satisfying

$$Y_t^\circ = Y_{t-}^\circ + \frac{Y_t^\circ + Y_{t-}^\circ}{2} \Delta Z_t,$$

which leads to $\big(2 - \Delta Z_t\big) Y_t^\circ = \big(2 + \Delta Z_t\big) Y_{t-}^\circ$ and $Y_t^\circ = \frac{2+\Delta Z_t}{2-\Delta Z_t} Y_{t-} = \prod_{0<s\leqslant t} \Big(1 + \frac{2\Delta Z_s}{2-\Delta Z_s}\Big)$. Neither $Y$ nor $Y^\circ$ coincide with $X_t$.

Other than that, $X_t$ solves a different type of SDE:
Obviously, we have $X_t = \prod_{0<s<t} \mathrm{e}^{\Delta Z_s} = \prod_{0<s<t} \prod_{i=1}^{K} \mathrm{e}^{\alpha_i \Delta N_s^i}$ by definition of $Z$, which can be rewritten by

$$\begin{aligned} X_t &= X_{t-} + \big(\mathrm{e}^{\Delta Z_t} - 1\big) X_{t-} \\ &= x + \sum_{0<s<t} \sum_{i=1}^{K} \Big(\mathrm{e}^{\alpha_i} - 1\Big) X_{s-} \Delta N_s^i, \end{aligned}$$

similar to the Itô case.

This gives us

$$X_t = x + \sum_{i=1}^{K} \int_0^t \Big( \Phi(\alpha_i) X_{s-} - X_{s-} \Big) \, \mathrm{d}N_s^i,$$

where $\Phi(t) = \mathrm{e}^t$. More importantly, $\Phi(t)z$ solves the ODE $\dot{z}_t = z_t$ with $z_0 = z$.

Roughly speaking, if there is a jump at time $t$, then solution jumps from $X_{t-}$ to $\Phi(\alpha_i)X_{s-}$ along the flow generated by this ODE. For the general case there will be two generalisation:

- If the SDE is generated by the noise $Z$ and function $f$ (in the example we have $f(x) = x$), then the underlying ODE depends on $f$ and $\Delta Z$ accordingly.
- Instead of solving the ODE for a random time, we solve an $\omega$-wise deterministic RDE (including the random jump) for the fixed time $t = 1$. This way we can deal with multi-dimensional noise.

### General Definition

Let $Z = (Z_1, Z_2, \ldots, Z_m)$, $m \in \mathbb{N}$ be a $\mathbb{R}^m$-valued semimartingale and $f_i \colon \mathbb{R}^n \supset D_{f_i} \to \mathbb{R}^n$, $n \in \mathbb{N}$ be a family of functions in $\mathscr{C}^1\left(\mathbb{R}^n, \mathbb{R}^n\right)$, $i = 1, \ldots, m$.

**Definition 2.6.** An adapted stochastic process $X = (X_t)_{t \geqslant 0}$ is called *solution of the Marcus type stochastic differential equation* (MSDE) generated by $f = (f_1, f_2, \ldots, f_m)$ and $Z$, if $X$ satisfies

$$\begin{aligned} X_t &= X_0 + \sum_{i=1}^{m} \int_0^t f_i\left(X_s\right) \diamond \mathrm{d}Z_s^i \\ &= X_0 + \sum_{i=1}^{m} \int_0^t f_i\left(X_{s-}\right) \circ \mathrm{d}Z_s^{c,i} + \sum_{i=1}^{m} \int_0^t f_i\left(X_{s-}\right) \, \mathrm{d}Z_s^{d,i} \\ &\quad + \sum_{0<s\leq t} \left\{ \varphi\left(f\Delta Z_s, X_{s-}, 1\right) - X_{s-} - f\left(X_{s-}\right)\Delta Z_s \right\}, \end{aligned} \tag{2.7}$$

where $\varphi\left(g, x, u\right)$ is the solution if the ODE

$$\begin{cases} \frac{\mathrm{d}y_t}{\mathrm{d}t} = g\left(y_t\right), \\ y_0 = x, \end{cases} \tag{2.8}$$

at time $t = u$, where $x \in \mathbb{R}^n$ and $\Delta Z_s = \left(\Delta Z_s^1, \ldots, \Delta Z_s^m\right)$.

Even if it looks like a new type of stochastic integral, it is not possible to define $\int X_s \diamond \mathrm{d}Z_s$ for semimartingales $X$ and $Z$ in general, see [KPP95, p. 352].
Let $X$ be the solution of (2.7) with respect to $f$ and $Z$ and let $g \in \mathscr{C}^1\big(\mathbb{R}^n, \mathbb{R}^m\big)$ be differentiable.

Then we can extend the definition of Marcus type SDEs to an integral of $g(X)$ with respect to $Z$:

$$\begin{aligned}\int_0^t g(X_s) \diamond \mathrm{d}Z_s := \int_0^t g(X_{s-}) \, \mathrm{d}Z_s + \frac{1}{2} \operatorname{tr} \int_0^t g'(X_s) \, \mathrm{d}[Z,Z]_s^c f(X_s)^* \\ + \sum_{0<s\leqslant t} \left( \int_0^1 \left\{ g\big(\varphi(f\Delta Z_s, X_{s-}, u)\big) - g\big(X_{s-}\big) \right\} \, \mathrm{d}u \right) \Delta Z_s. \end{aligned} \tag{2.9}$$

Indeed, if $g \equiv f$ then (2.9) coincides with the definition of Marcus type SDEs (2.7), since

$$\int_0^1 \left\{ f\big(\varphi(f\Delta Z_s, X_{s-}, u)\big)\Delta Z_s \right\} \, \mathrm{d}u = \varphi\big(f\Delta Z_s, X_{s-}, 1\big) - X_{s-} \tag{2.10}$$

by definition and

$$[g(X), Z]_t^c = \int_0^t g'(X_s) \, \mathrm{d}[X,Z]_s^c = \int_0^t g'(X_s) \, \mathrm{d}[Z,Z]_s^c f(X_s)^*.$$

**Theorem 2.7** (Change of Variables Formula – [KPP95], Proposition 4.2, p.363)**.**
*Let $X$ be the solution of* (2.7) *with respect to $f$ and $Z$ and $g \in \mathscr{C}^2(\mathbb{R}^n)$. Then we have*

$$g(X_t) - g(X_0) = \int_0^t g' f(X_s) \diamond \mathrm{d}Z_s. \tag{2.11}$$

For the sake of completeness we give an outline of the proof. However, we restrict ourselves to the one-dimensional case since there are already all ideas which are needed to prove the multi-dimensional case as well. The proof of the multi-dimensional case is given in [KPP95, Proposition 4.2, p. 363]. Here we capture the main ideas of the proof and add detailed calculations:

*Proof.* According to the definition of Marcus type SDEs we have

$$\begin{aligned} \mathrm{d}X_t &= f(X_{t-}) \, \mathrm{d}Z_t + \frac{1}{2} f'f(X_t) \, \mathrm{d}[Z,Z]^c \\ &\quad + \sum_{0<s\leqslant t} \left\{ \varphi\big(f\Delta Z_s, X_{s-}, 1\big) - X_{s-} - f\big(X_{s-}\big)\Delta Z_s \right\}, \\ \mathrm{d}[X,X]_t^c &= f^2\big(X_t\big) \, \mathrm{d}[Z,Z]_t^c, \\ \Delta X_t &= \varphi\big(f\Delta Z_t, X_{t-}, 1\big) - X_{t-}. \end{aligned}$$

Itô's formula implies

$$\begin{aligned} g\big(X_t\big) - g\big(X_0\big) = \int_0^t g'\big(X_{s-}\big) \, \mathrm{d}X_s + \frac{1}{2} \int_0^t g''\big(X_s\big) \, \mathrm{d}[X,X]_s^c \\ + \sum_{0<s\leqslant t} \left\{ g\big(X_{s-} + \Delta X_s\big) - g\big(X_{s-}\big) - g'\big(X_{s-}\big)\Delta X_s \right\}. \end{aligned}$$

Thus, we get

$$g\big(X_t\big) - g\big(X_0\big) = \int_0^t g'f\big(X_{s-}\big) \, \mathrm{d}Z_s$$

$$
\begin{aligned}
&+ \frac{1}{2}\int_0^t g'(f'f)\big(X_s\big)\,\mathrm{d}[Z,Z]^c_s + \frac{1}{2}\int_0^t g''f^2\big(X_s\big)\,\mathrm{d}[Z,Z]^c_s \\
&+ \sum_{0<s\leqslant t}\big\{g'\big(X_{s-}\big)\big(\varphi\big(f\Delta Z_s, X_{s-},1\big) - X_{s-} - f\big(X_{s-}\big)\Delta Z_s\big)\big\} \\
&+ \sum_{0<s\leqslant t}\big\{g\big(\varphi(f\Delta Z_s, X_{s-},1)\big) - g\big(X_{s-}\big) - g'\big(X_{s-}\big)\Delta X_s\big\} \\
= &\int_0^t g'f\big(X_{s-}\big)\,\mathrm{d}Z_s + \frac{1}{2}\int_0^t \big(g'f\big)'f\big(X_s\big)\,\mathrm{d}[Z,Z]^c_s \\
&+ \sum_{0<s\leqslant t}\big\{g'\big(X_{s-}\big)\big(\underbrace{\varphi\big(f\Delta Z_s, X_{s-},1\big) - X_{s-}}_{=\Delta X_s} - f\big(X_{s-}\big)\Delta Z_s\big)\big\} \\
&+ \sum_{0<s\leqslant t}\big\{g\big(\varphi(f\Delta Z_s, X_{s-},1)\big) - g\big(X_{s-}\big) - g'\big(X_{s-}\big)\Delta X_s\big\} \\
= &\int_0^t g'f\big(X_{s-}\big)\,\mathrm{d}Z_s + \frac{1}{2}\int_0^t \big(g'f\big)'f\big(X_s\big)\,\mathrm{d}[Z,Z]^c_s \\
&+ \sum_{0<s\leqslant t}\left\{\int_0^1 g'f\big(\varphi(f\Delta Z_s, X_{s-},u)\big)\Delta Z_s\,\mathrm{d}u - g'f\big(X_{s-}\big)\Delta Z_s\right\} \qquad (2.12) \\
= &\int_0^t g'f\big(X_{s-}\big)\diamond\mathrm{d}Z_s,
\end{aligned}
$$

where (2.12) holds because of

$$
\begin{aligned}
\frac{\partial g}{\partial u}\big(\varphi(f\Delta Z_s, X_{s-},u)\big) &= g'\big(\varphi(f\Delta Z_s, X_{s-},u)\big)\cdot\frac{\mathrm{d}}{\mathrm{d}u}\,\varphi(f\Delta Z_s, X_{s-},u) \\
&= g'f\big(\varphi(f\Delta Z_s, X_{s-},u)\big)\Delta Z_s \quad \text{and} \\
g\big(\varphi(f\Delta Z_s, X_{s-},0)\big) &= g\big(X_{s-}\big).
\end{aligned}
$$

□

**Assumption** (L)**.** To ensure existence and uniqueness of the solution of (2.7) we assume that $f=(f_1,\dots,f_m)$ and $f'f$ are Lipschitz continuous, cf. [KPP95, Theorem 3.2, p. 358].

It is worth mentioning, that $f' = \big(\frac{\partial f_i}{\partial x_j}\big)_{i,j=1}^n$ is a matrix-valued function on $\mathbb{R}^n$ and $f'f$ is vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Moreover, the solution of a Marcus type SDE is a semimartingale and we can always find a càdlàg modification.

In case of $Z$ is not a purely continuous semimartinale, we can emphasise the differnce of solutions of Itô, Stratonovich and Marcus type SDEs by investigating their change of variables formula. For instance, let $X$, $X^\circ$ and $X^\diamond$ the solution of the Itô, Stratonovich and Marcus type SDE generated by functions $f_i\in\mathscr{C}^2(\mathbb{R}^n,\mathbb{R}^n)$ and $Z$ respectively and $g\in\mathscr{C}^2(\mathbb{R}^n,\mathbb{R})$. Then $g(X)$,

$g(X^\circ)$ and $g(X\diamond)$ are semimartingales and satisfy

$$\begin{aligned}
g(X_t) &= g(X_0) + \sum_{i=1}^n \int_0^t \frac{\partial g}{\partial x_i}(X_{s-})\,\mathrm{d}X_s^i + \frac{1}{2}\sum_{i,j=1}^n \int_0^T\int_0^T \frac{\partial^2 g}{\partial x_i \partial x_j}(X_{s-})\,\mathrm{d}\left[X^i, X^j\right]_s^c \\
&\quad + \sum_{0<s\leqslant t}\left\{g(X_s) - g(X_{s-}) - \sum_{i=1}^n \frac{\partial g}{\partial x_i}(X_{s-})\,\Delta X_s^i\right\}, \\
g(X_t^\circ) &= g(X_0^\circ) + \sum_{i=1}^n \int_0^t \frac{\partial g}{\partial x_i}\left(X_{s-}^\circ\right) \circ \mathrm{d}X_s^{c,i} \\
&\quad + \sum_{0<s\leqslant t}\left\{g(X_s^\circ) - g\left(X_{s-}^\circ\right) - \sum_{i=1}^n \frac{\partial g}{\partial x_i}\left(X_{s-}^\circ\right)\Delta X_s^{\circ,i}\right\} \text{ or} \\
g(X_t^\diamond) &= g(X_0^\diamond) + \sum_{i=1}^n \int_0^t \frac{\partial g}{\partial x_i}\left(X_{s-}^\diamond\right) f_i(X_{s-}) \diamond \mathrm{d}Z_s^i,
\end{aligned}$$

cf. [Pro04, Corollary (Itô's Formula), p. 81] for $g(X)$ or $g(X^\circ)$ and Theorem 2.7 for $g(X^\diamond)$.

## Flow property and flow of diffeomorphism I.

Let $f$ satisfy assumption (L). Then there is a unique (strong) solution $X$ of the MSDE

$$X_t = x + \sum_{i=1}^m \int_0^t f_i(X_s) \diamond \mathrm{d}Z_s^i$$

for each fixed $x \in \mathbb{R}^n$.

In the Brownian case a stochastic flow is given by a family of mappings $\varphi_{s,t}\colon \mathbb{R}^n \times \Omega \to \mathbb{R}^n$, such that $\varphi_{s,t}(\cdot,\omega) = \varphi_{\tau,t}(\cdot,\omega) \circ \varphi_{s,\tau}(\cdot,\omega)$ for all $\omega \in \Omega_0$ with $\mathrm{P}(\Omega_0) = 1$ and $\Omega_0$ is independent of $s, t \in \mathbb{R}$. Usually we can apply Kolmogorov's continuity theorem to find a modification of the solution, which is continuous in $s, t$ and $x$.

In the càdlàg case, Kolmogorov's continuity theorem only deals with the spatial component $x \in \mathbb{R}^n$. Accordingly, we say the solution of the MSDE driven by a semimartingale $Z$ generates a stochastic flow, if $\varphi_{s,t}(\cdot,\omega) = \varphi_{\tau,t}(\cdot,\omega) \circ \varphi_{s,\tau}(\cdot,\omega)$ a.s. (for each $\omega \in \Omega_{s,t,\tau}$ depending on $s, t, \tau$) and the mapping $x \mapsto X(x,\omega)$ from $\mathbb{R}^n \to \mathscr{D}(\mathbb{R}^+, \mathbb{R}^n)$ is continuous with respect to the topology of **u**niform convergence on **c**ompacts in **p**robability (ucp), cf. [KPP95, Theorem 3.4, p. 359].

**Definition 2.8.** Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces. A mapping $X\colon \mathscr{X} \to \mathscr{Y}$ is called $\mathscr{C}^k$-*diffeomorphism*, if $\mathscr{X} \ni x \mapsto X(x) \in \mathscr{Y}$ is bijective and $k$-times continuously differentiable and its inverse is $k$-times continuously differentiable as well, $k \geqslant 0$.

If $f_i \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and all derivatives of $f = (f_1, f_2, \ldots, f_m)$ and ${f_i}' f_i$ are bounded, then the flow $x \mapsto X_t(x,\omega)$ is a diffeomorphism on $\mathbb{R}^n$ for each $t \geqslant 0$, cf. [KPP95, Theorem 3.9, p. 361].

These results are well-known for solutions of Itô type SDEs. In the following we describe a method to reformulate solutions of MSDEs to solutions of Itô SDEs:

**Lemma 2.9** ([KPP95] Lemma 2.1, p. 356 et. seqq.). *We consider the functions*

$$h_i\left(s,x\right)=\frac{\varphi\left(f_i\Delta Z_s^i,x,1\right)-x-f_i(x)\Delta Z_s^i}{|\Delta Z_s^i|^2}$$

*Then the solution of the Marcus type SDE* (2.7) *solves the Itô type SDE*

$$\begin{aligned}X_s=X_0&+\sum_{i=1}^m\int_0^t f_i\left(X_{s-}\right)\circ \mathrm{d}Z_s^{c,i}+\sum_{i=1}^m\int_0^t f_i\left(X_s\right)\,\mathrm{d}Z_s^{d,i}\\&+\sum_{i=0}^m\int_0^t h_i\left(s,X_{s-}\right)\,\mathrm{d}\left[Z^i,Z^i\right]_s^d\end{aligned}\tag{2.13}$$

*and converse.*

Thus, the solution of the MSDE generates a flow of diffeomorphism according to [Pro04, Theorem 37 – Theorem 39, p. 301 et seqq.] or [App09, Theorem 6.10.10, p. 424 et seqq.].

**Assumption** ($\mathrm{C}^\infty$)**.** We assume $f_1,\dots,f_m\in\mathscr{C}^\infty\left(\mathbb{R}^n,\mathbb{R}^n\right)$ and that all derivatives of $f$ and $f_i{}'f_i$ are bounded to ensure that the solution of the Marcus type SDE driven by a semimartingale $Z$ generates a flow of diffeomorphism.

## 2.4 Lévy processes

### Definition and Lévy–Itô decomposition

The following definition introduces one of the most important types of stochastic processes. These processes will, roughly speaking, represent the source of randomness for the stochastic equations in the later part of this work:

**Definition 2.10.** A $\mathbb{R}^n$-valued stochastic process $L=\left(L_t\right)_{t\in\mathbb{R}_+}$ is called

(a) process *with stationary increments*, if there is a family of probability measures $(\mu_t)_{t\geq 0}$ on $\left(\mathbb{R}^n,\mathscr{B}(\mathbb{R}^n)\right)$, such that

$$\mathrm{P}_{L_t-L_s}=\mu_{t-s},$$

for each $s,t>0$.

(b) process *with independent increments*, if the random variables

$$L_{t_0},L_{t_1}-L_{t_0},\dots,L_{t_k}-L_{t_{k-1}}$$

are stochastically independent for $0<t_0<t_1<\cdots<t_k<\infty$ and $k\in\mathbb{N}$.

(c) *Lévy process*, if it is a process with independent and stationary increments, $L_0=0$ (a.s.) and each trajectory is càdlàg.

Moreover, if $L_t - L_s \sim \mathscr{N}\big(0, |t-s|\mathrm{id}_n\big)$ is $n$-dimensional standard Gaussian distributed, then $L$ is called *Brownian motion* and we usually write $B$ instead of $L$.

To use Lévy processes $L$ for stochastic integration properly, we consider the famous *Lévy–Itô decomposition*, where we separate the continuous part and purely discontinuous part of Lévy processes separately.
The continuous part $L^c$ is given by a Brownian motion with drift. However, instead of working with $\Delta L$ by itself it will be more convenient to count jumps of a given size. Therefore we consider the family of (random) counting measures $N = \big(N(t,\cdot)\big)_{t>0}$ on $\mathscr{B}\big(\mathbb{R}^n \setminus \{0\}\big)$ given by

$$N(t,A)(\omega) := \sum_{0<s\leqslant t} \mathbb{1}_A\big(\Delta L_s(\omega)\big),$$

where we count the number of jumps $\Delta L_s(\omega)$ which belong to $A \in \mathscr{B}\big(\mathbb{R}^n \setminus \{0\}\big)$, such that $0 \notin \bar{A}$. The set function $A \mapsto \mathbb{E}\big[N(t,A)\big]$ is a Borel measure on $\mathscr{B}\big(\mathbb{R}^n \setminus \{0\}\big)$ and we can define the *intensity measure* $\nu(A) := \mathbb{E}\big[N(1,A)\big]$, see [App04, p. 87] for more details.
Now we have all the tools needed to formulate the Lévy–Itô decomposition:

**Theorem 2.11** (Lévy–Itô decomposition – [App04] Theorem 2.4.16, p. 108)**.**
*Let $L$ be a Lévy process. Then there exit a $\mathbb{R}^n$-valued Brownian motion $B$ with covariance matrix $Q$, a Poisson random measure $N$ and an $\alpha \in \mathbb{R}^n$, such that*

$$L_t = \alpha t + B_t + \int_{\big\{|x|<1\big\}} x\, \tilde{N}(t,\mathrm{d}x) + \int_{\big\{|x|\geq 1\big\}} x\, N(t,\mathrm{d}x) \quad \textit{a.s.}, \tag{2.14}$$

*where $\tilde{N}(t,\mathrm{d}x) := N(t,\mathrm{d}x) - t\nu(\mathrm{d}x)$ is the compensated Poisson random measure.*

*Remarks.*

- The constant $\alpha$ in (2.14) can be calculated explicitly according to

$$\alpha = \mathbb{E}\left[L_1 - \int_{\{|x|\geq 1\}} x\, N(1,\mathrm{d}x)\right].$$

- The Lévy–Itô decomposition holds almost surely. In general $L$ does not need to have finite moments of any order.
Nevertheless $\alpha$ is well defined, since the jumps of $X_t := L_t - \int_{\{|x|\geq 1\}} x\, N(t,\mathrm{d}x)$ are bounded by $1$ and each Lévy process with bounded jumps has finite moments of any order, see [Pro04, Theorem 34, p. 25].

- If we assume, that the jumps of $L$ are summable (a.s.) for each $t \in \mathbb{R}_+$, i.e.

$$\int_{\big\{|x|<1\big\}} |x|\, N(t,\mathrm{d}x) < \infty,$$

then the Lévy–Itô decomposition can be simplified. Instead of an integral with respect to a compensated Poisson random measure we just get

$$L_t = \tilde{\alpha} t + B_t + \int_{\mathbb{R}^n} x\, N(t, \mathrm{d}x), \text{ where } \tilde{\alpha} = \mathbb{E}\left[L_1 - \int_{\mathbb{R}^n} x\, N(1, \mathrm{d}x)\right].$$

## 2.5 Marcus type SDEs driven by Lévy processes

### Stochastic integration with respect to Lévy processes and Kunita's inequalities

Subsequently we will prove $\mathbf{L}^p$ estimates for solutions of SDEs driven by Lévy processes. These inequalities were first considered by Kunita in [Kun04, Theorem 2.11, p. 332] and they will play an important role to prove the flow property for solutions of SDEs driven by Lévy processes. However, the first proof was not completely correct (especially for $p > 2$). A complete proof can be found in [App09, Theorem 4.4.23 & Corollary 4.4.24, p. 265 et seqq.].

According to Lemma 2.9 and by Theorem 2.11 and the so-called interlacing technique it is sufficient to consider the class of stochastic processes given by

$$M_i(t) = \int_0^t b_i(s)\,\mathrm{d}s + \int_0^t f_i^j(s)\,\mathrm{d}B_j(s) + \int_0^t \int_E h_i(s,x)\,\tilde{N}(\mathrm{d}s, \mathrm{d}x),$$

where $E = \{y \in \mathbb{R}^n \colon 0 < |y| < c\}$ for some $c \in \mathbb{R}_+$, $\sqrt{|b_i|}$ and $f_i^j$ are square integrable and predictable and $h_i$ is a.s. square integrable on $E$ and predictable (with respect to the natural filtration generated by $B$ and $N$), for each $i, j = 1, \ldots, n$.
Kunita's first and second inequality can be seen as an extension of Burkholder's inequality for stochastic integrals with respect to compensated Poisson random measures:

**Theorem 2.12** (Kunita's first inequality). *Let $h_i \colon [0,t] \times \mathbb{R}^n \to \mathbb{R}$ be a.s. square integrable on $E$ and predictable for each $i = 1, \ldots, n$. Further let $I(t)$ be a $\mathbb{R}^n$-valued process given by*

$$I_i(t) := \int_0^t \int_E h_i(s,x)\,\tilde{N}(\mathrm{d}s, \mathrm{d}x),$$

*where $\tilde{N}$ is a compensated Poisson random measure with bounded jumps. Then for each $p \geqslant 2$ there is a constant $D(p) > 0$, such that*

$$\mathbb{E}\Big(\sup_{0<s\leqslant t} \big|I(s)\big|^p\Big) \leqslant D(p)\Bigg\{\mathbb{E}\Bigg[\Big(\int_0^t \int_E \big|h(s,x)\big|^2\,\nu(\mathrm{d}x)\mathrm{d}s\Big)^{p/2}\Bigg] + \mathbb{E}\Bigg[\Big(\int_0^t \int_E \big|h(s,x)\big|^p\,\nu(\mathrm{d}x)\mathrm{d}s\Big)\Bigg]\Bigg\}.$$

The proof can be found in [App09, p. 265 et seqq.]. For the sake of completeness we will present the proof here as well:

*Proof of Kunita's first inequality.* The case $p = 2$ can be proven similar to the Brownian case, see [App09, Lemma 4.2.2, p. 221] for instance.

Hence we assume $p > 2$: Ito's formula implies

$$\big|I(t)\big|^p = M(t) + A(t), \tag{2.15}$$

where

$$\begin{aligned} M(t) &= \int_0^t \int_E \Big( \big|I(s-) + h(s,x)\big|^p - \big|I(s-)\big|^p \Big)\, \tilde{N}(\mathrm{d}s, \mathrm{d}x) \quad \text{and} \\ A(t) &= \int_0^t \int_E \Big( \big|I(s-) + h(s,x)\big|^p - \big|I(s-)\big|^p \\ &\qquad\qquad - p\big|I(s-)\big|^{p-2} I(s-) h(s,x) \Big)\, \nu(\mathrm{d}x)\mathrm{d}s. \end{aligned}$$

With help of localisation we can assume that $M(t)$ is in fact a martingale.

Next we will apply Taylor's expansion. Let $J(I, h; \theta)$ be a $\mathbb{R}^n$-valued process given by

$$J(I, h; \theta)[s] := I(s-) + \theta \cdot h(s,x),$$

where $\theta \in (0,1)^n$. Using Taylor's expansion, there is a $\theta^* \in (0,1)^n$, such that

$$\begin{aligned} A(t) = \int_0^t \int_E \bigg[ &\frac{p}{2}(p-2)\big|J(I,h;\theta^*)[s]\big|^{p-4} \big(J(I,h;\theta^*)[s] \cdot h(s,x)\big)^2 \\ &+ p\big|J(I,h;\theta^*)\big|^{p-2} \big|h(s,x)\big|^2 \bigg]\, \nu(\mathrm{d}x)\mathrm{d}s. \end{aligned} \tag{2.16}$$

Indeed, we have

$$\underbrace{\big|J(I,h;1)\big|^p - \big|J(I,h;0)\big|^p}_{=\left|I(s-)+h(s,x)\right|^p - \left|I(s-)\right|^p} = p\big|J(I,h;\theta^*)\big|^{p-2} J(I,h;\theta^*) \cdot h(s,x) \qquad \text{and}$$

$$\begin{aligned} p\big|J(I,h;\theta_1)\big|^{p-2} &J(I,h;\theta_1)h(s,x) - p\big|J(I,h;0)\big|^{p-2} J(I,h;0)h(s,x) \\ &= p(p-2)\big|J(I,h;\theta^*)[s]\big|^{p-4} \big(J(I,h;\theta^*)[s] \cdot h(s,x)\big)^2, \end{aligned}$$

which lead to (2.16).

Since $|a+b|^p \leqslant (2^{p-1} \vee 1)\big(|a|^p + |b|^p\big)$ we get

$$\big|A(t)\big| \leqslant \mathrm{C}_1(p) \int_0^t \int_E \Big[ \big|I(s-)\big|^{p-2} \big|h(s,x)\big|^2 + \big|h(s,x)\big|^p \Big]\, \nu(\mathrm{d}x)\mathrm{d}s$$

where we also use the Cauchy–Schwarz inequality. Now Doob's martingale inequality implies

$$\mathbb{E}\Big(\sup_{0<s\leqslant t}\big|I(s)\big|^p\Big) \leqslant \mathrm{C}_2(p)\Big\{\mathbb{E}\int_0^t\int_E \big|I(s-)\big|^{p-2}\big|h(s,x)\big|^2\,\nu(\mathrm{d}x)\mathrm{d}s + \mathbb{E}\int_0^t\int_E \big|h(s,x)\big|^p\,\nu(\mathrm{d}x)\mathrm{d}s\Big\}. \tag{2.17}$$

For all $x, y > 0$ and $p, q > 1$ we have

$$xy \leqslant \frac{x^a}{a} + \frac{y^b}{b}, \tag{2.18}$$

where $\frac{1}{a} + \frac{1}{b} = 1$, see [HLP99, p. 61].
In the following we first apply Hölder's inequality and then use (2.18) with $a = p/p-2$ and $b = p/2$ to obtain for each $\alpha > 1$

$$\begin{aligned}
&\mathrm{C}_2(p)\,\mathbb{E}\int_0^t\int_E \big|I(s-)\big|^{p-2}\big|h(s,x)\big|^2\,\nu(\mathrm{d}x)\mathrm{d}s \\
&\leqslant \mathrm{C}_2(p)\,\mathbb{E}\Big(\sup_{0<s\leqslant t}\big|I(s-)\big|^{p-2}\frac{1}{\alpha}\int_0^t\int_E \alpha\big|h(s,x)\big|^2\,\nu(\mathrm{d}x)\mathrm{d}s\Big) \\
&\leqslant \mathrm{C}_2(p)\left[\mathbb{E}\Big(\sup_{0<s\leqslant t}\big|I(s-)\big|^p\alpha^{-p/p-2}\Big)\right]^{p-2/p}\left[\mathbb{E}\Big(\int_0^t\int_E \alpha\big|h(s,x)\big|^2\,\nu(\mathrm{d}x)\mathrm{d}s\Big)^{p/2}\right]^{2/p} \\
&\leqslant \mathrm{C}_3(p)\,\alpha^{-p/p-2}\,\mathbb{E}\Big(\sup_{0<s\leqslant t}\big|I(s-)\big|^p\Big) + \mathrm{C}_4(p)\,\alpha^{p/2}\,\mathbb{E}\Big(\int_0^t\int_E \big|h(s,x)\big|^2\,\nu(\mathrm{d}x)\mathrm{d}s\Big)^{p/2}.
\end{aligned} \tag{2.19}$$

Since $\alpha^{-p/p-2} \to 0$ as $\alpha \to \infty$ we can choose $\alpha$ sufficiently large, such that $\mathrm{C}_3(p)\,\alpha^{-p/p-2} < 1$. Hence Kunita's first inequality follows from (2.17) and (2.19) by rearranging terms. □

Kunita's second inequality combines the first inequality and Burgholder's inequality:

**Corollary 2.13.** *For each $p \geqslant 2$ and $t > 0$ there is a constant $\tilde{D}(p,t)$, such that*

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0<s\leqslant t}\big|M(s)\big|^p\Big) \leqslant \tilde{D}(p,t)\Big\{&\mathbb{E}\Big(\int_0^t |b|^p\,\mathrm{d}s\Big) + \mathbb{E}\Big(\big\{\operatorname{tr}\big[M^c, M^c\big](t)\big\}^{p/2}\Big) \\
&+ \mathbb{E}\Big[\Big(\int_0^t\int_E \big|h(s,x)\big|^2\,\nu(\mathrm{d}x)\mathrm{d}s\Big)^{p/2}\Big] \\
&+ \mathbb{E}\Big[\Big(\int_0^t\int_E \big|h(s,x)\big|^p\,\nu(\mathrm{d}x)\mathrm{d}s\Big)\Big]\Big\}.
\end{aligned}$$

### Flow property and flow of diffeomorphisms II.

The first results on continuity and differentiability of flows generated by MSDEs was proven by [KPP95] using several strong assumptions. There are two papers by Fujiwara and Kunita, where similar results are obtained under significantly weaker assumptions, see [FK99a] and [FK99b]. However, their main focus is on spatial noise. Nevertheless, their approach can be adapted to our setting as well. This was done by Applebaum [App09] (in the 2nd Version of his book). In the following we outline their proof for differentiability, where we can see the strength of Kunita's inequalities:

**Theorem 2.14.** *Let $f_1, \ldots, f_m \in \mathscr{C}_b^{k+2}\big(\mathbb{R}^n\big)$ and $L$ be a $\mathbb{R}^m$-valued Lévy process. Then the solution $X$ of the MSDE* (2.7) *generates a flow of $\mathscr{C}^k$-diffeomorphisms.*

For the sake of completeness we give an outline of the proof based on that in [App09, Theorem 6.10.10, p. 424].
For each $s, t \geqslant 0$ let $\Psi_{s,t}(x)$ satisfy $\Psi_{s,s}(x) = x$ and

$$\mathrm{d}\Psi_{s,t}(x) = \sum_{i=1}^{m} f_i\big(\Psi_{s,t}(x)\big) \diamond \mathrm{d}L_t^i \quad \text{for } t \geqslant s,$$

*Proof.* Let $e_j := (0, \ldots, 0, 1, 0, \ldots, 0)$ be the $j$-th unit vector, where the 1 is at the $j$-th position. Then we can define

$$\delta_j \Psi_{s,t}(y, h) := \frac{\Psi_{s,t}(y + h \cdot e_j) - \Psi_{s,t}(y)}{h}, \quad \text{where} \quad y \in \mathbb{R}^n, h > 0.$$

First we assume that $L$ is a Lévy process with bounded jumps:
Kunita's second inequality implies

$$\mathbb{E}\left(\sup_{s \leqslant r \leqslant t} \big|\delta_j \Psi_{s,t}(y_1, h_1) - \delta_j \Psi_{s,t}(y_2, h_2)\big|^p\right) \leqslant \mathrm{C}\big(|y_1 - y_2|^p + |h_1 - h_2|^p\big),$$

which implies continuity of $\partial\Psi$ with respect to $x$ according to Kolmogorov's continuity theorem.

