FERMIONS IN CURVED SPACETIMES

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Zusammenfassung

In dieser Promotionsschrift untersuchen wir eine Formulierung von Dirac-Fermionen in gekrümmten Raumzeiten, die sowohl eine allgemeine Koordinateninvarianz, sowie eine lokale Spinbasen-Invarianz besitzt. Wir beleuchten die Vorteile der Spinbasen-invarianten Formulierung aus konzeptioneller und praktischer Sicht. Dies legt nahe, dass die lokale Spinbasen-Invarianz auf die Liste der (effektiven) Eigenschaften der (Quanten-) Gravitation gehört. Wir finden Argumente für diesen Standpunkt durch die Konstruktion einer globalen Realisierung der Clifford-Algebra auf der 2-Sphäre, eine Struktur die es im (nicht Spinbasen-invarianten) Vielbein-Formalismus nicht gibt. Die natürlichen Variablen für diese Formulierung sind die raumzeitabhängigen Dirac-Matrizen, welche die Clifford-Algebra erfüllen müssen. Insbesondere benötigt man kein "Achsenkreuz", d. h. ein Vielbein. Wir decken die versteckte Spinbasen-Invarianz des Vielbein-Formalismus auf. Explizite Formeln für den Spin-Zusammenhang in Abhängigkeit der Dirac-Matrizen werden angegeben. Dieser Zusammenhang besteht aus einem kanonischen Teil, der vollständig durch die Dirac-Matrizen fixiert ist, und einem freien Teil, der als Spin-Torsion interpretiert werden kann. Die übliche Lorentz-symmetrische Eichung für das Vielbein wird auf die Dirac-Matrizen verallgemeinert, auch für Metriken die nicht linear verbunden sind. Nach bestimmten Kriterien bildet sie die einfachste Eichung – das zeigt warum diese Eichung so nützlich ist. Mit Hilfe des Spinbasen-Formalismus konstruieren wir eine Feldtheorie für quantisierte Gravitation and Materiefelder und zeigen, dass die Quantisierung der Metrik und der Materiefelder genügt. Für feldtheoretische Zugänge zur Quantengravitation ist diese Beobachtung von besonderer Relevanz, da sie ein rein metrikbasiertes Quantisierungsschema, auch in der Anwesenheit von Fermionen, nahelegt.

Daher untersuchen wir im zweiten Teil dieser Arbeit die Eich- und Feldparametrisierungsabhängigkeit des Renormierungsgruppenflusses in der Nähe von nicht-Gaußschen Fixpunkten in Quantengravitationstheorien. Während physikalische Observablen unabhängig von solchen Details der Rechnung sind, benötigt man für die Konstruktion von Quantengravitationstheorien typischerweise "off-shell" Größen wie Betafunktionen und erzeugende Funktionale. Daher begegnet man potenziellen Stabilitätsproblemen bei diesen verallgemeinerten Parametrisierungen. Wir analysieren eine zwei-Parameter Klasse von kovarianten Eichbedingungen, die Rolle der impulsabhängigen Feldreskalierung und eine Klasse von Feldparametrisierungen. Die Familie der Eichungen beinhaltet eine (nicht-harmonische) Verallgemeinerung der harmonischen Eichung (De-Donder-Eichung), dabei ist die letztere besonders nützlich für die Analyse von Gravitationswellen, die vermutlich die asymptotischen Zustände der Quantengravitation sind. Mit Hilfe des Prinzips der kleinsten Sensitivität identifizieren wir über das Produkt von Gravitations- und kosmologischer Konstante stationäre Punkte in diesem Parameterraum, diese zeigen eine beachtliche Insensitivität gegenüber den Details der Parametrisierung. In den insensitivsten Fällen zeigt das quantisierte Gravitationssystem einen nicht-Gaußschen ultravioletten, stabilen Fixpunkt. Das liefert weiteren Rückhalt für asymptotisch sichere Quantengravitation. Im

stationären Bereich der Parametrisierung des exponentiellen Splits weist der Renormierungsgruppenfluss einige bemerkenswerte Eigenschaften auf: (1) eine mögliche Abhängigkeit vom übrigen Eichparameter fällt heraus, dies impliziert einen vergrößerten Grad an Eichinvarianz, (2) der Renormierungsgruppenfluss wird besonders einfach, sodass das Phasendiagramm in der Ebene der Gravitations- und kosmologischen Konstante analytisch berechnet werden kann, (3) im Fluss tauchen keine Singularitäten auf, sodass eine große Klasse von Renormierungsgruppentrajektorien (inklusive derer mit klassischem Bereich) auf beliebig große und kleine Skalen ausgedehnt werden kann, (4) die ultravioletten, kritischen Exponenten sind reell und nah bei ihren kanonischen Gegenstücken, und (5) es zeigen sich Anzeichen, dass das Szenario der asymptotischen Sicherheit sich nicht direkt auf Dimensionen viel größer als $d = 4$ fortsetzen lässt.

Das letzte Kapitel dieser Promotion widmet sich den Fermionen in gekrümmten Hintergrundraumzeiten und im Besonderen der katalysierten Symmetriebrechung. Dieses Phänomen entsteht durch eine parametrische Vergrößerung der kritischen Fluktuationen unabhängig von der Kopplungsstärke. Aus Molekularfeldtheoriestudien ist bekannt, dass symmetriebrechende fermionische langreichweitige Fluktuationen so eine Vergrößerung auf negativ gekrümmten Räumen aufweisen. Wir untersuchen gravitative Katalyse aus der Sicht der funktionalen Renormierungsgruppe mit Hilfe des $3d$ Gross-Neveu-Modells als ein spezielles Beispiel. Wir beobachten gravitative Katalyse hin zu einer Phase gebrochener diskreter chiraler Symmetrie, sowohl auf einer maximal symmetrischen Raumzeit (AdS) als auch auf einer rein räumlich gekrümmten Mannigfaltigkeit (Lobachevsky Ebene) mit konstanter negativer Krümmung. Das resultierende Bild der gravitativen Katalyse, das wir aus dem Renormierungsgruppenfluss erhalten haben, ist eng verbunden mit dem der magnetischen Katalyse. Als eine Anwendung schätzen wir die nötige Krümmung für ein subkritisches System endlicher Länge ab, um eine gravitativ katalysierte Massenlücke auszubilden.

Summary

In this thesis we study a formulation of Dirac fermions in curved spacetime that respects general coordinate invariance as well as invariance under local spin base transformations. We emphasize the advantages of the spin base invariant formalism both from a conceptual as well as from a practical viewpoint. This suggests that local spin base invariance should be added to the list of (effective) properties of (quantum) gravity theories. We find support for this viewpoint by the explicit construction of a global realization of the Clifford algebra on a 2-sphere which is impossible in the spin-base non-invariant vielbein formalism. The natural variables for this formulation are spacetime-dependent Dirac matrices subject to the Clifford-algebra constraint. In particular, a coframe, i.e. vielbein field is not required. We disclose the hidden spin base invariance of the vielbein formalism. Explicit formulas for the spin connection as a function of the Dirac matrices are found. This connection consists of a canonical part that is completely fixed in terms of the Dirac matrices and a free part that can be interpreted as spin torsion. The common Lorentz symmetric gauge for the vielbein is constructed for the Dirac matrices, even for metrics which are not linearly connected. Under certain criteria, it constitutes the simplest possible gauge, demonstrating why this gauge is so useful. Using the spin base formulation for building a field theory of quantized gravity and matter fields, we show that it suffices to quantize the metric and the matter fields. This observation is of particular relevance for field theory approaches to quantum gravity, as it can serve for a purely metric-based quantization scheme for gravity even in the presence of fermions.

Hence, in the second part of this thesis we critically examine the gauge, and the field-parametrization dependence of renormalization group flows in the vicinity of non-Gaußian fixed points in quantum gravity. While physical observables are independent of such calculational specifications, the construction of quantum gravity field theories typically relies on off-shell quantities such as beta functions and generating functionals. Thus one faces potential stability issues with regard to such generalized parametrizations. We analyze a two-parameter class of covariant gauge conditions, the role of momentum-dependent field rescalings and a class of field parametrizations. The family of gauges includes a (non-harmonic) generalization of the harmonic gauge (De-Donder gauge), the latter being particularly useful for the analysis of gravitational waves which presumably are the asymptotic states of quantum gravity. Using the product of Newton and cosmological constant as an indicator, the principle of minimum sensitivity identifies stationary points in this parametrization space which show a remarkable insensitivity to the details of the parametrization choices. In the most insensitive cases, the quantized gravity system exhibits a non-Gaußian ultraviolet stable fixed point, lending further support to asymptotically free quantum gravity. In the stationary regime of the parametrization based on the exponential split, the resulting renormalization group flow exhibits several remarkable properties: (1) a possible dependence on the residual gauge parameter drops out implying an enhanced degree of gauge invariance, (2) the renormalization group flow becomes

particularly simple, so that the phase diagram in the plane of Newton and cosmological constant can be computed analytically, (3) no singularities arise in the flow, such that a large class of renormalization group trajectories (including those with a classical regime) can be extended to arbitrarily high and low scales, (4) the ultraviolet critical exponents are real and close to their canonical counterparts, and (5) indications are found that the asymptotic safety scenario may not extend straightforwardly to dimensions much higher than $d = 4$.

The final chapter of this thesis is devoted to fermions in curved background spacetimes and, in particular, catalyzed symmetry breaking. This phenomenon arises from a parametric enhancement of critical fluctuations independently of the coupling strength. Symmetry-breaking fermionic long-range fluctuations exhibit such an enhancement on negatively curved spaces, as is known from mean-field studies. We study gravitational catalysis from the viewpoint of the functional renormalization group using the $3d$ Gross-Neveu model as a specific example. We observe gravitational catalysis towards a phase of broken discrete chiral symmetry both on a maximally symmetric spacetime (AdS) and on a purely spatially curved manifold (Lobachevsky plane) with constant negative curvature. The resulting picture for gravitational catalysis obtained from the renormalization group flow is closely related to that of magnetic catalysis. As an application, we estimate the curvature required for subcritical systems of finite length to acquire a gravitionally catalyzed mass gap.

Contents

1 Introduction

To find out and understand the principles of nature in all her beauty (and difficulty) is the goal of physics. Thus, physics serves as a way to describe the explored nature qualitatively as well as quantitatively. In order to achieve this we use tools like the mathematical framework in general and in particular the choice of adapted coordinates or a specific gauge. As these are merely devices with the sole use of simplifying things[1] or keeping track of auxiliary quantities, it is quite reasonable to assume that observable quantities do not depend on these details. Hence, an important amount of work goes into a coordinate and gauge independent formulation of physical theories. This search brought up two frameworks highly suited for the description of nature. On the one hand there is the theory of general relativity, used for macroscopic scales (infrared).[2] And on the other hand we have quantum field theory, employed for the microscopic regime (ultraviolet). Both frameworks led to models of nature that are well tested in their domain of applicability (see [1] for an extensive overview).

Even though this sounds satisfactory, unfortunately so far we failed to fit these two frameworks into *one*. Literally *one* is key in this statement, as in fact there are many approaches to a unifying framework, including string theory [2–5], asymptotic safety [6–10], loop quantum gravity and spin foams [11–16], causal sets [17, 18], causal dynamical triangulations [19, 20], Hořava-Lifshitz gravity [21–26], and even more. See references [27, 28] for a recent review.

Hence, gravitation seems to be rather different from the forces appearing in the standard model of particle physics. This becomes already apparent by looking at the way gravity is described classically. Casually we say that gravity corresponds to the geometry of spacetime. But, in contrast to the other forces, there is a plethora of ways to implement this idea in formulas leading to very different fundamental degrees of freedom. Beginning with Einstein in 1915, the metric description of gravity was developed [29]. Already a few years later, in 1919, Palatini used the connection as additional, independent degree of freedom [30]. In 1929 Weyl wrote a paper [31] on the dynamics of electrons in gravitational fields and thereby introduced the "*Achsenkreuz*" (tetrad) and the spin connection, which are used very often nowadays, especially in string theory. Another thirty years later in 1961 Regge introduced the Regge calculus which was the first discrete approach suitable for numerical simulations [32]. This is used for the causal dynamical triangulations first described in [19, 33]. Furthermore in 1977 Plebanski found a pure connection formulation of gravity [34]. Later on in 1986 Ashtekar discovered the Ashtekar variables as alternative way of describing the spacetime structure [35]. These variables play an important role in the theory of loop quantum gravity. Rather recently Hořava put forward the idea of an anisotropy between space and time leading to a further candidate for a set of fundamental variables [21, 36].

While there are even more ideas, the aforementioned are the most common. In order to decide

[1] Sometimes "simplifying" means making the description or a calculation possible in the first place.

[2] In this work we use the term "general relativity" as a generic term for the concepts and guiding ideas of general relativity, but not the Einstein field equations as particular equations of motion.

which of these classically (infrared) equivalent parametrizations is realized in nature, we need to make predictions for their quantum (ultraviolet) behavior and compare these to experiments. Unfortunately so far we only have experimental access to the nonquantum regime of gravity. Still, people try to find arguments in favor (or disfavor) for one or the other approach. Recently some evidence has been collected that the existence of matter degrees of freedom can constrain the existence of certain quantum gravity theories [8, 10, 37–39]. This is analogous to QCD where the presence of too many dynamical fermions can destroy the high-energy completeness of the theory. The mutual interrelation of matter and spacetime ("matter curves spacetime – spacetime determines the paths of matter") is particularly apparent for fermions. For instance for Dirac fermions (i.e. observable matter), information about both spin as well as spacetime meets in the Clifford algebra, where the Dirac matrices and the metric generally are spacetime dependent. A common line of reasoning, which suggests that the mere existence of fermions should give preference to vielbein based theories of gravity, goes as follows: Whatever the correct fundamental realization of gravity may be, it has to accommodate the other interactions and matter degrees of freedom – in particular fermions. According to textbook knowledge the coupling of fermions to curved spacetimes makes the introduction of a vielbein necessary [31, 40–42]. Since the metric can be constructed from the vielbein, it is now tempting to argue that the vielbein language is at least better adapted to the description of fermions.

Surprisingly the common practice is to first write the action in terms of the vielbein, and then reexpress the vielbein as a function of the metric with the help of some gauge-fixing condition. While this is perfectly valid for classical dynamics, this is somewhat irritating for a quantum theory. If the vielbein were a fundamental variable, then the path integral measure $\mathcal{D}e$ should be defined in terms of the vielbein degrees of freedom. If so, one would have to take into account a nontrivial Jacobian coming from the variable transformation $\mathcal{D}e$ to $\mathcal{D}g$ (not to be confused with the Faddeev-Popov determinant from the gauge fixing). This Jacobian is usually disregarded.[3] Meanwhile, there are indications that a pure vielbein quantization will have at least quantitative differences compared to the case where one reexpresses the vielbein as a function of the metric [9, 10].

In fact, it is by no means obvious that one has to introduce a vielbein in order to describe fermions in curved spacetimes at all. In the first part of this thesis, we show that the introduction of a vielbein (or something similar) can be avoided completely in a very natural way. In the following we aim at working out the ideas of Schrödinger [44], Bargmann [45], Finster [46] and Weldon [47] on a spin-base invariant formulation of fermions on curved spacetimes. The central objects in the spin-base formalism are the Dirac matrices γ_μ, subject to the Clifford algebra constraint, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}$, where $g_{\mu\nu}$ are the components of the metric and I is the unit

[3] For perturbative quantum gravity one can argue that for ultra-local variable transformations the Jacobian has no influence on physical observables (loosely speaking, this is when no differential operators are involved) [43]. The situation is more subtle here: first, we are dealing with a nonperturbative renormalization. Second, the degrees of freedom of the metric and the vielbein are different, and hence only for a strict gauge fixing a generalization of the above argument could apply.

matrix in Dirac space. We stress that this approach completely covers the vielbein formalism. Hence, we encourage the reader to compare all our calculations with the standard description. Even though the spin-base invariant formulation has quite some advantages compared to its vielbein counterpart, it is rarely used in the literature [48–53].

The interrelation of gravity and fermions provided by the Clifford algebra has been interpreted in various, partly conflicting, directions: read in the one way, one is tempted to conclude that one first needs a spacetime metric in order to give a meaning to spinorial degrees of freedom and corresponding physical observables such as fermionic currents. On the other hand, representation theory of the Lorentz group in flat space suggests that all nontrivial representations can be composed out of the fundamental spinorial representation. If so, then also the metric might be a composite degree of freedom, potentially arising as an expectation value of composite spinorial operators (see, e.g., [54–56]). As a starting point to disentangle this hen-or-egg problem – fermions or metric first? – we consider the Clifford algebra as fundamental in this work. We emphasize that this is different from a conventional approach [31, 40–42], where one starts from the analogous Clifford algebra in flat (tangential) space, with fixed flat Dirac matrices and then uplifts the Clifford algebra to curved space with the aid of a vielbein. In addition to diffeomorphism invariance, the vielbein approach supports a local symmetry of Lorentz transformations in tangential space. By contrast, the Clifford algebra actually supports a bigger symmetry of local spin-base transformations in addition to general covariance. Spin base invariance ensures the independence of observables with respect to the choice of an actual representation of the Dirac matrices, exactly like diffeomorphism invariance ensures that observables do not depend on the chosen coordinate system. We have given a full account of the formalism in [57, 58]. Particular advantages are not only the inclusion and generalization of the vielbein formalism. In a quantized setting, it even justifies the widespread use of the vielbein as an auxiliary quantity and not as a fundamental entity. The gravitational quantization is then performed by a path integral over Dirac matrices. Because of the Clifford algebra constraint one cannot integrate over arbitrary matrix configurations. It turns out that the Dirac matrices naturally decouple into metric degrees of freedom and the choice of a spin-base. Hence, this formalism suggests to choose a gauge for the spin-base and then integrate over metrics. By construction, observables do not depend on the actual gauge choice. This procedure leads to the exact same results as one finds for the reexpression of the vielbein as a function of the metric, without keeping the Jacobian. In this way the above-mentioned common treatment of the vielbein as a function of the metric (without keeping the Jacobian) becomes fully justified [57]. Common quantization schemes relying on the metric as fundamental degree of freedom remain applicable also with fermionic matter.

Since the conventional vielbein formalism can always be recovered within the spin-base invariant formalism, it is tempting to think that the latter is merely a technical, perhaps overabundant generalization of the former. We have demonstrated that this is not the case by an explicit construction of a global spin-base on the 2-sphere – a structure which is not possible in the conventional formalism because of global obstructions from the Poincaré-Brouwer (hairy ball) theorem [59]. Additionally we have shown that it is at least inconvenient to construct the Dirac matrices in terms of vielbein degrees of freedom.

With the spin-base formulation settled we move on to the analysis of two specific quantum settings in the second part of this thesis. The first being metric quantum gravity in the asymptotic safety scenario [7, 60], since the Dirac matrix formulation advocates the metric as fundamental degree of freedom. Introductions to the asymptotic safety scenario for quantum gravity are given in [61–63]. Although physical observables should be independent of their computational derivation, many practical calculations are based on convenient choices for intermediate auxiliary tools such as coordinate systems, gauges, etc. Appropriate parametrizations of the details of a system simply decrease the computational effort. Beyond pure efficiency aspects, such suitable parametrizations can also be conceptually advantageous or even offer physical insight. This is similar to coordinate choices in classical mechanics where polar coordinates with respect to the ecliptic plane in celestial mechanics support a better understanding in comparison with, say, Cartesian coordinates with a z axis pointing towards Betelgeuse.

Appropriate parameterizations become particularly significant in quantum calculations. While on-shell quantities such as S-matrix elements are invariant observables [43, 64, 65], off-shell quantities generically feature parametrization dependencies [66–68]. Further ordering schemes such as perturbative expansions may defer the dependencies to higher orders (e.g., scheme dependence in mass-independent schemes), but these are merely special and not always useful limits. Approximation schemes that can also deal with non-perturbative regimes may even introduce further artificial parametrization dependencies which have to be carefully removed (e.g., discretization artefacts in lattice regularizations).

In an ideal situation, this parametrization dependence of a nonperturbative approximation could be quantified and proven to be smaller than the error of the truncated solution. However, as soon as a result is parametrization dependent, it is likely that some pathological parametrization can be constructed that modifies the result in an arbitrary fashion. This suggests to look for general criteria of *good parametrizations* that minimize the artificial dependence in approximation schemes which adequately capture the physical mechanisms.

A-priori criteria suggest the construction of parametrizations that support the identification of physically relevant degrees of freedom, such as the use of Coulomb-Weyl gauge in quantum optics, or the use of pole-mass regularization schemes in heavy-quark physics. Further a-priori criteria include symmetry preserving properties (covariant gauges, non-linear field parametrizations) or strict implementations of a parametrization condition such as the Landau-gauge limit $\alpha \to 0$. A major advantage of the latter is that some redundant degrees of freedom decouple fully from the dynamical equations in such a limit. Good parametrizations may also be identified *a posteriori* by allowing for a family of parametrizations and identifying stationary points in the parameter space. This realizes the principle of minimum sensitivity [69, 70] (originally advocated for regularization-scheme dependencies), suggesting those points as candidate parameters for minimizing the influence of parametrization dependencies.

In the second quantum setting, we look at quantized fermions in negatively curved background spacetimes. There the so-called gravitational catalysis can lead to a mass gap generation and correspondingly to (chiral) symmetry breaking. These effects can arise from a variety of mechanisms which are often related to certain couplings or interaction channels becoming dominant. This is different for catalyzed symmetry breaking, first studied in the context of magnetic catalysis [71–78], where mass gap generation is triggered by the presence of a magnetic field even for arbitrarily small values of the interaction strength. This phenomenon can be understood in various ways, the essence being that the long-range fluctuations driving the symmetry-breaking transitions are parametrically enhanced, see [79] for a recent review. Magnetic catalysis has found a rich variety of applications both in particle physics (chiral phases of QCD) [80–86] and condensed matter physics [87–96]. A simple picture for magnetic catalysis has rather recently been developed within the framework of the functional renormalization group in the context of the $3d$ Gross-Neveu model [97]. In line with the fact that symmetry-breaking phase transitions are often related to fixed points of renormalization group transformations, also magnetic catalysis can be related to the behavior of renormalization group fixed points as a function of the magnetic field. This renormalization group picture has already successfully been applied in the context of QCD [98].

In the last part of this thesis, we verify the underlying renormalization group picture of catalyzed symmetry breaking in the context of curved spacetimes. The fact that symmetry breaking and mass generation in fermionic systems can be influenced by negative curvature of the spacetime has been realized early [99, 100], and is meanwhile reviewed in textbooks [42]. The phenomenon is typically studied at mean-field level and occurs in many different fermionic models [101–116]. In [117, 118], the similarity to magnetic catalysis was realized in terms of an effective dimensional reduction mechanism of the spectral properties of the Dirac operator. This justifies the use of the terminology “gravitational catalysis” [119].

In fact, we find that the effective dimensional reduction and the corresponding enhancement of the density of states in the infrared is directly related to the fixed point structure as identified below. From this renormalization group viewpoint, symmetry breaking arises as a consequence of the fact that the coupling value required for criticality becomes arbitrarily small as a function of the curvature (the catalyzer). Hence, any finite value of the fermionic interactions ultimately becomes supercritical, typically driving the system towards the ordered phase.

We investigate this renormalization group mechanism within the simple $3d$ Gross-Neveu model. And we concentrate on two different curved backgrounds with constant negative curvature: a maximally symmetric spacetime (Anti de Sitter) and a purely spatially curved case (Lobachevsky plane). For both cases, mean-field studies are already available, see [105] and [106, 113] respectively. Whereas the former allows for an analytic treatment in terms of simple functions, the latter is potentially relevant for curved layered condensed matter systems. For instance, the exitonic or anti-ferromagnetic instabilities in graphite and graphene have been associated with quantum phase transitions falling into the $3d$ Gross-Neveu universality class [120, 121]. As catalyzed symmetry breaking is manifestly driven by the long-range modes, the renormalization group analysis allows us to estimate the required curvature in relation to the length scale of the sample.

This thesis is organized as follows. The first part deals with the description of fermions in curved spacetimes in general. In chapter 2, we give a collection of standard general relativity equations while also summarizing our conventions. Afterward, we introduce the vielbein formulation and concentrate on the arising differences. Chapter 3 is devoted to a detailed motivation of spin-base invariance. Particularly we describe the ideas leading to this new formulation of fermions in a curved spacetime and thereby find a hidden spin-base invariance within the vielbein formulation. In chapter 4 we summarize the precise mathematical prerequisites and subsequently develop the necessary technicalities, i.e. the spin metric and the spin connection. At the end of this chapter we derive a convenient gauge choice for the Dirac matrices, which turns out to be the analog of the Lorentz symmetric gauge for the vielbein.

The second part of this thesis is focused on quantum gravity and quantized fermions. In chapter 5, we shortly repeat the path integral formulation of quantum field theory. Then we give some details on the mathematics, which are helpful for the treatment of the Wetterich equation, i.e. the heat kernel and how we use it in the following. Chapter 6 is devoted to the analysis of the parametrization dependencies of quantum gravity in the asymptotic safety scenario. In chapter 7 we look at quantized fermions in the Gross-Neveu model on a curved background spacetime. Our particular interest lies in an interesting feature of this model, the gravitational catalysis. Conclusions are drawn in chapter 8. We have deferred the details of lengthy proofs and calculations to the appendices. The Weldon theorem for arbitrary integer dimensions $d \geq 2$ is proven in appendix A. Appendix B is devoted to the construction of the minimal group ensuring full spin-base invariance. Some important identities for the Dirac matrices are derived in appendices C and D. The existence and uniqueness of the canonical part of the spin connection is shown in appendix E. Appendix F demonstrates the existence and uniqueness (up to a sign) of the spin metric. We comment on reducible representations and the inclusion of a gauge field in appendices G and H. In appendix I we reveal that the metric arises as the only relevant degree of freedom, when quantizing the Dirac matrices. Furthermore, we point out that a similar construction involving the vielbein is at least more complicated. The quantum gravity flow equations for the analysis of the parametrization dependence are calculated in appendix J. We collect the mathematical details for the derivation and the analysis of the beta function of the Gross-Neveu model in appendices K, L and M.

The compilation of this thesis is solely due to the author. However, a large part of the work presented here has been published in a number of articles and in collaboration with other authors. Chapters 3, 4 and 7 rely on papers written together with Holger Gies [53, 57–59]. The presentation of chapter 6 is founded on a collaboration with Benjamin Knorr and Holger Gies [122].

2 Curved Spacetimes

This chapter is devoted to collecting the concepts and formulas used for the description of a curved spacetime. Additionally we clarify our conventions. The first section is a simple recapitulation of the standard metric formulation and hence basically a compilation of formulas for later reference. In the second section we give an introduction into the vielbein formalism and give special emphasis on the differences between the metric and the vielbein degrees of freedom. For books on the subjects covered here see, e.g., [123–125].

2.1 Metric Formulation

Let us consider a $d \in \{2, 3, \ldots\}$ dimensional orientable pseudo-Riemannian manifold $\mathcal{M}$ without boundary.[4] On this manifold we introduce a coordinate system $\{x^\mu\}$, where the lower case greek indices run over $0, \ldots, d-1$.[5] We assume to have a symmetric nondegenerate metric $g_{\mu\nu}$ with indefinite signature of $\mathfrak{p}$ space-like directions (positive sign) and $\mathfrak{m} = d - \mathfrak{p}$ time-like directions (negative sign). To be more precise: The metric is a tensor on the manifold and $g_{\mu\nu}(\boldsymbol{x})$ are the components of this tensor at the point $\boldsymbol{x}$ in the coordinate basis $\{\hat{\partial}_\mu, \mathrm{d}\hat{x}^\mu\}$ of the coordinate system $\{x^\mu\}$. For brevity we drop the argument $\boldsymbol{x}$ in the following and call the components of a tensor also tensor, unless there could be any confusion. We denote the inverse of the metric with upper indices $g^{\mu\nu}$, where $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$. The Einstein summation convention is understood. Here δ^μ_ν is the basis independent Kronecker delta, which gives 1 for $\mu = \nu$ and 0 otherwise. If we change the coordinate system from $\{x^\mu\}$ to another one, say $\{x'^\mu\}$, the components of the metric transform such that the distance of infinitesimally neighboring points does not change. Hence, the transformed components of the metric $g'_{\mu\nu}$ read

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} g_{\alpha\beta} \frac{\partial x^\beta}{\partial x'^\nu}. \tag{2.1}$$

This leads to the definition of a tensor. The defining property of a tensor is its transformation behavior under coordinate transformations, it is required to transform analogously to the metric. Hence, the transformation rule for a general tensor ${T_{\text{gen}}}_{\mu_1 \ldots \mu_p}{}^{\nu_1 \ldots \nu_q}$ with p lower and q upper indices is given by

$${T_{\text{gen}}}_{\mu_1 \ldots \mu_p}{}^{\nu_1 \ldots \nu_q} \to {T_{\text{gen}}}'_{\mu_1 \ldots \mu_p}{}^{\nu_1 \ldots \nu_q} = \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\alpha_p}}{\partial x'^{\mu_p}} {T_{\text{gen}}}_{\alpha_1 \ldots \alpha_p}{}^{\beta_1 \ldots \beta_q} \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x'^{\nu_q}}{\partial x^{\beta_q}}. \tag{2.2}$$

We call a tensor with n indices a tensor of rank n. The indices of a tensor can be lowered and raised with the metric and its inverse respectively. Using the metric we can define a scalar

[4] The case $d = 1$ is special, as there the Riemann curvature tensor vanishes identically (see equation (2.31)) and the irreducible representation of the Clifford algebra is not traceless (see section 3.1 and equation (3.1)).

[5] Often one needs several coordinate patches to cover the complete manifold. Details can be found in the books [123–125] and also here in section 3.3.

product between two vectors (a tensor with one upper index) ${T_{\mathrm{gen}_1}}^\mu$ and ${T_{\mathrm{gen}_2}}^\mu$

$$g(T_{\mathrm{gen}_1}, T_{\mathrm{gen}_2}) = {T_{\mathrm{gen}_1}}^\mu g_{\mu\nu} {T_{\mathrm{gen}_2}}^\nu, \tag{2.3}$$

which is by construction completely independent of any basis or choice of coordinates, i.e. it is a scalar. Note that this product actually defines what we mean when we talk about the components of the metric $g_{\mu\nu} = g(\hat{\partial}_\mu, \hat{\partial}_\nu)$, in the coordinate basis $\{\hat{\partial}_\mu, \mathrm{d}\hat{x}^\mu\}$ of the coordinate system $\{x^\mu\}$. There is another coordinate and basis independent product between two tensors, i.e. if one has a one-form (a tensor with one lower index) $T_{\mathrm{gen}_1 \mu}$ and a vector ${T_{\mathrm{gen}_2}}^\mu$,

$$T_{\mathrm{gen}_1}(T_{\mathrm{gen}_2}) = T_{\mathrm{gen}_1 \mu} {T_{\mathrm{gen}_2}}^\mu. \tag{2.4}$$

As one can raise and lower indices using the metric this can also be viewed as a product of the type (2.3), hence we will denote this as scalar product as well. With the Kronecker delta we can construct a unit operator ${\delta_{\mathrm{S}}}^{\nu_1 \ldots \nu_p}_{\mu_1 \ldots \mu_p}$ for totally symmetric tensors $T_{\mathrm{S}\nu_1 \ldots \nu_p}$ with p lower indices,

$${\delta_{\mathrm{S}}}^{\nu_1 \ldots \nu_p}_{\mu_1 \ldots \mu_p} = \delta^{\ \nu_1}_{(\mu_1} \ldots \delta^{\nu_p}_{\ \mu_p)}, \qquad {\delta_{\mathrm{S}}}^{\nu_1 \ldots \nu_p}_{\mu_1 \ldots \mu_p} T_{\mathrm{S}\nu_1 \ldots \nu_p} = T_{\mathrm{S}\mu_1 \ldots \mu_p}, \tag{2.5}$$

where (...) around indices corresponds to the normalized complete symmetrization of the enclosed indices of the same height, e.g. ${\delta_{\mathrm{S}}}^{\nu_1 \nu_2}_{\mu_1 \mu_2} = \frac{1}{2}(\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} + \delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2})$ and ${T_{\mathrm{gen}}}_{(\mu}{}^{\rho}{}_{\nu)} = \frac{1}{2}({T_{\mathrm{gen}}}_{\mu}{}^{\rho}{}_{\nu} + {T_{\mathrm{gen}}}_{\nu}{}^{\rho}{}_{\mu})$. Analogously we define a unit operator ${\delta_{\mathrm{A}}}^{\nu_1 \ldots \nu_p}_{\mu_1 \ldots \mu_p}$ for totally antisymmetric tensors $T_{\mathrm{A}\nu_1 \ldots \nu_p}$ with p lower indices,

$${\delta_{\mathrm{A}}}^{\nu_1 \ldots \nu_p}_{\mu_1 \ldots \mu_p} = \delta^{\ \nu_1}_{[\mu_1} \ldots \delta^{\nu_p}_{\ \mu_p]}, \qquad {\delta_{\mathrm{A}}}^{\nu_1 \ldots \nu_p}_{\mu_1 \ldots \mu_p} T_{\mathrm{A}\nu_1 \ldots \nu_p} = T_{\mathrm{A}\mu_1 \ldots \mu_p}, \tag{2.6}$$

where $[\ldots]$ around indices corresponds to the normalized complete antisymmetrization of the enclosed indices of the same height, e.g., ${\delta_{\mathrm{A}}}^{\nu_1 \nu_2}_{\mu_1 \mu_2} = \frac{1}{2}(\delta^{\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2} - \delta^{\nu_2}_{\mu_1}\delta^{\nu_1}_{\mu_2})$ and ${T_{\mathrm{gen}}}_{[\mu}{}^{\rho}{}_{\nu]} = \frac{1}{2}({T_{\mathrm{gen}}}_{\mu}{}^{\rho}{}_{\nu} - {T_{\mathrm{gen}}}_{\nu}{}^{\rho}{}_{\mu})$. These tensors are also called p-forms. Next we introduce the Levi-Civita pseudo tensor $\varepsilon_{\mu_1 \ldots \mu_d}$ defined by

$$\varepsilon_{\mu_1 \ldots \mu_d} = \sqrt{\mathfrak{g}}\, \epsilon_{\mu_1 \ldots \mu_d}, \tag{2.7}$$

where $\mathfrak{g}$ is the absolute value of the determinant of the metric and $\epsilon_{\mu_1 \ldots \mu_d}$ is the totally antisymmetric Levi-Civita symbol with $\epsilon_{0 \ldots d-1} = 1$. Using upper indices the Levi-Civita pseudo tensor reads $\varepsilon^{\mu_1 \ldots \mu_d} = g^{\mu_1 \nu_1} \ldots g^{\mu_d \nu_d} \varepsilon_{\nu_1 \ldots \nu_d} = \frac{(-1)^{\mathrm{m}}}{\sqrt{\mathfrak{g}}} \epsilon_{\mu_1 \ldots \mu_d}$. The pseudo tensor property means that it transforms under a coordinate transformation as

$$\varepsilon_{\mu_1 \ldots \mu_d} \to {\varepsilon'}_{\mu_1 \ldots \mu_d} = \mathrm{sign}\left(\det \frac{\partial x^\nu}{\partial x'^\mu}\right) \cdot \frac{\partial x^{\nu_1}}{\partial x'^{\mu_1}} \ldots \frac{\partial x^{\nu_d}}{\partial x'^{\mu_d}} \varepsilon_{\nu_1 \ldots \nu_d}. \tag{2.8}$$

It is straightforward to derive the following identities:

$$T_{\text{gen}}{}^{\mu_1}{}_{\alpha_1} \ldots T_{\text{gen}}{}^{\mu_n}{}_{\alpha_n} \varepsilon_{\mu_1 \ldots \mu_n} = \det T_{\text{gen}} \cdot \varepsilon_{\alpha_1 \ldots \alpha_n}, \tag{2.9}$$

$$\delta_{\mathrm{A}}{}^{\mu_1 \ldots \mu_n \beta_1 \ldots \beta_p}_{\mu_1 \ldots \mu_n \alpha_1 \ldots \alpha_p} = \frac{p!\,(d-p)!}{(n+p)!\,(d-n-p)!} \delta_{\mathrm{A}}{}^{\beta_1 \ldots \beta_p}_{\alpha_1 \ldots \alpha_p}, \tag{2.10}$$

$$\varepsilon_{\mu_1 \ldots \mu_n \alpha_1 \ldots \alpha_{d-n}} \varepsilon^{\mu_1 \ldots \mu_n \beta_1 \ldots \beta_{d-n}} = (-1)^{\mathfrak{m}} n!(d-n)! \delta_{\mathrm{A}}{}^{\beta_1 \ldots \beta_{d-n}}_{\alpha_1 \ldots \alpha_{d-n}}. \tag{2.11}$$

Since the partial derivative of a tensor usually does not transform like a tensor, we need a generalization thereof which does so. First we have the exterior derivative d, which does not need any additional structure, but is only defined for p-forms $T_{\text{gen}_{\mu_1 \ldots \mu_p}} = T_{\text{gen}_{[\mu_1 \ldots \mu_p]}}$ via

$$(\mathrm{d}T_{\text{gen}})_{\mu_1 \ldots \mu_{p+1}} = (p+1)\partial_{[\mu_1} T_{\text{gen}_{\mu_2 \ldots \mu_{p+1}]}}. \tag{2.12}$$

It is straightforward to check that this quantity transforms as a tensor.

The second generalization of the partial derivative is the spacetime covariant derivative for which we need the spacetime connection. Using the metric and its first derivatives we introduce the Christoffel symbols $\left\{ {\rho \atop \mu\nu} \right\}$ with

$$\left\{ {\rho \atop \mu\nu} \right\} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right), \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{2.13}$$

The Christoffel symbol is symmetric with respect to the interchange of the lower index pair $(\mu \leftrightarrow \nu)$. Note that it does not represent the components of a tensor,

$$\left\{ {\rho \atop \mu\lambda} \right\} \to \frac{\partial x^\nu}{\partial x'^\mu} \left[\frac{\partial x'^\rho}{\partial x^\alpha} \left\{ {\alpha \atop \nu\beta} \right\} \frac{\partial x^\beta}{\partial x'^\lambda} - \left(\partial_\nu \frac{\partial x'^\rho}{\partial x^\alpha} \right) \frac{\partial x^\alpha}{\partial x'^\lambda} \right]. \tag{2.14}$$

Next we define the metric compatible, spacetime covariant derivative D_μ of a tensor $T_{\text{gen}}{}^\rho$ of rank one. It has to satisfy

$$\begin{array}{rll} \text{(i)} & \text{linearity:} & D_\mu (T_{\text{gen}_1}{}^\rho + T_{\text{gen}_2}{}^\rho) = D_\mu T_{\text{gen}_1}{}^\rho + D_\mu T_{\text{gen}_2}{}^\rho, \\ \text{(ii)} & \text{product rule:} & D_\mu (f_{\text{gen}} \cdot T_{\text{gen}}{}^\rho) = (\partial_\mu f_{\text{gen}}) \cdot T_{\text{gen}}{}^\rho + f_{\text{gen}} \cdot (D_\mu T_{\text{gen}}{}^\rho), \\ \text{(iii)} & \text{metric compatibility:} & D_\mu T_{\text{gen}_\nu} = g_{\nu\rho} \cdot D_\mu T_{\text{gen}}{}^\rho, \\ \text{(iv)} & \text{covariance:} & D_\mu T_{\text{gen}}{}^\rho \to D'_\mu T_{\text{gen}}{}'^\rho = \dfrac{\partial x^\nu}{\partial x'^\mu} \cdot \dfrac{\partial x'^\rho}{\partial x^\lambda} \cdot (D_\nu T_{\text{gen}}{}^\lambda), \end{array} \tag{2.15}$$

where f_{gen} in (ii) is a general scalar function. Analogous relations hold for tensors of rank n. With this definition one can show that the covariant derivative is of the following form:

$$D_\mu T_{\text{gen}}{}^\rho = \partial_\mu T_{\text{gen}}{}^\rho + \Gamma_\mu{}^\rho{}_\lambda T_{\text{gen}}{}^\lambda, \qquad \Gamma_\mu{}^\rho{}_\lambda = \left\{ {\rho \atop \mu\lambda} \right\} + K_\mu{}^\rho{}_\lambda, \qquad K_\mu{}^\rho{}_\lambda = -K_{\mu\lambda}{}^\rho, \tag{2.16}$$

where we introduced the spacetime connection $\Gamma_\mu{}^\rho{}_\lambda$ and the contorsion tensor $K_\mu{}^\rho{}_\lambda$. Contrary to the Christoffel symbol and the spacetime connection the contorsion tensor really is a tensor with the corresponding transformation behavior. The contorsion tensor has to be antisymmetric in the last two indices in order to be metric compatible, but is arbitrary otherwise.[6] It is sometimes useful to rewrite the contorsion tensor $K_{\mu\ \lambda}^{\ \rho}$ in terms of the torsion tensor $C_{\mu\lambda}{}^{\rho}$,

$$C_{\mu\lambda}{}^{\rho} = 2K_{[\mu\ \lambda]}^{\ \rho}, \qquad K_{\mu\ \lambda}^{\ \rho} = \frac{1}{2}(C_{\mu\lambda}{}^{\rho} - C_{\lambda\ \mu}^{\ \rho} + C^{\rho}{}_{\mu\lambda}). \tag{2.17}$$

Additionally to the full covariant derivative D_μ we define the Levi-Civita covariant derivative $D_{(\mathrm{LC})_\mu}$ to be

$$D_{(\mathrm{LC})_\mu} T_{\mathrm{gen}}{}^{\rho} = \partial_\mu T_{\mathrm{gen}}{}^{\rho} + \left\{ {\rho \atop \mu\lambda} \right\} T_{\mathrm{gen}}{}^{\lambda} \equiv D_\mu T_{\mathrm{gen}}{}^{\rho} - K_{\mu\ \lambda}^{\ \rho} T_{\mathrm{gen}}{}^{\lambda}, \tag{2.18}$$

and analogously for tensors of rank n. Note that D_μ and $D_{(\mathrm{LC})_\mu}$ agree if we were to add the condition of vanishing torsion,

$$\text{(v)} \quad \text{vanishing torsion:} \qquad D_\mu T_{\mathrm{gen}_\nu} - D_\nu T_{\mathrm{gen}_\mu} = \partial_\mu T_{\mathrm{gen}_\nu} - \partial_\nu T_{\mathrm{gen}_\mu}, \tag{2.19}$$

to the definition of D_μ, equation (2.15). In the following we will not assume that torsion vanishes. Note that the covariant derivative of the Levi-Civita tensor vanishes, $D_\mu \varepsilon_{\mu_1 \ldots \mu_d} = D_{(\mathrm{LC})_\mu} \varepsilon_{\mu_1 \ldots \mu_d} = 0$.

Another important concept is the integration over a manifold. For this we need the volume pseudo d-form dV defined as

$$\mathrm{dV} = \sqrt{\mathfrak{g}}\, \mathrm{d}\hat{x}^0 \wedge \ldots \wedge \mathrm{d}\hat{x}^{d-1} = \frac{1}{d!} \varepsilon_{\mu_1 \ldots \mu_d}\, \mathrm{d}\hat{x}^{\mu_1} \wedge \ldots \wedge \mathrm{d}\hat{x}^{\mu_d}, \tag{2.20}$$

where we introduced the wedge product $\wedge$, given by the completely antisymmetrized tensor product between a p-form $T_{\mathrm{gen}1_{\mu_1 \ldots \mu_p}}$ and a q-form $T_{\mathrm{gen}2_{\mu_1 \ldots \mu_q}}$, i.e. the $p+q$-form $T_{\mathrm{gen}_1} \wedge T_{\mathrm{gen}_2}$

$$(T_{\mathrm{gen}_1} \wedge T_{\mathrm{gen}_2})_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} T_{\mathrm{gen}1_{[\mu_1 \ldots \mu_p}} T_{\mathrm{gen}2_{\mu_{p+1} \ldots \mu_{p+q}]}}. \tag{2.21}$$

Note that every pseudo d-form on $\mathcal{M}$ can be written as scalar function times the volume pseudo form dV. The integration of a pseudo d-form $T_{\mathrm{gen}} = f_{\mathrm{gen}} \mathrm{dV}$ over $\mathcal{M}$ is then defined as

$$\int_{\mathcal{M}} T_{\mathrm{gen}} = \int_{\mathcal{M}} f_{\mathrm{gen}} \mathrm{dV} = \int \mathrm{d}x^0 \ldots \int \mathrm{d}x^d \sqrt{\mathfrak{g}} f_{\mathrm{gen}}, \tag{2.22}$$

with an appropriate generalization if multiple coordinate patches are needed. As a short hand

[6] If we have a tensor $T_{\mathrm{gen}}{}^{\mu}{}_{\nu}$ with one upper and one lower index, then there is no natural notion of symmetric or antisymmetric in general. However, as we have a metric on $\mathcal{M}$ we can raise and lower indices. Hence, in our case $T_{\mathrm{gen}}{}^{\mu}{}_{\nu}$ is said to be (anti)symmetric if $T_{\mathrm{gen}_{\mu\nu}}$ is (anti)symmetric.

we write $\int_x = \int_{\mathcal{M}} \mathrm{dV}$. Let us also note Stokes' theorem,

$$\int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega, \tag{2.23}$$

where ω is a pseudo $(d-1)$-form and $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$. Since we assume $\mathcal{M}$ to be without a boundary, the right-hand side of equation (2.23) vanishes in all cases considered in this thesis. Of special interest are pseudo d-forms of the type $(D_{(\mathrm{LC})\mu} T_{\mathrm{gen}}{}^{\mu})\mathrm{dV}$, where $T_{\mathrm{gen}}{}^{\mu}$ is an arbitrary vector. We can write these as exterior derivative $\mathrm{d}\omega = (D_{(\mathrm{LC})\mu} T_{\mathrm{gen}}{}^{\mu})\mathrm{dV}$ of the pseudo $(d-1)$-form $\omega_{\mu_2 \dots \mu_d} = \frac{1}{d!} T_{\mathrm{gen}}{}^{\mu_1} \varepsilon_{\mu_1 \dots \mu_d}$. Using Stokes' theorem this implies

$$\int_x D_{(\mathrm{LC})\mu} T_{\mathrm{gen}}{}^{\mu} = \int_{\mathcal{M}} \mathrm{d}\omega = \int_{\partial\mathcal{M}} \omega = 0. \tag{2.24}$$

Finally, we turn to the gravitational field strength known as the Riemann tensor $R_{\mu\nu}{}^{\rho}{}_{\lambda}$. It is defined as the commutator of two covariant derivatives amended by torsion,

$$R_{\mu\nu}{}^{\rho}{}_{\lambda} T_{\mathrm{gen}}{}^{\lambda} = [D_\mu, D_\nu] T_{\mathrm{gen}}{}^{\rho} + C_{\mu\nu}{}^{\sigma} D_\sigma T_{\mathrm{gen}}{}^{\rho}. \tag{2.25}$$

Employing the explicit form of the spacetime covariant derivative, equation (2.16), we find

$$R_{\mu\nu}{}^{\rho}{}_{\lambda} = \partial_\mu \Gamma_\nu{}^{\rho}{}_{\lambda} - \partial_\nu \Gamma_\mu{}^{\rho}{}_{\lambda} + \Gamma_\mu{}^{\rho}{}_{\sigma} \Gamma_\nu{}^{\sigma}{}_{\lambda} - \Gamma_\nu{}^{\rho}{}_{\sigma} \Gamma_\mu{}^{\sigma}{}_{\lambda}. \tag{2.26}$$

One can decompose this into

$$R_{\mu\nu}{}^{\rho}{}_{\lambda} = R_{(\mathrm{LC})\mu\nu}{}^{\rho}{}_{\lambda} + D_{(\mathrm{LC})\mu} K_\nu{}^{\rho}{}_{\lambda} - D_{(\mathrm{LC})\nu} K_\mu{}^{\rho}{}_{\lambda} + K_\mu{}^{\rho}{}_{\sigma} K_\nu{}^{\sigma}{}_{\lambda} - K_\nu{}^{\rho}{}_{\sigma} K_\mu{}^{\sigma}{}_{\lambda}, \tag{2.27}$$

where $R_{(\mathrm{LC})\mu\nu}{}^{\rho}{}_{\lambda}$ is the curvature tensor induced by the Christoffel symbols,

$$R_{(\mathrm{LC})\mu\nu}{}^{\rho}{}_{\lambda} = \partial_\mu \left\{ {\rho \atop \nu\lambda} \right\} - \partial_\nu \left\{ {\rho \atop \mu\lambda} \right\} + \left\{ {\rho \atop \mu\sigma} \right\} \left\{ {\sigma \atop \nu\lambda} \right\} - \left\{ {\rho \atop \nu\sigma} \right\} \left\{ {\sigma \atop \mu\lambda} \right\}. \tag{2.28}$$

By contraction of the indices, we get the Ricci curvature tensor $R_{\mu\nu}$, the Ricci curvature scalar R and their Levi-Civita analogs $R_{(\mathrm{LC})\mu\nu}$ and $R_{(\mathrm{LC})}$,

$$R_{\mu\nu} = R_{\alpha\mu}{}^{\alpha}{}_{\nu} = R_{(\mathrm{LC})\mu\nu} + D_{(\mathrm{LC})\alpha} K_\mu{}^{\alpha}{}_{\nu} - D_{(\mathrm{LC})\mu} K_\alpha{}^{\alpha}{}_{\nu} + K_\alpha{}^{\alpha}{}_{\sigma} K_\mu{}^{\sigma}{}_{\nu} - K_\mu{}^{\alpha}{}_{\sigma} K_\alpha{}^{\sigma}{}_{\nu}, \tag{2.29}$$

$$R = g^{\mu\nu} R_{\mu\nu} = R_{(\mathrm{LC})} - 2 D_{(\mathrm{LC})\alpha} K_\mu{}^{\mu\alpha} - K_\mu{}^{\mu\sigma} K_\alpha{}^{\alpha}{}_{\sigma} + K^{\mu\alpha\sigma} K_{\sigma\alpha\mu}. \tag{2.30}$$

The various curvature tensors satisfy some useful identities,

$$R_{(\mu\nu)\rho\lambda} = 0, \quad R_{\mu\nu(\rho\lambda)} = 0, \quad R_{(\mathrm{LC})[\mu\nu\rho]\lambda} = 0, \quad R_{(\mathrm{LC})\mu\nu\rho\lambda} = R_{(\mathrm{LC})\rho\lambda\mu\nu}, \tag{2.31}$$

$$D_{(\mathrm{LC})[\kappa} R_{(\mathrm{LC})\mu\nu]\rho\lambda} = 0, \quad D_{(\mathrm{LC})\alpha} R_{(\mathrm{LC})}{}^{\alpha}{}_{\lambda\mu\nu} = 2 D_{(\mathrm{LC})[\mu} R_{(\mathrm{LC})\nu]\lambda}, \quad D_{(\mathrm{LC})\alpha} R_{(\mathrm{LC})}{}^{\alpha}_{\mu} = \frac{1}{2} D_{(\mathrm{LC})\mu} R_{(\mathrm{LC})}.$$

The quantities defined above are the usual variables considered in the metric based description of curved spacetimes. In the next section we turn to the formulation of these in terms of the vielbein based description.

2.2 Vielbein Formulation

The following section is largely inspired by chapter 3 of Carroll's lecture notes on general relativity [126]. The idea of the vielbein was first introduced by Weyl [31] and independently also by Fock and Ivanenko [40]. A more modern treatment can be found in the books [41, 42, 125, 126] and in this nice and short paper [127].