To complete the proof we have to obtain continuity for the inverse as well. This can be done in several ways. Since we already obtained continuity for the Jacobian $D := \partial\Psi$ it is sufficient to prove, that $D$ defines a regular matrix.
As stated in [Pro90, Theorem 49, p. 265] (along with Lemma 2.9) or [Kun04, p. 356] we get

$$D_t^{i,k} = \delta^{i,k} + \sum_{i=1}^{m} \sum_{l=1}^{n} \int_s^t \frac{\partial f_i}{\partial x_l}\big(\Psi_{s,r}(x)\big) D_r^{i,k} \diamond \mathrm{d}L_r^i.$$

Then [Pro90, Theorem 50, p. 265] implies regularity of $D$ straight away.

To complete the proof we have to consider Lévy processes $\hat{L}$ with unbounded jump sizes. According to the Lévy–Itô decomposition we get

$$\hat{L}(t) = \eta(t) + \xi(t) \quad \text{a.s.},$$

where $\eta(t) := \int_{\{|x| \geq 1\}} x\, N(t, \mathrm{d}x)$ and $\xi(t) := \hat{L}(t) - \eta(t)$. Moreover $\eta$ and $\xi$ are stochastically intependent.
Let $\big(\tau_n\big)_n$ be the sequence of jump times of the compound Poisson process $\eta$ and $\Delta_n = \Delta\hat{L}_{\tau_n}$.

According to [Kun04, p. 354 et seq.] we can decompose the flow $\hat{\Psi}$ generated by

$$\mathrm{d}\hat{\Psi}_{s,t}(x) = \sum_{i=1}^{m} f_i\big(\hat{\Psi}_{s,t}(x)\big) \diamond \mathrm{d}\hat{L}_t^i \quad \text{for } t \geqslant s.$$

Indeed, we obtain $\hat{\Psi}_{s,t}(x) = \hat{\Psi}_{s,\tau_m} \circ \phi_{\Delta_m} \circ \cdots \circ \phi_{\Delta_{n-1}} \circ \hat{\Psi}_{\tau_{n-1},t}(x)$ on the event $\Omega_{m,n}$, where $\phi_z(x) := x + \varphi(x,z)$ and $\Omega_{m,n} := \big\{\omega \in \Omega \colon \tau_{m-1} < s < \tau_m \text{ and } \tau_n < t < \tau_{n+1}\big\}$. Since both $\Psi$ and $\phi$ are differentiable, we get the differentiability of $\hat{\Psi}$ as well, which completes the proof. □

*Remark* 2.15.

- Instead of dealing with the Jacobian $\partial\Psi$ directly, we could have proven continuity of $\partial\Psi^{-1}$ by using inverse flows, see [Kun04, Theorem 3.13, p. 359]. Then we can prove estimates for

$$\delta_j \Psi_{s,t}^{-1}(y,h) = \frac{\Psi_{s,t}^{-1}(y + h \cdot e_j) - \Psi_{s,t}^{-1}(y)}{h}, \quad \text{where} \quad y \in \mathbb{R}^n, h > 0,$$

  similarly to those for $\delta_j \Psi_{s,t}(y,h)$.

- Other than that we could have used the fact, that each continuous and injective function $f \colon \mathbb{R}^n \to \mathbb{R}^n$ with $\big|f(x)\big| \to \infty$ as $|x| \to \infty$ is a homeomorphism, see [Pro90, Comment p. 263]. We will come back to this in Section 5.1.

**Assumption** ($\mathrm{C}^k$)**.** We assume $f_1, \ldots, f_m \in \mathscr{C}_b^{k+2}\big(\mathbb{R}^n\big)$, which ensures that the solution of the Marcus type SDE driven by a Lévy process $L$ generates a flow of $\mathscr{C}^k$-diffeomorphism.

## 2.6 Random Dynamical Systems

### Definitions

One of the first, if not the very first approach to dynamical systems was given by Birkhoff in [Bir27, Chapter VII, Section 2]:

**Definition 2.16.** A deterministic dynamical system is characterised by a time space $\mathbb{T}$ (basically $\mathbb{N}_0$ or $\mathbb{Z}$ for discrete time and $\mathbb{R}^+$ or $\mathbb{R}$ for continuous time respectively), a state space $E$ (basically $\mathbb{R}^n$ for finite dimensional systems) and a mapping $\xi \colon \mathbb{T} \times E \to E$, such that $\xi(0,x) = x$ for each $x \in E$ and $\xi\big(t, \xi(s,x)\big) = \xi(t+s,x)$ for each $t, s \in \mathbb{T}$ and $x \in E$.

The most important examples of deterministic dynamical systems are given by solutions of ordinary differential equations

$$\frac{d}{dt}\xi_t^i = F^i(\xi_t) \;\text{ for }\; i \in \big\{1, \ldots, n\big\} \;\text{ and }\; \xi_0 = x$$

on $[0,T]$, where $T \in \mathbb{R}^+$, $n \in \mathbb{N}$ and $F \colon \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently regular, ensuring uniqueness of $\xi_t$ and the flow-property $\xi_t\big(\xi_s(x)\big) = \xi_{t+s}(x)$ for $t, s \in \mathbb{R}^+$.

One of the main components of this work are given by equations who generate so-called *random dynamical systems* (RDEs). These systems are a natural extension of deterministic dynamical systems based on measure theoretical methods.
However, it is not as easy as simply adding an $\omega$ to each and every equation:
For instance, let $\varphi(t, \omega, x)$ be the random state of a particle as motivated in the introduction.
We consider the random state $\varphi(\tau, \omega, x)$ at time $\tau$ and use this state as the new starting point for an other movement. In doing so we have to compensate, that while we wait up to time $\tau$ in order that the particle moves from $x$ to $\varphi(\tau, \omega, x)$, the underlying $\omega$ potentially might change as well over time, i.e. instead of $\omega$ we have to consider $\theta_\tau\omega$ for the new movement, where at this point $\theta_\tau$ simply indicates the development of the underlying probability space over time. This gives us the position $\varphi(t, \theta_\tau\omega, \varphi(\tau, \omega, x))$. Accordingly, we arrive at the same position as given by $\varphi(t + \tau, \omega, x)$.

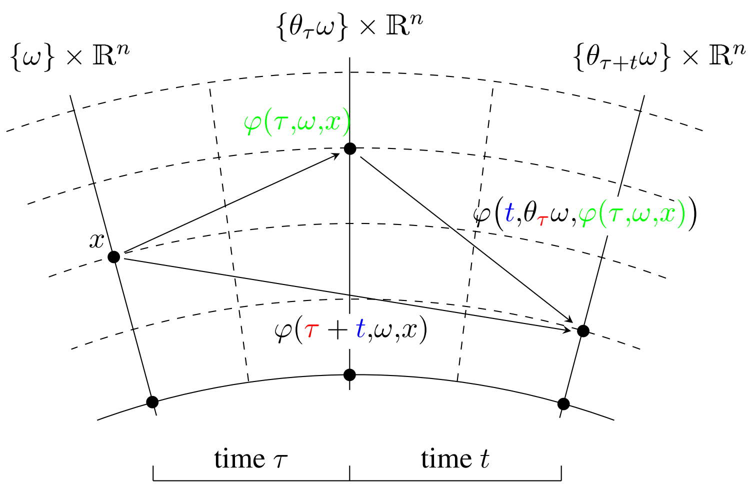



**Figure 2.1:** *(Perfect) Cocycle property*

In the following we give a complete definition of RDS. Therefore we define metric dynamical systems first, which will step in for $\theta$ in the definition of RDS:

**Definition 2.17** ([Arn98], p. 537)**.** Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a probability space. A family $(\theta_t)_{t\in\mathbb{T}}$ of endomorphisms on $(\Omega, \mathscr{F})$ is called *metric dynamical system* with time space $\mathbb{T}$ (which needs to have group structure at least), if

(i) *Measurability*: the mapping $(\omega, t) \mapsto \theta_t\omega$ is jointly $\mathscr{F} \otimes \mathscr{B}(\mathbb{T}) - \mathscr{F}$ measurable;

(ii) *(Semi)group Property*: $\theta\,(s + t) = \theta\,(s) \circ \theta\,(t)$ for all $s, t \in \mathbb{T}$ and $\theta_0 = \mathrm{id}_\Omega$ is the identity on $\Omega$.

The definition of RDS is strongly motivated by figure 2.1 combined with a measuability property:

**Definition 2.18** ([Arn98], p. 5)**.** A *random dynamical system (RDE)* with values in a measurable space $(E, \mathscr{E})$ over a metric dynamical system $\left(\Omega, \mathscr{F}, \mathrm{P}, (\theta_t)_{t\in\mathbb{T}}\right)$ with time $\mathbb{T}$ is given by a mapping

$$\varphi\colon \mathbb{T} \times \Omega \times E \to E$$

satisfying the following properties:

(i) *Measurability*: $\varphi$ is $\mathscr{B}(\mathbb{T}) \otimes \mathscr{F} \otimes \mathscr{E}$–$\mathscr{E}$-measurable; and

(ii) *(Perfect) Cocycle Property*: The mappings $\varphi(t, \omega, \cdot) : E \to E$ form a cocycle over $\theta(\cdot)$, i.e.,

$$\begin{aligned} \varphi(0, \omega, \cdot) &= \mathrm{id}_E \text{ for each } \omega \in \Omega; \\ \varphi(t+s, \omega, \cdot) &= \varphi(t, \theta_s \omega, \cdot) \circ \varphi(s, \omega, \cdot) \text{ for each } s, t \in \mathbb{T},\ \omega \in \Omega. \end{aligned}$$

*Remark* 2.19. If the mapping $\varphi(t, \omega, \cdot) : E \to E$ satisfies the cocylce property (only) for almost all $\omega \in \Omega_{s,t}$ (depending on $s, t$) instead of each $\omega \in \Omega$, then we say $\varphi$ forms a crude cocyle instead of a perfect cocycle.

### The Wiener space and Development over time

There is a well known approach to define MDS suitable to deal with Brownian motions:
Let $\left(B_t^+\right)_{t \geqslant 0}$ and $\left(B_t^-\right)_{t \geqslant 0}$ be independent real-valued Brownian motions defined on $\mathbb{R}^+$. Then the process $\left(B_t\right)_{t \in \mathbb{R}}$ given by

$$B_t = \begin{cases} B_t^+, & \text{if } t \geqslant 0, \\ -B_t^-, & \text{else,} \end{cases} \tag{2.20}$$

is a real-valued Brownian motion on $\mathbb{R}$, see [Arn98, Appendix A].

The space $\mathscr{C}_0$ endowed with the Borel $\sigma$-field $\mathscr{B}(\mathscr{C}_0)$ and the canonical probability measure corresponding to finite dimensional distributions given by $B$ is called *Wiener space*. Especially we have $B_t(\omega) = \omega(t)$.
The mapping $\theta_s\colon \mathscr{C}_0 \to \mathscr{C}_0$, $s \in \mathbb{R}$ given by $\theta_s f(t) = f(t+s) - f(s)$ for a function $f \in \mathscr{C}_0$ is an endomorphism on $\mathscr{C}_0$. The family of shift endomorphisms $(\theta_t)_{t \in \mathbb{R}}$ is called *Wiener shift*. It satisfies $\theta_s\big(\theta_r f(t)\big) = \big(f(t+s+r) - f(r)\big) - \big(f(s+r) - f(r)\big) = \theta_{s+r} f(t)$. We will use this family of shift endomorphis to a great extend later on.

The measurability property for the Wiener shift $\theta$ is satisfied according to [Arn98, Appendix A], which implies that $\left(\theta_s\right)_{s \in \mathbb{R}}$ generates a metric dynamical system with respect to the Wiener space.

In the subsequent chapter we modify this construction of MDS for càdlàg processes to get a similar approach capable to deal with Lévy processes as well. Therefore we have to carefully construct Lévy processes with two-sided time (which will differ from the contruction of Brownian motion with two-sided time in the first place).

# 3 Metric dynamical systems generated by Lévy processes with two-sided time

*“ ‘Obvious’ is the most dangerous word in mathematics."*

— ERIC TEMPLE BELL *(1883 - 1960)*

Brownian motions and (compound) Poisson processes are probably the most popular examples for Lévy processes on $\mathbb{R}^+$. To construct Poisson processes with positive time we can exploit the fact, that the inter-jump period of Poisson processes is exponential distributed. Given a sequence of independent and identically exponential distributed random variables $\tau_1, \tau_2, \ldots$ the stochastic process

$$N_t = \sum_{i=1}^{\infty} \mathbb{1}_{(T_{i-1}, T_i]}(t) \tag{3.1}$$

is a Poisson process, where $T_i = \sum_{k=1}^{i} \tau_k$ for $i > 0$ and $T_0 = 0$, see [Ç11].

It might be tempting to define Lévy processes similar to (2.20) given independent Lévy processes $L^+$ and $L^-$. Indeed, the combined process $L$ is a stochastic process on $\mathbb{R}$ with independent, stationary increments and $L_0 = 0$. However, let $\left(N_t^+\right)_{t \geqslant 0}$ and $\left(N_t^-\right)_{t \geqslant 0}$ be independent Poisson processes with parameter $\lambda > 0$ and define $N$ similarly to (2.20).

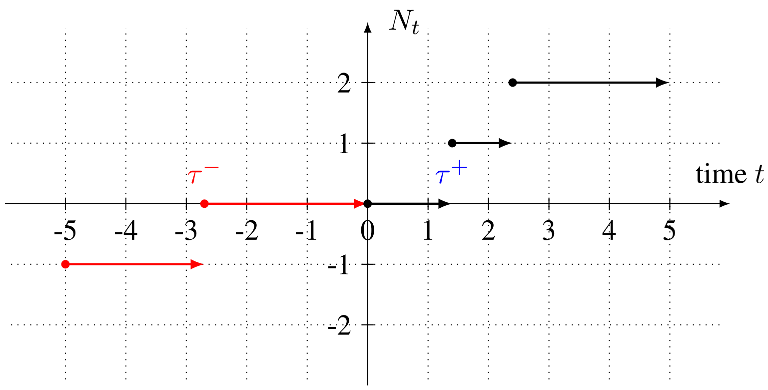



**Figure 3.1:** *Combination if two independent Poisson processes $N^+$ and $N^-$*

The inter-jump period between the last jump $\tau^-$ in negative time and the first jump $\tau^+$ in positive time (which are indeed consecutive jumps) is Gamma distributed with parameters 2 and $\lambda$ (as the sum of independent exponentially distributed random variables), i.e. the corresponding process with two-sided time does not satisfy (3.1). We give remarks on such properties in the end of this section.

In the following section we prove a construction of Lévy processes with two-sided time which is completely different to that of a two-sided Brownian motion, which we briefly discussed in the end of Chapter 2. Later we show, that both definitions lead to the same process, i.e. there is no difference between this approach and the combination of two independent Lévy processes (up to a modification). Nevertheless our approach allows us to prove the existence of an ergodic metric dynamical systems generated by Lévy processes with two-sided time in the sequel.

## 3.1 Canonical Construction of Stochastic processes – Kolmogorov's Existence theorem

Let $E$ be a separable Banach space endowed with the Borel $\sigma$-field $\mathscr{E}$ and let $(X_t)_{t\in\mathbb{T}}$ be a family of $E$-valued random variables with non-empty time space $\mathbb{T} \subset \mathbb{R}$. Each $X_t$ is defined on the probability space $(\Omega, \mathscr{F}, \mathrm{P})$. For a finite subset $\mathrm{S} = \{s_1, s_2, \dots, s_k\} \subset \mathbb{T}$ we set

$$X_{\mathrm{S}} := (X_{s_1}, X_{s_2}, \dots, X_{s_k}) \in E^k.$$

The probability measure induced by $X_S$ on $\left(E^k, \mathscr{E}^k\right)$ is denoted by $\mathrm{P}_{\mathrm{S}}$. Especially we define $\mathrm{P}_t = \mathrm{P}_S$ for $S = \{t\}$.

Further let $\mathrm{S}_1 \subset \mathrm{S}_2 \subset \mathbb{T}$ be non-empty subsets of $\mathbb{T}$, such that $S_2 = \{s_1, \dots, s_k\}$ and $S_1 = \{s_{i_1}, \dots, s_{i_l}\}$ for an injective mapping $i\colon \{1, 2, \dots, l\} \to \{1, 2, \dots, k\}$, $l, k \in \mathbb{N}$, $l < k$. Then we can define *measurable projections* $\mathrm{p}_{\mathrm{S}_1}^{\mathrm{S}_2}\colon E^k \to E^l$ given by

$$\mathrm{p}_{\mathrm{S}_1}^{\mathrm{S}_2}\left(B_{s_1} \times B_{s_2} \times \dots \times B_{s_k}\right) := B_{s_{i_1}} \times B_{s_{i_2}} \times \dots \times B_{s_{i_l}},$$

where $B_{s_i} \in \mathscr{E}$, $i = 1, 2, \dots, k$.
Let $\mathscr{S}$ be the set of all *non-empty, finite subsets* of $\mathbb{T}$. The family of probability measures $(\mathrm{P}_{\mathrm{S}})_{\mathrm{S}\in\mathscr{S}}$ is called *finite dimensional distributions*. It is called *projective*, if

$$\mathrm{p}_{\mathrm{S}_1}^{\mathrm{S}_2}\left(\mathrm{P}_{\mathrm{S}_2}\right)(\cdot) := \mathrm{P}_{\mathrm{S}_2}\left(\left(\mathrm{p}_{\mathrm{S}_1}^{\mathrm{S}_2}\right)^{-1}(\cdot)\right) = \mathrm{P}_{\mathrm{S}_1}(\cdot)$$

for each finite subsets $\mathrm{S}_1 \subset \mathrm{S}_2 \subset \mathbb{T}$.

**Theorem 3.1** (Kolmogorov's existence theorem – [Bau91], Theorem 35.3, p. 307)**.** *Let* $(E, \mathscr{E})$ *be a separable Banach space and* $\mathbb{T}$ *a non-empty time space. Then, for each projective family* $(\mathrm{P}_{\mathrm{S}})_{\mathrm{S}\in\mathscr{S}}$ *of probability measures on* $\left(E^{\mathrm{S}}, \mathscr{E}^{\mathrm{S}}\right)$ *there is a unique probability measure* $\mathrm{P}_{\mathbb{T}}$ *on*

$\left(E^{\mathbb{T}}, \mathscr{E}^{\mathbb{T}}\right)$ *satisfying*

$$\mathrm{p}_{\mathrm{S}}^{\mathbb{T}}\left(\mathrm{P}_{\mathbb{T}}\right) = \mathrm{P}_{\mathrm{S}},$$

*where*

$$\mathscr{E}^{\mathbb{T}} := \sigma\left(\left\{\left(\mathrm{p}_{\mathrm{S}}^{\mathbb{T}}\right)^{-1}\left(B_{s_1} \times B_{s_2} \times \cdots \times B_{s_k}\right) \subset E^{\mathbb{T}} : \mathrm{S} \in \mathscr{S}, S = \left\{s_1, s_2, \ldots, s_k\right\}, k \in \mathbb{N}\right\}\right).$$

Now we can set $\Omega = E^{\mathbb{T}}$, $\mathscr{F} = \mathscr{E}^{\mathbb{T}}$ and $\mathrm{P} = \mathrm{P}_{\mathbb{T}}$ to get the so-called *canonical probability space* with respect to the finite dimensional distributions $\left(\mathrm{P}_{\mathrm{S}}\right)_{\mathrm{S} \in \mathscr{S}}$. Every single path of the stochastic process $\left(X_t\right)_{t \in \mathbb{T}}$ equals a randomly choosen element from $\Omega$,

$$X_t\left(\omega\right) = \omega\left(t\right). \tag{3.2}$$

## Construction of Lévy processes with positive time

Let $\left(E_1, \mathscr{E}_1\right)$ and $\left(E_2, \mathscr{E}_2\right)$ be measurable spaces. A function

$$P\colon E_1 \times \mathscr{E}_2 \to [0, +\infty]$$

is called *kernel* from $\left(E_1, \mathscr{E}_1\right)$ to $\left(E_2, \mathscr{E}_2\right)$, if the following conditions are satisfied:

$$\begin{aligned} x &\mapsto P\left(x, A\right) \text{ is } \mathscr{E}_1\text{-measurable for every } A \in \mathscr{E}_2 \text{ and} \\ A &\mapsto P\left(x, A\right) \text{ is a measure on } \mathscr{E}_2 \text{ for every } x \in E_1. \end{aligned}$$

If $P\left(x, E_2\right) = 1$ for each $x \in E_1$, then $P$ is called *Markovian kernel*. If $E_1 = E_2 = E$ and

$$P\left(x, B\right) = P\left(x + z, B + z\right),$$

for $x, z \in E$, $B \in \mathscr{E}$, then $P$ is called *translation invariant*.

Moreover, a family of kernels $\left(P_t\right)_{t \geq 0}$ from $\left(E, \mathscr{E}\right)$ to itself and with time space $\mathbb{R}^+$ is called *semigroup of kernels*, if

$$P_{s+t}\left(x, A\right) = \int_A P_t\left(y, A\right) P_s\left(x, \mathrm{d}y\right), \text{ for each } A \in \mathscr{E} \text{ and } s, t > 0. \tag{3.3}$$

From now onwards we set $E = \mathbb{R}^n$, $n \in \mathbb{N}$, and we apply Kolmogorov's existence theorem to construct $\mathbb{R}^n$-valued Lévy processes:
A semigroup of Markovian kernels generates finite dimensional distributions according to

$$\begin{aligned} \mathrm{P}_{\mathrm{S}}\left(B\right) := \int\int\ldots\int \mathbb{1}_B\left(x_1, x_2, \ldots, x_n\right) \mathrm{P}_{s_k - s_{k-1}}\left(x_{k-1}, \mathrm{d}x_k\right) \\ \mathrm{P}_{s_{k-1} - s_{k-2}}\left(x_{k-2}, \mathrm{d}x_{k-1}\right) \ldots \mathrm{P}_{s_1}\left(x_0, \mathrm{d}x_1\right) \mu\left(\mathrm{d}x_0\right), \end{aligned}$$

where $\mu$ is the initial probability measure on $\left(E^k, \mathscr{E}^K\right)$, $\mathrm{S} = \{s_0, s_1, \ldots, s_k\} \in \mathscr{S}$ and $B \in \mathscr{E}^k$, see [Bau91, Theorem 36.4, p. 320].
Canonical processes $L_t$ generated by semigroups of translation invariant Markovian kernels are processes with independent and stationary increments, see [Bau91, Theorem 37.2, p. 327].

### Lévy processes with two-sided time

Let $L = (L_t)_{t\geq 0}$ with $L_0 = 0$ a.s. (by setting the Dirac measure $\delta_0$ on $\left(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n)\right)$ as initial measure $\mu$) be a Lévy process with values in $\mathbb{R}^n$ defined on the canonical space $(\Omega, \mathscr{F})$. To extend the time intervall from $\mathbb{R}_+$ to $\mathbb{R}$ we apply a technique described by Cornfeld et al. in [CFS82]. This approach differs from the usual construction using two independent copies of the same one-sided process (one for positive and one for negative time). This makes it easier for us to prove invariance and ergodicity, when we use metric dynamical systems generated by Lévy processes with two-sided time later on.

Let

$$\theta\colon \mathbb{R} \times E^{\mathbb{R}^+} \to E^{\mathbb{R}^+}$$
$$\theta_s\omega\left(\cdot\right) \mapsto \omega\left(\cdot + s\right) - \omega\left(s\right)$$

be a family of shift endomorphism on $E^{\mathbb{R}^+}$ similar to Wiener shifts defined in Chapter 2, where $s > 0$.
For sufficiently large $s > 0$ we can define finite dimensional distributions on $\left(E^{\mathbb{R}}, \mathscr{E}^{\mathbb{R}}\right)$ by

$$\bar{\mathrm{P}}_{s_1,\ldots,s_k}\Big(B_1, \ldots, B_k\Big) := \mathrm{P}\Big(\omega \in E^{\mathbb{R}^+} : \omega\left(s_1 + s\right) - \omega\left(s\right) \in B_1, \ldots, \omega\left(s_k + s\right) - \omega\left(s\right) \in B_k\Big), \tag{3.4}$$

where $S = \{s_1, \ldots, s_k\}$ is a finite subset of $\mathbb{R}$ and $B_1, \ldots, B_k \in E$.
Since $L$ is a process with stationary increments, the definition of $\bar{\mathrm{P}}_{s_1,\ldots,s_k}$ is independent of $s$, once $s_i + s > 0$ for each $i = 1, \ldots, k$. Given $-\infty < s_1 < s_2 < \cdots < s_n < \infty$, then (3.4) is well-defined by choosing any $s \in \mathbb{R}$ such that $s > -s_1$. Applying Kolmogorov's extension theorem we get a unique probability measure $\bar{\mathrm{P}}$ on $\left(E^{\mathbb{R}}, \mathscr{E}^{\mathbb{R}}\right)$, such that $\bar{L}_t\left(\omega\right) := \omega_t$ is a Lévy process on $\mathbb{R}$. For the sake of simplicity we write $\mathrm{P}$ instead of $\bar{\mathrm{P}}$ and $L$ instead of $\bar{L}$ subsequently. The process $L$ is called Lévy process with two-sided time.

## 3.2 Construction of Metric Dynamical Systems driven by Lévy Processes

To prove the jointly measurability of the Wiener shift we apply [AB07, Lemma 4.51, p. 153]. It says, given a metric and separable spaces $(\mathscr{X}, \mathfrak{X})$, a metric space $(\mathscr{Y}, \mathfrak{Y})$ and a measurable space $(\mathbb{T}, \mathfrak{T})$, then each *Carathéodory function* $f\colon \mathbb{T} \times \mathscr{X} \to \mathscr{Y}$ is jointly $\mathfrak{T} \otimes \mathfrak{X} - \mathfrak{Y}$ measurable.
A mapping $f\colon \mathbb{T} \times \mathscr{X} \to \mathscr{Y}$ is called Carathéodory, if $f(t, \cdot)\colon \mathscr{X} \to \mathscr{Y}$ is continuous for each fixed $t \in \mathbb{T}$ and $f(\cdot, x)\colon \mathbb{T} \to \mathscr{Y}$ is measurable for each fixed $x \in \mathscr{X}$. Thus, we have to find

suitable spaces such that the conditions above are satisfied. According to Kolmogorov's existence theorem the canonical probability space is given by $\left(E^{\mathbb{R}}, \mathscr{E}^{\mathbb{R}}\right)$. This is not the space we end up being in. Neither $\mathscr{C}_0$ nor $\mathscr{D}_0$ is an element of $\left(E^{\mathbb{R}}, \mathscr{E}^{\mathbb{R}}\right)$, see [Bau91, Lemma 38.4., p. 336 et seq.]. Thus, the Wiener shift $\theta$ cannot be a Carathéodory function on $\mathbb{R} \times E^{\mathbb{R}}$.

Each Lévy processes $L$ has a càdlàg modification $\tilde{L}$, i.e. $\mathrm{P}\left(L_t = \tilde{L}_t\right) = 1$ for each $t \in \mathbb{R}$, such that $t \mapsto \tilde{L}_t(\omega)$ is a.s. càdlàg, see [PZ07, Theorem 4.3, p. 39]. Moreover, by definition of $\bar{\mathrm{P}}$ we obtain $\bar{\mathrm{P}}_0(B) = \mathbb{1}_B(0)$, which entails $L_0 = 0$ a.s.. Because of that we can define Lévy processes on the space $\left(\mathscr{D}_0, \mathscr{E}^{\mathbb{R}} \cap \mathscr{D}_0, \tilde{\mathrm{P}}\right)$ instead of $\left(E^{\mathbb{R}}, \mathscr{E}^{\mathbb{R}}, \mathrm{P}\right)$, where $\tilde{\mathrm{P}}\left(\mathscr{D}_0 \cap B\right) := \mathrm{P}(B)$ for each $B \in \mathscr{E}^{\mathbb{R}}$, applying [Bau91, p. 335 et seq.]. Here we write $\mathscr{D}_0$ instead of $\mathscr{D}_0\left(\mathbb{R}, \mathbb{R}^n\right)$ for simplicity's sake.
This modification is necessary for the measurability property in the definition of metric dynamical systems and is needed for the measurability of $\theta$ (else it would be $\theta_t^{-1}\left(E^{\mathbb{R}}\right) = \mathscr{D}_0 \notin \mathscr{E}^{\mathbb{R}}$ for each $t \in \mathbb{R}$). In this regard we need that $\mathscr{B}\left(\mathbb{R}^n\right)^{\mathbb{R}} \cap \mathscr{D}_0 = \mathscr{B}\left(\mathscr{D}_0\right)$ with respect to a suitable topology. We use Skorokhod's $J_1$-topology for technical reasons.

We define *Skorokhod's $J_1$-topology* for $\mathscr{D}[0,1]$ first. Let $\Lambda$ be the family of all strictly monotonically increasing and continuous functions from $[0,1]$ into itselfs. For functions $x, y \in \mathscr{D}[0,1]$, we define a metric $d_0(x,y)$ given by the infimum over all $\varepsilon > 0$, such that there is a $\lambda \in \Lambda$ satisfying

$$\sup_{s \neq t} \left| \log \frac{\lambda t - \lambda s}{t - s} \right| =: \|\lambda\| \leq \varepsilon \qquad \text{and} \qquad \sup_{t \in [0,1]} \|x(t) - y(\lambda t)\|_E \leq \varepsilon, \tag{3.5}$$

see [Bil68, p. 113]. The topology generated by the metric $d_0$ is called Skorokhod's $J_1$-topology.

Now we consider $\mathscr{D}_0 = \mathscr{D}_0(\mathbb{R}, \mathbb{R}^n)$ and define the metric $d_{\mathbb{R}}$ as modification of $d_0$ by replacing $\|\lambda\|$ for $\lambda \in \Lambda$ with $\|\lambda\|_{\text{arctan}}$ for $\lambda \in \Lambda_{\mathbb{R}}$, where

$$\|\lambda\|_{\text{arctan}} := \left| \log \frac{\arctan(\lambda t) - \arctan(\lambda s)}{\arctan(t) - \arctan(s)} \right|$$

and $\Lambda_{\mathbb{R}}$ is the set of injective increasing functions $\lambda$ satisfying $\lim_{t \to -\infty} = -\infty$ and $\lim_{t \to +\infty} = +\infty$.

**Theorem 3.2** ([Lac92] – Theorem 1, p. 92)**.** *The space $\mathscr{D}_0$ endowed with the metric $d_{\mathbb{R}}$ is a Polish space.*

*Idea of the Proof.* Let $f \in \mathscr{D}_0$. The mapping $\Xi \colon \mathscr{D}_0\left(\mathbb{R}, \mathbb{R}^n\right) \to \mathscr{D}\left((0,1), \mathbb{R}^n\right)$ given by

$$\Xi(f)[t] := \lim_{s \to t+} f\left(\tan\left(\pi s - \frac{\pi}{2}\right)\right) \qquad \text{for each} \quad t \in (0,1) \tag{3.6}$$

is a homeomorphism, see [Lac92, p. 92]. Since $\mathscr{D}_0\left((0,1), \mathbb{R}\right)$ is a Polish space we get the same for $\mathscr{D}_0\left(\mathbb{R}, \mathbb{R}^n\right)$, see [Str69]. $\square$

**Lemma 3.3.** *We have* $\mathscr{E}^{\mathbb{R}} \cap \mathscr{D}_0 = \mathscr{B}(\mathscr{D}_0)$.

*Proof.* We consider measurable projections $\pi_{t_1,\dots,t_k}$ from $\mathscr{D}_0$ to $(\mathbb{R}^n)^k$ given by

$$\pi_{t_1,\dots,t_k}(\omega) := (\omega(t_1), \dots, \omega(t_k)),$$

where $t_1, \dots, t_n \in \mathbb{R}$ and $\omega \in \mathscr{D}_0$. The measurability of $\pi_{t_1,\dots,t_k}$ is proven in [Bil68, p. 120 et seq.] for $[0,1]$ instead of $\mathbb{R}$. The same arguments remain valid for $\mathscr{D}_0(\mathbb{R})$. Indeed, since the function

$$h_\varepsilon(x) := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} x(s)\ \mathrm{d}s$$

is continuous in the $J_1$-topology we get $h_\varepsilon(x) \to \pi_t(x)$ as $\varepsilon \to 0$ for each $x \in \mathscr{D}_0(\mathbb{R})$, which is the key observation in [Bil68] for $\mathscr{D}_0[0,1]$.

According to [Bil68, Theorem 14.5, p. 121] we get

$$\mathscr{B}(\mathscr{D}_0) = \sigma\left\{\pi^{-1}_{t_1,\dots,t_k} B \subset \mathscr{D}_0 \colon t_1, t_2, \dots, t_k \in \mathbb{R}, k \geq 1, B \in \mathscr{B}(\mathbb{R}^n)^k\right\}.$$

Since $\left\{\pi^{-1}_{t_1,\dots,t_k} B \subset \mathscr{D}_0 \colon t_1, t_2, \dots, t_k \in \mathbb{R}, k \geq 1, B \in \mathscr{B}(\mathbb{R}^n)^k\right\}$ is a generator of $\mathscr{B}(\mathbb{R}^n)^{\mathbb{R}} \cap \mathscr{D}_0$ as well, we have $\mathscr{B}(\mathbb{R}^n)^{\mathbb{R}} \cap \mathscr{D}_0 = \mathscr{B}(\mathscr{D}_0)$. □

In conclusion, we have

$$\begin{aligned}\theta_s^{-1}&\{\omega \in \mathscr{D}_0 \colon \omega_{t_1} \in B_1, \dots, \omega_{t_k} \in B_k, B_1, \dots, B_k \in \mathscr{E}\}\\ &= \{\omega \in \mathscr{D}_0 \colon \omega_{t_1} - \omega_s \in B_1, \dots, \omega_{t_k} - \omega_s \in B_k, B_1, \dots, B_k \in \mathscr{E}\}\end{aligned}$$

and the Wiener shift

$$\begin{aligned}\theta \colon \mathbb{R} \times \mathscr{D}_0 &\to \mathscr{D}_0\\ \theta_s\omega(\cdot) &\mapsto \omega(\cdot + s) - \omega(s)\end{aligned}$$

is a Carathéodory function. Moreover, $(\mathscr{D}_0, \mathscr{B}(\mathscr{D}_0))$ is a separable metric space, see [Lac92, Lemma 1 and Theorem 2, p. 93–95]. Thus we can apply [AB07, Lemma 4.51, p. 153] to prove, that $\theta$ is jointly measurable. This completes the construction of metric dynamical systems generated by Lévy processes with two-sided time.

## Properties

**Decomposition in independent Lévy processes on $\mathbb{R}_+$** Let $L^+$ and $L^-$ be independent and identically distributed Lévy processes on $\mathbb{R}^+$ and $L$ the canonical Lévy processes on $\mathbb{R}$ with finite dimensional distributions according to (3.4) given the finite dimensional distributions of $L^+$.

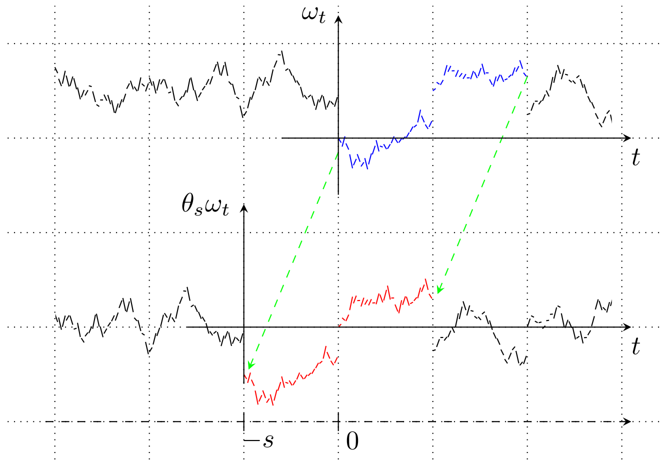



**Figure 3.2:** *(Semi)group Property for MDS generated by Lévy processes*

**Lemma 3.4.** *Let $\bar{L}$ be given by*

$$\bar{L}_t = \begin{cases} L_t^+, & \text{if } t \geqslant 0, \\ -L_{-t-}^-, & \text{else}. \end{cases} \tag{3.7}$$

*The finite dimensional distributions of $L$ and $\bar{L}$ are the same, which implies that $L$ and $\bar{L}$ are identically distributed.*

*Proof.* Let $r, s, t \in \mathbb{R}$ and $s < t$.

- If $0 < s < t$:

$$\mathrm{P}\big(\bar{L}_t - \bar{L}_s < r\big) = \mathrm{P}\big(L_t^+ - L_s^+ < r\big) = \mathrm{P}\big(L_t - L_s < r\big).$$

- Else, if $s < t < 0$:

$$\begin{aligned}\mathrm{P}\big(\bar{L}_t - \bar{L}_s < r\big) &= \mathrm{P}\big(-L_{-t-}^- + L_{-s-}^- < r\big) = \mathrm{P}\big(L_{-s+t+s} - L_{-t+t+s} < r\big) \\ &= \mathrm{P}\big(L_t - L_s < r\big).\end{aligned}$$

- Let $s < 0 < t$. Since

$$\mathrm{P}\big(\bar{L}_0 - \bar{L}_s < q\big) = \mathrm{P}\big(L_{-s-}^- - L_{-0-}^- < q\big) = \mathrm{P}\big(L_{-s+s} - L_{0+s} < q\big)$$

we obtain

$$\begin{aligned}\mathrm{P}\big(\bar{L}_t - \bar{L}_s < r\big) &= \int_{\mathbb{R}} \mathrm{P}\big(\bar{L}_t - \bar{L}_0 < r - q\big)\, \mathrm{P}_{\bar{L}_0 - \bar{L}_s}(\mathrm{d}q) \\ &= \int_{\mathbb{R}} \mathrm{P}\big(L_t^+ - L_0^+ < r - q\big)\, \mathrm{P}_{L^-_{-s-} - L^-_{-0-}}(\mathrm{d}q) \\ &= \int_{\mathbb{R}} \mathrm{P}\big(L_t - L_0 < r - q\big)\, \mathrm{P}_{L_0 - L_s}(\mathrm{d}q) = \mathrm{P}\big(L_t - L_s < r\big).\end{aligned}$$

□

**Markov Property** Lévy processes on $\mathbb{R}^+$ possess the Markov property, i.e. given the natural filtration $\mathbb{F} = \big(\mathscr{F}_s\big)$ of $L$, then we have $\mathrm{P}\big(L_t \in A \big| \mathscr{F}_s\big) = \mathrm{P}\big(L_t \in A \big| L_s\big)$ for each $A \in \mathscr{B}\big(\mathbb{R}^n\big)$ and $s < t$. On $\mathbb{R}$ a Lévy process does not satisfy the Markov property anymore (since we claim that $L_0 = 0$).
Indeed, let $L$ be a Lévy process on $\mathbb{R}$, $s < 0$, $B \in \mathscr{B}\big(\mathbb{R}\big)$ and assume that the Markov property is satisfied $(*)$ . Then we obtain

$$\begin{aligned}\mathrm{P}\left(L_0 - L_s \in B\right) &= \mathrm{P}\left(L_0 = 0, -L_s \in B\right) = \mathrm{P}\left(L_0 - L_s \in B, L_s \in -B\right) \\ &\overset{(*)}{=} \mathrm{P}\left(L_0 - L_s \in B\right) \mathrm{P}\left(L_s \in -B\right) = \mathrm{P}\left(L_0 - L_s \in B\right)^2,\end{aligned}$$

which implies $\mathrm{P}\left(L_0 - L_s \in B\right) \in \big\{0, 1\big\}$ for each $s < 0$ and each $B \in \mathscr{B}\big(\mathbb{R}\big)$, which implies $L$ is almost surely constant. This is not true in general.

**Inter jump periods** Let $N^+$ be a Poisson process with parameter $\lambda > 0$ on $\mathbb{R}^+$. Then the inter jump period of $N^+$ is stationary and identically, exponentially distributed.

Indeed, let $\tau_k^+ := \inf\big\{t \in \mathbb{R} \colon N_t > k\big\}$ and $\tau_k^- := \sup\big\{t \in \mathbb{R} \colon N_t < k\big\}$ for $k \in \mathbb{N}$. The detention time $\tau_k$ in the state $k$ is given by $\tau_k = \tau_k^+ - \tau_k^-$. Since $\tau_0^- = 0$ a.s. and $\mathrm{P}\big(\tau_0^+ > t\big) = \mathrm{P}\big(N_t^+ = 0\big) = 1 - \mathrm{e}^{-\lambda}$, we get $\tau_0 \sim \mathscr{E}xp(\lambda)$. Moreover, $N^+$ is (strong) Markovian on $\mathbb{R}^+$. Hence we get $\mathrm{P}\big(\tau_k^+ > t\big) = \mathrm{P}\big(\tau_0^+ > t\big)$, which implies $\tau_k \sim \mathscr{E}xp(\lambda)$, for each $k \in \mathbb{N}_0$.

Contrary to this, the inter jump periods of a Poisson process $N$ on $\mathbb{R}$ is not stationary distributed anymore. Since $\tau_0^- \sim \tau_0^+ \sim \mathscr{E}xp(\lambda)$ we obtain, that $\tau_0$ is $\Gamma$-distributed (as the sum of $\mathscr{E}xp$-distributed detention times in positive and negative time respectively) with parameters 2 and $\lambda$. Furthermore, there are independent and identically distributed Poisson processes $N^+$ and $N^-$ on $\mathbb{R}^+$ such that $N_t = N_t^+ \mathbb{1}_{(0,\infty)}(t) - N^-_{-t-} \mathbb{1}_{(-\infty,0]}(t)$. Hence we get $\tau_k \sim \mathscr{E}xp(\lambda)$ if $k \neq 0$.

## 3.3 Invariance of the Measure and Ergodicity

An essential property for Ergodic theory is $\theta$-invariance and ergodicity of the probability measure $\mathrm{P}$.

**Definition 3.5.** Let $\left(\Omega, \mathscr{F}, \mathrm{P}, (\theta_s)_{s\in\mathbb{R}}\right)$ be a metric dynamical system.

(i) The probability measure $\mathrm{P}$ is called *$\theta$-invariant*, if $\theta_t^{-1}\mathrm{P} = \mathrm{P}$ for each $t \in \mathbb{R}$.

(ii) A set $A \in \mathscr{F}$ is said to be *invariant* with respect to $\theta$, if $\theta_t^{-1}A = A$ for all $t \in \mathbb{R}$. Invariant sets form a sub-$\sigma$-field $\mathfrak{J} \subset \mathscr{F}$.

(iii) A metric dynamical system is called *ergodic*, if each set in $\mathfrak{J}$ has either probability $0$ or $1$.

**Theorem 3.6.** *The probability measure* $\mathrm{P}$ *is $\theta$-invariant and the metric dynamical system*

$$\left(\mathscr{D}_0(\mathbb{R}), \mathscr{B}(\mathscr{D}_0), \mathrm{P}, (\theta_t)_{t\in\mathbb{R}}\right) \tag{3.8}$$

*is ergodic.*

*Proof.* The proof is a generalization of [Box88, p. 35–38]. We show the invariance of P first:
For a finite subset $S = \{t_1, t_2, \ldots, t_k\} \subset \mathbb{R}$, $k \in \mathbb{N}$ and $A_S \in (\mathbb{R}^n)^k$, we define *generalised finite dimensional sets*

$$C(\tau, A_S) := \left\{\omega \in \mathscr{D}_0 \colon \left(\theta_\tau\omega(t_1), \ldots, \theta_\tau\omega(t_k)\right) \in A_S\right\}, \quad \tau \in \mathbb{R}.$$

Since $\mathscr{B}(\mathscr{D}_0)$ is generated by the finite dimensional sets, the generalised finite dimensional sets generate $\mathscr{B}(\mathscr{D}_0)$ as well (e.g. by setting $k = 1$). When we let $\theta$ operate on these sets, they are shifted in the sence that

$$\begin{aligned}\theta_{-t}C(\tau, A_S) &= \{\omega \colon \theta_t\omega(\cdot) \in C(\tau, A_S)\} \\ &= \left\{\omega \in \mathscr{D}_0 \colon (\omega_{\tau+t_i+t} - \omega_{\tau+t})_{i=1,\ldots,k} \in A_T\right\} = C(\tau + t, A_S),\end{aligned}$$

for all $t \in \mathbb{R}$. Lévy processes have stationary increments. Hence we get

$$\mathrm{P}(\theta_{-t}C(\tau, A_S)) = \mathrm{P}(C(\tau + t, A_S)) = \mathrm{P}(C(\tau, A_S)).$$

Before we can prove the ergodicity, we will need a few preliminaries. The same arguments was used by [Box88] for the Brownian motion: Without loss of generality, we choose $s_1 \le s_2$ and consider the sets $T_1 = \{t_1\}$ and $T_2 = \{t_2\}$.
Since Lévy processes have independent increments we can factorise the probability of the intersection of generalised finite dimensional sets by shifting one of them sufficiently strongly. Indeed, for $s_1, s_2 \in \mathbb{R}$ we set $t_0 := \max(0, s_1 - s_2 + |t_1| + |t_2|)$. Then for each $t \ge t_0$ we have

$$s_1 - t + |t_1| \le s_1 - s_1 + s_2 - |t_1| - |t_2| + |t_1| = s_2 - |t_2|.$$

Hence, we get:

$$\mathrm{P}\left(\theta_{-t} C\left(t_1, A_{T_1}\right) \cap C\left(t_2, A_{T_2}\right)\right)=\mathrm{P}\left(C\left(t_1, A_{T_1}\right)\right) \mathrm{P}\left(C\left(t_2, A_{T_2}\right)\right). \tag{3.9}$$

Now let $B \in \mathscr{B}\left(\mathscr{D}_0\right)$ be a $\theta$-invariant Borel set. Since $\mathscr{B}\left(\mathscr{D}_0\right)$ is generated by cylindrical sets, for each $\varepsilon>0$ we can find a finite union $B_0=B_0(\varepsilon)$ of cylindrical sets, such that $\mathrm{P}\left(B \triangle B_0\right) \leq \varepsilon$. Using (3.9) there is a $t>0$ sufficiently large, such that

$$\mathrm{P}\left(\left(\theta_{-t} B_0^c\right) \cap B_0\right)=\mathrm{P}\left(B_0\right) \mathrm{P}\left(B_0^c\right)=\mathrm{P}\left(\left(\theta_{-t} B_0\right) \cap B_0^c\right). \tag{3.10}$$

Since $\mathrm{P}(A \triangle B)$ defines a pseudo metric on $\mathscr{B}\left(\mathscr{D}_0\right)$, we get

$$\mathrm{P}\left(\left(\theta_{-t} B_0\right) \triangle B_0\right) \leq \mathrm{P}\left(\left(\theta_{-t} B_0\right) \triangle\left(\theta_{-t} B\right)\right)+\underbrace{\mathrm{P}\left(\left(\theta_{-t} B\right) \triangle B\right)}_{=0}+\mathrm{P}\left(B \triangle B_0\right).$$

The second term vanishes due to $B$ is invariant and we get

$$\mathrm{P}\left(\left(\theta_{-t} B_0\right) \triangle B_0\right) \leq 2 \mathrm{P}\left(B \Delta B_0\right) \leq 2 \varepsilon. \tag{3.11}$$

Apart from that we can use the definition of the symmetric difference and (3.10) to obtain

$$\begin{aligned}
\mathrm{P}\left(\left(\theta_{-t} B_0\right) \triangle B_0\right) & = \mathrm{P}\left(\left(\theta_{-t} B_0\right) \cap B_0^c\right)+\mathrm{P}\left(\left(\theta_{-t} B_0^c\right) \cap B_0\right) \\
& = 2 \mathrm{P}\left(B_0\right) \mathrm{P}\left(B_0^c\right)=2 \mathrm{P}\left(B_0\right)\left(1-\mathrm{P}\left(B_0\right)\right).
\end{aligned} \tag{3.12}$$

Now we put (3.11) and (3.12) together and get

$$\mathrm{P}\left(B_0\right)\left(1-\mathrm{P}\left(B_0\right)\right) \leq \varepsilon \quad \text{for each} \quad \varepsilon>0.$$

Since we can choose $\varepsilon$ arbitrarily small, we get $\mathrm{P}\left(B_0\right)\left(1-\mathrm{P}\left(B_0\right)\right)=0$, which implies ergodicity. $\square$

# 4 On the Conjugacy of solutions of Marcus type SDEs and RDEs

*“The elegance of a mathematical theorem is directly proportional to the number of independent ideas one can see in the theorem and inversely proportional to the effort it takes to see them."*

— GEORGE PÓLYA, *Mathematical Discovery on Understanding, Learning, and Teaching Problem Solving, Volume I*

## 4.1 Perfection of Cocycles

### Motivation and Definition

Let $L$ be a one-dimensional canonical Lévy process on $\big(\mathscr{D}_0, \mathscr{B}(\mathscr{D}_0), \mathrm{P}\big)$, i.e. $L_t(\omega) = \omega(t)$ for each $\omega \in \mathscr{C}_0$ and let $X$ be the (strong) solution of the Itô type SDE $\mathrm{d}Z_t = \lambda Z_t \mathrm{d}t + \mathrm{d}L_t$ with $Z_0 = z \in \mathbb{R}$. Then $X$ is called *Lévy type Ornstein–Uhlenbeck process* and satisfies

$$Z_t = z\,\mathrm{e}^{\lambda t} + \int_0^t \mathrm{e}^{\lambda(t-r)}\,\mathrm{d}L_r(\omega).$$

Especially,

$$Z_t := \int_{-\infty}^t \mathrm{e}^{\lambda(t-r)}\,\mathrm{d}L_r(\omega),$$

is the stationary solution of $\mathrm{d}Z_t = \lambda Z_t \mathrm{d}t + \mathrm{d}L_t$, see [JV83, Theorem 2.3, p. 250]. Moreover, we have

$$\begin{aligned}
Z_{t+s}(\omega) &= \int_{-\infty}^{t+s} \mathrm{e}^{\lambda(t+s-r)}\,\mathrm{d}L_r(\omega)\\
&= \mathrm{e}^{\lambda(t+s)} \int_{-\infty}^{t} \mathrm{e}^{-\lambda(r+s)}\,\mathrm{d}L_{r+s}(\omega) \quad \text{a.s.}\\
&= \mathrm{e}^{\lambda(t+s)} \int_{-\infty}^{t} \mathrm{e}^{-\lambda(r+s)}\,\mathrm{d}\big(L_{r+s}(\omega) - L_s(\omega)\big) \quad \text{a.s.}\\
&= \int_{-\infty}^{t} \mathrm{e}^{\lambda(t-r)}\,\mathrm{d}\big(\theta_s L_r(\omega)\big)\\
&= Z_t(\theta_s\omega),
\end{aligned}$$

which implies $Z_{t+s}(\omega) = Z_t(\theta_s\omega)$ almost surely for each fixed $s \in \mathbb{R}$, i.e. there is a null set $N_{s,t}$

(depending on $s, t$), such that the equation holds $\omega$-wise for each $\omega \in \Omega \setminus N_{s,t}$.

Accordingly $Z$ forms a crude cocyle, not a random dynamical system though. To get a random dynamical system, we have to find a modification of $Z$, such that the cocyle property is satisfied for each and every $\omega \in \Omega$. Therefore we have to collect all exceptional sets $N_{s,t}$ for each $s, t$, which is a uncountable union and is not null set in general:

**(Counter-)Example 4.1.** Let $\big(\Omega, \mathscr{F}\big) = \big(\mathbb{R}, \mathscr{B}(\mathbb{R})\big)$ and P be a probability measure which is equivalent to the Lebesgue-measure and $\theta_t(\omega) := \omega + t$ similar to [Kag96, Beispiel 3.36, p. 56]. Now we define

$$\varphi(t, \omega) := \begin{cases} \frac{\sin(\theta_t \omega)}{\sin(\omega)}, & \text{if } \sin(\omega) \neq 0; \\ 1, & \text{if } \sin(\omega) = 0, \end{cases}$$

according to [Kag96, Beispiel 2.16, p. 26]. Then $\varphi(0, \omega) = 1$ for each and every $\omega \in \Omega$ and $\varphi$ forms a measurable (since continuous) crude multiplicative cocycle.

Indeed, fix $s \in \mathbb{R}$ and $\omega \in \Omega \setminus \big(\pi\mathbb{Z} \cup (\pi\mathbb{Z} - s)\big)$. Then we get

$$\varphi(t + s, \omega) = \frac{\sin(\omega + t + s)}{\sin(\omega + s)} \cdot \frac{\sin(\omega + s)}{\sin(\omega)} = \varphi(t, \theta_s \omega)\varphi(s, \omega)$$

for each $t \in \mathbb{R}$.
Now we set $\omega \in (0, \pi)$, $s := \pi - \omega$ and $t := \pi/2$, which implies $\varphi(s, \omega) = 0$ and $\varphi(t + s, \omega) \neq 0$. In this case there is no perfected cocycle $\psi$ which is still indistinguishable from $\varphi$:

Assuming that $\varphi(\cdot, \omega) = \psi(\cdot, \omega)$ for each $\omega \in \Omega_1 \subset \Omega$. Then

$$\varphi(t + s, \omega) = \psi(t + s, \omega) = \psi(t, \theta_s \omega)\psi(s, \omega) = \psi(t, \theta_s \omega)\varphi(s, \omega) = 0$$

for each $t \in \mathbb{R}$ and $\omega \in \Omega_1$. Since $\varphi(t + s, \omega) \neq 0$ for $t = \pi/2$ and $\omega \in (0, \pi)$ we get $(0, \pi) \cap \Omega_1 = \emptyset$ and $\mathrm{P}\big(\Omega_1\big) < 1$, which implies that there is no indistinguishable perfect cocycle for $\varphi$. $\square$

The procedure to find a suitable modification with time-independent exclusion set is frequently called *perfection*. The following result covers the perfection problem for all processes of this work:

**Theorem 4.2.** *Let $E$ be a separable Banach space and $(X_t)_{t \in \mathbb{R}}$ a $E$-valued and $\mathscr{F}$-measurable stochastic process with càdlàg paths generating a crude cocyle with respect to the metric dynamical system $\big(\Omega, \mathscr{F}, \mathrm{P}, (\theta_t)_{t \in \mathbb{R}}\big)$, i.e. for all $t \in \mathbb{R}$ we have*

$$X_t = X_0 \circ \theta_t \quad \text{P–}a.s. \tag{4.1}$$

*Then there is an $E$-valued process $\widehat{X} = \big(\widehat{X}_t\big)_{t \in \mathbb{R}}$, such that:*

*(i) The processes $X$ and $\widehat{X}$ are undistinguishable.*

*(ii) The process $\widehat{X}$ is strictly stationary, i.e.*

$$\widehat{X}_t(\omega) = \widehat{X}_0(\theta_t\omega) \tag{4.2}$$

*for all $t \in \mathbb{R}$, $\omega \in \Omega$.*

The proof is a modification of a general theorem in [Arn98]. For continuous processes this theorem was already proven in [Led01]. The results are based on Arnold and Scheutzow [AS95, Theorem 31, p. 85].

*Proof.* Let $\nu$ be a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ equivalent to the Lebesgue measure $\lambda$. We define

$$\begin{aligned}\Omega_0 &:= \big\{\omega \in \Omega\colon \exists N^0 \subset \mathbb{R} \text{ with } \nu\left(N^0\right) = 0, \text{ such that} \\ &\qquad X_t(\omega) = X_0(\theta_t\omega) \text{ for each } t \in \mathbb{R}\setminus N^0\big\}, \qquad (4.3)\\ \Omega_1 &:= \big\{\omega \in \Omega\colon \exists N^1 \subset \mathbb{R} \text{ with } \nu\left(N^1\right) = 0, \text{ such that} \\ &\qquad \theta_t\omega \in \Omega_0 \text{ for each } t \in \mathbb{R}\setminus N^1\big\}, \qquad (4.4)\end{aligned}$$

The probability measure $\nu$ is needed for technical reasons (see Step 3) and can be choosen fairly freely. The only important property we need later on is the equivalence to the Haar measure of the underlying time space.

**Step 1.** We show, that $\Omega_0$ is measurable (i.e. $\Omega_0 \in \mathscr{F}$) and $\mathrm{P}(\Omega_0) = 1$:

Since $X$ has càdlàg path, $(t,\omega) \mapsto X_t(\omega)$ is a Carathéodory function and jointly $\mathscr{B}(\mathbb{R}) \otimes \mathscr{F} - \mathscr{B}(E)$ measurable. The mapping $(t,\omega) \mapsto \theta_t\omega$ is jointly measurable by definition. Hence the set

$$A := \{(t,\omega) \in \mathbb{R}\times\Omega\colon X_t(\omega) \neq X_0(\theta_t\omega)\} \in \mathscr{B}(\mathbb{R}) \otimes \mathscr{F} \tag{4.5}$$

is measurable. By the cross section theorem [DM79] the $\omega$-section $A_\omega = \{t \in \mathbb{R}\colon (t,\omega) \in A\}$ is measurable for each $\omega \in \Omega$ and $\Omega \setminus \Omega_0 = \{\omega \in \Omega\colon \lambda(A_\omega) > 0\} = \{\omega \in \Omega\colon \nu(A_\omega) > 0\}$. Now we can apply Fubini's theorem and get

$$\int_\Omega \lambda(A_\omega)\, \mathrm{dP}(\omega) = \int_{\mathbb{R}} \int_\Omega \mathbb{1}_A(t,\omega)\, \mathrm{dP}(\omega)\, \lambda(\mathrm{d}t) = \int_{\mathbb{R}} \mathrm{P}(X_t \neq X_0 \circ \theta_t)\, \lambda(\mathrm{d}t) = 0,$$

i.e., $\lambda(A_\omega) = 0 = \nu(A_\omega)$ for P-a.a. $\omega \in \Omega$ and we get $\mathrm{P}(\Omega\setminus\Omega_0) = 0$.

**Step 2.** By replacing $A$ with

$$B := \{(t,\omega) \in \mathbb{R}\times\Omega\colon \theta_t\omega \notin \Omega_0\} \in \mathscr{B}(\mathbb{R}) \otimes \mathscr{F}, \tag{4.6}$$

we still have

$$P\big(\theta_t\omega \in \Omega_0\big) = \mathrm{P}\big(X_s\left(\theta_t\omega\right) = X_0\left(\theta_s \circ \theta_t\omega\right) \text{ for } \nu\text{–almost every } s \in \mathbb{R}\big) = 1 \tag{4.7}$$

for a fixed $t \in \mathbb{R}$, according to (4.1). Thus we get

$$\int_\Omega \lambda\left(B_\omega\right) \,\mathrm{dP}\left(\omega\right) = \int_\mathbb{R}\int_\Omega \mathbb{1}_B\left(t,\omega\right) \,\mathrm{dP}\left(\omega\right)\lambda\left(\mathrm{d}t\right) = \int_\mathbb{R} \mathrm{P}\left(\theta_t\omega \notin \Omega_0\right)\lambda\left(\mathrm{d}t\right) = 0,$$

analogously to Step 1.
Furthermore, for each $\omega \in \Omega_1$ it is $X_s\left(\omega\right) = X_0\left(\theta_s\omega\right)$ for $\nu$ a.a. $s \in \mathbb{R}$. Since $\nu$ is equivalent to $\lambda$, which is a Haar measure (i.e. translation invariant), it is also $X_{s-t}\left(\theta_t\omega\right) = X_0\left(\theta_{s-t}\circ\theta_t\omega\right)$ for $\nu$ a.a. $s, t \in \mathbb{R}$. Hence we get $\theta_t\omega \in \Omega_1$. In other words, $\Omega_1$ is $\theta$-invariant.

**Step 3.** We choose $x_0 \in S$ arbitrarily and define

$$\widehat{X}_t\left(\omega\right) = \begin{cases} X_{t-s}\left(\theta_s\omega\right), & \text{if } \omega \in \Omega_1 \text{ and } s \in \mathbb{R} \text{ such that } \theta_s\omega \in \Omega_0, \\ x_0, & \text{if } \omega \in \Omega \setminus \Omega_1. \end{cases} \tag{4.8}$$

We prove, that $\widehat{X}$ is well-defined (i.e. independent from the choice of $s$) and measurable. After that we show, that $X$ and $\widehat{X}$ are undistinguishable.

Let $\omega \in \Omega$ and $s_1 \neq s_2 \in \mathbb{R}$ with $\theta_{s_1}\omega \in \Omega_0$ and $\theta_{s_2}\omega \in \Omega_0$.
Then there are measurable sets $G\left(\theta_{s_0}\omega\right), G\left(\theta_{s_1}\omega\right) \in \mathscr{B}\left(\mathbb{R}\right)$ and $\nu\left(G\left(\theta_{s_i}\omega\right)\right) = 1$, $i = 1, 2$, such that

$$X_u\left(\theta_{s_i}\omega\right) = X_0\left(\theta_{u+s_i}\omega\right), \quad u \in G\left(\theta_{s_i}\omega\right).$$

Further $\nu\left(\bigcap_{i=1,2}\left(G\left(\theta_{s_i}\omega\right) + s_i\right)\right) = 1$ and thus we find a sequence $\left(t_n\right)_{n\in\mathbb{N}}$, such that $t_n \in \bigcap_{i=1,2}\left(G\left(\theta_{s_i}\omega\right) + s_i\right)$ and $t_n \searrow t$ as $n \to \infty$. This leads to

$$X_{t-s_1}\left(\theta_{s_1}\omega\right) = \lim_{n\to\infty} X_{t_n-s_1}\left(\theta_{s_1}\omega\right) = \lim_{n\to\infty} X_0\left(\theta_{t_n}\right) = \lim_{n\to\infty} X_{t_n-s_2}\left(\theta_{s_2}\omega\right) = X_{t-s_2}\left(\theta_{s_2}\omega\right).$$

Since $X$ has càdlàg paths the same applies to $\widehat{X}$. To prove measurability of $\widehat{X}$ we consider

$$\Psi\left(t,s,\omega\right) = \begin{cases} X_{t-s}\left(\theta_s\omega\right), & \text{if } \omega \in \Omega_1 \text{ and } s \in \mathbb{R} \text{ such that } \theta_s\omega \in \Omega_0, \\ x_0, & \text{if } \omega \in \Omega \setminus \Omega_1. \end{cases} \tag{4.9}$$

Then $\Psi$ is $\mathscr{B}\left(\mathbb{R}\right)\otimes\mathscr{B}\left(\mathbb{R}\right)\otimes\mathscr{F}$–$\mathscr{B}\left(S\right)$-measurable, since $\Omega_0, \Omega_1$ are measurable and $\theta$ is jointly measurable. Analogously to [Arn98, Theorem 1.3.2, Step 6] it is

$$\widehat{X}_t\left(\omega\right) = \int_\mathbb{R} \Psi\left(t,s,\omega\right) \,\mathrm{d}\nu\left(s\right),$$

since $\nu$ is a probability measure equivalent to $\lambda$ and $\lambda$ is translation invariant. Hence $\widehat{X}$ is measurable due to Fubini's theorem.

If $\omega \in \Omega_0 \cap \Omega_1$, it is $\widehat{X}_t(\omega) = X_t(\omega)$ and $\mathrm{P}(\Omega_0 \cap \Omega_1) = 1$. Thus $X$ and $\widehat{X}$ are undistinguishable.

**Step 4.** We show that $\widehat{X}$ is strictly stationary:

If $\omega \notin \Omega_1$ we have $\widehat{X}_t(\omega) = x_0 = \widehat{X}_0(\theta_t\omega)$ for each $t \in \mathbb{R}$, since $\Omega_1$ is $\theta$-invariant. Let $\omega \in \Omega_1$ and $t \in \mathbb{R}$: Then there is a measurable set $H(\omega) \in \mathscr{B}(\mathbb{R})$, such that $\nu(H(\omega)) = \nu(H(\theta_t\omega)) = 1$ and

$$\begin{aligned}\widehat{X}_t(\omega) &= X_{t-r}(\theta_r\omega) \text{ for each } r \in H(\theta_t\omega) \text{ and}\\ \widehat{X}_0(\theta_t\omega) &= X_{-s}(\theta_{s+t}\omega) \text{ for each } s \in H(\omega).\end{aligned}$$

Now we can choose $s \in H(\theta_t\omega) \cap (H(\omega)+t)$ (where $\nu(H(\theta_t\omega) \cap (H(\omega)+t)) = 1$), to get

$$\widehat{X}_t(\omega) = X_{t-s}(\theta_s\omega) = X_{-(s-t)}\left(\theta_{(s-t)+t}\omega\right) = \widehat{X}_0(\theta_t\omega). \tag{4.10}$$

□

The process $X_t(\omega)$ in Theorem 4.2 is assumed to be measurable with respect to the non-completed $\sigma$-field $\mathscr{F}$. Later on we focus on stochastic processes, which are given by solutions of stochastic differential equations. These processes are measurable with respect to the completed filtration $\bar{\mathscr{F}}$, not measurable with respect to the non-completed $\mathscr{F}$ though.