In the previous section all tensor components referred to the so-called coordinate basis $\{\hat{\partial}_\mu, \mathrm{d}\hat{x}^\mu\}$. However, we are free to choose any other basis we like. Another very common choice is the vielbein basis $\{\hat{e}_a, \hat{\theta}^a\}$, which is chosen such that the scalar product of two basis elements satisfies

$$g(\hat{e}_a, \hat{e}_b) = \eta_{ab}, \qquad \hat{\theta}^a(\hat{e}_b) = \delta^a_b, \qquad a, b \in \{0, \ldots, d-1\}, \tag{2.32}$$

where $(\eta_{ab}) = \mathrm{diag}(-1, \ldots, -1, +1, \ldots, +1)$ is the flat metric with signature corresponding to the signature of the metric $g_{\mu\nu}$.[7] Note that the vielbein basis is not uniquely fixed by the above requirements. They enjoy a local Lorentz symmetry $\mathrm{O}(\mathfrak{m}, \mathfrak{p})$,

$$\hat{e}'_a = \hat{e}_b {\Lambda^{-1}_{\mathrm{Lor}}}^b{}_a, \quad \hat{\theta}'^a = {\Lambda_{\mathrm{Lor}}}^a{}_b \hat{\theta}^b, \quad g(\hat{e}'_a, \hat{e}'_b) = \eta_{ab}, \quad \hat{\theta}'^a(\hat{e}'_b) = \delta^a_b, \quad \Lambda_{\mathrm{Lor}} \in \mathrm{O}(\mathfrak{m}, \mathfrak{p}), \tag{2.33}$$

where

$$\mathrm{O}(\mathfrak{m}, \mathfrak{p}) = \{\Lambda_{\mathrm{Lor}} \in \mathrm{Mat}(d \times d) : \eta_{ab}\, {\Lambda_{\mathrm{Lor}}}^a{}_c\, {\Lambda_{\mathrm{Lor}}}^b{}_d = \eta_{cd}\}. \tag{2.34}$$

The vielbein $e_\mu{}^a$ then corresponds to the components of the coordinate basis elements $\{\hat{\partial}_\mu, \mathrm{d}\hat{x}^\mu\}$ in terms of the vielbein basis $\{\hat{e}_a, \hat{\theta}^a\}$,

$$e_\mu{}^a = \hat{\theta}^a(\hat{\partial}_\mu), \quad \hat{\partial}_\mu = e_\mu{}^a \hat{e}_a, \quad \mathrm{d}\hat{x}^\mu = e^\mu{}_a \hat{\theta}^a, \quad g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab}, \quad e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}, \tag{2.35}$$

where the Latin bein indices are lowered and raised with the flat metric η_{ab}. We can transform the components of any tensor T_{gen} with respect to the coordinate basis ${T_{\mathrm{gen}}}_{\mu_1 \ldots \mu_p}{}^{\nu_1 \ldots \nu_q}$ into the components with respect to the vielbein basis ${T_{\mathrm{gen}}}_{a_1 \ldots a_p}{}^{b_1 \ldots b_q}$,

$${T_{\mathrm{gen}}}_{\mu_1 \ldots \mu_p}{}^{\nu_1 \ldots \nu_q} = e_{\mu_1}{}^{a_1} \ldots e_{\mu_p}{}^{a_p} {T_{\mathrm{gen}}}_{a_1 \ldots a_p}{}^{b_1 \ldots b_q} e^{\nu_1}{}_{b_1} \ldots e^{\nu_q}{}_{b_q}. \tag{2.36}$$

Sometimes it can be helpful to consider components with mixed indices. This is particularly

[7] The $\{\hat{e}_a, \hat{\theta}^a\}$ do not form a coordinate basis in general. That is, usually there is no coordinate system on $\mathcal{M}$, such that the $\{\hat{e}_a, \hat{\theta}^a\}$ correspond to the coordinate basis of that coordinate system.

the case for the covariant derivative. Since the covariant derivative is a generalization of the partial derivative, it is somehow tied to coordinates. However, as the vielbein basis is explicitly not a coordinate basis one usually does not transform the index of the covariant derivative, even though this is possible in principle. During this thesis we use the symbol D_μ for the spacetime covariant derivative, respecting all the indices in the coordinate basis, but acting as a partial derivative for all other types of indices, like bein or spin indices. Let us calculate the components of the covariant derivative of a vector in a mixed basis,

$$(D_\mu {T_{\text{gen}}}^\nu)\mathrm{d}\hat{x}^\mu \otimes \hat{\partial}_\nu = \left[D_\mu({e^\nu}_b {T_{\text{gen}}}^b)\right] {e_\nu}^a \mathrm{d}\hat{x}^\mu \otimes \hat{e}_a = \left[(\partial_\mu {T_{\text{gen}}}^a) + {e_\nu}^a (D_\mu {e^\nu}_b) {T_{\text{gen}}}^b\right] \mathrm{d}\hat{x}^\mu \otimes \hat{e}_a. \tag{2.37}$$

We take this to define the vielbein covariant derivative $D_{(e)\mu}$, respecting the bein indices as well as the coordinate indices, by

$$D_{(e)\mu} {T_{\text{gen}}}^a = \partial_\mu {T_{\text{gen}}}^a + {\omega_{(e)\mu}}^a{}_b {T_{\text{gen}}}^b, \qquad {\omega_{(e)\mu}}^a{}_b = {e_\nu}^a D_\mu {e^\nu}_b = {e_\nu}^a \partial_\mu {e^\nu}_b + {e_\nu}^a {\Gamma_\mu}^\nu{}_\rho {e^\rho}_b, \tag{2.38}$$

with the vielbein connection ${\omega_{(e)\mu}}^a{}_b$.[8] This directly implies the vielbein postulate

$$D_{(e)\mu} {e_\nu}^a = 0, \tag{2.39}$$

and hence allows us to freely interchange between bein and coordinate indices. Hence, we can either express the vielbein connection ${\omega_{(e)\mu}}^a{}_b$ in terms of the spacetime connection ${\Gamma_\mu}^\rho{}_\lambda$ or vice versa. Like for the spacetime connection we can define a Levi-Civita part of the vielbein connection ${\omega_{(\mathrm{LC}e)\mu}}^a{}_b$,

$${\omega_{(\mathrm{LC}e)\mu}}^a{}_b = {e_\nu}^a D_{(\mathrm{LC})\mu} {e^\nu}_b, \qquad {\omega_{(e)\mu}}^a{}_b = {\omega_{(\mathrm{LC}e)\mu}}^a{}_b + {K_\mu}^a{}_b, \qquad {K_\mu}^a{}_b = {e_\rho}^a {K_\mu}^\rho{}_\lambda {e^\lambda}_b. \tag{2.40}$$

In particular the metric compatibility then implies that the vielbein connection is antisymmetric in the bein indices,

$$0 = D_\mu g_{\rho\lambda} \quad \Rightarrow \quad 0 = D_{(e)\mu}\eta_{ab} \quad \Rightarrow \quad 0 = \omega_{(e)\mu ab} + \omega_{(e)\mu ba}. \tag{2.41}$$

The curvature, ${R_{\mu\nu}}^a{}_b$, in this formalism is then defined by

$${R_{\mu\nu}}^a{}_b {T_{\text{gen}}}^b = [D_{(e)\mu}, D_{(e)\nu}] {T_{\text{gen}}}^a + {C_{\mu\nu}}^\sigma D_{(e)\sigma} {T_{\text{gen}}}^a. \tag{2.42}$$

[8] This connection is often called "spin connection" in the literature. Unfortunately this terminology is ambiguous in the present context. Hence, we want to distinguish between the "spacetime connection" ${\Gamma_\mu}^\rho{}_\lambda$, used for coordinate indices, the "vielbein connection" ${\omega_{(e)\mu}}^a{}_b$, used for the bein indices, the "vielbein spin connection" $\Gamma_{(e)\mu}$, used for the fermions in the vielbein formulation, and the "spin connection" Γ_μ, used for fermions in the new spin base formulation. The vielbein spin connection and the spin connection are introduced in sections 3.2 and 4.2 respectively.

In an explicit form it reads

$$R_{\mu\nu}{}^{a}{}_{b} = e_{\rho}{}^{a} R_{\mu\nu}{}^{\rho}{}_{\lambda} e^{\lambda}{}_{b} = \partial_{\mu}\omega_{(e)\nu}{}^{a}{}_{b} - \partial_{\nu}\omega_{(e)\mu}{}^{a}{}_{b} + \omega_{(e)\mu}{}^{a}{}_{c}\omega_{(e)\nu}{}^{c}{}_{b} - \omega_{(e)\nu}{}^{a}{}_{c}\omega_{(e)\mu}{}^{c}{}_{b}. \tag{2.43}$$

The last missing operation is the integration over a manifold, for which we need the volume pseudo form dV, cf. equation (2.20). In terms of the vielbein it reads

$$\mathrm{dV} = \frac{\operatorname{sign}(\det e_{\mu}{}^{a})}{d!} \epsilon_{a_1 \dots a_d} \hat{\theta}^{a_1} \wedge \dots \wedge \hat{\theta}^{a_d} = \left|\det e_{\mu}{}^{a}\right| \mathrm{d}\hat{x}^{0} \wedge \dots \wedge \mathrm{d}\hat{x}^{d-1}. \tag{2.44}$$

Now that we have established the concepts of the vielbein formalism we can compare it to the metric formulation. First of all we note that the vielbein has d^2 independent components, whereas the metric only has $\frac{d(d+1)}{2}$. These additional degrees of freedom are compensated by the $\frac{d(d-1)}{2}$ generators of the local Lorentz symmetry. As we have seen, the vielbein connection and the spacetime connection can be converted into each other with the aid of equation (2.39). Because of the distinction of the bein indices and the spacetime indices the interpretation of the curvature as the field strength of geometry becomes more apparent and the definition directly resembles the field strength of standard gauge theory. Furthermore the vielbein is often called the more fundamental quantity compared to the metric, since it accommodates the spin degrees of freedom and one can derive the metric from the vielbein. Nevertheless we want to stress that the vielbein together with the local Lorentz symmetry is completely artificial, as we have seen there was no reason to introduce it at any step so far. Contrary to the common literature [41, 42] this remains true even when we are going to introduce fermions in the next section.

Additionally, note that on generic manifolds there will be no global set of smooth basis vector fields, serving as a global vielbein. One has to cover the manifold by multiple patches, similar to the multiple coordinate patches one needs in order to cover the whole manifold. However, this is not the case for the metric. There is one globally well defined metric on the manifold, but there are infinitely many (usually nonglobal) vielbeins. From this point of view the vielbein is just another specific basis, which we can choose but do not have to.

3 Motivation of Spin Base Invariance

The main results of this thesis are covered in the following two chapters, which are based on our publications [57–59]. We present the spin-base invariant formulation of fermions in curved spacetimes in full detail. For the sake of generality we added the discussion of arbitrary signatures.

In this chapter we motivate the spin-base invariance on quite general grounds. We especially consider the behavior of fermions under coordinate transformations. Further we find a hidden spin-base invariance in the vielbein formulation, supporting the idea of this invariance being a fundamental symmetry of nature. As an application we construct a global spin-base on the 2-sphere, a manifold where it is not possible to find a global vielbein.

3.1 How Fermions Transform under Coordinate Transformations

In this section we give a motivation for the spin-base invariant formalism. We aim at describing fermions in a curved spacetime with $\mathfrak{m}$ timelike and $\mathfrak{p} = d - \mathfrak{m}$ spacelike directions. The Dirac structure, present in any description of Dirac fermions, is introduced with the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathrm{I}, \qquad \gamma_\mu \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C}), \qquad d_\gamma = 2^{\lfloor d/2 \rfloor}, \tag{3.1}$$

where $\mathrm{I} \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ is the $d_\gamma \times d_\gamma$ unit matrix. The Dirac matrices γ_μ are complex $d_\gamma \times d_\gamma$ matrices and constitute an irreducible[9] representation of the Clifford algebra. It is important to note that the Clifford algebra enjoys an invariance with respect to similarity transformations $\gamma_\mu \to \mathcal{S} \gamma_\mu \mathcal{S}^{-1}$, where $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ [128, 129]. As we change the representation of the Clifford algebra by such a similarity transformation, this corresponds to a change of the spin-base. Fermions are then represented as Graßmann valued vectors ψ in Dirac space with d_γ components (i.e. objects with upper spin indices). The corresponding dual vectors $\bar{\psi}$ are denoted with a bar (i.e. objects with lower spin indices). Vectors ψ are related to their dual vectors $\bar{\psi}$ via the spin metric h,

$$\bar{\psi} = \psi^\dagger h. \tag{3.2}$$

We will give a precise definition of the spin metric later. For the moment it suffices to know that we need a spin metric in order to define a product between two fermionic fields ψ and χ which results in a scalar with respect to coordinate transformations,

$$\bar{\psi} \chi = \psi^\dagger h \chi. \tag{3.3}$$

[9] The irreducible representation ensures that we do not introduce any redundant degrees of freedom. We comment on reducible representations in appendix G.

This is completely analogous to the scalar product using the spacetime metric $g_{\mu\nu}$, cf. equations (2.3) and (2.4). Additionally we require this spin metric to not introduce any scale and therefore demand

$$|\det h| = 1, \tag{3.4}$$

which is reminiscent to the fixed determinant of the spacetime metric in unimodular gravity, cf. [130–134]. As is well known, in four dimensional flat spacetime we can choose Cartesian coordinates and the Dirac matrices in Dirac representation [129]. There the spin metric turns out to be $h = \gamma^0$. In other representations of the Dirac matrices and different coordinate systems the spin metric is in general not equal to γ^0. We will see how this comes about later on.

First we have to understand what fermions are. From the view point of a theoretical physicist this means that we need to know how they transform under which symmetry group. Since we deal with curved spacetimes we have to know how the fermionic fields behave under coordinate transformations. To this end one usually looks at $\bar\psi\gamma^\mu\psi$ and demands that this object transforms like a usual contravariant spacetime vector since the complete Dirac structure is eliminated,

$$\bar\psi\gamma^\mu\psi \to \frac{\partial x'^\mu}{\partial x^\rho}\bar\psi\gamma^\rho\psi = \bar\psi\frac{\partial x'^\mu}{\partial x^\rho}\gamma^\rho\psi. \tag{3.5}$$

In flat spacetimes in Cartesian coordinates, where one usually restricts oneself to Lorentz transformations $\Lambda_{\mathrm{Lor}}{}^a{}_b$ as coordinate transformations, we are used to a nice property of the flat Dirac matrices $\gamma_{(\mathrm{f})}{}^a$, namely

$$\mathcal{S}_{\mathrm{Lor}}^{-1}\gamma_{(\mathrm{f})}{}^a\mathcal{S}_{\mathrm{Lor}} = \Lambda_{\mathrm{Lor}}{}^a{}_b\gamma_{(\mathrm{f})}{}^b, \tag{3.6}$$

where $\mathcal{S}_{\mathrm{Lor}} \in \mathrm{Spin}(\mathfrak{m},\mathfrak{p})$ and $\Lambda_{\mathrm{Lor}}{}^a{}_b \in \mathrm{SO}^+(\mathfrak{m},\mathfrak{p})$ is the corresponding Lorentz transformation [129].[10] Therefore we can write

$$\bar\psi\gamma_{(\mathrm{f})}{}^a\psi \to \bar\psi'\gamma_{(\mathrm{f})}{}^a\psi' \stackrel{!}{=} \Lambda_{\mathrm{Lor}}{}^a{}_b\bar\psi\gamma_{(\mathrm{f})}{}^b\psi = \bar\psi\mathcal{S}_{\mathrm{Lor}}^{-1}\gamma_{(\mathrm{f})}{}^a\mathcal{S}_{\mathrm{Lor}}\psi, \tag{3.7}$$

which suggests that fermions transform under Lorentz transformations according to

$$\psi \to \psi' = \mathcal{S}_{\mathrm{Lor}}\psi, \quad \bar\psi \to \bar\psi' = \bar\psi\mathcal{S}_{\mathrm{Lor}}^{-1}. \tag{3.8}$$

However, this is rather a group theoretical accident for Lorentz transformations than a rule for general coordinate transformations which enjoy no such relation. One simple counterexample is the stretching of one of the axes, $x^3 \to x'^3 = \frac{1}{\alpha}x^3$. Then the Minkowski metric in $d = 4$

[10] With $\mathrm{SO}^+(\mathfrak{m},\mathfrak{p})$ we denote the identity component of $\mathrm{SO}(\mathfrak{m},\mathfrak{p})$, preserving the orientations of the negative and the positive signature subspaces.

spacetime dimensions changes to

$$(\eta_{ab}) = \mathrm{diag}(-1,1,1,1) \to (\eta'_{ab}) = \mathrm{diag}(-1,1,1,\alpha^2), \tag{3.9}$$

and therefore the transformed Dirac matrix ${\gamma_{(\mathrm{f})}}'^3$ would have to square to $\alpha^2\mathrm{I}$. This cannot be achieved via a similarity transformation, since $\left(\mathcal{S}{\gamma_{(\mathrm{f})}}^3\mathcal{S}^{-1}\right)^2 = \mathrm{I}$ for all $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$. This example illustrates why it is in general not possible to pass on a coordinate transformation to a similarity transformation and why we should start rethinking.

We will give an intuitive introduction to spin-base invariance in the following. If we perform a general coordinate transformation we have to transform the metric, unlike for Lorentz transformations, in a nontrivial way,

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} g_{\rho\lambda}. \tag{3.10}$$

Therefore we also have to transform the Dirac matrices nontrivially, $\gamma_\mu \to \gamma'_\mu$. Taking the Clifford algebra as a guideline, we find

$$\{\gamma'_\mu, \gamma'_\nu\} = 2g'_{\mu\nu}\mathrm{I} = 2\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} g_{\rho\lambda}\mathrm{I} = \left\{\frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho, \frac{\partial x^\lambda}{\partial x'^\nu}\gamma_\lambda\right\}. \tag{3.11}$$

This equation implies that

$$\gamma_\mu \to \gamma'_\mu = \begin{cases} \frac{\partial x^\rho}{\partial x'^\mu}\mathcal{S}\gamma_\rho\mathcal{S}^{-1} & , d \text{ even}, \\ \pm\frac{\partial x^\rho}{\partial x'^\mu}\mathcal{S}\gamma_\rho\mathcal{S}^{-1} & , d \text{ odd}, \end{cases} \tag{3.12}$$

where $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ is arbitrary. The proof of this relation uses that every irreducible representation of the Clifford algebra for a given metric is connected to each other via a similarity transformation and in odd dimensions, if necessary, via an additional sign change (since there are two connected components) [128, 129]. This sign flip has to be global if we want the Dirac matrices to be differentiable.

We can rephrase our findings of equation (3.12) by saying that a coordinate transformation for Dirac matrices is a combination of the usual transformation of the vector part $\frac{\partial x^\rho}{\partial x'^\mu}$, a similarity transformation $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ and if necessary a sign flip. Since we still have a solution to the Clifford algebra if we perform a similarity transformation or a sign flip on the Dirac matrices, we should distinguish two kinds of coordinate transformations [57, 59].

First we have the usual spacetime coordinate transformations,

$$\gamma_\mu \to \gamma'_\mu = \frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho. \tag{3.13}$$

These transformations change the spacetime coordinate bases and are called diffeomorphisms, cf. equation (2.2). Second we have the similarity transformations $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$, and in odd dimensions also the sign flip, which are the Dirac (or spin) coordinate transformations,

$$\gamma_\mu \to {\gamma'}_\mu = \begin{cases} \mathcal{S}\gamma_\mu \mathcal{S}^{-1} & , d \text{ even}, \\ \pm\mathcal{S}\gamma_\mu \mathcal{S}^{-1} & , d \text{ odd}. \end{cases} \tag{3.14}$$

They change the spin basis and therefore we will call them spin-base transformations in the following.

At the moment the choice of $\mathrm{SL}(d_\gamma, \mathbb{C})$ as the transformation group[11] for the spin-base transformations seems a little arbitrary. For example, we could also take $\mathrm{GL}(d_\gamma, \mathbb{C})$ or $\mathrm{SL}(d_\gamma, \mathbb{C})/\mathbb{Z}_{d_\gamma}$. However, it turns out that $\mathrm{SL}(d_\gamma, \mathbb{C})$ is special. In order to formalize this choice, we have to clarify what we need from the spin-base transformations.

First of all we are dealing with different choices for a spin-base, therefore we need a group $\mathrm{SB}_{\mathrm{min}}$ to connect these. As the different spin-bases are connected via similarity transformations, this group should be a subgroup of $\mathrm{GL}(d_\gamma, \mathbb{C})$, with the usual matrix multiplication as the group law, $\mathrm{SB}_{\mathrm{min}} \leq \mathrm{GL}(d_\gamma, \mathbb{C})$. Next we have to ensure that we do not miss any spin-base, i.e. every two sets γ_μ and ${\gamma'}_\mu$ compatible with the Clifford algebra for a given metric have to be connected via equation (3.14) where $\mathcal{S} \in \mathrm{SB}_{\mathrm{min}}$. Finally we want to keep $\mathrm{SB}_{\mathrm{min}}$ minimal in order not to artificially inflate the symmetry. In other words we have to minimize the cardinality of the set $\{\mathcal{S} \in \mathrm{SB}_{\mathrm{min}} : \mathcal{S}\gamma_\mu\mathcal{S}^{-1} = \gamma_\mu\}$. In appendix B it is shown that $\mathrm{SB}_{\mathrm{min}} = \mathrm{SL}(d_\gamma, \mathbb{C})$ is the unique group satisfying the preceding conditions.

A general coordinate transformation of the Dirac matrices is therefore given by an independent change of the spacetime base and the spin-base. Here independent means that we can in principle perform one of them without the other, as long as we stay on one fixed patch of the manifold. Anyway, we have to keep in mind that there might be some topological obstructions similar to those encountered in the vielbein formalism. There it can happen that one has to change the orthonormal frame while changing the patch on the manifold. For the vielbein this is already true on the 2-sphere due to the Poincaré-Brouwer (hairy ball) theorem. The Dirac matrices on the other hand do have a global spin-base on the 2-sphere, rendering the complete decoupling of spacetime coordinates and spin-bases obvious. A detailed analysis of the situation on the 2-sphere will be given in section 3.3. Whether a global spin-base exists on all metrizable manifolds is unclear so far.

Now we can turn back to the question of how the fermionic fields behave under spacetime coordinate transformations and spin-base transformations. For the description of dynamics we need a kinetic fermion term. If we additionally want to have covariance we need this term to be invariant under all types of coordinate transformations. We assume the kinetic term to be

[11] In fact we are dealing with the fundamental representation of $\mathrm{SL}(d_\gamma, \mathbb{C})$ and not the group itself. Still, we will keep this terminology in the following for simplicity, as we are exclusively working with the representations of the groups throughout this work. By fundamental representation we mean the defining matrix representation of $\mathrm{SL}(d_\gamma, \mathbb{C})$, which is $\{\mathcal{S} \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C}) : \det \mathcal{S} = 1\}$ together with the matrix multiplication as the group law.

of the form $\bar{\psi}\nabla\!\!\!/\psi$ where $\nabla\!\!\!/ = \gamma^\mu\nabla_\mu$ is the Dirac operator with ∇_μ the covariant derivative. Again we postpone the precise definition of ∇_μ, but for the moment it is sufficient to know that this derivative has to have two important properties: First, if ψ is a fermionic Dirac spinor, then $\nabla_\mu\psi$ is also a fermionic Dirac spinor, i.e. it transforms in the same way under spin-base transformations.[12] Second, if ψ is a spacetime tensor, then $\nabla_\mu\psi$ is a spacetime tensor of one rank higher, i.e. the additional spacetime index μ transforms as a covariant vector index under spacetime coordinate transformations. At the moment we do not assume anything about the tensorial rank of ψ. Since $\nabla_\mu\psi$ acts exactly like ψ under spin-base transformations and as a tensor of one rank higher than ψ under spacetime coordinate transformations, we can investigate $\bar{\psi}\gamma^\mu\psi$ instead of the original kinetic operator, demanding that it transforms as a scalar under spin-base transformations and as a contravariant vector under spacetime coordinate transformations.

The discussion straightforwardly generalizes to fermions with further internal (flavor, color) symmetries. As we are dealing with complex degrees of freedom, we expect to find a U(1) symmetry for $\bar{\psi}\gamma^\mu\psi$. If we dealt with N_f families of fermions we would find a $\mathrm{U}(N_\mathrm{f})$ symmetry, similar to the gauge symmetries of the standard model of particle physics. We are going to ignore these symmetries for the most part of the thesis, as we could always regain them by adding a gauge field with an appropriate charge to the covariant derivative, cf. appendix H. The even- and the odd-dimensional case are structurally very different, therefore we will discuss them separately.

3.1.1 Odd-Dimensional Case

First we look at the behavior under spin-base transformations $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$. To this end we remind ourselves that the Dirac matrices and their antisymmetric combinations form a complete basis in $\mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$, the set of all $d_\gamma \times d_\gamma$ matrices [129]. In the odd-dimensional case we need only the antisymmetric combinations with an even number of Dirac matrices to decompose an arbitrary $M_\mathrm{gen} \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$, $M_\mathrm{gen} = \sum_{n=0}^{\frac{d-1}{2}} {m_\mathrm{gen}}^{\mu_1\cdots\mu_{2n}}\gamma_{\mu_1\ldots\mu_{2n}}$, with the "coordinates" ${m_\mathrm{gen}}^{\mu_1\cdots\mu_{2n}} \in \mathbb{C}$, whose indices are completely antisymmetrized. The antisymmetric combinations of the Dirac matrices are given by

$$\gamma_{\mu_1\ldots\mu_n} = \begin{cases} \mathrm{I} & , n = 0 \\ \gamma_{[\mu_1}\ldots\gamma_{\mu_n]} & , n \geq 1 \end{cases}. \tag{3.15}$$

Since in odd dimensions the basis elements are the $\gamma_{\mu_1\ldots\mu_{2n}}$, they transform homogeneously under spin-base transformations because the possible sign flip drops out. In appendix C and D

[12] "Spinors" in mathematics are well defined objects, and are intimately related to the vielbein formulation. We will use the term "spinor" for the fermions in our new spin-base formulation as well. On the one hand this keeps the used language familiar, and on the other hand the fermions in the spin-base formulation are closely related to the fermions in the vielbein formulation.

we have collected some important properties of the Dirac matrices and the basis elements.

Now we look at the behavior under spin-base transformations $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ of $\bar{\psi}\gamma_\mu\psi$,

$$\bar{\psi}\gamma_\mu\psi \to \bar{\psi}'\gamma'_\mu\psi' = \pm\bar{\psi}'\mathcal{S}\gamma_\mu\mathcal{S}^{-1}\psi' \stackrel{!}{=} \bar{\psi}\gamma_\mu\psi, \tag{3.16}$$

and demand invariance. Without loss of generality we make the ansatz

$$\psi' = \mathcal{S}\mathcal{B}\psi, \qquad h' = (\mathcal{S}^\dagger)^{-1}(\mathcal{B}^\dagger)^{-1}h\mathcal{C}\mathcal{S}^{-1}, \tag{3.17}$$

where $\mathcal{B}, \mathcal{C} \in \mathrm{GL}(d_\gamma, \mathbb{C})$ are arbitrary invertible matrices. Note that the invertibility of $\mathcal{B}$ and $\mathcal{C}$ is mandatory because otherwise we would violate the reversibility of spin-base transformations and they would not form a group. Plugging in our ansatz we get $\pm\bar{\psi}\mathcal{C}\gamma_\mu\mathcal{B}\psi = \bar{\psi}\gamma_\mu\psi$. Because of the independence of ψ and $\psi^\dagger$ we conclude

$$\pm\mathcal{C}\gamma_\mu\mathcal{B} = \gamma_\mu. \tag{3.18}$$

By multiplying with $\pm(d\mathcal{B})^{-1}\gamma^\mu$ from the right, we can read off $\mathcal{C} = \pm\gamma_\rho(d\mathcal{B})^{-1}\gamma^\rho$. Inserting this back into equation (3.18) we get

$$\left(\gamma_\rho(d\mathcal{B})^{-1}\gamma^\rho\right)\gamma_\mu\mathcal{B} = \gamma_\mu. \tag{3.19}$$

If we multiply with $d^{-1}\gamma^\mu$ from the right we infer $d^2(\gamma_\rho\mathcal{B}^{-1}\gamma^\rho)^{-1} = \gamma_\lambda\mathcal{B}\gamma^\lambda$. Therefore we can rewrite equation (3.19) as $d\gamma_\mu\mathcal{B} = (\gamma_\lambda\mathcal{B}\gamma^\lambda)\gamma_\mu$. We finally multiply with $d^{-1}\gamma^\mu$ from the left and find

$$d^2\mathcal{B} = \gamma_\mu(\gamma_\lambda\mathcal{B}\gamma^\lambda)\gamma^\mu. \tag{3.20}$$

Now we use that we can write $\mathcal{B}$ as $\mathcal{B} = \sum_{n=0}^{\frac{d-1}{2}} b_{\rho_1\dots\rho_{2n}}\gamma^{\rho_1\dots\rho_{2n}}$ and use the identity (C.11) from App. C to calculate

$$\gamma_\lambda\mathcal{B}\gamma^\lambda = \sum_{n=0}^{\frac{d-1}{2}}(d-4n)b_{\rho_1\dots\rho_{2n}}\gamma^{\rho_1\dots\rho_{2n}}, \qquad \gamma_\mu(\gamma_\lambda\mathcal{B}\gamma^\lambda)\gamma^\mu = \sum_{n=0}^{\frac{d-1}{2}}(d-4n)^2 b_{\rho_1\dots\rho_{2n}}\gamma^{\rho_1\dots\rho_{2n}}. \tag{3.21}$$

Together with equation (3.20) and a comparison of coefficients we conclude

$$b_{\rho_1\dots\rho_{2n}} = \left(1-4\frac{n}{d}\right)^2 b_{\rho_1\dots\rho_{2n}}, \qquad n \in \left\{0,\dots,\frac{d-1}{2}\right\}. \tag{3.22}$$

These equations imply

$$\mathcal{B} = b\cdot\mathrm{I}, \qquad \mathcal{C} = \pm\frac{1}{b}\cdot\mathrm{I}, \qquad b \in \mathbb{C}\backslash\{0\}. \tag{3.23}$$

Since equation (3.4) has to be a spin-base independent statement, also $|\det h'| = 1$ has to hold. Therefore b is restricted to a U(1) phase, $b = \mathrm{e}^{\mathrm{i}\varphi} \in \mathrm{U}(1)$. Summing up, we found $\psi \to \mathrm{e}^{\mathrm{i}\varphi}\mathcal{S}\psi$. The $\mathrm{SL}(d_\gamma, \mathbb{C})$ part is the nontrivial spin-base transformation, whereas the U(1) phase is the aforementioned gauge symmetry which we are going to ignore. Note the reminiscence to the $\mathrm{Spin}^{\mathbb{C}}$ group [135]. The transformation law for spin-base transformations $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ in odd dimensions then reads

$$\gamma_\mu \to \pm\mathcal{S}\gamma_\mu\mathcal{S}^{-1}, \quad \psi \to \mathcal{S}\psi, \quad \bar{\psi} \to \pm\bar{\psi}\mathcal{S}^{-1}, \quad h \to \pm(\mathcal{S}^\dagger)^{-1}h\mathcal{S}^{-1}. \tag{3.24}$$

Actually the U(1) phase would drop out of the transformation law of γ_μ and h, confirming that this symmetry is independent of the spin-base transformations.

Next we investigate the behavior under diffeomorphisms. Again we look at $\bar{\psi}\gamma_\mu\psi$ and demand that it behaves like a covariant vector,

$$\bar{\psi}\gamma_\mu\psi \to \bar{\psi}'\gamma'_\mu\psi' = \bar{\psi}'\frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho\psi' \overset{!}{=} \frac{\partial x^\rho}{\partial x'^\mu}\bar{\psi}\gamma_\rho\psi. \tag{3.25}$$

Now we can go through the same steps as for the spin-base transformations and we find that the fermions have to transform like scalars under spacetime coordinate transformations again with an additional arbitrary phase transformation, which we neglect. Therefore we find the transformation law under diffeomorphisms in odd dimensions as

$$\gamma_\mu \to \frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho, \quad \psi \to \psi, \quad \bar{\psi} \to \bar{\psi}, \quad h \to h. \tag{3.26}$$

An important remark is in order here. Since the Clifford algebra has two connected components in odd dimensions we had to introduce the sign flip for the spin-base transformations. This sign flip spoils full spin-base invariance of a mass term $\bar{\psi}\psi$, since this sign flip does not drop out as for $\bar{\psi}\gamma_\mu\psi$. This implies that $\bar{\psi}\psi$ transforms as a scalar under the continuous part $\mathcal{S}$, but as a pseudo scalar under the discrete sign flip.

3.1.2 Even-Dimensional Case

To find the transformation behavior in even dimensions we proceed in a similar way as for the odd-dimensional case. First we introduce the complete basis [129] in $\mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ in terms of the $\gamma_{\mu_1 \dots \mu_n}$. We can then express an arbitrary $M_{\mathrm{gen}} \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ as $M_{\mathrm{gen}} = \sum_{n=0}^{d} {m_{\mathrm{gen}}}^{\mu_1\dots\mu_n}\gamma_{\mu_1\dots\mu_n}$, where the ${m_{\mathrm{gen}}}^{\mu_1\dots\mu_n}$ are the "coordinates" with respect to this basis, whose indices are completely antisymmetrized. Additionally we introduce the matrix γ_* defined solely in even dimensions as

$$\gamma_* = \frac{\mathrm{i}^{(\mathfrak{m}-\mathfrak{p})/2}}{d!}\varepsilon_{\mu_1\dots\mu_d}\gamma^{\mu_1}\dots\gamma^{\mu_d} \equiv \frac{\mathrm{i}^{(\mathfrak{m}-\mathfrak{p})/2}}{d!}\varepsilon_{\mu_1\dots\mu_d}\gamma^{\mu_1\dots\mu_d}. \tag{3.27}$$

The defining properties (up to a sign) of γ_* are

$$\{\gamma_*, \gamma_\mu\} = 0, \qquad \operatorname{tr}\gamma_* = 0, \qquad \gamma_*^2 = \mathrm{I}. \tag{3.28}$$

Again we start with the spin-base transformations and analyze the behavior of $\bar\psi\gamma_\mu\psi$,

$$\bar\psi\gamma_\mu\psi \to \bar\psi'\gamma'_\mu\psi' = \bar\psi'\mathcal{S}\gamma_\mu\mathcal{S}^{-1}\psi' \overset{!}{=} \bar\psi\gamma_\mu\psi \tag{3.29}$$

demanding that it behaves like a scalar. We employ again the general ansatz

$$\psi' = \mathcal{S}\mathcal{B}\psi, \qquad h' = (\mathcal{S}^\dagger)^{-1}(\mathcal{B}^\dagger)^{-1}h\mathcal{C}\mathcal{S}^{-1}, \tag{3.30}$$

with $\mathcal{B}, \mathcal{C} \in \mathrm{GL}(d_\gamma, \mathbb{C})$ arbitrary. Following the same route as before we find $\mathcal{C}\gamma_\mu\mathcal{B} = \gamma_\mu$ and from there with similar manipulations $\mathcal{C} = \gamma_\rho(d\mathcal{B})^{-1}\gamma^\rho$ and $d^2\mathcal{B} = \gamma_\mu(\gamma_\lambda\mathcal{B}\gamma^\lambda)\gamma^\mu$. Here we use the convenient basis $\gamma^{\mu_1\cdots\mu_n}$ for $\mathcal{B}$, $\mathcal{B} = \sum_{n=0}^{d} b_{\mu_1\cdots\mu_n}\gamma^{\mu_1\cdots\mu_n}$. With the aid of the identity (C.11) from appendix C we then calculate

$$\gamma_\lambda\mathcal{B}\gamma^\lambda = \sum_{n=0}^{d}(-1)^n(d-2n)b_{\rho_1\cdots\rho_n}\gamma^{\rho_1\cdots\rho_n}, \quad \gamma_\mu(\gamma_\lambda\mathcal{B}\gamma^\lambda)\gamma^\mu = \sum_{n=0}^{d}(d-2n)^2 b_{\rho_1\cdots\rho_n}\gamma^{\rho_1\cdots\rho_n}. \tag{3.31}$$

By comparison of the coefficients we can read off

$$b_{\rho_1\cdots\rho_n} = \left(1 - 2\frac{n}{d}\right)^2 b_{\rho_1\cdots\rho_n}, \qquad n \in \{0, \ldots, d\}. \tag{3.32}$$

This time the general solution is

$$\mathcal{B} = b_1 \mathrm{e}^{b_2\gamma_*} = b_1(\cosh b_2 \cdot \mathrm{I} + \sinh b_2 \cdot \gamma_*), \qquad \mathcal{C} = \frac{1}{b_1}\mathrm{e}^{b_2\gamma_*}, \qquad b_1 \in \mathbb{C}\backslash\{0\}, \qquad b_2 \in \mathbb{C}. \tag{3.33}$$

Since $\det \mathrm{e}^{b_2\gamma_*} = 1$, the implementation of equation (3.4) restricts b_1 to a U(1) phase, $b_1 = \mathrm{e}^{\mathrm{i}\varphi} \in \mathrm{U}(1)$. That means by solely demanding that the kinetic term is invariant under spin-base transformations we have another degree of freedom. We cannot only have a phase transformation $\mathrm{e}^{\mathrm{i}\varphi}$ but also a nontrivial chiral transformation $\mathrm{e}^{b_2\gamma_*}$. As usual, the chiral symmetry can be broken explicitly by a mass term $\bar\psi\psi$. We demand that it transforms as a scalar under all spin-base transformations since the Clifford algebra has only one connected component in even dimensions. If we thus also demand that

$$\bar\psi\psi \to \bar\psi'\psi' = \bar\psi\mathcal{C}\mathcal{B}\psi \overset{!}{=} \bar\psi\psi, \tag{3.34}$$

we find that

$$\mathcal{C}\mathcal{B} = \mathrm{e}^{2b_2\gamma_*} = \cosh(2b_2)\cdot\mathrm{I} + \sinh(2b_2)\cdot\gamma_* \overset{!}{=} \mathrm{I}, \qquad b_2 \in \mathbb{C}. \tag{3.35}$$

This equation leads to only two solutions for $\mathrm{e}^{b_2\gamma_*}$, $\mathrm{e}^{b_2\gamma_*} = \pm\mathrm{I}$. The sign ambiguity can be compensated by a phase conversion,

$$\mathcal{B} = \pm\mathrm{e}^{\mathrm{i}\varphi}\mathrm{I} = \mathrm{e}^{\mathrm{i}\varphi'}\mathrm{I}, \qquad \mathcal{C} = \pm\mathrm{e}^{-\mathrm{i}\varphi}\mathrm{I} = \mathrm{e}^{-\mathrm{i}\varphi'}\mathrm{I}, \tag{3.36}$$

with an appropriately chosen $\mathrm{e}^{\mathrm{i}\varphi'} \in \mathrm{U}(1)$. Now we can apply the same arguments as before and ignore the phase again. Therefore we conclude that spin-base transformations $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ in even dimensions act as

$$\gamma_\mu \to \mathcal{S}\gamma_\mu\mathcal{S}^{-1}, \quad \psi \to \mathcal{S}\psi, \quad \bar\psi \to \bar\psi\mathcal{S}^{-1}, \quad h \to (\mathcal{S}^\dagger)^{-1}h\mathcal{S}^{-1}. \tag{3.37}$$

Finally, we investigate the diffeomorphisms by demanding that $\bar\psi\gamma_\mu\psi$ transforms as a covariant spacetime vector,

$$\bar\psi\gamma_\mu\psi \to \bar\psi'{\gamma'}_\mu\psi' = \bar\psi'\frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho\psi' \stackrel{!}{=} \frac{\partial x^\rho}{\partial x'^\mu}\bar\psi\gamma_\rho\psi. \tag{3.38}$$

Once again we find the phase transformation $\mathrm{e}^{\mathrm{i}\varphi}$ and the chiral transformation $\mathrm{e}^{b_2\gamma_*}$. If we then proceed analogously to the spin-base transformations and demand that $\bar\psi\psi$ is a scalar,

$$\bar\psi\psi \to \bar\psi'\psi' \stackrel{!}{=} \bar\psi\psi, \tag{3.39}$$

the chiral transformation turns out to be just a sign, $\mathrm{e}^{b_2\gamma_*} = \pm\mathrm{I}$. This sign can be absorbed into the phase $\pm\mathrm{e}^{\mathrm{i}\varphi}\mathrm{I} = \mathrm{e}^{\mathrm{i}\varphi'}\mathrm{I}$, which we drop. We summarize the behavior under diffeomorphisms as

$$\gamma_\mu \to \frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho, \quad \psi \to \psi, \quad \bar\psi \to \bar\psi, \quad h \to h. \tag{3.40}$$

In even dimensions it is possible to demand that the kinetic term as well as the mass term is invariant under all types of coordinate transformations. If we do so, the behavior under spin-base transformations is given by equation (3.37) and under spacetime coordinate transformations by (3.40).

3.2 Relation to Flat Spacetime and Vielbein Formulation

To define fermions more formally one usually starts in flat space with the Lorentz group $\mathrm{SO}(\mathfrak{m}, \mathfrak{p})$ and investigates its representations. In four spacetime dimensions, with $\mathfrak{m} = 1$ and $\mathfrak{p} = 3$, fermions are objects transforming under the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of $\mathrm{Spin}(1, 3)$ which is the double cover of $\mathrm{SO}^+(1, 3)$. With $\mathrm{SO}^+(1, 3)$ we denote the identity component of $\mathrm{SO}(1, 3)$. Already at this stage it is apparent that a similar construction for the diffeomorphisms will be difficult. This is because of two reasons: First, the Lorentz transformations leave the metric invariant and thus the explicit form of the Clifford algebra. Second, the fermions do not correspond to representations of the Lorentz group $\mathrm{SO}(1,3)$ but to the double cover of the to the identity connected component of the Lorentz group $\mathrm{SO}^+(1,3)$, which is the spin group $\mathrm{Spin}(1,3)$. One may expect that something similar, probably more complicated holds for the diffeomorphisms. In fact Ogievetsky and Polubarinov found a highly nonlinear way of assigning a diffeomorphism to transformations in spinor space [136–138]. The standard way, however, to recover the Lorentz group is by introducing the vielbein, which then has the bein index carrying the Lorentz symmetry. In order to make contact with the spin group the *flat* Clifford algebra, $\{\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}\} = 2\eta_{ab}\mathrm{I}$, is then introduced in tangential space at every point of the manifold. Now only those realizations of the *curved* Clifford algebra $\gamma_{(e)\mu}$ are considered that can be spanned by a fixed set of Dirac matrices, $\gamma_{(e)\mu} = e_\mu{}^a\gamma_{(\mathrm{f})a}$. A local Lorentz transformation with respect to the bein index can then be rewritten in terms of

$$\gamma_{(e)\mu} = e_\mu{}^a\gamma_{(\mathrm{f})a} \to \gamma'_{(e)\mu} = e'_\mu{}^a\gamma_{(\mathrm{f})a} = e_\mu{}^a\Lambda_{\mathrm{Lor}}{}^b{}_a\gamma_{(\mathrm{f})b} = e_\mu{}^a\mathcal{S}_{\mathrm{Lor}}\gamma_{(\mathrm{f})a}\mathcal{S}_{\mathrm{Lor}}^{-1} = \mathcal{S}_{\mathrm{Lor}}\gamma_{(e)\mu}\mathcal{S}_{\mathrm{Lor}}^{-1}, \tag{3.41}$$

where $\mathcal{S}_{\mathrm{Lor}} \in \mathrm{Spin}(\mathfrak{m},\mathfrak{p}) \subset \mathrm{SL}(d_\gamma,\mathbb{C})$, cf. equation (3.6). Conventionally, the $\mathcal{S}_{\mathrm{Lor}}$ factors are interpreted as Lorentz transformations of Dirac spinors, e.g., $\psi \to \mathcal{S}_{\mathrm{Lor}}\psi$, cf. equation (3.8).[13] These Lorentz (or spin) transformations are local, hence one needs a covariant derivative in order to construct a well defined kinetic term for fermions. We already have the vielbein covariant derivative $D_{(e)\mu}$ for the bein indices, cf. equation (2.38). The vielbein connection $\omega_{(e)\mu}{}^a{}_b$ can be promoted to a vielbein spin connection $\Gamma_{(e)\mu}$ using the flat Dirac matrices $\gamma_{(\mathrm{f})a}$,

$$\Gamma_{(e)\mu} = \frac{1}{8}\omega_{(e)\mu}{}^{ab}[\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}]. \tag{3.42}$$

This is motivated by the fact that the matrices $\frac{1}{4\mathrm{i}}[\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}]$ satisfy the Lorentz algebra,

$$\left[\frac{1}{4\mathrm{i}}[\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}], \frac{1}{4\mathrm{i}}[\gamma_{(\mathrm{f})c}, \gamma_{(\mathrm{f})d}]\right] = \mathrm{i}\left(\frac{\eta_{ac}}{4\mathrm{i}}[\gamma_{(\mathrm{f})b}, \gamma_{(\mathrm{f})d}] + \frac{\eta_{bd}}{4\mathrm{i}}[\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})c}] - \frac{\eta_{ad}}{4\mathrm{i}}[\gamma_{(\mathrm{f})b}, \gamma_{(\mathrm{f})c}] - \frac{\eta_{bc}}{4\mathrm{i}}[\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})d}]\right). \tag{3.43}$$

Then we can define the vielbein spin covariant derivative $\nabla_{(e)\mu}$ for fermions ψ,

$$\nabla_{(e)\mu}\psi = \partial_\mu\psi + \Gamma_{(e)\mu}\psi. \tag{3.44}$$

Similarly to the spacetime connection $\Gamma_\mu{}^\rho{}_\lambda$ and the vielbein connection $\omega_{(e)\mu}{}^a{}_b$ we can decompose the vielbein spin connection $\Gamma_{(e)\mu}$ into a Levi-Civita part $\hat{\Gamma}_{(e)\mu}$ and a torsion part $\Delta\Gamma_{(e)\mu}$,

$$\hat{\Gamma}_{(e)\mu} = \frac{1}{8}\omega_{(\mathrm{LC}e)\mu}{}^{ab}[\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}], \qquad \Delta\Gamma_{(e)\mu} = \frac{1}{8}K_\mu{}^{ab}[\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}]. \tag{3.45}$$

[13] Since $\mathrm{Spin}(\mathfrak{m},\mathfrak{p})$ is the double cover of $\mathrm{SO}^+(\mathfrak{m},\mathfrak{p})$ one thinks of performing a spin transformation $\mathcal{S}_{\mathrm{Lor}}$ and then projects it onto the corresponding Lorentz transformation $\Lambda_{\mathrm{Lor}}{}^a{}_b$.

It is easy to check that the connection $\Gamma_{(e)\mu}$ transforms under spin transformations $\mathcal{S}_{\text{Lor}} \in \text{Spin}(\mathfrak{m}, \mathfrak{p})$ inhomogeneously,

$$\Gamma_{(e)\mu} \to \mathcal{S}_{\text{Lor}} \Gamma_{(e)\mu} \mathcal{S}_{\text{Lor}}^{-1} - (\partial_\mu \mathcal{S}_{\text{Lor}}) \mathcal{S}_{\text{Lor}}^{-1}, \tag{3.46}$$

such that $\nabla_{(e)\mu}\psi$ transforms covariantly,

$$\nabla_{(e)\mu}\psi \to \mathcal{S}_{\text{Lor}} \nabla_{(e)\mu}\psi. \tag{3.47}$$

Note that the transformation behavior of $\Gamma_{(e)\mu}$ is the standard transformation behavior of any connection, cf. equation (2.14) and

$$\begin{aligned} \Gamma_{\mu}{}^{\rho}{}_{\lambda} &\xrightarrow{x \to x'} \frac{\partial x^\nu}{\partial x'^\mu} \left[\frac{\partial x'^\rho}{\partial x^\alpha} \Gamma_{\nu}{}^{\alpha}{}_{\beta} \frac{\partial x^\beta}{\partial x'^\lambda} - \left(\partial_\nu \frac{\partial x'^\rho}{\partial x^\alpha} \right) \frac{\partial x^\alpha}{\partial x'^\lambda} \right], \\ \omega_{(e)\mu}{}^{a}{}_{b} &\xrightarrow{\hat{e} \to \hat{e}'} \Lambda_{\text{Lor}}{}^{a}{}_{c} \omega_{(e)\mu}{}^{c}{}_{d} (\Lambda_{\text{Lor}}^{-1})^{d}{}_{b} - (\partial_\mu \Lambda_{\text{Lor}}{}^{a}{}_{c}) \Lambda_{\text{Lor}}^{-1}{}^{c}{}_{b}. \end{aligned} \tag{3.48}$$

The interpretation of the Spin$(\mathfrak{m}, \mathfrak{p})$ ($\sim$ Lorentz) subgroup of spin-base transformations SL$(d_\gamma, \mathbb{C})$ is at the heart of understanding fields as representations of the Lorentz algebra. This viewpoint is held to argue that higher-spin fields (such as the metric) may eventually be composed out of a fundamental spinorial representation [54–56]. In view of the hen-or-egg problem, our symmetry analysis does not single out a specific viewpoint. On the one hand, the representation theory of the Lorentz algebra suggesting "spinors first" should be embedded into the larger spin-base invariant framework; while this presumably does not change the result for the classification of fields, there is no analog of equation (3.41) for general spin-base transformations. On the other hand, the fact that we need a metric to define the Clifford algebra does not link spinors closer to the metric as other fields; diffeomorphisms leave spinors untouched and the transformed Dirac matrices satisfy the Clifford algebra automatically. Instead, our analysis rather suggests that not only local Lorentz invariance, but full local spin-base invariance should be a requirement for possible underlying quantum theories of gravity. If not at the fundamental level, local spin-base invariance should at least be emergent for the long-range effective description.

We want to stress that spin-base invariance is in some sense already present in the vielbein construction. It is usually assumed that the flat Dirac matrices $\gamma_{(\mathrm{f})a}$ are chosen to be the same in every tangential space. However, there is no reason to do this, as every point of the manifold has its own tangential space, with its own base. In fact this is the reason why the SO$(\mathfrak{p}, \mathfrak{m})$ is local in the vielbein formulation, cf. equation (2.33). If we allow the flat Dirac matrices to be different for the different tangential spaces, we find the SL$(d_\gamma, \mathbb{C})$ again as the corresponding transformation group between the different choices of the bases. From the sheer size of the spin-base group, it is obvious that this is a larger set of Dirac matrices satisfying the Clifford algebra than can be spanned by the vielbein construction. We can now observe that neither the vielbein $e_\mu{}^a$ nor the flat Dirac matrices $\gamma_{(\mathrm{f})a}$ appear alone in the usual terms of the gravitational and matter action. Instead, it is exclusively the combination $e_\mu{}^a \gamma_{(\mathrm{f})a}$, i.e. the full Dirac matrices γ_μ.[14] Therefore it seems rather artificial to decouple the Dirac matrices γ_μ into a vielbein $e_\mu{}^a$ and the flat Dirac matrices $\gamma_{(\mathrm{f})a}$.

Finally we can explain what it means that spinors transform under Lorentz transformations as in equation (3.8) from our new point of view. We have to read this transformation as a coordinate transformation composed of a spin-base transformation $\mathcal{S} = \mathcal{S}_{\mathrm{Lor}}$ and a diffeomorphism $\frac{\partial x'^a}{\partial x^b} = \Lambda_{\mathrm{Lor}}{}^a{}_b$ such that

$$\gamma_{(\mathrm{f})}{}^a \to \frac{\partial x'^a}{\partial x^b} \mathcal{S} \gamma_{(\mathrm{f})}{}^b \mathcal{S}^{-1} = \Lambda_{\mathrm{Lor}}{}^a{}_b \mathcal{S}_{\mathrm{Lor}} \gamma_{(\mathrm{f})}{}^b \mathcal{S}_{\mathrm{Lor}}^{-1} \equiv \gamma_{(\mathrm{f})}{}^a, \quad \psi \to \mathcal{S}\psi = \mathcal{S}_{\mathrm{Lor}}\psi, \quad \bar{\psi} \to \bar{\psi}\mathcal{S}^{-1} = \bar{\psi}\mathcal{S}_{\mathrm{Lor}}^{-1}. \tag{3.49}$$

By contrast, if we only perform a spacetime coordinate transformation the components of the fermions do not change. Strictly speaking there is no sense in saying that fermions change sign under a spatial rotation of 360°. The standard sign change becomes only visible if also the spin-base is transformed in a specific way. However, the spin-base can be rotated without the spacetime and vice versa.

The spin-base transformations and especially the invariance of the action with respect to these have an intuitive interpretation. If we start with the Clifford algebra for a given metric we have many different sets of Dirac matrices we can choose from. All these different sets are connected to each other via a similarity transformation and in odd dimensions additionally via a sign flip. With this in mind we can read the invariance under spin-base transformations as an invariance of the choice of Dirac matrices, i.e. for any choice of compatible γ_μ we get the same physical answer. In order for the invariance to hold for all compatible representations of the Clifford algebra for a given metric we have to respect the complete $\mathrm{SL}(d_\gamma, \mathbb{C})$ as shown in appendix B. This consideration also tells us that in odd dimensions physical results can depend on the choice of the connected component of the Clifford algebra. We have an invariance with respect to $\mathrm{SL}(d_\gamma, \mathbb{C})$, but if we, e.g., include a mass term we lose invariance under the sign flip. Therefore the choice of the connected component can be an integral part of the theory. This is, for instance, familiar from fermion-induced Chern-Simons terms [139, 140].

To summarize, spinors should be viewed as objects that transform as scalars under diffeomorphisms and as "vectors" under spin-base transformations. In flat space, Lorentz transformations and spin-base transformations may be combined in order to keep the Dirac matrices fixed. We emphasize that the latter is merely a convenient choice and by no means mandatory. In fact, the freedom not to link the two transformations can have significant advantages as shown in the next section.

[14] This becomes most apparent by comparing the later formulas for the spin connection $\hat{\Gamma}_\mu$, cf. equation (4.19), and the spin metric h, cf. equation (4.29), with their standard vielbein formalism analogs.

3.3 Global Surpluses of Spin-Base Invariant Fermions

On the following pages we want to emphasize the advantages of the spin-base invariant formalism both from a conceptual as well as from a practical viewpoint. Since symmetry-breaking perturbations typically contain relevant components which inhibit symmetry emergence, local symmetries are expected to be fundamental. Hence, we consider the local symmetries of the Clifford algebra as fundamental. These are diffeomorphisms, cf. equations (3.26), (3.40), and local spin-base transformations, cf. equations (3.24), (3.37). The corresponding transformation of spinors ensures that typical fermion bilinears and higher-order interaction terms serving as building blocks for a relativistic field theory are also invariant, provided a suitable covariant derivative ∇_μ exists.[15]

For simplicity, let us confine ourselves to the case $d = 2$ for the remainder of section 3. Here, the dimension of the irreducible representation of the Clifford algebra is $d_\gamma = 2$. As explained above, a natural choice for the group of spin-base transformations maintaining all invariance properties is then given by $\mathrm{SL}(d_\gamma, \mathbb{C})$. The choice of the local spin-base group only becomes relevant once a dynamics is associated with the connection. In section 4 it is shown that for the choice of $\mathrm{SL}(d_\gamma, \mathbb{C})$ and vanishing torsion, the corresponding field strength $\Phi_{\mu\nu}$ satisfies the identity [47, 57]

$$\Phi_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \frac{1}{8} R_{\mu\nu\lambda\kappa} [\gamma^\lambda, \gamma^\kappa]. \tag{3.50}$$

It is somewhat surprising as well as reassuring that – out of the large number of degrees of freedom in the connection – only those acquire a nontrivial dynamics which can be summarized in the Christoffel symbols and hence lead to the Riemann tensor on the right-hand side of equation (3.50). As a consequence, spin-base invariance is also a (hidden) local symmetry of any special relativistic fermionic theory in flat space with an automatically trivial dynamics for the connection, even if kinetic terms of the form $\sim \mathrm{tr}\, \gamma^\mu \Phi_{\mu\nu} \gamma^\nu \sim R$ (Einstein-Hilbert action) or $\sim \mathrm{tr}\, \Phi_{\mu\nu} \Phi^{\mu\nu} \sim R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda}$ would be added.

This is different if spin-base transformations are associated with $\mathrm{GL}(d_\gamma, \mathbb{C})$. Then two additional abelian field strengths corresponding to the $\mathrm{U}(1)$ and the noncompact $\mathbb{R}_+$ factors appear on the right-hand side of equation (3.50) and thus introduce further physical degrees of freedom. These correspond to the imaginary and real part of the trace of the connection.[16] Hence, the identification of the spin-base group is in principle an experimental question to be addressed by verifying the interactions of fermions. In this sense, one might speculate whether the hypercharge $\mathrm{U}(1)$ factor of the standard model could be identified with the spin-base group provided proper charge assignments are chosen for the different fermions. The inclusion of the $\mathrm{U}(1)$ factor is particularly natural on manifolds that do not permit a spin structure (e.g., $\mathbb{C}\mathrm{P}^2$) [135], as it provides exactly for the necessary ingredient to define the more general $\mathrm{Spin}^{\mathbb{C}}$ structure.

[15] The existence and the properties of this derivative and the arising connection are discussed in detail in section 4.2.

[16] The noncompact factor (real part of the trace) was removed in the preceding discussion by fixing the determinant of the spin metric, see also [47, 57, 58].

It is a legitimate question as to whether spin-base invariance introduces an overabundant symmetry structure without gaining any advantages or further insights. In fact, already the vielbein formalism with much less symmetry has been criticized for its redundancy. For instance, the Ogievetsky-Polubarinov spinors [136] not only remove the SO(1,3) redundancy of the vielbein formulation (analogously to the Lorentz symmetric gauge for the vielbein [141]), but make spinors compatible with tensor calculus, see, e.g., [137, 138]. Nevertheless, $\mathrm{SL}(d_\gamma,\mathbb{C})$ spin-base invariance is not a symmetry that may or may not be constructed on top of existing symmetries. On the contrary, global spin-base invariance is present in any relativistic fermionic theory. Its local version does not need an additional new compensator field, but the connection is built from the Dirac matrices which are present anyway, see section 4.2. We will now present an example which demonstrates the advantages of full spin-base invariance.