This might lead to problems, since given a metric dynamical system $(\theta_t)_t$ over $(\Omega, \mathscr{F}, \mathrm{P})$, then it is not a metric dynamical system over the completed space $(\Omega, \bar{\mathscr{F}}, \mathrm{P})$ in gerneral:

**(Counter-)Example 4.3.** Let $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and P be a probability measure which is equivalent to the Lebesgue-measure. Now we define $\theta\colon \mathbb{R} \times \Omega \to \Omega$ according to $\theta_t(\omega) := \omega + t$ similar to Example 4.1. Then $\theta$ is a $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F})$–$\mathscr{F}$-measurable shift operator due to continuity. Now let $A \in \bar{\mathscr{F}} \setminus \mathscr{F}$ and consider $\theta^{-1}(A)_\omega := \{t \in \mathbb{R}\colon (t,\omega) \in \theta^{-1}(A)\}$. Then we have $\theta^{-1}(A)_\omega\big|_{\omega=0} = A \notin \mathscr{B}(\mathbb{R})$. Hence $\theta$ is not $(\mathscr{B}(\mathbb{R}) \otimes \bar{\mathscr{F}})$–$\bar{\mathscr{F}}$-measurable.

Thus, given a $\bar{\mathscr{F}}$-measurable process $X$ we need the existence of a process $Y$ indistinguishable from $X$, which is measurable with respect to the non-completed $\sigma$-field $\mathscr{F}$:

**Lemma 4.4** (Decompletion – cf. Scheutzow [Sch96], Lemma 2.7, p. 242)**.** *Let $G$ be a Hausdorff topological group and $H$ be a Hausdorff second-countable (or completely separable) topological group, whose $\sigma$-fields are denoted by $\mathscr{G}$ or $\mathscr{H}$ respectively and let $\varphi\colon G \times \Omega \to H$ be a $(\mathscr{G} \otimes \bar{\mathscr{F}})$–$\mathscr{H}$-measurable mapping. Then there is a $(\mathscr{G} \otimes \mathscr{F})$–$\mathscr{H}$-measurable mapping $\bar{\varphi}\colon G \times \Omega \to H$ which is indistinguishable from $\varphi$.*

Later on, we have $G = \mathbb{R}$ and $H = \mathrm{Diff}\left(\mathbb{R}^n, \mathbb{R}^n\right)$, which denotes the space of diffeomorphism from $\mathbb{R}^n$ to itself. Then both $G$ and $H$ satisfy the conditions of Lemma 4.4, see [Kun90, p. 115].

The proof can be found in [Sch96, p. 242]. Since the result is an essential part of the perfection, we present the proof here as well. It is an application of the monotone class theorem.

*Proof.* According to [Zim84, Proposition A.1 & Theorem A.3, p. 194 et seq.] we can choose $H \subset [0,1]$ and $\mathscr{H} = \mathscr{B}(\mathbb{R})|_H$ (i.e. $\mathscr{H}$ is the restriction of $\mathscr{B}(\mathbb{R})$ on $H$). There we use that $H$ is completely separable.

Now we define

$$V := \big\{ f\colon G \times \Omega \to \mathbb{R} \mid f \text{ is } (\mathscr{G} \otimes \bar{\mathscr{F}})\text{–}\mathscr{B}(\mathbb{R})\text{-measurable and bounded, such that there is} \\ \text{a } (\mathscr{G} \otimes \mathscr{F})\text{–}\mathscr{B}(\mathbb{R})\text{-measurable mapping } g \text{ which is indistinguishable from } f \big\}.$$

Then $V$ is a linear space of real-valued functions. Moreover, given a sequence $\big(f_n\big)_n \subset V$ and $0 \leqslant f_n \uparrow f$ (pointwise), such that $f$ is bounded, then (according to the definition of $V$) there is a sequence $\big(g_n\big)_n$ of $(\mathscr{G} \otimes \mathscr{F})$–$\mathscr{B}(\mathbb{R})$-measurable mappings and $g_n \uparrow g$, where $g$ is $(\mathscr{G} \otimes \mathscr{F})$–$\mathscr{B}(\mathbb{R})$-measurable, bounded and indistinguishable from $f$. Hence we have $f \in V$, i.e. $V$ satisfies the conditions of the (functional) monotone class theorem, see [RW00, Theorem 3.1 or Theorem 3.2, p. 90 et seq.]. Let

$$S := \big\{ f\colon G \times \Omega \to \mathbb{R} \mid f(t,\omega) = \mathbb{1}_{A \times B}(t,\omega), A \in \mathscr{G}, B \in \bar{\mathscr{F}} \big\}$$

be the space of simple $(\mathscr{G} \otimes \bar{\mathscr{F}})$–$\mathscr{B}(\mathbb{R})$-measurable mappings. Then $S$ is closed under multiplication and $S \subset V$, which implies that the set of bounded $(\mathscr{G} \otimes \bar{\mathscr{F}})$–$\mathscr{B}(\mathbb{R})$-measurable mappings (which are generated by $S$) is a subset of $V$ as well.

For a $H$-valued mapping $f$ we can modify $g$, such that $g$ is $H$-valued as well (by redefining $g$ to be the unit element $e_H$ of $H$ on the null set $\mathscr{N}$ on which $f$ and $g$ differs). Moreover, a $H$-valued mapping is $(\mathscr{G} \otimes \bar{\mathscr{F}})$–$\mathscr{B}(\mathbb{R})$-measurable if and only if it is $(\mathscr{G} \otimes \bar{\mathscr{F}})$–$\mathscr{H}$-measurable, which completes the proof. $\square$

## 4.2 Conjugacy of Cocycles

Subsequently we discuss the first main result of this work. We give conditions under which we can proof the existence of a random transformation, such that a solution of a Marcus type SDE can be represented as a transformed solution of a random ordinary differential equation (RDE) and vice versa. More precisely, we proof *conjugacy* of the corresponding cocycles. The following approach is motivated by the inspiring work of Imkeller & Schmalfuß [IS01] and Lederer [Led01], both covering the Brownian case.
However, first we prove a useful property of Lévy processes. Then we motivate the following definitions with help of an elementary example.

**Lemma 4.5** ([JV83], Theorem 2.3, p. 250)**.** *Let $L$ be a Lévy process satisfying*

$$\mathbb{E}\,\log\left(1+|L_1|\right)<\infty. \tag{4.11}$$

*Then the stationary Ornstein–Uhlenbeck type process*

$$Z_t := \int_{-\infty}^{t} \mathrm{e}^{-(t-s)}\,\mathrm{d}L_s \tag{4.12}$$

*is a well-defined stationary semimartingale (convergence a.s.).*

*Idea of the proof.* Let $\delta_1, \delta_2, \ldots$ be a sequence of iid random variables and $c \in (0,1)$.
If $\mathbb{E}\log\left(1+|\delta_1|\right)<\infty$ then we obtain $\left|\sum_{i=1}^{n} c^i\delta_i\right| \to 0$ with probability 1 as $n\to\infty$ according to [JV83, Lemma 2.1, p. 250].

Indeed, if $\sum_{i=1}^{n} c^i\delta_i$ converge, it is $\mathrm{P}\big(\limsup\left\{\,|\delta_i|^{1/i} > d\right\}\big) = 0$, for each $d > 1/c$. Then we get

$$\mathrm{P}\left(\limsup\left\{\log^+|\delta_i| > i\log d\right\}\right) = 0 \text{ if and only if } \sum_{i=1}^{\infty}\mathrm{P}\left(\log^+|\delta_i| > i\log^+ d\right) < \infty$$

due to the Borel–Cantelli lemma, cf. [JV83, Proof of Lemma 2.1, p. 250]. Finally we observe, that $\mathbb{E}\log^+|\delta_1| < \infty$ if and only if $\sum_{i=1}^{\infty}\mathrm{P}\left(\log^+|\delta_i| > i\log^+ d\right) < \infty$. □

*Remark* 4.6. Let $L$ be a Lévy processes in $\mathbb{R}^n$. If for each $a>0$ there are $b>0$ and $c\in\mathbb{R}^n$, such that $L_{at} \overset{d}{=} bL_t + c$, then we have $b = a^H$ for some $H \geqslant 1/2$, see [Sat99, Theorem 13.11, p. 73]. In this case, $L$ is called $\alpha$-stable, where $\alpha = 1/H$ and $L$ satisfies (4.11).

*Proof of Remark 4.6.* Let $\nu$ be the Lévy measure of $L_1$.
Then we have $\nu\left(\mathrm{d}y\right) = 1/|y|^{n+\alpha}\mathbb{1}(y\neq 0)\,\mathrm{d}y$ on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ and according to [JV83, Theorem 2.3, p. 250] it is sufficient to show, that $\int_{|x|\geq 1}\log\left(1+|y|\right)\nu\left(\mathrm{d}y\right)<\infty$.

By using hyperspherical coordinates we get:

$$
\begin{aligned}
\int_{|x|>1} \log\left(1+|y|\right)\, \nu\left(\mathrm{d}y\right) &= \int_{|x|>1} \log\left(1+|y|\right) \frac{1}{|y|^{n+\alpha}}\,\mathrm{d}y \\
&= \sigma_n \int_1^\infty \log\left(1+r\right) \frac{1}{r^{1+\alpha}}\,\mathrm{d}r \\
&\leqslant 2^{1+\alpha}\,\sigma_n \int_1^\infty \underbrace{\log\left(1+r\right)}_{=:u} \frac{1}{\left(1+r\right)^{1+\alpha}}\,\mathrm{d}r \\
&= 2^{1+\alpha}\,\sigma_n \int_{\log 2}^\infty u\,\mathrm{e}^{-\alpha u}\,\mathrm{d}u < \infty,
\end{aligned}
$$

where $\sigma_n$ is the surface area of a $n$-dimensional unit sphere for $\alpha > 0$. $\square$

**Example 4.7.** Let $L$ be a Lévy processes on $\mathbb{R}$ satisfying (4.11) and define

$$
\tilde{L}_t^\tau := \int_{-\infty}^t \mathrm{e}^{-(\tau-s)}\,\mathrm{d}L_s, \qquad t,\tau \in \mathbb{R}.
$$

Next we consider flow $\Phi_t(\omega,x)$ generated by the so-called stationary linear (Marcus type) power equation

$$
X_t(x) = x + \int_{-\infty}^t \mathrm{e}^{-(t-s)} X_s \diamond \mathrm{d}L_s.
$$

To solve this equation we can alternatively solve the SDE

$$
\hat{X}_t^\tau(x) = x + \int_0^t \hat{X}_s^\tau \diamond \mathrm{d}\hat{L}_s^\tau, \qquad t,\tau \in \mathbb{R},
$$

and obtain $X_t = \hat{X}_{\mathrm{e}^t}^t$, where $\hat{L}_t^\tau = \tilde{L}_{\log t}^\tau$ (especially $\hat{L}_0^\tau = 0$ as we will see later). Its solution is given by

$$
\hat{X}_t^\tau(x) = x\,\mathrm{e}^{\hat{L}_t^\tau}.
$$

This can be seen by using Theorem 2.7. Then we get

$$
X_t(x) = \hat{X}_{\mathrm{e}^t}^t(x) = x\,\mathrm{e}^{\tilde{L}_t},
$$

where $\tilde{L}_t := \tilde{L}_t^\tau\big|_{\tau=t}$. Now consider the solution of the RDE $\dot{Y}_t(y) = -\tilde{L}_t\,Y_t(y)$ and $Y_0 = y$, which is given by $Y_t = y\,\mathrm{e}^{-\bar{L}_t}$, where $\bar{L}_t := \int_0^t \tilde{L}_s\,\mathrm{d}s$.

Since $\tilde{L}$ satisfies $\mathrm{d}\tilde{L}_t = \tilde{L}_t\,\mathrm{d}t + \mathrm{d}L_t$, we obtain that the process $\Phi_0$ given by

$$
\Phi_0\big(\theta_t\omega, x\big) := x\,\mathrm{e}^{\tilde{L}_t}\,\mathrm{e}^{-\tilde{L}_0-\bar{L}_t}
$$

satisfies $\mathrm{d}\Phi_0\big(\theta_t\omega, Y_t\big(\Phi_0(\omega,x)^{-1}\big)\big) = \Phi_0\big(\theta_t\omega, Y_t\big(\Phi_0(\omega,x)^{-1}\big)\big) \diamond \mathrm{d}L_t$. Thus, we get

$$\Phi_0\big(\theta_t\omega, Y_t\big(\Phi_0(\omega,x)^{-1}\big)\big) = x + \int_0^t \Phi_0\big(\theta_s\omega, Y_s\big(\Phi_0(\omega,x)^{-1}\big)\big) \diamond \mathrm{d}L_s.$$

□

What we can see in this example, is that we are able to construct the solution of the Marcus type SDE $X_t = X_0 + \int_0^t X_t \diamond \mathrm{d}L_t$ by using a certain RDE and a transformation which is given by some sort of SDE driven by stationary Lévy-type Ornstein–Uhlenbeck noise (which works $\omega$-wise).

This type of transformations is called *conjugacy*.

**Definition 4.8.** Let $\xi$ and $\psi$ be two random dynamical systems. Then $\xi$ and $\psi$ are called *conjugated*, if there is a random mapping $\Phi\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, such that $(t,x) \mapsto \Phi(\theta_t\omega, x)$ is a Carathéodory function for each $\omega \in \Omega$, $x \mapsto \Phi(\theta_t\omega, x)$ is homeomorphic for each $t \in \mathbb{R}$ and $\omega \in \Omega$, and

$$\xi_t(x) = \Phi\left(\theta_t\omega, \psi_t\big(\Phi^{-1}(\omega,x)\big)\right) \text{ for each } x \in \mathbb{R}^n. \tag{4.13}$$

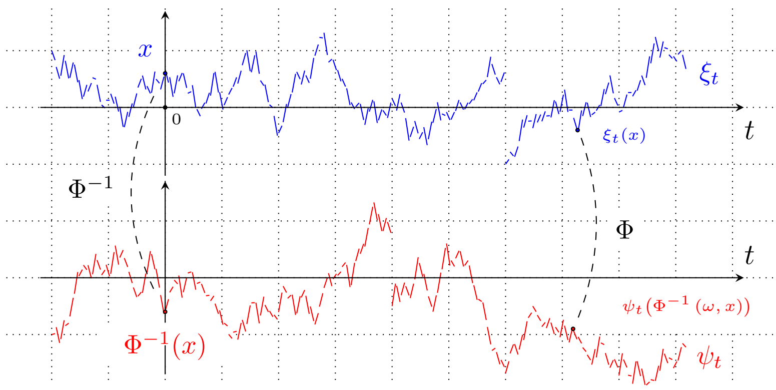



**Figure 4.1:** *Conjugacy of $\xi$ and $\psi$*

*Remark* 4.9. Let $\xi$ and $\psi$ be conjugated due to $\Phi$, and $\psi$ and $\phi$ be conjugated due to $\Psi$. Then $\psi_t(x) = \Psi\left(\theta_t\omega, \phi_t\big(\Psi^{-1}(\omega,x)\big)\right)$ and $\xi_t(x) = \left(\Phi \circ \Psi\right)\left(\theta_t\omega, \phi_t\big(\big(\Psi^{-1} \circ \Phi^{-1}\big)(\omega,x)\big)\right)$. Thus, $\xi$ and $\phi$ are conjugated due to $\Phi \circ \Psi$.

**Theorem 4.10.** *Let $L$ be a Lévy process, such that* (4.11) *holds and $f_1, f_2, \dots, f_m$ satisfy the assumption (L). Then there is a unique stationary solution of the Marcus type SDE*

$$X_t^\tau(x) = x + \mathrm{e}^{-\tau} \sum_{j=1}^m \int_{-\infty}^t \mathrm{e}^s f_j\left(X_s^\tau(x)\right) \diamond \mathrm{d}L_s^j, \qquad t \in \mathbb{R}, \tag{4.14}$$

*for each $\tau \in \mathbb{R}$.*

The solution of (4.14) is denoted by $\xi^\tau$. It is one building block for the proof of the conjugacy between the solution of Marcus type SDEs and RDEs. The proof is separated in several parts.

*Proof of Theorem 4.10.* We define the process $\tilde{L}_t^{\tau,j}$ according to

$$\tilde{L}_t^{\tau,j} := \int_{-\infty}^{t} \mathrm{e}^{-(\tau-s)} \, \mathrm{d}L_s^j, \tag{4.15}$$

where $\tau \in \mathbb{R}$.
Since there is a version of the Lévy process with $\omega$-wise finite $p$-variation for each $p > 2$, we can apply [MN87, Theorem 2.9, p. 411] to define (4.15) as ($\omega$-wise deterministic) Young intgral. Hence we have $\tilde{L}_t^{\tau,j} \to 0$ ($\omega$-wise) as $t \to -\infty$.

Next we extend $\tilde{L}_t^{\tau,j}$ on $\mathbb{R} \cup \{-\infty\}$ by $\tilde{L}_{-\infty}^{\tau,j} := 0$. According to [Mon78, Theorem 1, p. 44] we can define the semimartingale $\hat{L}_t^{\tau,j}$ on $[0,\infty)$ by using a transformation of time:

$$\hat{L}_t^{\tau,j} = \tilde{L}_{\log t}^{\tau,j} \qquad\qquad t > 0$$

and $\hat{L}_0^{\tau,j} = \tilde{L}_{-\infty}^{\tau,j} = 0$ with respect to the filtration $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $\mathscr{F}_t = \sigma\left\{\tilde{L}_s^{\tau,j} \colon s \leqslant \log t\right\}$.
Let $\hat{X}^\tau$ satisfy the Marcus type SDE

$$\hat{X}_t^\tau(x) = x + \sum_{j=1}^{m} \int_0^t f_j\left(\hat{X}_s^\tau(x)\right) \diamond \mathrm{d}\hat{L}_s^{\tau,j}. \tag{4.16}$$

Hence there is a unique solution for the MSDE (4.16) as stated in [KPP95, Theorem 3.9, p. 361].

We define $\xi_t^\tau(x) := \hat{X}_{\mathrm{e}^t}^\tau(x)$. Then $\xi_t^\tau(x)$ satisfies the Marcus type SDE (4.14). Indeed, using [Pro04, Theorem 45, p. 190] and the change of variables formula Theorem 2.7 we get

$$\begin{aligned}
\xi_t^\tau(\omega, x) = \hat{X}_{\mathrm{e}^t}^\tau(x) &= x + \sum_{j=1}^{m} \int_0^{\mathrm{e}^t} f_j\left(\hat{X}_s^\tau(x)\right) \diamond \mathrm{d}\hat{L}_s^{\tau,j} \\
&= x + \sum_{j=1}^{m} \int_{-\infty}^{t} f_j\left(\underbrace{\hat{X}_{\mathrm{e}^s}^\tau(x)}_{=\xi_s^\tau(x)}\right) \diamond \mathrm{d}\, \underbrace{\hat{L}_{\mathrm{e}^s}^{\tau,j}}_{=\tilde{L}_s^{\tau,j}} \\
&= x + \sum_{j=1}^{m} \int_{-\infty}^{t} \mathrm{e}^{-(\tau-s)} f_j\left(\xi_s^\tau(x)\right) \diamond \mathrm{d}L_s^j.
\end{aligned} \tag{4.17}$$

To prove stationarity, we consider $\xi_{t-r}^{\tau-r}(\theta_r\omega)$, for each $r \in \mathbb{R}$. Since

$$\begin{aligned}
\xi_{t-r}^{\tau-r}(\theta_r\omega, x) &= x + \mathrm{e}^{-\tau+r} \sum_{j=1}^{m} \int_{-\infty}^{t-r} \mathrm{e}^s f_j\left(\xi_s^{\tau-r}(\theta_r\omega)\right) \diamond \mathrm{d}L_s^j(\theta_r\omega) \\
&= x + \mathrm{e}^{-\tau+r} \sum_{j=1}^{m} \int_{-\infty}^{t-r} \mathrm{e}^s f_j\left(\xi_s^{\tau-r}(\theta_r\omega)\right) \diamond \mathrm{d}L_{s+r}^j(\omega) \\
&= x + \mathrm{e}^{-\tau} \sum_{j=1}^{m} \int_{-\infty}^{t} \mathrm{e}^s f_j\left(\xi_{s-r}^{\tau-r}(\theta_r\omega)\right) \diamond \mathrm{d}L_s^j(\omega),
\end{aligned}$$

the process $\xi_{t-r}^{\tau-r}(\theta_r\omega)$ satisfies the Marcus type SDE (4.14). Due to assumption (L) the solution is unique. Thus, we get $\xi_{t-r}(\theta_r\omega) = \xi_t(\omega)$ almost surely for each $r \in \mathbb{R}$. □

**Proposition 4.11.** *The solution process $\xi_t^\tau$ of the Marcus type SDE* (4.14) *is differentiable with respect to $\tau$.*

*Proof.* We set $\tilde{L}_t^j := \tilde{L}_t^{\tau,j}\Big|_{\tau=0}$ and $\hat{L}_t^j = \tilde{L}_{\log t}^j$. Then we can rewrite (4.16) by

$$\hat{X}_t^\tau(x) = x + \sum_{j=1}^m \int_0^t \mathrm{e}^{-\tau} f_j\left(\hat{X}_s^\tau(x)\right) \diamond \mathrm{d}\hat{L}_s^j. \tag{4.18}$$

We generalise the $n$-dimensional SDE (with parameter $\tau$) to a $n+1$-dimensional SDE in $(X,\tau)$ by adding the equation $\mathrm{d}\tau = 0$ (formally) and consider the mapping $(\tau, x) \mapsto \hat{X}_t^\tau(x)$. Since the function $\mathrm{e}^{-\tau} f_j(x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is not bounded (as $\tau \to \infty$), we cannot apply the result in [KPP95, Theorem 3.9, p. 361] right away.

We consider the restrictions $f_K := \mathrm{e}^{-\tau} f_j(x)|_{[-K,+\infty)\times\mathbb{R}^n} : [-K,+\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ first. This function satisfies the condition of [KPP95, Theorem 3.9, p. 361] for each $K \in \mathbb{N}$. Hence we get the existence of a flow of diffeomorphism on $[-K,+\infty) \times \mathbb{R}^n$ (with an exclusion set $N_K \subset \Omega$ depending on $K$). Now we set $N = \bigcup_{m\in\mathbb{N}} N_m$ with $\mathrm{P}(N) = 0$. Let $(\tau, x) \mapsto \hat{\xi}_t^K(\tau, x)$ be the solution of the system (4.18) of Marcus type SDEs extended by the equation $\mathrm{d}\tau = 0$ on $[-K,+\infty) \times \mathbb{R}^n$. Since $\hat{\xi}^{K'} = \hat{\xi}^K\Big|_{[-K',+\infty)\times\mathbb{R}^n}$ for $K' < K$, we can define

$$\hat{\xi}_t(\tau, x) := \hat{\xi}_t^K(\tau, x) \text{ for some } K \in \mathbb{N} \text{ such that } -K < \tau.$$

Thus $\hat{\xi}_t^\tau(x) = \hat{\xi}_t(\tau, x)$ generates a flow of diffeomorphism $\hat{\xi}_t(\cdot,\cdot)$ on $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ for $\omega \in \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m) \setminus N$ and $\xi_t(\tau, x) := \hat{\xi}_{\mathrm{e}^t}(\tau, x)$ still satisfies (4.17). □

Now we can formulate and prove the main result of this section. The ideas are based on [IS01, Section 1]. We change the notation slightly according to [Led01] by including the formula for the stationary Ornstein–Uhlenbeck type process. The existence of the processes was already proven in Theorem 4.10.

According to Proposition 4.11 we define

$$\Phi_t(x) := \xi_t(\tau, x)|_{\tau=t} \quad \text{and} \quad \Gamma_t(x) := \frac{\partial}{\partial\tau}\, \xi_t(\tau, x)|_{\tau=t}. \tag{4.19}$$

Hence it is

$$\begin{aligned}\Phi_t(x) &= x + \sum_{j=1}^{m} \int_{-\infty}^{t} \mathrm{e}^{-(t-s)} f_j(\Phi_s(x)) \diamond \mathrm{d}L_s^j, \qquad (4.20)\\ \Gamma_t(x) &= -\sum_{j=1}^{m} \int_{-\infty}^{t} \mathrm{e}^{-(t-s)} f_j(\Phi_s(x)) \diamond \mathrm{d}L_s^j \\ &\quad + \sum_{j=1}^{m} \int_{-\infty}^{t} \mathrm{e}^{-(t-s)} f_j{}'(\Phi_s(x))\,\Gamma_s(x) \diamond \mathrm{d}L_s^j.\end{aligned}$$

**Theorem 4.12** (Conjugacy of solutions of Marcus type SDEs and RDEs)**.** *Let $g \in \mathscr{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ be a function with bounded derivatives and $\xi$ be the flow generated by the solution of the Marcus type SDE*

$$\mathrm{d}X_t = g(X_t)\,\mathrm{d}t + \sum_{j=1}^{m} f_j(X_t) \diamond \mathrm{d}L_t^j, \qquad (4.21)$$

*where $L$ is a Lévy process such that $\mathbb{E}\,\log(1 + \|L_1\|) < \infty$ and $f_1, f_2, \ldots, f_m$ satisfy the assumtions* (L) *and* ($\mathrm{C}^\infty$)*. Moreover let $\psi$ be the flow generated by the solution of the RDE*

$$\dot{Y} = \left\{ \frac{\partial}{\partial x} \Phi_0(\theta_t\omega, Y_t) \right\}^{-1} \left[ g(\Phi_0(\theta_t\omega, Y_t)) - \Gamma_0(\theta_t\omega, Y_t) \right] \qquad (4.22)$$

*for all $\omega \in \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m)$, where $\Phi_0(\theta_t\omega, x)$ and $\Gamma_0(\theta_t\omega, x)$ are strict stationary versions corresponding to $\Phi_t(\omega, x)$ and $\Gamma_t(\omega, x)$, respectively.*
*Then $\xi$ and $\psi$ are conjugated.*

*Proof.* By the change of variables formula for MSDEs (Theorem 2.7) we get

$$\begin{aligned}\mathrm{d}\Phi_t(x) &= \left.\frac{\mathrm{d}\xi_t(\tau, x)}{\mathrm{d}t}\,\mathrm{d}t\right|_{\tau=t} + \left.\frac{\mathrm{d}\xi_t(\tau, x)}{\mathrm{d}\tau}\,\mathrm{d}t\right|_{\tau=t} \\ &= \sum_{j=1}^{m} f_j(\Phi_s(x)) \diamond \mathrm{d}L_t^j + \Gamma_t(x)\,\mathrm{d}t. \qquad (4.23)\end{aligned}$$

Moreover, by using Itô's formula we get

$$\begin{aligned}\mathrm{d}\Phi_t(Y_t) &= \underbrace{\frac{\mathrm{d}\Phi_t(Y_t)}{\mathrm{d}t}\,\mathrm{d}t}_{=\mathrm{d}\Phi_t(x)|_{x=Y_t}} + \left.\frac{\mathrm{d}\Phi_t(x)}{\mathrm{d}x}\right|_{x=Y_t} \mathrm{d}Y_t \\ &= \sum_{j=1}^{m} f_j(\Phi_t(Y_t)) \diamond \mathrm{d}L_t^j + \Gamma_t(Y_t)\,\mathrm{d}t + \underbrace{\Phi_t{}'(Y_t)\,\mathrm{d}Y_t}_{=[g(\Phi_t(Y_t))-\Gamma_t(Y_t)]\,\mathrm{d}t} \\ &= \sum_{j=1}^{m} f_j(\Phi_t(Y_t)) \diamond \mathrm{d}L_t^j + g(\Phi_t(Y_t))\,\mathrm{d}t. \qquad (4.24)\end{aligned}$$

The process

$$\begin{pmatrix} \Phi_t(\omega, x) \\ \Gamma_t(\omega, x) \end{pmatrix} \tag{4.25}$$

is stationary due to Theorem 4.10. According to the Decompletion lemma 4.4 we can apply the Perfection theorem 4.2 on

$$\begin{pmatrix} \Phi_t(\omega, x) \\ \Gamma_t(\omega, x) \end{pmatrix} = \begin{pmatrix} \Phi_0(\theta_t\omega, x) \\ \Gamma_0(\theta_t\omega, x) \end{pmatrix} \quad \text{P-a.s.}$$

to get strict stationary processes.

Thus, in combination with (4.24) we obtain

$$\mathrm{d}\Phi_0(\theta_t\omega, Y_t) = g(\Phi_0(\theta_t\omega, Y_t))\,\mathrm{d}t + \sum_{j=1}^{m} f_j(\Phi_0(\theta_t\omega, Y_t)) \diamond \mathrm{d}L_t^j, \text{ for all } \omega \in \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m),$$

i.e. $\xi$ and $\psi$ are conjugated by $\Phi_0$. □

In [QD12] the authors construct a (random) functional dependency between the solution of Marcus type SDEs (in the setting of [Kun95]) and RDEs (similar to the random homeomorphism $\Phi_0$ from Theorem 4.12). They use a totally different approach to construct this mapping by using the concrete definition of Marcus SDEs in the setting of [Kun95] and applying a generalised Itô's formula.

Moreover, the work from [QD12] requires stricter restrictions on the driving Lévy process, e.g. they assume bounded jumps of L and lacks results about perfection and stationarity of the solution of the corresponding Marcus type SDE, which is crucial for the definition of conjugacy. Indeed, there is no such topics in [QD12] at all. Hence their results can not be applied on flows generated by Marcus type SDEs.

Conclusively, we consider the following example to show that solutions of Itô and Stratonovich SDEs driven by Lévy noise do not lead to a random dynamical system in general, which implies that Theorem 4.12 does not hold for Itô and Stratonovich SDEs:

**(Counter-)Example 4.13.** Let $Z$ be a semimartingale. We consider the Stratonovich type SDE

$$X_t = x + \int_0^t X_{s-} \circ \mathrm{d}Z_s.$$

According to [Pro04, Theorem 23, p. 280] the solution $X$ is explicitely given by

$$X_t = x\,\mathrm{e}^{Z_t} \prod_{0<s\leqslant t} \big(1 + \Delta Z_s\big)\mathrm{e}^{-\Delta Z_s},$$

which follows from the Itô formula.

Now let $N$ be a standard Poisson process, i.e. $\Delta N \in \{0, 1\}$, and set $Z_s := -N_s$. Then $X_t = 0$ for each $t > \tau$, where $\tau = \inf\{s > 0 \colon \Delta N_s \neq 0\}$ is the first jump point of $N$ (independently of $x$). By the definition of RDS, the cocycle property needs to hold for each $t \in \mathbb{R}$, especially for $t < 0$. Since $X_\tau = 0$ independently of $x$, we cannot identify the starting point $x$ once $N$ jumps for the first time. Hence the cocycle property cannot hold for $X$.

# 5 Local Linearization of Marcus type SDEs

*“If you can’t solve a problem, then there is an easier problem you can solve: find it."*
— GEORGE PÓLYA, *Mathematical Discovery on Understanding, Learning, and Teaching Problem Solving, Volume I*

In 1959 and 1960 both Philip Hartman and David Grobman proved independently of each other, that the dynamics generated by an ODE and the dynamics of the linearised ODE are basically the same in the neighbourhood of hyperbolic fixed points.
In the following section we will deal with the same question for MSDEs, by investigating the local behavior of solutions of MSDEs.
To prove the result for a wide class of functions and driving noises we use the conjugacy result of the previous section, which allows us to separate the problem in several parts. Then we can solve each part independently.

## 5.1 Hartman–Grobman type theorem

Before we can formulate the main result of this section, we have to prove an existence result first:

**Theorem 5.1** (Existence theorem – extension of [Kun04], Theorem 2.11, p. 332)**.**
*Let $L = \left(L_t\right)_{t\in\mathbb{R}}$ be a $\mathbb{R}^m$-valued Lévy process with bounded jumps given by its Lévy–Itô decomposition*

$$L_t = \beta t + \sigma B_t + \int_{-\infty}^{t} \int_{\{z:\ \|z\|<c\}} z\, \tilde{N}(\mathrm{d}z, \mathrm{d}s), \tag{5.1}$$

*where $\beta \in \mathbb{R}^m$, $\sigma \in \mathbb{R}^{m\times m}$ and $\tilde{N}$ is the compensated jump measure. Furthermore let $\Phi_t(x)$ be given by*

$$\Phi_t\left(x\right) = x + \sum_{j=1}^{m} \int_{-\infty}^{t} \mathrm{e}^{-(t-s)} f_j\left(\Phi_s\left(x\right)\right) \diamond \mathrm{d}L_s^j, \tag{5.2}$$

*where $x \in \mathbb{R}^n$, $f = (f_1, \ldots, f_m)$ satisfies assumptions $(\mathrm{L})$ and $(\mathrm{C}^1)$ as well as $\|D_x f\|_\infty < \infty$ and $\|D_x f' f\|_\infty < \infty$ and fix $T \in \mathbb{R}$. Then we have*

$$\mathbb{E}\left[\sup_{-\infty<s\leqslant t} \left|D_x\Phi_s(x)\right|^2\right] < \infty \quad \textit{for each} \quad x \in \mathbb{R}^n \textit{ and } t \leqslant T.$$

The following proof is motivated by [Kun04, Proof of Theorem 3.3, p. 342], where the author considers solution of SDEs on finite time intervals. We modify the proof for stationary solutions of SDEs which – roughly speaking – start at $-\infty$.

*Proof.* Since $f$ satisfies assumption ($\mathrm{C}^1$) we know that $\Phi_t'(x)$ exist and it is sufficient to prove, that

$$\sup_{h>0} \mathbb{E}\left[\sup_{-\infty<s\leqslant t} |N_s(x,h)|^2\right] < \infty,$$

where

$$N_t(x,h) := \frac{\Phi_t(x+he_i) - \Phi_t(x)}{\lambda}, \text{ for } h>0, x\in\mathbb{R}^n \text{ and } e_i = (0,\dots,1,\dots,0)\,. \tag{5.3}$$

According to equation (5.1), we can rewrite (5.3) as

$$\begin{aligned} N_t(x,h) = e_i &+ \int_{-\infty}^t \left(\tilde f(s)\,\mathrm{e}^{-(t-s)}\beta - \frac{1}{2}\widetilde{f'f}(s)\,\mathrm{e}^{-2(t-s)}\sigma^2\right)\mathrm{d}s + \int_{-\infty}^t \tilde f(s)\,\mathrm{e}^{-(t-s)}\sigma\,\mathrm{d}B_s \\ &+ \int_{-\infty}^t \int_{\{z:\,\|z\|\leqslant c\}} \left(\tilde f(s)\,z + \tilde g(z,s)\right)\mathrm{e}^{-(t-s)}\,\tilde N(\mathrm{d}z,\mathrm{d}s)\,, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} \tilde f(s) &:= \frac{f(\Phi_s(x+he_i)) - f(\Phi_s(x))}{h}, \\ \widetilde{f'f}(s) &:= \frac{f'f(\Phi_s(x+he_i)) - f'f(\Phi_s(x))}{h} \qquad \text{and} \\ \tilde g(z,s) &:= \frac{g(\Phi_s(x+he_i)) - g(\Phi_s(x))}{h}. \end{aligned}$$

Hence we get

$$\begin{aligned} \mathbb{E}\,|N_t(x,h)| \leqslant 1 &+ \mathbb{E}\int_{-\infty}^t \left(\left|\tilde f(s)\,\beta\right|\mathrm{e}^{-(t-s)} + \frac{1}{2}\left|\widetilde{f'f}(s)\,\sigma^2\right|\mathrm{e}^{-2(t-s)}\right)\mathrm{d}s \\ &+ \mathbb{E}\left|\int_{-\infty}^t \int_{\{z:\,|z|\leqslant c\}} \left(\tilde f(s)\,z + \tilde g(z,s)\right)\mathrm{e}^{-(t-s)}\,\tilde N(\mathrm{d}z,\mathrm{d}s)\right| \\ &+ \mathbb{E}\left|\int_{-\infty}^t \tilde f(s)\,\sigma\,\mathrm{e}^{-(t-s)}\,\mathrm{d}B_s\right|. \end{aligned}$$

Since $(x_1+\cdots+x_n)^2 \leqslant n(x_1^2+\cdots+x_n^2)$ we obtain

$$\begin{aligned} \mathbb{E}\,|N_t(x,h)|^2 \leqslant 4\Bigg(1 &+ \mathbb{E}\left[\int_{-\infty}^t \left(\left|\tilde f(s)\,\beta\right|\mathrm{e}^{-(t-s)} + \frac{1}{2}\left|\widetilde{f'f}(s)\,\sigma^2\right|\mathrm{e}^{-2(t-s)}\right)\mathrm{d}s\right]^2 \\ &+ \mathbb{E}\left|\int_{-\infty}^t \int_{\{z:\,\|z\|\leqslant c\}} \left(\tilde f(s)\,z + \tilde g(z,s)\right)\mathrm{e}^{-(t-s)}\,\tilde N(\mathrm{d}z,\mathrm{d}s)\right|^2 \\ &+ \mathbb{E}\left|\int_{-\infty}^t \tilde f(s)\,\sigma\,\mathrm{e}^{-(t-s)}\,\mathrm{d}B_s\right|^2\Bigg). \end{aligned} \tag{5.5}$$

We will deal with each integral separately.
Let $Y_t^N := \int_{-N}^t \tilde{f}(s)\,\sigma\,\mathrm{e}^s\,\mathrm{d}B_s$ and consider the mapping $|x|^2 \colon \mathbb{R}^n \to \mathbb{R}$, where $\nabla |x|^2 = 2x$ and $\nabla^2 |x|^2 = 2 \cdot \mathrm{id}_n$, where $\mathrm{id}_n$ is the identity matrix in $\mathbb{R}^n$. Itô's formula implies

$$\left|Y_t^N\right|^2 = 2\int_{-N}^t Y_s^N \,\mathrm{d}Y_s^N + \frac{1}{2}\cdot 2\int_{-N}^t \left|\tilde{f}(s)\sigma\right|^2 \mathrm{e}^{2s}\,\mathrm{d}s.$$

By localisation we can assume that the first integral is a zero-mean martingale with respect to the natural filtration. Else there is a sequence of stopping times $\tau_n$ and $\mathrm{P}(\tau_n < T) \to 0$ as $n \to \infty$, such that $\int_{-N}^{t\wedge\tau_n} Y_s^N \,\mathrm{d}Y_s^N$ is a zero-mean martingale for each $n \in \mathbb{R}$, see [Kun04, Proof of Theorem 2.11, p. 333]. Thus, we get

$$\mathbb{E}\left|Y_{t\wedge\tau_n}^N\right|^2 = \mathbb{E}\left[\int_{-N}^{t\wedge\tau_n} \left|\tilde{f}(s)\sigma\right|^2 \mathrm{e}^{2s}\,\mathrm{d}s\right].$$

Applying *Doob's inequality* we get

$$\mathbb{E}\left[\sup_{-N<s\leqslant t\wedge\tau_n} \left|Y_s^N\right|^2\right] \leqslant 2\,\mathbb{E}\left|Y_{t\wedge\tau_n}\right|^2 \leqslant 2\,\mathbb{E}\left[\int_{-N}^{t\wedge\tau_n} \left|\tilde{f}(s)\sigma\right|^2 \mathrm{e}^{2s}\,\mathrm{d}s\right],$$

which leads to

$$\mathbb{E}\left[\sup_{-\infty<s\leqslant t} \left|Y_s\right|^2\right] \leqslant 2\,\mathbb{E}\left[\int_{-\infty}^{t} \left|\tilde{f}(s)\sigma\right|^2 \mathrm{e}^{2s}\,\mathrm{d}s\right] \quad \text{as} \quad n, N \to \infty. \tag{5.6}$$

Now let $Y_t^N := \int_{-N}^t \int H(z,s)\,\mathrm{e}^s\,\tilde{N}(\mathrm{d}z,\mathrm{d}s)$ be a pure jump process, where $H(z,s)$ is square integrable with respect to $\nu\otimes\mathrm{e}^s\mathrm{d}s$ on $\left(\mathbb{R}^{m+1}, \mathscr{B}(\mathbb{R}^{m+1})\right)$, where $\nu$ is the Lévy measure associated with $L$. Similarly to the Brownian case it is sufficient to consider the one-dimensional case. Applying Itô's formula leads to

$$\begin{aligned}\left|Y_t^N\right|^2 &= \int_{-N}^t \int \left(\left|Y_s^N + H(z,s)\,\mathrm{e}^s\right|^2 - \left|Y_s^N\right|^2\right)\,\tilde{N}(\mathrm{d}z,\mathrm{d}s)\\ &\quad + \int_{-N}^t \int \left(\left|Y_s^N + H(z,s)\,\mathrm{e}^s\right|^2 - \left|Y_s^N\right|^2 - 2\,Y_t^N\big(H(z,s)\,\mathrm{e}^s\big)\right)\,\nu(\mathrm{d}z)\mathrm{d}s.\end{aligned}$$

Since $\int_{-N}^t \int \left(\left|Y_s^N + H(z,s)\,\mathrm{e}^s\right|^2 - \left|Y_s^N\right|^2\right)\,\tilde{N}(\mathrm{d}z,\mathrm{d}s)$ is a local martingale (localised by a sequence of stopping times $\tau_n$) and since

$$\underbrace{\left|Y_s^N + H(z,s)\,\mathrm{e}^s\right|^2}_{(x+y)^2} - \underbrace{\left|Y_s^N\right|^2}_{x^2} - \underbrace{2\,Y_t^N\left(H(z,s)\,\mathrm{e}^s\right)}_{2xy} = \underbrace{|H(z,s)|^2\,\mathrm{e}^{2s}}_{y^2}$$

we get

$$\mathbb{E}\left|Y_{t\wedge\tau_n}^N\right|^2 = \mathbb{E}\int_{-N}^{t\wedge\tau_n}\int |H(z,s)|^2\,\mathrm{e}^{2s}\,\nu(\mathrm{d}z)\mathrm{d}s.$$

Finally Doob's inequality implies

$$\mathbb{E}\left[\sup_{-N<s\leqslant t\wedge\tau_n}\left|Y_s^N\right|^2\right]\leqslant 2\,\mathbb{E}\left|Y_{t\wedge\tau_n}\right|^2\leqslant 2\,\mathbb{E}\int_{-N}^{t\wedge\tau_n}\int|H(z,s)|^2\,\mathrm{e}^{2s}\,\nu(\mathrm{d}z)\mathrm{d}s$$

and

$$\mathbb{E}\left[\sup_{-\infty<s\leqslant t}\left|Y_s\right|^2\right]\leqslant 2\,\mathbb{E}\int_{-\infty}^{t}\int|H(z,s)|^2\,\mathrm{e}^{2s}\,\nu(\mathrm{d}z)\mathrm{d}s\quad\text{as}\quad n,N\to\infty. \tag{5.7}$$

Since $\|D_x f\|_\infty < c_1$ and $\|D_x f' f\|_\infty < c_2$ it is

$$\frac{f\left(\Phi_s\left(x+he_i\right)\right)-f\left(\Phi_s\left(x\right)\right)}{h}=\underbrace{\left(\int_0^1 D_x f\left(\Phi_s\left(x\right)+\theta h N_s\left(x,h\right)\right)\,\mathrm{d}\theta\right)}_{|\cdot|\leqslant c_1}N_s\left(x,h\right), \tag{5.8}$$

and

$$\left\|\widetilde{f'f}\left(s\right)\right\|_\infty\leqslant c_2\,\left|N_s\left(x,h\right)\right|. \tag{5.9}$$

Indeed, let $F=(F_1,\dots,F_n)\colon\mathbb{R}^n\to\mathbb{R}^n$ be a vector-valued function $F=F(x_1,\dots,x_n)$, and consider $\theta\mapsto F_i(a+\theta b)$ for fixed $a,b\in\mathbb{R}^n$ and $1\leqslant i\leqslant n$ mapping from $[0,1]$ to $\mathbb{R}$. Then the chain rule implies

$$\frac{\partial F_i}{\partial\theta}(a+\theta b)=\sum_{k=1}^n\frac{\partial F_i}{\partial x_k}(a+\theta b)b_k,$$

where $b=(b_1,\dots,b_n)$. Hence we obtain

$$\begin{aligned}\left(\int_0^1 D_x F_i(a+\theta b)\,\mathrm{d}\theta\right)\cdot b&=\int_0^1\left(\sum_{k=1}^n\frac{\partial F_i}{\partial x_k}(a+\theta b)b_k\right)\mathrm{d}\theta\\&=\int_0^1\frac{\partial F_i}{\partial\theta}(a+\theta b)\,\mathrm{d}\theta=F_i(a+b)-F_i(a),\end{aligned}$$

or equivalently

$$\left(\int_0^1 D_x F(a+\theta b)\,\mathrm{d}\theta\right)\cdot b=F(a+b)-F(a).$$

Especially, we get

$$\begin{aligned}
\left(\int_0^1 D_x f\left(\Phi_s\left(x\right)+\theta h N_s\left(x,h\right)\right)\,\mathrm{d}\theta\right)\cdot N_s\left(x,h\right)& \\
&= \frac{f\left(\Phi_s(x)+hN_s\left(x,h\right)\right)-f\left(\Phi_s\left(x\right)\right)}{h}\\
&= \frac{f\left(\Phi_s(x)+h\left(\frac{\Phi_s(x+he_i)-\Phi_s(x)}{h}\right)\right)-f\left(\Phi_s\left(x\right)\right)}{h}\\
&= \frac{f\left(\Phi_s\left(x+he_i\right)\right)-f\left(\Phi_s\left(x\right)\right)}{h}.
\end{aligned}$$

Using the Taylor expansion for $y_1$ in the definition of Marcus type SDEs (2.8), we get

$$\left\|D_x g\left(x,z\right)\right\|_\infty = \left\|\frac{1}{2}D_x f' f\left(x\right)\right\|_\infty |z|^2.$$

Since $\|D_x f' f\|_\infty < c_2$ we get $|D_x g\left(x,z\right)| \leqslant c_3\,|z|^2$ and $|\widetilde{g}\left(s\right)| \leqslant c_3\,\left|N_s\left(x,\lambda\right)\right|\cdot|z|^2$ similarly to (5.8).

Using (5.8) and (5.9) leads us to

$$\begin{aligned}
\mathbb{E}\Bigg[\int_{-\infty}^t\left(\left|\tilde{f}\left(s\right)\beta\right|\mathrm{e}^{-(t-s)}+\frac{1}{2}\left|\widetilde{f'f}\left(s\right)\sigma^2\right|\mathrm{e}^{-2(t-s)}\right)\mathrm{d}s\Bigg]^2 & \\
&\leqslant \mathbb{E}\Bigg[\int_{-\infty}^t\left(c_1\left|N_{r-}(x,h)\right|\left|\beta\right|\mathrm{e}^{-(t-s)}+{}^{c_2}\!/\!_2\left|N_{r-}(x,h)\right|\left|\sigma\right|^2\mathrm{e}^{-2(t-s)}\right)\mathrm{d}s\Bigg]^2\\
&\leqslant \left(c_1\left|\beta\right|+{}^{c_2}\!/\!_2\left|\sigma\right|^2\right)\left(\int_{-\infty}^t\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r-}(x,h)\right|\right]\mathrm{e}^{-(t-s)}\mathrm{d}s\right)^2 \qquad (5.10)\\
&= \left(c_1\left|\beta\right|+{}^{c_2}\!/\!_2\left|\sigma\right|^2\right)\times\\
&\quad\times\left(\int_{-\infty}^t\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r-}(x,h)\right|\right]\left(\mathrm{e}^{-1/2(t-s)}\right)\cdot\left(\mathrm{e}^{-1/2(t-s)}\right)\mathrm{d}s\right)^2\\
&\leqslant \left(c_1\left|\beta\right|+{}^{c_2}\!/\!_2\left|\sigma\right|^2\right)\int_{-\infty}^t\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r-}(x,h)\right|\right]^2\mathrm{e}^{-(t-s)}\,\mathrm{d}s, \qquad (5.11)
\end{aligned}$$

where (5.11) holds because of Hölder's inequality and (5.10) since $0 < \mathrm{e}^{-(t-s)} \leqslant 1$ for $-\infty < s \leqslant t$.

According to (5.6) and (5.8) we obtain

$$\mathbb{E}\left|\int_{-\infty}^t\tilde{f}\left(s\right)\,\mathrm{e}^{-(t-s)}\sigma\,\mathrm{d}B_s\right|^2 \leqslant 2\,c_1^2\,\left|\sigma^2\right|\int_{-\infty}^t\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_s(x,h)\right|\right]^2\mathrm{e}^{-(t-s)}\,\mathrm{d}s. \qquad (5.12)$$

Analogously we get

$$
\begin{aligned}
\mathbb{E}&\left|\int_{-\infty}^{t}\int_{\{y\colon \|y\|\leqslant c\}}\left(\tilde{f}z+\tilde{g}\left(z\right)\right)\mathrm{e}^{-(t-s)}\,\tilde{N}\left(\mathrm{d}z,\mathrm{d}s\right)\right|^{2}\\
&\leqslant 2\left(\int_{-\infty}^{t}\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r-}(x,h)\right|\right]^{2}\mathrm{e}^{-2(t-s)}\int_{\{y\colon |y|\leqslant c\}}\left(c_1\left|z\right|+c_3\left|z\right|^{2}\right)^{2}\nu\left(\mathrm{d}z\right)\mathrm{d}s\right)\\
&\leqslant c_4\int_{-\infty}^{t}\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r-}(x,h)\right|\right]^{2}\mathrm{e}^{-(t-s)}\mathrm{d}s, \qquad (5.13)
\end{aligned}
$$

due to inequality (5.7) and since

$$
\int_{\{y\colon |y|\leqslant c\}}\underbrace{\left(c_1\left|z\right|+c_3\left|z\right|^{2}\right)^{2}}_{\leqslant 2\left(c_1^2|z|^2+c_3^2|z|^4\right)}\nu\left(\mathrm{d}z\right)\leqslant 2\left(c_1^2+c^2c_3^2\right)\left|z\right|^{2}\int_{\{y\colon |y|\leqslant c\}}\left|z\right|^{2}\nu\left(\mathrm{d}z\right)<\infty.
$$

We use (5.5) and combine the estimates (5.11)–(5.13) to obtain

$$
\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r}(x,h)\right|\right]^{2}\leqslant 4+\mathrm{C}\int_{-\infty}^{t}\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r}(x,h)\right|\right]^{2}\mathrm{e}^{-(t-s)}\,\mathrm{d}s, \qquad (5.14)
$$

for a constant $\mathrm{C}$ depending on $c, c_1, \ldots, c_4$, $\beta$ and $\sigma$.

Let $\varepsilon>0$. Since $\mathbb{E}\left|N_{r-}(x,h)\right|^{2}\to 1$ as $r\to-\infty$ we get $\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_r(x,h)\right|\right]^{2}\to 1$ as $s\to-\infty$ due to Doob's inequality again (as $N_{r-}(x,h)$ is the sum of Lebesgue integrals and stochastic integrals with respect to martingales). Hence there is a $N\in\mathbb{R}$, such that

$$
\int_{-\infty}^{N}\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r}(x,h)\right|\right]^{2}\mathrm{e}^{s}\,\mathrm{d}s<\varepsilon.
$$

Thus, (5.14) implies

$$
\begin{aligned}
\mathrm{e}^{t}\,\mathbb{E}\left[\sup_{-\infty<r<t}\left|N_{r}(x,h)\right|\right]^{2}&\leqslant\left(4\,\mathrm{e}^{t}+\mathrm{C}\int_{-\infty}^{N}\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r}(x,h)\right|\right]^{2}\mathrm{e}^{s}\,\mathrm{d}s\right)\\
&\quad+\mathrm{C}\int_{N}^{t}\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r}(x,h)\right|\right]^{2}\mathrm{e}^{s}\,\mathrm{d}s\\
&\leqslant\left(4\,\mathrm{e}^{t}+\mathrm{C}\cdot\varepsilon\right)+\mathrm{C}\int_{N}^{t}\mathbf{1}\cdot\mathrm{e}^{s}\,\mathbb{E}\left[\sup_{-\infty<r<s}\left|N_{r}(x,h)\right|\right]^{2}\mathrm{d}s.
\end{aligned}
$$

Finally, we apply *Gronwall's lemma* and obtain

$$
\begin{aligned}
\mathrm{e}^{t}\,\mathbb{E}\left[\sup_{-\infty<r<t}\left|N_{r}(x,h)\right|\right]^{2}&\leqslant\left(\mathrm{e}^{t}+\mathrm{C}\cdot\varepsilon\right)+\int_{N}^{t}\left(\mathrm{e}^{s}+\mathrm{C}\cdot\varepsilon\right)\cdot\mathrm{e}^{\mathrm{C}\int_{s}^{t}\mathbf{1}\,\mathrm{d}r}\,\mathrm{d}s\\
&=\left(\mathrm{e}^{t}+\mathrm{C}\cdot\varepsilon\right)+\int_{N}^{t}\left(\mathrm{e}^{s}+\mathrm{C}\cdot\varepsilon\right)\cdot\mathrm{e}^{\mathrm{C}(t-s)}\,\mathrm{d}s
\end{aligned}
$$

$$\leqslant \underbrace{\left(\mathrm{e}^t + \mathrm{C}\cdot\varepsilon\right)}_{<\infty} + \underbrace{\mathrm{e}^{\mathrm{C}t}(t-N) + \varepsilon\mathrm{e}^{\mathrm{C}t}\left(\mathrm{e}^{-\mathrm{C}N} - \mathrm{e}^{-\mathrm{C}t}\right)}_{<\infty} < \infty, \quad (5.15)$$

for all $t \leqslant T$ and $h > 0$. □

### Lyapunov Exponents and Oseledets Spaces

To formulate the main result of this section, we have to fix several definitions first. In the deterministic Hartman–Grobman theorem it is crucial to consider a neighbourhood of a hyperbolic fixed point, which is characterised by eigenvalues of the Jacobian. We will use Lyapunov exponents instead. Roughly speaking we measure the rate of separation of different trajectories starting close to each other:

**Theorem 5.2** (Oseledets theorem – [Arn98], Theorem 3.4.1 & Theorem 3.4.11, p. 134 & p. 153)**.** *Let $\phi(t,\omega) \in \mathbb{R}^{n\times n}$ be a linear random dynamical system over $\left(\Omega, \mathscr{F}, \mathrm{P}, (\theta_t)_{t\in\mathbb{R}}\right)$ with two-sided time. Assume, that $\mathbb{E}\left[\alpha^{\pm}(\omega)\right] < \infty$, where*

$$\alpha^{\pm}(\omega) := \sup_{0\leqslant t\leqslant 1} \log^{+}\left|\phi(t,\omega)^{\pm 1}\right|.$$

*Then there exists a invariant set $\tilde{\Omega}$ with $\mathrm{P}\left(\tilde{\Omega}\right) = 1$, such that for each $\omega \in \tilde{\Omega}$:*

*(i) There is a splitting of $\mathbb{R}^n$*

$$\mathbb{R}^n = E_1(\omega) \oplus \cdots \oplus E_{p(\omega)}(\omega),$$

*into random closed subspaces $E_i(\omega)$ with dimensions $\dim E_i(\omega) = d_i(\omega)$ for each $i = 1, \ldots, p(\omega)$.*

*(ii) Let $P_i(\omega)\colon \mathbb{R}^d \to E_i(\omega)$ be the corresponding random projections onto $E_i(\omega)$. Then we have*

$$\phi(t,\omega)P_i(\omega) = P_i(\theta_t\omega)\phi(t,\omega),$$

*or equivalently*

$$\phi(t,\omega)E_i(\omega) = E_i(\theta_t\omega).$$

*(iii) For each $x \in \mathbb{R}^n \setminus \{0\}$, the Lyapunov exponents*

$$\lambda(\omega, x) := \lim_{t\to\pm\infty} \frac{1}{t} \log\left|\phi(t,\omega)x\right|$$

*are well defined and*

$$\lambda(\omega, x) = \lambda_i(\omega) \Leftrightarrow x \in E_i(\omega) \setminus \{0\}.$$

*(iv) The function $p(\cdot)$ is constant on $\tilde{\Omega}$, and the functions $\lambda_i(\cdot)$ and $d_i(\cdot)$ are constant on $\{\omega \in \tilde{\Omega}\colon p(\omega) > i\}$ (solely depending on $i$), if $\left(\Omega, \mathscr{F}, \mathrm{P}, (\theta_t)_{t\in\mathbb{R}}\right)$ is ergodic.*

**Definition 5.3.** The linear RDS $\varphi$ is called *hyperbolic*, if $\lambda_i(\omega) \neq 0$ for each $i \in \{1, \ldots, p(\omega)\}$.

A main characteristic of a hyperbolic RDS is the gap between the smallest positive Lyapunov exponent $\lambda^+(\omega)$ and largest negative Lyapunov exponent $\lambda^-(\omega)$ and the existence of this gap is a key component of the following proofs.

**Theorem 5.4** (Local Linearization)**.** *Let $g \in \mathscr{C}^{1+\delta}(\mathbb{R}^n, \mathbb{R}^n)$ with bounded derivatives and let $f = (f_1, \ldots, f_m)$ satisfy assumptions (L) and ($\mathrm{C}^\infty$). Further let $f_j(0) = g(0) = 0$, $A_0 = g'(0)$ and $A_j = {f_j}'(0)$ for $1 \leqslant j \leqslant m$.*
*Let $\xi(x)$ be the flow generated by the (non-linear) SDE*

$$\begin{cases} \mathrm{d}X_t = g(X_t)\,\mathrm{d}t + \sum_{j=1}^m f_j(X_t) \diamond \mathrm{d}L_t^j, \\ X_0 = x, \end{cases} \tag{5.16}$$

*and let $D\xi(x)$ be the flow generated by the linearised SDE*

$$\begin{cases} \mathrm{d}X_t = A_0 X_t \mathrm{d}t + \sum_{j=1}^m A_j X_t \diamond \mathrm{d}L_t^j, \\ X_0 = x. \end{cases} \tag{5.17}$$

*Furthermore we assume that $D\xi$ is hyperbolic, i.e. all its Lyapunov exponents are non-zero. Then there exists a mapping $\Psi\colon \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m) \times \mathbb{R}^n \to \mathbb{R}^n$ and a measurable set $\Lambda(\omega)$ with $0 \in \Lambda(\omega)$, such that:*

*(i) The mapping $\Psi$ is a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $\Psi(\omega, 0) = 0$, for each $\omega \in \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m)$.*

*(ii) There are stopping times $\tau_- < 0 < \tau_+$, such that*

$$\xi_t(x) = \Psi\left(\theta_t\omega, D\xi_t\left(\Psi^{-1}(\omega, x)\right)\right), \tag{5.18}$$

*for each $\omega \in \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m)$, $x \in \mathbb{R}^n$ and $\tau_-(\omega, x) < t < \tau_+(\omega, x)$.*

*Remark.* In the definition of $D\xi(x)$ the '$D$' is part of the notation, whereas by $D_x f = f'$ we mean the derivative of the function $f$. Apart from that, the function $f'\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a matrix given by $f' = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1}^n$, such that $f'f\colon \mathbb{R}^n \to \mathbb{R}^n$ is the product of the matrix-valued function $f'$ and the vector-valued function $f$.

The main idea of the proof is illustrated in the following figure:
The arrows indicate conjugacy. The dashed arrow is equivalent to the statement of Theorem 5.4. According to Remark 4.9 and Theorem 4.12 it is sufficient to prove the conjugacy of $\psi$ and $D\xi$ (indicated by the solid arrows). Therefore we prove the conjugacy of $\psi$ and $D\psi$ as well as the conjugacy of $D\psi$ and $D\xi$.

$$\begin{tikzcd}
\xi_t \arrow[r, leftrightarrow, "\text{Theorem 4.12}"] \arrow[d, leftrightarrow, dashed, "\text{Theorem 5.4}"'] & \psi_t \arrow[d, leftrightarrow, "\text{Part 2}"', "\text{Lemma 5.5}"] \\
D\xi_t \arrow[r, leftrightarrow, "\text{Part 1}", "\text{Differentiation}"'] & D\psi_t
\end{tikzcd}$$

The idea of the proof is motivated by [Led01, Satz 3.4, Satz 3.6 & Satz 3.7]. We separate the proof in two parts.

*Proof of Theorem 5.4 (Part 1).* Theorem 4.12 gives us $\xi_t(x) = \Phi\left(\theta_t\omega, \psi_t\left(\Phi^{-1}(\omega, x)\right)\right)$. This yields to

$$D\xi_t = \Phi'\left(\theta_t\omega, \psi_t\left(\Phi^{-1}(\omega, x)\right)\right) \cdot D\psi_t\left(\Phi^{-1}(\omega, x)\right) \cdot \left(\Phi'\right)^{-1}(\omega, x). \tag{5.19}$$

Hence $D\xi_t$ and $D\psi_t$ are conjugated due to $\Phi'$, where $\Phi'(\omega, x) = \frac{\partial}{\partial z}\Phi(\omega, z)\big|_{z=x}$.

□

To continue the proof, we use the following lemma:

**Lemma 5.5** (Lederer – Lemma 3.1, p. 25, [Led01]). *Let $\rho$ be a linear cocyle on $\mathbb{R}^d$ with respect to the ergodic dynamical system $\left(\Omega, \mathscr{F}, \mathrm{P}, (\theta_t)_{t\in\mathbb{R}}\right)$ satisfying the integrability condition of the Oseledets theorem*

$$\sup_{0\leqslant t\leqslant 1}\left(\log^+|\rho_t| + \log^+\left|\rho_t^{-1}\right|\right) \in L_1. \tag{5.20}$$

*Moreover, let $\rho$ be hyperbolic with Lyapunov exponents $\lambda_1(\omega), \dots, \lambda_d(\omega)$ and corresponding Oseledets spaces $E_1(\omega), \dots, E_d(\omega)$.*
*Further define $E^+(\omega) := \bigoplus_{\lambda_i>0} E_i(\omega)$ and $E^-(\omega) := \bigoplus_{\lambda_i<0} E_i(\omega)$ with projections $P^+(\omega)$ and $P^-(\omega)$ related to the decomposition $\mathbb{R}^d = E^+(\omega) \oplus E^-(\omega)$. Then there is an $\alpha > 0$ and a random variable $R_\varepsilon\colon \Omega \to [1, \infty)$ with $\varepsilon \in (0, \alpha)$, such that*

*(i) $R_\varepsilon$ satisfies*

$$R_\varepsilon(\theta_t\omega) \leqslant \mathrm{e}^{\varepsilon|t|} R_\varepsilon(\omega). \tag{5.21}$$

*(ii) the following estimates hold true:*

$$\begin{aligned}
\left|\rho_t(\omega) P^+(\omega) \rho_s^{-1}(\theta_s\omega)\right| &\leqslant R_\varepsilon(\theta_t\omega)\,\mathrm{e}^{-\alpha|t-s|}, &\text{if } s \geq t; \qquad (5.22)\\
\left|\rho_t(\omega) P^-(\omega) \rho_s^{-1}(\theta_s\omega)\right| &\leqslant R_\varepsilon(\theta_t\omega)\,\mathrm{e}^{-\alpha|t-s|}, &\text{if } t \geq s. \qquad (5.23)
\end{aligned}$$

*Remark* 5.6. Normally there are different exponential growth/decay rates $\lambda^+$ and $\lambda^-$ being used for the stable and unstable spaces $E^+$ and $E^-$ respectively and indeed, we will distinguish them later on (cf. Section 5.2). For this section it is more convenient to use the same value $\alpha$ for growth and decay.

We define

$$h(\omega, x) := \Phi_0(\omega, x)^{-1}\left[g(\Phi_0(\omega, x)) - \Gamma_0(\omega, x)\right].$$

Then we can rewrite RDE (4.22) according to

$$\dot{Y}_t = h(\theta_t\omega, Y_t).$$

Since $\Phi_0(0) = 0 = \Gamma_t(0)$ we get $h(\theta_t\omega, 0) = 0$. This can be easly seen, due to the uniqueness of the solution of the SDE and since $\Phi_0(0) \equiv 0$ solves equation (4.21). Let $A\colon \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m) \to L(\mathbb{R}^n, \mathbb{R}^n)$ be defined by $A(\omega) := D_x h(\omega, 0)$, $\omega \in \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m)$. Then the solution $D_x\psi_t$ of the linearised RDE solves

$$\dot{D\psi_t} = A(\theta_t\omega) D\psi_t, \tag{5.24}$$

i.e. $D\psi_t$ is the solution of the linearization of RDE (4.22) in the equilibrium state $\psi_t \equiv 0$.