Rather generically, smooth orientable manifolds may not be parametrizable with a single coordinate system, but may require several overlapping coordinate patches. In the vielbein formalism, where $g_{\mu\nu} = e_\mu{}^a \eta_{ab} e_\nu{}^b$, it is natural to expect that patches with different coordinates and corresponding metrics $g_{\mu\nu}$ also require different vielbeins $e_\mu{}^a$. This becomes most obvious for the simple example of a 2-sphere which requires at least two overlapping coordinate patches to be covered. The same is true for the vielbein: for each fixed bein index, $e_\mu{}^a$ is a spacetime vector which has to satisfy the Poincaré-Brouwer (hairy ball) theorem. This implies that it has to vanish at least on one point of the 2-sphere, thus also the determinant of the vielbein vanishes, $\det e_\mu{}^a = 0$. Hence, at least two sets of vielbeins and corresponding transition functions are required to cover the 2-sphere without singularities. For the spin-base invariant formalism, however, the independence of diffeomorphisms and spin-base transformations suggests that a change of the coordinate patch and metric does not necessarily require a change of the spin-base patch. More constructively, two sets of spin-bases on two neighboring coordinate patches may be smoothly connected by a suitable spin-base transformation. We now show that this is possible for the 2-sphere S^2 resulting in a global spin-base.

To keep this discussion transparent, we use the pair of polar and azimuthal angles $(\boldsymbol{\theta},\boldsymbol{\phi})$ to label all points on the sphere (not as coordinates). This labeling is overcomplete, as all the elements of $\{(\boldsymbol{\theta},\boldsymbol{\phi}+2\pi n) : n\in\mathbb{Z}\}$ for a given pair $(\boldsymbol{\theta},\boldsymbol{\phi}) \in [0,\pi]\times\mathbb{R}$ describe the same point. However, this is no matter here, since we only need a surjective map from $[0,\pi]\times\mathbb{R}$ onto S^2 in order to label all the points uniquely. For the polar coordinates, we use the notation (ϑ,φ), i.e., $(x^\mu)|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = (\vartheta,\varphi)|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = (\boldsymbol{\theta},\boldsymbol{\phi})$. These are legitimate coordinates except for the poles at $\boldsymbol{\theta}\in\{0,\pi\}$.[17] In polar coordinates the metric reads

$$(g_{\mu\nu})\big|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} g\big(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^1}\big) & g\big(\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2}\big) \\ g\big(\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^1}\big) & g\big(\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^2}\big) \end{pmatrix}\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{pmatrix}\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\boldsymbol{\theta} \end{pmatrix}. \tag{3.51}$$

[17] To be precise, the whole circular segment $\boldsymbol{\phi}=0$ is not covered by this coordinate system. Still, the transition from a coordinate system with $(\vartheta,\varphi)\in(0,\pi)\times(0,2\pi)$ to a coordinate system with $(\vartheta',\varphi')\in(0,\pi)\times(-\pi,\pi)$ is not problematic.

Obviously, the components of the metric become degenerate at the poles, rendering the coordinates ill-defined there. In these coordinates, one suitable choice for the vielbein ${e_\mu}^a$ is

$$\left({e_\mu}^a\right)\big|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} \hat{\theta}^1\big(\frac{\hat{\partial}}{\partial x^1}\big) & \hat{\theta}^2\big(\frac{\hat{\partial}}{\partial x^1}\big) \\ \hat{\theta}^1\big(\frac{\hat{\partial}}{\partial x^2}\big) & \hat{\theta}^2\big(\frac{\hat{\partial}}{\partial x^2}\big) \end{pmatrix}\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} 1 & 0 \\ 0 & \sin\vartheta \end{pmatrix}\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} 1 & 0 \\ 0 & \sin\boldsymbol{\theta} \end{pmatrix}. \tag{3.52}$$

This choice is perfectly smooth everywhere, but is not appropriate at the poles. In order to cover the poles $\boldsymbol{\theta} \in \{0, \pi\}$, we need to change coordinates. For simplicity, we choose Cartesian coordinates $(x'^\mu)|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = (x, y)|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = (\cos\boldsymbol{\phi}\sin\boldsymbol{\theta}, \sin\boldsymbol{\phi}\sin\boldsymbol{\theta})$, these are well defined at the poles but ill defined at the equator $\boldsymbol{\theta} = \pi/2$. For the coordinate transformation, we need the Jacobian

$$\left(\mathrm{d}\hat{x}^\nu\Big(\frac{\hat{\partial}}{\partial x'^\mu}\Big)\right)\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} \frac{\partial x^1}{\partial x'^1} & \frac{\partial x^2}{\partial x'^1} \\ \frac{\partial x^1}{\partial x'^2} & \frac{\partial x^2}{\partial x'^2} \end{pmatrix}\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} \frac{\cos\boldsymbol{\phi}}{\cos\boldsymbol{\theta}} & -\frac{\sin\boldsymbol{\phi}}{\sin\boldsymbol{\theta}} \\ \frac{\sin\boldsymbol{\phi}}{\cos\boldsymbol{\theta}} & \frac{\cos\boldsymbol{\phi}}{\sin\boldsymbol{\theta}} \end{pmatrix}. \tag{3.53}$$

We emphasize again that the pair $(\boldsymbol{\theta}, \boldsymbol{\phi})$ is used only for convenience to label a point on the sphere and not as a coordinate system. The metric for the primed (Cartesian) coordinates x'^μ reads

$$\left(g'_{\mu\nu}\right)\big|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \frac{1}{1-x^2-y^2}\begin{pmatrix} 1-y^2 & xy \\ xy & 1-x^2 \end{pmatrix}\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \frac{1}{\cos^2\boldsymbol{\theta}}\begin{pmatrix} 1-\sin^2\boldsymbol{\phi}\sin^2\boldsymbol{\theta} & \sin\boldsymbol{\phi}\cos\boldsymbol{\phi}\sin^2\boldsymbol{\theta} \\ \sin\boldsymbol{\phi}\cos\boldsymbol{\phi}\sin^2\boldsymbol{\theta} & 1-\cos^2\boldsymbol{\phi}\sin^2\boldsymbol{\theta} \end{pmatrix}. \tag{3.54}$$

The transformation of the vielbein yields

$$\left({e'_\mu}^a\right)\big|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \left(\frac{\partial x^\nu}{\partial x'^\mu}{e_\nu}^a\Bigg|_{(\boldsymbol{\theta},\boldsymbol{\phi})}\right) = \frac{1}{\cos\boldsymbol{\theta}}\mathcal{R}_{\boldsymbol{\phi}}\begin{pmatrix} 1 & 0 \\ 0 & \cos\boldsymbol{\theta} \end{pmatrix}, \quad \mathcal{R}_{\boldsymbol{\phi}} = \begin{pmatrix} \cos\boldsymbol{\phi} & -\sin\boldsymbol{\phi} \\ \sin\boldsymbol{\phi} & \cos\boldsymbol{\phi} \end{pmatrix}. \tag{3.55}$$

First, we observe a coordinate singularity at the equator as expected. Moreover, we obtain a $\boldsymbol{\phi}$ dependence which seems to render the vielbein ill-defined at the poles. Nevertheless, this can be cured by performing a corresponding (counter-)rotation in tangential space with respect to the bein index. The hairy ball theorem manifests itself here by the fact that one pole needs a rotation, while the other needs a combination of the same rotation and a reflection. These are elements of the two different connected components of the rotation group O(2), the proper and improper rotations. Since we cannot perform a continuous transformation from proper to improper rotations, we cannot cure the residual $\boldsymbol{\phi}$ dependence at both poles at the same time in a continuous way (independently of the expected coordinate singularity at the equator). Incidentally, an inverse rotation would also cure the problematic $\boldsymbol{\phi}$ dependence at the south pole, but because of the required 2π periodicity in $\boldsymbol{\phi}$, the direction of the rotation cannot be changed continuously from north to south pole.

The same conclusion remains true for those sets of Dirac matrices which are constructed via the vielbein $\gamma'_{(\mathrm{e})\mu}|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = {e'_\mu}^a|_{(\boldsymbol{\theta},\boldsymbol{\phi})}\gamma_{(\mathrm{f})a}$. By contrast, the spin-base invariant formalism allows to continuously connect all representations of the two-dimensional Clifford algebra, i.e., proper and improper rotations of O(2) should be continuously connectable on the level of $\mathrm{SL}(2,\mathbb{C})$ spin-base transformations of the Dirac matrices. For this, we first define conventional *constant* flat Dirac matrices $\tilde{\gamma}_{(\mathrm{f})a}$,

$$\left(\tilde{\gamma}_{(\mathrm{f})a}\right) = \begin{pmatrix} \tilde{\gamma}_{(\mathrm{f})1} \\ \tilde{\gamma}_{(\mathrm{f})2} \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ -\sigma_2 \end{pmatrix}, \tag{3.56}$$

where σ_1 and σ_2 are the first two Pauli matrices. The $\tilde{\gamma}_{(\mathrm{f})a}$ fulfill the two dimensional flat Euclidean Clifford algebra $\{\tilde{\gamma}_{(\mathrm{f})a}, \tilde{\gamma}_{(\mathrm{f})b}\} = 2\delta_{ab}\mathrm{I}$. Next, we construct auxiliary spacetime *dependent* flat Dirac matrices,

$$\gamma_{(\mathrm{f})a}|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \mathcal{S}(\boldsymbol{\theta},\boldsymbol{\phi})\tilde{\gamma}_{(\mathrm{f})a}\mathcal{S}^{-1}(\boldsymbol{\theta},\boldsymbol{\phi}), \qquad \mathcal{S}(\boldsymbol{\theta},\boldsymbol{\phi}) = \mathrm{e}^{-\mathrm{i}\frac{\boldsymbol{\phi}}{2}\sigma_3}\mathrm{e}^{-\mathrm{i}\frac{\boldsymbol{\theta}-\pi}{2}\sigma_1}, \tag{3.57}$$

which also satisfy the Euclidean Clifford algebra as equation (3.57) is a spin-base transformation. We emphasize that the spin-base transformation $\mathcal{S}(\boldsymbol{\theta},\boldsymbol{\phi})$ of equation (3.57) goes beyond the subgroup of Spin(2) transformations because of the second exponential factor. The new flat Dirac matrices read explicitly

$$\left(\gamma_{(\mathrm{f})a}\right)|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} \cos\boldsymbol{\phi}\sigma_1 + \sin\boldsymbol{\phi}\sigma_2 \\ \cos\boldsymbol{\theta}(-\sin\boldsymbol{\phi}\sigma_1 + \cos\boldsymbol{\phi}\sigma_2) + \sin\boldsymbol{\theta}\sigma_3 \end{pmatrix}. \tag{3.58}$$

Here it becomes manifest that these Dirac matrices smoothly vary from a proper rotation at $\boldsymbol{\theta} = 0$ to an improper rotation at $\boldsymbol{\theta} = \pi$, while maintaining 2π-periodicity in $\boldsymbol{\phi}$. Based on this special set of flat-space Dirac matrices, the Dirac matrices on the 2-sphere in Cartesian coordinates $\gamma'_\mu = {e'_\mu}^a\gamma_{(\mathrm{f})a}$ read

$$\left(\gamma'_\mu\right)|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \frac{1}{\cos\boldsymbol{\theta}}\mathcal{R}_{\boldsymbol{\phi}}\begin{pmatrix} 1 & 0 \\ 0 & \cos^2\boldsymbol{\theta} \end{pmatrix}\mathcal{R}_{\boldsymbol{\phi}}^{-1}\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} + \sin\boldsymbol{\theta}\mathcal{R}_{\boldsymbol{\phi}}\begin{pmatrix} 0 \\ \sigma_3 \end{pmatrix}. \tag{3.59}$$

These Dirac matrices are obviously well-behaved at the poles $\boldsymbol{\theta} \in \{0,\pi\}$, since there are no singularities and no $\boldsymbol{\phi}$ dependence is left. Of course, the singularity at the equator remains, where the Cartesian coordinates are ill-defined. This singularity is not present in polar coordinates, where we obtain the Dirac matrices $\gamma_\mu = \frac{\partial x'^\rho}{\partial x^\mu}\gamma'_\rho$,

$$\left(\gamma_\mu\right)|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2}\sin 2\boldsymbol{\theta} \end{pmatrix}\mathcal{R}_{\boldsymbol{\phi}}^{-1}\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} + \sin^2\boldsymbol{\theta}\begin{pmatrix} 0 \\ \sigma_3 \end{pmatrix}. \tag{3.60}$$

Note that γ_μ and γ'_μ are connected solely by a diffeomorphism – no change of the spin-base is involved. Whereas the vielbein construction given above actually proceeded via ill-defined intermediate objects,[18] the resulting spin-base chosen for the curved Dirac matrices given by equation (3.59) in Cartesian coordinates (i.e. except for the equator) and by equation (3.60) in polar coordinates (i.e. except for the poles) holds globally all over the 2-sphere. No additional patch for spin-base coordinates is required to cover the whole 2-sphere. In particular the limit towards the poles in equation (3.59) is unique and smooth in contrast to the vielbein case.

It is interesting to see how the spin-base invariant formalism evades the hairy ball theorem: the important point is that γ_μ does not represent a globally nonvanishing vector field (which would be forbidden), but is a vector of Dirac matrix fields, $(\gamma_\mu)^I{}_J$. For every fixed pair $(I, J) \in \{1, \dots, d_\gamma\}^2$, we have a complex vector field. It is easy to check that each of the real subcomponent vector fields has at least one zero on the sphere, being therefore compatible with the hairy ball theorem. The zeros of these vector fields are however distributed such that the Dirac matrices γ_μ satisfy the Clifford algebra all over the 2-sphere.

We expect that the construction above generalizes to all $2n$-spheres, since the corresponding spin-base group $\mathrm{SL}(d_\gamma, \mathbb{C})$ with $d_\gamma = 2^n$ is connected and all representations of the Dirac matrices are connected to each other via a spin-base transformation. The problem of the disconnected components of the orthogonal group should then be resolvable in the same way as shown above. Incidentally, the hairy ball theorem applies to the $2n$-spheres, implying that vielbeins cannot be defined globally on these spheres.

As an application of this global spin-base, let us study the eigenfunctions of the Dirac operator on the 2-sphere. Using the vielbein $e_\mu{}^a$ of equation (3.52) and the flat Dirac matrices $\tilde{\gamma}_{(\mathrm{f})a}$ of equation (3.56) the eigenfunctions have been calculated in [142] within the vielbein formalism. The vielbein spin connection $\Gamma_{(e)\mu}$, cf. equation (3.42), in spherical coordinates is then given by

$$(\Gamma_{(e)\mu})\big|_{(\boldsymbol{\theta},\boldsymbol{\phi})} = \begin{pmatrix} 0 \\ \frac{\mathrm{i}}{2}\cos\boldsymbol{\theta}\sigma_3 \end{pmatrix}. \tag{3.61}$$

The eigenfunctions of the Dirac operator $\nabla\!\!\!/ = \gamma_{(e)}{}^\mu \nabla_{(e)\mu}$ satisfy $\nabla\!\!\!/ \psi^{(s)}_{\pm,n,l} = \pm\mathrm{i}(n+1)\psi^{(s)}_{\pm,n,l}$, $s \in \{-1, 1\}$ and read [142]

$$\psi^{(s)}_{\pm,n,l}(\boldsymbol{\theta},\boldsymbol{\phi}) = \frac{c_2(n,l)}{\sqrt{16\pi}} \mathrm{e}^{s\mathrm{i}(l+\frac{1}{2})\boldsymbol{\phi}} \begin{pmatrix} (1-s)\Phi_{n,l}(\boldsymbol{\theta}) + \mathrm{i}(1+s)\Psi_{n,l}(\boldsymbol{\theta}) \\ \pm(1+s)\Phi_{n,l}(\boldsymbol{\theta}) \pm \mathrm{i}(1-s)\Psi_{n,l}(\boldsymbol{\theta}) \end{pmatrix}, \tag{3.62}$$

where $n \in \mathbb{N}_0$, $l \in \{0, \dots, n\}$, and

$$\Phi_{n,l}(\boldsymbol{\theta}) = \cos^{l+1}\frac{\boldsymbol{\theta}}{2}\, \sin^l\frac{\boldsymbol{\theta}}{2}\, P^{(l,l+1)}_{n-l}(\cos\boldsymbol{\theta}),\ \ \Psi_{n,l}(\boldsymbol{\theta}) = (-1)^{n-l}\Phi_{n,l}(\pi-\boldsymbol{\theta}),\ \ c_2(n,l) = \frac{\sqrt{(n-l)!\,(n+l+1)!}}{n!}, \tag{3.63}$$

[18] In hindsight, the vielbein $e_\mu{}^a|_{(\boldsymbol{\theta},\boldsymbol{\phi})}$ and the flat Dirac matrices $\gamma_{(\mathrm{f})a}|_{(\boldsymbol{\theta},\boldsymbol{\phi})}$ are ill-defined at the poles in a complementary way. Hence, the Dirac matrices $\gamma_\mu|_{(\boldsymbol{\theta},\boldsymbol{\phi})}$ are well-defined.

with the Jacobi polynomials $P_n^{(\alpha,\beta)}$. We pay particular attention to the following properties of the eigenfunctions:

$$\psi^{(s)}_{\pm,n,l}(\boldsymbol{\theta},\boldsymbol{\phi}+2\pi) = -\psi^{(s)}_{\pm,n,l}(\boldsymbol{\theta},\boldsymbol{\phi}), \tag{3.64}$$

$$\psi^{(s)}_{\pm,n,l=0}(\boldsymbol{\theta}=0,\boldsymbol{\phi}) = \sqrt{\frac{n+1}{16\pi}}\,\mathrm{e}^{s\mathrm{i}\frac{\phi}{2}}\begin{pmatrix}1-s\\ \pm(1+s)\end{pmatrix}, \tag{3.65}$$

$$\psi^{(s)}_{\pm,n,l=0}(\boldsymbol{\theta}=\pi,\boldsymbol{\phi}) = \mathrm{i}(-1)^n\sqrt{\frac{n+1}{16\pi}}\mathrm{e}^{s\mathrm{i}\frac{\phi}{2}}\begin{pmatrix}1+s\\ \pm(1-s)\end{pmatrix}. \tag{3.66}$$

The first equation (3.64) shows that the eigenspinors pick up a minus sign upon a 2π rotation. In the second (3.65) and third (3.66) equation it is revealed that the eigenspinors are not well-defined at the poles, as an ambiguous $\boldsymbol{\phi}$ dependence remains. This is similar to the residual $\boldsymbol{\phi}$ dependence of the vielbein at the poles.

Let us now study these properties with the global spin-base constructed above. The Dirac matrices γ_μ of equation (3.60) and $\gamma_{(e)\mu}$ are connected via the spin-base transformation of equation (3.57), $\gamma_\mu = \mathcal{S}\gamma_{(e)\mu}\mathcal{S}^{-1}$. The corresponding spin connection Γ_μ can be calculated from the usual behavior of a connection, cf. equations (2.14) and (3.48),

$$\Gamma_\mu = \mathcal{S}\Gamma_{(e)\mu}\mathcal{S}^{-1} - (\partial_\mu\mathcal{S})\mathcal{S}^{-1} = \frac{\mathrm{i}}{2}\gamma_\mu. \tag{3.67}$$

Note that we would have found the same connection if we had used equations (4.19 - 4.18) from section 4.2. The eigenfunctions $\hat{\psi}^{(s)}_{\pm,n,l}$ of the Dirac operator $\nabla\!\!\!/ = \gamma^\mu(\partial_\mu + \Gamma_\mu)$ in the global spin-base are then given by

$$\hat{\psi}^{(s)}_{\pm,n,l}(\boldsymbol{\theta},\boldsymbol{\phi}) = \mathcal{S}(\boldsymbol{\theta},\boldsymbol{\phi})\psi^{(s)}_{\pm,n,l}(\boldsymbol{\theta},\boldsymbol{\phi}). \tag{3.68}$$

It is now straightforward to check that these eigenfunctions are globally well behaved, in particular

$$\hat{\psi}^{(s)}_{\pm,n,l}(\boldsymbol{\theta},\boldsymbol{\phi}+2\pi) = \hat{\psi}^{(s)}_{\pm,n,l}(\boldsymbol{\theta},\boldsymbol{\phi}), \tag{3.69}$$

$$\hat{\psi}^{(s)}_{\pm,n,l=0}(\boldsymbol{\theta}=0,\boldsymbol{\phi}) = \mathrm{i}\sqrt{\frac{n+1}{16\pi}}\begin{pmatrix}\pm(1+s)\\ 1-s\end{pmatrix}, \tag{3.70}$$

$$\hat{\psi}^{(s)}_{\pm,n,l=0}(\boldsymbol{\theta}=\pi,\boldsymbol{\phi}) = \mathrm{i}(-1)^n\sqrt{\frac{n+1}{16\pi}}\begin{pmatrix}1+s\\ \pm(1-s)\end{pmatrix}. \tag{3.71}$$

Not only has the ambiguous $\boldsymbol{\phi}$ dependence disappeared at the poles, but also the spinors have become 2π-periodic. Since the eigenfunctions form a complete set of spinor functions on the 2-sphere, we have found a spin-base that permits to span functions on the sphere in terms of globally defined smooth base spinors. This can serve as a convenient starting point for the construction of functional integrals for quantized fermion fields.

4 Spin Base Formulation

In this chapter we will present the technical details of the spin-base invariant formulation and discuss the appearing degrees of freedom. Special emphasis is put on the construction of the spin metric and the spin connection in terms of the Dirac matrices. In this construction the spin torsion arises naturally as a completely new field, which is similar to the contorsion tensor of the metric formulation. At the end we focus on a particularly convenient gauge choice for the Dirac matrices analogously to the Lorentz symmetric gauge of the vielbein.

4.1 General Requirements

When considering curved spacetimes and fermions, we have to care about covariance with respect to coordinate transformations, i.e. in particular: spacetime coordinate transformations and spin-base transformations. To describe spinors we need Dirac structure, defined via the Clifford algebra in irreducible representation

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}, \qquad\qquad \gamma_\mu \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C}). \tag{4.1}$$

Fermions ψ are then complex Graßmann valued fields transforming as "vectors" under the fundamental representation of the special linear group $\mathrm{SL}(d_\gamma, \mathbb{C})$. The dual vector $\bar{\psi}$ is related to the vector ψ via the spin metric h

$$\bar{\psi} = \psi^\dagger h \tag{4.2}$$

whose determinant has to satisfy

$$|\det h| = 1, \tag{4.3}$$

such that h does not introduce any scale between ψ and $\bar{\psi}$. The transformation law for the Dirac matrices, the spin metric and the fermions under a spin-base transformation $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ reads

$$\gamma_\mu \to \mathcal{S}\gamma_\mu\mathcal{S}^{-1}, \qquad h \to h' = (\mathcal{S}^\dagger)^{-1}h\mathcal{S}^{-1}, \qquad \psi \to \psi' = \mathcal{S}\psi, \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\mathcal{S}^{-1}, \tag{4.4}$$

and under diffeomorphisms

$$\gamma_\mu \to \frac{\partial x^\rho}{\partial x'^\mu}\gamma_\rho, \qquad h \to h' = h, \qquad \psi \to \psi' = \psi, \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}. \tag{4.5}$$

In odd dimensions there are two connected components for the γ_μ such that there exists additionally the possibility of a change of the component via a sign flip,

$$\gamma_\mu \to -\gamma_\mu, \qquad h \to -h, \qquad \psi \to \psi, \qquad \bar{\psi} \to -\bar{\psi}, \qquad d \text{ odd}. \tag{4.6}$$

In even dimensions there is only one connected component and therefore there is no such discrete transformation. Since we aim at describing dynamics we also have to introduce a covariant derivative ∇_μ. The defining equations are chosen analogously to the spacetime covariant derivative, cf. equation (2.15),

$$\begin{aligned}
&\text{(i)} && \text{linearity:} && \nabla_\mu(\psi_1+\psi_2) = \nabla_\mu\psi_1 + \nabla_\mu\psi_2, \\
&\text{(ii)} && \text{product rule:} && \nabla_\mu(\psi\bar{\psi}) = (\nabla_\mu\psi)\bar{\psi} + \psi(\nabla_\mu\bar{\psi}), \\
&\text{(iii)} && \text{metric compatibility:} && \nabla_\mu\bar{\psi} = \overline{\nabla_\mu\psi}, \\
&\text{(iv)} && \text{covariance I:} && \nabla_\mu\psi \to \nabla'_\mu\psi' = \frac{\partial x^\nu}{\partial x'^\mu}\cdot\mathcal{S}\cdot(\nabla_\nu\psi), \\
&\text{(v)} && \text{covariance II:} && \nabla_\mu(\bar{\psi}\gamma^\nu\psi) = D_\mu(\bar{\psi}\gamma^\nu\psi).
\end{aligned} \tag{4.7}$$

The first two properties are quite intuitive. Requirement (iii) is the analog of metric compatibility of D_μ, see (iii) of equation (2.15). The first covariance condition ensures the "correct" transformation behavior under all types of coordinate transformations. In the last equation a connection of the spacetime covariant derivative D_μ to the spin covariant derivative ∇_μ is established. Finally the action of a unitary dynamical theory containing fermions should be real. Therefore we demand that the kinetic and the mass term in their usual forms are real,

$$\left(\mathrm{i}^{\varepsilon_{\mathfrak{p},d}}\int_x \bar{\psi}\nabla\!\!\!/\,\psi\right)^* = \mathrm{i}^{\varepsilon_{\mathfrak{p},d}}\int_x \bar{\psi}\nabla\!\!\!/\,\psi, \qquad\qquad \left(\int_x \bar{\psi}\psi\right)^* = \int_x \bar{\psi}\psi. \tag{4.8}$$

Here $\nabla\!\!\!/$ denotes the Dirac operator $\nabla\!\!\!/ = \gamma^\mu\nabla_\mu$ and $\int_x$ is a shorthand for the spacetime integral $\int_{\mathcal{M}} \mathrm{dV}$, see equation (2.22).[19] The exponent of the imaginary unit $\varepsilon_{\mathfrak{p},d}$ is assumed to be an element of $\{0,1\}$, hence switching on and off the imaginary unit of the kinetic term. As explained in the next section, the presence of the imaginary unit can be necessary in odd dimensions, since the Clifford algebra there has two connected components. It turns out that $\varepsilon_{\mathfrak{p},d} \in \{0,1\}$ is arbitrary in even dimensions, but is fixed by the signature in odd dimensions. In the next section we construct the spin metric and the spin connection which will ensure the spin-base covariance.

4.2 Spin Metric and Spin Connection

Using our assumptions from the previous section, we can analyze the properties of the necessary spin metric and spin connection. Beginning with the mass term of equation (4.8) and the Graßmann nature of fermions,

$$(\psi^\dagger h\psi)^* = \psi^{\mathrm{T}}h^*\psi^* = -\psi^\dagger h^\dagger\psi, \tag{4.9}$$

[19] We tacitly assume that the considered manifolds and the fermionic fields allow us to freely integrate by parts under the integral without the occurrence of any boundary terms, see chapter 2.

it turns out that the spin metric has to be anti-Hermitean,

$$h^\dagger = -h. \tag{4.10}$$

Additionally we define the Dirac conjugation of a matrix $M_{\text{gen}} \in \text{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ analogously to the Dirac conjugation of a vector ψ as

$$\bar{M}_{\text{gen}} = h^{-1} M_{\text{gen}}^\dagger h. \tag{4.11}$$

This Dirac conjugation is of particular interest for the complex conjugation of objects like $(\bar{\psi} M_{\text{gen}} \psi)^* = \bar{\psi} \bar{M}_{\text{gen}} \psi$. For the next step in our analysis we use the properties (i) - (iv) of equation (4.7) to deduce

$$(\partial_\mu \bar{\psi})\psi + \bar{\psi}(\partial_\mu \psi) = \partial_\mu \bar{\psi}\psi = \nabla_\mu \bar{\psi}\psi = (\nabla_\mu \bar{\psi})\psi + \bar{\psi}(\nabla_\mu \psi). \tag{4.12}$$

From here we conclude that the covariant derivative must carry a connection Γ_μ,

$$\nabla_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi, \qquad \nabla_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi}\Gamma_\mu. \tag{4.13}$$

From the transformation laws under spin-base transformations and diffeomorphisms of spinors ψ we find the transformation law of the connection Γ_μ, cf. equations (2.14), (3.46) and (3.48),

$$\Gamma_\mu \to \mathcal{S}\Gamma_\mu \mathcal{S}^{-1} - (\partial_\mu \mathcal{S})\mathcal{S}^{-1}, \qquad \Gamma_\mu \to \frac{\partial x^\rho}{\partial x'^\mu}\Gamma_\rho. \tag{4.14}$$

From (iii) we infer

$$\nabla_\mu \bar{\psi} = \overline{\nabla_\mu \psi} = (\nabla_\mu \psi)^\dagger h = \partial_\mu \bar{\psi} - \bar{\psi} h^{-1} \partial_\mu h + \bar{\psi}\bar{\Gamma}_\mu, \tag{4.15}$$

and deduce the metric compatibility equation

$$h^{-1}\partial_\mu h = \Gamma_\mu + \bar{\Gamma}_\mu. \tag{4.16}$$

The auxiliary matrices $\hat{\Gamma}_\mu$ turn out to be useful for our analysis. They are explicitly defined by

$$\hat{\Gamma}_\mu = \sum_{n=1}^{d} \hat{m}_{\mu\rho_1\dots\rho_n}\gamma^{\rho_1\dots\rho_n}, \quad \hat{m}^{\mu\rho_1\dots\rho_n} = \frac{(-1)^{\frac{n(n+1)}{2}} \operatorname{tr}\left(\gamma_{\rho_1\dots\rho_n}[(D_{(\text{LC})\mu}\gamma^\nu), \gamma_\nu]\right)}{2\cdot n!\cdot\left((1-(-1)^n)d - 2n\right)\cdot d_\gamma}, \quad d \text{ even}, \tag{4.17}$$

$$\hat{\Gamma}_\mu = \sum_{n=1}^{\frac{d-1}{2}} \hat{m}_{\mu\rho_1\dots\rho_{2n}}\gamma^{\rho_1\dots\rho_{2n}}, \quad \hat{m}^{\mu\rho_1\dots\rho_n} = \frac{(-1)^{n+1} \operatorname{tr}\left(\gamma_{\rho_1\dots\rho_{2n}}[(D_{(\text{LC})\mu}\gamma^\nu), \gamma_\nu]\right)}{8\cdot(2n)!\cdot n\cdot d_\gamma}, \quad d \text{ odd}, \tag{4.18}$$

where $D_{(\text{LC})\mu}$ is the (torsionless) Levi-Civita spacetime covariant derivative, cf. equation (2.18).

Equivalently one can define them in an implicit manner,

$$D_{(\mathrm{LC})_\mu}\gamma^\nu = \partial_\mu\gamma^\nu + \left\{{\nu \atop \mu\rho}\right\}\gamma^\rho = -[\hat{\Gamma}_\mu, \gamma^\nu], \qquad \operatorname{tr}\hat{\Gamma}_\mu = 0. \tag{4.19}$$

The proof is found in appendix E. Note that the $\hat{\Gamma}_\mu$ are completely determined in terms of the γ_μ and their first derivatives. Since for $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ we have $\operatorname{tr}\big((\partial_\mu\mathcal{S})\mathcal{S}^{-1}\big) = 0$, the matrices $\hat{\Gamma}_\mu$ transform exactly like the spin connection Γ_μ under coordinate transformations

$$\hat{\Gamma}_\mu \to \mathcal{S}\hat{\Gamma}_\mu\mathcal{S}^{-1} - (\partial_\mu\mathcal{S})\mathcal{S}^{-1}, \qquad \hat{\Gamma}_\mu \to \frac{\partial x^\nu}{\partial x'^\mu}\hat{\Gamma}_\nu. \tag{4.20}$$

It is worth mentioning that the existence of these matrices is guaranteed by the generalized Weldon theorem [47, 58]

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu], \qquad \operatorname{tr}\delta\mathcal{S}_\gamma = 0, \tag{4.21}$$

where $\delta\gamma_\mu$ is an arbitrary variation of the Dirac matrices compatible with the Clifford algebra, $\delta g_{\mu\nu}$ is the corresponding infinitesimal variation of the metric and $\delta\mathcal{S}_\gamma \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ parametrizes an arbitrary infinitesimal similarity transformation. We use this theorem as a tool and a guideline throughout this thesis rather often. Details and a proof are given in appendix A.

In order to investigate equation (4.8) we calculate

$$\begin{aligned} D_{(\mathrm{LC})_\mu}\bar{\gamma}^\nu =& D_{(\mathrm{LC})_\mu}(h^{-1}\gamma^{\nu\dagger}h) = [\bar{\gamma}^\nu, h^{-1}(\partial_\mu h)] + h^{-1}(D_{(\mathrm{LC})_\mu}\gamma^{\nu\dagger})h \\ =& [\bar{\gamma}^\nu, \Gamma_\mu + \bar{\Gamma}_\mu] + h^{-1}(D_{(\mathrm{LC})_\mu}\gamma^\nu)^\dagger h = [\bar{\gamma}^\nu, \Gamma_\mu + \bar{\Gamma}_\mu - \bar{\hat{\Gamma}}_\mu] \end{aligned} \tag{4.22}$$

and note that, cf. equation (2.24),

$$0 = \int_x D_{(\mathrm{LC})_\mu}(\bar{\psi}\gamma^\mu\psi) = \int_x \big((\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi) + \bar{\psi}(D_{(\mathrm{LC})_\mu}\gamma^\mu)\psi\big). \tag{4.23}$$

With that in mind it is easy to evaluate

$$\begin{aligned} \mathrm{i}^{\varepsilon_{\mathfrak{p},d}}\int_x \bar{\psi}\nabla\!\!\!/\psi =& (-\mathrm{i})^{\varepsilon_{\mathfrak{p},d}}\int_x (\bar{\psi}\nabla\!\!\!/\psi)^* = (-\mathrm{i})^{\varepsilon_{\mathfrak{p},d}}\int_x (\overline{\nabla_\mu\psi})\bar{\gamma}^\mu\psi = (-\mathrm{i})^{\varepsilon_{\mathfrak{p},d}}\int_x (\nabla_\mu\bar{\psi})\bar{\gamma}^\mu\psi \\ =& -(-\mathrm{i})^{\varepsilon_{\mathfrak{p},d}}\int_x \bar{\psi}\big((D_{(\mathrm{LC})_\mu}\bar{\gamma}^\mu) + \Gamma_\mu\bar{\gamma}^\mu + \bar{\gamma}^\mu\partial_\mu\big)\psi \\ =& \mathrm{i}^{\varepsilon_{\mathfrak{p},d}}\int_x \Big[\bar{\psi}\big((-1)^{1+\varepsilon_{\mathfrak{p},d}}\cdot\bar{\gamma}^\mu\big)\nabla_\mu\psi + \bar{\psi}[\bar{\Gamma}_\mu - \bar{\hat{\Gamma}}_\mu, \bar{\gamma}^\mu]\psi\Big]. \end{aligned} \tag{4.24}$$

Since this statement has to be true for all spinors ψ we identify

$$\bar{\gamma}^\mu = (-1)^{1+\varepsilon_{\mathfrak{p},d}}\cdot\gamma^\mu, \qquad [\Delta\Gamma_\mu, \gamma^\mu] = 0. \tag{4.25}$$

Here we have decomposed the spin connection Γ_μ without loss of generality into

$$\Gamma_\mu = \mathrm{i}\mathcal{A}_\mu \cdot \mathrm{I} + \hat{\Gamma}_\mu + \Delta\Gamma_\mu. \tag{4.26}$$

Apart from $\hat{\Gamma}_\mu$ defined above, we find a trace part $\mathcal{A}_\mu$ and the spin torsion $\Delta\Gamma_\mu$ [57],

$$\mathcal{A}_\mu = -\frac{\mathrm{i}}{d_\gamma}\,\mathrm{tr}(\Gamma_\mu), \qquad \Delta\Gamma_\mu = \Gamma_\mu - \hat{\Gamma}_\mu - \frac{1}{d_\gamma}\,\mathrm{tr}(\Gamma_\mu)\cdot \mathrm{I}. \tag{4.27}$$

The transformation law under spin-base transformations for the components of the spin connection reads

$$\mathcal{A}_\mu \to \mathcal{A}_\mu, \qquad \Delta\Gamma_\mu \to \mathcal{S}\Delta\Gamma_\mu\mathcal{S}^{-1}. \tag{4.28}$$

We found the three important algebraic equations for the spin metric

$$\gamma^\dagger{}_\mu = (-1)^{1+\varepsilon_{\mathfrak{p},d}} \cdot h\gamma_\mu h^{-1}, \qquad h^\dagger = -h, \qquad |\det h| = 1. \tag{4.29}$$

For a given set of Dirac matrices there is a unique spin metric (up to a sign) as proven in appendix F. Further it turns out that in odd dimensions $\varepsilon_{\mathfrak{p},d}$ is fixed to, cf. equation (F.9),

$$\varepsilon_{\mathfrak{p},d} = \mathfrak{p} \mod 2, \qquad d \text{ odd}. \tag{4.30}$$

In even dimensions $\varepsilon_{\mathfrak{p},d}$ is arbitrary as there is only one connected component of the Clifford algebra. However, a convenient choice can be found by noting that $\gamma_*^\dagger = (-1)^{\mathfrak{p}} h\gamma_* h^{-1}$, cf. equation (3.27). This is true independent of the choice of $\varepsilon_{\mathfrak{p},d}$. Hence, we can demand that γ_* behaves in the same manner under Hermitean conjugation as the Dirac matrices,

$$\gamma_*^\dagger \overset{!}{=} (-1)^{1+\varepsilon_{\mathfrak{p},d}} \cdot h\gamma_* h^{-1} \qquad \Rightarrow \qquad \varepsilon_{\mathfrak{p},d} = (\mathfrak{p}+1) \mod 2, \qquad d \text{ even}. \tag{4.31}$$

As this is not necessary, we are keeping $\varepsilon_{\mathfrak{p},d}$ in even dimensions arbitrary. Next we use the equations (F.27) and (F.28) from appendix F to infer

$$\overline{\Delta\Gamma_\mu} = -\Delta\Gamma_\mu, \qquad \mathrm{Im}\,\mathcal{A}_\mu = 0. \tag{4.32}$$

If we compare the spin covariant derivative ∇_μ with the spacetime covariant derivative D_μ we note a similar structure,

$$\begin{array}{rcccccccc} \nabla_\mu\psi &=& \partial_\mu\psi &+& \hat{\Gamma}_\mu\psi &+& \Delta\Gamma_\mu\psi &+& \mathrm{i}\mathcal{A}_\mu\psi, \\ D_\mu T_{\mathrm{gen}}{}^\nu &=& \partial_\mu T_{\mathrm{gen}}{}^\nu &+& \left\{{\nu \atop \mu\rho}\right\} T_{\mathrm{gen}}{}^\rho &+& K^\mu{}_{\nu\rho} T_{\mathrm{gen}}{}^\rho. && \end{array} \tag{4.33}$$

The first part is the ordinary partial derivative. Next we have the canonical (Levi-Civita) connection, i.e. the Christoffel symbols $\left\{ {\rho \atop \mu\nu} \right\}$ and the Levi-Civita spin connection $\hat{\Gamma}_\mu$ which are determined in terms of the Dirac matrices and the metric, respectively. In the third part we find a possible torsion term, whose dynamics is essentially independent of the Dirac matrices (or the metric) and has to be determined by other means, e.g., an action principle. For the fermionic fields there is another, for the moment unrestricted, contribution, $\mathcal{A}_\mu$, without analog in the spacetime covariant derivative. This vector field is reminiscent of a U(1) gauge field from the standard model. If we included a U(1) symmetry transformation for the fermions, then this field would behave exactly like a usual gauge field. As discussed above, we ignore this gauge field in the following and comment on an inclusion in appendix H.

Now we are in a very comfortable situation. Given a set of Dirac matrices we can calculate everything we need to describe fermions in a curved spacetime. There is a (up to a sign) unique spin metric h and a unique canonical (Levi-Civita) part of the connection $\hat{\Gamma}_\mu$. Furthermore there is a rather undetermined object $\Delta\Gamma_\mu$, which we call spin torsion and whose dynamics we are going to investigate in the next section.

Let us first justify the name "spin torsion" by comparing it to spacetime torsion. The spacetime torsion is the part of the spacetime connection, that even in locally inertial coordinates at an arbitrary point $\boldsymbol{z}$ is nonvanishing, it cannot be transformed away with a spacetime coordinate transformation. In order to be more precise, we need a notion of "locally inertial coordinates" in our setup. We want locally inertial spacetime coordinates as well as locally inertial spin-bases. For the spacetime coordinates we demand that the spacetime metric acquires flat form and its first derivatives vanish,

$$g_{\mu\nu}|_{\boldsymbol{z}} = \eta_{\mu\nu}, \qquad\qquad \partial_\lambda g_{\mu\nu}|_{\boldsymbol{z}} = 0, \tag{4.34}$$

i.e. the spacetime coordinate base is "constant" and "orthonormal" in a vicinity around $\boldsymbol{z}$. Since there is no preferred set of Dirac matrices compatible with the Clifford algebra, there is no standard "flat" form of the $\gamma_\mu|_{\boldsymbol{z}}$.[20] Still, we can analogously demand that the spin-base is adjusted in the same fashion around a vicinity of $\boldsymbol{z}$,

$$\partial_\lambda \gamma_\mu|_{\boldsymbol{z}} = 0. \tag{4.35}$$

These coordinates are by no means unique, e.g. for the spacetime coordinates we can always perform constant Lorentz transformations and for the spin-bases we can perform constant similarity transformations. However, the essential property of local inertial coordinates is that in these coordinates at the point $\boldsymbol{z}$ the Christoffel symbols vanish, but the contorsion tensor $K_\mu{}^\rho{}_\lambda|_{\boldsymbol{z}}$ only vanishes if there is no torsion at this point. We observe now the same behavior for

[20] In fact the flat form of $g_{\mu\nu}|_{\boldsymbol{z}}$ is not important. We could change the spacetime coordinates in a nontrivial but constant way and we would lose the flat form, but still the Christoffel symbols would vanish. With "constant way" we mean the vanishing of the first derivatives of the Jacobian at $\boldsymbol{z}$, $\partial_\mu \frac{\partial x'^\rho}{\partial x^\nu}\Big|_{\boldsymbol{z}} = 0$. The important point is the "constant" spacetime base, i.e. the vanishing of the first derivatives of the metric at $\boldsymbol{z}$.

the spin connection. The canonical (Levi-Civita) part $\hat{\Gamma}_\mu|_z$ vanishes, whereas the spin torsion $\Delta\Gamma_\mu|_z$ would only vanish if it was zero also before the coordinate transformation, i.e. if there was no spin torsion at all.

The dynamics of the spin torsion $\Delta\Gamma_\mu$ is still missing, as well as the actual degrees of freedom of $\Delta\Gamma_\mu$. For example the contorsion $K_\mu{}^\rho{}_\lambda$ is not an arbitrary tensor, but it has to be antisymmetric in the last indices, cf. equation (2.17), in order to satisfy the metric compatibility condition. A similar statement holds for the spin torsion which has to be antisymmetric with respect to Dirac conjugation, cf. equation (4.32), so that the spin-metric compatibility is satisfied. Additionally we found the constraint equation (4.25), which ensures that the kinetic term is real. A perfectly valid, but quite simple solution to this equation is $\Delta\Gamma_\mu \overset{!}{=} 0$. However, it is obvious that this is not the most general choice compatible with the constraints.

4.3 Dynamics of Spin Torsion

This section is devoted to the spin torsion and its degrees of freedom. In particular, we compare it to its vielbein based counterpart and thereby show that the new spin torsion is a more general concept with additional degrees of freedom, which are not covered by the vielbein formalism. We also present the construction of a possible action governing the dynamics of $\Delta\Gamma_\mu$.

First, we decompose the spin torsion into the basis of Dirac matrices,

$$\Delta\Gamma_\mu = \sum_{n=1}^{d} \varrho_{\mu\rho_1\ldots\rho_n}\gamma^{\rho_1\cdots\rho_n}, \quad d \text{ even}; \qquad \Delta\Gamma_\mu = \sum_{n=1}^{\frac{d-1}{2}} \varrho_{\mu\rho_1\ldots\rho_{2n}}\gamma^{\rho_1\cdots\rho_{2n}}, \quad d \text{ odd}. \tag{4.36}$$

Next, we use the identities from appendix D to implement equation (4.25) and (4.32). The odd-dimensional case is simpler, we employ equation (D.1) and find

$$0=[\Delta\Gamma_\mu,\gamma^\mu]=\sum_{n=1}^{\frac{d-1}{2}}\varrho_{\mu\rho_1\ldots\rho_{2n}}[\gamma^{\rho_1\cdots\rho_{2n}},\gamma^\mu]=-4\sum_{n=1}^{\frac{d-1}{2}}n\varrho_{\mu\rho_1\ldots\rho_{2n}}g^{\mu[\rho_1}\gamma^{\rho_2\cdots\rho_{2n}]}=-4\sum_{n=1}^{\frac{d-1}{2}}n\varrho^{\rho_1}{}_{\rho_1\rho_2\ldots\rho_{2n}}\gamma^{\rho_2\cdots\rho_{2n}}.$$

From this we conclude

$$0 = \varrho^{\rho_1}{}_{\rho_1\rho_2\ldots\rho_{2n}}, \qquad n \in \left\{1, \ldots, \frac{d-1}{2}\right\}. \tag{4.37}$$

In even dimensions we plug in our ansatz

$$0 = [\Delta\Gamma_\mu, \gamma^\mu] = \sum_{n=1}^{d} \varrho_{\mu\rho_1\ldots\rho_n}[\gamma^{\rho_1\cdots\rho_n}, \gamma^\mu], \tag{4.38}$$

and calculate for $k \in \{1, \ldots, d\}$

$$0 = \sum_{n=1}^{d} \varrho_{\mu\rho_1\ldots\rho_n} \frac{1}{d_\gamma} \operatorname{tr}\big([\gamma^{\rho_1\cdots\rho_n}, \gamma^\mu]\gamma_{\nu_1\ldots\nu_k}\big). \tag{4.39}$$

Since the trace over a product of an odd number of Dirac matrices vanishes in even dimensions, $0 = \operatorname{tr}(\gamma_{\mu_1}\ldots\gamma_{\mu_{2l+1}})$, $l \in \mathbb{N}_0$, we have to distinguish two cases, k even and k odd. Then we can neglect half of the sum for the respective choice of k. For even k we write $k = 2m$, $m \in \{1, \ldots, \frac{d}{2}\}$ and find with the identity (D.2) from appendix D

$$\begin{aligned} 0 =& \sum_{l=1}^{\frac{d}{2}} \varrho_{\mu\rho_1\ldots\rho_{2l-1}} \frac{1}{d_\gamma} \operatorname{tr}\big([\gamma^{\rho_1\cdots\rho_{2l-1}}, \gamma_\mu]\gamma_{\nu_1\ldots\nu_{2m}}\big) = \sum_{l=1}^{\frac{d}{2}} \varrho^\mu{}_{\rho_1\ldots\rho_{2l-1}} \frac{1}{d_\gamma} \operatorname{tr}\big([\gamma_{\nu_1\ldots\nu_{2m}}, \gamma^{\rho_1\cdots\rho_{2l-1}}]\gamma_\mu\big) \\ =& \sum_{l=1}^{\frac{d}{2}} \varrho^\mu{}_{\rho_1\ldots\rho_{2l-1}} (-1)^{l-1} \cdot 2 \cdot (2l)! \cdot g_{\mu[\nu_1} \delta_{\mathrm{A}}{}^{\rho_1\ldots\rho_{2l-1}}_{\nu_2\ldots\nu_{2l}]} \cdot \delta^l_m. \end{aligned} \tag{4.40}$$

Since $m \in \{1, \ldots, \frac{d}{2}\}$ is arbitrary, we infer

$$0 = \varrho_{[\mu\rho_1\ldots\rho_{2m-1}]}. \tag{4.41}$$

Next we choose k odd and write $k = 2m - 1$ with $m \in \{1, \ldots, \frac{d}{2}\}$. Then the trace evaluates to

$$\begin{aligned} 0 =& \sum_{l=1}^{\frac{d}{2}} \varrho^\mu{}_{\rho_1\ldots\rho_{2l}} \frac{1}{d_\gamma} \operatorname{tr}\big([\gamma^{\rho_1\cdots\rho_{2l}}, \gamma_\mu]\gamma_{\nu_1\ldots\nu_{2m-1}}\big) = \sum_{l=1}^{\frac{d}{2}} \varrho^\mu{}_{\rho_1\ldots\rho_{2l}} \frac{1}{d_\gamma} \operatorname{tr}\big(-4l\delta^{[\rho_1}_\mu \gamma^{\rho_2\ldots\rho_{2l}]}\gamma_{\nu_1\ldots\nu_{2m-1}}\big) \\ =& 2\sum_{l=1}^{\frac{d}{2}} \varrho^\mu{}_{\rho_1\ldots\rho_{2l}} \cdot (2m)! \cdot (-1)^m \delta^{[\rho_1}_\mu \delta_{\mathrm{A}}{}^{\rho_2\ldots\rho_{2l}]}_{\nu_1\ldots\nu_{2m-1}} \cdot \delta^l_m, \end{aligned} \tag{4.42}$$

where we have made use of equations (D.1) and (C.6). Again since $m \in \{1, \ldots, \frac{d}{2}\}$ is arbitrary, we deduce

$$0 = \varrho^{\rho_1}{}_{\rho_1\rho_2\ldots\rho_{2m}}. \tag{4.43}$$

Equation (4.32) reexpresses the metric compatibility and tells us whether the coefficients $\varrho_{\mu\rho_1\ldots\rho_n}$ are purely real or purely imaginary. We introduce the new variables

$$\tilde{\varrho}_{\mu\rho_1\ldots\rho_n} = \mathrm{i}^{-\varepsilon_{\mathfrak{p},d}n - \frac{n(n+1)+2}{2}} \cdot \varrho_{\mu\rho_1\ldots\rho_n}, \qquad n \in \{1, \ldots, d\}, \qquad d \text{ even}, \tag{4.44}$$

$$\tilde{\varrho}_{\mu\rho_1\ldots\rho_{2n}} = (-1)^{\varepsilon_{\mathfrak{p},d}n} \cdot \mathrm{i}^{n-1} \cdot \varrho_{\mu\rho_1\ldots\rho_{2n}}, \qquad n \in \left\{1, \ldots, \frac{d-1}{2}\right\}, \qquad d \text{ odd}, \tag{4.45}$$

and find that these have to be purely real employing the metric compatibility together with equation (D.1) from appendix D, $\tilde{\varrho}_{\mu\rho_1\ldots\rho_n} \in \mathbb{R}$.

Summing up, the spin torsion is given in even dimensions by

$$\Delta\Gamma_\mu = \sum_{n=1}^{d} \mathrm{i}^{\varepsilon_{\mathfrak{p},d}n+\frac{n(n+1)+2}{2}} \cdot \tilde{\varrho}_{\mu\rho_1\dots\rho_n}\gamma^{\rho_1\dots\rho_n}, \quad 0 = \tilde{\varrho}^{\rho_1}{}_{\rho_1\rho_2\dots\rho_{2m}}, \quad 0 = \tilde{\varrho}_{[\mu\rho_1\dots\rho_{2m-1}]}, \quad d \text{ even}, \tag{4.46}$$

and in odd dimensions by

$$\Delta\Gamma_\mu = \sum_{n=1}^{\frac{d-1}{2}} (-1)^{(1-\varepsilon_{\mathfrak{p},d})n} \cdot \mathrm{i}^{n+1} \cdot \tilde{\varrho}_{\mu\rho_1\dots\rho_{2n}}\gamma^{\rho_1\dots\rho_{2n}}, \qquad 0 = \tilde{\varrho}^{\rho_1}{}_{\rho_1\rho_2\dots\rho_{2m}}, \qquad d \text{ odd}, \tag{4.47}$$

where in both equations $m \in \{1, \dots, \lfloor d/2 \rfloor\}$. Further, we can count the degrees of freedom and compare $\Delta\Gamma_\mu$ with $\Delta\Gamma_{(e)\mu}$. In even dimensions, for each $\tilde{\varrho}_{\mu\rho_1\dots\rho_n}$ we have $d \cdot \binom{d}{n}$ real components. For even n there are $\binom{d}{n-1}$ constraints and for odd n there are $\binom{d}{n+1}$ constraints,

$$d \cdot \sum_{n=1}^{d} \binom{d}{n} - \sum_{n=1}^{\frac{d}{2}} \binom{d}{2n-1} - \sum_{n=1}^{\frac{d}{2}} \binom{d}{2n} = (d-1)(d_\gamma^2 - 1). \tag{4.48}$$

Therefore we have in total $(d-1)(d_\gamma^2 - 1)$ real degrees of freedom for spin torsion. In odd dimensions, for each $\tilde{\varrho}_{\mu\rho_1\dots\rho_{2n}}$ we have $d \cdot \binom{d}{2n}$ real components and $\binom{d}{2n-1}$ constraints,

$$d \cdot \sum_{n=1}^{\frac{d-1}{2}} \binom{d}{2n} - \sum_{n=1}^{\frac{d-1}{2}} \binom{d}{2n-1} = (d-1)(d_\gamma^2 - 1). \tag{4.49}$$

Hence, we also have $(d-1)(d_\gamma^2 - 1)$ real degrees of freedom for spin torsion in odd dimensions. For even dimensions this number decreases if we also demand chiral invariance ($\psi \to \gamma_*\psi$, $\bar{\psi} \to -\bar{\psi}\gamma_*$) of the kinetic operator, $\bar{\psi}\nabla\!\!\!/\,\psi \to -\bar{\psi}\gamma_*\nabla\!\!\!/\,\gamma_*\psi \overset{!}{=} \bar{\psi}\nabla\!\!\!/\,\psi$. This requirement leads to

$$0 = \gamma^\mu(\nabla_\mu\gamma_*) = \gamma^\mu(\partial_\mu\gamma_* + [\hat{\Gamma}_\mu, \gamma_*]) + \gamma^\mu[\Delta\Gamma_\mu, \gamma_*] = \gamma^\mu[\Delta\Gamma_\mu, \gamma_*]. \tag{4.50}$$

In order to implement this constraint we insert our expansion for $\Delta\Gamma_\mu$ and use equation (4.46),[21]

$$\begin{aligned}
0 &= \mathrm{i}^{\varepsilon_{\mathfrak{p},d}-1} \sum_{n=1}^{d} \mathrm{i}^{\varepsilon_{\mathfrak{p},d}n+\frac{n(n+1)+2}{2}} \cdot \tilde{\varrho}_{\mu\rho_1\dots\rho_n}\gamma^\mu[\gamma^{\rho_1\dots\rho_n}, \gamma_*] = 2 \cdot \gamma_* \sum_{n=1}^{\frac{d}{2}} (-1)^{n\varepsilon_{\mathfrak{p},d}} \cdot \mathrm{i}^n \cdot \tilde{\varrho}_{\mu\rho_1\dots\rho_{2n-1}}\gamma^\mu\gamma^{\rho_1\dots\rho_{2n-1}} \\
&= \gamma_* \sum_{n=1}^{\frac{d}{2}} (-1)^{n\varepsilon_{\mathfrak{p},d}} \cdot \mathrm{i}^n \cdot \tilde{\varrho}_{\mu\rho_1\dots\rho_{2n-1}}[\gamma^\mu, \gamma^{\rho_1\dots\rho_{2n-1}}] + \gamma_* \sum_{n=1}^{\frac{d}{2}} (-1)^{n\varepsilon_{\mathfrak{p},d}} \cdot \mathrm{i}^n \cdot \tilde{\varrho}_{\mu\rho_1\dots\rho_{2n-1}}\{\gamma^\mu, \gamma^{\rho_1\dots\rho_{2n-1}}\} \\
&= \gamma_* \sum_{n=1}^{\frac{d}{2}} (-1)^{n\varepsilon_{\mathfrak{p},d}} \cdot \mathrm{i}^n \cdot \tilde{\varrho}_{\mu\rho_1\dots\rho_{2n-1}}\{\gamma^\mu, \gamma^{\rho_1\dots\rho_{2n-1}}\}.
\end{aligned} \tag{4.51}$$

[21] Note that if we decompose $[\gamma^\mu, \gamma^{\rho_1\dots\rho_{2n-1}}]$ into our standard basis, we find $\mathrm{tr}([\gamma^\mu, \gamma^{\rho_1\dots\rho_{2n-1}}]\gamma_{\nu_1\dots\nu_{2m-1}}) = 0$, since the trace of an odd number of Dirac matrices vanishes in even dimensions. Using equations (D.2) and (C.6) this leads to $[\gamma^\mu, \gamma^{\rho_1\dots\rho_{2n-1}}] = -2\gamma^{\rho_1\dots\rho_{2n-1}\mu}$.