To prove the conjugacy of $\psi$ and $D_x\psi$ we use the following lemma:

**Lemma 5.7.** *Let $f\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function with $f(\omega, 0) = 0$. Furthermore let the mapping $x \mapsto f(\omega, x)$ be bounded and Lipschitz continuous with Lipschitz constant $L(\omega)$, such that*

$$L(\omega) R_\varepsilon(\omega) \leqslant c < \frac{\alpha}{2}, \tag{5.25}$$

*where both $R_\varepsilon$ and $\alpha$ are given by Lemma 5.5. Then $D\psi_t$ and the flow $\phi$ generated by*

$$\dot{Z}_t = A(\theta_t\omega) Z_t + f(\theta_t\omega, Z_t) \tag{5.26}$$

*are conjugated.*

For the sake of clarity we write $\dot{v}(t)$ instead of $\frac{\mathrm{d}v}{\mathrm{d}t}(t)$. Since the right-hand side of (5.26) is possibliy discontinuous, we always consider the integrated version of the RDE. The following proof is based on [Led01, Proof of Theorem 3.4., p. 28 et seq.]:

*Proof of Lemma 5.7.* We define the process $\Psi(t, \omega, x) : \mathbb{R} \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\begin{aligned}\Psi(t, \omega, x) := x &+ \int_t^\infty D\psi_t P^+ \left(D\psi_s\right)^{-1} f\left(\theta_s\omega, \phi_s\phi_t^{-1}x\right)\,\mathrm{d}s \\ &- \int_{-\infty}^t D\psi_t P^- \left(D\psi_s\right)^{-1} f\left(\theta_s\omega, \phi_s\phi_t^{-1}x\right)\,\mathrm{d}s.\end{aligned} \tag{5.27}$$

This process is well-defined due to Lemma 5.5 and (5.25) and it is a Carathéodory function itself. Moreover, $\Psi(t, \omega, 0) = 0$ and

$$\begin{aligned}|\Psi(t, \omega, x) - x| &\leqslant R_\varepsilon(\theta_t\omega)\,\|f\|_\infty \left(\int_t^\infty \mathrm{e}^{-\alpha|t-s|}\,\mathrm{d}s + \int_{-\infty}^t \mathrm{e}^{-\alpha|t-s|}\,\mathrm{d}s\right) \\ &= \frac{2}{\alpha} \cdot R_\varepsilon(\theta_t\omega)\,\|f\|_\infty.\end{aligned} \tag{5.28}$$

Since

$$\Psi(t,\omega,\phi_t) = \phi_t + \big(D\psi_t\big)\Bigg[\int_t^\infty P^+\big(D\psi_s\big)^{-1} f(\theta_s\omega,\phi_s)\,\mathrm{d}s \\ - \int_{-\infty}^t P^-\big(D\psi_s\big)^{-1} f(\theta_s\omega,\phi_s)\,\mathrm{d}s\Bigg] \tag{5.29}$$

we get

$$\begin{aligned}\dot\Psi(t,\omega,\phi_t) &= \dot\phi_t + \big(D\psi_t\big)\Big\{-P^+\big(D\psi_t\big)^{-1} f(\theta_t\omega,\phi_t) - P^-\big(D\psi_t\big)^{-1} f(\theta_t\omega,\phi_t)\Big\}\\ &\quad + \big(\dot{D\psi}\big)_t\left\{\int_t^\infty P^+\big(D\psi_s\big)^{-1} f(s,\phi_s)\,\mathrm{d}s - \int_{-\infty}^t P^-\big(D\psi_s\big)^{-1} f(\theta_s\omega,\phi_s)\,\mathrm{d}s\right\}\\ &= A(\theta_t\omega)\,\phi_t + f(\theta_t\omega,\phi_t) - f(\theta_t\omega,\phi_t)\\ &\quad \underbrace{+A(\theta_t\omega)\big(D\psi_t\big)\left\{\int_t^\infty P^+\big(D\psi_s\big)^{-1} f(\theta_s\omega,\phi_s)\,\mathrm{d}s - \int_{-\infty}^t P^-\big(D\psi_s\big)^{-1} f(\theta_s\omega,\phi_s)\,\mathrm{d}s\right\}}_{=A(\theta_t\omega)\Psi(t,\omega,\phi_t)-A(\theta_t\omega)\phi_t}\\ &= A(\theta_t\omega)\,\Psi(t,\omega,\phi_t)\,.\end{aligned}$$

Now we observe, that

$$\begin{aligned}&\int_t^\infty D\psi_t(\omega)P^+(\omega)\big(D\psi_s\big)^{-1}(\omega) f\left(\theta_s\omega,\phi_s\phi_t^{-1}(\omega)x\right)\,\mathrm{d}s\\ &\quad= \int_0^\infty D\psi_t(\omega)P^+(\omega)\big(D\psi_{t+s}\big)^{-1}(\omega) f\left(\theta_{t+s}\omega,\phi_{t+s}(\omega)\phi_t^{-1}(\omega)x\right)\,\mathrm{d}s\\ &\quad= \int_0^\infty D\psi_t(\omega)P^+(\omega)\big(D\psi_t\big)^{-1}(\omega)\big(D\psi_s\big)^{-1}(\theta_t\omega) f\left(\theta_s\theta_t\omega,\phi_s(\theta_t\omega)\phi_t(\omega)\phi_t^{-1}(\omega)x\right)\,\mathrm{d}s\\ &\quad= \int_0^\infty D\psi_0(\theta_t\omega)P^+(\theta_t\omega)\big(D\psi_s\big)^{-1}(\theta_t\omega) f\left(\theta_s(\theta_t\omega),\phi_s(\theta_t\omega)x\right)\,\mathrm{d}s,\end{aligned}$$

and similarly to this we get the same result for the integral from $-\infty$ to $t$.

Thus we can define $\Psi(\theta_t\omega,x) := \Psi\big(0,\theta_t\omega,x\big)$ by perfection and obtain $D\psi_t = \Psi(\theta_t\omega,\phi_t)$ due to the uniqueness of the solution of the RDE (5.24).

According to [Pro90, Theorem 46 & subsequent Comment, p. 262 et seq.] the mapping $x \mapsto \Psi(\theta_t\omega,x)$ is homeomorphic, if it is injective and moreover $\big|\Psi(\theta_t\omega,x)\big| \to \infty$ as $|x| \to \infty$. The growth condition is satisfied due to equation (5.28). To complete the proof we show injectivity, which then ensures the homeomorphy property of $\Psi$.

Now assume $\Psi(\omega,x_1) = \Psi(\omega,x_2)$ for $x_1,x_2 \in \mathbb{R}^d$ and let $z(t) := z_2(t) - z_1(t)$, where $z_i$ is the solution of (5.26) starting in $x_i$ for $i = 1,2$.

By definition, $z$ solves

$$\dot{z}(t,\omega) = A(\theta_t\omega) + e(t,\theta_t\omega), \text{ where } e(t,\omega) := f(\omega, z_2(t)) - f(\omega, z_1(t)).$$

Further we get

$$\big|e(t,\theta_t\omega)\big| \leqslant L(\theta_t\omega)\big|z(t,\omega)\big|$$

due to the Lipschitz condition on $f$.

Variation of constants leads to

$$z(t) = D\psi_t(\omega)\Big[D\psi_{t_0}(\theta_{t_0}\omega)^{-1}z(t_0,\omega) + \int_{t_0}^{t} D\psi_s(\theta_s\omega)^{-1}e(s,\theta_s\omega)\,\mathrm{d}s\Big].$$

Then we obtain

$$\begin{aligned}\big|D\psi_t(\omega)P^-(\omega)D\psi_t(\theta_t\omega)^{-1}z(t_0)\big| &\leqslant \big|D\psi_t(\omega)P^-(\omega)D\psi_{t_0}(\theta_{t_0}\omega)^{-1}\big|\big|z(t_0)\big| \\ &\quad + \int_{t_0}^{t}\big|D\psi_t(\omega)P^-(\omega)D\psi_s(\theta_s\omega)^{-1}e(s,\theta_s\omega)\big|\,\mathrm{d}s.\end{aligned}$$

By (5.28) we observe, that

$$\begin{aligned}\big|z(t)\big| = \big|z_2(t) - z_1(t)\big| &\leqslant \big|z_2(t) - \Psi(\theta_t\omega, x_2)\big| + \big|z_1(t) - \Psi(\theta_t\omega, x_1)\big| \\ &\leqslant \frac{4}{\alpha}\cdot R_\varepsilon(\theta_t\omega)\|f\|_\infty.\end{aligned}$$

This leads to

$$\begin{aligned}\big|D\psi_t(\omega)P^-(\omega)D\psi_{t_0}(\theta_{t_0}\omega)^{-1}\big|\big|z(t_0)\big| &\leqslant R_\varepsilon(\theta_t\omega)\mathrm{e}^{-\alpha|t-t_0|}\cdot\frac{4}{\alpha}\cdot R_\varepsilon(\theta_{t_0}\omega)\|f\|_\infty \\ &\leqslant \mathrm{e}^{-\alpha|t-t_0|}\cdot\mathrm{e}^{\varepsilon|t-t_0|}\frac{4}{\alpha}\cdot R_\varepsilon(\omega)\|f\|_\infty \to 0 \text{ as } t_0\to-\infty,\end{aligned}$$

since $\varepsilon \in (0,\alpha)$, see Lemma 5.5.

Similarly to (5.28) and (5.29) (with $e$ instead of $f$) we get

$$\begin{aligned}\big|z(t)\big| &\leqslant \big|D\psi_t(\omega)P^+(\omega)D\psi_t^{-1}(\theta_s\omega)z(t)\big| + \big|D\psi_t(\omega)P^-(\omega)D\psi_t^{-1}(\theta_s\omega)z(t)\big| \\ &\leqslant 2R_\varepsilon(\theta_t\omega)\int_{-\infty}^{t}\mathrm{e}^{-\alpha|t-s|}\underbrace{L(\theta_s\omega)R_\varepsilon(\theta_s\omega)}_{\leqslant c}\frac{\big|z(s)\big|}{R_\varepsilon(\theta_s\omega)}\,\mathrm{d}s \leqslant 2R_\varepsilon(\theta_t\omega)\left\|\frac{z(\cdot)}{R(\theta.\omega)}\right\|_\infty\frac{c}{\alpha}.\end{aligned}$$

Conclusively, due to $c < \alpha/2$ we obtain

$$\frac{|z(t)|}{R(\theta_t\omega)} \leqslant \underbrace{\frac{2c}{\alpha}}_{<1}\left\|\frac{z(\cdot)}{R(\theta.\omega)}\right\|_\infty,$$

which implies $z(t) \equiv 0$ and especially $0 = z(0) = x_1 - x_2$, i.e. $\Psi$ is a homeomorphism, and $D\psi$ and $\phi$ are conjugated due to $\Psi$. $\square$

To finish the proof of Theorem 5.4 it is sufficient to apply Lemma 5.7 on the flow generated by (4.22), for $f$ explicitly given by

$$f(\omega, x) := h(\omega, x) - A(\omega)\, x. \tag{5.30}$$

Therefore we have to show, that

(i) the cocycle $D\psi$ is hyperbolic and satisfies the integrability condition of the MET,

(ii) the conditions of Lemma 5.7 are satisfied.

According to [Kun04, Theorem 3.3, p. 342] $D_x\xi$ satisfies the integrability condition of the MET and is hyperbolic as stated in Theorem 5.4.

**Lemma 5.8.** *The mapping $\Phi$ given by Theorem 4.12 satisfies*

$$\mathbb{E}\left[\sup_{0<t\leqslant 1} \log^+ \left|\Phi_t'(x)^{\pm 1}\right|\right] < \infty, \quad \textit{for each } x \in \mathbb{R}^n.$$

*Proof of Lemma 5.8.* As stated in [Kun04, Theorem 3.18, p. 369] the inverse flow $\Phi_t^{-1}(y)$ satisfies the *backward Marcus type SDE*

$$\Phi_t^{-1}(y) = y - \sum_{j=1}^m \int_0^t f_j\left(\Phi_t^{-1}(y)\right) \diamond_b \mathrm{d}Z_s^j \tag{5.31}$$

where

$$Z_t = \int_{-\infty}^t \mathrm{e}^{-(t-s)}\, \mathrm{d}L_s.$$

We get

$$y - \sum_{j=1}^m \int_0^t f_j\left(\Phi_t^{-1}(y)\right) \diamond_b \mathrm{d}Z_s^j = y - \sum_{j=1}^m \int_{-\infty}^t \mathrm{e}^{-(t-s)} f_j\left(\Phi_t^{-1}(y)\right) \diamond_b \mathrm{d}L_s^j$$

similarly to the proof of Theorem 4.10. Hence it is sufficient to show that

$$\mathbb{E}\left[\sup_{0<t\leqslant 1} \log^+ \left|\Phi_t'(x)\right|\right] < \infty, \quad \text{for each } x \in \mathbb{R}^n.$$

Since $\log^+(x) \leqslant |x|^2$ we obtain

$$\mathbb{E}\left[\sup_{0<t\leqslant 1} \log^+ \left|\Phi_t'(x)\right|\right] < \mathbb{E}\left[\sup_{-\infty<s\leqslant 1} \left|\Phi_s'(x)\right|^2\right] < \infty,$$

according to Theorem 5.1. $\square$

Now we can complete the proof of Theorem 5.4:

*Proof of Theorem 5.4 (Part 2).* Since

$$D\xi_t(x) = \Phi'\left(\theta_t\omega, \psi_t\left(\Phi^{-1}(\omega,x)\right)\right) \cdot D\psi_t\left(\Phi^{-1}(\omega,x)\right) \cdot \left(\Phi'\right)^{-1}(\omega,x),$$

we get

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\log|D\xi_t(x)| &= \lim_{t\to\infty}\left[\frac{1}{t}\log\left|\Phi'\left(\theta_t\omega, \psi_t\left(\Phi^{-1}(\omega,x)\right)\right) \cdot D\psi_t\left(\Phi^{-1}(\omega,x)\right) \cdot \left(\Phi'\right)^{-1}(\omega,x)\right|\right] \\ &= \lim_{t\to\infty}\left[\frac{1}{t}\log\left|\Phi'\left(\theta_t\omega, \psi_t\left(\Phi^{-1}(\omega,x)\right)\right)\right|\right] \\ &\quad + \lim_{t\to\infty}\left[\frac{1}{t}\log\left|D\psi_t\left(\Phi^{-1}(\omega,x)\right)\right|\right] + \lim_{t\in\infty}\left[\frac{1}{t}\log\left|\left(\Phi'\right)^{-1}(\omega,x)\right|\right].\end{aligned}$$

According to Theorem 5.1 it is $\mathbb{E}\left[\sup_{-\infty<s\leqslant t}|\Phi'_s(x)|^2\right] < \infty$, for all $x\in\mathbb{R}^n$ and $t\leqslant T$ for some $T\in\mathbb{R}$. Similarly to Lemma 5.8 we can extend this result to $\left|\Phi'_s(x)^{-1}\right|^2$ as well. Since $\Phi'$ is stationary, it is

$$\begin{aligned}\mathrm{P}\left(\left|\frac{1}{n}\log\left|\Phi'(\theta_n\omega,x)^{\pm 1}\right|\right| > \varepsilon\right) &\leqslant \mathrm{P}\left(\left|\Phi'(\theta_n\omega,x)\right|^2 > \mathrm{e}^{2\varepsilon n}\right) \\ &\quad + \mathrm{P}\left(\left|\Phi'(\theta_n\omega,x)^{-1}\right|^2 > \mathrm{e}^{2\varepsilon n}\right) \\ &\leqslant \frac{\mathbb{E}\,|\Phi'(\theta_n\omega,x)|^2}{\mathrm{e}^{2\varepsilon n}} + \frac{\mathbb{E}\left|\Phi'(\theta_n\omega,x)^{-1}\right|}{\mathrm{e}^{2\varepsilon n}} \leqslant \mathrm{c}\mathrm{e}^{-2\varepsilon n}.\end{aligned}$$

Hence we get

$$\sum_{i=1}^{n}\mathrm{P}\left(\left|\frac{1}{n}\log\left|\Phi'(\theta_n\omega,x)^{\pm 1}\right|\right| > \varepsilon\right) \leqslant \mathrm{c}\sum_{n=1}^{\infty}\big(\underbrace{\mathrm{e}^{-2\varepsilon}}_{<1}\big)^n < \infty,$$

for each $\varepsilon > 0$.

Thus, the *Borel–Cantelli lemma* implies that

$$\lim_{t\to\infty}\frac{1}{t}\ln\left|\Phi'\left(\theta_t\omega, Y_t\left(\Phi^{-1}(\omega,x)\right)\right)\right| = 0 = \lim_{t\to\infty}\ln\left|\frac{1}{t}\left(\Phi'\right)^{-1}(\omega,x)\right| \text{ almost surely,}$$

which proves that $D\psi$ satisfies the integrability condition of the MET and is hyperbolic.

The approximation error $f$ given by (5.30) possibly fails to satisfy the conditions of Lemma 5.7 globally. This problem can by avoided by localisation. We localise similar to that in [Led01, Proof of Theorem 3.6]:

We define

$$\chi_r(x) := \begin{cases} x & \text{if } |x| \leqslant r; \\ \frac{r}{|x|}x & \text{if } |x| > r. \end{cases} \tag{5.32}$$

The mapping $\chi_r$ is a orthogonal projection from $\mathbb{R}^n$ onto $B_r(0) := \{x \in \mathbb{R}^n : |x| \leqslant r\}$, which is Lipschitz continuous with Lipschitz constant 1. Now we set

$$\Lambda_r(\omega) := \sup_{|x|<r} \left|D_x f(\omega, x)\right| \quad \text{and} \quad \rho_c := \sup\{r > 0 \colon \Lambda_r(\omega) R_\varepsilon(\omega) \leqslant c\}.$$

We observe, that $\Lambda_r(\omega) \to 0$ as $r \to 0$ for each $\omega \in \Omega$, since $D_x f(\omega, 0) = 0$.
Then the mapping

$$x \mapsto f(\omega, \cdot) \circ \chi_{\rho_c(\omega)}(x) \tag{5.33}$$

is globally Lipschitz with random Lipschitz constant $L(\omega)$, and $L(\omega) \leqslant \Lambda_{\rho_c}(\omega)$ since $\Lambda_{\rho_c}(\omega)$ is an upper bound for the Lipschitz constant thanks to the mean value theorem. Hence we obtain $L(\omega) R_\varepsilon(\omega) \leqslant \Lambda_{\rho_c}(\omega) R_\varepsilon(\omega) \leqslant c$ by definition.
Hence we can apply Lemma 5.7 (locally) on the flow generated by (4.22) and $F(x) := f(\omega, \cdot) \circ \chi_{\rho_c(\omega)}(x)$ instead of $f$, which implies that $D\xi$ and $\psi$ are locally conjugated.

Since $\chi_{\rho_c(\omega)}$ is a random variable and $\Phi_0$ is measurable, the random set $\Theta$ given by

$$\Theta(\omega) = \Phi_0\left(\left\{x \in \mathbb{R}^n \colon |x|^2 \leqslant \rho_c(\omega)\right\}\right) \tag{5.34}$$

is a measurable set, such that $\xi$ and $D\xi$ are conjugated, for each $\omega \in \mathscr{D}_0(\mathbb{R}, \mathbb{R}^m)$ and $x \in \mathbb{R}^n$ locally on $\tau_-(\omega, x) < 0 < \tau_+(\omega, x)$, where

$$\tau_-(\omega, x) = \inf\{t < 0 \colon \xi_s(\omega, x) \in \Theta(\theta_s\omega) \text{ for each } t \leqslant s < 0\} \tag{5.35}$$

$$\tau_+(\omega, x) = \sup\{t > 0 \colon \xi_s(\omega, x) \in \Theta(\theta_s\omega) \text{ for each } 0 < s \leqslant t\}. \tag{5.36}$$

This completes the proof of Theorem 5.4. □

*Remark.* The previous result was motivated by [Led01, Kapitel 3, p. 22 et seqq.]. The main difficulty in the previous proof was to apply Kunitas inequalities in such a way, that we can apply a modified Gronwall's lemma afterwards. In the Brownian setting, these inequalities are satisfied thanks to the Burkholder–Davis–Gundy inequality. We can see, that by the help of Kunita's inequalities we obtain results comparable to those in the Brownian setting.

## 5.2 Random Invariant Manifolds

### Global Manifolds

In the following section we introduce stable and unstable manifolds:

**Definition 5.9.** Let $\varphi$ be a random dynamical system.

(i) A random set $\mathscr{M}(\omega) \subset \mathbb{R}^n$ is called invariant, if

$$\varphi\big(t, \omega, \mathscr{M}(\omega)\big) \subseteq \mathscr{M}(\theta_t\omega) \text{ for each } t \geqslant 0.$$

An invariant set $\mathscr{M}(\omega)$ with graph structure

$$\mathscr{M}(\omega) = \big\{p^+(\omega, u^-) + u^- : u^- \in E^-(\omega)\big\}, \quad p^+ : \Omega \times E^-(\omega) \to E^+(\omega)$$

is called *stable manifold*, if it is exponentially attracting at $\infty$, i.e. $\big|\varphi(t, \omega, x)\big| \to 0$ exponentially fast as $t \to \infty$ for each $x \in \mathscr{M}(\omega)$.

(ii) Similar we call an invariant set $\mathscr{M}(\omega) \subset \mathbb{R}^n$ *unstable manifold*, if

$$\mathscr{M}(\omega) = \big\{u^+ + p^-(\omega, u^+) : u^+ \in E^+(\omega)\big\}, \quad p^- : \Omega \times E^+(\omega) \to E^-(\omega)$$

and $\big|\varphi(t, \omega, x)\big| \to 0$ exponentially fast as $t \to -\infty$ for each $x \in \mathscr{M}(\omega)$ instead.

Roughly speaking, a stable or unstable manifold consists of all points, such that the flow starting in these points stay in the manifold and moreover converge to 0 as $t \to \infty$ or $t \to -\infty$, respectively.

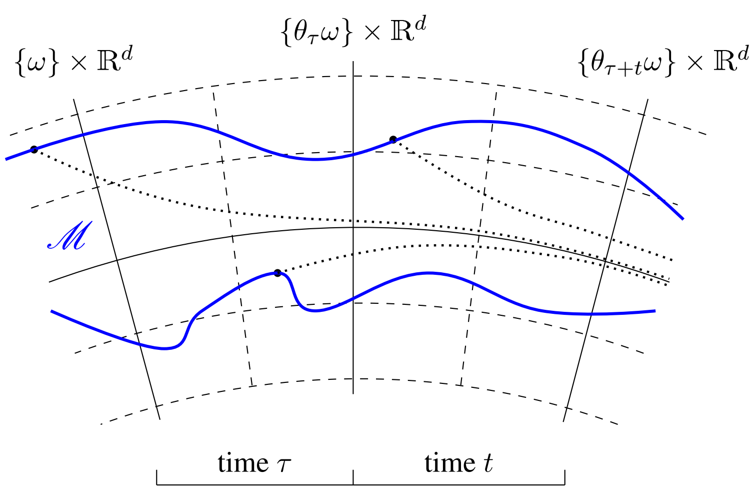



**Figure 5.1:** *Stable Manifold covered by curved thick lines*

We consider the equation (5.26) and the same setting as given in Lemma 5.7, i.e.

$$\dot{Z}_t = A(\theta_t\omega) Z_t + f(\theta_t\omega, Z_t) \tag{5.26}$$

and let $S\colon \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be the solution operator corresponding to $A$.
Based on Theorem 5.5 of the previous section, we can decompose the domain $\mathbb{R}^d$ in the direct sum of the stable and unstable subspaces $E^+$ and $E^-$ with (not necessarily orthogonal) projections $P^+$ and $P^-$. Roughly speaking we separate Oseledets spaces with positive and negative Lyapunov exponents.

To construct invariant manifolds (both stable and unstable) we will apply the Lyapunov–Perron method. It is a very powerful tool, that allows us to solve a fixed point equation, which will then provide a function we can use to characterise the (un)stable manifold. To motivate the Lyapunov–Perron transform we can think of the definition of mild solutions for ODEs or PDEs. To separate the stable and unstable part of the solution, we consider the projections on $E^+$ or $E^-$, respectively. Hence the stable or unstable part of the RDE (5.26) can be represented as fixed point of the projected RDE.

According to the Oseledets theorem 5.2 and Lemma 5.5 there are $\lambda^+ > 0 > \lambda^-$, such that

$$\begin{aligned} \left|S^+(t-s,\theta_s\omega)\xi\right| &\leqslant R_\varepsilon(\theta_t\omega)\,\mathrm{e}^{\lambda^+|t-s|}\,|\xi| \quad \text{if } t<s,\\ \left|S^-(t-s,\theta_s\omega)\xi\right| &\leqslant R_\varepsilon(\theta_t\omega)\,\mathrm{e}^{\lambda^-|t-s|}\,|\xi| \quad \text{if } t>s, \end{aligned} \tag{5.37}$$

where $S^\pm := P^\pm S$ and $\varepsilon > 0$ satisfies $\lambda_N + \varepsilon < \lambda^- < 0 < \lambda^+ < \lambda^+ < \lambda_{N+1} - \varepsilon$. Hence we get

$$\begin{aligned} \left|S^+(t-s,\theta_s\omega)\xi\right| &\to 0 \quad \text{as } s\to\infty,\\ \left|S^-(t-s,\theta_s\omega)\xi\right| &\to 0 \quad \text{as } s\to-\infty. \end{aligned}$$

These results are closely linked to Lemma 5.5, expect for the symmetry privided by Lemma 5.5, which is not convenient in the following anymore. Especially, we set $\gamma := \frac{\lambda^+ + \lambda^-}{2}$ and assume $\gamma > 0$.

Let $u(t) = u^+(t) + u^-(t)$ be a solution of (5.26), where $u^\pm(t) := P^\pm u(t)$. Then we have

$$u^\pm(t) = S^\pm(t-s,\theta_s\omega)u(s) + \int_s^t S^\pm(t-r,\theta_r\omega) f\big(\theta_r\omega, u(r)\big)\,\mathrm{d}r, \tag{5.38}$$

which implies

$$\begin{aligned} u^+(t) &= -\int_t^\infty S^+(t-r,\theta_r\omega) f\big(\theta_r\omega,u(r)\big)\,\mathrm{d}r \text{ as } s\to\infty \quad \text{if} \quad u(0)\in E^- \quad \text{and}\\ u^-(t) &= \int_{-\infty}^t S^-(t-r,\theta_r\omega) f\big(\theta_r\omega,u(r)\big)\,\mathrm{d}r \text{ as } s\to-\infty \quad \text{if} \quad u(0)\in E^+. \end{aligned} \tag{5.39}$$

**Theorem 5.10.** *Let the conditions of Lemma 5.7 be satisfied and moreover, assume $L(\omega)R_\varepsilon(\omega) \leqslant c < \frac{\lambda^+ - \lambda^-}{4}$, then there is an unstable manifold for equation* (5.26) *given by*

$$M^-(\omega) := \left\{u^+ + m(\omega, u^+) \colon u^+ \in E^+\right\}, \tag{5.40}$$

*where $(\omega, u^+) \mapsto m(\omega, u^+) \colon \mathbb{R} \times E^+ \to E^-$ is a Carathéodory function.*

The following proof is based on the Lyapunov–Perron transform, which can be found for example in [Ogr11, p. 58 et seqq.].

*Proof.* Let $\mathscr{H}_\gamma^- := \left\{u \in \mathscr{D}(\mathbb{R}^-, \mathbb{R}) \colon \|u\|_{\mathscr{H}_\gamma^-} < \infty\right\}$ be a Banach space, where $\|u\|_{\mathscr{H}_\gamma^-}$ is given by

$$\|u\|_{\mathscr{H}_\gamma^-} := \sup_{t \in \mathbb{R}^-} \left|\mathrm{e}^{-\gamma t} u(t)\right|.$$

**Step 1.** Let $t \in \mathbb{R}^-$ and $\xi \in E^+$. Then we define the Lyapunov–Perron transform

$$T_{\xi,\omega}^-(y)[t] := S^+(t,\omega)\xi - \int_t^0 S^+(t-r, \theta_r\omega) f\big(\theta_r\omega, y(r)\big)\,\mathrm{d}r \tag{5.41}$$

$$+ \int_{-\infty}^t S^-(t-r, \theta_r\omega) f\big(\theta_r\omega, y(r)\big)\,\mathrm{d}r. \tag{5.42}$$

**Step 2.** The mapping $T_{\xi,\omega}^- \colon \mathscr{H}_\gamma^- \to \mathscr{H}_\gamma^-$ is a contraction:
Using (5.37) and the fact that $f$ is a Carathéodory function, we obtain

$$\begin{aligned}
\left\|T_{\xi,\omega}^-(y_1) - T_{\xi,\omega}^-(y_2)\right\|_{\mathscr{H}_\gamma^-} &= \sup_{t \in \mathbb{R}^-} \mathrm{e}^{-\gamma t} \Big| - \int_t^0 S^+(t-r, \theta_r\omega)\big[f\big(\theta_r\omega, y_1(r)\big) - f\big(\theta_r\omega, y_2(r)\big)\big]\,\mathrm{d}r \\
&\qquad + \int_{-\infty}^t S^-(t-r, \theta_r\omega)\big[f\big(\theta_r\omega, y_1(r)\big) - f\big(\theta_r\omega, y_2(r)\big)\big]\,\mathrm{d}r \Big| \\
&\leqslant \|y_1 - y_2\|_{\mathscr{H}_\gamma^-} \\
&\quad \sup_{t \in \mathbb{R}^-} L(\theta_t\omega) R_\varepsilon(\theta_t\omega) \underbrace{\left(\int_t^0 \mathrm{e}^{(\lambda^+ - \gamma)(t-r)}\,\mathrm{d}r + \int_{-\infty}^t \mathrm{e}^{(\lambda^- - \gamma)(t-r)}\,\mathrm{d}r\right)}_{\leqslant \frac{1}{\lambda^+ - \gamma} - \frac{1}{\lambda^- - \gamma}} \\
&\leqslant \frac{4c}{\lambda^+ - \lambda^-} \|y_1 - y_2\|_{\mathscr{H}_\gamma^-}.
\end{aligned}$$

Since $L(t)R_\varepsilon(\theta_t\omega) \leqslant c < \frac{\lambda^+ - \lambda^-}{4}$ we have $\frac{4c}{\lambda^+ - \lambda^-} < 1$, which implies that $T_{\xi,\omega}^-$ is a contraction for each $\xi \in E^+$ and $\omega \in \Omega$.

**Step 3.** Let $\Gamma(\omega, \xi) \in \mathscr{H}_\gamma^-$ be the (unique) fixed point according to the Banach fixed-point theorem. Now we define $m(\omega, u^+) := P^-\Gamma(\omega, u^+)[0]$ and $M^+$ accordingly. Then $m$ satisfies

$$m(\omega, u^+) = \int_{-\infty}^0 S^-(-r, \theta_r\omega) f\big(\theta_r\omega, \Gamma(\omega, u^+)[r]\big)\,\mathrm{d}r.$$

and (by definition) we have

$$\begin{aligned}\Gamma\big(\omega,u^+\big)[t] &= T_{u^+}\Big(\Gamma\big(\omega,u^+\big)\Big)[t]\\ &= S^+(t,\omega)u^+ - \int_t^0 S^+(t-r,\theta_r\omega)f\big(\theta_r\omega,\Gamma\big(\omega,u^+\big)[r]\big)\,\mathrm{d}r\\ &\quad+\int_{-\infty}^t S^-(t-r,\theta_r\omega)f\big(\theta_r\omega,\Gamma\big(\omega,u^+\big)[r]\big)\,\mathrm{d}r,\end{aligned}$$

which implies that, given $u(\omega,0)=u^+ + m(\omega,u^+)\in M^-$ then $\Gamma(\omega,u^+)[t]\in M^-$ is the unique solution of (5.26) for each $t\in\mathbb{R}^-$. Indeed, according to (5.38) and (5.39) the solution of (5.26) is in $\mathscr{H}_\gamma^-$ if and only if it is a fixed point of the Lyapunov–Perron transform (5.41).

**Step 4.** Let $x\mapsto\phi(t,\omega,x)$ be the flow generated by equation (5.26). To complete the proof it is sufficient to show, that

$$\phi\big(t,\omega,\Gamma(\omega,u^+)[0]\big)=\Gamma\big(\theta_t\omega,P^+\Gamma(\omega,u^+)[0]\big)\,[0], \tag{5.43}$$

for each $t\geqslant 0$.
Let $t\geqslant 0$, $u=u^+ + u^-\in E^+\oplus E^-$ and define

$$\Lambda^u_{t,\omega}[s]:=\begin{cases}\Gamma(\omega,u^+)[t+s], & \text{if } s+t<0;\\ \phi\big(t+s,\omega,\Gamma(\omega,u^+)[0]\big), & \text{if } s+t>0.\end{cases}$$

Then we have to show

$$\Lambda^u_{t,\omega}[s]=\Gamma\Big(\theta_t\omega,P^+\phi\big(t,\omega,\Gamma(\omega,u^+)\big)\Big)[s] \text{ for each } s\leqslant 0, \tag{5.44}$$

which implies $\phi\big(t,\omega,\Gamma(\omega,u^+)\big)\in M^-(\theta_t\omega)$ for each $t\in\mathbb{R}$.

By the definition of $\Gamma$ we obtain

$$\begin{aligned}\Gamma\big(\omega,u^+\big)[s] &= T_{u^+}\big(\Gamma(\omega,u^+)\big)[s]\\ &= S^+(s,\omega)u^+ - \int_s^0 S^+(s-r,\theta_r\omega)f\big(\theta_r\omega,\Gamma(\omega,u^+)[r]\big)\,\mathrm{d}r\\ &\quad+\int_{-\infty}^s S^-(s-r,\theta_r\omega)f\big(\theta_r\omega,\Gamma(\omega,u^+)[r]\big)\,\mathrm{d}r,\end{aligned}$$

where $s\in\mathbb{R}$.

If $-\infty < s < -t$ then we obtain

$$\begin{aligned} P^-\Lambda^u_{t,\omega}[s] &= P^-\Gamma(\omega,u^+)[t+s] \\ &= \int_{-\infty}^{t+s} S^-(t+s-r,\theta_r\omega) f\big(\theta_r\omega, \Gamma\big(\omega,u^+\big)[r]\big)\,\mathrm{d}r \\ &= \int_{-\infty}^{s} S^-(s-r,\theta_{t+r}\omega) f\big(\theta_{r+t}\omega, \underbrace{\Gamma\big(\omega,u^+\big)[r+t]}_{=\Lambda^u_{t,\omega}[r]}\big)\,\mathrm{d}r. \end{aligned} \tag{5.45}$$

Apart from that, we observe

$$\begin{aligned} P^+\Lambda^u_{t,\omega}[s] &= P^+\Gamma(\omega,u^+)[t+s] \\ &= S^+(t+s,\omega)u^+ - \int_{t+s}^{0} S^+(t+s-r,\theta_r\omega) f\big(\theta_r\omega, \Gamma\big(\omega,u^+\big)[r]\big)\,\mathrm{d}r \\ &= S^+(s,\theta_t\omega)\bigg(S(t,\omega)u^+ + \int_0^t S^+(t-r,\theta_r\omega) f\Big(\theta_r\omega, \phi\big(r,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\,\mathrm{d}r\bigg) \\ &\quad - \int_{t+s}^{0} S^+(t+s-r,\theta_r\omega) f\big(\theta_r\omega, \Gamma\big(\omega,u^+\big)[r]\big)\,\mathrm{d}r \\ &\quad - \int_0^t S^+(t+s-r,\theta_r\omega) f\Big(\theta_r\omega, \phi\big(r,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\,\mathrm{d}r \\ &= S^+(s,\theta_t\omega)P^+\phi\big(t,\omega,\Gamma(\omega,u^+)[0]\big) \qquad\qquad (5.46) \\ &\quad - \int_s^{-t} S^+(s-r,\theta_r\omega) f\big(\theta_{r+t}\omega, \underbrace{\Gamma\big(\omega,u^+\big)[r+t]}_{=\Lambda^u_{t,\omega}[r]}\big)\,\mathrm{d}r \\ &\quad - \int_{-t}^{0} S^+(s-r,\theta_r\omega) f\Big(\theta_{r+t}\omega, \underbrace{\phi\big(r+t,\omega,\Gamma(\omega,u^+)[0]\big)}_{=\Lambda^u_{t,\omega}[r]}\Big)\,\mathrm{d}r \\ &= S^+(s,\theta_t\omega)P^+\phi\big(t,\omega,\Gamma(\omega,u^+)[0]\big) - \int_s^0 S^+(s-r,\theta_{t+r}\omega) f\big(\theta_{r+t}\omega, \Lambda^u_{t,\omega}[r]\big)\,\mathrm{d}r. \end{aligned} \tag{5.47}$$

Now assume $-t < s < 0$. According to (5.39) we obtain

$$\begin{aligned}
P^-\Lambda^u_{t,\omega}[s] &= P^-\phi\big(t+s,\omega,\Gamma(\omega,u^+)[0]\big)\\
&= S(t+s,\omega)P^-\Gamma(\omega,u^+)[0] \qquad (5.48)\\
&\quad + \int_0^{t+s} S^-(t+s-r,\theta_r\omega) f\Big(\theta_r\omega, \phi\big(r,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\,\mathrm{d}r\\
&= S(t+s,\omega)\int_{-\infty}^0 S^-(-r,\theta_r\omega) f\big(\theta_r\omega,\Gamma(\omega,u^+)[r]\big)\,\mathrm{d}r\\
&\quad + \int_0^{t+s} S^-(t+s-r,\theta_r\omega) f\Big(\theta_r\omega, \phi\big(r,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\,\mathrm{d}r\\
&= \int_{-\infty}^{-t} S^-(s-r,\theta_{t+r}\omega) f\big(\theta_{r+t}\omega, \underbrace{\Gamma(\omega,u^+)[r+t]}_{=\Lambda^u_{t,\omega}[r]}\big)\,\mathrm{d}r\\
&\quad + \int_{-t}^{s} S^-(s-r,\theta_{t+r}\omega) f\Big(\theta_{r+t}\omega, \underbrace{\phi\big(r+t,\omega,\Gamma(\omega,u^+)[0]\big)}_{=\Lambda^u_{t,\omega}[r]}\Big)\,\mathrm{d}r\\
&= \int_{-\infty}^{s} S^-(s-r,\theta_{t+r}\omega) f\big(\theta_{r+t}\omega, \Lambda^u_{t,\omega}[r]\big)\,\mathrm{d}r. \qquad (5.49)
\end{aligned}$$

Similar to (5.48) we get

$$\begin{aligned}
P^+\Lambda^u_{t,\omega}[s] &= P^+\phi\big(t+s,\omega,\Gamma(\omega,u^+)[0]\big)\\
&= S(t+s,\omega)P^+\Gamma(\omega,u^+)[0] \qquad (5.50)\\
&\quad + \int_0^{t+s} S^+(t+s-r,\theta_r\omega) f\Big(\theta_r\omega, \phi\big(r,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\,\mathrm{d}r\\
&= S(t+s,\omega)u^+ + \int_0^{t+s} S^+(t+s-r,\theta_r\omega) f\Big(\theta_r\omega, \phi\big(r,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\,\mathrm{d}r\\
&= S^+(s,\theta_t\omega)\bigg(S(t,\omega)u^+ + \int_0^t S^+(t-r,\theta_r\omega) f\Big(\theta_r\omega, \phi\big(r+t,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\bigg)\\
&\quad + \int_t^{t+s} S^+(t+s-r,\theta_r\omega) f\Big(\theta_r\omega, \phi\big(r,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\,\mathrm{d}r\\
&= S^+(s,\theta_t\omega)P^+\phi\big(t,\omega,\Gamma(\omega,u^+)[0]\\
&\quad - \int_s^0 S^+(s-r,\theta_{t+r}\omega) f\Big(\theta_{r+t}\omega, \phi\big(r+t,\omega,\Gamma(\omega,u^+)[0]\big)\Big)\,\mathrm{d}r. \qquad (5.51)
\end{aligned}$$

Using (5.45)–(5.50) we derive, that $\Lambda^u_{t,\omega}$ is a fixed point of the Lyapunov–Perron transform $T^-_{\xi,\theta_t\omega}$, where $\xi = \phi\big(t,\omega,\Gamma(\omega,u^+)[0]\big)$, which implies (5.44).

Now let $u \in M^-$ be a solution of 5.26. Then equation (5.44) implies $\phi(t,\omega,u) \in \mathscr{H}^-_\gamma$ and moreover, $\big|\varphi(t,\omega,u)\big| \to 0$ exponentially fast as $t \to -\infty$. Hence $M^-$ is the unstable manifold and the proof is done. $\square$

Similar to Theorem 5.10 we can obtain the stable manifold as well:

**Theorem 5.11.** *Let the conditions of Theorem 5.10 be satisfied and moreover, assume $\gamma < 0$. Then the stable manifold for equation* (5.26) *is given by*

$$M^+(\omega) := \left\{p(\omega, u^-) + u^- : u^- \in E^-\right\}, \tag{5.52}$$

*where $p$ is a Carathéodory function.*

The proof is closely related to that of Theorem 5.10.

*Proof.* Let $\mathscr{H}_\gamma^+ := \left\{u \in \mathscr{D}(\mathbb{R}^+, \mathbb{R}) : \|u\|_{\mathscr{H}_\gamma^+} < \infty\right\}$ be a Banach space, where $\|u\|_{\mathscr{H}_\gamma^+}$ is given by

$$\|u\|_{\mathscr{H}_\gamma^+} := \sup_{t\in\mathbb{R}^+} \left|\mathrm{e}^{|\gamma|t} u(t)\right|.$$

**Step 1.** Let $t \in \mathbb{R}^+$ and $\xi \in E^-$. Then we define the Lyapunov–Perron transform

$$T_{\xi,\omega}^+(y)[t] := S^-(t,\omega)\xi + \int_0^t S^-(t-r, \theta_r\omega) f\big(\theta_r\omega, y(r)\big)\,\mathrm{d}r \tag{5.53}$$

$$- \int_t^\infty S^+(t-r, \theta_r\omega) f\big(\theta_r\omega, y(r)\big)\,\mathrm{d}r. \tag{5.54}$$

**Step 2.** The mapping $T_{\xi,\omega}^+ : \mathscr{H}_\gamma^+ \to \mathscr{H}_\gamma^+$ is a contraction. The calculations are the same as for $T_{\xi,\omega}^-$ in **Step** 2 from the proof of Theorem 5.10:

$$\begin{aligned}
\left\|T_{\xi,\omega}^+(y_1) - T_{\xi,\omega}^+(y_2)\right\|_{\mathscr{H}_\gamma^+} &= \sup_{t\in\mathbb{R}^+} \mathrm{e}^{|\gamma|t} \bigg| \int_0^t S^-(t-r,\theta_r\omega)\big[f\big(\theta_r\omega, y_1(r)\big) - f\big(\theta_r\omega, y_2(r)\big)\big]\,\mathrm{d}r \\
&\qquad - \int_t^\infty S^+(t-r,\theta_r\omega)\big[f\big(\theta_r\omega, y_1(r)\big) - f\big(\theta_r\omega, y_2(r)\big)\big]\,\mathrm{d}r \bigg| \\
&\leqslant \|y_1 - y_2\|_{\mathscr{H}_\gamma^+} \\
&\qquad \sup_{t\in\mathbb{R}^+} L(\theta_t\omega) R_\varepsilon(\theta_t\omega) \underbrace{\left(\int_0^t \mathrm{e}^{(\lambda^- - \gamma)(t-r)}\,\mathrm{d}r - \int_t^\infty \mathrm{e}^{(\lambda^+ - \gamma)(t-r)}\,\mathrm{d}r\right)}_{\leqslant -\frac{1}{\lambda^- - \gamma} + \frac{1}{\lambda^+ - \gamma}} \\
&\leqslant \underbrace{\frac{4c}{\lambda^+ - \lambda^-}}_{<1} \|y_1 - y_2\|_{\mathscr{H}_\gamma^+}.
\end{aligned}$$

Now let $\Gamma(\omega, \xi) \in \mathscr{H}_\gamma^+$ denote the unique fixed point of $T_{\xi,\omega}^+$ once again. Similarly to **Step** 3 & **Step** 4 from the proof of Theorem 5.10 we obtain

$$\phi\big(t, \omega, \Gamma(\omega, u^-)\big) = \Gamma\big(\theta_t\omega, P^-\Gamma(\omega, u^-)[0]\big)[0], \tag{5.55}$$

for each $t < 0$.
Moreover, assume $u \in M^+$ is a solution of (5.26), then we have $\left|\varphi\big(t, \omega, u\big)\right| \to 0$ exponentially fast as $t \to \infty$, which completes the proof. □

**Local Manifolds**

Let $f\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function, such that $x \mapsto f(\omega, x)$ is continuously differentiable with $D_x f(\omega, 0) = 0$ and Lipschitz continuous with Lipschitz constant $L(\omega)$, for each $\omega \in \Omega$. Then we cannot apply Theorem 5.10 or Theorem 5.11 directly. We consider a cut-off instead, which ensures existence of a global manifold for the modified system. Then we show, that the global manifold for the cut-off system is a local manifold for the original system:

We focus on the stable case. Definitions, results and proofs for the unstable case are similar.

**Definition 5.12.** Let $\varphi$ be a random dynamical system. The random set $\mathscr{M}(\omega)$ is called local stable manifold, if the following properties are satisfied:

(i) The set $\mathscr{M}(\omega)$ is locally invariant, i.e.

$$\lim_{|x|\to 0} \tau(\omega, x) = \infty,$$

for each $\omega \in \Omega$ and $x \in \mathscr{M}(\omega)$, where $\tau(\omega, x) := \inf\big\{t > 0\colon \varphi(t, \omega, x) \notin \mathscr{M}(\theta_t\omega)\big\}$.

(ii) There is a random radius $r(\omega) > 0$, such that

(a) for each $x \in \mathscr{M}(\omega) \cap B_{r(\omega)}(0)$ we have

$$\lim_{t\to\infty} \big|\varphi(t, \omega, x)\big| \to 0,$$

where $B_{r(\omega)}(0) = \big\{x \in \mathbb{R}^n\colon |x| < r(\omega)\big\}$.

(b) $\mathscr{M}$ has a local graph structure

$$\mathscr{M}(\omega) = \big\{p^+(\omega, u^-) + u^-\colon u^- \in B_{r(\omega)}(0) \cap E^-\big\},$$

where $p^+(\omega)\colon B_{r(\omega)}(0) \cap E^-(\omega) \to E^+(\omega)$.

To apply Theorem 5.10 we have to cut off the function $f$ accordingly to obtain the gap condition $L(\omega)R_\varepsilon(\omega) \leqslant c < \frac{\lambda^+ - \lambda^-}{4}$. Similar to (5.33) we consider $f \circ \chi_{\rho_c^{\mathrm{loc}}(\omega)}$, where

$$\Lambda_r(\omega) := \sup_{|x|<r} \big|D_x f(\omega, x)\big| \quad \text{and} \quad \rho_c^{\mathrm{loc}} := \sup\big\{r > 0\colon \Lambda_r(\omega)R_\varepsilon(\omega) \leqslant c\big\}.$$

We observe, that $\Lambda_r(\omega) \to 0$ as $r \to 0$ for each $\omega \in \Omega$, since $D_x f(\omega, 0) = 0$.
Then the mapping

$$x \mapsto f(\omega, \cdot) \circ \chi_{\rho_c^{\mathrm{loc}}(\omega)}(x) \tag{5.56}$$

is globally Lipschitz with random Lipschitz constant $L(\omega) \leqslant \Lambda_{\rho_c^{\mathrm{loc}}}(\omega)$, where $L(\omega)R_\varepsilon(\omega) \leqslant c$.

**Theorem 5.13.** *Assume that $\rho_c^{\mathrm{loc}}$ is subexponentially growing from below, i.e. for each $\epsilon > 0$ we have*

$$\mathrm{C}_\epsilon(\omega)\mathrm{e}^{-\epsilon|t|} \leqslant \rho_c^{\mathrm{loc}}(\theta_t\omega).$$

*Let $\mathscr{M}^{\mathrm{loc}}$ be the global stable manifold for the system $\phi_c$ generated by $f \circ \chi_{\rho_c^{\mathrm{loc}}}$ given by Theorem 5.11. Then $\mathscr{M}^{\mathrm{loc}}$ is a local manifold for the system $\phi$ generated by $f$.*

*Proof.* We have $\left|\varphi\big(t,\omega,u^-\big)\right| \to 0$ exponentially fast as $t \to \infty$ for each $u^- \in \mathscr{M}^{\mathrm{loc}}$. Other than that, $\rho_c^{\mathrm{loc}}$ is subexponentially growing from below, which implies $\rho_c^{\mathrm{loc}}(\theta_t\omega) \geqslant \mathrm{C}_\epsilon(\omega)\mathrm{e}^{-\epsilon|t|}$ for each $\epsilon > 0$. Hence we find a radius $r(\omega) > 0$ (sufficiently small), such that $\phi(t+\tau,\omega,u^-) \in B_{\rho_c(\theta_{t+\tau}\omega)}(0)$ for each $\tau > 0$ if $u^- \in \mathscr{M}^{\mathrm{loc}} \cap B_{r(\omega)}(0)$. On $B_{\rho_c^{\mathrm{loc}}}$ the systems $\phi$ and $\phi_c$ coincide. This gives us $\phi\big(t,\omega,\Gamma(\omega,u^-)[0]\big) \in \mathscr{M}^{\mathrm{loc}}(\theta_t\omega)$ for each $u^- \in \mathscr{M}^{\mathrm{loc}} \cap B_{r(\omega)}(0)$, which proves (i) and (ii) of Definition 5.12 for local manifolds. □

## 5.3 Random Foliations

In the final part of this of this section we prove the existence of a *folitation* of a stable manifold, i.e. a decomposition into disjoint submanifolds (called *leaves*), where we collect all points of the stable manifold, such that the solutions starting on the same leaf approach each other exponentially fast. Therefore we can use the Lyapunov–Perron method again:

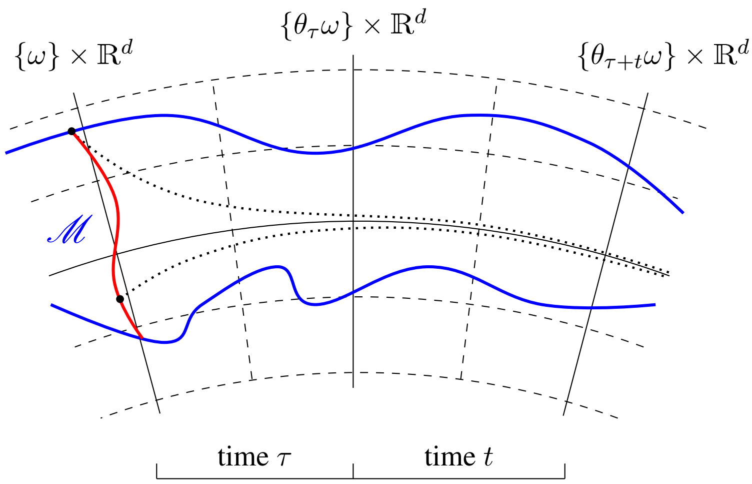



**Figure 5.2:** *A single leaf of the foliated stable manifold*

We fix $u_0 = u_0^+ + u_0^- \in E^+ \oplus E^-$ and consider the solution $\phi(t,\omega,u_0)$ of (5.26). Let

$$W^+(\omega,u_0) := \Big\{u \in \mathbb{R}^d \colon \phi(\cdot,\omega,u) - \phi(\cdot,\omega,u_0) \in \mathscr{H}_\gamma^+\Big\} \tag{5.57}$$

be the set of starting points $u$, such that $\left|\phi(t,\omega,u) - \phi(t,\omega,u_0)\right| \to 0$ as $t \to \infty$ exponentially fast with rate $\gamma$.

**Theorem 5.14.** *Let the conditions of Theorem 5.11 be satisfied. Moreover let* $L(\omega)R_\varepsilon(\omega) \leqslant c < \frac{\lambda^+ - \lambda^-}{6}$*. Then there is a random invariant folitation for* (5.26) *and the corresponding leaves are given by* $W^+(\omega, u_0)$*. Moreover, we have*

$$W^+(\omega, u_0) = \Big\{ p^{\mathrm{fol}}(u^-, \omega, u_0) + u^- \colon u^- \in E^- \Big\}, \tag{5.58}$$

*where* $p^{\mathrm{fol}}$ *is a Carathéodory function.*

*Proof.* First we collect all solution $\phi(t)$ of (5.26), such that

$$y(t) := \phi(t) - \phi(t, \omega, u_0) \in \mathscr{H}_\gamma^+, \tag{5.59}$$

from which we deduce $u \in W^+$ if and only if $u = y(0) + u_0$.

**Step 1.** We use the Lyapunov–Perron transform once again. Let

$$\begin{aligned} T^{\mathrm{fol}}_{\xi,\omega}(y)[t] := {} & S^-(t,\omega)(\xi - u_0^-) \\ & + \int_0^t S^-(t-r, \theta_r\omega) \Big[ f\big(\theta_r\omega, y(r) + \phi(r,\omega,u_0)\big) - f\big(\theta_r\omega, \phi(r,\omega,u_0)\big) \Big] \,\mathrm{d}r \\ & - \int_t^\infty S^+(t-r, \theta_r\omega) \Big[ f\big(\theta_r\omega, y(r) + \phi(r,\omega,u_0)\big) - f\big(\theta_r\omega, \phi(r,\omega,u_0)\big) \Big] \,\mathrm{d}r \end{aligned} \tag{5.60}$$

Similar to **Step** 2 in the proof of Theorem 5.11 the mapping $T^{\mathrm{fol}}_{\xi,\omega} \colon \mathscr{H}_\gamma^+ \to \mathscr{H}_\gamma^+$ is a contraction, and thus there is a unique fixed point $\Gamma^{\mathrm{fol}}(u_0^-, \omega, \xi)$.

Thus we get the equality of (5.57) and (5.58) similar to **Step** 3 & **Step** 4 from the proof of Theorem 5.10, where $p^{\mathrm{fol}}(u^-, \omega, u_0)$ is now given by

$$p^{\mathrm{fol}}(u^-, \omega, u_0) := u_0^+ + P^+ \Gamma^{\mathrm{fol}}(u^- - u_0^-, \omega, u_0)[0].$$

**Step 2.** To complete the proof, we show that $W^+(\omega, u_0) \cap M^-(\omega)$ is a singleton, i.e. it consists of exactly one point:

Assume $u \in W^+(\omega, u_0) \cap M^-(\omega)$. Then we have $u^+ + m(\omega, u^+) = u = p^{\mathrm{fol}}(u^-, \omega, u_0) + u^-$, which implies $u^+ = p^{\mathrm{fol}}(u^-, \omega, u_0)$ and $u^- = m(\omega, u^+)$. Due to uniqueness we obtain $u^+ = p^{\mathrm{fol}}\big(m(\omega, u^+), \omega, u_0\big)$. Hence it is sufficient to prove, that the mapping $u^+ \mapsto p^{\mathrm{fol}}\big(m(\omega, u^+), \omega, u_0\big)$ is a contraction.

Let $\Gamma(\omega, \xi)$ be the fixed point according to the Lyapunov–Perron transform $T^-_{\xi,\omega}$ and consider

$u_1^+, u_2^+ \in E^+$. Then we have

$$\begin{aligned}
&\left\|\Gamma(\omega, u_1^+) - \Gamma(\omega, u_2^+)\right\|_{\mathscr{H}_\gamma^-} \\
&\quad\leqslant \left\| T^-_{u_1^+,\omega}\big(\Gamma(\omega, u_1^+)\big) - T^-_{u_2^+,\omega}\big(\Gamma(\omega, u_1^+)\big) \right\|_{\mathscr{H}_\gamma^-} \\
&\qquad\qquad + \left\| T^-_{u_2^+,\omega}\big(\Gamma(\omega, u_1^+)\big) - T^-_{u_2^+,\omega}\big(\Gamma(\omega, u_2^+)\big) \right\|_{\mathscr{H}_\gamma^-} \\
&\quad\leqslant \left\| T^-_{u_1^+,\omega}\big(\Gamma(\omega, u_1^+)\big) - T^-_{u_2^+,\omega}\big(\Gamma(\omega, u_1^+)\big) \right\|_{\mathscr{H}_\gamma^-} \\
&\qquad\qquad + \frac{4c}{\lambda^+ - \lambda^-} \left\|\Gamma(\omega, u_1^+) - \Gamma(\omega, u_2^+)\right\|_{\mathscr{H}_\gamma^-} .
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
\left\|\Gamma(\omega, u_1^+) - \Gamma(\omega, u_2^+)\right\|_{\mathscr{H}_\gamma^-} &\leqslant \frac{\lambda^+ - \lambda^-}{\lambda^+ - \lambda^- - 4c} \left\| T^-_{u_1^+,\omega}\big(\Gamma(\omega, u_1^+)\big) - T^-_{u_2^+,\omega}\big(\Gamma(\omega, u_1^+)\big) \right\|_{\mathscr{H}_\gamma^-} \\
&\leqslant \frac{\lambda^+ - \lambda^-}{\lambda^+ - \lambda^- - 4c} \left|u_1^+ - u_2^+\right| ,
\end{aligned}$$

which leads to

$$\begin{aligned}
\left|m(\omega, u_1^+) - m(\omega, u_2^+)\right| &= \left| \int_{-\infty}^0 S^-(-r, \theta_r\omega)\Big[ f\big(\theta_r\omega, \Gamma(\omega, u_1^+)[r]\big) - f\big(\theta_r\omega, \Gamma(\omega, u_2^+)[r]\big)\Big] \,\mathrm{d}r \right| \\
&\leqslant c \int_{-\infty}^0 \mathrm{e}^{-(\lambda^- - \gamma)r} \,\mathrm{d}r \left\|\Gamma(\omega, u_1^+) - \Gamma(\omega, u_2^+)\right\|_{\mathscr{H}_\gamma^-} \\
&\leqslant \frac{2c}{\lambda^+ - \lambda^-} \left\|\Gamma(\omega, u_1^+) - \Gamma(\omega, u_2^+)\right\|_{\mathscr{H}_\gamma^-} \\
&\leqslant \frac{2c}{\lambda^+ - \lambda^- - 4c} \left|u_1^+ - u_2^+\right| .
\end{aligned}$$

Since $c < \frac{\lambda^+ - \lambda^-}{6}$ we have $\frac{\lambda^+ - \lambda^-}{2c} - 2 > 1$, which implies $\frac{2c}{\lambda^+ - \lambda^- - 4c} < 1$ and $u^+ \mapsto m(\omega, u^+)$ is Lipschitz continuous with constant $\frac{2c}{\lambda^+ - \lambda^- - 4c} < 1$. Analogously we get a similar result for $u^- \mapsto p^{\mathrm{fol}}(u^-, \omega, u_0)$. Hence $u^+ \mapsto p^{\mathrm{fol}}\big(m(\omega, u^+), \omega, u_0\big)$ is a contraction and the proof is done. $\square$

# 6 Marcus type SDEs with memory

*“Reductio ad absurdum, which Euclid loved so much, is one of a mathematician’s finest weapons. It is a far finer gambit than any chess play: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game."*
— GODFREY HAROLD HARDY, *A Mathematician’s Apology*

## 6.1 Marcus type Stochastic Delay Differential Equations

Let $x \in \mathscr{D}\big([-\alpha,\infty),\mathbb{R}^n\big)$. Then the *memory* $x_t\colon [-\alpha,0] \to \mathbb{R}^n$ of $x$ is given by $x_t(s) := x(t+s)$, where $s \in [-\alpha,0]$. In this work we focus on delay with on a finite interval $[-\alpha,0]$ for $\alpha > 0$.

Up to this point we occasionally wrote $X_s$ or $L_s$ with subscript time variable $s$ instead of $X(s)$ or $L(s)$, respectively. It was convenient to separate the time variable from other parameters for the sake of clarity. From now onwards $X(s)$ and $X_s$ are strictly different objects in different spaces.

A Marcus type stochastic delay differential equation (MSDDE) is an extension of the ordinary MSDE (2.7), now depending on its memory as well. Even the slightest delay in the diffusion part $f$ will change the behavior of the solution drastically, see [Moh84, Section 5.3, p. 144 et seqq.]. Hence we consider MSDDEs without delay in the stochastic integral, more precisely

$$\begin{aligned} X(t) &= X(0) + \int_0^t f\big(s, X(s)\big) \diamond \mathrm{d}L(s) + \int_0^t g\big(X(s), X_s\big)\,\mathrm{d}s \quad \text{for} \quad t \in [0,T], \\ X(t) &= \eta(t) \quad \text{for} \quad t \in [-\alpha,0], \end{aligned} \tag{6.1}$$

where $f\colon \mathbb{R}^n \times \mathscr{D} \to \mathbb{R}^{n\times m}$, $g\colon \mathbb{R}^n \times \mathscr{D} \to \mathbb{R}^n$ and $\eta \in \mathscr{D}\big([-\alpha,0],\mathbb{R}^n\big)$.

### Linear Delay

Introductory we consider linear delay, which is somehow the most regular delay we can get:
Let $X$ be the solution of the MSDDE

$$\begin{aligned} X(t) &= X(0) + \int_0^t f\big(X(s), X_s\big) \diamond \mathrm{d}L(s) + \int_0^t \int_{[-\alpha,0]} X(s+\tau)\,\mu(\mathrm{d}\tau)\mathrm{d}s \quad \text{for} \quad t \in [0,T], \\ X(t) &= \eta(t) \quad \text{for} \quad t \in [-\alpha, 0], \end{aligned} \tag{6.2}$$

where $\mu$ is a signed measure.
Lévy driven Itô type stochastic delay differential equations with linear delay was deeply analysed by Reis et al. in [RRvG06] (existence, uniqueness, stationarity, Feller property). Based on Lemma 2.9 the same techniques can be applied for MSDDEs as well. We will illustrate these methods without going into details.

The approach of Reis et al. is based on the methods for deterministic delay differential equation using the fundamental solution, see [HL93, Section 1.5, p. 18 et seqq.]. The basic idea goes back to solving ordinary differential equation. First we solve a homogeneous part of an ODE and then use the variation-of-constants formula to get a general solution:
Accordingly, we first solve the homogenous part of the delay equation, which is given by (6.3). Then we apply a modification of the variation-of-constants formula with help of a modification of Fubini's theorem for stochastic integrals and finite signed measures, cf. [RRvG07, Theorem 2.5 & Lemma 6.1].

In particular, let $r\colon \mathbb{R} \to \mathbb{R}$ be the fundamental solution of the delay part of (6.2), i.e.

$$\begin{aligned} r(t) &= r(0) + \int_0^t \int_{[-\alpha,0]} r(s+\tau)\,\mu(\mathrm{d}\tau)\,\mathrm{d}s \quad \text{for} \quad t \in [0,T], \\ r(t) &= 0 \quad \text{for} \quad t \in [-\alpha, 0) \quad \text{and} \quad r(0) = 1. \end{aligned} \tag{6.3}$$

Then we can solve (6.3) for general initial sequence $\eta$ by the use of

$$x(t) := \eta(0) r(t) + \int_{[-\alpha,0]} \int_0^t r(t+\tau-s)\eta(s)\,\mathrm{d}s\,\mu(\mathrm{d}\tau),$$

i.e. $x$ satisfies

$$\begin{aligned} x(t) &= x(0) + \int_0^t \int_{[-\alpha,0]} x(s+\tau)\,\mu(\mathrm{d}\tau)\mathrm{d}s \quad \text{for} \quad t \in [0,T], \\ x(t) &= \eta(t) \quad \text{for} \quad t \in [-\alpha, 0]. \end{aligned}$$

Finally we can reformulate the variation-of-constants formula [RRvG06, Theorem 3.1, p. 1414] for MSDDEs instead of Itô type SDDEs. More precisely, $X$ solves (6.2) if and only if

$$\begin{aligned} X(t) &= x(t) + \int_0^t r(t-s) f\big(X(s)\big) \diamond \mathrm{d}L(s) \quad \text{for} \quad t \in [0,T], \\ X(t) &= \eta(t) \quad \text{for} \quad t \in [-\alpha, 0]. \end{aligned} \tag{6.4}$$

This way we can get existence and uniqueness for sufficiently smooth functions $f$ and regular measures $\mu$.

### General Delay

Opposite to the approach by Reis et al., there is a different method based on Mohammed and Scheutzow [MS03], which ensures existence and uniqueness for non-linear delays in the case of Stratonovich type stochastic delay differential equations. This technique does not work for Itô type SDDEs in general. We adapt this method for MSDDEs. From now onwards we consider the following MSDDE:

$$\begin{aligned} X(t) &= x + \int_0^t G\big(s, X(s)\big) \diamond \mathrm{d}L(s) + \int_0^t H\big(s, X(s), X_s\big)\,\mathrm{d}s \quad \text{for} \quad t \in [0,T], \\ X(t) &= \eta(t) \quad \text{for each} \quad t \in [-\alpha, 0]. \end{aligned} \tag{6.5}$$

To deal with this type of delay we will represent the solution of the MSDDE as the composition of the flow generated by a Lévy driven MSDE without delay and the solution of a delayed RDE, which ensures existence and uniqueness. Moreover, we can prove that MSDDEs generate a stochastic semiflow as well.

To prove existence and uniqueness we will use a fixed point argument. Therefore we have to verify several estimates. To obtain the flow property for MSDEs without delay, let $G$ satisfy the assumptions (L) and ($\mathrm{C}^1$). According to [KPP95, Theorem 3.7, p. 361], the flow $x \mapsto \psi(t,x,\omega)$ given by $\mathrm{d}\psi(t) = G\big(t,\psi(t)\big) \diamond \mathrm{d}L(t)$ with $\psi(0) = x \in \mathbb{R}^n$ defines a diffeomorphism. We define the inverse $\zeta(t,x,\omega) := \psi(t,\cdot,\omega)^{-1}[x]$. Both $\psi$ and $\zeta$ are $\mathscr{B}(\mathbb{R}^+) \otimes \mathscr{B}\big(\mathbb{R}^n\big) \otimes \mathscr{F}$–$\mathscr{B}\big(\mathbb{R}^n\big)$-measurable mappings. Finally we fix the assumptions on $H$.

**Assumption** (D). We assume, that

[i] $\mathbb{R}^n \times \mathscr{D} \ni (x,\eta) \mapsto H(\cdot,x,\eta) \in \mathscr{D}$ is a jointly continuous mapping,

[ii] $\mathbb{R}^n \times \mathscr{D} \ni (x,\eta) \mapsto H(t,x,\eta) \in \mathbb{R}^n$ is uniformly (w.r.t. $t$) Lipschitz continuous and bounded

[iii] $\big|H(t,x,\eta)\big| \leqslant \mathrm{C}\big(1 + \big\|(x,\eta)\big\|_{\mathscr{H}_2}\big)$ for $\mathrm{C} > 0$ (independent of $t$), $x \in \mathbb{R}^n$ and $\eta \in \mathscr{D}$, where $\mathscr{H}_2 := \mathbb{R}^n \times \mathscr{D}$ and $\|(x,\eta)\|_{\mathscr{H}_2} = \big(|x|^2 + \|\eta\|_\infty^2\big)^{1/2}$.