Next we employ the result (D.1) from appendix D and find

$$
\begin{aligned}
0=&\sum_{n=1}^{\frac{d}{2}}(-1)^{n\varepsilon_{\mathfrak{p},d}}\cdot\mathrm{i}^n\cdot\tilde{\varrho}_{\mu\rho_1\dots\rho_{2n-1}}\{\gamma^\mu,\gamma^{\rho_1\dots\rho_{2n-1}}\}=2\sum_{n=1}^{\frac{d}{2}}(-1)^{n\varepsilon_{\mathfrak{p},d}}\cdot\mathrm{i}^n(2n-1)\tilde{\varrho}_{\mu\rho_1\dots\rho_{2n-1}}g^{\mu[\rho_1}\gamma^{\rho_2\dots\rho_{2n-1}]}\\
=&2\sum_{n=1}^{\frac{d}{2}}(-1)^{n\varepsilon_{\mathfrak{p},d}}\cdot\mathrm{i}^n(2n-1)\tilde{\varrho}^{\rho_1}{}_{\rho_1\rho_2\dots\rho_{2n-1}}\gamma^{\rho_2\dots\rho_{2n-1}}.
\end{aligned}
\tag{4.52}
$$

Since the $\gamma^{\rho_1\cdots\rho_n}$ form a basis we can read off

$$
0=\tilde{\varrho}^{\rho_1}{}_{\rho_1\rho_2\dots\rho_{2n-1}}, \qquad\qquad n\in\left\{1,\dots,\frac{d}{2}\right\}.
\tag{4.53}
$$

These additional constraints are independent of the first set (4.46). Again we can count the new constraints, $\sum\limits_{n=1}^{d/2}\binom{d}{2n-2}=\frac{1}{2}d_\gamma^2-1$, leaving us with $(d-2)(d_\gamma^2-1)+\frac{d_\gamma^2}{2}$ real degrees of freedom for chiral spin torsion in even dimensions. Note that the coefficients $\tilde{\varrho}_{\mu\rho_1\dots\rho_n}$ are spin-base independent: in accordance with the preceding discussion we cannot transform away any of these coefficients with a spin-base transformation. One can check, that if we choose the Dirac matrices to be given by the vielbein construction, cf. section 3.2, then the Levi-Civita parts $\hat{\Gamma}_\mu$ and $\hat{\Gamma}_{(e)\mu}$ agree. However, this is not the case for the torsion parts $\Delta\Gamma_\mu$ and $\Delta\Gamma_{(e)\mu}$. This becomes most apparent by looking at the degrees of freedom. The vielbein spin torsion is constructed via the contorsion tensor, $\Delta\Gamma_{(e)\mu}=\frac{1}{4}K_{\mu\rho_1\rho_2}\gamma^{\rho_1\rho_2}$, cf. equation (3.45). Hence, it possesses $\frac{d^2(d-1)}{2}$ real degrees of freedom. For the spin torsion $\Delta\Gamma_\mu$ this corresponds to the component $(-1)^{\varepsilon_{\mathfrak{p},d}}\tilde{\varrho}_{\mu\rho_1\rho_2}\gamma^{\rho_1\rho_2}$, where $\tilde{\varrho}_{\mu\rho_1\rho_2}$ is traceless, $\tilde{\varrho}^{\rho_1}{}_{\rho_1\rho_2}$ (see the previous discussion). Therefore this particular component carries $\frac{d^2(d-1)}{2}-d$ real degrees of freedom, i.e. less than the vielbein spin torsion. However, for $d>3$ the spin torsion $\Delta\Gamma_\mu$ carries further components, which lead in total to more degrees of freedom than the vielbein spin torsion has. In $d=2$, they are balanced and in $d=3$ the vielbein spin torsion possesses more degrees of freedom. Anyway, even if the degrees of freedom were the same, there is no reason to relate the vielbein spin torsion $\Delta\Gamma_{(e)\mu}$ with the "new" spin torsion $\Delta\Gamma_\mu$, as they come from very different concepts. Note however, that the spin covariant derivative ∇_μ with the spin torsion $\Delta\Gamma_\mu$ leads to a real kinetic fermion term by construction. This is not the case for the vielbein spin covariant derivative $\nabla_{(e)\mu}$ if the contorsion tensor has a nonvanishing trace, $K^{\rho_1}{}_{\rho_1\rho_2}\neq 0$. This relation is spoiled by the vielbein spin torsion $\Delta\Gamma_{(e)\mu}$, cf. equations (4.8), (4.25), (4.46) and (4.47).

With the covariant derivative ∇_μ at hand we can turn to a construction of an action similar to the Einstein-Hilbert action. This action is constructed from the field strength tensor, which in general relativity is the Riemann tensor $R_{\mu\nu\rho\lambda}$. Following the same route as for the Riemann tensor, cf. equations (2.25) and (2.42), we define the spin curvature $\Phi_{\mu\nu}$ to be

$$
\Phi_{\mu\nu}\psi=[\nabla_\mu,\nabla_\nu]\psi+C_{\mu\nu}{}^{\sigma}\nabla_\sigma\psi.
\tag{4.54}
$$

More precisely $\Phi_{\mu\nu}$ reads

$$\Phi_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] = \hat{\Phi}_{\mu\nu} + 2\partial_{[\mu}\Delta\Gamma_{\nu]} + 2[\hat{\Gamma}_{[\mu}, \Delta\Gamma_{\nu]}] + [\Delta\Gamma_\mu, \Delta\Gamma_\nu], \tag{4.55}$$

where $\hat{\Phi}_{\mu\nu}$ is the curvature induced by $\hat{\Gamma}_\mu$, cf. equation (2.28),

$$\hat{\Phi}_{\mu\nu} = \partial_\mu \hat{\Gamma}_\nu - \partial_\nu \hat{\Gamma}_\mu + [\hat{\Gamma}_\mu, \hat{\Gamma}_\nu] = \frac{1}{8} R_{(\mathrm{LC})\mu\nu\alpha\beta}[\gamma^\alpha, \gamma^\beta]. \tag{4.56}$$

Then the simplest first order invariant without introduction of any new fields is $\frac{1}{d_\gamma}\operatorname{tr}(\Phi_{\mu\nu}\gamma^{\nu\mu})$. With this invariant we can construct an action,

$$\mathcal{S}_\Phi = \frac{1}{8\pi \mathrm{G_N}} \int_x \mathfrak{L}_\Phi, \qquad\qquad \mathfrak{L}_\Phi = \frac{1}{d_\gamma}\operatorname{tr}(\Phi_{\mu\nu}\gamma^{\nu\mu}), \tag{4.57}$$

which turns out to be reminiscent of the usual Einstein-Hilbert action, where $\mathrm{G_N}$ is then the gravitational constant. For the explicit evaluation of $\mathfrak{L}_\Phi$, we first calculate in even dimensions

$$D_{(\mathrm{LC})\mu}\Delta\Gamma_\nu = D_{(\mathrm{LC})\mu} \sum_{n=1}^{d} \varrho_{\nu\rho_1\ldots\rho_n}\gamma^{\rho_1\ldots\rho_n} = \sum_{n=1}^{d} (D_{(\mathrm{LC})\mu}\varrho_{\nu\rho_1\ldots\rho_n})\gamma^{\rho_1\ldots\rho_n} - [\hat{\Gamma}_\mu, \Delta\Gamma_\nu]. \tag{4.58}$$

Analogously in odd dimensions,

$$D_{(\mathrm{LC})\mu}\Delta\Gamma_\nu = \sum_{n=1}^{\frac{d-1}{2}} (D_{(\mathrm{LC})\mu}\varrho_{\nu\rho_1\ldots\rho_{2n}})\gamma^{\rho_1\ldots\rho_{2n}} - [\hat{\Gamma}_\mu, \Delta\Gamma_\nu] \tag{4.59}$$

holds. Additionally it helps to rewrite the complete antisymmetrization of $\tilde{\varrho}_{\mu\rho_1\ldots\rho_n}$ as

$$\frac{1}{n}\sum_{l=1}^{n}(-1)^{l-1}\tilde{\varrho}_{\rho_l\mu\rho_1\ldots\rho_{l-1}\rho_{l+1}\ldots\rho_n} = \frac{1}{n}\tilde{\varrho}_{\mu\rho_1\ldots\rho_n} - \frac{n+1}{n}\tilde{\varrho}_{[\mu\rho_1\ldots\rho_n]}. \tag{4.60}$$

With the aid of the identities from appendix C and D and the constraints for the $\varrho_{\mu\rho_1\ldots\rho_n}$ it is then straightforward to evaluate $\mathfrak{L}_\Phi$ in even dimensions,

$$\mathfrak{L}_\Phi = \frac{1}{2}R_{(\mathrm{LC})} + 2\sum_{n=1}^{d}(-1)^{n(1-\varepsilon_{\mathfrak{p},d})}\cdot n!\cdot\Big[\tilde{\varrho}^{\mu\rho_1\ldots\rho_n}\tilde{\varrho}_{\mu\rho_1\ldots\rho_n} - (n+1)\tilde{\varrho}^{[\mu\rho_1\ldots\rho_n]}\tilde{\varrho}_{[\mu\rho_1\ldots\rho_n]} - n\tilde{\varrho}^{\mu}{}_{\mu}{}^{\rho_2\ldots\rho_n}\tilde{\varrho}^{\nu}{}_{\nu\rho_2\ldots\rho_n}\Big], \tag{4.61}$$

and in odd dimensions,

$$\mathfrak{L}_\Phi = \frac{1}{2}R_{(\mathrm{LC})} + 2\sum_{n=1}^{\frac{d-1}{2}}(2n)!\cdot\Big[\tilde{\varrho}^{\mu\rho_1\ldots\rho_{2n}}\tilde{\varrho}_{\mu\rho_1\ldots\rho_{2n}} - (2n+1)\tilde{\varrho}^{[\mu\rho_1\ldots\rho_{2n}]}\tilde{\varrho}_{[\mu\rho_1\ldots\rho_{2n}]}\Big]. \tag{4.62}$$

While this is a compact form of the Lagrangian in terms of the $\tilde{\varrho}_{\mu\rho_1\ldots\rho_n}$ it is more convenient to rewrite it into a form which is respecting the constraints of the $\tilde{\varrho}_{\mu\rho_1\ldots\rho_n}$ explicitly. We are dealing with tensors $T_{\mathrm{gen}_{\mu\rho_1\ldots\rho_n}}$ of the form $T_{\mathrm{gen}_{\mu\rho_1\ldots\rho_n}} = T_{\mathrm{gen}_{\mu[\rho_1\ldots\rho_n]}}$. Hence, it is helpful to introduce the projectors onto the trace,

$$(\mathbb{P}^n_{\mathrm{Tr}})_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n} = \frac{n(-1)^{n-1}}{d-(n-1)} g_{\mu[\rho_1}\delta_{\mathrm{A}}{}_{\rho_2\ldots\rho_n]\nu}^{\lambda_1\ldots\lambda_n} g^{\nu\alpha}, \tag{4.63}$$

as well as onto the totally antisymmetric part,

$$(\mathbb{P}^n_{\mathrm{A}})_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n} = \delta_{\mathrm{A}}{}_{\mu\rho_1\ldots\rho_n}^{\alpha\lambda_1\ldots\lambda_n}, \tag{4.64}$$

and onto the traceless part,

$$(\mathbb{P}^n_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n} = (\mathbb{1}^n)_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n} - (\mathbb{P}^n_{\mathrm{Tr}})_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n} - (\mathbb{P}^n_{\mathrm{A}})_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n}, \tag{4.65}$$

of such a tensor. The identity projector reads

$$(\mathbb{1}^n)_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n} = \delta^\alpha_\mu \delta_{\mathrm{A}}{}_{\rho_1\ldots\rho_n}^{\lambda_1\ldots\lambda_n}. \tag{4.66}$$

The projector properties such as idempotence,[22]

$$(\mathbb{P}^n_{\mathrm{Tr}})^2 = (\mathbb{P}^n_{\mathrm{Tr}}), \qquad (\mathbb{P}^n_{\mathrm{A}})^2 = (\mathbb{P}^n_{\mathrm{A}}), \qquad (\mathbb{P}^n_{\mathrm{TL}})^2 = (\mathbb{P}^n_{\mathrm{TL}}), \tag{4.67}$$

orthogonality,

$$(\mathbb{P}^n_{\mathrm{Tr}})(\mathbb{P}^n_{\mathrm{A}}) = (\mathbb{P}^n_{\mathrm{A}})(\mathbb{P}^n_{\mathrm{Tr}}) = 0, \quad (\mathbb{P}^n_{\mathrm{Tr}})(\mathbb{P}^n_{\mathrm{TL}}) = (\mathbb{P}^n_{\mathrm{TL}})(\mathbb{P}^n_{\mathrm{Tr}}) = 0, \quad (\mathbb{P}^n_{\mathrm{A}})(\mathbb{P}^n_{\mathrm{TL}}) = (\mathbb{P}^n_{\mathrm{TL}})(\mathbb{P}^n_{\mathrm{A}}) = 0, \tag{4.68}$$

and the partition of unity,

$$(\mathbb{P}^n_{\mathrm{Tr}}) + (\mathbb{P}^n_{\mathrm{A}}) + (\mathbb{P}^n_{\mathrm{TL}}) = (\mathbb{1}^n), \tag{4.69}$$

are easily checked with the aid of equation (2.10). We denote the trace of $\tilde{\varrho}_{\mu\rho_1\ldots\rho_n}$ by $\tilde{\phi}_{\rho_2\ldots\rho_n}$,

$$\tilde{\phi}_{\rho_2\ldots\rho_n} = \tilde{\varrho}^\mu{}_{\mu\rho_2\ldots\rho_n}, \tag{4.70}$$

the antisymmetric part by $(\tilde{\varrho}_{\mathrm{A}})_{\mu\rho_1\ldots\rho_n}$,

$$(\tilde{\varrho}_{\mathrm{A}})_{\mu\rho_1\ldots\rho_n} = (\mathbb{P}^n_{\mathrm{A}})_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n} \tilde{\varrho}_{\alpha\lambda_1\ldots\lambda_n}, \tag{4.71}$$

[22] The product $(\mathbb{P}^n_{\mathrm{Tr}})(\mathbb{P}^n_{\mathrm{A}})$ of two projectors $(\mathbb{P}^n_{\mathrm{Tr}})$ and $(\mathbb{P}^n_{\mathrm{A}})$ is defined as $[(\mathbb{P}^n_{\mathrm{Tr}})(\mathbb{P}^n_{\mathrm{A}})]_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n} = (\mathbb{P}^n_{\mathrm{Tr}})_{\mu\rho_1\ldots\rho_n}{}^{\beta\kappa_1\ldots\kappa_n}(\mathbb{P}^n_{\mathrm{A}})_{\beta\kappa_1\ldots\kappa_n}{}^{\alpha\lambda_1\ldots\lambda_n}$.

and the traceless part by $(\tilde{\varrho}_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_n}$,

$$(\tilde{\varrho}_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_n} = (\mathbb{P}^n_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_n}{}^{\alpha\lambda_1\ldots\lambda_n}\,\tilde{\varrho}_{\alpha\lambda_1\ldots\lambda_n}. \tag{4.72}$$

Then we can decompose $\tilde{\varrho}_{\mu\rho_1\ldots\rho_n}$ as

$$\tilde{\varrho}_{\mu\rho_1\ldots\rho_n} = (\tilde{\varrho}_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_n} + (\tilde{\varrho}_{\mathrm{A}})_{\mu\rho_1\ldots\rho_n} + \frac{n}{d-(n-1)} g_{\mu[\rho_1}\tilde{\phi}_{\rho_2\ldots\rho_n]}. \tag{4.73}$$

In particular, the square of $\tilde{\varrho}_{\mu\rho_1\ldots\rho_n}$ then reads

$$\tilde{\varrho}^{\mu\rho_1\ldots\rho_n}\tilde{\varrho}_{\mu\rho_1\ldots\rho_n} = (\tilde{\varrho}_{\mathrm{TL}})^{\mu\rho_1\ldots\rho_n}(\tilde{\varrho}_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_n} + (\tilde{\varrho}_{\mathrm{A}})^{\mu\rho_1\ldots\rho_n}(\tilde{\varrho}_{\mathrm{A}})_{\mu\rho_1\ldots\rho_n} + \frac{n}{d-(n-1)}\tilde{\phi}^{\rho_2\ldots\rho_n}\tilde{\phi}_{\rho_2\ldots\rho_n}. \tag{4.74}$$

In other words, the trace, the antisymmetric part and the traceless part decouple from each other. The Lagrangian in these variables in even dimensions is then given by

$$\begin{aligned}\mathfrak{L}_\Phi = {} & \frac{1}{2}R_{(\mathrm{LC})} + 2\sum_{n=1}^{d}(-1)^{n(1-\varepsilon_{\mathfrak{p},d})}n!\cdot(\tilde{\varrho}_{\mathrm{TL}})^{\mu\rho_1\ldots\rho_n}(\tilde{\varrho}_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_n} - 2\sum_{n=1}^{\frac{d}{2}}2n\cdot(2n)!\cdot(\tilde{\varrho}_{\mathrm{A}})^{\mu\rho_1\ldots\rho_{2n}}(\tilde{\varrho}_{\mathrm{A}})_{\mu\rho_1\ldots\rho_{2n}}\\ & + (-1)^{1-\varepsilon_{\mathfrak{p},d}}2\sum_{n=1}^{\frac{d}{2}}(2n-1)\cdot(2n-1)!\cdot\frac{d-2n+1}{d-2n+2}\tilde{\phi}^{\rho_2\ldots\rho_n}\tilde{\phi}_{\rho_2\ldots\rho_n},\end{aligned} \tag{4.75}$$

and in odd dimensions we have

$$\mathfrak{L}_\Phi = \frac{1}{2}R_{(\mathrm{LC})} + 2\sum_{n=1}^{\frac{d-1}{2}}(2n)!\cdot(\tilde{\varrho}_{\mathrm{TL}})^{\mu\rho_1\ldots\rho_{2n}}(\tilde{\varrho}_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_{2n}} - 2\sum_{n=1}^{\frac{d-1}{2}}2n\cdot(2n)!\cdot(\tilde{\varrho}_{\mathrm{A}})^{\mu\rho_1\ldots\rho_{2n}}(\tilde{\varrho}_{\mathrm{A}})_{\mu\rho_1\ldots\rho_{2n}}. \tag{4.76}$$

In this form it is apparent that the resulting classical equations of motion after varying with respect to the spin torsion degrees of freedom are purely algebraic in the fields $(\tilde{\varrho}_{\mathrm{TL}})_{\mu\rho_1\ldots\rho_n}$, $(\tilde{\varrho}_{\mathrm{A}})_{\mu\rho_1\ldots\rho_n}$ and $\tilde{\phi}_{\rho_2\ldots\rho_n}$. Therefore the spin torsion vanishes classically in the absence of, e.g., spinorial sources. The variation with respect to the metric gives us the usual Einstein field equations. Note that the spacetime torsion does not enter in this action.

4.4 Lorentz Symmetric Gauge

In the usual vielbein setup one often needs the vielbein $e_\mu{}^a$ as a function of the metric $g_{\mu\nu}$ with respect to some fixed but arbitrary background metric $\boldsymbol{g}_{\mu\nu}$ and background vielbein $\boldsymbol{e}_\mu{}^a$. Such relations define a gauge for the vielbein. It is known that the Lorentz-symmetric gauge is very useful and minimizes in practice the calculational effort [37, 38, 143]. In particular, corresponding SO(1,3) Faddeev-Popov ghosts do not contribute in perturbation theory [141, 144]. An interesting application of the generalized Weldon theorem (4.21) is the derivation of the analog of the Lorentz symmetric gauge for the Dirac matrices as the *simplest*[23] possible choice (or gauge) of the Dirac matrices $\gamma_\mu = \gamma_\mu(g)$. We show how this is done in the following.

With $h_{\mu\nu}$ we denote a metric fluctuation which parametrizes the full metric $g_{\mu\nu}$ with respect to an unspecified (arbitrary) background metric $\boldsymbol{g}_{\mu\nu}$,

$$g_{\mu\nu} = \boldsymbol{g}_{\mu\nu} + h_{\mu\nu}. \tag{4.77}$$

The background Dirac matrices are denoted with $\boldsymbol{\gamma}_\mu$.

We assume that we can expand[24] $\gamma_\mu(g)$ in powers of the fluctuation $h_{\mu\nu}$,

$$\gamma_\mu(g) = \sum_{n=0}^{\infty} \frac{\partial^n \gamma_\mu(g)}{n! \partial g_{\nu_1\nu_2} \ldots \partial g_{\nu_{2n-1}\nu_{2n}}}\bigg|_{g=\boldsymbol{g}} h_{\nu_1\nu_2} \ldots h_{\nu_{2n-1}\nu_{2n}}. \tag{4.78}$$

Using the generalized Weldon theorem (4.21) we can write

$$\frac{\partial \gamma_\mu(g)}{\partial g_{\nu_1\nu_2}} = \frac{1}{2} \delta\mathrm{s}_{\mu\rho}^{\nu_1\nu_2} \gamma^\rho(g) + [G^{\nu_1\nu_2}(g), \gamma_\mu(g)], \tag{4.79}$$

where $G^{\nu_1\nu_2}$ is a Dirac valued function of the metric encoding the gauge choice. That is, by fixing $G^{\nu_1\nu_2}(g)$ and $\boldsymbol{\gamma}_\mu = \gamma_\mu(\boldsymbol{g})$ we completely fix the function $\gamma_\mu = \gamma_\mu(g)$. Since there is no preferred choice of Dirac matrices for a given metric, we can leave the background Dirac matrices $\boldsymbol{\gamma}_\mu$ arbitrary while compatible with the Clifford algebra. We aim at optimizing the function $G^{\nu_1\nu_2}$ such that equation (4.78) becomes as simple as possible.

In order to do so we expand $G^{\nu_1\nu_2}(g)$ in powers of the metric fluctuations,

$$G^{\nu_1\nu_2} = \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{G}^{\nu_1\nu_2\lambda_1\ldots\lambda_{2n}} h_{\lambda_1\lambda_2} \ldots h_{\lambda_{2n-1}\lambda_{2n}}, \tag{4.80}$$

where the $\boldsymbol{G}^{\nu_1\nu_2\lambda_1\ldots\lambda_{2n}}$ are the expansion coefficients to be determined. Since we aim at simplifying the function (4.78) we have to simplify the derivatives of the Dirac matrices. Looking at the first nontrivial term we find

$$\frac{\partial \gamma_\mu(g)}{\partial g_{\nu_1\nu_2}}\bigg|_{g=\boldsymbol{g}} = \frac{1}{2} \delta\mathrm{s}_{\mu\rho}^{\nu_1\nu_2} \boldsymbol{\gamma}^\rho + [\boldsymbol{G}^{\nu_1\nu_2}, \boldsymbol{\gamma}_\mu]. \tag{4.81}$$

Taking into account that the symmetric part and the commutator part are completely independent, it is obvious that the best simplification we can find is $\boldsymbol{G}^{\nu_1\nu_2} = 0$. With this we can

[23] The notion of "*simplest*" here will become apparent below. It means the least possible change of the Dirac structure while going from the background Dirac matrices $\boldsymbol{\gamma}_\mu$ to γ_μ, see below.
[24] We discuss the situation for a nonexpandable metric at the end of this section.

go on to the second derivative,

$$\left.\frac{\partial^2 \gamma_\mu(g)}{\partial g_{\nu_1\nu_2}\partial g_{\nu_3\nu_4}}\right|_{g=\boldsymbol{g}} = -\omega^{\nu_1\dots\nu_4}{}_{\mu\rho}\boldsymbol{\gamma}^\rho + [\boldsymbol{G}^{\nu_1\dots\nu_4}, \boldsymbol{\gamma}_\mu], \tag{4.82}$$

where $\omega^{\nu_1\dots\nu_4}{}_{\mu\rho} = \frac{1}{4}\delta\mathrm{s}^{\nu_1\nu_2}_{\mu\kappa}\boldsymbol{g}^{\kappa\sigma}\delta\mathrm{s}^{\nu_3\nu_4}_{\sigma\rho}$. The tensor $\omega^{\nu_1\dots\nu_4}{}_{\mu\rho}$ has a symmetric and an antisymmetric part concerning the pair (μ,ρ). Using that we can rewrite the antisymmetric part as a commutator. We find

$$\left.\frac{\partial^2 \gamma_\mu(g)}{\partial g_{\nu_1\nu_2}\partial g_{\nu_3\nu_4}}\right|_{g=\boldsymbol{g}} = -\omega^{\nu_1\dots\nu_4}{}_{(\mu\rho)}\boldsymbol{\gamma}^\rho + \left[\boldsymbol{G}^{\nu_1\dots\nu_4} + \frac{1}{8}\omega^{\nu_1\dots\nu_4}{}_{[\lambda_1\lambda_2]}[\boldsymbol{\gamma}^{\lambda_1}, \boldsymbol{\gamma}^{\lambda_2}], \boldsymbol{\gamma}_\mu\right]. \tag{4.83}$$

Once again we have two independent terms and we conclude that the simplest choice is $\boldsymbol{G}^{\nu_1\dots\nu_4} = -\frac{1}{8}\omega^{\nu_1\dots\nu_4}{}_{[\lambda_1\lambda_2]}[\boldsymbol{\gamma}^{\lambda_1}, \boldsymbol{\gamma}^{\lambda_2}]$. We can iterate this process of identifying the symmetric part and the commutator part and eliminate the commutator part by appropriate choices of the $\boldsymbol{G}^{\nu_1\nu_2\lambda_1\dots\lambda_{2n}}$. By doing this we end up with an expansion of the Dirac matrices $\gamma_\mu(g)$ which is directly proportional to the background Dirac matrices $\boldsymbol{\gamma}_\mu$,

$$\gamma_\mu = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!}(\omega_n)^{\nu_1\dots\nu_{2n}}{}_{\mu\rho}\boldsymbol{\gamma}^\rho h_{\nu_1\nu_2}\dots h_{\nu_{2n-1}\nu_{2n}}. \tag{4.84}$$

The $(\omega_n)^{\nu_1\dots\nu_{2n}}{}_{\mu\rho} = (\omega_n)^{\nu_1\dots\nu_{2n}}{}_{(\mu\rho)}$ can be calculated recursively applying the above given construction of the $\boldsymbol{G}^{\nu_1\nu_2\lambda_1\dots\lambda_{2n}}$, where we already know the first three of them,

$$(\omega_0)_{\mu\rho} = -\boldsymbol{g}_{\mu\rho}, \qquad (\omega_1)^{\nu_1\nu_2}{}_{\mu\rho} = \frac{1}{2}\delta\mathrm{s}^{\nu_1\nu_2}_{\mu\rho}, \qquad (\omega_2)^{\nu_1\dots\nu_4}{}_{\mu\rho} = \omega^{\nu_1\dots\nu_4}{}_{(\mu\rho)}. \tag{4.85}$$

Unfortunately it is difficult to perform this iteration to all orders. To circumvent this problem we remind ourselves that equation (4.79) is a consequence of the Clifford algebra and insert the simplified ansatz (4.84) directly,

$$2(\boldsymbol{g}_{\mu\nu}+h_{\mu\nu})\mathrm{I} = \sum_{n,m=0}^{\infty}\frac{(-1)^{n+m}}{n!\cdot m!}\{\boldsymbol{\gamma}^\alpha,\boldsymbol{\gamma}^\beta\}(\omega_n)^{\rho_1\dots\rho_{2n}}{}_{\mu\alpha}h_{\rho_1\rho_2}\dots h_{\rho_{2n-1}\rho_{2n}}(\omega_m)^{\lambda_1\dots\lambda_{2m}}{}_{\mu\beta}h_{\lambda_1\lambda_2}\dots h_{\lambda_{2m-1}\lambda_{2m}}. \tag{4.86}$$

We can reorder the sums such that we sum over the powers of the fluctuations in increasing order. For this we introduce the new summation variables $(s,l) = (n+m,m)$, where $l \in \{0,\dots,s\}$ and $s \in \{0,\dots,\infty\}$ and the shorthand $(\omega_m h^m)^\rho{}_\lambda = \boldsymbol{g}^{\rho\alpha}(\omega_m)^{\nu_1\dots\nu_{2m}}{}_{\alpha\lambda}h_{\nu_1\nu_2}\dots h_{\nu_{2m-1}\nu_{2m}}$. Then we find

$$\boldsymbol{g}_{\mu\nu}+h_{\mu\nu} = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\sum_{l=0}^{s}\binom{s}{l}\boldsymbol{g}_{\mu\kappa}(\omega_{s-l}h^{s-l})^\kappa{}_\alpha(\omega_l h^l)^\alpha{}_\nu. \tag{4.87}$$

Since this equation has to be true for each power of $h_{\mu\nu}$ individually the equation splits into three parts. First we have from $s = 0$:

$$\delta^\mu_\nu = (\omega_0 h^0)^\mu{}_\alpha (\omega_0 h^0)^\alpha{}_\nu, \tag{4.88}$$

which is of course satisfied, cf. equation (4.85). For $s = 1$ we find again just a trivially satisfied equation,

$$\boldsymbol{g}^{\mu\alpha} h_{\alpha\nu} = -(\omega_1 h^1)^\mu{}_\alpha (\omega_0 h^0)^\alpha{}_\nu - (\omega_0 h^0)^\mu{}_\alpha (\omega_1 h^1)^\alpha{}_\nu, \tag{4.89}$$

cf. equation (4.85). The last part is $s \geq 2$

$$0 = \sum_{l=0}^{s} \binom{s}{l} (\omega_{s-l} h^{s-l})^\mu{}_\alpha (\omega_l h^l)^\alpha{}_\nu, \tag{4.90}$$

which we can rewrite as

$$(\omega_s h^s)^\mu{}_\nu = \frac{1}{2} \sum_{l=1}^{s-1} \binom{s}{l} (\omega_{s-l} h^{s-l})^\mu{}_\alpha (\omega_l h^l)^\alpha{}_\nu \tag{4.91}$$

by splitting off the $l = 0$ and $l = s$ parts and using equation (4.85). In other words we have a recursion relation with initial conditions (4.85). This recursion obviously has a unique solution. With the initial conditions we can show by induction that

$$(\omega_s h^s)^\mu{}_\nu = c_s \cdot h^\mu{}_{\rho_1} \dots h^{\rho_{s-1}}{}_\nu, \qquad s \geq 1, \tag{4.92}$$

where $h^\mu{}_\nu = \boldsymbol{g}^{\mu\rho} h_{\rho\nu}$ and c_s are just numbers to be determined. Plugging our result into equation (4.91) we get a recursion for the c_s

$$c_s = \frac{1}{2} \sum_{l=1}^{s-1} \binom{s}{l} c_{s-l} c_l, \quad s \geq 2, \tag{4.93}$$

with initial condition $c_1 = \frac{1}{2}$. The explicit solution of this recursion reads

$$c_s = (-1)^{s-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - s\right)}, \tag{4.94}$$

which can be shown by induction again. As a result we have

$$\gamma_\mu(g) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n!} \boldsymbol{g}_{\mu\kappa} (\omega_n h^n)^\kappa{}_\lambda \boldsymbol{\gamma}^\lambda = \boldsymbol{g}_{\mu\kappa} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(n+1)\,\Gamma\left(\frac{3}{2} - n\right)} (h^n)^\kappa{}_\lambda \boldsymbol{\gamma}^\lambda, \tag{4.95}$$

where $(h^n)^\kappa{}_\lambda = h^\kappa{}_{\rho_1} h^{\rho_1}{}_{\rho_2} \dots h^{\rho_{n-2}}{}_{\rho_{n-1}} h^{\rho_{n-1}}{}_\lambda$. This sum is exactly the series representation of the square root and we can write formally

$$\gamma_\mu(g) = \boldsymbol{g}_{\mu\kappa}\big[\sqrt{\delta + h}\big]^{\kappa}{}_{\lambda}\boldsymbol{\gamma}^\lambda. \tag{4.96}$$

This is precisely the representation given by Woodard for the vielbein [141]

$$e_\mu{}^a = \boldsymbol{g}_{\mu\kappa}\big[\sqrt{\delta + h}\big]^{\kappa}{}_{\lambda}\boldsymbol{e}^{\lambda a}. \tag{4.97}$$

This calculation illustrates *why* the Lorentz symmetric gauge proved so useful.

In view of contemporary nonperturbative quantum gravity calculations, an urgent question arises [145–150]: Is there a way to fix the gauge without assuming that $\gamma_\mu(g)$ is expandable in the metric fluctuation $h_{\mu\nu}$ around the background metric $\boldsymbol{g}_{\mu\nu}$? We will outline a possibility here.

Let us assume two given metrics $g_{\mu\nu}$ and $\boldsymbol{g}_{\mu\nu}$, with the same notation as before. We have seen that we can tune the spin-base such that the full Dirac matrices $\gamma_\mu(g)$ and the background Dirac matrices $\boldsymbol{\gamma}_\mu$ are related in a linear way $\gamma_\mu \sim \boldsymbol{\gamma}_\nu$. Let us take such a form as an ansatz to find a "nonperturbative" gauge

$$\gamma_\mu(g) = \mathcal{E}_{\mu\nu}(g,\boldsymbol{g})\boldsymbol{\gamma}^\nu, \qquad\qquad \mathcal{E}_{\mu\nu}(g,\boldsymbol{g}) = \mathcal{E}_{\nu\mu}(g,\boldsymbol{g}), \tag{4.98}$$

where we have to determine the complex functions $\mathcal{E}_{\mu\nu}(g,\boldsymbol{g})$. The symmetry of $\mathcal{E}_{\mu\nu}(g,\boldsymbol{g})$ is in the same spirit as our construction from above and is supposed to ensure the simplicity. Plugging this ansatz into the Clifford algebra we find

$$g_{\mu\nu} = \mathcal{E}_{\mu\rho}(g,\boldsymbol{g})\boldsymbol{g}^{\rho\kappa}\mathcal{E}_{\nu\kappa}(g,\boldsymbol{g}). \tag{4.99}$$

For clarity, we switch to an intuitive matrix formulation $g_{\mu\nu} \to g$, $\mathcal{E}_{\mu\nu}(g,\boldsymbol{g}) \to \mathcal{E}$ and additionally drop the arguments $(g,\boldsymbol{g})$ from now on. By using the symmetry $\mathcal{E} = \mathcal{E}^{\mathrm{T}}$ we can rewrite equation (4.99)

$$g = \mathcal{E}\boldsymbol{g}^{-1}\mathcal{E}^{\mathrm{T}} = \mathcal{E}\boldsymbol{g}^{-1}\mathcal{E} = \boldsymbol{g}(\boldsymbol{g}^{-1}\mathcal{E})^2. \tag{4.100}$$

Therefore $\boldsymbol{g}^{-1}\mathcal{E}$ has to be a square root of $\boldsymbol{g}^{-1}g$, compatible with the symmetry condition.[25] To simplify the structure we use that $\boldsymbol{g}$ is a real symmetric matrix and therefore has a (nonunique) symmetric square root χ,

$$\boldsymbol{g} = \chi^2, \qquad\qquad \chi^{\mathrm{T}} = \chi. \tag{4.101}$$

Depending on the signature χ can be complex. Then it follows that $\chi^{-1}\mathcal{E}\chi^{-1}$ is symmetric as

[25] Note that for $g = \boldsymbol{g} + h$ we get the perturbative result from above, $\mathcal{E} = \boldsymbol{g}\sqrt{\mathrm{I} + \boldsymbol{g}^{-1}h}$.

long as $\mathcal{E}$ is symmetric. Hence, we arrive at

$$(\chi^{-1}\mathcal{E}\chi^{-1})^2 = \chi^{-1}g\chi^{-1}, \tag{4.102}$$

where $\chi^{-1}g\chi^{-1}$ is obviously a symmetric matrix as well. This is a quite comfortable situation, as we are looking for a symmetric square root of a symmetric matrix. If we suppose there is a symmetric square root κ of $\chi^{-1}g\chi^{-1}$,

$$\chi^{-1}g\chi^{-1} = \kappa^2, \qquad \kappa^{\mathrm{T}} = \kappa, \tag{4.103}$$

then we have a solution $\mathcal{E}$ with

$$\mathcal{E} = \chi\kappa\chi, \tag{4.104}$$

as can be checked easily. In particular, for the recently become prominent exponential parametrization [134, 151–153],

$$g = \boldsymbol{g}\mathrm{e}^{\boldsymbol{g}^{-1}h}, \qquad h^{\mathrm{T}} = h, \tag{4.105}$$

we find

$$\kappa = \mathrm{e}^{\frac{1}{2}\chi^{-1}h\chi^{-1}}, \qquad \mathcal{E} = \boldsymbol{g}\mathrm{e}^{\frac{1}{2}\boldsymbol{g}^{-1}h}. \tag{4.106}$$

Unfortunately in general there is no guarantee that for a complex symmetric matrix a symmetric square root exists. Still, any Euclidean metric corresponds to a symmetric, positive definite matrix. Hence, there is a unique, symmetric, positive definite χ. Therefore we also have a unique, symmetric, positive definite κ, leading to a unique $\mathcal{E}$ given by equation (4.104). As proven in appendix A of [152] one can uniquely parametrize any Euclidean metric g by equation (4.105), hence κ and $\mathcal{E}$ are given by equation (4.106).

In general dimensions the situation for the Lorentzian signature is unclear so far. The problem stems from the minus sign in the signature of the metric leading to a complex χ. The first nontrivial dimension is $d = 2$. One can show, however, that the only complex symmetric 2×2 matrices without symmetric square root are of the form $c \cdot \begin{pmatrix} \pm\mathrm{i} & 1 \\ 1 & \mp\mathrm{i} \end{pmatrix}$, with $c \in \mathbb{C}\backslash\{0\}$. Fortunately these matrices have vanishing determinant guaranteeing the existence of the symmetric square root of $\chi^{-1}g\chi^{-1}$ at least in two dimensions independent of the signature. One can hope that this generalizes somehow to arbitrary integer dimensions $d \geq 2$, but this is beyond the scope of this thesis.

5 Quantum Field Theory and Heat Kernel

So far we we have dealt with the classical description of gravity and the conceptual formulation of fermions in curved spacetimes. The remainder of this thesis is dedicated to the quantum aspects of gravity on the one hand and to quantized fermions in a curved spacetime on the other hand. In this section we start with a very short recapitulation of the path integral and the functional renormalization group [154] and then concentrate on our main tool for its investigation: the heat kernel [155]. We will not go into the details of standard quantum field theory, as there are plenty of books on that subject, see, e.g., [156–158]. For a nice and recent review of the Wilsonian idea of renormalization see [159]. An introduction to the functional renormalization is found in [160–162]. Here we present the concepts in a way which does not claim to be mathematically rigorous, but is rather meant to be a point of view familiar from quantum mechanics and infinite dimensional Hilbert spaces. In this way ideas, which seem complicated at first, fit well into a very simple language. Many things can be made mathematically rigorous in the language of fiber bundles, see [155] and references therein.

5.1 Path Integral and Functional Renormalization Group

One of the central objects of modern quantum field theory is the partition sum in terms of the path integral. It is the generating functional of the n-point correlation functions, which can be related to observables, i.e. experiments. In order to define the path integral we first need to set the stage. For illustrative purposes we restrict ourselves to euclidean signature in this chapter ($\mathfrak{m} = 0$, $\mathfrak{p} = d$). Suppose for the moment we only have one field $\phi^{(i)}(\boldsymbol{x})$, which has internal indices $i \in \Omega_\phi$ (e.g. spacetime, spin, color and/or flavor indices). It is important to note that it is usually hard to define the space of functions in which the $\phi^{(i)}(\boldsymbol{x})$ are supposed to live. Hence, one often lacks a precise knowledge of the properties of these fields. Nevertheless, they are supposed to transform under a specific representation $\boldsymbol{\varrho} : \boldsymbol{\mathcal{G}} \to \mathrm{Mat}(|\Omega_\phi| \times |\Omega_\phi|, \mathbb{C})$ of a group $\boldsymbol{\mathcal{G}}$, as indicated by the index "i".[26] With ${\mathfrak{T}^{(i)}}_{(j)}(\boldsymbol{x})$ we denote the representation of a generic symmetry transformation. The components of the transformed field ${\phi'}^{(i)}(\boldsymbol{x})$ are then given by ${\mathfrak{T}^{(i)}}_{(j)}(\boldsymbol{x})\phi^{(j)}(\boldsymbol{x})$. Here we made use of the Einstein summation convention for the repeated index "j". Further we define the conjugate fields $\bar{\phi}_{(i)}(\boldsymbol{x})$, i.e. with "opposite" index, such that the conjugate fields transform with the inverse of ${\mathfrak{T}^{(i)}}_{(j)}(\boldsymbol{x})$, $\bar{\phi}_{(i)}(\boldsymbol{x}) \to \bar{\phi}_{(j)}(\boldsymbol{x}){\mathfrak{T}^{-1(j)}}_{(i)}(\boldsymbol{x})$, and the product $\bar{\phi}_{(i)}(\boldsymbol{x})\phi^{(i)}(\boldsymbol{x})$ is real. In general one needs a nondegenerate metric $\mathfrak{h}_{(i)(j)}(\boldsymbol{x})$ for the lowering of the index "i", $\bar{\phi}_{(i)}(\boldsymbol{x}) = \left(\phi^{(j)}(\boldsymbol{x})\right)^* \mathfrak{h}_{(j)(i)}(\boldsymbol{x})$. For example for spacetime indices we have the spacetime metric $g_{\mu\nu}$, and for spinorial indices we have the spin metric h. As one complex degree of freedom corresponds to two real degrees of freedom, we treat $\bar{\phi}_{(i)}(\boldsymbol{x})$

[26] As the considered internal structure may especially correspond to that of a spacetime tensor, we use the term "symmetry transformation" for a combination of a spacetime coordinate transformation and a possible internal symmetry transformation (e.g., a spin base transformation). That is, the diffeomorphisms on $\mathcal{M}$ are part of $\boldsymbol{\mathcal{G}}$.

as independent of $\phi^{(i)}(\boldsymbol{x})$ in the complex case. The collection of independent variables is then $\Phi = \big(\phi^{(i)}(\boldsymbol{x})\big)_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}}$ in the real case, and $\Phi = \big(\phi^{(i)}(\boldsymbol{x}), \bar{\phi}_{(i)}(\boldsymbol{x})\big)_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}}$ in the complex case. Here we interpret the points $\boldsymbol{x}$ on the manifold as a "continuous index" of $\phi^{(i)}(\boldsymbol{x})$ and the spacetime integral $\int_x$ as the summation over that index.[27] Furthermore we use $\bar{\Phi}$ as the collection conjugated to Φ, $\bar{\Phi} = \big(\bar{\phi}_{(i)}(\boldsymbol{x})\big)_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}}$ or $\bar{\Phi} = \big(\bar{\phi}_{(i)}(\boldsymbol{x}), \phi^{(i)}(\boldsymbol{x})\big)_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}}$. In J we collect the corresponding sources, i.e. auxiliary fields $J_{(i)}(\boldsymbol{x})$ with an "opposite" index placing compared to $\phi^{(i)}(\boldsymbol{x})$, $J = \big(J_{(i)}(\boldsymbol{x})\big)_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}}$ or $J = \big(J_{(i)}(\boldsymbol{x}), \bar{J}^{(i)}(\boldsymbol{x})\big)_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}}$. The product $J\cdot\Phi$ gives a real number and is defined by $\int_x J_{(i)}(\boldsymbol{x})\phi^{(i)}(\boldsymbol{x})$ in the real case or by $\int_x \big(J_{(i)}(\boldsymbol{x})\phi^{(i)}(\boldsymbol{x}) + \bar{\phi}_{(i)}(\boldsymbol{x})\bar{J}^{(i)}(\boldsymbol{x})\big)$ in the complex case.[28]

The path integral gives us the partition sum $Z[J]$ as a functional of the source fields J,

$$Z[J] = \int_{\Lambda_{\mathrm{UV}}} \mathcal{D}\Phi\, \exp\big(-S_{\Lambda_{\mathrm{UV}}}[\Phi] + J\cdot\Phi\big), \tag{5.1}$$

where $S_{\Lambda_{\mathrm{UV}}}[\Phi]$ is the action of the quantum fields Φ at an ultraviolet scale Λ_{UV}. Generically the results are highly divergent. Physicists needed years to understand the arising divergencies and to convert the results into a *finite* physical interpretation, which is nowadays known as renormalization [157]. For our purposes it is sufficient to think of the functional integral as a suitably regularized integral over all field configurations $\mathcal{D}\Phi = \prod_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}} \mathrm{d}\phi^{(i)}(\boldsymbol{x})$ or $\mathcal{D}\Phi = \prod_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}} \mathrm{d}\phi^{(i)}(\boldsymbol{x})\mathrm{d}\bar{\phi}_{(i)}(\boldsymbol{x})$.[29] The vacuum expectation value $\langle\mathcal{O}_{\mathrm{Ob}}[\,\cdot\,]\rangle$ of an observable $\mathcal{O}_{\mathrm{Ob}}[\Phi]$ is then given by

$$\langle\mathcal{O}_{\mathrm{Ob}}[\,\cdot\,]\rangle = \frac{\int\mathcal{D}\Phi\ \mathcal{O}_{\mathrm{Ob}}[\Phi]\,\mathrm{e}^{-S_{\Lambda_{\mathrm{UV}}}[\Phi]}}{\int\mathcal{D}\Phi'\ \mathrm{e}^{-S_{\Lambda_{\mathrm{UV}}}[\Phi']}} = \left(\frac{1}{Z[J]}\,\mathcal{O}_{\mathrm{Ob}}\big[\tfrac{\delta}{\delta J}\big]\,Z[J]\right)\bigg|_{J=0}. \tag{5.2}$$

In particular, for the vacuum expectation value φ_{vac} of the quantum fields Φ we find

$$\varphi_{\mathrm{vac}} = \frac{\int\mathcal{D}\Phi\ \Phi\,\mathrm{e}^{-S_{\Lambda_{\mathrm{UV}}}[\Phi]}}{\int\mathcal{D}\Phi'\ \mathrm{e}^{-S_{\Lambda_{\mathrm{UV}}}[\Phi']}} = \left(\frac{\delta}{\delta J}\ln Z[J]\right)\bigg|_{J=0}. \tag{5.3}$$

Hence, the Legendre transform of $\ln Z[\Phi]$,

$$\Gamma[\varphi] = \sup_J\,(J\cdot\varphi - \ln Z[J]), \tag{5.4}$$

[27] Sometimes it can be helpful to think of $\mathcal{M}$ as a discrete space with a finite number of points. This is exactly the same point of view one has in lattice quantum field theory. However, this picture has to be handled with care.

[28] Note that when considering fermions one has to be particularly careful with the order of the fields in $J\cdot\Phi$ (because of the Graßmann nature).

[29] When considering gauge fields special care is necessary as the unphysical gauge degrees of freedom typically lead to further divergencies. To cure this problem one usually introduces a gauge condition and so-called Faddeev-Popov ghosts, see, e.g., [162, 163].

serves as an effective action functional $\Gamma[\varphi]$, whose (classical) Euler-Lagrange equations have the solution φ_{vac}, $\left.\frac{\delta\Gamma[\varphi]}{\delta\varphi}\right|_{\varphi=\varphi_{\text{vac}}} = 0$. In fact the effective action $\Gamma[\varphi]$ stores all the information of the partition sum in an efficient way, as it is also the generator of the one-particle irreducible correlation functions. The calculation of the effective action, however, is a delicate business.

A nonperturbative approach is the functional renormalization group which follows the Wilsonian viewpoint on renormalization [164, 165]. Instead of calculating the path integral in one step one introduces a regulator $\mathcal{R}_k$ which facilitates an interpolating action functional $\Gamma_k[\varphi]$ flowing from the ultraviolet action $\Gamma_{k=\Lambda_{\text{UV}}}[\varphi] = S_{\Lambda_{\text{UV}}}[\varphi]$ at the ultraviolet scale $k = \Lambda_{\text{UV}}$ to the effective action $\Gamma_{k=0}[\varphi] = \Gamma[\varphi]$ in the infrared at $k = 0$. The role of the path integral is then played by the Wetterich equation [154],

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)}[\varphi] + \mathcal{R}_k(\boldsymbol{\Delta}_E)\right)^{-1} \partial_k \mathcal{R}_k(\boldsymbol{\Delta}_E)\right], \tag{5.5}$$

with the initial condition $\Gamma_{k=\Lambda_{\text{UV}}}[\varphi] = S_{\Lambda_{\text{UV}}}[\varphi]$. By solving this equation we find the full quantum effective action $\Gamma[\varphi]$. Let us explain the quantities and operations appearing in (5.5) in some detail. However, for this we need some preparatory definitions. As the field $\phi^{(i)}(\boldsymbol{x})$ carries an internal structure indicated by the index "i", we assume to have a metric compatible covariant derivative $\mathbf{D}_\mu$,

$$\mathbf{D}_\mu \phi^{(i)}(\boldsymbol{x}) = \partial_\mu \phi^{(i)}(\boldsymbol{x}) + \mathcal{A}_\mu{}^{(i)}{}_{(j)}(\boldsymbol{x})\phi^{(j)}(\boldsymbol{x}). \tag{5.6}$$

Here $\mathcal{A}_\mu{}^{(i)}{}_{(j)}(\boldsymbol{x})$ is the connection, ensuring that $\mathbf{D}_\mu\phi^{(i)}(\boldsymbol{x})$ transforms covariantly under a symmetry transformation, $\mathbf{D}_\mu\phi^{(i)}(\boldsymbol{x}) \to \frac{\partial x^\nu}{\partial x'^\mu}\mathfrak{T}^{(i)}{}_{(j)}\mathbf{D}_\nu\phi^{(j)}(\boldsymbol{x})$. The conjugate field, $\bar\phi_{(i)}(\boldsymbol{x})$, together with the covariant derivative, $\mathbf{D}_\mu$ induces a conjugated covariant derivative, $\bar{\mathbf{D}}_\mu$, with $\partial_\mu\big(\bar\phi_{(i)}(\boldsymbol{x})\phi^{(i)}(\boldsymbol{x})\big) \stackrel{!}{=} \big(\bar{\mathbf{D}}_\mu\bar\phi_{(i)}(\boldsymbol{x})\big)\phi^{(i)}(\boldsymbol{x}) + \bar\phi_{(i)}(\boldsymbol{x})\big(\mathbf{D}_\mu\phi^{(i)}(\boldsymbol{x})\big)$ we find

$$\bar{\mathbf{D}}_\mu\bar\phi_{(i)}(\boldsymbol{x}) = \partial_\mu\bar\phi_{(i)}(\boldsymbol{x}) - \bar\phi_{(j)}(\boldsymbol{x})\mathcal{A}_\mu{}^{(j)}{}_{(i)}(\boldsymbol{x}). \tag{5.7}$$

Then "metric compatible" means $\big(\mathbf{D}_\mu\phi^{(j)}(\boldsymbol{x})\big)^*\,\mathfrak{h}_{(j)(i)}(\boldsymbol{x}) = \bar{\mathbf{D}}_\mu\bar\phi_{(i)}(\boldsymbol{x})$ and leads to

$$\mathfrak{h}^{(i)(k)}(\boldsymbol{x})\partial_\mu\mathfrak{h}_{(k)(j)}(\boldsymbol{x}) = \mathcal{A}_\mu{}^{(i)}{}_{(j)}(\boldsymbol{x}) + \mathfrak{h}^{(i)(l)}(\boldsymbol{x})\big(\mathcal{A}_\mu{}^{(k)}{}_{(l)}(\boldsymbol{x})\big)^*\,\mathfrak{h}_{(k)(j)}(\boldsymbol{x}), \tag{5.8}$$

where $\mathfrak{h}^{(i)(j)}(\boldsymbol{x})$ denotes the inverse metric, $\mathfrak{h}^{(i)(l)}(\boldsymbol{x})\mathfrak{h}_{(l)(j)}(\boldsymbol{x}) = \delta^{(i)}_{(j)}$. Note that $\mathbf{D}_\mu\phi^{(i)}(\boldsymbol{x})$ has the internal indices "μ" and "i", hence the second covariant derivative reads

$$\mathbf{D}_\mu\mathbf{D}_\nu\phi^{(i)}(\boldsymbol{x}) = \frac{\partial}{\partial x^\mu}\mathbf{D}_\nu\phi^{(i)}(\boldsymbol{x}) - \Gamma_\mu{}^\rho{}_\nu(\boldsymbol{x})\mathbf{D}_\rho\phi^{(i)}(\boldsymbol{x}) + \mathcal{A}_\mu{}^{(i)}{}_{(j)}(\boldsymbol{x})\mathbf{D}_\nu\phi^{(j)}(\boldsymbol{x}), \tag{5.9}$$

and analogously for $\bar{\mathbf{D}}_\mu\bar{\mathbf{D}}_\nu\bar\phi_{(i)}(\boldsymbol{x})$. The Laplacean $\boldsymbol{\Delta}$ is then defined as usual,

$$\boldsymbol{\Delta}\phi^{(i)}(\boldsymbol{x}) = -\mathbf{D}^\mu\mathbf{D}_\mu\phi^{(i)}(\boldsymbol{x}), \qquad \bar{\boldsymbol{\Delta}}\bar\phi_{(i)}(\boldsymbol{x}) = -\bar{\mathbf{D}}^\mu\bar{\mathbf{D}}_\mu\bar\phi_{(i)}(\boldsymbol{x}). \tag{5.10}$$

In particular, we can check that

$$\bar{\phi}_{(i)}(\boldsymbol{x})\boldsymbol{\Delta}\phi^{(i)}(\boldsymbol{x}) = \big(\bar{\mathbf{D}}_\mu\bar{\phi}_{(i)}(\boldsymbol{x})\big)\big(\mathbf{D}^\mu\phi^{(i)}(\boldsymbol{x})\big) - D_{(\mathrm{LC})}{}^\mu\Big[\bar{\phi}_{(i)}(\boldsymbol{x})\big(\mathbf{D}_\mu\phi^{(i)}(\boldsymbol{x})\big)\Big] \\ + C^\rho{}_\mu{}^\mu(\boldsymbol{x})\bar{\phi}_{(i)}(\boldsymbol{x})\big(\mathbf{D}_\rho\phi^{(i)}(\boldsymbol{x})\big). \tag{5.11}$$

Hence, in the following we assume the spacetime torsion to be traceless, $C^\rho{}_\mu{}^\mu = K_\mu{}^{\mu\rho} = 0$, to ensure that the Laplacean is selfadjoint in the sense

$$\int_x \bar{\phi}_{(i)}(\boldsymbol{x})\big(\boldsymbol{\Delta}\phi^{(i)}(\boldsymbol{x})\big) = \int_x \big(\bar{\boldsymbol{\Delta}}\bar{\phi}_{(i)}(\boldsymbol{x})\big)\phi^{(i)}(\boldsymbol{x}), \tag{5.12}$$

cf. equation (2.24).[30] We further add a "Hermitean" endomorphism $E^{(i)}{}_{(j)}(\boldsymbol{x})$ and define

$$\boldsymbol{\Delta}_E\phi^{(i)}(\boldsymbol{x}) = \boldsymbol{\Delta}\phi^{(i)}(\boldsymbol{x}) + E^{(i)}{}_{(j)}(\boldsymbol{x})\phi^{(j)}(\boldsymbol{x}), \quad E^{(i)}{}_{(j)}(\boldsymbol{x}) = \mathfrak{h}^{(i)(k)}(\boldsymbol{x})\big(E^{(l)}{}_{(k)}\big)^*\mathfrak{h}_{(l)(j)}(\boldsymbol{x}). \tag{5.13}$$

The endomorphism is sometimes used as a tool to ensure that $\boldsymbol{\Delta}_E$ is a positive operator. Then one can analyze the properties of $\boldsymbol{\Delta}_E$ and via an analytic continuation also the properties of $\boldsymbol{\Delta}$.[31] Using that $\boldsymbol{\Delta}_E$ is selfadjoint, we can choose an orthonormal eigenbasis $\{\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}{}^{(i)}(\boldsymbol{x})\}$ in the space of functions,

$$\boldsymbol{\Delta}_E\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}{}^{(i)}(\boldsymbol{x}) = \boldsymbol{\lambda}\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}{}^{(i)}(\boldsymbol{x}), \qquad \int_x \bar{\mathfrak{f}}_{\boldsymbol{\lambda},\mathfrak{l}(i)}(\boldsymbol{x})\ \mathfrak{f}_{\boldsymbol{\lambda}',\mathfrak{l}'}{}^{(i)}(\boldsymbol{x}) = \delta^{\boldsymbol{\lambda},\mathfrak{l}}_{\boldsymbol{\lambda}',\mathfrak{l}'}, \qquad \sum_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}}\sum_{\mathfrak{l}\in\mathrm{D}_{\boldsymbol{\lambda}}}\delta^{\boldsymbol{\lambda},\mathfrak{l}}_{\boldsymbol{\lambda}',\mathfrak{l}'} = 1, \tag{5.14}$$

where $\mathfrak{l} \in \mathrm{D}_{\boldsymbol{\lambda}}$ accounts for the degeneracy of the eigenvalue $\boldsymbol{\lambda} \in \sigma_{\boldsymbol{\Delta}_E}$. The "sum" over the eigenvalues and their degeneracy is just a formal expression, which roughly corresponds to a sum for the point spectrum and an integral with an appropriate measure for the continuous spectrum. Hence, the shorthand $\delta^{\boldsymbol{\lambda},\mathfrak{l}}_{\boldsymbol{\lambda}',\mathfrak{l}'}$ also corresponds to the Kronecker delta for the point spectrum, and a delta distribution for the continuous spectrum, which is normalized according to (5.14). With these eigenfunctions we can define a unit operator in the space of functions, i.e. the delta distribution $\delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}$,

$$\delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)} = \sum_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}}\sum_{\mathfrak{l}\in\mathrm{D}_{\boldsymbol{\lambda}}}\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}{}^{(i)}(\boldsymbol{x})\bar{\mathfrak{f}}_{\boldsymbol{\lambda},\mathfrak{l}(j)}(\boldsymbol{y}), \qquad \phi^{(i)}(\boldsymbol{x}) = \int_y \delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}\phi^{(j)}(\boldsymbol{y}). \tag{5.15}$$

Note that the combination $\delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}\phi^{(j)}(\boldsymbol{y})$ is a perfect scalar at the point $\boldsymbol{y}$, but behaves under symmetry transformations exactly like $\phi^{(i)}(\boldsymbol{x})$ at the point $\boldsymbol{x}$. Every field configuration can be decomposed in the eigenbasis, $\phi^{(i)}(\boldsymbol{x}) = \sum\limits_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}}\sum\limits_{\mathfrak{l}\in\mathrm{D}_{\boldsymbol{\lambda}}}\mathfrak{a}^\phi_{\boldsymbol{\lambda},\mathfrak{l}}\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}{}^{(i)}(\boldsymbol{x})$, giving rise to the functional

[30] One can relax this condition a little: if the trace of the torsion tensor can be written as the derivative of a scalar, then one can define an integral measure such that the Laplacean is selfadjoint, see [166].

[31] A method that does so explicitly is the zeta function regularization, see [167, 168]. In some cases the Laplacean $\boldsymbol{\Delta}_E$ can still have negative or zero modes, these then have to be removed from the space of functions, or treated carefully in a different way.

derivative acting to the left,

$$\frac{\overleftarrow{\delta}}{\delta\phi^{(j)}(\boldsymbol{y})} = \sum_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}} \sum_{\mathfrak{l}\in\mathrm{D}_{\boldsymbol{\lambda}}} \frac{\overleftarrow{\partial}}{\partial\mathfrak{a}^{\phi}_{\boldsymbol{\lambda},\mathfrak{l}}} \bar{\mathfrak{f}}_{\boldsymbol{\lambda},\mathfrak{l}(j)}(\boldsymbol{y}), \qquad \phi^{(i)}(\boldsymbol{x})\frac{\overleftarrow{\delta}}{\delta\phi^{(j)}(\boldsymbol{y})} = \delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}, \tag{5.16}$$

and the functional derivative acting to the right,[32]

$$\frac{\overrightarrow{\delta}}{\delta\bar{\phi}_{(i)}(\boldsymbol{x})} = \sum_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}} \sum_{\mathfrak{l}\in\mathrm{D}_{\boldsymbol{\lambda}}} {\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}}^{(i)}(\boldsymbol{x}) \frac{\overrightarrow{\partial}}{\partial\mathfrak{a}^{\phi*}_{\boldsymbol{\lambda},\mathfrak{l}}}, \qquad \frac{\overrightarrow{\delta}}{\delta\bar{\phi}_{(i)}(\boldsymbol{x})}\bar{\phi}_{(j)}(\boldsymbol{y}) = \delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}. \tag{5.17}$$

With $\frac{\overleftarrow{\delta}}{\delta\Phi}$ we denote the functional derivative with respect to all independent degrees of freedom, i.e. $\frac{\overleftarrow{\delta}}{\delta\Phi} = \big(\frac{\overleftarrow{\delta}}{\delta\phi^{(i)}(\boldsymbol{x})}\big)_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}}$ in the real case and $\frac{\overleftarrow{\delta}}{\delta\Phi} = \big(\frac{\overleftarrow{\delta}}{\delta\phi^{(i)}(\boldsymbol{x})}, \frac{\overrightarrow{\delta}}{\delta\bar{\phi}_{(i)}(\boldsymbol{x})}\big)_{i\in\Omega_\phi,\boldsymbol{x}\in\mathcal{M}}$ in the complex case and analogously for $\frac{\overrightarrow{\delta}}{\delta\bar{\Phi}}$. Using the Delta-Distribution we can interpret the covariant derivatives as well as the Laplaceans as "matrices", e.g., ${\mathbf{D}_\mu}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}$ with

$${\mathbf{D}_\mu}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)} = \mathbf{D}_\mu\delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}, \quad {\big(\mathbf{D}\phi\big)_\mu}^{(i)}(\boldsymbol{x}) = \int_y {\mathbf{D}_\mu}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}\phi^{(j)}(\boldsymbol{y}) = \mathbf{D}_\mu\phi^{(i)}(\boldsymbol{x}). \tag{5.18}$$

With hindsight, one could say that the space of functions depends on the Laplacean $\boldsymbol{\Delta}_E$ appearing in the kinetic operator of the action. As the average effective action $\Gamma_k[\varphi]$ interpolates from $S_{\Lambda_{\mathrm{UV}}}[\Phi]$ to $\Gamma[\varphi]$ the kinetic operator can change. In particular, this can lead to a change of the space of functions during the flow of k.[33] Even though this is possible in principle, we will assume that φ and Φ live in the same functional space.

Now we can turn to the quantities and operations in (5.5). First we have the supertrace "STr", which is a trace over all fields, internal indices and also an integration over the manifold (i.e. the trace over the continuous index $\boldsymbol{x}$). This trace inherits an additional minus sign for the trace over fermionic degrees of freedom as indicated by the "S". Then we have $\Gamma_k^{(2)}[\varphi]$ as a shorthand for the second functional derivative of the effective action, $\Gamma_k^{(2)}[\varphi] = \frac{\overrightarrow{\delta}}{\delta\bar{\varphi}}\Gamma_k[\varphi]\frac{\overleftarrow{\delta}}{\delta\varphi}$. Next there is the regulator $\mathcal{R}_k(\boldsymbol{\Delta}_E)$, which has to satisfy

$$\lim_{\boldsymbol{\lambda}/k^2\to 0}\mathcal{R}_k(\boldsymbol{\lambda}) > 0, \qquad \lim_{k^2/\boldsymbol{\lambda}\to 0}\mathcal{R}_k(\boldsymbol{\lambda}) = 0, \qquad \lim_{k\to\Lambda_{\mathrm{UV}}\to\infty}\mathcal{R}_k(\boldsymbol{\lambda}) \to \infty. \tag{5.19}$$

Using the Laplacean $\boldsymbol{\Delta}_E$ we can lift the regulator to an operator itself $\mathcal{R}_k(\boldsymbol{\Delta}_E)$.[34] The regulator is also supposed to be such that the combination $\int_x\int_y \bar{\phi}_{(i)}\,\mathcal{R}_k(\boldsymbol{\Delta}_E)^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}\,\phi^{(j)}(\boldsymbol{y})$ is real. Last we have $\big(\Gamma_k^{(2)}[\varphi]+\mathcal{R}_k(\boldsymbol{\Delta}_E)\big)^{-1}$, this has to be understood as the inverse of $\big(\Gamma_k^{(2)}[\varphi]+\mathcal{R}_k(\boldsymbol{\Delta}_E)\big)$

[32] The distinction of a functional derivative acting to the "left" or to the "right" only becomes important when considering (Graßmann-valued) fermions.

[33] Note that Φ is the quantum field which is pictorially highly fluctuating, while φ corresponds to the effective action $\Gamma[\varphi]$ where all the quantum fluctuations are integrated out. Hence, it is reasonable to think that the fields φ are "smoothed".

[34] For fermionic fields the Regulator is usually not only a function of the Laplacean $\boldsymbol{\Delta}_E = -\nabla^\mu\nabla_\mu + E$ but also of the Dirac operator $\not\nabla = \gamma^\mu\nabla_\mu$.

as a matrix in field space (respecting also the internal indices and their symmetries), and the inverse of this object as a differential operator. That is, if we consider the decomposition of a field into the eigenbasis, $\phi^{(i)}(\boldsymbol{x}) = \sum\limits_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}} \sum\limits_{\mathfrak{l}\in \mathrm{D}_{\boldsymbol{\lambda}}} \mathfrak{a}^{\phi}_{\boldsymbol{\lambda},\mathfrak{l}} \, {\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}}^{(i)}(\boldsymbol{x})$, then we have

$$\boldsymbol{\Delta}_E^{-1}\phi^{(i)}(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}} \sum_{\mathfrak{l}\in \mathrm{D}_{\boldsymbol{\lambda}}} \boldsymbol{\lambda}^{-1}\mathfrak{a}^{\phi}_{\boldsymbol{\lambda},\mathfrak{l}} \, {\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}}^{(i)}(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}} \sum_{\mathfrak{l}\in \mathrm{D}_{\boldsymbol{\lambda}}} \int_0^\infty \mathrm{d}\mathfrak{s}\, \mathrm{e}^{-\mathfrak{s}\boldsymbol{\lambda}} \, \mathfrak{a}^{\phi}_{\boldsymbol{\lambda},\mathfrak{l}} \, {\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}}^{(i)}(\boldsymbol{x}) = \int_0^\infty \mathrm{d}\mathfrak{s}\, \mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E}\phi^{(i)}(\boldsymbol{x}), \tag{5.20}$$

where we assume the Laplacean $\boldsymbol{\Delta}_E$ to be positive $\boldsymbol{\lambda} > 0$. The arising operator $\mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E}$ is the so-called heat kernel. Due to the complexity of the Wetterich equation it is not possible to calculate the average effective action $\Gamma_k[\varphi]$ in general. Hence, one often makes an ansatz (truncation), which then satisfies the flow equation only up to a given order of the fields or their derivatives. Even for a truncated functional $\Gamma_k[\varphi]$ the actual evaluation of the right-hand side of the Wetterich equation is a rather involved task. During the next two sections we collect some tools for the treatment of the flow equation.

5.2 Heat Kernel

The Heat kernel ${\mathfrak{K}_{\boldsymbol{\Delta}_E}}^{(i)}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})_{(j)}$ of the Laplacean ${\boldsymbol{\Delta}_E}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}$ is the unique solution of the equations

$$\partial_{\mathfrak{s}}{\mathfrak{K}_{\boldsymbol{\Delta}_E}}^{(i)}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})_{(j)} = -\int_z {\boldsymbol{\Delta}_E}^{(i)}(\boldsymbol{x},\boldsymbol{z})_{(k)}{\mathfrak{K}_{\boldsymbol{\Delta}_E}}^{(k)}(\boldsymbol{z},\boldsymbol{y};\mathfrak{s})_{(j)}, \quad \lim_{\mathfrak{s}\searrow 0}{\mathfrak{K}_{\boldsymbol{\Delta}_E}}^{(i)}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})_{(j)} = \delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}, \tag{5.21}$$

where $\mathfrak{s} \in (0,\infty)$. One can straightforwardly derive

$$\begin{aligned} {\mathfrak{K}_{\boldsymbol{\Delta}_E}}^{(i)}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})_{(j)} &= \left(\mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E}\right)^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)} = \sum_{n=0}^{\infty}\frac{(-\mathfrak{s})^n}{n!}\boldsymbol{\Delta}_E^{n\,(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)} = \sum_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}} \sum_{\mathfrak{l}\in \mathrm{D}_{\boldsymbol{\lambda}}} \mathrm{e}^{-\boldsymbol{\lambda}\mathfrak{s}} \, {\mathfrak{f}_{\boldsymbol{\lambda},\mathfrak{l}}}^{(i)}(\boldsymbol{x})\bar{\mathfrak{f}}_{\boldsymbol{\lambda},\mathfrak{l}(j)}(\boldsymbol{y}), \\ \boldsymbol{\Delta}_E^{n\,(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)} &= \begin{cases} \delta^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}, & n=0, \\ \int_{x_1}\dots\int_{x_{n-1}} {\boldsymbol{\Delta}_E}^{(i)}(\boldsymbol{x},\boldsymbol{x}_1)_{(j_1)}\dots{\boldsymbol{\Delta}_E}^{(j_{n-1})}(\boldsymbol{x}_{n-1},\boldsymbol{y})_{(j)}, & n\geq 1. \end{cases} \end{aligned} \tag{5.22}$$

The task is to rewrite this solution in a more direct form (i.e. without differential operator or integration).

Let us first look at the heat kernel for the complex scalar field $\phi(\boldsymbol{x})$ (i.e. no internal indices) in flat space. There we can choose Cartesian coordinates, such that

$$g_{\mu\nu} = \delta_{\mu\nu}, \qquad \left\{ {\rho \atop \mu\nu} \right\} = 0, \qquad \boldsymbol{\Delta}\phi(\boldsymbol{x}) = -\sum_{\mu=0}^{d-1}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\mu}\phi(\boldsymbol{x}). \tag{5.23}$$

Then it is easy to check, that one suitable eigenbasis from equation (5.14) is given by the plane waves, $\mathfrak{f}_{\vec{p}}(\boldsymbol{x}) = \frac{\mathrm{e}^{\mathrm{i}p_\mu x^\mu}}{(2\pi)^{d/2}}$. Here $\vec{p} \in \mathbb{R}^d$ encodes the eigenvalue $\boldsymbol{\lambda} = |\vec{p}|^2 \in [0, \infty)$ and the degeneracy $\mathfrak{l} = \frac{\vec{p}}{|\vec{p}|} \in S^{d-1}$ with $\sum\limits_{\boldsymbol{\lambda}\in\sigma_{\boldsymbol{\Delta}_E}} \sum\limits_{\mathfrak{l}\in \mathrm{D}_{\boldsymbol{\lambda}}} = \int \mathrm{d}^d\vec{p}$. This leads to the heat kernel $\mathfrak{K}_{\boldsymbol{\Delta}}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})$

$$\mathfrak{K}_{\boldsymbol{\Delta}}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s}) = \int \mathrm{d}^d\vec{p}\, \frac{\mathrm{e}^{-|p|^2\mathfrak{s}+\mathrm{i}p_\mu(x^\mu-y^\mu)}}{(2\pi)^d} = \frac{\exp\left(\frac{-|x-y|^2}{4\mathfrak{s}}\right)}{(4\pi\mathfrak{s})^{d/2}} = \frac{\exp\left(\frac{-\sigma(\boldsymbol{x},\boldsymbol{y})}{2\mathfrak{s}}\right)}{(4\pi\mathfrak{s})^{d/2}}, \tag{5.24}$$

where $\sigma(\boldsymbol{x},\boldsymbol{y})$ is one half the square of the geodesic distance from $\boldsymbol{x}$ to $\boldsymbol{y}$.[35] This expression is also present in the DeWitt iterative procedure for the derivation of the heat kernel of Laplaceans with further internal structure and on arbitrary Riemannian manifolds, see, e.g., [155] and the end of this section.

Now let us consider the one-dimensional circle S^1 of radius r. As coordinate we use the angle $\varphi \in (0, 2\pi)$. The metric only has one component, $g_{00} = r^2$. On the circle the Laplacean reads $\boldsymbol{\Delta}\phi(\boldsymbol{x}) = -\frac{1}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\varphi_{\boldsymbol{x}}^2}\phi(\boldsymbol{x})$. Then it is straight forward to check that the eigenbasis $\mathfrak{f}_{n,\mathfrak{l}}(\boldsymbol{x})$ from equation (5.14) is given by

$$\mathfrak{f}_{\boldsymbol{\lambda}_n,\mathfrak{l}}(\boldsymbol{x}) = \frac{\mathrm{e}^{(-1)^{\mathfrak{l}}\cdot \mathrm{i}n\varphi_{\boldsymbol{x}}}}{\sqrt{2\pi r}}, \qquad \boldsymbol{\lambda}_n = \frac{n^2}{r^2}, \qquad n \in \mathbb{N}_0, \qquad \mathfrak{l} \in \mathrm{D}_{\boldsymbol{\lambda}_n} = \begin{cases} \{1\}, & \boldsymbol{\lambda}_n = 0, \\ \{1,2\}, & \boldsymbol{\lambda}_n \geq 1/r^2, \end{cases} \tag{5.25}$$

where n counts the different eigenvalues $\boldsymbol{\lambda}_n$. We find for the heat kernel

$$\begin{aligned} \mathfrak{K}_{\boldsymbol{\Delta}}(\boldsymbol{x},\boldsymbol{y},\mathfrak{s}) &= \sum_{n=0}^{\infty}\sum_{\mathfrak{l}\in\mathrm{D}_{\boldsymbol{\lambda}_n}} \frac{1}{2\pi r}\mathrm{e}^{-\mathfrak{s}\frac{n^2}{r^2}}\mathrm{e}^{(-1)^{\mathfrak{l}}\cdot\mathrm{i}n(\varphi_{\boldsymbol{x}}-\varphi_{\boldsymbol{y}})} = \frac{1}{2\pi r} + \frac{1}{\pi r}\sum_{n=1}^{\infty}\mathrm{e}^{-\mathfrak{s}\frac{n^2}{r^2}}\cos\left(n(\varphi_{\boldsymbol{x}}-\varphi_{\boldsymbol{y}})\right) \\ &= \frac{1}{2\pi r}\left[1 + 2\sum_{n=1}^{\infty}\mathrm{e}^{-\mathfrak{s}\frac{n^2}{r^2}}\cos\left(\frac{n}{r}\cdot\sqrt{2\sigma(\boldsymbol{x},\boldsymbol{y})}\right)\right], \end{aligned} \tag{5.26}$$

where we used the periodicity and the symmetry of the cosine in the last step. The term in the bracket is exactly the definition of the Jacobi theta function $\vartheta\left(\frac{\sqrt{2\sigma(\boldsymbol{x},\boldsymbol{y})}}{2\pi r}, \mathrm{i}\frac{\mathfrak{s}}{\pi r^2}\right)$, see, e.g., [169]. Using the Jacobi identity, $\vartheta\left(z, -\frac{1}{\tau}\right) = \sqrt{-\mathrm{i}\tau}\cdot\mathrm{e}^{\mathrm{i}\pi z^2\tau}\cdot\vartheta(z\tau,\tau)$, where $\sqrt{-\mathrm{i}\tau}$ is the principal square root, one can reexpress the heat kernel as

$$\mathfrak{K}_{\boldsymbol{\Delta}}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s}) = \frac{\exp\left(-\frac{\sigma(\boldsymbol{x},\boldsymbol{y})}{2\mathfrak{s}}\right)}{\sqrt{4\pi\mathfrak{s}}}\left[1 + 2\sum_{n=1}^{\infty}\mathrm{e}^{-\frac{(n\cdot\pi r)^2}{\mathfrak{s}}}\cosh\left(n\cdot\frac{(\pi r)^2}{\mathfrak{s}}\cdot\frac{\sqrt{2\sigma(\boldsymbol{x},\boldsymbol{y})}}{\pi r}\right)\right]. \tag{5.27}$$

Here we see again the characteristic exponential part, cf. equation (5.24). The sum in the bracket corresponds to a topological contribution coming from the possibility to close a geodesic on the circle by going around it n times. Such topological terms are not covered by local (early time) heat kernel expansions, as, e.g., $\mathrm{e}^{-\frac{(n\cdot\pi r)^2}{\mathfrak{s}}}$ is not Taylor expandable in powers of $\mathfrak{s}$ at $\mathfrak{s} = 0$.

The previous examples show that it is possible to calculate the full heat kernel for some

[35] This object is usually called world function [41].

special cases. However, already on such a simple space as S^1 the heat kernel acquires a rather complicated form, making it hard to use it in an actual calculation. Fortunately, in the quantum field theory context one is often interested in the so-called coincidence limit, i.e. $\lim_{\boldsymbol{y}\to\boldsymbol{x}} \mathbf{D}_{\mu_1}\dots\mathbf{D}_{\mu_n}\mathfrak{K}_{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})_{(j)}$, up to a given (finite) order of derivatives and field invariants (e.g., curvatures). If there is no derivative acting on the heat kernel, this can be achieved by the (early time) Seeley-DeWitt expansion [170, 171]. For higher derivatives one can employ the geodesic expansion of the off-diagonal heat kernel [172, 173]. In both cases, the idea is to expand the heat kernel in the following way

$$\mathfrak{K}_{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})_{(j)} = \frac{\mathrm{e}^{-\frac{\sigma(\boldsymbol{x},\boldsymbol{y})}{2\mathfrak{s}}}}{(4\pi\mathfrak{s})^{d/2}}\sum_{n=0}^{\infty}\mathfrak{s}^n A_n^{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}. \tag{5.28}$$

As already mentioned this expansion does not cover contributions which are not expandable in $\mathfrak{s}$ (e.g., nonanalytic topological contributions, cf. equation (5.27)). Since we are interested in covariant derivatives of the heat kernel in the coincidence limit, we need these coincidence limits for the world function $\sigma(\boldsymbol{x},\boldsymbol{y})$ and the off-diagonal heat kernel coefficients $A_n^{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}$. The coincidence limits of the former can be derived from the identities [41, 174]

$$\begin{aligned}
&\frac{1}{2}g^{\mu\nu}(\boldsymbol{x})\big(D_{(\mathrm{LC})}{}^{(x)}_{\mu}\sigma(\boldsymbol{x},\boldsymbol{y})\big)\big(D_{(\mathrm{LC})}{}^{(x)}_{\nu}\sigma(\boldsymbol{x},\boldsymbol{y})\big) = \sigma(\boldsymbol{x},\boldsymbol{y}),\\
&\lim_{\boldsymbol{y}\to\boldsymbol{x}}\sigma(\boldsymbol{x},\boldsymbol{y}) = 0, \quad \lim_{\boldsymbol{y}\to\boldsymbol{x}} D_{(\mathrm{LC})}{}^{(x)}_{\mu}\sigma(\boldsymbol{x},\boldsymbol{y}) = 0, \quad \lim_{\boldsymbol{y}\to\boldsymbol{x}} D_{(\mathrm{LC})}{}^{(x)}_{\mu}D_{(\mathrm{LC})}{}^{(x)}_{\nu}\sigma(\boldsymbol{x},\boldsymbol{y}) = g_{\mu\nu}(\boldsymbol{x}),\\
&\lim_{\boldsymbol{y}\to\boldsymbol{x}} D_{(\mathrm{LC})}{}^{(x)}_{(\alpha_1}\dots D_{(\mathrm{LC})}{}^{(x)}_{\alpha_n)}D_{(\mathrm{LC})}{}^{(x)}_{\mu}\sigma(\boldsymbol{x},\boldsymbol{y}) = 0, \quad n\geq 2.
\end{aligned} \tag{5.29}$$

Then one can insert the ansatz (5.28) into equation (5.21) and finds the recursion relation for the $A_n^{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}$,

$$\begin{aligned}
0 = &\left[n - \frac{d}{2} + \frac{1}{2}\big(D^{\mu}_{(x)}D^{(x)}_{\mu}\sigma(\boldsymbol{x},\boldsymbol{y})\big)\right]\cdot A_n^{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)} + \big(D^{\mu}_{(x)}\sigma(\boldsymbol{x},\boldsymbol{y})\big)\cdot\mathbf{D}^{(x)}_{\mu}A_n^{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)}\\
&- \mathbf{D}^{\mu}_{(x)}\mathbf{D}^{(x)}_{\mu}A_{n-1}^{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)} + E^{(i)}{}_{(k)}(\boldsymbol{x})A_{n-1}^{\boldsymbol{\Delta}_E}{}^{(k)}(\boldsymbol{x},\boldsymbol{y})_{(j)}, \quad A_{-1}^{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y})_{(j)} = 0, \quad n\geq 0,
\end{aligned} \tag{5.30}$$

with the initial condition, $A_0^{\boldsymbol{\Delta}_E}{}^{(i)}(\boldsymbol{x},\boldsymbol{y}\to\boldsymbol{x})_{(j)} = \delta^{(i)}_{(j)}$, coming from the boundary condition of the heat kernel (5.21).[36]

5.3 Using the Heat Kernel

With the previous discussion of the heat kernel and its properties we can look at applications. As we have been very explicit concerning indices and arguments in the preceding sections, we

[36] Due to the fast increasing complexity of the heat kernel coefficients $A_n^{\boldsymbol{\Delta}_E}$ the use of computer algebra is inevitable.

will drop this detail for brevity from now on. First of all we can use the heat kernel to give a well-defined meaning to functions of the Laplacean $\boldsymbol{\Delta}_E$. We have already seen how this works for the inverse of the Laplacean, cf. equation (5.20). For a more general treatment let us consider a *scalar* function $F : \mathbb{R}_0^+ \to \mathbb{R}$ from the nonnegative reals to the reals. F is supposed to have a Laplace transform $\tilde{F}$ with

$$F(z) = \int_0^\infty \mathrm{d}\mathfrak{s}\, \tilde{F}(\mathfrak{s})\, \mathrm{e}^{-\mathfrak{s}z}, \qquad z \in \mathbb{R}_0^+. \tag{5.31}$$

Instead of just a number we can insert $\boldsymbol{\Delta}_E$ as the argument, $F(\boldsymbol{\Delta}_E)$, and thereby lift the function to a "matrix" in field space,

$$F(\boldsymbol{\Delta}_E) = \int_0^\infty \mathrm{d}\mathfrak{s}\, \tilde{F}(\mathfrak{s})\, \mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E} = \int_0^\infty \mathrm{d}\mathfrak{s}\, \tilde{F}(\mathfrak{s})\, \mathfrak{K}_{\boldsymbol{\Delta}_E}(\mathfrak{s}). \tag{5.32}$$

In particular we can define the "m-th derivative" of such a function as

$$F^{(m)}(\boldsymbol{\Delta}_E) = \left[\frac{\mathrm{d}^m}{\mathrm{d}x^m} F(x)\right]_{x=\boldsymbol{\Delta}_E} = \int_0^\infty \mathrm{d}\mathfrak{s}\, \tilde{F}(\mathfrak{s})\, (-\mathfrak{s})^m\, \mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E}. \tag{5.33}$$

Another important application of the heat kernel is the commutation of a function of the Laplacean $F(\boldsymbol{\Delta}_E)$ with some other object Q, e.g., fields or a covariant derivative. For this let us first look at Hadamard's lemma,

$$\mathrm{e}^X Y \mathrm{e}^{-X} = \sum_{m=0}^\infty \frac{1}{m!}[X,Y]_m, \qquad [X,Y]_{m+1} = \big[X,[X,Y]_m\big], \qquad [X,Y]_0 = Y, \tag{5.34}$$

where X and Y are some linear operators. We can reorder the expressions in the following way

$$\mathrm{e}^{-\mathfrak{s}X} Y = \sum_{m=0}^\infty \frac{(-\mathfrak{s})^m}{m!}[X,Y]_m \mathrm{e}^{-\mathfrak{s}X} = Y\mathrm{e}^{-\mathfrak{s}X} + \sum_{m=1}^\infty \frac{(-\mathfrak{s})^m}{m!}[X,Y]_m \mathrm{e}^{-\mathfrak{s}X}. \tag{5.35}$$

Using this general relation we find [175, 176]

$$F(\boldsymbol{\Delta}_E)Q = \int_0^\infty \mathrm{d}\mathfrak{s}\, \tilde{F}(\mathfrak{s})\, \mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E} Q = QF(\boldsymbol{\Delta}_E) + \sum_{m=1}^\infty \frac{1}{m!}[\boldsymbol{\Delta}_E, Q]_m F^{(m)}(\boldsymbol{\Delta}_E). \tag{5.36}$$

Hence, we have an explicit formula for such commutations. In particular, if we only want to keep invariants up to a given order of derivatives or curvatures, the series in m "terminates" (i.e. every commutator adds at least one derivative or endomorphism on Q).

By applying commutations of this kind one can rearrange things on the right-hand side of the Wetterich equation (5.5) such that the functional trace is a linear combination of terms of the type $\mathrm{STr}\left[\mathcal{F}^{(\mu_1\ldots\mu_n)}[\varphi]\,\mathbf{D}_{\mu_1}\ldots\mathbf{D}_{\mu_n}F(\mathbf{\Delta}_E)\right]$, where the vertex $\mathcal{F}^{(\mu_1\ldots\mu_n)}[\varphi]$ is supposed to be an insertion of fields and their derivatives, but is no differential operator.[37] Applying the formulas of the preceding section together with the Laplace transform representation of $F(\mathbf{\Delta}_E)$ we can boil down these traces to integrals over the proper time $\mathfrak{s}$,

$$\mathrm{STr}\Big[\mathcal{F}^{(\mu_1\ldots\mu_n)}\mathbf{D}_{\mu_1}\ldots\mathbf{D}_{\mu_n}F(\mathbf{\Delta}_E)\Big]=\sum_{n=0}^{\infty}\int_0^\infty \mathrm{d}\mathfrak{s}\,\frac{\tilde{F}(\mathfrak{s})\mathfrak{s}^n}{(4\pi\mathfrak{s})^{d/2}}\,\mathrm{STr}\Big[\mathcal{F}^{(\mu_1\ldots\mu_n)}\mathbf{D}_{\mu_1}\ldots\mathbf{D}_{\mu_n}\mathrm{e}^{-\frac{\sigma}{2\mathfrak{s}}}A_n^{\mathbf{\Delta}_E}\Big]. \tag{5.37}$$

For the evaluation of the proper time integrals we can use the following identities

$$\int_0^\infty \mathrm{d}\mathfrak{s}\,\frac{\tilde{F}(\mathfrak{s})}{\mathfrak{s}^x}=\frac{1}{\Gamma[x]}\int_0^\infty \mathrm{d}z\,F(z)\,z^{x-1},\qquad \int_0^\infty \mathrm{d}\mathfrak{s}\,(-\mathfrak{s})^n\tilde{F}(\mathfrak{s})=\lim_{z\to0}\frac{\mathrm{d}^n}{\mathrm{d}z^n}F(z),\quad x\in(0,\infty),\quad n\in\mathbb{N}_0. \tag{5.38}$$

Furthermore, a rather convenient choice of the regulator $\mathcal{R}_k$ is the so-called Litim regulator specified by the regulator shape function [177, 178]

$$\mathcal{R}_k(\mathbf{\Delta}_E)\sim r_k\left(\frac{\mathbf{\Delta}_E}{k^2}\right)=\left(\frac{k^2}{\mathbf{\Delta}_E}-1\right)\cdot\theta\left(1-\frac{\mathbf{\Delta}_E}{k^2}\right), \tag{5.39}$$

where θ is the Heaviside step function. As this regulator is distributional one should think of it as a family of smooth regulators $r_k{}^\varepsilon(z)=\left(\frac{1}{z}-1\right)\theta_\varepsilon(1-z)$ converging to the Litim regulator, $r_k{}^\varepsilon\xrightarrow{\varepsilon\to0}r_k$. This point of view becomes particularly important, when derivatives of the regulator appear. They arise due to the necessary commutations (cf. equation (5.36)) and the scale derivative of the regulator (cf. equation (5.5)). As is pointed out in [179], in such a case one has

$$\int_0^\infty \mathrm{d}z\,\delta(1-z)\,g\big(\theta(1-z)\big)\,f(z)=\lim_{\varepsilon\to0}\int_0^\infty \mathrm{d}z\left[\frac{\mathrm{d}}{\mathrm{d}z}\theta_\varepsilon(1-z)\right]g\big(\theta_\varepsilon(1-z)\big)\,f(z)=\int_0^1 \mathrm{d}t\,g(t)\cdot f(1), \tag{5.40}$$

where g and f are smooth functions from the nonnegative reals to the reals.[38]

[37] For $n=0$ (i.e. no derivatives on $F(\mathbf{\Delta}_E)$) and $\mathcal{F}^{(i)}{}_{(j)}[\varphi]\sim\delta^{(i)}_{(j)}$ the trace "STr" corresponds to a sum over the eigenvalues of $\mathbf{\Delta}_E$ (and over the internal indices). Such a sum can be treated using the Euler-Maclaurin formula or its generalization the Darboux formula.

[38] Note that this agrees with the standard (physicists) interpretation of $\theta(0)=\frac{1}{2}$ for $g(t)=t$, but differs for other functions.

6 Parametrization Dependence in Quantum Gravity

The technical goal of quantum gravity is to construct a functional integral over suitable integration variables which in the long-range limit can be described by a diffeomorphism-invariant effective field theory of metric variables approaching a classical regime for a wide range of macroscopic scales. The fact that the first part of this statement is rather unspecific is reflected by the large number of legitimate quantization proposals [28, 180, 181]. However, let us start our considerations with the following observation concerning the Dirac matrices γ_μ. The Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I} \tag{6.1}$$

by construction is present in any description of Dirac fermions. We treat this relation as a fundamental equation, valid in the classical as well as the quantum regime. This suggests to construct everything we need for the description of fermions in curved spacetimes in terms of the Dirac matrices γ_μ, cf. chapter 4. Since the metric is also determined by the γ_μ, it is tempting to use them as the fundamental variables of gravity. If we now aim at a functional integral over the Dirac matrices, the metric arises naturally as the only relevant degree of freedom. To see this we have to keep in mind that we cannot integrate over arbitrary γ_μ, but they have to satisfy a Clifford algebra at every spacetime point. The most general infinitesimal variation $\delta\gamma_\mu$ of the Dirac matrices (one integration step within a path integral) can be decomposed as, cf. equation (4.21),

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu], \quad \mathrm{tr}\,\delta\mathcal{S}_\gamma = 0. \tag{6.2}$$

In $d = 4$, this has been shown by Weldon [47]. A general proof for arbitrary integer $d \geq 2$ is given in appendix A. Here $\delta g_{\mu\nu}$ corresponds to a metric fluctuation and $\delta\mathcal{S}_\gamma$ to a spin-base fluctuation.[39] Note, that this is a bijective mapping. In other words, given an allowed variation of the Dirac matrices $\delta\gamma_\mu$ (compatible with the Clifford algebra), then there is a unique metric fluctuation $\delta g_{\mu\nu}$ and a unique spin-base fluctuation $\delta\mathcal{S}_\gamma$, satisfying equation (4.21). On the other hand for an arbitrary metric fluctuation $\delta g_{\mu\nu}$ and an arbitrary spin-base fluctuation $\delta\mathcal{S}_\gamma$ we can calculate the corresponding Dirac matrix fluctuation from equation (4.21). Hence, we can give the restricted integral over Dirac matrices compatible with the Clifford algebra a meaning by an unrestricted integral over metrics and spin-bases. As we have argued in favor of spin-base invariance (see chapter 3), the integration over spin-bases turns out to be just a trivial normalization constant for the path integral, leaving us with a pure metric quantization, see the first part of appendix I. We stress that it is more complicated and inconvenient to integrate over Dirac matrices in terms of vielbeins. This is mainly because the vielbein alone does not cover all

[39] A spin-base fluctuation $\delta\mathcal{S}_\gamma$ corresponds to an element of the Lie algebra $\mathfrak{sl}(d_\gamma, \mathbb{C})$ of the group of spin-base transformations $\mathrm{SL}(d_\gamma, \mathbb{C})$.

possible Dirac matrices. Hence we need some additional quantity to integrate over. It turns out that in the simplest form this additional quantity cannot form a group. Whereas for the metric decomposition this additional quantity is the integration over the spin-base transformations, and hence forms a group. Details are found in the second part of appendix I. The remainder of this chapter is founded on our paper [122].

6.1 Quantum Gravity and Parametrizations

Independently of the precise choice of integration variables, a renormalization group approach (see chapter 5) appears useful in order to facilitate a scale-dependent description of the system and a matching to the long-range classical limit which is given at least to a good approximation by an (effective) action of Einstein-Hilbert type:

$$\Gamma_k = -\int_x \mathcal{Z}_R (R - 2\Lambda). \tag{6.3}$$

Here, we have already introduced a momentum scale k, expressing the fact that this effective description should a priori hold only for a certain range of classical scales. In this regime, we have $\mathcal{Z}_R = 1/(16\pi \mathrm{G_N})$ with the Newton constant $\mathrm{G_N}$, and Λ parametrizing the cosmological constant. In a quantum setting, $\mathcal{Z}_R$ plays the role of a (dimensionful) wave-function renormalization, and $\mathrm{G_N}$ and Λ are expected to be replaced by their running counterparts depending on the scale k.

According to the previous discussion we confine ourselves to a quantum gravity field theory assuming that the metric itself is already a suitable integration variable. That is, we also set any kind of torsion to zero, $K_{\mu}{}^{\rho}{}_{\lambda} = 0 = \Delta\Gamma_{\mu}$. A first step towards a diffeomorphism-invariant functional integral then proceeds via the Faddeev-Popov method involving a gauge choice for intermediate steps of the calculation. In this work, we use the background-field gauge with the gauge-fixing quantity,

$$F_\mu = \left(\delta^\beta_\mu \bar{D}^\alpha - \frac{1+\beta}{d} \boldsymbol{g}^{\alpha\beta} \bar{D}_\mu \right) g_{\alpha\beta}, \tag{6.4}$$

which should vanish if the gauge condition is exactly matched. Here, $g_{\alpha\beta}$ is the full (fluctuating) metric, whereas $\boldsymbol{g}_{\alpha\beta}$ denotes a fiducial background metric which remains unspecified, but assists to keep track of diffeomorphism invariance within the background-field method. Gauge-fixing is implemented in the functional integral by means of the gauge-fixing action

$$\Gamma_{\mathrm{gf}} = \frac{\mathcal{Z}_R}{2\alpha} \int_x \boldsymbol{g}^{\mu\nu} F_\mu F_\nu. \tag{6.5}$$

More precisely, this gauge choice defines a two-parameter (α, β) family of covariant gauges. For instance, the choice $\beta = 1$ corresponds to the harmonic/De-Donder gauge which together with $\alpha = 1$ (Feynman gauge) yields a variety of technical simplifications, being used in standard effective field theory calculations [182–184] as well as in functional renormalization group studies

[7, 62] of quantum gravity. More conceptually, the Landau-gauge limit $\alpha \to 0$ appears favorable, as it implements the gauge condition in a strict fashion and thus should be a fixed point under renormalization group evolution [185, 186].

In the Euclidean formulation considered here, $\mathfrak{m} = 0$ and $\mathfrak{p} = d$, the parameter α is bound to be non-negative to ensure the positivity of the gauge-fixing part of the action (this restriction may not be necessary for a Lorentzian formulation). The parameter β can be chosen arbitrarily except for the singular value $\beta_{\text{sing}} = d - 1$. To elucidate this singularity, let us take a closer look at the induced Faddeev-Popov ghost term:

$$\Gamma_{\text{gh}} = -\int_x \bar{C}_\mu \mathcal{M}^\mu{}_\nu C^\nu, \qquad \mathcal{M}^\mu{}_\nu = \frac{\delta F^\mu}{\delta v^\nu}, \tag{6.6}$$

where v^ν characterizes the vector field along which we study the Lie derivative generating the coordinate transformations,

$$\frac{\delta g_{\alpha\beta}}{\delta v^\nu} = \frac{\delta}{\delta v^\nu} \mathcal{L}_v g_{\alpha\beta} = 2 \frac{\delta}{\delta v^\nu} D_{(\alpha} v_{\beta)}. \tag{6.7}$$

The corresponding variation of the gauge-fixing condition yields

$$\delta F^\mu = 2 \left(\boldsymbol{g}^{\mu\alpha} \bar{D}^\beta - \frac{(1+\beta)}{d} \boldsymbol{g}^{\alpha\beta} \bar{D}^\mu \right) D_{(\alpha} \delta v_{\beta)}. \tag{6.8}$$

Let us decompose the vector δv_β into a transversal part $\delta v_\beta^{\text{T}}$ and a longitudinal part $D_\beta \delta\chi$. For the following argument, it suffices to study the limit of the quantum metric approaching the background metric $g_{\mu\nu} \to \boldsymbol{g}_{\alpha\beta}$, which diagrammatically corresponds to studying the inverse ghost propagator ignoring higher vertices,

$$\delta F^\mu = (\delta^\mu_\nu \bar{D}^2 + \bar{R}^\mu_\nu) \delta v^{\text{T}\nu} + \frac{1}{2} \big((d - 1 - \beta) \bar{D}^\mu \bar{D}_\nu + 4 \bar{R}^\mu_\nu \big) \bar{D}^\nu \delta\chi + \mathcal{O}(g - \boldsymbol{g}). \tag{6.9}$$

In this form it is obvious that the longitudinal direction $\bar{D}^\nu \delta\chi$ is not affected by the gauge fixing for $\beta = d - 1$ to zeroth order in the curvature. In other words, the gauge fixing is not complete for this singular case $\beta_{\text{sing}} = d - 1$. This singularity is correspondingly reflected by the ghost propagator. The Faddeev-Popov operator in equation (6.6) reads

$$\mathcal{M}^\mu{}_\nu = 2 \boldsymbol{g}^{\mu\beta} \bar{D}^\alpha D_{(\alpha} g_{\beta)\nu} - 2 \frac{1+\beta}{d} \boldsymbol{g}^{\alpha\beta} \bar{D}^\mu D_\alpha g_{\beta\nu}. \tag{6.10}$$

Decomposing the ghost fields $\bar{C}_\mu$, C^ν also into transversal $\bar{C}^{\text{T}}_\mu$, $C^{\text{T}\nu}$ and longitudinal parts $\bar{D}^\mu \bar{\eta}$, $\bar{D}^\nu \eta$ we find for the ghost Lagrangian,

$$\bar{C}_\mu \mathcal{M}^\mu{}_\nu C^\nu = \bar{C}^{\text{T}}_\mu \left(\delta^\mu_\nu \bar{D}^2 + \bar{R}^\mu_\nu \right) C^{\text{T}\nu} - \bar{\eta} \left(\frac{d - 1 - \beta}{2} \bar{D}^4 + \bar{R}^{\mu\nu} \bar{D}_\mu \bar{D}_\nu \right) \eta + \mathcal{O}(g - \boldsymbol{g}). \tag{6.11}$$

where we have performed partial integrations in order to arrive at a convenient form and dropped covariant derivatives of the curvature. This form of the inverse propagator of the ghosts makes it obvious that a divergence of the form $\frac{1}{d-1-\beta}$ arises in the longitudinal parts. This divergence at $\beta_{\text{sing}} = d - 1$, related to an incomplete gauge fixing, will be visible in all our results below.

Let us now turn to the metric modes. As a technical tool, we parametrize the fully dynamical metric $g_{\mu\nu}$ in terms of a fiducial background metric $\boldsymbol{g}_{\mu\nu}$ and fluctuations $h_{\mu\nu}$ about the background. Background independence is obtained by keeping $\boldsymbol{g}_{\mu\nu}$ arbitrary and requiring that physical quantities such as scattering amplitudes are independent of $\boldsymbol{g}_{\mu\nu}$. Still, these requirements do not completely fix the parametrization of the dynamical field $g = g[\boldsymbol{g}; h]$. Several parametrizations have been used in concrete calculations. The most commonly used parametrization is the *linear split* [187]

$$g_{\mu\nu} = \boldsymbol{g}_{\mu\nu} + h_{\mu\nu}. \tag{6.12}$$

By contrast, the exponential split [151, 188–191]

$$g_{\mu\nu} = \boldsymbol{g}_{\mu\rho}\left(e^{h}\right)^{\rho}{}_{\nu}, \tag{6.13}$$

is a parametrization that has been discussed more recently to a greater extent [152, 153, 192, 193]. In both cases, h is considered to be a symmetric matrix field (with indices raised and lowered by the background metric). If a path integral of quantum gravity is now defined by some suitable measure $\mathcal{D}h$, it is natural to expect that the space of dynamical metrics g is sampled differently by the two parametrizations, implying different predictions at least for off-shell quantities – unless the variable change from (6.12) to (6.13) is taken care of by suitable (ultralocal) Jacobians. While a parametrization (and gauge-condition) independent construction of the path integral has been formulated in a geometric setting [187, 194–197], its usability is hampered by the problem of constructing the full decomposition of h in terms of fluctuations between physically inequivalent configurations and fluctuations along the gauge orbit. Geometric functional renormalization group flows have been conceptually developed in [145], with first results for asymptotic safety obtained in [198], and recently to a leading-order linear-geometric approximation in [199]. The relation between the geometric approach and the exponential parametrization was discussed in [192].

In this thesis, we take a more pragmatic viewpoint, and consider the different parameterizations of equations (6.12) and (6.13) as two different approximations of an ideal parametrization. Since the functional renormalization group actually requires the explicit form of $g[\boldsymbol{g}; h]$ only to second order in h (in the single-metric approximation, see below), we mainly consider a one-parameter class of parametrizations of the type

$$g_{\mu\nu} = \boldsymbol{g}_{\mu\nu} + h_{\mu\nu} + \frac{\tau}{2} h_{\mu\rho} h^{\rho}_{\nu} + \mathcal{O}(h^3). \tag{6.14}$$

For $\tau = 0$, we obtain the linear split, whereas $\tau = 1$ is *exactly* related to the exponential split within our truncation. Incidentally, it is straightforward to write down the most general, ultra-local parametrization to second order that does not introduce a scale,

$$g_{\mu\nu} = \boldsymbol{g}_{\mu\nu} + h_{\mu\nu} + \frac{1}{2}\left(\tau h_{\mu\rho}h^{\rho}_{\nu} + \tau_2 h h_{\mu\nu} + \tau_3 \boldsymbol{g}_{\mu\nu} h_{\rho\sigma}h^{\rho\sigma} + \tau_4 \boldsymbol{g}_{\mu\nu} h^2\right) + \mathcal{O}(h^3). \tag{6.15}$$

Here, $h = h^{\mu}_{\mu}$ is the trace of the fluctuation. As mentioned above, third and higher-order terms will not contribute to our present study anyway. Instead of exploring the full parameter dependence, we will highlight some interesting results in this more general framework below.

The key ingredient for a quantum computation is the propagator of the dynamical field. In our setting, its inverse is given by the second functional derivative (Hessian) of the action (6.3) including the gauge fixing (6.5) with respect to the fluctuating field h,

$$\begin{aligned}\frac{1}{\mathcal{Z}_R}\,\Gamma^{(2)\kappa\nu}_{hh\alpha\beta}\Big|_{h=0,C=0} = {}& \frac{1}{16\alpha}\left(8\alpha\delta^{\kappa\nu}_{\alpha\beta} - [8\alpha - (1+\beta)^2]\boldsymbol{g}^{\kappa\nu}\boldsymbol{g}_{\alpha\beta}\right)(-\bar{D}^2) - \frac{1-\alpha}{\alpha}\delta^{(\kappa}_{(\alpha}\bar{D}^{\nu)}\bar{D}_{\beta)} \\ &+ \frac{1+\beta-2\alpha}{4\alpha}\left(\boldsymbol{g}^{\kappa\nu}\bar{D}_{(\alpha}\bar{D}_{\beta)} + \boldsymbol{g}_{\alpha\beta}\bar{D}^{(\kappa}\bar{D}^{\nu)}\right) + \frac{1}{2}(1-\tau)\delta^{\kappa\nu}_{\alpha\beta}(\bar{R}-2\lambda_k) \\ &- \frac{1}{4}\boldsymbol{g}^{\kappa\nu}\boldsymbol{g}_{\alpha\beta}(\bar{R}-2\lambda_k) - (1-\tau)\bar{R}^{(\kappa}_{(\alpha}\delta^{\nu)}_{\beta)} + \frac{1}{2}(\bar{R}^{\kappa\nu}\boldsymbol{g}_{\alpha\beta} + \bar{R}_{\alpha\beta}\boldsymbol{g}^{\kappa\nu}) - \bar{R}^{\kappa}{}_{(\alpha}{}^{\nu}{}_{\beta)},\end{aligned} \tag{6.16}$$

Here and in the following, we specialize to $d = 4$, except if stated otherwise. A standard choice for the gauge parameters is harmonic DeDonder gauge with $\alpha = 1 = \beta$ for which the first and the second lines simplify considerably. Simplifications also arise for the exponential split $\tau = 1$; in particular, a dependence on the cosmological constant λ_k remains only in the trace mode $\sim \boldsymbol{g}^{\kappa\nu}\boldsymbol{g}_{\alpha\beta}$.

A standard tool for dealing with the tensor structure of the propagator is the York decomposition of the fluctuations $h_{\mu\nu}$ into transverse traceless tensor modes, a transverse vector mode and two scalar modes,

$$h_{\mu\nu} = h^{\mathrm{T}}_{\mu\nu} + 2\bar{D}_{(\mu}\xi^{\mathrm{T}}_{\nu)} + \left(2\bar{D}_{(\mu}\bar{D}_{\nu)} - \frac{1}{2}\boldsymbol{g}_{\mu\nu}\bar{D}^2\right)\sigma + \frac{1}{4}\boldsymbol{g}_{\mu\nu}h,\ \ \bar{D}^{\mu}h^{\mathrm{T}}_{\mu\nu} = 0,\ \ \boldsymbol{g}^{\mu\nu}h^{\mathrm{T}}_{\mu\nu} = 0,\ \ \bar{D}^{\mu}\xi^{\mathrm{T}}_{\mu} = 0. \tag{6.17}$$

It is convenient to split $\Gamma^{(2)}_k$ into a pure kinetic part $\mathcal{P}_k$ which has a nontrivial flat-space limit, and a curvature-dependent remainder $\mathcal{F}_k = \mathcal{O}(\bar{R})$. This facilitates an expansion of the propagator $(\Gamma^{(2)}_k)^{-1} = (\mathcal{P}_k + \mathcal{F}_k)^{-1} = \sum_{n=0}^{\infty}(-\mathcal{P}_k{}^{-1}\mathcal{F}_k)^n\mathcal{P}_k{}^{-1}$. Let us first concentrate on the kinetic part $\mathcal{P}_k$:

$$\mathcal{P}^{h^{\mathrm{T}}\mu\nu}_{k\alpha\beta} = \frac{\mathcal{Z}_R}{2}\delta^{\mu\nu}_{\alpha\beta}\big(\Delta - 2(1-\tau)\lambda_k\big), \qquad \mathcal{P}^{\xi^{\mathrm{T}}\mu}_{k\alpha} = \frac{\mathcal{Z}_R}{\alpha}\delta^{\mu}_{\alpha}\Delta\big(\Delta - 2\alpha(1-\tau)\lambda_k\big), \tag{6.18}$$

$$\mathcal{P}^{(\sigma h)}_k = \mathcal{Z}_R\begin{pmatrix} 3\frac{(3-\alpha)\Delta - 4\alpha(1-\tau)\lambda_k}{4\alpha}\Delta^2 & \frac{3}{8\alpha}(\beta-\alpha)\Delta^2 \\ \frac{3}{8\alpha}(\beta-\alpha)\Delta^2 & \frac{(\beta^2-3\alpha)\Delta + 4\alpha(1+\tau)\lambda_k}{16\alpha}\end{pmatrix}, \tag{6.19}$$

where $\Delta = -\bar{D}^2$. In this form it is straightforward to calculate the propagator $\mathcal{P}_k{}^{-1}$. In particular, the transverse traceless mode h^{T} does not exhibit any dependence on the gauge parameters. As discussed in the introduction, a-priori criteria suggest the Landau-gauge limit $\alpha \to 0$ as a preferred choice for the gauge fixing, as it strictly implements the gauge-fixing condition. It thus should also be a fixed point of the renormalization group flow [185, 186]. Whereas the choice of α and β, in principle, are independent, there can arise a subtle interplay with certain regularization strategies as will be highlighted in the following.