*Remark.* When we know $\eta(t)$ for each $t \in [-\alpha, 0]$, then it is not necessary to consider $x = \eta(0)$ separately. There are technical reasons to keep both $x$ and $\eta$. It is possible to consider $(x,\eta)$

for $\eta \in \mathscr{D}\big([-\alpha,0),\mathbb{R}^n\big)$ to remove the redundancy of $x$, but then $\mathscr{D}\big([-\alpha,0),\mathbb{R}^n\big)$ endowed with Skorokhods $J_1$-topology is not separable anymore. To avoid problems with the lack of separability, we consider $(x,\eta)$ for $\eta \in \mathscr{D}\big([-\alpha,0],\mathbb{R}^n\big)$. As we will see later, the redundancy of $x$ will not cause any problems, since the corresponding fixed point operator acts on $\mathscr{D}\big([-\alpha,T],\mathbb{R}^n\big)$, which (endowed with the $J_1$-topology) is a separable, complete metric space.

Now we can formulate the first main result of this section: The following definitions are based on [MS03, p. 277 et seqq.], where Mohammed and Scheutzow obtain similar results for Stratonovich type SDDEs driven by Brownian noise. Let

$$F(t,z,\omega,x,\eta) := \big\{D_z\psi(t,z,\omega)\big\}^{-1}H(t,x,\eta), \tag{6.6}$$

where $t\in[0,T]$, $z,x\in\mathbb{R}^n$, $\eta \in \mathscr{D}\big([-\alpha,0],\mathbb{R}^n\big)$. To preserve a clear overview we also introduce the operator $\xi\colon \mathbb{R}\times\mathscr{D}\big([-\alpha,0],\mathbb{R}^n\big)\times\Omega\to\mathbb{R}^d$ given by

$$\xi\big(t,\nu,\omega\big) := \zeta(0,x,\omega) + \int_0^t F\Big(s,\zeta\big(s,\nu(s,\omega),\omega\big),\omega,\nu\big(s,\omega\big),\nu_s(\cdot,\omega)\Big)\,\mathrm{d}s.$$

**Lemma 6.1.** *The process $\bar{x}=\bar{x}(t,\omega)$ solves the MSDDE* (6.5) *with initial sequence $(x,\eta)$ if and only if $\bar{x}$ satisfies the fixed point equation*

$$\bar{x}(t,\omega) = \begin{cases}\psi\big(t,\xi(t,\bar{x},\omega),\omega\big) & \textit{for} \quad t\in[0,T],\\ \eta(t) & \textit{for} \quad t\in[-\alpha,0].\end{cases} \tag{6.7}$$

*Proof.* Assume $\bar{x}$ solves (6.7) for a.a. $\omega\in\Omega$. Then we can apply Itô's formula to obtain

$$\begin{aligned}\mathrm{d}\bar{x}(t,\omega) &= \frac{\partial}{\partial t}\psi\big(t,\xi(t,\bar{x},\omega),\omega\big)\,\mathrm{d}t\\ &\quad + D_x\psi\big(t,\xi(t,\bar{x},\omega),\omega\big)F\big(t,\zeta\big(t,\bar{x}(t,\omega),\omega\big),\omega,\bar{x}(t,\omega),\bar{x}_t(\cdot,\omega)\big)\,\mathrm{d}t\\ &= G\Big(t,\psi\big(t,\xi(t,\bar{x},\omega),\omega\big)\Big)\diamond\mathrm{d}L(t)\\ &\quad + D_x\psi\big(t,\xi(t,\bar{x},\omega),\omega\big)\big\{D_x\psi(t,\zeta\big(t,\bar{x}(t,\omega),\omega\big),\omega)\big\}^{-1}H\Big(t,\bar{x}(t,\omega),\bar{x}_t\Big)\,\mathrm{d}t\end{aligned} \tag{6.8}$$

for a.a. $\omega\in\Omega$.
According to (6.7) we have $\bar{x}(t,\omega)=\psi\big(t,\xi(t,\bar{x},\omega),\omega\big)$, which implies $\psi^{-1}\big(t,\bar{x}(t,\omega),\omega\big)=\xi(t,\bar{x},\omega)$. Especially we get

$$\begin{aligned}\psi\Big(t,\xi\big(t,\bar{x},\omega\big),\omega\Big) &= \psi\Big(t,\psi^{-1}\big(t,\bar{x}(t,\omega),\omega\big),\omega\Big)\\ &= \bar{x}(t,\omega)\\ &= \psi\Big(t,\zeta\big(t,\bar{x}(t,\omega),\omega\big),\omega\Big).\end{aligned} \tag{6.9}$$

Hence (6.8) becomes

$$
\begin{aligned}
\mathrm{d}\bar{x}(t,\omega) &= G\Big(t, \psi\big(t, \xi(t,\bar{x},\omega),\omega\big)\Big) \diamond \mathrm{d}L(t) \\
&\quad + \underbrace{D_x\psi\big(t,\xi(t,\bar{x},\omega),\omega\big)\big\{D_x\psi(t,\zeta\big(t,\bar{x}(t,\omega),\omega\big),\omega)\big\}^{-1}}_{=\mathrm{id}} H\Big(t,\bar{x}(t,\omega),\bar{x}_t\Big)\,\mathrm{d}t \\
&= G\big(t,\bar{x}(t,\omega)\big) \diamond \mathrm{d}L(t) + H\big(t,\bar{x}(t,\omega),\bar{x}_t\big)\,\mathrm{d}t, \qquad (6.10)
\end{aligned}
$$

which proves the only if-case.

Now assume $\bar{x}$ solves the MSDDE (6.5) with initial sequence $\eta$. Then we can define

$$
\xi\big(t,\omega\big) := \zeta(0,\bar{x},\omega) + \int_0^t F\Big(s, \zeta\big(s,\bar{x}(s,\omega),\omega\big),\omega,\bar{x}\big(s,\omega\big),\bar{x}_s(\cdot,\omega)\Big)\,\mathrm{d}s
$$

explicitly. Then we can prove similar to the only if-case, that the process

$$
\tilde{x}(t,\omega) := \begin{cases} \psi\big(t,\xi(t,\omega),\omega\big) & \text{for} \quad t\in[0,T], \\ \eta(t) & \text{for} \quad t\in[-\alpha,0] \end{cases}
$$

satisfies $\mathrm{d}\tilde{x}(t) = G\big(t,\tilde{x}(t)\big) \diamond \mathrm{d}L(t) + H\big(t,\tilde{x}(t),\tilde{x}_t\big)\,\mathrm{d}t$ with initial sequence $\eta$, which completes the proof. □

Subsequently, we apply the preceding lemma to prove existence and uniqueness of solutions for MSDDEs:

**Theorem 6.2.** *Let assumptions* ($\mathrm{C}^1$) *and (D) hold. Then there exists a $\omega$-wise unique solution $X = X(t,x,\eta,\omega)$ to the MSDDE* (6.5).

*Proof.* Motivated by (6.7) we consider the following operator:
Let $U_T\colon \mathbb{R}^d \times \mathscr{D}\big([-\alpha,0],\mathbb{R}^d\big) \times \mathscr{D}_0 \to \mathscr{D}_0$ be given by

$$
U_T(x,\eta,\bar{x})[t] := \begin{cases} \psi\big(t,\zeta(0,x,\omega)+V(x,\eta,\bar{x})[t]\big) - x, & \text{for } t\in[0,T], \\ 0, & \text{for } t\in[-\alpha,0], \end{cases} \qquad (6.11)
$$

where

$$
V(x,\eta,\bar{x})[t] := \int_0^t F\Big(u,\zeta\big(u,\bar{x}(u)+x,\omega\big),\bar{x}(u)+x,\bar{x}_u + \big(x\mathbb{1}_{\{u\geqslant 0\}} + \eta(u)\mathbb{1}_{\{-\alpha\leqslant u<0\}}\big)\Big)\,\mathrm{d}u.
$$

For the sake of overview we define the mapping

$$
\widetilde{(x,\eta)}\colon [-\alpha,T]\to\mathbb{R}^d \quad \text{by} \quad \widetilde{(x,\eta)}(u) := x\mathbb{1}_{\{u\geqslant 0\}} + \eta(u)\mathbb{1}_{\{-\alpha\leqslant u<0\}}. \qquad (6.12)
$$

These definitions are similar to those in [MS03, p. 280].

According to Lemma 6.1, the process $X$ solves (6.5) if and only if $X$ is a fixed point of (6.11). Thus, there is a unique solution to the MSDDE (6.5) if and only if there is a $\omega$-wise unique fixed point for equation (6.11).

To obtain existence of such a fixed point we will apply Banach's fixed point theorem. Accordingly we need a contractive mapping from some complete metric space into itself. Therefore we will show, that

$$d_0\Big(U_{\tilde{T}}(x,\eta,\bar{x}_1)[t], U_{\tilde{T}}(x,\eta,\bar{x}_2)[t]\Big) \leqslant \mathrm{q}\cdot d_0\big(\bar{x}_1,\bar{x}_2\big) \tag{6.13}$$

for some $\tilde{T} \in (0,T]$, where $\mathrm{q} \in (0,1)$. Here $d_0$ denotes Skorokhod's $J_1$-metric. Indeed, then we get a fixed point $\tilde{x} \in \mathscr{D}_0\big([0,\tilde{T}],\mathbb{R}^n\big)$ which we can extend to $\mathscr{D}\big([0,T],\mathbb{R}^n\big)$ with help of a concatenation technique.

We consider a ball $\mathrm{B}(r) := \big\{\vartheta \in \mathscr{D}_0 \colon d_0(\vartheta,0) \leqslant r\big\} \subset \mathscr{D}_0\big([-\alpha,T],\mathbb{R}^n\big)$. Then we observe, that $d_0(\vartheta,0) = \sup_{-\alpha\leqslant t\leqslant T}\big|\vartheta(t)\big|$, i.e. the $J_1$-ball centered at a constant function coincides with the ball centered at the same constant function using the sup-norm. Nevertheless, the $J_1$-topology cannot be generated by any norm on $\mathscr{D}_0$ (especially, it cannot be generated by the sup-norm). The space $\mathscr{D}_0$ endowed with the sup-norm is not separable, which would cause problems with measurability. Hence it is important to apply Banach's fixed point theorem for $U_{\hat{T}}$ acting on $\big(\mathscr{D}_0\big([-\alpha,\hat{T}],\mathbb{R}^n\big),d_0\big)$:

First we prove that the contraction property holds locally:
Therefore we seek for a complete ball $\mathrm{B}(r^*) \subset \mathscr{D}_0$ of càdlàg processes, such that

(1) $U_{\hat{T}}(x,\eta,x_1)[\cdot] \in \mathrm{B}(r^*)$ for each $x_1 \in \mathrm{B}(r^*)$ and for some $\hat{T} < T$ and $r^* > 0$,

(2) $d_0\Big(U_{\hat{T}}(x,\eta,x_1)[t], U_{\hat{T}}(x,\eta,x_2)[t]\Big) \leqslant \mathrm{q}\cdot d_0\big(x_1,x_2\big)$ for each $x_1, x_2 \in \mathrm{B}(r^*)$,

where $\hat{T}$ does not depend on $x$.
By definition we get

$$\|U_{\hat{T}}(x,\eta,x_1)[\,\cdot\,]\|_\infty = \sup_{t\leqslant\hat{T}}\big|\psi\big(t,\zeta(0,x,\omega)+V(x,\eta,\bar{x})[t]\big)-x\big|,$$

where $\psi\big(0,\zeta(0,x,\omega)+V(x,\eta,\bar{x})[0]\big)-x=0$ and $t\mapsto\psi\big(t,\zeta(0,x,\omega)+V(x,\eta,\bar{x})[t]\big)$ is càdlàg. Moreover, the mapping $\bar{x} \mapsto U_T(x,\eta,\bar{x})[t]$ is continuous (by definition of $V$ and assumption (D)[i]). This gives us

$$\sup_{t\leqslant\hat{T}}\big|U_{\hat{T}}(x,\eta,\bar{x})[t]\big| \leqslant \mathrm{C}(x,\eta,r^*,\hat{T}) \text{ for each } \bar{x}\in\mathrm{B}(r^*),$$

where $\mathrm{C}(x,\eta,r^*,\hat{T})$ as $\hat{T}\to 0$ for fixed $(x,\eta)$ and $r^*$. Now we set $r^* := \sup_{t<T}\big|U_T\big(x,\eta,0\big)[t]\big|$. Then we have $U_T\big(x,\eta,\mathrm{B}(r^*)\big) \subset \mathrm{B}(r^{**})$ for some $r^{**} = r^{**}(T) > 0$ with $r^{**}(T) \to 0$ as $T \to 0+$. Thus we can find some $\hat{T} > 0$ (sufficiently small) such that $U_{\hat{T}}\big(x,\eta,\mathrm{B}(r^*)\big) \subset \mathrm{B}(r^*)$ on $\mathscr{D}\big([-\alpha,\hat{T}],\mathbb{R}^n\big)$, i.e. (1) holds, where $\hat{T}$ is independent on $(x,\eta)$ according to assumption (D)[ii] (mainly by the boundedness of $H$).

The flow $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism ($\omega$-wise for each fixed $t \in [0,T]$) according to assumption ($\mathrm{C}^1$). Hence, we obtain for each $\bar{x}^1, \bar{x}^2 \in \mathrm{B}(r^*)$:

$$
\begin{aligned}
&\Big|U_T(x,\eta,\bar{x}^1)[t] - U_T(x,\eta,\bar{x}^2)[t]\Big| \\
&\quad = \Big|\psi\big(t, \zeta(0,x,\omega) + V(x,\eta,\bar{x}^1)[t]\big) - \psi\big(t, \zeta(0,x,\omega) + V(x,\eta,\bar{x}^2)[t], \omega\big)\Big| \\
&\quad \leqslant \sup_{y\in \mathrm{B}(r)\subset\mathbb{R}^n} \big|D_x\psi(t,y,\omega)\big| \cdot \big|V(x,\eta,\bar{x}^1)[t] - V(x,\eta,\bar{x}^2)[t]\big|, \\
&\qquad\quad \Big(\text{where } r = \sup_{\bar{x}\in \mathrm{B}(r^*)} \big|\zeta(0,x,\omega) + V(x,\eta,\bar{x})[t]\big)\big| < \infty \text{ according to assumption (D)[ii]}\Big) \\
&\quad \leqslant \mathrm{c}\cdot \bigg| \int_0^t F\Big(u, \zeta\big(u, \bar{x}^1(u) + x, \omega\big), \bar{x}^1(u) + x, \bar{x}^1_u + \widetilde{(x,\eta)}_u\Big) \\
&\qquad\qquad - F\Big(u, \zeta\big(u, \bar{x}^2(u) + x, \omega\big), \bar{x}^2(u) + x, \bar{x}^2_u + \widetilde{(x,\eta)}_u\Big)\,\mathrm{d}u\bigg| \\
&\quad = \mathrm{c}\cdot \bigg| \int_0^t \big\{D_x\psi\big(u,\zeta\big(u,\bar{x}^1(u)+x,\omega\big),\omega\big)\big\}^{-1} H(t, \bar{x}^1(u) + x, \bar{x}^1_u + \widetilde{(x,\eta)}_u) \\
&\qquad\qquad - \big\{D_x\psi\big(u,\zeta\big(u,\bar{x}^2(u)+x,\omega\big),\omega\big)\big\}^{-1} H(t, \bar{x}^2(u) + x, \bar{x}^2_u + \widetilde{(x,\eta)}_u)\,\mathrm{d}u\bigg| \\
&\quad = \mathrm{c}\cdot \bigg| \int_0^t \big\{D_x\psi\big(u,\zeta\big(u,\bar{x}^1(u)+x,\omega\big),\omega\big)\big\}^{-1} H(t, \bar{x}^1(u) + x, \bar{x}^1_u + \widetilde{(x,\eta)}_u) \\
&\qquad\qquad \mp \big\{D_x\psi\big(u,\zeta\big(u,\bar{x}^2(u)+x,\omega\big),\omega\big)\big\}^{-1} H(t, \bar{x}^1(u) + x, \bar{x}^1_u + \widetilde{(x,\eta)}_u) \\
&\qquad\qquad - \big\{D_x\psi\big(u,\zeta\big(u,\bar{x}^2(u)+x,\omega\big),\omega\big)\big\}^{-1} H(t, \bar{x}^2(u) + x, \bar{x}^2_u + \widetilde{(x,\eta)}_u)\,\mathrm{d}u\bigg| \\
&\quad \leqslant \mathrm{c}\cdot \bigg( \int_0^t \Big| H\big(u, \bar{x}^1(u) + x, \bar{x}^1_u + \widetilde{(x,\eta)}_u\big) - H\big(u, \bar{x}^2(u) + x, \bar{x}^2_u + \widetilde{(x,\eta)}_u\big)\Big|\,\mathrm{d}u \\
&\qquad + \sup_{u\in[0,t]} \Big| \big\{D_x\psi\big(u,\zeta(u,\bar{x}^1(u)+x,\omega),\omega\big)\big\}^{-1} - \big\{D_x\psi\big(u,\zeta(u,\bar{x}^2(u)+x,\omega),\omega\big)\big\}^{-1}\Big| \times \\
&\qquad\qquad \times \int_0^t \Big| H\big(u, x + \bar{x}^1(u), \bar{x}^1_u + \widetilde{(x,\eta)}_u\big)\Big|\,\mathrm{d}u \bigg),
\end{aligned}
\tag{6.14}
$$

where $c := \sup_{y\in \mathrm{B}(r)} \big|D_x\psi(t,y,\omega)\big|$.

According to assumption (D)[ii]+[iii], (6.14) implies

$$\begin{aligned}
&\Big|U_T(x,\eta,\bar{x}^1)[t] - U_T(x,\eta,\bar{x}^2)[t]\Big| \\
&\quad\leqslant \mathrm{c}\cdot\bigg(\int_0^t \Big|H\big(u,\bar{x}^1(u)+x,\bar{x}^1_u+\widetilde{(x,\eta)}_u\big) - H\big(u,\bar{x}^2(u)+x,\bar{x}^2_u+\widetilde{(x,\eta)}_u\big)\Big|\,\mathrm{d}u \\
&\qquad + \sup_{u\in[0,t]}\Big|\big\{\nabla\psi\big(u,\zeta(u,\bar{x}^1(u)+x)\big)\big\}^{-1} - \big\{\nabla\psi\big(u,\zeta(u,\bar{x}^2(u)+x)\big)\big\}^{-1}\Big|\times \\
&\qquad\qquad\times\int_0^t \Big|H\big(u,x+\bar{x}^1(u),\bar{x}^1_u+\widetilde{(x,\eta)}_u\big)\Big|\,\mathrm{d}u\bigg) \\
&\quad\leqslant \mathrm{c}\cdot\bigg(\mathrm{L}\int_0^t \big\|(\bar{x}^1(u),\bar{x}^1_u)-(\bar{x}^2(u),\bar{x}^2_u)\big\|_{\mathscr{H}_2}\,\mathrm{d}u \\
&\qquad + \sup_{u\in[0,t]}\Big|\big\{\nabla\psi\big(u,\zeta(u,\bar{x}^1(u)+x)\big)\big\}^{-1} - \big\{\nabla\psi\big(u,\zeta(u,\bar{x}^2(u)+x)\big)\big\}^{-1}\Big|\times \\
&\qquad\qquad\times\,\mathrm{C}\int_0^t 1+\big\|(\bar{x}^1(u),\bar{x}^1_u)\big\|_{\mathscr{H}_2}\,\mathrm{d}u\bigg).
\end{aligned} \tag{6.15}$$

Furthermore, we have

$$\sup_{t\in[0,T]}\big|D_x\psi(t,x,\omega)\big| \leqslant \mathrm{K}_+\big(1+|x|^\varepsilon\big) \quad \text{and}$$

$$\sup_{t\in[0,T]}\Big|\big\{D_x\psi(t,x,\omega)\big\}^{-1}\Big| \leqslant \mathrm{K}_-\big(1+|x|^\varepsilon\big) \quad \text{for each } \varepsilon>0\text{, where } \mathrm{K}_\pm=\mathrm{K}_\pm(T,\varepsilon,\omega)>0$$

see [MS03, p. 291 et seq.]. These estimates are a minor extension of Theorem 5.1, cf. [App09, Proposition 6.6.2, p. 396 et seqq.] for instance. Thus, (6.15) implies

$$\begin{aligned}
&\Big|U_T(x,\eta,\bar{x}^1)[t] - U_T(x,\eta,\bar{x}^2)[t]\Big| \\
&\quad\leqslant t\cdot\mathrm{c}\cdot\mathrm{L}\sup_{u\in[0,t]}\big\|(\bar{x}^1(u),\bar{x}^1_u)-(\bar{x}^2(u),\bar{x}^2_u)\big\|_{\mathscr{H}_2} \\
&\qquad + t\cdot\mathrm{c}\cdot\mathrm{C}\sup_{u\in[0,t]}\Big(1+\big\|(\bar{x}^1(u),\bar{x}^1_u)\big\|_{\mathscr{H}_2}\Big)\times \\
&\qquad\times\sup_{u\in[0,t]}\Big|\big\{D_x\psi\big(u,\zeta(u,\bar{x}^1(u)+x,\omega),\omega\big)\big\}^{-1} - \big\{D_x\psi\big(u,\zeta(u,\bar{x}^2(u)+x,\omega),\omega\big)\big\}^{-1}\Big|.
\end{aligned}$$

Both the flow $\zeta$ and the inverse $\big\{D_x\psi\big\}^{-1}$ are $\mathscr{C}^1$-diffeomorphism according to assumption $(\mathrm{C}^1)$ and Theorem 2.14. This implies

$$\begin{aligned}
&\Big|U_T(x,\eta,\bar{x}^1)[t] - U_T(x,\eta,\bar{x}^2)[t]\Big| \\
&\quad\leqslant t\cdot\mathrm{c}\cdot\mathrm{L}\sup_{u\in[0,t]}\big\|(\bar{x}^1(u),\bar{x}^1_u)-(\bar{x}^2(u),\bar{x}^2_u)\big\|_{\mathscr{H}_2} \\
&\qquad + t\cdot\mathrm{c}\cdot\mathrm{C}\sup_{u\in[0,t]}\Big(1+\big\|(\bar{x}^1(u),\bar{x}^1_u)\big\|_{\mathscr{H}_2}\Big)\cdot\mathrm{c}\cdot\bar{\mathrm{c}}\sup_{u\in[0,t]}\big|\bar{x}^1(u)-\bar{x}^2(u)\big| \\
&\quad\leqslant T\cdot\mathrm{c}\cdot\big(\mathrm{L}+\mathrm{C}\cdot\bar{\mathrm{c}}\big)\sup_{u\in[0,t]}\big\|(\bar{x}^1(u),\bar{x}^1_u)\big\|_{\mathscr{H}_2}.
\end{aligned} \tag{6.16}$$

Now we choose $\tilde{T} < \alpha \wedge \hat{T}$ sufficiently small, such that $\tilde{T} \cdot \mathrm{c} \cdot \big(\mathrm{L} + \mathrm{C} \cdot \bar{\mathrm{c}}\big) < 1$ to obtain, that

$$U_{\tilde{T}}(x, \eta, \cdot) \colon \mathscr{D}_0\big([0, \tilde{T}], \mathbb{R}^n\big) \to \mathscr{D}_0\big([0, \tilde{T}], \mathbb{R}^n\big) \tag{6.17}$$

is contractive. Hence there is a unique fixed point $\tilde{x}_1 \in \mathrm{B}(r)$ according to Banach's fixed point theorem.

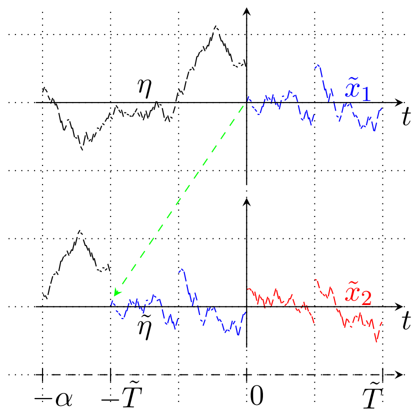
We complete the proof by applying a concatenation technique:
Due to assumption (D) the constants L and C are independent of $t$ and $T$ by which the same arguments for (6.17) work for

$$U_{\tilde{T}}(\tilde{x}_1(\tilde{T}), \tilde{\eta}, \cdot) \colon B(r^*) \to B(r^*), \tag{6.18}$$

where
$\tilde{\eta}(t) = \eta(t + \tilde{T})\mathbb{1}_{[-\alpha, -\tilde{T}]}(t) + \tilde{x}_1(t + \tilde{T})\mathbb{1}_{(-\tilde{T}, 0]}(t)$.
Hence we get a fixed point $\tilde{x}_2 \in \mathscr{D}_0\big([0, \tilde{T}], \mathbb{R}^n\big)$ for (6.18). This approach can be extended recursively to obtain fixed points $\tilde{x}_i$, $i = 1, \dots, N$ respectively, where $N := \lceil T/\tilde{T} \rceil$.
Now we define

$$\tilde{x}(t) := \sum_{i=1}^{N} \tilde{x}_i\big(t - (i-1)\tilde{T}\big)\mathbb{1}_{\big((i-1)\tilde{T}, i\tilde{T}\big]}(t).$$

Then $\tilde{x}$ is the unique fixed point of (6.11).




To complete the proof, we show uniqueness for solutions outside of the ball $\mathrm{B}\big(r^*\big)$:
Let $x$ and $y$ be solutions for (6.5) and let $t^* \in [0, T]$ be the latest time point, such that $x\big(t^*\big) = y\big(r^*\big)$ and $x(t) \neq y(t)$ for some $t > t^*$. Then we can apply the same arguments as before using a slightly larger ball $\mathrm{B}\big(t^{**}\big)$ with $x(t), y(t) \in \mathrm{B}\big(t^{**}\big)$ instead of $\mathrm{B}\big(r^*\big)$, which implies $x(t) = y(t)$. Hence we get uniqueness outside of $\mathrm{B}\big(r^*\big)$ by contradiction. □

## 6.2 Cocycle Property for MSDDEs

Conclusively, we apply the fixed point argument to prove the cocycle property for solutions of MSDDEs. Since solutions of MSDDEs depend on the actual state and the history, we have to modify the cocycle property accordingly. Therefore we consider cocycles in $\mathscr{H}_2$:

**Theorem 6.3.** *Let $x(t) = x\big(t; (x, \eta), \omega\big)$ be the solution of (6.5) given $(x, \eta) \in \mathscr{H}_2$ and $\omega \in \Omega$. We define the mapping $X \colon \mathbb{R}_+ \times \mathscr{H}_2 \times \Omega \to \mathscr{H}_2$ by*

$$X\big(t, (x, \eta), \omega\big) := \big(x(t), x|_{[t-\alpha, t]}\big).$$

*Then we have $X\big(t + \tau, (x, \eta), \omega\big) = X\big(t, \cdot, \theta_\tau \omega\big) \circ X\big(\tau, (x, \eta), \omega\big)$, for each $t, \tau > 0$ and $\omega \in \Omega$.*

*Proof.* We set $y(t) := x\Big(t+\tau,(x,\eta),\omega\Big)$ and $z(t) := x\Big(t, X\big(\tau,(x,\eta),\omega\big),\theta_\tau\omega\Big)$. By the definition of $\zeta$ and according to the proof of Lemma 6.1 we get

$$\xi\big(t, y|_{[-\alpha,t]},\omega\big) = \zeta(0, x(\tau),\omega) + \int_0^t F\Big(u, \zeta\big(u, y(u),\omega\big),\omega, y(u), y_u\Big)\,\mathrm{d}u,$$

where $F(t,z,\omega,x,\eta) := \big\{D_x\psi(t,z,\omega)\big\}^{-1}H(t,x,\eta)$ for each $t\in[0,T]$, $z,x\in\mathbb{R}^n$ and $\eta\in\mathscr{D}_0\big([-\alpha,0],\mathbb{R}^n\big)$.
Similar to this, we obtain

$$\xi\big(t, z|_{[-\alpha,t]},\omega\big) = \xi(0, z|_{[-\alpha,0]},\omega) + \int_\tau^{\tau+t} F\Big(u, \zeta\big(u, z(u-\tau),\omega\big),\omega, z(u-\tau), z_{u-\tau}\Big)\,\mathrm{d}u.$$

According to equation (6.9) we observe, that

$$\xi\big(0, z|_{[-\alpha,0]},\omega\big) = \zeta\big(0, z(0),\omega\big) = \zeta\big(0, x(\tau),\omega\big),$$

which implies

$$\xi\big(t, z|_{[-\alpha,t]},\omega\big) = \zeta(0, x(\tau),\omega) + \int_0^t F\Big(u, \zeta\big(u, z(u),\omega\big),\omega, z(u), z_u\Big)\,\mathrm{d}u.$$

Hence both $y(t)$ and $z(t)$ are fixed points for

$$U_T(x,\eta,\bar{x})[t] := \begin{cases} \psi\big(t, \zeta(0,x) + V(x,\eta,\bar{x})[t]\big) - x, & \text{for } t\in[0,T],\\ 0, & \text{for } t\in[-\alpha,0], \end{cases} \tag{6.11}$$

which follows from Lemma 6.1.
The fixed point of equation (6.11) is unique according to Theorem 6.2. Hence $y(t) = z(t)$ for each $t\in[-\alpha,T]$ which moreover implies, that $y|_{[t-\alpha,t]} = z|_{[t-\alpha,t]}$ for each $t\in[0,T]$ and the proof is done. □

# 7 Conclusion and further perspectives

In summary, the focus of the thesis lies in the study of MSDEs driven by Lévy noise regarding their dynamical behavior. Based on the study of continuous Stratonovich SDEs we already know various results for Brownian driven dynamics, such as existence of manifolds or conjugacy with respect to solutions of RDEs. To prove these results, the self-similarity property provided by the Brownian motion can be exploited to a great extent. In the Lévy case there is no self-similarity in general. Besides that, Stratonovich SDEs driven by Lévy processes are not suitable to generate dynamics in general. Other than Stratonovich SDEs we consider Marcus type SDEs, which provide useful properties to study their dynamics. Nevertheless, to obtain similar results for Lévy driven dynamics there are several estimates and inequalities to verify using other methods than applied in the continuous case.
The first major accomplishment of this work is given by the accumulation of different techniques, which enables the study of Lévy driven dynamics, such as

(i) the perfection theorem for càdlàg processes, or

(ii) adapting Kunita's inequalities to prove the conditions for the multiplicative ergodic theorem, or

(iii) existence of stable and unstable manifolds and folitated manifolds by applying a modified Lyapunov–Perron method.

These methods are capable to adapt more dynamical properties known from Stratonovich SDEs to dynamics generated by Lévy driven SDEs. Especially the construction of Lévy processes with two-sided time and the connection to metric dynamical system in the space $\mathscr{D}_0$ of càdlàg processes is a basic requirement to study any dynamic property.
The other important result is the proof of the flow property of MSDDEs. In doing so, the fixed point approach from Mohammed and Scheutzow was generalised to diffusions with jumps. Moreover, it can be seen that this technique is capable to cover an even wider class of stochastic processes with memory, as long as the undelayed equation generates a RDS.

In this work we mainly considered finite-dimensional processes. Considering MSDDEs is the first step to introduce (in some sence) infinite-dimensional equations. As part of ongoing studies it seems reasonable to study MSDEs in Hilbert spaces (or even Banach spaces) to study infinite dimensional dynamics with jumps. However, there are difficulties to generalise the Stratonovich correction term including the quadratic variation in the Hilbert space case. When Hilbert space-values MSDEs are well-defined, then adapting the methods used in this work such as Lyapunov–

Perron or Kunita's inequalities might be as useful to prove dynamical properties as they are in the finite dimensional case.

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# List of Symbols

| | |
|---|---|
| $B = \big(B_t\big)_t$ | Brownian motion |
| $\mathscr{C}^k$ | Space of continuously differentiable functions |
| $\mathscr{D}$ | Space of càdlàg functions |
| $\mathscr{D}_0$ | Space of càdlàg functions starting at 0 |
| $\mathbb{E}$ | Expectation |
| $L = \big(L_t\big)_t$ | Lévy process |
| **MSDE** | Marcus type stochastic differential equation |
| **MSDDE** | Marcus type stochastic delay differential equation |
| $\tilde{N}$ | compensated Poisson random measure |
| $\nu$ | Poisson random measure |
| $\varphi$ | Random dynamical system |
| **RDE** | Random differential equation |
| **SDE** | Stochastic differential equation |
| **SDDE** | Stochastic delay differential equation |
| $\theta = \big(\theta_t\big)_t$ | Family of Wiener shifts |
| $Z = \big(Z_t\big)_t$ | Semimartingale |

# Index

# Ehrenwörtliche Erklärung

Hiermit erkläre ich,

- dass mir die Promotionsordnung der Fakultät bekannt ist,
- dass ich die Dissertation selbst angefertigt habe, keine Textabschnitte oder Ergebnisse eines Dritten oder eigener Prüfungsarbeiten ohne Kennzeichnung übernommen und alle von mir benutzten Hilfsmittel, persönliche Mitteilungen und Quellen in meiner Arbeit angegeben habe,
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- dass ich die Dissertation noch nicht als Prüfungsarbeit für eine staatliche oder eine andere wissenschaftliche Prüfung eingereicht habe,
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Jena, 13.04.2016 Kai Kümmel