By taking the limit $\alpha \to 0$ while keeping β finite, we make the gauge fixing explicit, especially we find for the gauge-dependent modes

$$\mathcal{P}_k^{\xi^{\mathrm{T}}\,-1}{}_{\alpha}{}^{\mu} \to \alpha \frac{1}{\mathcal{Z}_R \Delta^2}\delta^{\mu}_{\alpha}, \tag{6.20}$$

$$\mathcal{P}_k^{(\sigma h)\,-1} \to \frac{-\frac{1}{3\mathcal{Z}_R}\Delta^{-2}}{\frac{(3-\beta)^2}{4}\Delta - (3-\beta^2+(3+\beta^2)\tau)\lambda_k}\begin{pmatrix} \beta^2 & -6\beta\Delta \\ -6\beta\Delta & 36\Delta^2 \end{pmatrix}. \tag{6.21}$$

The transverse mode ξ^{T}_{μ} decouples linearly with $\alpha \to 0$ and hence is pure gauge in the present setting. Whereas finite parts seem to remain in the (σh) subspace, we observe that the matrix $\mathcal{P}_k^{(\sigma h)\,-1}$ in (6.21) becomes degenerate in this limit (e.g., the determinant of the matrix in equation (6.21) is zero). Effectively, only one scalar mode remains in the propagator. The nature of this scalar mode is a function of the second gauge parameter: taking the limit $\beta \to \infty$, the remaining scalar mode can be identified with σ, while the limit $\beta \to 0$ leaves us with a pure h mode.

Whereas the transverse modes in equation (6.20) decouple smoothly in the limit $\alpha \to 0$, the decoupling of the scalar mode in equation (6.21) is somewhat hidden in the degeneracy of the scalar sector with the corresponding eigenmode depending on β. This can lead to a subtle interplay with regularization techniques for loop diagrams as can be seen on rather general grounds by the following argument. Structurally, the propagator in the (σh) sector has the following form in the limit $\alpha \to 0$ and for small but finite β, cf. equation (6.21),

$$\mathcal{P}_k^{(\sigma h)\,-1} \to \begin{pmatrix} \mathcal{O}(\beta^2) & \mathcal{O}(\beta) \\ \mathcal{O}(\beta) & \mathcal{O}(1) \end{pmatrix}. \tag{6.22}$$

Regularizations of traces over loops built from this propagator are typically adjusted to the spectrum of the involved operators. Let us formally write this as $\mathrm{Tr}\left[\mathcal{L}_{\mathcal{R}}\,\mathcal{P}_k{}^{-1}(\dots)\right]$, where $\mathcal{L}_{\mathcal{R}}$ denotes a regularizing operator and the ellipsis stands for further vertices and propagators. Now, it is often useful to regularize all fluctuation operators at the same scale, e.g., the spectrum of all Δ's should be cut off at one and the same scale k^2. Therefore, the regularizing operator $\mathcal{L}_{\mathcal{R}}$ inherits its tensor structure from the Hessian $\Gamma_k^{(2)}$ of equation (6.16). In the (σh) sector, the regularizing operator can hence acquire the same dependence on the gauge-parameters as in equation (6.19),

$$\mathcal{L}_{\mathcal{R}}^{(\sigma h)} \to \frac{1}{\alpha}\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\beta) \\ \mathcal{O}(\beta) & \mathcal{O}(\beta^2) \end{pmatrix}, \tag{6.23}$$

for $\alpha \to 0$ and small β. The complete scalar contribution to traces of the considered type would then be of the parametric form,

$$\operatorname{Tr}\left[\mathcal{L}_{\mathcal{R}}^{(\sigma h)}\,\mathcal{P}_k^{(\sigma h)\,-1}(\dots)\right] \to \frac{1}{\alpha}\mathcal{O}(\beta^2). \tag{6.24}$$

For finite β, such regularized traces can thus be afflicted with divergencies in the Landau-gauge limit $\alpha \to 0$. If this happens, we still have the option to choose suitable values of β. In fact, equation (6.24) suggest that still a whole one-parameter family of gauges exists in the Landau-gauge limit, if we set $\beta = \gamma \cdot \sqrt{\alpha}$, with arbitrary real but finite gauge parameter γ distinguishing different gauges.

We emphasize that this is a rather qualitative analysis. Since the limit of products is not necessarily equal to the product of limits, the trace over the matrix structure of the above operator products can still eliminate this $1/\alpha$ divergence, such that any finite value of β remains admissible.

In the following we observe that the appearance of the $1/\alpha$ divergence depends on the explicit choice of the regularization procedure, as expected. Still, as this discussion shows, even if this divergence occurs, it can perfectly well be dealt with by choosing $\beta = \gamma\sqrt{\alpha}$ and still retaining a whole one-parameter family of gauges in the Landau gauge limit.

6.2 Gravitational Renormalization Group Flow

For our study of generalized parametrization dependencies of gravitational renormalization group flows, we use the functional renormalization group in terms of a flow equation for the effective average action (see chapter 5) amended by the background-field method [200–202] and formulated for gravity [7]

$$\partial_t \Gamma_k[g, \boldsymbol{g}] = \frac{1}{2}\operatorname{STr}\left[\big(\partial_t \mathcal{R}_k(\Delta)\big)\big(\Gamma_k^{(2,0)}[g,\boldsymbol{g}] + \mathcal{R}_k(\Delta)\big)^{-1}\right], \qquad \partial_t = k\partial_k. \tag{6.25}$$

Equation (6.25) describes the flow of an action functional Γ_k as a function of a renormalization group scale k that serves as a regularization scale for the infrared fluctuations. Here, $\Gamma_k^{(2,0)}[g,\boldsymbol{g}]$ denotes the Hessian of the action with respect to the fluctuation field g, at fixed background $\boldsymbol{g}$. The details of the regularization are encoded in the choice of the regulator $\mathcal{R}_k$. Suitable choices of $\mathcal{R}_k$ guarantee that Γ_k becomes identical to the full quantum effective action in the limit $k \to 0$, and approaches the bare action for large scales $k \to \Lambda_{\text{UV}} \to \infty$ (where Λ_{UV} denotes an ultraviolet cutoff). For reviews in the present context, see [62, 162, 203–207].

First approaches towards the asymptotic safety scenario in quantum gravity using the Wetterich equation were done using the Einstein-Hilbert action in terms of the metric [7]. These were extended to higher derivative gravity (action functionals depending on higher powers of the curvature), i.e. squared curvature terms [208–210], polynomials up to the 34th power of the Ricci scalar [211, 212] and even functions of the Ricci scalar [213–219]. Furthermore, different degrees of freedom and symmetries were tested, such as the vielbein [9, 10, 220], the vielbein together with the spin connection [221, 222], the metric together with the spacetime torsion and non-metricity [223, 224], as well as the variables of conformal gravity [225–227], a foliated spacetime (Hořava-Lifshitz gravity) [23, 24, 228–230], and matrix models [231]. In addition, the inclusion of matter is checked quite extensively [8, 37, 38, 143, 153, 193, 232–241] and also the influence of the necessary ghost fields [242–245].

A conceptual difficulty lies in the fact that $\Gamma_k[g, \boldsymbol{g}]$ should be computed on a subspace of action functionals that satisfy the constraints imposed by diffeomorphism invariance and background independence. In general, this requires to work with g and $\boldsymbol{g}$ independently during large parts of the computation [146, 246–248]. Such bi-metric approaches can, for instance, be organized in the form of a vertex expansion on a flat space as put forward recently in [149, 249, 250], or via a level expansion as developed in [148], see [146, 147, 239, 246, 251–253] for further bi-metric results. For the present study of parametrization dependencies, we confine ourselves to a single-metric approximation, defined by identifying g with $\boldsymbol{g}$ on both sides of the flow equation, after the Hessian has been analytically determined. In the following, we therefore do no longer have to distinguish between the background field and the fluctuation field as far as the presentation is concerned, and hence drop the boldface notation for simplicity.

Whereas exact solutions of the flow equation so far have only been found for simple models, approximate nonperturbative flows can be constructed with the help of systematic expansion schemes. In the case of gravity, a useful scheme is given by expanding Γ_k in powers of curvature invariants. The technical difficulties then lie in the construction of the inverse of the regularized Hessian $\big(\Gamma_k^{(2,0)}[g, \boldsymbol{g}] + \mathcal{R}_k(\Delta)\big)^{-1}$, corresponding to the regularized propagator, and performing the corresponding traces.

Spanning the action in terms of the Einstein-Hilbert truncation (6.3) and neglecting the flow of the gauge-fixing and ghost sector [242–244], we use the *universal RG machine* [175, 176, 254] as our computational strategy. The key idea is to subdivide the Hessian $\Gamma_k^{(2)}[g]$ into a kinetic part and curvature parts with a subsequent expansion in the curvature. This is complicated by terms containing uncontracted covariant derivatives which could invalidate the counting scheme. Within the present truncation, this problem is solved with the aid of the York decomposition (6.17). This helps both to set up the curvature expansion as well as to invert the kinetic terms in the corresponding subspaces of TT, T and scalar modes. From a technical point of view, we use the package xAct [255–260] to handle the extensive tensor calculus.

Schematically, the flow equation for the Einstein-Hilbert truncation can then be written as

$$\partial_t \Gamma_k = \int_x \left(\mathcal{S}^{\mathrm{TT}} + \mathcal{S}^{\mathrm{T}} + \mathcal{S}^{\sigma h} + \mathcal{S}^{\mathrm{gh}} + \mathcal{S}^{\mathrm{Jac}}\right), \tag{6.26}$$

where the first three terms denote the contributions from the graviton fluctuations as parametrized by the York decomposition (6.17). The fourth term $\mathcal{S}^{\mathrm{gh}}$ arises from the Faddeev-Popov ghost fluctuations, cf. equation (6.11). The last term $\mathcal{S}^{\mathrm{Jac}}$ comes from the use of transverse decompositions of the metric (6.17) and the ghost fields (6.11). The corresponding functional integral measure over the new degrees of freedom involves Jacobians which – upon analogous regularization – contribute to the flow of the effective average action.

At this point, we actually have a choice that serves as another source of parametrization dependencies studied in this work: one option is to formulate the regularized path integral in terms of the decomposed fields as introduced above. In that case, the Jacobians are nontrivial and their contribution $\mathcal{S}^{\mathrm{Jac}}$ is listed in equation (J.13). Alternatively, we can reintroduce canonically normalized fields by means of a nonlocal field redefinition [261, 262],

$$\sqrt{\Delta - \mathrm{Ric}}\; \xi^\mu \to \xi^\mu, \qquad \sqrt{\Delta^2 + \frac{4}{3} D_\mu R^{\mu\nu} D_\nu}\; \sigma \to \sigma, \qquad \sqrt{\Delta}\; \eta \to \eta, \tag{6.27}$$

and analogously for the longitudinal anti-ghost field $\bar{\eta}$. (Here, we have used $(\mathrm{Ric}\,\xi)^\mu = R^{\mu\nu}\xi_\nu$.) This field redefinition goes along with another set of Jacobians contributing to the measure of the rescaled fields. As shown in [262], the Jacobians for the original York decomposition and the Jacobians from the field redefinition (6.27) cancel at least on maximally symmetric backgrounds. The latter choice of backgrounds is sufficient for identifying the flows in the Einstein-Hilbert truncation. Therefore, if we set up the flow in terms of the redefined fields (6.27), the last term in equation (6.26) vanishes, $\mathcal{S}^{\mathrm{Jac}}_{\mathrm{fr}} = 0$.

For an exact solution of the flow, it would not matter whether or not a field redefinition of the type (6.27) is performed. Corresponding changes in the full propagators would be compensated for by the (dis-)appearance of the Jacobians. For the present case of a truncated nonperturbative flow, a dependence on the precise choice will, however, remain, which is another example for a parametrization dependence. This dependence also arises from the details of the regularization. The universal RG machine suggests to construct a regulator $\mathcal{R}_k$ such that the Laplacians Δ appearing in the kinetic parts are replaced by

$$\Delta \to \Delta + R_k(\Delta), \tag{6.28}$$

where $R_k(x)$ is a (scalar) regulator function that provides a finite mass-like regularization for the long-range modes, e.g., $R_k(x) \to k^2$, for $x \ll k^2$, but leaves the ultraviolet modes unaffected, $R_k(x) \to 0$ for $x \gg k^2$. Since the field redefinition (6.27) is nonlocal, it also affects the kinetic terms and thus takes influence on the precise manner of how modes are regularized via equation (6.28). In other words, the dependence of our final results on using or not using the field redefinition (6.27) is an indirect probe of the regularization-scheme dependence and thus of the generalized parametrization dependence we are most interested in here.

We focus on the renormalization group flow of the effective average action parametrized by the operators of the Einstein-Hilbert truncation (6.3). For this, we introduce the dimensionless versions of the gravitational coupling and the cosmological constant,

$$g := \frac{k^2}{16\pi \mathcal{Z}_R} \equiv k^2 \mathrm{G_N}, \qquad \lambda = \frac{\Lambda}{k^2}, \tag{6.29}$$

and determine the corresponding renormalization group β functions for g and λ, by computing the $\mathcal{S}$ terms on the right hand side of the flow (6.26) to order R in the curvature. Many higher-order computations have been performed by now [208, 210, 211, 213, 219, 233, 263–268], essentially confirming and establishing the simple picture visible in the Einstein-Hilbert truncation.

We are particularly interested in the existence of fixed points g_* and λ_* of the β functions, defined by

$$\partial_t g = \dot{g} \equiv \beta_g(g_*, \lambda_*) = 0, \qquad \partial_t \lambda = \dot{\lambda} \equiv \beta_\lambda(g_*, \lambda_*) = 0. \tag{6.30}$$

In addition to the Gaußian fixed point $g_* = 0 = \lambda_*$, we search for a non-Gaußian interacting fixed point, the existence of which is a prerequisite for the asymptotic-safety scenario. Physically viable fixed points should have a positive value for the Newton coupling and should be connectable by a renormalization group trajectory with the classical regime, where the dimensionful couplings are approximately constant, i.e., the dimensionless versions should scale as $g \sim k^2$, $\lambda \sim 1/k^2$. The asymptotic-safety scenario also requires that a possible non-Gaußian fixed point has finitely many ultraviolet attractive directions. This is quantified by the number of positive critical exponents θ_i which are defined as (-1) times the eigenvalues of the stability matrix $\partial(\beta_g, \beta_\lambda)/\partial(g, \lambda)$.

Whereas the fixed-point values g_* and λ_* are renormalization group scheme-dependent, the critical exponents θ_i are universal and thus should be parametrization independent in an exact calculation. Also, the product $g_*\lambda_*$ has been argued to be physically observable in principle and thus should be universal [262]. Testing the parametrization dependence of the critical exponents θ_i and $g_*\lambda_*$ therefore provides us with a quantitative criterion for the reliability of approximative results.

6.3 Generalized Parametrization Dependence

With these prerequisites, we now explore the parametrization dependencies of the following scenarios: we consider the linear (6.12) and the exponential (6.13) split, both with and without field redefinition (6.27), and study the corresponding dependencies on the gauge parameters,

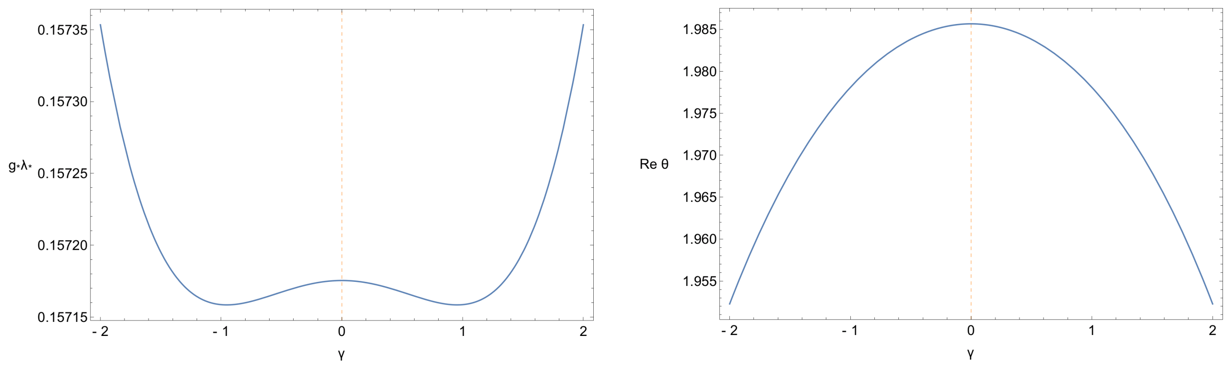


Figure 6.1: Linear split without field redefinition: residual dependence of our estimates for the universal quantities on gauge parameter γ in the limit $\alpha \to 0$. We find a common stationary point at $\gamma = 0$ and a remarkably small variation of the results on the level of 0.1% for $g_*\lambda_*$ and 1.6% for Reθ in the range $\gamma \in [-2, 2]$.

focusing on a strict implementation of the gauge-fixing condition $\alpha \to 0$ (Landau gauge). As suggested by the principle of minimum sensitivity, we look for stationary points as a function of the remaining parameter(s) where universal results become most insensitive to these generalized parametrizations. For the following quantitative studies, we exclusively use the piecewise linear regulator [178, 269], $R_k(x) = (k^2 - x^2)\theta(k^2 - x^2)$, for reasons of simplicity, cf. section 5.3. Studies of regulator-scheme dependencies which can also quantify parametrization dependencies have first been performed, e.g., in [262, 267]. The details of the calculation are deferred to appendix J.

6.3.1 Linear Split without Field Redefinition

Let us start with the case of the linear split (6.12) without field redefinition (6.27). Here, the degeneracy in the sector of scalar modes interferes with the regularization scheme, as illustrated in equation (6.24). Hence, in the Landau-gauge limit $\alpha \to 0$, we choose $\beta = \gamma\sqrt{\alpha}$, which removes any artificial divergence, but keeps γ as a real parameter that allows for a quantification of remaining parametrization/gauge dependence. We indeed find a non-Gaußian fixed point g_*, λ_* for wide range of values of γ. The critical exponents form a complex conjugate pair. The estimates for the universal quantities $g_*\lambda_*$ and the real part of the θ's (being the measure for the renormalization group relevance of perturbations about the fixed point) are depicted in figure 6.1. We observe a common point of minimum sensitivity at $\gamma = 0$. In a rather wide range of gauge parameter values $\gamma \in [-2, 2]$, our estimates for $g_*\lambda_*$ and Reθ vary only very mildy on the level of 0.1% and 1.6%. Given the limitations of the present simple approximation, this is a surprising degree of gauge independence lending further support to the asymptotic-safety scenario. The extremizing values at $\gamma = 0$ are near the results of [175, 213, 266] where the same gauge choice ($\alpha = \beta = 0$) was used. The main difference can be traced back to the fact that our inclusion of the (dimensionful) wave-function renormalization in the

parametrization	g_*	λ_*	$g_*\lambda_*$	θ
nfr $\tau=\alpha=\gamma=0$	0.879	0.179	0.157	$1.986 \pm \mathbf{i}\, 3.064$
nfr $\tau=0,\ \alpha=\beta=1$	0.718	0.165	0.119	$1.802 \pm \mathbf{i}\, 2.352$
fr $\tau=\alpha=0,\ \beta=1$	0.893	0.164	0.147	$2.034 \pm \mathbf{i}\, 2.691$
fr $\tau=0,\ \alpha=\beta=1$	0.701	0.172	0.120	$1.689 \pm \mathbf{i}\, 2.486$
fr $\tau=\alpha=0,\ \beta=\infty$	0.983	0.151	0.148	$2.245 \pm \mathbf{i}\, 2.794$
fr $\tau=1,\ \beta=\infty$	3.120	0.331	1.033	4, 2.148
fr $\tau=1.22,\ \alpha=0,\ \beta=\infty$	3.873	0.389	1.508	3.957, 1.898

Table 1: Non-Gaußian fixed-point properties for several parametrizations, characterized by the gauge parameters α, β or γ, as well as by the choice of the parametrization split parameter τ with $\tau = 0$ corresponding to the linear split (6.12) and $\tau = 1$, being the exponential split (6.13). Whether or not a field redefinition (6.27) is performed is labeled by "fr" or "nfr", respectively.

gauge fixing term (6.5) renders the gauge parameter α dimensionless as is conventional. If we ignored the resulting dimensional scaling, our extremizing result would be exactly that of [175] and in close agreement with [213, 266] with slight differences arising from the regularization scheme. It is also instructive to compare with [270], where the on-shell contributions to the flow have been singled out yielding a gauge-independent fixed-point value for the cosmological constant of $\lambda_* = 0.261$. Though the calculation also employs the linear split without field redefinition, the on-shell projection requires special choices for the field decomposition, the ghost sector, and the regularization scheme. The quantitative differences to our results which includes also off-shell contributions can be taken as a measure for the influence of all these sectors. We summarize a selection of our quantitative results in table 1.

6.3.2 Exponential Split without Field Redefinition

As a somewhat contrary example, let us now study the case of the exponential split (6.13) also without field redefinition (6.27). Again, we find a non-Gaußian fixed point. The corresponding estimates for the universal quantities at this fixed point in the Landau gauge limit $\alpha = 0$ are displayed in Fig. 6.2. At first glance, the results seem similar to the previous ones with a stationary point at $\gamma = 0$. However, the product $g_*\lambda_*$ shows a larger variation on the order of 5% and the critical exponent even varies by a factor of more than 40 in the range $\gamma \in [-2, 2]$. We interpret the strong dependence on the gauge parameter γ as a clear signature that these estimates based on the exponential split without field redefinition should not be trusted. In fact, the real parts of the critical exponents, Reθ, have even changed sign compared to the previous case implying that the non-Gaußian fixed point has turned ultraviolet repulsive. Similar observations have been made in [152] for the harmonic Feynman-type gauge $\alpha = 1 = \beta$ and an additional strong dependence on the regulator profile function $R_k(x)$ has been found. We have verified that our results agree with those of [152] for the corresponding gauge choice. In summary, this parametrization serves as an example that non-perturbative estimates can depend strongly on the details of the parametrization (even for seemingly reasonable parametrizations)

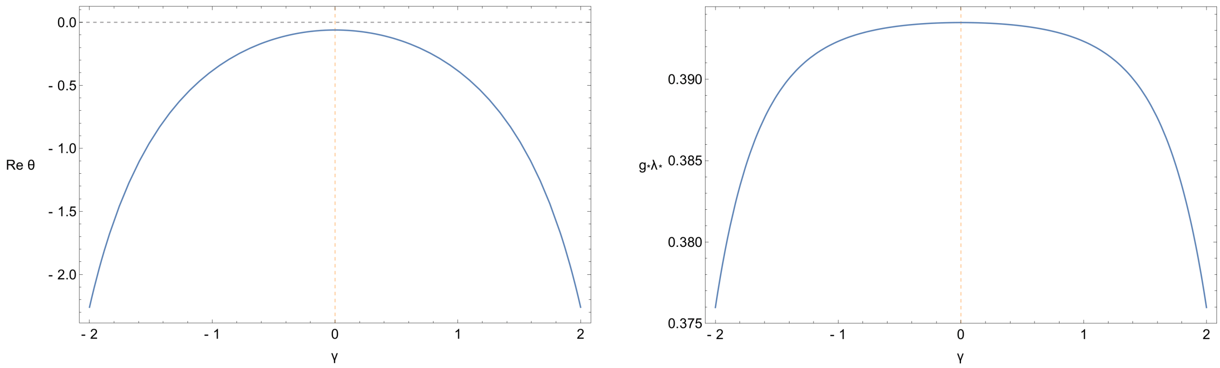


Figure 6.2: Exponential split without field redefinition: residual dependence of our estimates for the universal quantities on gauge parameter γ in the limit $\alpha \to 0$. A common stationary point is again present at $\gamma = 0$, but the estimates for the universal quantities exhibit a substantial variation in the range $\gamma \in [-2, 2]$: $g_*\lambda_*$ varies by $\sim 5\%$ and Reθ even by more than a factor of 40. The latter is a clear signal for the insufficiency of the parametrization.

and the results can be misleading. The good news is that a study of the parametrization dependence can – and in this case does – reveal the insufficiency of the parametrization through its strong dependence on a gauge parameter.

6.3.3 Linear Split with Field Redefinition

For the remainder, we consider parametrizations of the fluctuation field which include field redefinitions (6.27). The canonical normalization achieved by these field redefinitions has not merely aesthetical reasons. An important aspect is that the nonlocal field redefinition helps to regularize the modes in a more symmetric fashion: the kinetic parts of the propagators then become linear in the Laplacian which are all equivalently treated by the regulator (6.28). A practical consequence is that the interplay of the degeneracy in the scalar sector no longer interferes with the regularization, i.e., the gauge parameter β can now be chosen independently of α. Concentrating again on the Landau-gauge limit $\alpha \to 0$, we observe for generic split parameter τ that $\beta = 0$ no longer is an extremal point.

Our estimates for the universal quantities for the case of the linear split (6.12) with field redefinition (6.27) and $\alpha \to 0$ are plotted in Fig. 6.3. In order to stay away from the singularity at $\beta = 3$, cf. equation (6.11), we consider values for $\beta < 3$ down to $\beta \to -\infty$. As is obvious, e.g., from equation (6.21), the dependence of the propagator of the scalar modes and thus on β is such that the limits of large positive or negative $\beta \to \pm\infty$ yield identical results. Also the longitudinal ghost mode decouples in the limit $\beta \to \pm\infty$ such that the whole flow in the large $|\beta|$-limit is independent of the sign of β. A non-Gaußian fixed point exists, and a common extremum of $g_*\lambda_*$ and Re θ occurs for $\beta \to -\infty$. Near $\beta = 1$ marking the harmonic gauge condition, both quantities are also close to an extremum (which does not occur at exactly the same β value for both quantities). All fixed-point quantities for this case are listed in Tab. 1 ("fr $\tau = \alpha = 0$, $\beta = 1$"). These values agree with the results of [198]. They are remarkably close,

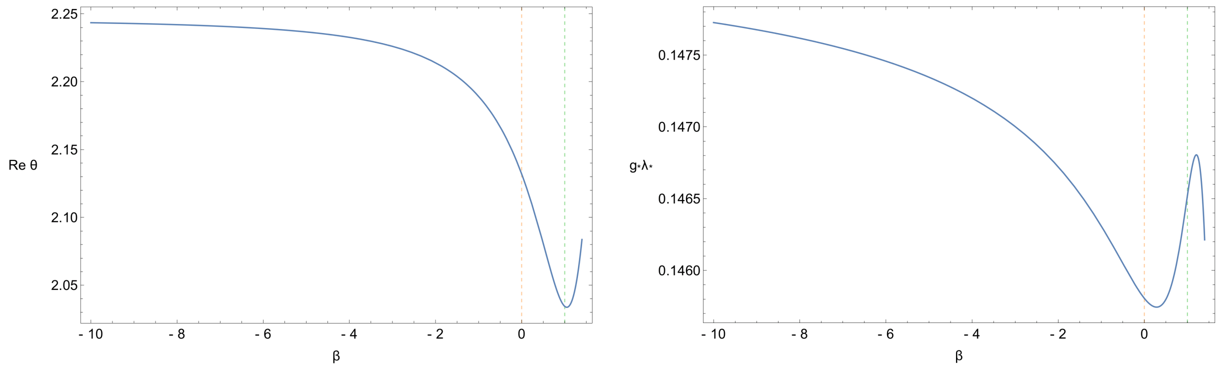


Figure 6.3: Linear split with field redefinition: residual dependence of our estimates for the universal quantities on the gauge parameter β in the limit $\alpha \to 0$. A common stationary point is approached for $|\beta| \to \infty$. Near the harmonic gauge $\beta = 1$ (green dashed vertical line), both quantities have an extremum. For the whole range of β values, the estimates for the universal quantities exhibit rather small variations of 1% for $g_*\lambda_*$ and 10% for the more sensitive critical exponent Reθ.

e.g., to those for the linear split without field redefinition. The situation is similar for the other extremum $|\beta| \to \infty$ ("fr $\tau = \alpha = 0$, $\beta = \infty$" in Tab. 1). For the whole infinite β range studied for this parametrization, $g_*\lambda_*$ varies on the level of 1%. The more sensitive critical exponent Re θ varies by 10% which is still surprisingly small given the simplicity of the approximation. Let us emphasize again that varying β from infinity to zero corresponds to a complete exchange of the scalar modes from σ (longitudinal vector component) to h (conformal mode) and hence to a rather different parametrization of the fluctuating degrees of freedom.

6.3.4 Exponential Split with Field Redefinition

Finally, we consider the exponential split (6.12), $\tau = 1$, with field redefinition (6.27). Having performed the latter has a strong influence on the stability of the estimates of the universal quantities at the non-Gaußian fixed point, as is visible in Fig. 6.4. Contrary to the linear split, we do not find a common extremum near small values of β: neither $\beta = 0$ nor the harmonic gauge $\beta = 1$ seem special, but, e.g., the product $g_*\lambda_*$ undergoes a rapid variation in this regime.

Rather, a common extremal point is found in the limit $\beta \to \infty$. In fact, $g_*\lambda_*$ becomes insensitive to the precise value of β for $\beta \lesssim -2$ (with a local maximum near $\beta \simeq -3$, and an asymptotic value of $g_*\lambda_* \simeq 1.033$ for $\beta \to \infty$. This estimate for $g_*\lambda_*$ is significantly larger than for the other parametrizations. The deviation may thus be interpreted as the possible level of accuracy that can be achieved in this simple Einstein-Hilbert truncation.

As an interesting feature, the critical exponents become real for $\beta \lesssim -2$, and approach the asymptotic values $\theta = \{4, 2.148\}$ for $\beta \to \infty$. The leading exponent $\theta = 4$ reflects the power-counting dimension of the cosmological term. This is a straightforward consequence of the fact that the λ dependence in this parametrization $\tau = 1$, $\beta \to \infty$ disappears from the propagators

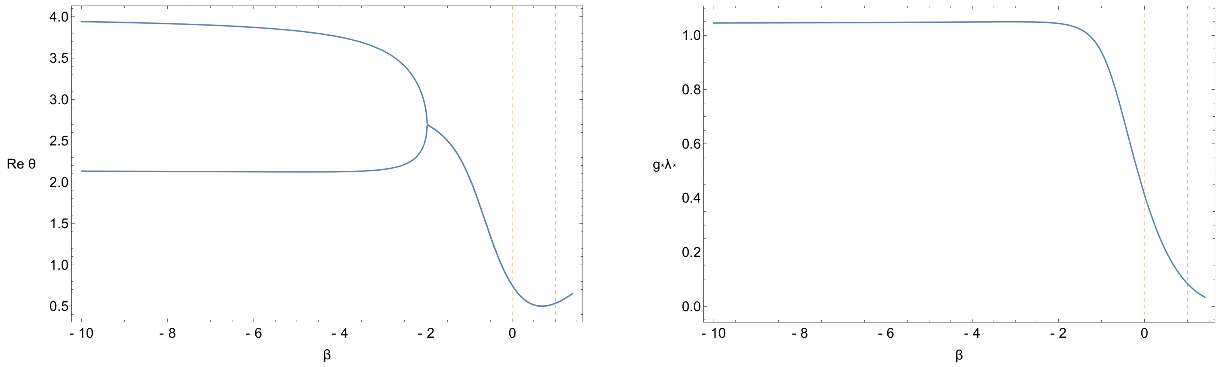


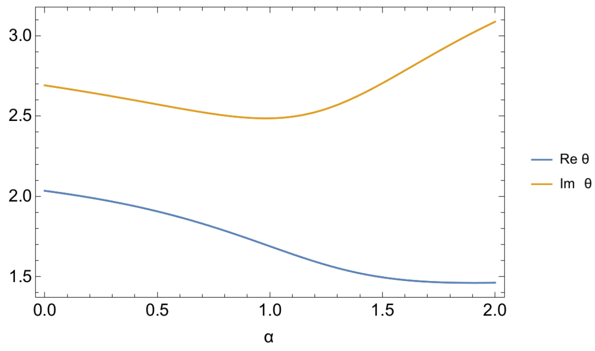
Figure 6.4: Exponential split with field redefinition: residual dependence of our estimates for the universal quantities on the gauge parameter β in the limit $\alpha \to 0$. A common stationary point is approached for $|\beta| \to \infty$, whereas no common minimum-sensitivity point is found near the harmonic gauge $\beta = 1$ or $\beta = 0$ (dashed vertical lines). Below $\beta \lesssim -2$, the critical exponents become real with the non-Gaußian fixed point remaining ultraviolet attractive. For $|\beta| \to \infty$, the results become independent of the gauge parameter α.

of the contributing modes. The leading nontrivial exponent $\theta = 2.148$ hence is associated with the scaling of the Newton constant near the fixed point, which is remarkably close to minus the power-counting dimension of the Newton coupling. The latter is a standard result for non-Gaußian fixed points which are described by a quadratic fixed-point equation [271, 272]. The small difference to the value $\theta = 2$ arises from the renormalization group improvement introduced by the anomalous dimension in the threshold functions ("η-terms" as discussed in appendix J). Neglecting these terms, the estimate of the leading critical exponents in dimension d is d and $d-2$, as first discussed in [153]. Also our other quantitative results for the fixed-point properties are in agreement with those of [153] within the same approximation.

The significance of the results within this parametrization is further underlined by the observation that the results in the limit $\beta \to \infty$ become completely independent of the gauge parameter α. In other words, the choice of the transverse traceless mode and the σ mode ($\beta \to \infty$) as a parametrization of the physical fluctuations removes any further gauge dependence.

The present parametrization has also some relation to [273, 274], where in addition to the exponential split the parametrization was further refined to remove the gauge-parameter dependence completely on the semi-classical level. More specifically, the parametrization of the fluctuations was chosen so that only fluctuations contribute that also have an on-shell meaning. In essence, this removes any contribution from the scalar modes to the ultraviolet running. At the semi-classical level [273], the nontrivial critical exponent is 2 as in [153] and increases upon inclusion of renormalization group improvement as in the present work. The increase determined in [274] is larger than in the present parametrization and yields $\theta \simeq 3$ which is remarkably close to results from simulations based on Regge calculus [275, 276].

The present parametrization with $|\beta| \to \infty$ is also loosely related to unimodular gravity, as



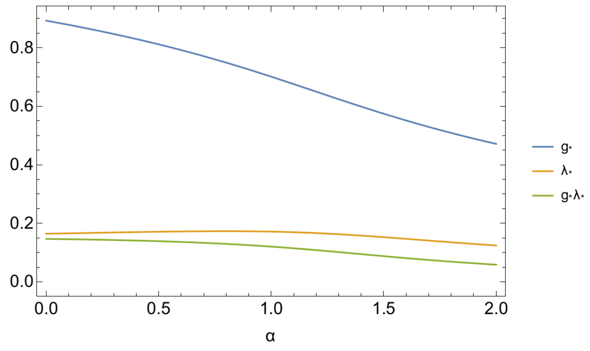


Figure 6.5: Linear split with field redefinition: dependence of estimates for the critical exponents (left panel) and the fixed-point values (right panel) on the gauge parameter α and harmonic gauge condition $\beta = 1$. No qualitative and only minor quantitative differences are found for the Feynman gauge $\alpha = 1$ in comparison to the Landau gauge $\alpha = 0$.

the conformal mode is effectively removed from the fluctuation spectrum. Still, differences to unimodular gravity remain in the gauge-fixing and ghost sector as unimodular gravity is only invariant under transversal diffeomorphisms. It is nevertheless interesting to observe that corresponding functional renormalization group calculations yield critical exponents of comparable size [39, 134].

In fact, the present parametrization allows for a closed form solution of the renormalization group flow as will be presented in section 6.3.7.

6.3.5 Landau vs. Feynman Gauge

Many of the pioneering computations in quantum gravity have been and still are performed within the harmonic gauge $\beta = 1$ and with $\alpha = 1$ corresponding to Feynman gauge. This is because this choice leads to a number of technical simplifications such as the direct diagonalization of the scalar modes as is visible from the off-diagonal terms in equation (6.19). Concentrating on the linear split with field redefinition, we study the α dependence for the harmonic gauge $\beta = 1$ in the vicinity of the Landau and Feynman gauges. The results for the non-Gaußian fixed point values are shown in the right panel of Fig. 6.5. In essence, the fixed-point values show only a mild variation during the transition from the Landau gauge $\alpha = 0$ to the Feynman gauge $\alpha = 1$. In particular, the decrease of g_* is slightly compensated for by a mild increase of λ_*. Effectively, the observed variation is only on a level which is quantitatively similar to other parametrization dependencies, cf. Table 1.

A similar conclusion holds for the more sensitive critical exponents. Real and imaginary parts of the complex pair are shown in the left panel of figure 6.5. Starting from larger values of α, it is interesting to observe that the imaginary part $\operatorname{Im}\theta$ decreases with decreasing α. This may be taken as an indication for a tendency towards purely real exponents; however, at about $\alpha = 1$ this tendency is inverted and the exponents remain a complex pair in between Feynman gauge and Landau gauge within the present estimate.

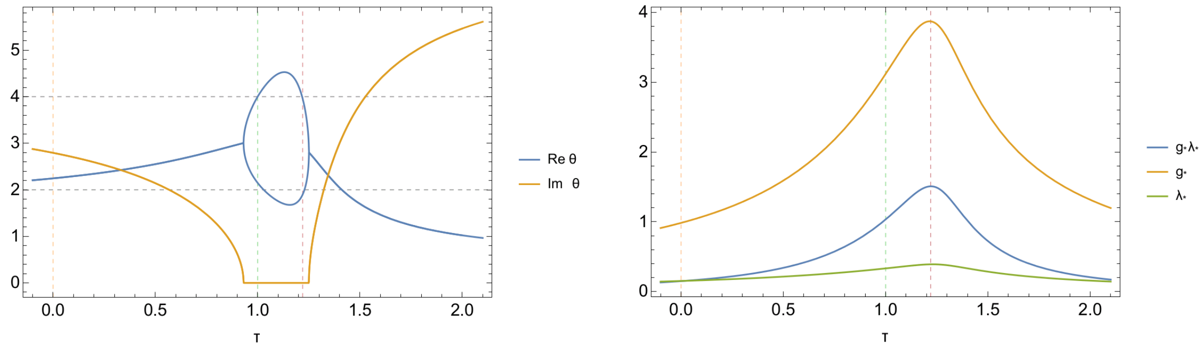


Figure 6.6: Parametrization dependence of critical exponents (left panel) and fixed-point values (right panel) as a function of the split parameter τ for the Landau gauge $\alpha = 0$ and $|\beta| \to \infty$. The fixed-point values exhibit extrema near $\tau \simeq 1.22$, for the product of fixed-point values, this occurs at $\tau = 1 + \frac{\sqrt{3}}{24}\left(\frac{278}{\pi}\right)^{1/4}$ (red dashed vertical line). In this regime, the critical exponents are real and close to their values for the exponential split $\tau = 1$ (green dashed vertical line).

In summary, we observe no substantial difference between the results in Feynman gauge $\alpha = 1$ and those of Landau gauge $\alpha = 0$ in any of the quantities of interest for the linear split and with field redefinition. Our results show an even milder dependence on the gauge parameter in comparison to the pioneering study of Ref. [262], where the regulator was chosen such as to explicitly lift the degeneracy in the sector of scalar modes in the limit $\alpha \to 0$. The present parametrization hence shows a remarkable degree of robustness against deformations away from the a-priori preferable Landau gauge. Hence, we conclude that the use of Feynman gauge is a legitimate option to reduce the complexity of computations.

6.3.6 Generalized Parametrizations

Having focused so far mainly on the gauge-parameter dependencies for fixed values of the split parameter τ, we now explore the one-parameter family of parametrizations for general τ. For this, we use the Landau gauge $\alpha = 0$ and take the limit $|\beta| \to \infty$, where the fixed-point estimates of all parametrizations used so far showed a large degree of stability. Figure 6.6 exhibits the results for the non-Gaußian fixed-point values (right panel) and the corresponding critical exponents (left panel).

A comparison of the results for $\tau = 0$ and $\tau = 1$ reveals the differences already discussed above: an increase of the fixed-point values and the occurrence of real critical exponents for the exponential split $\tau = 1$. From the perspective of the principle of minimum sensitivity, it is interesting to observe that the fixed-point values develop extrema near $\tau \simeq 1.22$. The product $g_*\lambda_*$ is maximal for $\tau = 1 + \frac{\sqrt{3}}{24}\left(\frac{278}{\pi}\right)^{1/4}$. Also for this parametrization, the critical exponents of the fixed point are real and still close to the values for the exponential split, cf. Table 1. For even larger values of τ, the critical exponents form complex pairs again.

To summarize, in the full three-parameter space defined by τ, β and $\alpha \geq 0$, we find a local extremum, i.e., a point of minimum sensitivity, at $\alpha = 0$, $\beta \to \infty$ and τ near the exponential split value $\tau = 1$. From this a-posteriori perspective, our results suggest that the exponential split (with field redefinition) in the limit where the scalar sector is represented by the σ mode may be viewed as a "best estimate" for the ultraviolet behavior of quantum Einstein gravity. Of course, due to the limitations imposed by the simplicity of our truncation, this conclusion should be taken with reservations. The resulting renormalization group flow for $\tau = 1$ is in fact remarkably simple and will be discussed next.

6.3.7 Analytical Solution for the Phase Diagram

Let us now analyze more explicitly the results for the renormalization group flow for the exponential split with field redefinition in the limit $|\beta| \to \infty$. Several simplifications arise in this case. The exponential split removes any dependence of the transverse traceless and vector components of the propagator on the cosmological constant. The remaining dependence on λ in the conformal mode is finally removed by the limit $|\beta| \to \infty$. As a consequence, the cosmological constant does not couple into the flows of the Newton coupling nor into any other higher-order coupling. Still, the cosmological constant is driven by graviton fluctuations. As emphasized above in subsection 6.3.4, any remaining gauge dependence on the gauge parameter α drops out of the flow equations. For the renormalization group flow of Newton coupling and cosmological constant, we find the simple set of equations:

$$\dot{g} \equiv \beta_g = 2g - \frac{135g^2}{72\pi - 5g}, \qquad \dot{\lambda} \equiv \beta_\lambda = \left(-2 - \frac{135g}{72\pi - 5g}\right)\lambda - g\left(\frac{43}{4\pi} - \frac{810}{72\pi - 5g}\right). \tag{6.31}$$

In addition to the Gaußian fixed point, these flow equations support a fixed point at

$$g_* = \frac{144\pi}{145}, \qquad \lambda_* = \frac{48}{145}, \qquad g_*\lambda_* = \frac{6912\pi}{21025}, \tag{6.32}$$

cf. table 1. Also the critical exponents θ_i being (-1) times the eigenvalues of the stability matrix $\partial\beta_{(g,\lambda)}/\partial(g,\lambda)$ can be determined analytically,

$$\theta_0 = 4, \quad \theta_1 = \frac{58}{27}. \tag{6.33}$$

The fact that the largest critical exponent corresponds to the power-counting canonical dimension of the cosmological term is a straightforward consequence of the structure of the flow equations within this parametrization: as we have $\dot{g} = (2+\eta(g))g$ and $\dot{\lambda} = (-2+\eta(g))\lambda + \mathcal{O}(g)$, the existence of a non-Gaußian fixed point requires $\eta(g_*) = -2$. As the stability matrix is triangular, the eigenvalue associated with the cosmological term must be -4 and thus $\theta_0 = 4$. Rather generically, other parametrizations lead to a dependence of η also on λ and thus to a more involved stability matrix.

In the physically relevant domain of positive gravitational coupling $g > 0$, the fixed point g_* separates a "weak" coupling phase with $g < g_*$ from a "strong" coupling phase $g > g_*$. Only

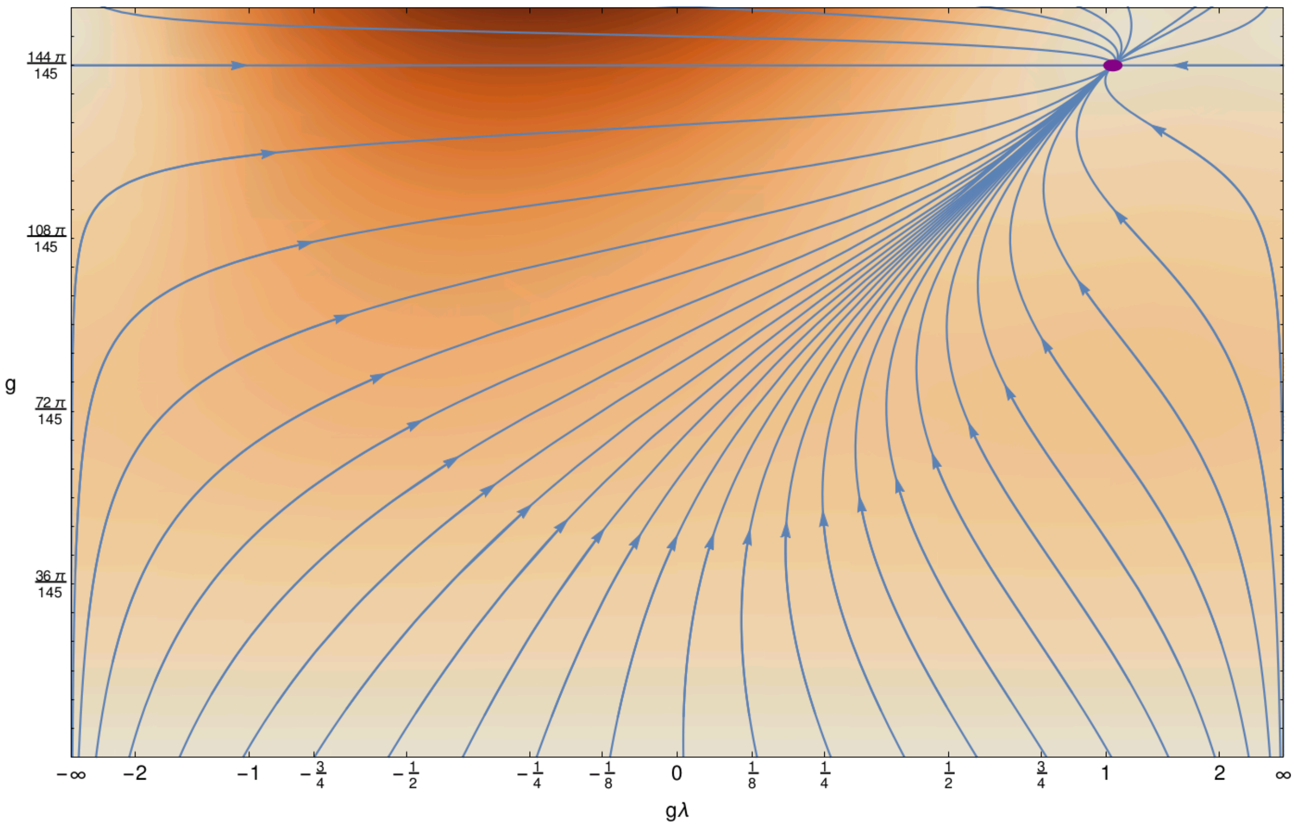


Figure 6.7: Global phase diagram in the $(g, g\lambda)$ plane for the exponential split with field redefinition and $|\beta| \to \infty$. Arrows point from infrared to ultraviolet indicating the approach to the ultraviolet fixed point at $g_* = 144\pi/145$ and $\lambda_* = 48/145$. The color indicates a measure for the flow velocity, $(\partial_t g)^2 + (\partial_t (g\lambda/\sqrt{1+g^2\lambda^2}))^2$.

the former allows for trajectories that can be interconnected with a classical regime where the dimensionless g and λ scale classically, i.e., $\dot{g} \simeq 2g$ and $\dot{\lambda} \simeq -2\lambda$ such that their dimensionful counterparts approach their observed values. Trajectories in the strong-coupling phase run to larger values of g and terminate in a singularity of β_g at $g_{\text{sing}} = 72\pi/5$ indicating the break-down of the truncation.

All trajectories in the weak coupling phase with $g < g_*$ run towards the Gaußian fixed point for g and thus, also the flow of λ in the infrared is dominated by the Gaußian fixed point. This implies that all trajectories emanating from the non-Gaußian fixed point with $g \leq g_*$ can be continued to arbitrarily low scales, i.e., are infrared complete. They can thus be labeled by their deep infrared value of $g\lambda$ approaching a constant, which may be identified with the product of Newton coupling and cosmological constant as observed at present. A plot of the resulting renormalization group flow in the plane $(g, g\lambda)$ is shown in figure 6.7. It represents a global phase diagram of quantum gravity as obtained in the present truncation/parametrization. We emphasize that no singularities appear towards the infrared contrary to conventional single-metric calculations based on the linear split.

The flows (6.31) can be integrated analytically. Converting back to dimensionful couplings, the flow of the running Newton coupling $G(k)$ satisfies the implicit equation,

$$\mathrm{G_N} = \frac{G(k)}{\left(1 - \frac{145}{144\pi}k^2 G(k)\right)^{\frac{27}{29}}}, \tag{6.34}$$

where $\mathrm{G_N}$ is the Newton coupling measured in the deep infrared $k \to 0$. Expanding the solution at low scales about the Newton coupling yields

$$G(k) \simeq \mathrm{G_N}\left(1 - \frac{15}{16\pi}k^2\mathrm{G_N} + \mathcal{O}\big((k^2\mathrm{G_N})^2\big)\right) \tag{6.35}$$

exhibiting the anti-screening property of gravity.

The flow of the dimensionful running cosmological constant $\Lambda(k)$ can be given explicitly in terms of that of the running Newton coupling,

$$\Lambda(k) = \frac{162k^2}{25} - \frac{43G(k)k^4}{16\pi} + \ell k^2\left(144\pi - 145G(k)k^2\right)^{\frac{25}{29}} - \frac{144\pi\left(87 + 25\ell\left(144\pi - 145G(k)k^2\right)^{\frac{25}{29}}\right)}{3625G(k)}. \tag{6.36}$$

Here, $\ell = -\frac{29}{86400}\left(2^{-13}3^{-21}\pi^{-54}\right)^{\frac{1}{29}}(125\Lambda\mathrm{G_N} + 432\pi)$, and Λ is the value of the classical cosmological constant in the deep infrared $k \to 0$. The low-scale expansion about $k = 0$ yields

$$\Lambda(k) \simeq \Lambda\left(1 - \frac{15}{16\pi}k^2\mathrm{G_N} + \mathcal{O}\left(\frac{k^4}{\Lambda^2}\Lambda\mathrm{G_N}, (k^2\mathrm{G_N})^2\right)\right) \tag{6.37}$$

Thus, $\Lambda(k)/G(k) = \Lambda/\mathrm{G_N} + \mathcal{O}(k^4)$, implying a comparatively slow running of the ratio towards the ultraviolet. This explicit solution of the renormalization group flow might be useful for an analysis of "RG-improved" cosmologies along the lines of [63, 277–282].

6.3.8 Generalized Ultra-Local Parametrizations

For the most general, ultra-local parametrization (6.15), it turns out that the flow equation in our truncation does only depend on the linear combinations $T_1 := \tau/4 + \tau_3$ and $T_2 := \tau_2/4 + \tau_4$, leaving only two independent split parameters. Instead of exploring the full high-dimensional parameter space, we try to identify relevant points as inspired by our preceding results. For instance for the choice $T_1 = 1/4$ and $T_2 = -1/8$, any dependence on α drops out, indicating an enhanced insensitivity to the gauge choice. The resulting flow equations are

$$\dot{g} = 2g + \frac{135(\beta-3)g^2}{(5\beta-3)g - 72(\beta-3)\pi}, \quad \dot{\lambda} = -2\lambda + \frac{g\big((-669+215\beta)g + 36(\beta-3)\pi(4-15\lambda)\big)}{4\pi\big((3-5\beta)g + 72(\beta-3)\pi\big)}. \tag{6.38}$$

In the limit $|\beta| \to \infty$, these are identical to the exponential split in the same limit. The non-Gaußian fixed point occurs at

$$g_* = \frac{144\pi(\beta-3)}{145\beta-411}, \qquad \lambda_* = \frac{48(\beta-3)}{145\beta-411}, \qquad g_*\lambda_* = 6912\pi\left(\frac{\beta-3}{145\beta-411}\right)^2. \tag{6.39}$$

Apart from the pathological choice $\beta_{\text{sing}} = 3$ (incomplete gauge fixing) where this fixed point merges with the Gaußian fixed point, no further extremal point is observed except for the limit $|\beta| \to \infty$. The critical exponents are

$$\theta_0 = 4, \quad \theta_1 = \frac{58}{27} + \frac{16}{45(\beta-3)}. \tag{6.40}$$

Also the exponents become minimally sensitive to the choice of β for $|\beta| \to \infty$.

As an oddity, we mention the particular case $\beta = 3/5$, where the flow equations acquire a pure one-loop form. In this case, the second critical exponent is exactly 2 as it must, since the slope of a parabolic β function at the interacting fixed point is minus the slope at the Gaußian fixed point [271].

More importantly, the interdependence of gauge and parametrization choices is also visible in the following fact: we observe that the choice of the gauge parameter $|\beta| \to \infty$ removes any dependence of our flow on the parameter T_2 independently of the value of α. In other words, this limit brings us back exactly to the case which we discussed above in Sect. 6.3.6, such that the seemingly much larger class of parametrizations (6.15) collapses to a one-parameter family.

6.3.9 Arbitrary Dimensions

Finally, we discuss the stability of the ultraviolet fixed-point scenario and its parametrization dependence in arbitrary dimensions, focusing on $d > 2$ (for a discussion of $d = 2$ in the present context, see [152, 153, 273]). In fact, there are some indications in the literature that the parametrization dependence is pronounced in higher dimensions. Whereas standard calculations based on the linear split generically find an ultraviolet fixed point in any dimension $d > 2$ and gauge-fixing parameter α, see e.g. [283, 284], a recent refined choice of the parametrization to remove gauge-parameter dependence on the semi-classical level arrives at a different result [273, 274]: the ultraviolet fixed point can be removed from the physical region if the number of physical gravity degrees of freedom becomes too large. As the latter increases with the dimensionality, there is a critical value d_{cr} above which asymptotically safe gravity does not exist. The resulting scenario is in line with the picture of paramagnetic dominance [239, 285], which is also at work for the QED and QCD β functions: the dominant sign of the β function coefficient arises from the paramagnetic terms in the Hessian which can be reversed if too many diamagnetically coupled degrees of freedom contribute.

Our results extend straightforwardly to arbitrary dimensions. Starting, for instance, with the most general parametrization (6.15) in d dimensions, the flows of g and λ depend only on

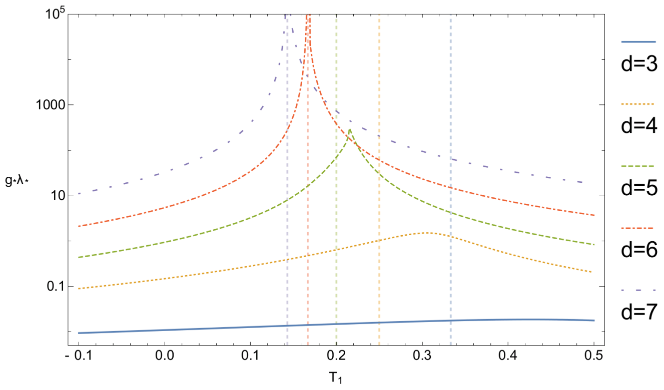


Figure 6.8: Parametrization dependence of fixed-point value for $g_*\lambda_*$ as a function of the split parameter T_1 in the Landau gauge $\alpha = 0$ and $|\beta| \to \infty$ for different dimensions $d = 3, 4, 5, 6, 7$ (from bottom to top). Vertical lines mark the value of the parameter $T_1 = 1/d$ preferred by gauge-parameter α independence. For $d \geq d_{\rm cr} \simeq 5.731$, the fixed-point product $g_*\lambda_*$ develops a singularity at $T_1 = 1/d$.

the linear combinations $T_1 = \tau/d + \tau_3$ and $T_2 = \tau_2/d + \tau_4$. Comparable results as in $d = 4$ dimensions apply: in the limit of $|\beta| \to \infty$, also T_2 drops out such that a one-parameter family remains. In turn, a complete independence of the gauge parameter α can be realized with the parametrization specified by $T_1 = 1/d$ and $T_2 = -1/(2d)$.

We illustrate the stability properties of the asymptotic-safety scenario in arbitrary dimensions by choosing the Landau-gauge limit $\alpha \to 0$ as well as $|\beta| \to \infty$, keeping T_1 as a free parameter. Then, we know a priori that $T_1 = 1/d$ would be a preferred choice from the view point of gauge invariance; it would also correspond to the exponential parametrization $\tau = 1$, $\tau_3 = 0$. Fig. 6.8 displays the fixed-point values for $g_*\lambda_*$ as a function of T_1 for various dimensions $d = 3, \ldots, 7$. While $d = 3$ exhibits a rather small parametrization dependence, $d = 4$ reproduces the earlier results of Fig. 6.6 (right panel) now as a function of T_1 with an extremum not far above $T_1 = 1/4$. By contrast, $g_*\lambda_*$ develops a kink for $d = 5$ that turns into a singularity for $d = 6$ and larger. For increasing d, the kink approaches the preferred parametrization $T_1 = 1/d$ (vertical dashed lines in Fig. 6.8). The singularity in $g_*\lambda_*$ occurs for a critical dimension $d_{\rm cr} \simeq 5.731$. This observation suggests the following interpretation: whereas we can identify an ultraviolet fixed point for any dimension as long as we choose T_1 sufficiently far away from $T_1 = 1/d$, we find a stable fixed-point scenario only for $d = 3$ and $d = 4$ integer dimensions. Already for $d = 5$, the fixed-point product $g_*\lambda_*$ can change by two orders of magnitude by varying the parametrization, which is at least a signature for the insufficiency of the truncation. For $d \geq d_{\rm cr} \simeq 5.731$, $g_*\lambda_*$ can become unboundedly large as a function of the parametrization, signaling the instability of the fixed point.

If these features persist also beyond our truncation, they suggest that the asymptotic safety scenario may not exist far beyond the spacetime dimension $d = 4$. Whereas this does not offer a dynamical explanation of our spacetime dimension, it may serve to rule out the mutual co-existence of extra dimensions and asymptotically safe quantum gravity.

7 Gross-Neveu Model in Curved Spacetime

In the final chapter we aim at investigating the $3d$ Gross-Neveu model [286] in curved spacetime with signature $(-,+,+)$ using functional renormalization group methods. This part of the thesis is based on our work in [53]. The microscopic action functional $S_{\Lambda_{\mathrm{UV}}}$ at some ultraviolet scale Λ_{UV} depends on the bare coupling constant $\bar{\lambda}_{\Lambda_{\mathrm{UV}}}$, the N_{f} Grassmann-valued fields $\psi = (\psi^i)$ and the N_{f} conjugated fields $\bar{\psi} = (\bar{\psi}^i)$,

$$S_{\Lambda_{\mathrm{UV}}}[\bar{\psi}^i, \psi^i] = \int_x \left[\sum_{i=1}^{N_{\mathrm{f}}} \bar{\psi}^i \nabla\!\!\!\!/\, \psi^i + \frac{\bar{\lambda}_{\Lambda_{\mathrm{UV}}}}{2N_{\mathrm{f}}} \left(\sum_{i=1}^{N_{\mathrm{f}}} \bar{\psi}^i \psi^i \right)^2 \right] = \int_x \left[\bar{\psi} \nabla\!\!\!\!/\, \psi + \frac{\bar{\lambda}_{\Lambda_{\mathrm{UV}}}}{2N_{\mathrm{f}}} (\bar{\psi}\psi)^2 \right]. \tag{7.1}$$

We use the irreducible representation of the Dirac matrices, such that $d_\gamma = 2$. Furthermore, we set any torsion to zero, $K_\mu{}^\rho{}_\lambda = 0$, $\Delta\Gamma_\mu = 0$. In the present chapter, we are interested in a discrete "chiral" $\mathbb{Z}_2$ symmetry, where the nontrivial transformation is defined by $\psi(x) \to -\psi(-x)$, $\bar{\psi}(x) \to \bar{\psi}(-x)$ [287]. This symmetry acts simultaneously on all flavors. It can spontaneously be broken by a chiral condensate $\langle \bar{\psi}\psi \rangle \neq 0$, which for finite interactions goes along with a mass gap generation. Incidentally, the $3d$ Gross-Neveu model actually has a much larger continuous $\mathrm{U}(N_{\mathrm{f}})$ flavor symmetry also allowing for more complicated breaking patterns [288, 289].[40]

7.1 Fermionic RG Flows in Curved Spacetime

In the following, we use the functional renormalization group to compute the RG flow of the Gross-Neveu coupling as a function of the (negative) curvature. We employ the Wetterich equation [154] with signature $(-,+,+)$, cf. section 5.1,

$$\partial_k \Gamma_k[\bar{\psi}^i, \psi^i] = \frac{\mathrm{i}}{2} \,\mathrm{STr} \left[\left(\Gamma_k^{(2)}[\bar{\psi}^i, \psi^i] + \mathcal{R}_k(\nabla\!\!\!\!/\,) \right)^{-1} \partial_k \mathcal{R}_k(\nabla\!\!\!\!/\,) \right]. \tag{7.2}$$

For reviews of the functional RG adapted to the present context, see references [160, 162, 204, 292–296]. Here we evaluate the flow within a rather simple approximation for the effective action. For this, we truncate the effective action to

$$\Gamma_k[\bar{\psi}^i, \psi^i] = \int_x \left[\bar{\psi} \nabla\!\!\!\!/\, \psi + \frac{\bar{\lambda}_k}{2N_{\mathrm{f}}} (\bar{\psi}\psi)^2 \right], \tag{7.3}$$

[40] In many $3d$ condensed matter systems where the Gross-Neveu model is considered as an effective theory, the low-energy degrees of freedom can be arranged into $N_{4\mathrm{f}}$ 4-component Dirac spinors, corresponding to a reducible representation of the Dirac algebra. This reducible representation can be constructed from a suitable combination of 2-component spinors such that $N_{\mathrm{f}} = 2N_{4\mathrm{f}}$ in terms of the counting of fermions of the present work, see, e.g., [290] for a review. Note that the Gross-Neveu interaction term considered in this work $\sim (\bar{\psi}\psi)^2$ corresponds to $\sim (\bar{\psi}\gamma_{45}\psi)^2$ in the reducible 4-component notation of [288, 289] (or to $\sim (\bar{\psi}\gamma_{35}\psi)^2$ in the notation of [97, 113]) for even N_{f}. The critical properties of the discrete chiral transition are, however, identical to a 4-component Gross-Neveu model with a $(\bar{\psi}\psi)^2$ interaction as considered in [291].

where the only scale dependence lies in the four fermion coupling $\bar{\lambda}_k$. Furthermore, $\bar{\lambda}_k$ parametrically depends on the curvature of the background manifold. The IR regularization is ensured by a chirally symmetric regulator of the form

$$[\mathcal{R}_k(\nabla\!\!\!/)](\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} \nabla\!\!\!/ r_k^\psi(\tau) & 0 \\ 0 & \nabla\!\!\!/^{\mathrm{T}} r_k^\psi(\tau^{\mathrm{T}}) \end{pmatrix} \mathbb{1}(\boldsymbol{x},\boldsymbol{y}), \quad \mathbb{1}(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} \delta(\boldsymbol{x},\boldsymbol{y}) & 0 \\ 0 & \delta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}} \end{pmatrix}, \quad \tau = -\frac{\boldsymbol{\Delta}_E}{k^2}, \tag{7.4}$$

where $\boldsymbol{\Delta}_E = -\nabla\!\!\!/^2 = -\nabla^\mu\nabla_\mu + \frac{R}{4}\mathrm{I}$, the superscript "T" denotes transposition in Dirac space, and $\delta(\boldsymbol{x},\boldsymbol{y})$ represents a spin-valued delta distribution, keeping track of the spinor or conjugate-spinor transformation properties associated with the spacetime arguments, cf. equation (5.15). For practical computations, we use a Callan-Symanzik type regulator, that facilitates the use of proper time representations (i.e. via the heat kernel),

$$r_k^\psi(x) = \sqrt{\frac{1+x}{x}} - 1, \qquad x \in \mathbb{R}\backslash\{0\}. \tag{7.5}$$

Within our investigations we restrict ourselves to negative curvature, giving rise to gravitational catalysis. It is intuitively clear, that positive curvature (e.g., a sphere) generically suppresses IR modes and thus reduces the density of states of low lying modes [42, 99, 100]. We also consider the system mainly in the large-N_f limit. At finite N_f, further pointlike fermionic self-interactions are generated which correspond to operators with an explicit curvature dependence.[41] We expect no qualitative modifications from these operators and hence ignore them in the following. This approximation becomes exact in the limit $N_\mathrm{f} \to \infty$.

It is straightforward (cf. appendix K) to calculate the flow of the coupling as an implicit functional of the choice of the manifold, which enters via the spectrum of the Dirac operator

$$\partial_k \bar{\lambda}_k = -\mathrm{i}\frac{2\bar{\lambda}_k^2}{\Omega N_\mathrm{f} k^3}\,\mathrm{STr}\left[(\mathrm{I}+\tau)^{-2}\,\delta(\boldsymbol{x},\boldsymbol{y})\right]. \tag{7.6}$$

For this calculation, it suffices to project the flow onto constant fields $\psi^i(\boldsymbol{x}) \equiv \Psi^i$, with $\partial_\mu \Psi^i = 0$. Here, $\Omega = \int_x 1$ denotes the spacetime volume. The operator occurring in the trace is related to the square of the regularized fermionic Green's function in curved spacetime. This has a direct correspondence to a Feynman diagram representation, see figure 7.1, as the flow in the present simple truncation is driven by a single fermion bubble (and RG-improved resummations thereof). In the following we distinguish between the cases of a maximally symmetric spacetime, i.e. the three dimensional anti-de Sitter space AdS_3, in section 7.1.1, which can be treated fully analytically, and a negatively curved space, i.e. the Lobachevsky plane, in section 7.1.2, which is a more interesting case in view of two-dimensional condensed matter systems.

[41] A similar mechanism has been observed in [97] for the case of a magnetic field, and the corresponding operators have been classified.

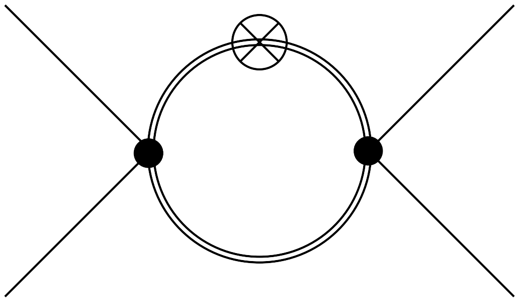


Figure 7.1: The diagram schematically exemplifies the fluctuation contributions to the flow of the Gross-Neveu coupling. The double lines represent the fermion propagator on the curved manifold. The full circles denote the RG-improved couplings, and the crossed circle marks the regulator insertion.

7.1.1 Maximally Symmetric Spacetime

The use of a Callan-Symanzik regulator shape function, cf. equation (7.5), facilitates to rewrite the right-hand side of equation (7.6) in terms of a simple proper time representation. Other shape functions would still permit to use a proper time representation but would lead to more intricate k dependencies. The Laplace transform of the operator of (7.6) reads

$$(\mathrm{I}+\tau)^{-2}\,\delta(\boldsymbol{x},\boldsymbol{y}) = -k^4\int_0^\infty \mathrm{d}\mathfrak{s}\,\mathfrak{s}\mathrm{e}^{-\mathrm{i}\mathfrak{s}k^2}\mathrm{e}^{-\mathrm{i}\mathfrak{s}\boldsymbol{\Delta}_E}\delta(\boldsymbol{x},\boldsymbol{y}), \tag{7.7}$$

which upon insertion into (7.6) yields

$$\partial_k\bar{\lambda}_k = \mathrm{i}\frac{2\bar{\lambda}_k^2 k}{\Omega N_\mathrm{f}}\int_0^\infty \mathrm{d}\mathfrak{s}\,\mathfrak{s}\,\mathrm{e}^{-\mathrm{i}\mathfrak{s}k^2}\,\mathrm{STr}\left[\mathrm{e}^{-\mathrm{i}\mathfrak{s}\boldsymbol{\Delta}_E}\delta(\boldsymbol{x},\boldsymbol{y})\right]. \tag{7.8}$$

The expression inside the super trace is the heat kernel $\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s}) = \mathrm{e}^{-\mathrm{i}\mathfrak{s}\boldsymbol{\Delta}_E}\delta(\boldsymbol{x},\boldsymbol{y})$ of the (squared) Dirac operator in mixed signature. It satisfies, cf. equation (5.21),

$$\text{(i)}\quad \partial_{\mathfrak{s}}\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s}) = -\mathrm{i}\boldsymbol{\Delta}_E\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s}),\quad \text{(ii)}\quad \lim_{\mathfrak{s}\searrow 0}\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s}) = \delta(\boldsymbol{x},\boldsymbol{y}), \tag{7.9}$$

and was calculated in [297] for any maximally symmetric space, with Euclidean signature. For our case the solution to this equation can be obtained in an easier way with a special ansatz, cf. appendix L:

$$\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s}) = \frac{\mathrm{e}^{\mathrm{i}\frac{\sigma(\boldsymbol{x},\boldsymbol{y})}{2\mathfrak{s}}}}{(4\pi\mathfrak{s})^{\frac{3}{2}}}\left(\frac{w(\boldsymbol{x},\boldsymbol{y})}{\sinh\left(w(\boldsymbol{x},\boldsymbol{y})\right)} + \mathrm{i}\frac{\mathfrak{s}\,|R|}{12\cosh\left(w(\boldsymbol{x},\boldsymbol{y})\right)}\right)\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{\cosh\left(w(\boldsymbol{x},\boldsymbol{y})\right)}\mathfrak{U}(\boldsymbol{x},\boldsymbol{y}), \tag{7.10}$$

where $\big(w(\boldsymbol{x},\boldsymbol{y})\big)^2 = \frac{|R|\sigma(\boldsymbol{x},\boldsymbol{y})}{12}$, and $\sigma(\boldsymbol{x},\boldsymbol{y})$ is one half the square of the geodesic distance from $\boldsymbol{x}$ to $\boldsymbol{y}$, cf. section 5.2. The parallel propagator $\mathfrak{U}(\boldsymbol{x},\boldsymbol{y})$ (Wegner-Wilson line) is defined by

$$\mathfrak{U}(\boldsymbol{x},\boldsymbol{y}) = \mathrm{P}\exp\left(-\int_0^1 \mathrm{d}t\,\frac{\mathrm{d}z^\mu(t)}{\mathrm{d}t}\hat{\Gamma}_\mu\big(\boldsymbol{z}(t)\big)\right), \tag{7.11}$$

where P denotes the path ordering prescription and $\boldsymbol{z}(t)$ is the geodesic between $\boldsymbol{x} = \boldsymbol{z}(t=0)$ and $\boldsymbol{y} = \boldsymbol{z}(t=1)$. Now we are able to calculate the supertrace as

$$\mathrm{STr}\left[\mathrm{e}^{-\mathrm{i}\mathfrak{s}\boldsymbol{\Delta}_E}\delta(\boldsymbol{x},\boldsymbol{y})\right] = -\frac{2\Omega N_\mathrm{f}}{(4\pi\mathfrak{s})^{\frac{3}{2}}}\left(1+\mathrm{i}\mathfrak{s}\frac{|R|}{12}\right)\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}, \tag{7.12}$$

and finally get

$$\partial_k\bar{\lambda}_k = -\frac{\bar{\lambda}_k^2}{2\pi}\left(1+\frac{|R|}{24k^2}\right). \tag{7.13}$$

In terms of the dimensionless coupling, $\lambda_k = k\bar{\lambda}_k$, we obtain the beta function β_λ

$$\beta_\lambda = k\partial_k\lambda_k = \lambda_k - \frac{\lambda_k^2}{2\pi}\left(1+\frac{|R|}{24k^2}\right). \tag{7.14}$$

This is an ordinary differential equation that parametrically depends on the curvature. It can be solved by straightforward integration:

$$\lambda_k = \lambda_{\Lambda_\mathrm{UV}}\cdot\frac{k}{\Lambda_\mathrm{UV}}\left(1-\frac{\lambda_{\Lambda_\mathrm{UV}}}{2\pi}\cdot\left(1-\frac{k}{\Lambda_\mathrm{UV}}\right)\left(1+\frac{|R|}{24k\Lambda_\mathrm{UV}}\right)\right)^{-1}. \tag{7.15}$$

The initial value is given by the dimensionless coupling $\lambda_{\Lambda_\mathrm{UV}}$ which in terms of the initial bare Gross-Neveu coupling reads $\lambda_{\Lambda_\mathrm{UV}} = \Lambda_\mathrm{UV}\bar{\lambda}_{\Lambda_\mathrm{UV}}$.

7.1.2 Negatively Curved Space

For the case of a manifold where the spatial part has a constant negative curvature, we choose a special set of coordinates such that the metric can be expressed as

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{\hat{g}_{ij}} \end{pmatrix}, \tag{7.16}$$

where $\hat{g}_{ij}$ (Latin indices running from 1 to 2) represents the metric of a two dimensional maximally symmetric space and therefore only depends on the spatial coordinates. Hence, the Dirac matrices are time independent as well. The Christoffel symbols vanish for every time component $\left\{{\rho \atop \mu 0}\right\} = \left\{{0 \atop \mu\nu}\right\} = 0$ and also the curvature tensor vanishes if any index is zero $R_{\mu\nu\rho 0} = 0$. From the first part of equations (4.19), $0 = \nabla_0 \gamma^\nu = [\hat{\Gamma}_0, \gamma^\nu]$, we infer that the time component of the spin connection $\hat{\Gamma}_0$ has to be proportional to I. Moreover, from the second part of equations (4.19), $\operatorname{tr} \hat{\Gamma}_0 = 0$, we conclude that $\hat{\Gamma}_0$ even has to vanish completely. This implies that the operator $\boldsymbol{\Delta}_E$ is separable into

$$\boldsymbol{\Delta}_E \psi^i = -\nabla\!\!\!/^2 \psi^i = \partial_0^2 \psi^i - \vec{\nabla\!\!\!/}^2 \psi^i = \partial_0^2 \psi^i - \vec{\nabla}^2 \psi^i + \frac{R}{4} \psi^i, \qquad \vec{\nabla\!\!\!/} \psi^i = \gamma^k \nabla_k \psi^i, \tag{7.17}$$

where the curvature is induced by the spatial components only.

For a simpler evaluation of the beta function, we perform a Wick rotation $x^0 \to -\mathrm{i}x^0$ and again a Laplace transformation of the operator occurring in equation (7.6),

$$(\mathrm{I} + \tau)^{-2} \delta(\boldsymbol{x}, \boldsymbol{y}) = k^4 \int_0^\infty \mathrm{d}\mathfrak{s}\, \mathfrak{s} \mathrm{e}^{-\mathfrak{s}k^2} \mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E} \delta(\boldsymbol{x}, \boldsymbol{y}). \tag{7.18}$$

We arrive at the Euclidean analogue of equation (7.8)

$$\partial_k \bar{\lambda}_k = \frac{2\bar{\lambda}_k^2 k}{\Omega N_\mathrm{f}} \int_0^\infty \mathrm{d}\mathfrak{s}\, \mathfrak{s} \mathrm{e}^{-\mathfrak{s}k^2} \operatorname{STr}\left[\mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E} \delta(\boldsymbol{x}, \boldsymbol{y})\right]. \tag{7.19}$$

Here, we calculate the super trace again with the aid of the heat kernel. Using that the differential operator $\boldsymbol{\Delta}_E$ is separable, we find that the delta distribution also factorizes into a time like and a spatial part, cf. equation (5.15). Hence, the heat kernel reads

$$\mathrm{e}^{-\mathfrak{s}\boldsymbol{\Delta}_E} \delta(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{e}^{\mathfrak{s}\partial_0{}^2} \delta(x^0 - y^0) \cdot \mathrm{e}^{\mathfrak{s}\vec{\nabla\!\!\!/}^2} \delta(\vec{x}, \vec{y}), \qquad \delta(\vec{x}, \vec{y}) = \frac{\delta(x^1 - y^1)\delta(x^2 - y^2)}{\sqrt{\hat{\mathfrak{g}}}}, \tag{7.20}$$

where $\hat{\mathfrak{g}} = \det \hat{g}_{ij}$. The quantity $\vec{x} = (x^1, x^2)$ denotes the spatial coordinates of $\boldsymbol{x}$ and should be treated as a set of coordinates and not as a vector. Both factors in the previous equation satisfy a heat-kernel equation individually. These can be solved analytically, cf. [297],

$$\mathrm{e}^{\mathfrak{s}\partial_0{}^2} \delta(x^0 - y^0) = \frac{\mathrm{e}^{-\frac{(x^0 - y^0)^2}{4\mathfrak{s}}}}{\sqrt{4\pi s}}, \quad \mathrm{e}^{\mathfrak{s}\vec{\nabla\!\!\!/}^2} \delta(\vec{x}, \vec{y}) = \frac{2\cosh^{-1}\frac{\hat{w}(\vec{x},\vec{y})}{2}}{(4\pi\mathfrak{s})^{\frac{3}{2}}\sqrt{|R|}} \int_{\hat{w}(\vec{x},\vec{y})}^\infty \mathrm{d}v\, \frac{v \mathrm{e}^{-\frac{v^2}{2\mathfrak{s}|R|}} \cosh\frac{v}{2}}{\sqrt{\cosh v - \cosh \hat{w}(\vec{x}, \vec{y})}} \hat{\mathfrak{U}}(\vec{x}, \vec{y}), \tag{7.21}$$

where $\left(\hat{w}(\vec{x}, \vec{y})\right)^2 = |R|\, \hat{\sigma}(\vec{x}, \vec{y})$, $\hat{\mathfrak{U}}(\vec{x}, \vec{y})$ is the parallel transporter for the spatial part and $\hat{\sigma}(\vec{x}, \vec{y})$ is one half the squared (nonnegative) spatial geodesic distance between $\vec{x}$ and $\vec{y}$ with $\sigma(\boldsymbol{x}, \boldsymbol{y}) = \hat{\sigma}(\vec{x}, \vec{y}) - \frac{1}{2}(x^0 - y^0)^2$. Plugging these relations into equation (7.19) gives

$$\partial_k \bar{\lambda}_k = -\frac{\bar{\lambda}_k^2}{2\pi} \cdot \mathfrak{I}(\alpha_k), \qquad \alpha_k = \sqrt{\frac{|R|}{2k^2}}, \qquad \mathfrak{I}(\alpha) = \frac{\alpha}{2\pi} \int_0^\infty \mathrm{d}\mathfrak{s} \int_0^\infty \mathrm{d}v\, \frac{\mathrm{e}^{-\mathfrak{s} - \frac{v^2}{4\mathfrak{s}}}}{\mathfrak{s}} v \coth\frac{\alpha v}{2}. \tag{7.22}$$

The $\mathfrak{s}$ integral is an integral representation of the modified Bessel function of the second kind $K_0(v)$ [169]. Therefore, we have

$$\mathfrak{I}(\alpha) = \frac{\alpha}{\pi}\int_0^\infty \mathrm{d}v\, v K_0(v) \coth\frac{\alpha v}{2}. \tag{7.23}$$

With these results we are again able to derive the beta function for the dimensionless coupling $\lambda_k = k\bar{\lambda}_k$,

$$\beta_\lambda = k\partial_k\lambda_k = \lambda_k - \frac{\lambda_k^2}{2\pi}\cdot\mathfrak{I}(\alpha_k). \tag{7.24}$$

The integration of this ordinary differential equation, depending parametrically on the curvature, can be cast into an integral representation,

$$\lambda_k = \lambda_{\Lambda_{\mathrm{UV}}}\cdot\frac{k}{\Lambda_{\mathrm{UV}}}\left(1 - \frac{\lambda_{\Lambda_{\mathrm{UV}}}}{2\pi}\cdot\alpha_{\Lambda_{\mathrm{UV}}}\int_{\alpha_{\Lambda_{\mathrm{UV}}}}^{\alpha_k}\frac{\mathfrak{I}(\alpha)}{\alpha^2}\mathrm{d}\alpha\right)^{-1}. \tag{7.25}$$

This result is qualitatively similar to the maximally symmetric case of equation (7.15).

7.2 Gravitational Catalysis

Let us now analyze the consequences of the RG flows for the long-range properties of the Gross-Neveu model. For both background manifolds, β_λ considered as a function of λ_k is a parabola where the prefactor of the quadratic part is scale and curvature dependent, see figure 7.2. For vanishing curvature, the β_λ function vanishes at the two fixed points $\lambda_k = 0$ (Gaußian) and $\lambda_*(R=0) = \lambda_{\mathrm{cr}} = 2\pi$ which corresponds to the well-known critical coupling of the Gross-Neveu model in flat space in this regularization scheme [291, 296]. This critical coupling separates the symmetric phase for $\lambda_{\Lambda_{\mathrm{UV}}} < \lambda_{\mathrm{cr}}$ where the long range behavior is controlled by the non-interacting Gaußian fixed point from the chiral symmetry broken phase for $\lambda_{\Lambda_{\mathrm{UV}}} > \lambda_{\mathrm{cr}}$. In the latter case, λ_k runs to large values towards the infrared. In the present simple truncation, λ_k in fact diverges at a finite scale k_{SB} signaling the transition into the ordered regime. The scale k_{SB} is thus characteristic for the physical scales in the ordered phase. In [298] it has been shown that k_{SB} actually agrees with the value of the dynamically generated fermion mass m_{f} as obtained in mean-field approximation. Since we are working in the large-N_{f} limit anyway, we will use this mean-field identification in the following: $m_{\mathrm{f}} = k_{\mathrm{SB}}$.

The existence of the non-Gaußian fixed point λ_{cr} can be attributed to the competition between the power-counting scaling (the linear coupling term in β_λ) and the interaction terms $\sim \lambda_k^2$. In our RG picture, the interaction terms are enhanced by negative curvature as soon as the wavelength of the fluctuations becomes of the order of the curvature scale. As a consequence, the interacting second zero of the β_λ function no longer is a true fixed point but becomes scale

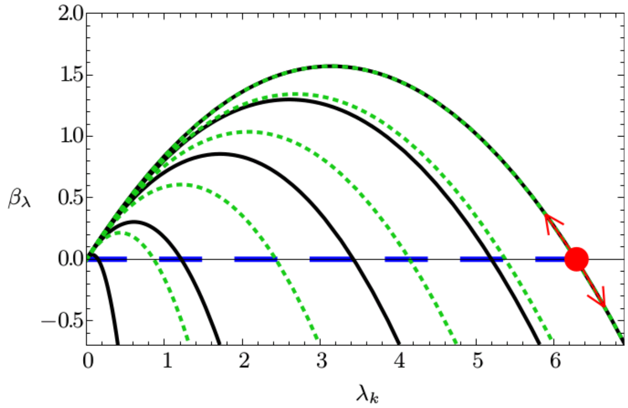


Figure 7.2: Plot of the RG β_λ function of the coupling λ_k for different values of the scale dependent negative curvature $\frac{|R|}{k^2}$ (from top to bottom: 0; 5; 20; 100; 1000). The black lines depict the β_λ functions for the maximally symmetric spacetime (AdS), cf. equation (7.14); arrows indicate the flow towards the IR. The green dotted graphs show the flows for the case of purely spatial curvature (Lobachevsky plane), cf. equation (7.24). In addition to the Gaußian fixed point, there exists a non Gaußian fixed point (full red circle at $\lambda_{\rm cr} = 2\pi$ for the present regulator scheme), separating the symmetric phase for $\lambda_{\Lambda_{\rm UV}} < \lambda_{\rm cr}$ from the broken phase for $\lambda_\Lambda > \lambda_{\rm cr}$ for zero curvature. For finite curvature, this critical point becomes scale-dependent and moves towards the Gaußian fixed point for increasing scale dependent curvature, i.e. with decreasing IR scale for fixed curvature. In the case of vanishing curvature, the symmetry is preserved and no mass is generated for initial values $\lambda_{\Lambda_{\rm UV}}$ in the blue dashed region.

dependent. This "pseudo-critical coupling" $\lambda_{\rm p} = \lambda_*(|R|/k^2)$ moves towards the Gaußian fixed point for decreasing scale k, see figure 7.2.

Any finite initial coupling strength $\lambda_{\Lambda_{\rm UV}}$ will eventually become larger than $\lambda_*(|R|/k^2)$ for small RG scales $k \to 0$. By this mechanism, the system is forced into the symmetry-broken phase even at the weakest initial coupling. We observe this mechanism in both cases of negative curvature, the maximally symmetric as well as the purely spatial curvature case. As we see in figure 7.2 the influence of the curvature is somewhat stronger in the maximally symmetric case.

As discussed above, we calculate the symmetry breaking scale $k_{\rm SB}$ by searching for a zero of the inverse coupling. The fermion mass $m_{\rm f}$ corresponding to this scale where the RG flow enters the symmetry-broken regime can thus be computed from the criterion

$$\lambda^{-1}_{k=m_{\rm f}}(|R|/m_{\rm f}^2) = 0. \tag{7.26}$$

Upon partial bosonization (Hubbard-Stratonovich transformation), λ_k^{-1} is related to the mass parameter of a composite bosonic field. Hence, the divergence of the fermionic self-interaction simply corresponds to an onset of the order parameter [296, 299, 300]. Let us now analyze the two different backgrounds under consideration in detail.

7.2.1 Maximally Symmetric Spacetime

In the maximally symmetric case, the fermion mass defined by the criterion (7.26) can be straightforwardly computed from the running coupling (7.15), yielding

$$\frac{m_{\mathrm{f}}}{\Lambda_{\mathrm{UV}}} = \frac{1}{2} - \frac{|R|}{48\Lambda_{\mathrm{UV}}^2} - \frac{\pi}{\lambda_{\Lambda_{\mathrm{UV}}}} + \sqrt{\left(\frac{1}{2} - \frac{|R|}{48\Lambda_{\mathrm{UV}}^2} - \frac{\pi}{\lambda_{\Lambda_{\mathrm{UV}}}}\right)^2 + \frac{|R|}{24\Lambda_{\mathrm{UV}}^2}}. \tag{7.27}$$

Plots of this gravitationally catalyzed fermion mass are shown as a function of the curvature as solid lines in figure 7.3. Let us discuss this result in various limits. In the zero-curvature limit $R = 0$, we find $m_{\mathrm{f}} = 0$ for $\lambda_{\Lambda_{\mathrm{UV}}} \leq 2\pi$. For super-critical couplings $\lambda_{\Lambda_{\mathrm{UV}}} > 2\pi$, we rediscover the standard mean-field result in $3d$,

$$m_{\mathrm{f},0} \equiv m_{\mathrm{f}}(R = 0) = \Lambda_{\mathrm{UV}} \left(1 - \frac{2\pi}{\lambda_{\Lambda_{\mathrm{UV}}}}\right). \tag{7.28}$$

This is in perfect agreement with the known behavior in flat spacetime.

Provided the fermion system is initially weakly coupled, $\lambda_{\Lambda_{\mathrm{UV}}} \ll \lambda_{\mathrm{cr}} = 2\pi$, a leading order expansion can be performed for any value of the curvature, resulting in

$$m_{\mathrm{f}} \simeq \frac{\Lambda}{1 + \frac{48\pi\Lambda_{\mathrm{UV}}}{|R|\bar{\lambda}_{\Lambda}}}, \qquad \bar{\lambda}_{\Lambda_{\mathrm{UV}}}\Lambda_{\mathrm{UV}} \ll 1, \tag{7.29}$$

where we have reinserted the dimensionful initial coupling $\bar{\lambda}_{\Lambda_{\mathrm{UV}}} = \lambda_{\Lambda_{\mathrm{UV}}}/\Lambda_{\mathrm{UV}}$. If we additionally consider the limit of small curvature, we find a linear dependence of the fermion mass on both the curvature as well as the coupling,

$$m_{\mathrm{f}} \simeq \frac{1}{48\pi} |R| \, \bar{\lambda}_{\Lambda}. \tag{7.30}$$

By contrast, in the limit of large curvature, $|R|/(48\pi\Lambda_{\mathrm{UV}}^2) \gg \pi/\lambda_{\Lambda_{\mathrm{UV}}} \gg 1$, we find that $m_{\mathrm{f}} \to \Lambda_{\mathrm{UV}}$. In other words, large curvature induces immediate chiral symmetry breaking, such that the induced mass becomes of the order of the cutoff. Incidentally, this result is similar for the large-coupling limit: for $\lambda_{\Lambda_{\mathrm{UV}}} \gg 2\pi$, we again find that $m_{\mathrm{f}} \to \Lambda_{\mathrm{UV}}$ to leading order independently of the curvature.

The above results display explicit ultraviolet cutoff and regularization-scheme dependencies. Since the $3d$ Gross-Neveu model is asymptotically safe and thus non-perturbatively renormalizable [97], we can remove the ultraviolet cutoff by keeping an infrared obvservable fixed while sending $\Lambda_{\mathrm{UV}} \to \infty$. This "line of constant physics" defines a renormalized trajectory. Most conveniently this can be done in the super-critical regime where $\lambda_{\Lambda_{\mathrm{UV}}} > 2\pi$ such that the fermion mass in flat-space $m_{\mathrm{f},0}$ of equation (7.28) defines a natural infrared renormalization point.[42] In

[42] In the sub-critical regime, the model is quasi conformal and can be renormalized, e.g., by fixing the coupling λ_k at a suitable renormalization point $k = \mu$ to a specific value.

this case, the generated fermion mass in the limit $\Lambda_{\mathrm{UV}} \to \infty$ can be written as

$$\frac{m_{\mathrm{f}}}{m_{\mathrm{f},0}} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{|R|}{24 m_{\mathrm{f},0}^2}} \simeq 1 + \frac{|R|}{24 m_{\mathrm{f},0}^2}, \tag{7.31}$$

where the relation holds for arbitrary curvature, and the second relation represents a weak-curvature expansion being in perfect agreement with [105].

We emphasize that the fermions acquire a mass $m_{\mathrm{f}} > 0$ for any given $\lambda_{\Lambda_{\mathrm{UV}}}$ as long as the curvature is nonvanishing. While we expect the tendency to drive the fermion system towards the broken phase through gravitational catalysis to remain also beyond our truncation, fluctuations of bosonic composites entering beyond the large-N_{f} limit typically provide for an opposite tendency. Hence, the status of gravitational catalysis beyond mean-field remains an interesting question. Analogously, the effects of beyond-mean-field fluctuations on magnetic catalysis are under active current investigation [98, 301].

7.2.2 Negatively Curved Space

In the case of pure spatial curvature, the criterion (7.26) cannot be resolved analytically, but we can give an implicit equation for the induced fermion mass m_{f},

$$0 = \frac{2\pi}{\lambda_{\Lambda_{\mathrm{UV}}}} - \alpha_{\Lambda_{\mathrm{UV}}} \int\limits_{\alpha_{\Lambda_{\mathrm{UV}}}}^{\alpha_{m_{\mathrm{f}}}} \frac{\mathfrak{I}(\alpha)}{\alpha^2} \mathrm{d}\alpha, \qquad \alpha_k = \sqrt{\frac{|R|}{2k^2}}, \tag{7.32}$$

which can be solved numerically. Though the basic picture does not differ much from the maximally symmetric case, there are some interesting differences. As can already be inferred from the beta function in figure 7.2 (dotted lines), the curvature induced mass in the spatially curved case is smaller than in the maximally symmetric case, cf. figure 7.3. Several limits can be discussed in an analytic fashion, using the series representation of $\mathfrak{I}(\alpha)$ developed in appendix M. Furthermore, we need the integral $\mathfrak{F}(\alpha)$ defined by

$$\text{(i)} \quad \mathfrak{F}(\alpha) = \int \frac{\mathfrak{I}(\alpha)}{\alpha^2} \mathrm{d}\alpha, \quad \text{(ii)} \quad \lim_{\alpha \to \infty} \left(\mathfrak{F}(\alpha) - \frac{\ln \alpha}{\pi} \right) = 0, \tag{7.33}$$

where (ii) fixes the constant of integration. The explicit calculation is done in appendix M. The defining equation for the fermion mass in terms of $\mathfrak{F}$ is

$$\mathfrak{F}(\alpha_{m_{\mathrm{f}}}) = \mathfrak{F}(\alpha_{\Lambda_{\mathrm{UV}}}) + \frac{2\pi}{\lambda_{\Lambda_{\mathrm{UV}}} \alpha_{\Lambda_{\mathrm{UV}}}}. \tag{7.34}$$

This representation provides us with some interesting insight. First, we can show that there exists a unique solution to this equation with $0 < m_{\mathrm{f}} < \Lambda_{\mathrm{UV}}$ for any given negative curvature $|R| > 0$ and $\lambda_{\Lambda_{\mathrm{UV}}} > 0$. This can be seen in two steps. The uniqueness is because the function

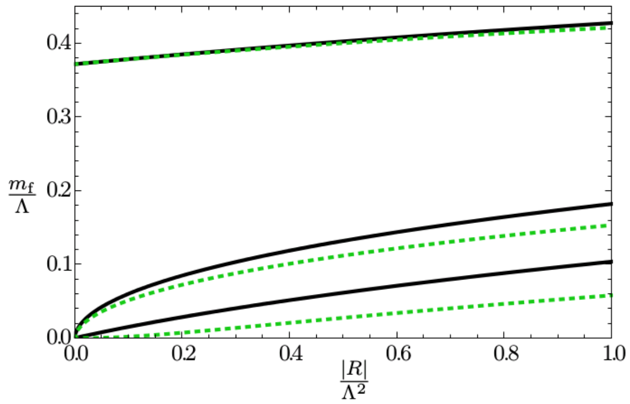


Figure 7.3: Gravitationally catalyzed fermion masses (mean-field level) as a function of negative curvature in units of the UV cutoff. The solid black lines display the maximally symmetric case, whereas the purely spatially curved case is shown as green dotted lines. The sets of three different lines correspond to super-critical, critical, and sub-critical bare fermion couplings, $\lambda_{\Lambda_{\rm UV}} \simeq 1.6\lambda_{\rm cr}, \lambda_{\rm cr}, 0.8\lambda_{\rm cr}$ from top to bottom ($\lambda_{\rm cr} = 2\pi$). As long as the background manifold is negatively curved, $|R| > 0$, a finite fermion mass is generated.

$\mathfrak{F}(\alpha) \in (-\infty, \infty)$ for $\alpha \in (0, \infty)$ is bijective. This property holds, because $\mathfrak{I}(\alpha)$ is positive and therefore $\mathfrak{F}$ is strictly monotonically increasing. Owing to $\mathfrak{I}(\alpha)/\alpha^2 \to 1/\alpha^2$ for small $\alpha \to 0$ and $\mathfrak{I}(\alpha)/\alpha^2 \to 1/(\pi\alpha)$ for large $\alpha \to \infty$, the function $\mathfrak{F}$ is not bounded. Since by assumption $2\pi/(\lambda_{\Lambda_{\rm UV}}\alpha_{\Lambda_{\rm UV}}) > 0$, it follows from equation (7.34) that $\mathfrak{F}(\alpha_{m_{\rm f}}) > \mathfrak{F}(\alpha_{\Lambda_{\rm UV}})$ has to hold, which implies that $\alpha_{m_{\rm f}} > \alpha_{\Lambda_{\rm UV}}$ because of the monotonic behavior. This demonstrates that $0 < m_{\rm f} < \Lambda_{\rm UV}$ as claimed above.

Let us first check the flat spacetime limit $|R| \to 0$. In complete agreement with equation (7.28), we find again that

$$\frac{2\pi}{\lambda_{\Lambda_{\rm UV}}} = \lim_{|R|\to 0} \left[\alpha_{\Lambda_{\rm UV}}\mathfrak{F}(\alpha_{m_{\rm f}}) - \alpha_{\Lambda}\mathfrak{F}(\alpha_{\Lambda_{\rm UV}})\right] = 1 - \frac{m_{{\rm f},0}}{\Lambda_{\rm UV}}, \tag{7.35}$$

where we have used equation (M.9).

In the weak coupling regime, we expect the fermion mass to be small compared to the curvature scale, cf. equation (7.30). Hence, we need $\mathfrak{F}(\alpha)$ for large argument as provided by equation (M.10), $\mathfrak{F}(\alpha \gg 1) \simeq \frac{\ln\alpha}{\pi}$. Using this asymptotic behavior for $\mathfrak{F}(\alpha_{m_{\rm f}})$, we infer from equation (7.34) that

$$m_{\rm f} \simeq \sqrt{\frac{|R|}{2}} \exp\left[-\frac{\pi}{\alpha_{\Lambda_{\rm UV}}}\left(\frac{2\pi}{\lambda_{\Lambda_{\rm UV}}} + \alpha_{\Lambda_{\rm UV}}\mathfrak{F}(\alpha_{\Lambda_{\rm UV}})\right)\right], \qquad \alpha_{\Lambda_{\rm UV}} = \sqrt{\frac{|R|}{2\Lambda_{\rm UV}^2}}. \tag{7.36}$$

If in addition, the curvature is small, i.e. $\alpha_{\Lambda_{\rm UV}} \ll 1$, we can use the corresponding expansion of equation (M.9) and replace $\alpha_{\Lambda_{\rm UV}}\mathfrak{F}(\alpha_{\Lambda_{\rm UV}}) \to -1$. Taking differences arising from the Dirac representation into account, the exponential inverse-coupling dependence is in perfect agreement with the results of [113].

Despite the overall similarities to the maximally symmetric case, we observe that the weak-coupling and weak-curvature limit of the case with pure spatial curvature shows some distinct differences. In particular, there is an exponential non-analytic dependence of the fermion mass on the coupling as well as on the curvature.

This difference is reminiscent of magnetic catalysis in $d = 2+1$ and $d = 3+1$ [72, 73], where the fermion gap is analytic in $d = 2+1$, but shows an essential singularity in the coupling in $d = 3+1$. Also in the present case, such a singularity shows up similar to BCS gap formation, as a consequence of the effective dimensional reduction of the fermionic fluctuation spectrum to $d \to 1+1$ [117, 118]. Also in this respect, our functional RG picture is in perfect agreement with previous studies [113, 119].

7.3 Pseudo-Critical Coupling and Probe Size

At zero curvature, the $3d$ Gross-Neveu model exhibits a critical coupling strength corresponding to a quantum critical point above which chiral symmetry is broken at large length scales. This critical coupling manifests itself as a non-Gaußian fixed point of the RG flow. In the present regularization scheme, we identified $\bar{\lambda}_{\mathrm{cr}} = 2\pi/\Lambda_{\mathrm{UV}}$, or $\lambda_{\mathrm{cr}} = 2\pi$ in dimensionless conventions. As illustrated in figure 7.2, the fixed point strictly speaking no longer exists at finite negative curvature. The nontrivial zero of the β_λ function becomes scale dependent and eventually merges with the Gaußian fixed point in the deep infrared for $k \to 0$, such that only the chirally broken branch of the β_λ function remains.

Let us call the nontrivial zero of β_λ a pseudo-critical coupling λ_{p}. We find

$$\lambda_{\mathrm{p}} = \frac{\lambda_{\mathrm{cr}}}{1 + \frac{|R|}{24k^2}} \tag{7.37}$$

for the maximally symmetric case, cf. equation (7.14), and

$$\lambda_{\mathrm{p}} = \frac{\lambda_{\mathrm{cr}}}{\mathfrak{I}\left(\sqrt{\frac{|R|}{2k^2}}\right)} \tag{7.38}$$

for the spatially curved case, cf. equation (7.24). Since $\lambda_{\mathrm{p}} = \lambda_{\mathrm{p}}(k)$ is a monotonically decreasing function of scale k, the coupling λ_k can eventually exceed λ_{p} such that the system becomes critical and runs towards the ordered phase. In this sense,

$$\lambda_{k_{\mathrm{c}}} = \lambda_{\mathrm{p}}(k_{\mathrm{c}}) \tag{7.39}$$

can be viewed as a criticality condition [302], defining a scale k_{c}, where the system becomes critical. For lower scale, the system is driven towards the symmetry broken regime which is ultimately entered at $k_{\mathrm{SB}} < k_{\mathrm{c}}$ defined above. The value of k_{c} depends on the curvature as well as the initial coupling $\lambda_{\Lambda_{\mathrm{UV}}}$ (the latter is considered as initially subcritical here and in the following).

The preceding discussion implicitly assumed that k can run over all scales down to $k = 0$, such that the criticality condition (7.39) can eventually always be satisfied. However, if, for instance, the system has a finite volume characterized by a finite length scale L, also the fluctuation momenta are restricted, typically leading to an infrared cutoff $k_L = \pi/L$.[43] One may think of a finite probe length, such as, e.g., the size of a layer of graphene. This finite probe length L can lead to a screening of the gravitationally catalyzed ordered regime if k_L is larger than the would-be critical scale k_c. Hence, $\lambda_\mathrm{p}(k_L)$ can be viewed as a lower bound for the coupling strength required for symmetry breaking in a real system of finite length. It thus generalizes the critical coupling at infinite volume and zero curvature to the situation of finite volume and finite curvature.

It is instructive to study λ_p as a function of the length scale L measured in units of a typical curvature length scale which we define by $r = 1/\sqrt{|R|}$. For finite systems, this gives an estimate for how strongly a probe has to be curved in order to exhibit gravitational catalysis. In turn, for a given curvature of the probe, λ_p provides an estimate for the initial coupling strength required for symmetry breaking in the finite system. For instance, for the maximally symmetric case, we read off from equation (7.37) that

$$\lambda_\mathrm{p}(k_L = \pi/L) = \frac{\lambda_\mathrm{cr}}{1 + \frac{L^2}{24\pi^2 r^2}}. \tag{7.40}$$

For the spatially curved case, we incidentally find the same result in the limit of small curvature, i.e. $L/r \ll 1$, using the expansion (M.3) which is accurate even for values of $L/r \simeq \mathcal{O}(1)$. By contrast, the large curvature limit for the spatially curved case is different, cf. equation (M.7):

$$\lambda_\mathrm{p}(k_L = \pi/L) = \sqrt{2}\pi^2 \frac{r}{L}\lambda_\mathrm{cr}, \qquad\qquad \frac{r}{L} \ll 1. \tag{7.41}$$

From equation (7.40), it is obvious that probe length to curvature ratios up to $L/r \simeq \mathcal{O}(1)$ lead to pseudo-critical couplings λ_p which deviate from the zero-curvature critical coupling of the Gross-Neveu model only below the 1% level. Significant deviations only occur for L being an order of magnitude larger than the curvature scale r. From the viewpoint of curvature-deformed condensed matter systems, a ratio of $L/r \simeq \mathcal{O}(1)$ appears to be "large" in the sense that – loosely speaking – a spatially curved $2d$ planar probe embedded in $3d$ Euclidean space would rather look like a $3d$ object.

Another way to interpret these results is the following: consider a finite-probe system with a subcritical bare coupling, $\lambda_{\Lambda_\mathrm{UV}} < \lambda_\mathrm{cr}$, thus being in the symmetric (e.g. semimetal) phase. In order to gravitationally catalyze a transition to a broken (gapped or insulating) phase, the criticality condition (7.39) has to be met for a sufficiently large $k_\mathrm{c} > k_L$. In view of equation (7.40) this requires comparatively strong curvature, i.e. a small curvature length scale r compared with the probe length L.

[43] Here, we tacitly assume that the boundary conditions are such that zero modes do not occur.

8 Conclusion

In this thesis we have explored a first-principles approach to a local spin-base invariant formulation of fermions in curved spacetimes. We have demonstrated how the concept of spin-base invariance arises naturally from completely standard considerations. Whereas general covariance and spin-base invariance seem hardwired to each other via the Clifford algebra, we have stressed in this work that the associated symmetry transformations can be used fully independent of each other. We have made this explicit by the construction of a global spin-base on the 2-sphere, which does not permit an equally globally well-defined choice of space coordinates. In other words, the spatial coordinate patches, required to cover a manifold, do not have to be in one-to-one correspondence with the spin-base patches that cover the spinor space at all points of the manifold. We consider this mutual independence of general covariance and spin-base invariance to be an indication for the fact that the spatial geometric structure should not be viewed as more fundamental than the spin structure or vice versa. Both symmetries should therefore be a direct or emergent property of a more fundamental theory for matter and gravity.

We have pointed out the hidden (local) $\mathrm{SL}(d_\gamma, \mathbb{C})$ spin-base invariance in the vielbein formulation. Thereby we have shown how the vielbein formulation artificially splits the full Dirac matrices into a vielbein and flat Dirac matrices. It is obvious that every manifold that admits a global vielbein also admits global Dirac matrices, but as we have revealed the converse is not true. The 2-sphere serves as a simple example for how our approach generalizes the usual vielbein formalism. We stress that the vielbein formalism, if applicable, is always a special choice of Dirac matrices and therefore completely covered by our approach, i.e. the conventional vielbein formalism can be viewed as "spin-base gauge-fixed" version of the invariant formalism. Furthermore, we have constructed all relevant quantities for the description of fermions in curved spacetimes from the Dirac matrices. While such a spin-base invariant formalism already has been discussed and successfully used at several instances in the literature, our presentation carefully distinguishes between assumptions and consequences, paving the way to generalizations and possibly quantization.

One such generalization is the inclusion of torsion. In addition to spacetime torsion, which can be included rather straightforwardly in the formalism, the spin connection admits further degrees of freedom which we interpret as spin torsion. The name spin torsion is motivated by the fact that this part of the spin connection transforms homogeneously under coordinate transformations and therefore cannot be transformed away locally by adjusting the coordinates. Similarly to spacetime torsion we can impose conditions like metric compatibility for the spin torsion leading to some algebraic constraints. These constraints have been resolved completely, so that we have been able to count the actual degrees of freedom of spin torsion. Some of these degrees of freedom are similar to the torsion part of the vielbein spin connection. However, in the general case the vielbein spin connection induces an imaginary part in the fermion kinetic term, whereas our spin covariant derivative leaves it purely real.

Further generalizations include the construction of spin curvature which can be used to define classical field theories of gravity (and fermions) in terms of the Dirac matrices (and Dirac spinors) as elementary degrees of freedom. We showed that the simplest possible field theory contains Einstein's theory of general relativity and predicts zero spin-torsion in the absence of explicit sources or boundary conditions.

We further have found that the analog of the commonly used Lorentz symmetric gauge in terms of the Dirac matrices is in fact the simplest possible choice of Dirac matrices for a given background metric and metric fluctuation. This explains why this gauge choice is so useful for explicit calculations. Furthermore, we have presented a possibility for an explicit gauge fixing of the Dirac matrices for two general metrics, which do not have to be linearly connected.

In addition, we used the formalism to show that a possible path integral quantization of gravity and fermionic matter fields can be solely based on an integration over metric and matter fluctuations. Despite the fact that the Dirac matrices appear to be the more fundamental degrees of freedom, their fluctuations can be parametrized by metric as well as spin-base fluctuations. We observe that the latter do not contribute to spin-base invariant observables and hence the spin-base fluctuations can be factored out of the quantum theory. Hence, a quantization of the spin-base degrees of freedom is not necessary, though certainly possible and legitimate. In view of the increasing complexity of quantization schemes based on vielbeins and/or spin connections [9, 10], the legitimation of a metric-based scheme (though still an open and frighteningly hard challenge) is good news.

This includes, for instance, the asymptotic safety scenario [6, 7, 62, 205, 206] for metric quantum Einstein gravity [7]. There we have reexamined generalized parametrization dependencies of non-perturbative computations based on the functional renormalization group. Whereas parametrically-ordered expansion schemes such as perturbation theory for on-shell quantities are free from such dependencies, off-shell quantities and non-perturbative expansions rather generically exhibit dependencies on, e.g., the choice of the regularization, the gauge fixing or the field parametrization. In the second part of this thesis, we have dealt with these dependencies in a pragmatic manner, analyzing the sensitivity and stability of the UV behavior of metric quantum gravity with respect to variations of such generalized parametrizations. We have focused on the question of the existence and the properties of a non-Gaußian UV fixed point, facilitating metric quantum gravity to be asymptotically safe. We have also concentrated on a widely studied and rather well-understood computing scheme, the Einstein-Hilbert truncation in the single-metric formulation.

Our results show a remarkable stability in a variety of qualitative aspects: for all parametrizations that exhibit rather large stationary regimes in the space of all parameters, we have found a non-Gaußian ultraviolet fixed point with two renormalization group relevant directions, corresponding to the Newton coupling and the cosmological constant being physical parameters. For most parametrizations, the universal quantities show a remarkably mild (given the simplicity of the approximation) variation and thus a high degree of stability. Our scan of parametrization dependencies can also help identifying less robust parametrizations, and thus help judging the physical relevance of results.

Some features, however, appear to depend more strongly on the parametrization or are even visible only in specific parametrizations. Moreover, a nontrivial interplay between various aspects of parametrizations, e.g., gauge choice vs. field parametrization, can arise. With hindsight, the results obtained within the exponential split with field redefinition in the limit where the graviton degrees of freedom are spanned by the transverse traceless and a scalar mode ($|\beta| \to \infty$) exhibit the highest degree of comprehensiveness: complete independence of the gauge parameter α, fully analytical and integrable global flows with a classical infrared limit in the physical parameter regime, real critical exponents at the ultraviolet fixed point, and the existence of an upper critical dimension for the asymptotic safety scenario. The exploration of higher-order truncations [218] and the inclusion of matter degrees of freedom [8, 37, 38] in this parametrization appears highly worthwhile, c.f. [153, 193] for scalar matter.

In the final chapter we have investigated the phenomenon of gravitational catalysis in the $3d$ Gross-Neveu model on specific manifolds with constant negative curvature. While the mechanism had already been studied frequently with mean-field methods as well as from the viewpoint of the fluctuation spectrum of the Dirac operator, we have added a new renormalization group picture to the comprehensive understanding of this phenomenon. The essence of this picture is that the critical coupling of the fermionic system, corresponding to a quantum phase transition in flat spacetime, is transmuted into a scale-dependent pseudo-critical coupling that flows to zero as a consequence of long-wavelength fluctuations (compared to the curvature scale). In this manner, the infinite-volume fermion system becomes critical for any arbitrarily weak coupling.

We have identified the renormalization group (pseudo-) fixed point mechanism for two example manifolds of constant negative curvature: a maximally symmetric spacetime (Anti de Sitter) and a purely spatially curved case (Lobachevsky plane). Both manifolds support the mechanism of gravitational catalysis, but exhibit a rather different behavior as far as the dependence of chiral symmetry breaking on the coupling and the curvature are concerned. The maximally symmetric case shows a linear dependence (to leading order) on both quantities which makes clear that the phenomenon is essentially perturbative, being reminiscent of the "quantum anomaly" for fermions in a magnetic field [303]. By contrast, order parameters indicating the symmetry broken state such as the induced fermion mass exhibit an essential singularity in both the coupling and the curvature for the case of the purely spatially curved case. This is in many respects similar to BCS-type gap formation. Again, our renormalization group picture goes hand in hand here with properties of the fermionic fluctuation spectrum, as analyzed in [117, 118].

As a benefit, the functional renormalization group also gives simple access to systems of finite extent by identifying the renormalization group infrared cutoff with an inverse length scale. In this manner, we can estimate the fate of gravitational catalysis in finite systems. In fact, the phenomenon only occurs as long as the curvature radius is sufficiently small compared to the systems length scale. We have been able to phrase this statement quantitatively by introducing a pseudo-critical coupling. Thinking in terms of curved layered condensed matter systems, rather large curvatures are needed compared to a realizable probe length in order to drive a sub-critical system into a phase dominated by gravitational catalysis.

We would like to emphasize that an immediate application of our results to condensed matter systems would only be possible for reparametrization invariant systems such as fluid membranes [304], curvature effects of which can be mapped onto the language of Riemannian geometry. For tethered membranes or general lattice systems, further phenomena, connected to extrinsic curvature or curvature related defects, can become relevant. In this context, it is interesting to note that an external strain exerted on a graphene sheet in flat space induces a pseudo-magnetic field [305–307], that may also support (pseudo-)magnetic catalysis. For such systems, we hence expect an interesting interplay between these various effects if we expose them to negative curvature inducing strain.

Finally, gravitational catalysis may become relevant in the context of asymptotically safe quantum gravity. In conjunction with fermionic degrees of freedom [8–10], the ultraviolet fixed point determining the shape of the universe at highest energies might go along with a negative (though scale-dependent) curvature [308]. Whether or not gravitational catalysis in connection with gravitionally modified critical fermion interactions [37] could become active and impose constraints on the matter content of the universe then is a highly involved question that deserves to be investigated in greater depth.

A Weldon Theorem in Arbitrary Integer Dimensions

An essential ingredient for our investigations is the Weldon theorem [47, 58]. It states that the most general infinitesimal variation of the Dirac matrices compatible with the Clifford algebra can be written as

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu], \qquad \operatorname{tr}\delta\mathcal{S}_\gamma = 0, \tag{A.1}$$

where $\delta g_{\mu\nu}$ is the infinitesimal variation of the metric and $\delta\mathcal{S}_\gamma \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ parametrizes an arbitrary infinitesimal similarity transformation. With $\mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ we denote the $d_\gamma \times d_\gamma$ matrices. Especially there is a one-to-one mapping between $\delta\gamma_\mu$ on the one hand and $\delta g_{\mu\nu}$ and $\delta\mathcal{S}_\gamma$ on the other hand. With this theorem we can proof the existence of $\hat{\Gamma}_\mu$ in appendix E, give a way of parametrizing all possible Dirac matrices and perform derivatives of the γ_μ with respect to the metric.

Weldon has proven this theorem in $d = 4$ spacetime dimensions. We give a general proof for arbitrary integer dimensions $d \geq 2$. Starting with the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathrm{I}, \tag{A.2}$$

we perform an infinitesimal variation and arrive at

$$\{\gamma_\mu + \delta\gamma_\mu, \gamma_\nu + \delta\gamma_\nu\} = 2(g_{\mu\nu} + \delta g_{\mu\nu})\mathrm{I}. \tag{A.3}$$

Now instead of solving this equation in general in one step, we start with one special solution namely

$$(\delta\gamma_\mu)_{\mathrm{special}} = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu. \tag{A.4}$$

This solution solves equation (A.3) not exactly but only to the first order in $\delta g_{\mu\nu}$, which is of course sufficient since we are only interested in infinitesimal variations. Now we employ the well known theorem that every solution to the Clifford algebra to a given metric is connected to each other via a similarity transformation, and in odd dimensions via a sign flip (if necessary) [129]. Since we only deal with infinitesimal variations, we cannot leave the connected component. This excludes the sign flip also in odd dimensions. Therefore the most general solution $\delta\gamma_\mu$ must be connected to $(\delta\gamma_\mu)_{\mathrm{special}}$ via a similarity transformation and actually this transformation has to be an infinitesimal one $\mathrm{e}^{\delta\mathcal{S}_\gamma} \simeq \mathrm{I} + \delta\mathcal{S}_\gamma$

$$\gamma_\mu + \delta\gamma_\mu \stackrel{!}{=} \mathrm{e}^{\delta\mathcal{S}_\gamma}\big(\gamma_\mu + (\delta\gamma_\mu)_{\mathrm{special}}\big)\mathrm{e}^{-\delta\mathcal{S}_\gamma}. \tag{A.5}$$

By expanding this equation we can read off

$$\delta\gamma_\mu = (\delta\gamma_\mu)_{\text{special}} + [\delta\mathcal{S}_\gamma, \gamma_\mu] = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu]. \tag{A.6}$$

Since the trace part completely drops out of the commutator it is sufficient to restrict $\delta\mathcal{S}_\gamma$ to be traceless. The last relation proves that we can decompose every Dirac matrix fluctuation compatible with the Clifford algebra as in equation (A.1).

We still have to proof the uniqueness of $\delta g_{\mu\nu}$ and $\delta\mathcal{S}_\gamma$ for a given $\delta\gamma_\mu$, where we impose that any metric fluctuation has to be symmetric $\delta g_{\mu\nu} = \delta g_{\nu\mu}$ and that any spin-base fluctuation has to be traceless $\operatorname{tr}\delta\mathcal{S}_\gamma = 0$. Now let us suppose we have two sets of compatible metric fluctuations and spin-base fluctuations $\delta g_{\mu\nu}, \delta\mathcal{S}_\gamma$ (unprimed decomposition) and $\delta g'_{\mu\nu}, \delta\mathcal{S}'_\gamma$ (primed decomposition) for a given $\delta\gamma_\mu$,

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu] = \frac{1}{2}(\delta g'_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}'_\gamma, \gamma_\mu]. \tag{A.7}$$

By calculating the trace of $\gamma_\mu\delta\gamma_\nu + \gamma_\nu\delta\gamma_\mu$ first in the unprimed decomposition

$$\frac{1}{d_\gamma}\operatorname{tr}(\gamma_\mu\delta\gamma_\nu + \gamma_\nu\delta\gamma_\mu) = \delta g_{\mu\nu} \tag{A.8}$$

and then again in the primed decomposition

$$\frac{1}{d_\gamma}\operatorname{tr}(\gamma_\mu\delta\gamma_\nu + \gamma_\nu\delta\gamma_\mu) = \delta g'_{\mu\nu} \tag{A.9}$$

we find that the two metric fluctuations have to be equal $\delta g_{\mu\nu} = \delta g'_{\mu\nu}$. From here it is obvious that $[\delta\mathcal{S}_\gamma, \gamma_\mu] = [\delta\mathcal{S}'_\gamma, \gamma_\mu]$, implying that

$$\big[[\delta\mathcal{S}_\gamma, \gamma_\mu], \gamma^\mu\big] = \big[[\delta\mathcal{S}'_\gamma, \gamma_\mu], \gamma^\mu\big]. \tag{A.10}$$

Now we can use that the $\gamma^{\mu_1\cdots\mu_n}$ in even dimensions or respectively the $\gamma^{\mu_1\cdots\mu_{2n}}$ in odd dimensions form a basis in $\mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ [129].[44] Next we observe that the contracted commutator of Dirac matrices $\big[[\cdot, \gamma_\mu], \gamma^\mu\big]$ does not mix the base elements, and only eliminates the part proportional to the identity, cf. equation (C.5) from appendix C. Hence the two matrices $\delta\mathcal{S}_\gamma$ and $\delta\mathcal{S}'_\gamma$ are equal up to a trace term. Since we know that they are traceless they have to be equal, $\delta\mathcal{S}_\gamma = \delta\mathcal{S}'_\gamma$. This proves the uniqueness of $\delta g_{\mu\nu}$ and $\delta\mathcal{S}_\gamma$.

[44] The $\gamma_{\mu_1\ldots\mu_n}$ are the normalized and antisymmetrized combinations of the Dirac matrices, cf. equation (3.15).

B Minimal Spin Base Group

In this section we will show, that $\mathrm{SL}(d_\gamma, \mathbb{C})$ is the unique group[45] $\mathrm{SB}_{\mathrm{min}} \leq \mathrm{GL}(d_\gamma, \mathbb{C})$ satisfying

(i) $\forall \gamma_\mu, \gamma'_\mu$ compatible with the Clifford algebra for the metric $g_{\mu\nu}$

$$\exists \mathcal{S} \in \mathrm{SB}_{\mathrm{min}} : \gamma'_\mu = \begin{cases} \mathcal{S}\gamma_\mu\mathcal{S}^{-1} & , d \text{ even}, \\ \pm\mathcal{S}\gamma_\mu\mathcal{S}^{-1} & , d \text{ odd}, \end{cases} \tag{B.1}$$

(ii) $\forall \gamma_\mu$ compatible with the Clifford algebra, it holds

$$\left|\{\mathcal{S} \in \mathrm{SB}_{\mathrm{min}} : \mathcal{S}\gamma_\mu\mathcal{S}^{-1} = \gamma_\mu\}\right| = \min_{\substack{\mathrm{SB}_{\mathrm{test}} \leq \mathrm{GL}(d_\gamma, \mathbb{C}) \\ \text{compatible with (i)}}} \left|\{\mathcal{S} \in \mathrm{SB}_{\mathrm{test}} : \mathcal{S}\gamma_\mu\mathcal{S}^{-1} = \gamma_\mu\}\right|, \tag{B.2}$$

where we denote the cardinality of a set $\mathfrak{S}$ with $|\mathfrak{S}|$.

The existence of a group satisfying (i) is guaranteed by the Clifford algebra and is independent of the metric [129]. In addition, condition (ii) is independent of the actual choice of the Dirac matrices, i.e. if it is satisfied for a specific set γ_μ compatible with the Clifford algebra, then it is satisfied for any. This follows from Schur's lemma[46] and $\mathcal{S}\gamma_\mu\mathcal{S}^{-1} = \gamma_\mu \Leftrightarrow \mathcal{S} = \frac{1}{d_\gamma}(\operatorname{tr}\mathcal{S}) \cdot \mathrm{I}$ for $\mathcal{S} \in \mathrm{GL}(d_\gamma, \mathbb{C})$.

Now let us construct the group $\mathrm{SB}_{\mathrm{min}}$. We start by observing that every element of $\mathrm{GL}(d_\gamma, \mathbb{C})$ can be written as e^M for some $M \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$. Next we can split M into its trace part $\frac{1}{d_\gamma}(\operatorname{tr} M) \cdot \mathrm{I}$ and the traceless part $\hat{M} = M - \frac{1}{d_\gamma}(\operatorname{tr} M) \cdot \mathrm{I}$. Since the trace part is proportional to the identity matrix it commutes with every element of $\mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$. Therefore the trace part is trivial for the similarity transformations. By the use of Jacobi's formula we find $\det \mathrm{e}^{\hat{M}} = 1$, leading us to $\mathrm{SB}_{\mathrm{min}} \leq \mathrm{SL}(d_\gamma, \mathbb{C})$.

If we calculate the set of trivial elements (condition (ii)) for $\mathrm{SL}(d_\gamma, \mathbb{C})$, we find

$$\{\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C}) : \mathcal{S}\gamma_\mu\mathcal{S}^{-1} = \gamma_\mu\} = \mathrm{Cen}\left(\mathrm{SL}(d_\gamma, \mathbb{C})\right), \tag{B.3}$$

where $\mathrm{Cen}\left(\mathrm{SL}(d_\gamma, \mathbb{C})\right) = \left\{\mathrm{e}^{\mathrm{i}\frac{2\pi}{d_\gamma}n} \cdot \mathrm{I} : n \in \{0, \dots, d_\gamma - 1\}\right\}$ is the center of $\mathrm{SL}(d_\gamma, \mathbb{C})$ and has finite cardinality $\left|\mathrm{Cen}\left(\mathrm{SL}(d_\gamma, \mathbb{C})\right)\right| = d_\gamma$.

In order to determine which elements of $\mathrm{SL}(d_\gamma, \mathbb{C})$ we definitely need, we use condition (i). Let us consider two different transformations $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{SL}(d_\gamma, \mathbb{C})$ connecting a given pair γ_μ, γ'_μ compatible with the Clifford algebra. It turns out that they have to be related by a center

[45] In fact we are dealing with the fundamental representation of $\mathrm{SL}(d_\gamma, \mathbb{C})$ and not the group itself. But we will keep this terminology in the following for simplicity. By fundamental representation we mean the defining matrix representation of $\mathrm{SL}(d_\gamma, \mathbb{C})$, which is $\{\mathcal{S} \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C}) : \det \mathcal{S} = 1\}$ together with the matrix multiplication as the group law.

[46] Schur's lemma basically says that a matrix $M \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ which commutes with every base element is proportional to the identity matrix. Since we can construct a basis in $\mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ from the $\gamma_{\mu_1 \dots \mu_n}$ it suffices that M commutes with the γ_μ, as it then obviously also commutes with the $\gamma_{\mu_1 \dots \mu_n}$.

element

$$\mathcal{S}_1\gamma_\mu\mathcal{S}_1^{-1} = \gamma'_\mu = \mathcal{S}_2\gamma_\mu\mathcal{S}_2^{-1} \Rightarrow [\mathcal{S}_2^{-1}\mathcal{S}_1, \gamma_\mu] = 0. \tag{B.4}$$

With this observation we can define an equivalence relation $\sim$, $\mathcal{S}_1 \sim \mathcal{S}_2$ iff $\exists \mathcal{C} \in \mathrm{Cen}\big(\mathrm{SL}(d_\gamma, \mathbb{C})\big)$ so that $\mathcal{S}_1 = \mathcal{C}\mathcal{S}_2$. For a given γ_μ every equivalence class generates a different γ'_μ, and we already know that all γ'_μ are generated in this way.

In the next step we will show that the center gets generated by a specific equivalence class. Since we need at least one representative of each equivalence class the whole $\mathrm{SL}(d_\gamma, \mathbb{C})$ gets generated as well by applying the generated center elements to the representatives of the equivalence classes. Let us define the matrix M as

$$M = \mathrm{i}\frac{2\pi}{d_\gamma}\mathrm{I} - \mathrm{i}2\pi A, \tag{B.5}$$

where A can be any matrix satisfying

$$A^2 = A, \quad \mathrm{tr}\, A = 1. \tag{B.6}$$

One such A is the matrix with a 1 in the upper left corner and 0 everywhere else. The matrix M is by construction traceless and satisfies

$$\mathrm{e}^M = \mathrm{e}^{\mathrm{i}\frac{2\pi}{d_\gamma}} \cdot \mathrm{I}, \tag{B.7}$$

i.e. it generates the center of $\mathrm{SL}(d_\gamma, \mathbb{C})$ and belongs to the equivalence class of the identity element. This relation can be verified by observing

$$\mathrm{e}^{a\cdot A} = \mathrm{I} + \sum_{n=1}^{\infty}\frac{a^n}{n!}A = \mathrm{I} + (\mathrm{e}^a - 1)A, \quad a \in \mathbb{C}. \tag{B.8}$$

Next we calculate

$$\mathrm{e}^{\frac{1}{d_\gamma}M} = \mathrm{e}^{\mathrm{i}\frac{2\pi}{d_\gamma^2}}\mathrm{I} + \mathrm{e}^{\mathrm{i}\frac{2\pi}{d_\gamma^2}}(\mathrm{e}^{-\mathrm{i}\frac{2\pi}{d_\gamma}} - 1)A. \tag{B.9}$$

The determinant of this matrix is equal to 1 and additionally it is not proportional to the identity matrix and therefore is a nontrivial spin-base transformation, i.e. it belongs to a nontrivial equivalence class. Hence, there has to be at least one $n \in \{0, \ldots, d_\gamma - 1\}$, so that $\mathrm{e}^{\mathrm{i}\frac{2\pi}{d_\gamma}n}\mathrm{e}^{\frac{1}{d_\gamma}M} \in \mathrm{SB}_{\mathrm{min}}$. Because $\mathrm{SB}_{\mathrm{min}}$ is supposed to be a group, it has to be closed under the group law. This implies that also $(\mathrm{e}^{\mathrm{i}\frac{2\pi}{d_\gamma}n}\mathrm{e}^{\frac{1}{d_\gamma}M})^{d_\gamma}$ has to be an element of $\mathrm{SB}_{\mathrm{min}}$, as d_γ is an integer. By calculating

$$(\mathrm{e}^{\mathrm{i}\frac{2\pi}{d_\gamma}n}\mathrm{e}^{\frac{1}{d_\gamma}M})^{d_\gamma} = \mathrm{e}^M = \mathrm{e}^{\mathrm{i}\frac{2\pi}{d_\gamma}} \cdot \mathrm{I} \tag{B.10}$$

we see that $e^{i\frac{2\pi}{d_\gamma}} \cdot \mathrm{I} \in \mathrm{SB}_{\min}$, implying that the whole center gets generated. Therefore we have $\mathrm{SL}(d_\gamma, \mathbb{C}) \leq \mathrm{SB}_{\min}$. With this finding we conclude that $\mathrm{SB}_{\min} = \mathrm{SL}(d_\gamma, \mathbb{C})$.

C Special Relations for the Dirac Matrices – Part I

In order to proof the uniqueness of $\hat{\Gamma}_\mu$ and give the explicit expressions (4.17) and (4.18) we will need some identities for the Dirac matrices. Let us introduce the shorthands

$$(A^n_m)^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m} = \frac{1}{d_\gamma}\,\mathrm{tr}(\gamma^{\mu_1\cdots\mu_n}\gamma_{\nu_1\dots\nu_m}), \qquad (A^{n,k}_{m,r})^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m} = (A^{n-1}_{m-1})^{\mu_1\dots\mu_{k-1}\mu_{k+1}\dots\mu_n}_{\nu_1\dots\nu_{r-1}\nu_{r+1}\dots\nu_m}. \tag{C.1}$$

Some of these (A^n_m) are easily calculated directly from their definition (C.1)

$$d \geq 2: \quad (A^0_0) = 1, \qquad (A^1_1)^\mu_\nu = \delta^\mu_\nu; \tag{C.2}$$

$$d \text{ even}: \quad (A^n_0)^{\mu_1\dots\mu_n} = 0,\ n > 0; \qquad (A^n_1)^{\mu_1\dots\mu_n}_\nu = 0,\ n > 1; \tag{C.3}$$

$$d \text{ odd}: \quad (A^{2n}_0)^{\mu_1\dots\mu_{2n}} = 0,\ n > 0; \quad (A^{2n}_2)^{\mu_1\dots\mu_{2n}}_{\nu_1\nu_2} = 0,\ n > 1; \quad (A^{2n-1}_1)^{\mu_1\dots\mu_{2n-1}}_\nu = 0,\ n > 1. \tag{C.4}$$

The identities to be proven are

$$\big[[\gamma^{\mu_1\cdots\mu_n}, \gamma^\nu], \gamma_\nu\big] = 2\big((1-(-1)^n)d + (-1)^n 2n\big)\gamma^{\mu_1\cdots\mu_n}, \tag{C.5}$$

and the traces of the base elements for even dimensions with $n, m \in \{1, \dots, d\}$

$$(A^n_m)^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m} = (-1)^{\frac{n(n-1)}{2}}\ n!\ \delta^n_m \delta_{\mathrm{A}}{}^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n} \tag{C.6}$$

as well as the traces of the base elements for odd dimensions with $n, m \in \{0, \dots, \frac{d-1}{2}\}$

$$(A^{2n}_{2m})^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2m}} = (-1)^n\ (2n)!\ \delta^n_m \delta_{\mathrm{A}}{}^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2n}}, \quad (A^{2n+1}_{2m+1})^{\mu_1\dots\mu_{2n+1}}_{\nu_1\dots\nu_{2m+1}} = (-1)^n\ (2n+1)!\ \delta^n_m \delta_{\mathrm{A}}{}^{\mu_1\dots\mu_{2n+1}}_{\nu_1\dots\nu_{2n+1}}. \tag{C.7}$$

As a first step we rewrite the Clifford algebra as

$$\gamma_\nu\gamma^\mu = -\gamma^\mu\gamma_\nu + 2\delta^\mu_\nu \mathrm{I} \tag{C.8}$$

to find that for $n \in \mathbb{N}^*$[47]

$$\gamma_\nu\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma^\nu = -\gamma^{\mu_1}\gamma_\nu\gamma^{\mu_2}\dots\gamma^{\mu_n}\gamma^\nu + 2\gamma^{\mu_2}\dots\gamma^{\mu_n}\gamma^{\mu_1}. \tag{C.9}$$

[47] We denote the natural numbers including zero with $\mathbb{N}_0$ and the natural numbers excluding zero with $\mathbb{N}^* = \mathbb{N}_0\backslash\{0\}$.

Now we can iterate this process n times to get

$$\begin{aligned}\gamma_\nu\gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma^\nu =&(-1)^n\gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma_\nu\gamma^\nu - 2\sum_{l=1}^{n}(-1)^l\gamma^{\mu_1}\ldots\gamma^{\mu_{l-1}}\gamma^{\mu_{l+1}}\ldots\gamma^{\mu_n}\gamma^{\mu_l}\\ =&(-1)^n\, d\,\gamma^{\mu_1}\ldots\gamma^{\mu_n} - 2\sum_{l=1}^{n}(-1)^l\gamma^{\mu_1}\ldots\gamma^{\mu_{l-1}}\gamma^{\mu_{l+1}}\ldots\gamma^{\mu_n}\gamma^{\mu_l}.\end{aligned} \tag{C.10}$$

With this equation we can infer

$$\gamma_\nu\gamma^{\mu_1\cdots\mu_n}\gamma^\nu = (-1)^n d\gamma^{\mu_1\cdots\mu_n} - 2\sum_{l=1}^{n}(-1)^l\gamma^{\mu_1\cdots\mu_n}(-1)^{n-l} = (-1)^n(d-2n)\gamma^{\mu_1\cdots\mu_n}, \tag{C.11}$$

and from there we deduce the first of two necessary results to give an explicit expression of $\hat{\Gamma}_\mu$

$$\big[[\gamma^{\mu_1\cdots\mu_n},\gamma^\nu],\gamma_\nu\big] = 2\big((1-(-1)^n)d + (-1)^n 2n\big)\gamma^{\mu_1\cdots\mu_n}. \tag{C.12}$$

Note that we did not assume d to be even, this result holds in any integer dimension $d \geq 2$.

The second result is concerning the trace of two basis elements $\gamma^{\mu_1\cdots\mu_n}$ in even dimensions and $\gamma^{\mu_1\cdots\mu_{2n}}$ or $\gamma^{\mu_1\cdots\mu_{2n+1}}$ in odd dimensions. At first we leave d without restrictions and look at $n, m \in \mathbb{N}^*$

$$\frac{1}{d_\gamma}\mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma_{\nu_1}\ldots\gamma_{\nu_m}) = -\frac{1}{d_\gamma}\mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{n-1}}\gamma_{\nu_1}\gamma^{\mu_n}\gamma_{\nu_2}\ldots\gamma_{\nu_m}) + \frac{2}{d_\gamma}\delta^{\mu_n}_{\nu_1}\mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{n-1}}\gamma_{\nu_2}\ldots\gamma_{\nu_m}). \tag{C.13}$$

This time it is a little more difficult to iterate and antisymmetrize the indices in equation (C.13). For the first term we get after iterating

$$\frac{(-1)^m}{d_\gamma}\,\mathrm{tr}(\gamma^{\mu_n}\gamma^{\mu_1}\ldots\gamma^{\mu_{n-1}}\gamma_{\nu_1}\ldots\gamma_{\nu_m}),$$

and after antisymmetrization

$$-(-1)^{n+m}(A^n_m)^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m}.$$

The iterated second term becomes

$$-\frac{2}{d_\gamma}\sum_{l=1}^{m}(-1)^l\delta^{\mu_n}_{\nu_l}\,\mathrm{tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_{n-1}}\gamma_{\nu_1}\ldots\gamma_{\nu_{l-1}}\gamma_{\nu_{l+1}}\ldots\gamma_{\nu_m}). \tag{C.14}$$

If we now perform the antisymmetrization we can split it into the antisymmetrization of the indices inside the trace and the indices outside the trace to reach

$$-\frac{2}{nm}\sum_{l=1}^{m}\sum_{k=1}^{n}\sum_{r=1}^{m}(-1)^{n+k+r}\delta^{\mu_k}_{\nu_r}(A^{n,k}_{m,r})^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m}. \tag{C.15}$$

Plugging this into equation (C.13) we find

$$(A^n_m)^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} = -(-1)^{n+m}(A^n_m)^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} - \sum_{k=1}^{n}\sum_{r=1}^{m}\frac{2(-1)^{n+k+r}}{n}\delta^{\mu_k}_{\nu_r}(A^{n,k}_{m,r})^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m}. \tag{C.16}$$

Because the even- and the odd-dimensional case are conceptually a little different we will discuss them separately now starting with the even-dimensional one.

It is obvious that

$$(A^n_m)^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} = 0, \qquad\qquad (n+m) \text{ odd}, \tag{C.17}$$

since the trace then contains an odd number of Dirac matrices and hence always vanishes in even dimensions. Therefore we can restrict ourselves to the case where $(n+m)$ is even. In this case we conclude from equation (C.16)

$$(A^n_m)^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m} = -\sum_{k=1}^{n}\sum_{r=1}^{m}\frac{(-1)^{n+k+r}}{n}\delta^{\mu_k}_{\nu_r}(A^{n,k}_{m,r})^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_m}. \tag{C.18}$$

Therefore the (A^n_m) are directly proportional to the (A^{n-1}_{m-1}). Via iteration and the conditions (C.3) we find that

$$(A^n_m) = 0, \qquad\qquad n \neq m. \tag{C.19}$$

For $n = m$ we get the recursion relation

$$(A^n_n)^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n} = -n(-1)^n\delta^{[\mu_1}_{[\nu_1}(A^{n-1}_{n-1})^{\mu_2\ldots\mu_n]}_{\nu_2\ldots\nu_n]} \tag{C.20}$$

with the initial condition

$$(A^1_1)^\mu_\nu = \delta^\mu_\nu. \tag{C.21}$$

This relation can easily be solved explicitly and we find

$$(A^n_n) = (-1)^{\frac{n(n-1)}{2}}\; n!\; \delta_{\mathrm{A}}{}^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}, \tag{C.22}$$

where $\delta_{\mathrm{A}}{}^{\mu_1\ldots\mu_n}_{\nu_1\ldots\nu_n}$ is the normalized and antisymmetrized Kronecker delta. Together with equation

(C.19) this proves the relation (C.6)

$$(A^n_m)^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m} = (-1)^{\frac{n(n-1)}{2}}\; n!\; \delta^n_m \delta_{\mathrm{A}}{}^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_n}. \tag{C.23}$$

To show the last relation we restrict ourselves to odd dimensions. Therefore we can shift $n \to 2n$ and $m \to 2m$ in equation (C.16) and find

$$(A^{2n}_{2m})^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2m}} = -\sum_{k=1}^{2n}\sum_{r=1}^{2m} \frac{(-1)^{k+r}}{2n} \delta^{\mu_k}_{\nu_r} (A^{2n,k}_{2m,r})^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2m}}. \tag{C.24}$$

Again we find a directly proportional relation from (A^{2n}_{2m}) to (A^{2n-1}_{2m-1}). Note that (A^{2n-1}_{2m-1}) are not the traces we are looking for since they have an odd number of upper and an odd number of lower indices. But we can further relate the (A^{2n-1}_{2m-1}) directly proportional to $(A^{2(n-1)}_{2(m-1)})$ because equation (C.16) is true for all $n, m \in \mathbb{N}^*$ and $(2n-1+2m-1)$ is an even number. Therefore we deduce a direct proportionality between (A^{2n}_{2m}) and $(A^{2(n-1)}_{2(m-1)})$ and with the iteration of that and the conditions (C.4) we get

$$(A^{2n}_{2m}) = (A^{2n+1}_{2m+1}) = 0, \qquad\qquad n \neq m. \tag{C.25}$$

Here we note that equation (C.22) uses only $n = m$ in equation (C.16) and the initial condition (C.2), with both of them valid in even and odd dimensions. Therefore equation (C.22) is also valid in odd dimensions. Hence, we easily conclude

$$(A^{2n}_{2n})^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2n}} = (-1)^n\; (2n)!\; \delta_{\mathrm{A}}{}^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2n}}, \quad (A^{2n+1}_{2n+1})^{\mu_1\dots\mu_{2n+1}}_{\nu_1\dots\nu_{2n+1}} = (-1)^n\; (2n+1)!\; \delta_{\mathrm{A}}{}^{\mu_1\dots\mu_{2n+1}}_{\nu_1\dots\nu_{2n+1}}. \tag{C.26}$$

The last two relations prove equations (C.7),

$$(A^{2n}_{2m})^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2m}} = (-1)^n\; (2n)!\; \delta^n_m \delta_{\mathrm{A}}{}^{\mu_1\dots\mu_{2n}}_{\nu_1\dots\nu_{2n}}, \quad (A^{2n+1}_{2m+1})^{\mu_1\dots\mu_{2n+1}}_{\nu_1\dots\nu_{2m+1}} = (-1)^n\; (2n+1)!\; \delta^n_m \delta_{\mathrm{A}}{}^{\mu_1\dots\mu_{2n+1}}_{\nu_1\dots\nu_{2n+1}}. \tag{C.27}$$

D Special Relations for the Dirac Matrices – Part II

For the explicit implementation of some conditions concerning the spin torsion $\Delta\Gamma_\mu$ we need more identities for the γ_μ. We have to prove

$$\bar{\gamma}_{\mu_1\dots\mu_n} = (-1)^{\frac{n(n+1)}{2}} \gamma_{\mu_1\dots\mu_n},\; [\gamma_\mu, \gamma^{\rho_1\dots\rho_{2n}}] = 4n\delta^{[\rho_1}_\mu \gamma^{\rho_2\dots\rho_{2n}]},\; \{\gamma_\mu, \gamma^{\rho_1\dots\rho_{2n+1}}\} = 2(2n+1)\delta^{[\rho_1}_\mu \gamma^{\rho_2\dots\rho_{2n+1}]}, \tag{D.1}$$

where $n \in \mathbb{N}_0$ is arbitrary. Additionally we need

$$\frac{1}{d_\gamma} \operatorname{tr}([\gamma_{\nu_1\dots\nu_{2m}}, \gamma^{\rho_1\dots\rho_{2n-1}}]\gamma_\mu) = (-1)^{n-1} \cdot 2 \cdot (2n)! \cdot g_{\mu[\nu_1} \delta_{\mathrm{A}}{}^{\rho_1\dots\rho_{2n-1}}_{\nu_2\dots\nu_{2n}]} \cdot \delta^n_m, \tag{D.2}$$

where $n, m \in \mathbb{N}^*$. The last important identity reads

$$\frac{1}{d_\gamma}\operatorname{tr}([\gamma^{\rho_1\dots\rho_n}, \gamma_{\lambda_1\dots\lambda_m}]\gamma_{\mu\nu}) = -4\cdot n\cdot n!\cdot(-1)^{\frac{n(n-1)}{2}}\cdot\delta^{[\rho_1}_{[\mu}\, g_{\nu][\lambda_1}\delta\mathrm{A}^{\rho_2\dots\rho_n]}_{\lambda_2\dots\lambda_n]}\cdot\delta^n_m, \tag{D.3}$$

where $(n+m)$ has to be even and $n, m \in \mathbb{N}^*$, i.e. either both have to be even or both have to be odd.

The proof of the first three identities is rather simple. First we use that $\bar{\gamma}_\mu = h^{-1}\gamma^\dagger_\mu h = -\gamma_\mu$ to show

$$h^{-1}(\gamma_{\mu_1}\dots\gamma_{\mu_n})^\dagger h = h^{-1}\gamma^\dagger_{\mu_n}\dots\gamma^\dagger_{\mu_1}h = (-1)^n\gamma_{\mu_n}\dots\gamma_{\mu_1}. \tag{D.4}$$

Next we antisymmetrize the indices on both sides to prove the first identity

$$\bar{\gamma}_{\mu_1\dots\mu_n} = (-1)^n(-1)^{\sum_{l=1}^{n-1} l}\gamma_{\mu_1\dots\mu_n} = (-1)^{\frac{n(n+1)}{2}}\gamma_{\mu_1\dots\mu_n}. \tag{D.5}$$

The second proof follows a similar track, we start with

$$\gamma_\mu\gamma^{\rho_1}\dots\gamma^{\rho_{2n}} = 2\delta^{\rho_1}_\mu\gamma^{\rho_2}\dots\gamma^{\rho_{2n}} - \gamma^{\rho_1}\gamma_\mu\gamma^{\rho_2}\dots\gamma^{\rho_{2n}}. \tag{D.6}$$

Again we iterate $2n$ times

$$\gamma_\mu\gamma^{\rho_1}\dots\gamma^{\rho_{2n}} = 2\sum_{l=1}^{2n}(-1)^{l-1}\delta^{\rho_l}_\mu\gamma^{\rho_1}\dots\gamma^{\rho_{l-1}}\gamma^{\rho_{l+1}}\dots\gamma^{\rho_{2n}} + \gamma^{\rho_1}\dots\gamma^{\rho_{2n}}\gamma_\mu. \tag{D.7}$$

If we now also antisymmetrize the indices we can read off

$$\gamma_\mu\gamma^{\rho_1\dots\rho_{2n}} = 2\sum_{l=1}^{2n}\delta^{[\rho_1}_\mu\gamma^{\rho_2\dots\rho_{2n}]} + \gamma^{\rho_1\dots\rho_{2n}}\gamma_\mu = 4n\delta^{[\rho_1}_\mu\gamma^{\rho_2\dots\rho_{2n}]} + \gamma^{\rho_1\dots\rho_{2n}}\gamma_\mu. \tag{D.8}$$

The last relation proves the identity. In order to show the third statement we perform analogous steps

$$\gamma_\mu\gamma^{\rho_1\dots\rho_{2n+1}} = 2\sum_{l=1}^{2n+1}\delta^{[\rho_1}_\mu\gamma^{\rho_2\dots\rho_{2n+1}]} - \gamma^{\rho_1\dots\rho_{2n+1}}\gamma_\mu = 2(2n+1)\delta^{[\rho_1}_\mu\gamma^{\rho_2\dots\rho_{2n+1}]} - \gamma^{\rho_1\dots\rho_{2n+1}}\gamma_\mu. \tag{D.9}$$

With the identities (C.6) and (C.7) from appendix C it is straightforward to calculate

$$\begin{aligned}\frac{1}{d_\gamma}\operatorname{tr}([\gamma_{\nu_1\dots\nu_{2m}}, \gamma^{\rho_1\dots\rho_{2n-1}}]\gamma_\mu) &= \frac{1}{d_\gamma}\operatorname{tr}([\gamma_\mu, \gamma_{\nu_1\dots\nu_{2m}}]\gamma^{\rho_1\dots\rho_{2n-1}}) = \frac{4m}{d_\gamma}g_{\mu[\nu_1}\operatorname{tr}(\gamma_{\nu_2\dots\nu_{2m}]}\gamma^{\rho_1\dots\rho_{2n-1}})\\ &= (-1)^{n-1}\cdot 2\cdot(2n)!\cdot g_{\mu[\nu_1}\delta\mathrm{A}^{\rho_1\dots\rho_{2n-1}}_{\nu_2\dots\nu_{2n}]}\cdot\delta^n_m,\end{aligned} \tag{D.10}$$

which proves the fourth identity.

Now we are left with the proof of the last identity, which is only true for $(n+m)$ even and $n,m \in \mathbb{N}^*$. Employing our usual trick we get

$$\frac{1}{d_\gamma}\operatorname{tr}(\gamma^{\rho_1\dots\rho_n}\gamma_{\lambda_1\dots\lambda_m}\gamma_{\mu\nu}) = n\delta_\nu^{[\rho_1}\frac{1}{d_\gamma}\operatorname{tr}(\gamma^{\rho_2\dots\rho_n]}\gamma_{\lambda_1\dots\lambda_m}\gamma_\mu) + m(-1)^n g_{\nu[\lambda_1}\frac{1}{d_\gamma}\operatorname{tr}(\gamma_{\lambda_2\dots\lambda_m]}\gamma_\mu\gamma^{\rho_1\dots\rho_n}) \tag{D.11}$$

and

$$\frac{1}{d_\gamma}\operatorname{tr}(\gamma_{\lambda_1\dots\lambda_m}\gamma^{\rho_1\dots\rho_n}\gamma_{\mu\nu}) = mg_{\nu[\lambda_1}\frac{1}{d_\gamma}\operatorname{tr}(\gamma_{\lambda_2\dots\lambda_m]}\gamma^{\rho_1\dots\rho_n}\gamma_\mu) + n(-1)^m\delta_\nu^{[\rho_1}\frac{1}{d_\gamma}\operatorname{tr}(\gamma^{\rho_2\dots\rho_n]}\gamma_\mu\gamma_{\lambda_1\dots\lambda_m}). \tag{D.12}$$

There are two distinct cases, n,m even and n,m odd. Starting with n,m even we shift $n \to 2l$ and $m \to 2k$ and find

$$\begin{aligned}\frac{1}{d_\gamma}\operatorname{tr}([\gamma^{\rho_1\dots\rho_{2l}},\gamma_{\lambda_1\dots\lambda_{2k}}]\gamma_{\mu\nu}) &= 2l\delta_\nu^{[\rho_1}\frac{1}{d_\gamma}\operatorname{tr}([\gamma^{\rho_2\dots\rho_{2l}]},\gamma_{\lambda_1\dots\lambda_{2k}}]\gamma_\mu) - 2kg_{\nu[\lambda_1}\frac{1}{d_\gamma}\operatorname{tr}([\gamma_{\lambda_2\dots\lambda_{2k}]},\gamma^{\rho_1\dots\rho_{2l}}]\gamma_\mu)\\ &= -4\cdot 2l\cdot(2l)!\cdot(-1)^{\frac{2l(2l-1)}{2}}\cdot\delta^{[\rho_1}_{[\mu}g_{\nu][\lambda_1}\delta\mathrm{A}^{\rho_2\dots\rho_{2l}]}_{\lambda_2\dots\lambda_{2l}]}\cdot\delta^l_k.\end{aligned} \tag{D.13}$$

This gives us the relation (D.3) for n,m even. On the other hand we now can take n,m odd and therefore shift $n \to 2l-1$ and $m \to 2k-1$. Now the commutator reads

$$\begin{aligned}&\frac{1}{d_\gamma}\operatorname{tr}([\gamma^{\rho_1\dots\rho_{2l-1}},\gamma_{\lambda_1\dots\lambda_{2k-1}}]\gamma_{\mu\nu})\\ &\quad= (2l-1)\delta_\nu^{[\rho_1}\frac{1}{d_\gamma}\operatorname{tr}(\gamma^{\rho_2\dots\rho_{2l-1}]}\{\gamma_{\lambda_1\dots\lambda_{2k-1}},\gamma_\mu\}) - (2k-1)g_{\nu[\lambda_1}\frac{1}{d_\gamma}\operatorname{tr}(\gamma_{\lambda_2\dots\lambda_{2k-1}]}\{\gamma^{\rho_1\dots\rho_{2l-1}},\gamma_\mu\})\\ &\quad= -4\cdot(2l-1)\cdot(2l-1)!\cdot(-1)^{\frac{(2l-1)((2l-1)-1)}{2}}\delta^{[\rho_1}_{[\mu}g_{\nu][\lambda_1}\delta\mathrm{A}^{\rho_2\dots\rho_{2l-1}]}_{\lambda_2\dots\lambda_{2l-1}]}\cdot\delta^l_k,\end{aligned} \tag{D.14}$$

proving the last identity for n,m odd.

E Spin Connection

In this appendix we prove the existence and the uniqueness of the spin connection $\hat{\Gamma}_\mu$ implicitly defined as

$$\partial_\mu\gamma^\nu + \left\{{\nu \atop \mu\rho}\right\}\gamma^\rho = -[\hat{\Gamma}_\mu,\gamma^\nu], \qquad \operatorname{tr}\hat{\Gamma}_\mu = 0. \tag{E.1}$$

We follow the idea of Weldon in [47] to prove the existence. First we expand the γ_μ and the metric around some arbitrary spacetime point $\boldsymbol{x}$ with coordinates x^μ

$$\gamma^\nu(x+\mathrm{d}x) \simeq \gamma^\mu(x) + \mathrm{d}x^\mu\partial_\mu\gamma^\nu(x), \qquad g^{\nu\lambda}(x+\mathrm{d}x) \simeq g^{\nu\lambda}(x) + \mathrm{d}x^\mu\partial_\mu g^{\nu\lambda}(x). \tag{E.2}$$

Next we plug the variations of the metric and the Dirac matrices into the Weldon theorem to get

$$\mathrm{d}x^\mu \partial_\mu \gamma^\nu = \frac{1}{2}\mathrm{d}x^\mu (\partial_\mu g^{\nu\lambda})\gamma_\lambda + [\delta S_\gamma, \gamma^\nu]. \tag{E.3}$$

Since this equation has to be fulfilled for all infinitesimal changes of the coordinates $\mathrm{d}x^\mu$, we can also expand $\delta\mathcal{S}_\gamma = \mathrm{d}x^\mu(\mathcal{S}_\gamma)_\mu$, where the $(\mathcal{S}_\gamma)_\mu$ are specified by the explicit choice $\gamma_\mu(x)$ (as a function of spacetime) and will therefore transform non homogeneously under coordinate transformations, i.e. spacetime coordinate as well as spin-base transformations.

Additionally we employ the metric compatibility of the Christoffel symbol

$$\partial_\mu g^{\nu\lambda} = -\left\{{\nu \atop \mu\rho}\right\} g^{\rho\lambda} - \left\{{\lambda \atop \mu\rho}\right\} g^{\rho\nu} \tag{E.4}$$

to conclude

$$\partial_\mu \gamma^\nu + \left\{{\nu \atop \mu\rho}\right\}\gamma^\rho = -\left[\frac{1}{8}\left\{{\alpha \atop \mu\rho}\right\} g^{\rho\beta}[\gamma_\alpha, \gamma_\beta] - (\mathcal{S}_\gamma)_\mu, \gamma^\nu\right], \tag{E.5}$$

where we took advantage of the identity

$$\big[[\gamma_\alpha, \gamma_\beta], \gamma^\nu\big] = 4\delta^\nu_\beta \gamma_\alpha - 4\delta^\nu_\alpha \gamma_\beta. \tag{E.6}$$

This means that $\partial_\mu \gamma^\nu + \left\{{\nu \atop \mu\rho}\right\}\gamma^\rho$ can be written as a commutator. Furthermore at least one $\hat{\Gamma}_\mu$ fulfilling equation (E.1) exists.

Since we know that there exists a solution to equation (E.1), we can expand $\hat{\Gamma}_\mu$ with the Dirac matrix basis elements, cf. sections 3.1.1 and (3.1.2),

$$\hat{\Gamma}_\mu = \sum_{n=1}^{d} \hat{m}_{\mu\rho_1\dots\rho_n}\gamma^{\rho_1\cdots\rho_n}, \quad d \text{ even}; \qquad \hat{\Gamma}_\mu = \sum_{n=1}^{\frac{d-1}{2}} \hat{m}_{\mu\rho_1\dots\rho_{2n}}\gamma^{\rho_1\cdots\rho_{2n}}, \quad d \text{ odd}. \tag{E.7}$$

From equation (E.1) we infer by calculating the commutator with γ_ν

$$\big[[\hat{\Gamma}_\mu, \gamma^\nu], \gamma_\nu\big] = -[({D_{(\mathrm{LC})}}_\mu \gamma^\nu), \gamma_\nu]. \tag{E.8}$$

Plugging in our ansatz for $\hat{\Gamma}_\mu$ and using the identity (C.5) from appendix C we get for the left hand side in the even dimensional case

$$\big[[\hat{\Gamma}_\mu, \gamma^\nu], \gamma_\nu\big] = \sum_{n=1}^{d} 2\big((1-(-1)^n)d + (-1)^n 2n\big)\hat{m}_{\mu\rho_1\dots\rho_n}\gamma^{\rho_1\cdots\rho_n} \tag{E.9}$$

and in the odd dimensional case

$$[[\hat{\Gamma}_\mu, \gamma^\nu], \gamma_\nu] = \sum_{n=1}^{\frac{d-1}{2}} 8n\ \hat{m}_{\mu\rho_1\ldots\rho_{2n}} \gamma^{\rho_1\cdots\rho_{2n}}. \tag{E.10}$$

The right hand side can be expanded into the Dirac matrix basis as well, cf. sections 3.1.1 and (3.1.2),

$$-[(D_{(\mathrm{LC})_\mu}\gamma^\nu), \gamma_\nu] = \sum_{n=1}^{d} \hat{a}_{\mu\rho_1\ldots\rho_n}\gamma^{\rho_1\cdots\rho_n},\ d \text{ even};\quad -[(D_{(\mathrm{LC})_\mu}\gamma^\nu), \gamma_\nu] = \sum_{n=1}^{\frac{d-1}{2}} \hat{a}_{\mu\rho_1\ldots\rho_{2n}}\gamma^{\rho_1\cdots\rho_{2n}},\ d \text{ odd}. \tag{E.11}$$

The coefficients $\hat{a}_{\mu\rho_1\ldots\rho_n}$ or respectively $\hat{a}_{\mu\rho_1\ldots\rho_{2n}}$ can be calculated employing the orthogonality of the trace (C.6) and (C.7)

$$\hat{a}_{\mu\rho_1\ldots\rho_n} = -\frac{(-1)^{\frac{n(n-1)}{2}}}{n!\, d_\gamma} \operatorname{tr}\left(\gamma_{\rho_1\ldots\rho_n}[D_{(\mathrm{LC})_\mu}\gamma^\nu, \gamma_\nu]\right),\quad \hat{a}_{\mu\rho_1\ldots\rho_{2n}} = -\frac{(-1)^n}{(2n)!\, d_\gamma} \operatorname{tr}\left(\gamma_{\rho_1\ldots\rho_{2n}}[D_{(\mathrm{LC})_\mu}\gamma^\nu, \gamma_\nu]\right). \tag{E.12}$$

Since the $\gamma^{\rho_1\cdots\rho_n}$ or respectively the $\gamma^{\rho_1\cdots\rho_{2n}}$ form a basis we are allowed to compare the coefficients and find

$$\hat{m}_{\mu\rho_1\ldots\rho_n} = \frac{\hat{a}_{\mu\rho_1\ldots\rho_n}}{2\big((1-(-1)^n)d + (-1)^n 2n\big)},\quad d \text{ even};\qquad \hat{m}_{\mu\rho_1\ldots\rho_{2n}} = \frac{\hat{a}_{\mu\rho_1\ldots\rho_{2n}}}{8n},\quad d \text{ odd}. \tag{E.13}$$

With the last equations we have shown the uniqueness and have given an explicit expression for $\hat{\Gamma}_\mu$ in terms of the γ_μ and their first derivatives.

F Spin Metric

The spin metric is an important quantity in our investigations. We found that it is restricted to satisfy

$$\text{(i)}\quad {\gamma_\mu}^\dagger = (-1)^{1+\varepsilon_{\mathfrak{p},d}} \cdot h\gamma_\mu h^{-1},\qquad \text{(ii)}\quad |\det h| = 1,\qquad \text{(iii)}\quad h^\dagger = -h. \tag{F.1}$$

We will show that these conditions are sufficient to determine (up to a sign) the spin metric h in terms of the Dirac matrices γ_μ. As a first step we show the uniqueness (up to a sign) of the spin metric. Let us assume that there is at least one spin metric h_1, which satisfies all three conditions. Then we know, if there is another spin metric h_2, they must be related via

$$[h_2^{-1}h_1, \gamma_\mu] = 0, \tag{F.2}$$

because both spin metrics have to fulfill

$$h_2 \gamma_\mu h_2^{-1} = (-1)^{1+\varepsilon_{\mathfrak{p},d}} \cdot {\gamma_\mu}^\dagger = h_1 \gamma_\mu h_1^{-1}. \tag{F.3}$$

Therefore, using Schur's Lemma,

$$h_2 = z h_1, \qquad z \in \mathbb{C}, \tag{F.4}$$

has to hold. With (ii), it follows that

$$|z| = 1. \tag{F.5}$$

But if both spin metrics satisfy the condition (iii), then

$$z^* h_1 = -z^* h_1^\dagger = -h_2^\dagger = h_2 = z h_1 \tag{F.6}$$

has to hold. Therefore both spin metrics have to be identical up to a sign,

$$h_2 = \pm h_1. \tag{F.7}$$

This demonstrates the uniqueness (up to a sign) of the spin metric.

Now we only need to prove the existence of one such spin metric h. For this, we first introduce the Matrix $\hat{M}$ satisfying

$${\gamma_\mu}^\dagger = (-1)^{1+\varepsilon_{\mathfrak{p},d}} \cdot \mathrm{e}^{\hat{M}} \gamma_\mu \mathrm{e}^{-\hat{M}}, \qquad \operatorname{tr} \hat{M} = 0. \tag{F.8}$$

The existence of such a matrix in every dimension is guaranteed by the Clifford algebra and our choice of $\varepsilon_{\mathfrak{p},d}$. In even dimensions the existence is obvious since ${\gamma_\mu}^\dagger$ and $-\gamma_\mu$ satisfy the Clifford algebra and therefore there must exist a connecting similarity transformation. For odd dimensions we use that the hermitean conjugation can change the connected component of the representation of the Clifford algebra depending on the signature of the metric. According to [129] the number $\mathfrak{p}$ of "+" signs in the signature tells us whether the connected component is changed or not. For an even number of plus signs the connected component is changed, whereas for an odd number it is not. In our case we have $\mathfrak{p}$ plus signs in the signature, i.e. in odd dimensions we have to choose

$$\varepsilon_{\mathfrak{p},d} = \mathfrak{p} \mod 2, \qquad d \text{ odd}. \tag{F.9}$$

The trace of $\hat{M}$ can always be set to zero, because the trace part commutes with all matrices and therefore drops out of equation (F.8). The hermitean conjugate of equation (F.8) is

$$\gamma_\mu = (-1)^{1+\varepsilon_{\mathfrak{p},d}} \mathrm{e}^{-\hat{M}^\dagger} {\gamma_\mu}^\dagger \mathrm{e}^{\hat{M}^\dagger}. \tag{F.10}$$

Therefore, also

$$\mathrm{e}^{\hat{M}} \gamma_\mu \mathrm{e}^{-\hat{M}} = (-1)^{1+\varepsilon_{\mathfrak{p},d}} {\gamma_\mu}^\dagger = \mathrm{e}^{\hat{M}^\dagger} \gamma_\mu \mathrm{e}^{-\hat{M}^\dagger} \tag{F.11}$$

has to hold. Schur's Lemma again implies that there exists a φ such that

$$\mathrm{e}^{\hat{M}^\dagger} = \mathrm{e}^{\mathrm{i}\varphi} \mathrm{e}^{\hat{M}}, \qquad \varphi \in \mathbb{R}. \tag{F.12}$$

This equation fixes $\mathrm{e}^{\mathrm{i}\varphi}$ once we have chosen a specific $\hat{M}$. Now we also know, that $\det \mathrm{e}^{\hat{M}} = 1$ and therefore the same has to hold for $\det \mathrm{e}^{\hat{M}^\dagger} = 1$. From this, we conclude that φ is limited to

$$\varphi \in \left\{ n \frac{2\pi}{d_\gamma} : n \in \{0, \ldots, d_\gamma - 1\} \right\}. \tag{F.13}$$

The desired spin metric h is then given by

$$h = \mathrm{i} \mathrm{e}^{\mathrm{i}\frac{\varphi}{2}} \mathrm{e}^{\hat{M}}. \tag{F.14}$$

It is straightforward to show, that this metric satisfies (i) - (iii).

Note that the determinant of the spin metric is also fixed and even independent of the set of Dirac matrices. To show this we just use that d_γ is even for $d \geq 2$ and therefore the sign ambiguity of the spin metric is not important for the determinant. In odd dimensions this also ensures the independence on the connected component of the Clifford algebra. Additionally the determinant of a spin-base transformation $\mathcal{S} \in \mathrm{SL}(d_\gamma, \mathbb{C})$ is equal to one, hence the chosen representation does not alter the determinant either. At last we could perform a spacetime coordinate transformation, but such a transformation has no effect on the spin metric and hence the determinant is invariant. With our previous investigations there are only two possibilities, namely $\det h = \pm 1$. Since we always can choose local inertial coordinates in an arbitrary point $\boldsymbol{x}$ of the manifold it is sufficient to calculate the determinant in this frame with a special chosen set of Dirac matrices compatible with the Clifford algebra. We are going to calculate the determinant stepwise. First we consider the case $\mathfrak{p} = d$. Hence, we can take a hermitean representation ${\gamma_\mu}^\dagger = \gamma_\mu$.[48] Then there are two cases, either $\varepsilon_{\mathfrak{p},d} = 0$, implying that d is even,

[48] An explicit example is $\gamma_{2j} = (\sigma_3 \otimes)^j \sigma_1 (\otimes \sigma_0)^{\lfloor \frac{d}{2} \rfloor - 1 - j}$, $\gamma_{2j+1} = (\sigma_3 \otimes)^j \sigma_2 (\otimes \sigma_0)^{\lfloor \frac{d}{2} \rfloor - 1 - j}$, $j \in \{0, \ldots, \lfloor \frac{d}{2} \rfloor - 1\}$ and for odd dimensions we additionally need $\gamma_{d-1} = \sigma_3 (\otimes \sigma_3)^{\lfloor \frac{d}{2} \rfloor - 1}$. We suppress the explicit reference to the considered point $\boldsymbol{x}$.

cf. equation (F.9). It is easy to check that then $h = \pm \mathrm{i}\gamma_*$ implying

$$\det h = \det\left(\mathrm{i}\gamma_*\right) = \det(\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\gamma_*}) = 1, \qquad \mathfrak{p} = d, \quad \varepsilon_{\mathfrak{p},d} = 0, \tag{F.15}$$

where we used the general relation $\gamma_* = \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}\gamma_*}$. The other case is $\varepsilon_{\mathfrak{p},d} = 1$. There we have $h = \pm\mathrm{i}\mathrm{I}$, leading to

$$\det h = (-1)^{\frac{d_\gamma}{2}}, \qquad \mathfrak{p} = d, \quad \varepsilon_{\mathfrak{p},d} = 1. \tag{F.16}$$

In the next step we investigate what happens when we change the signature from $\eta_{\mu\nu}$ with $(\mathfrak{m}, \mathfrak{p})$ to $\eta'_{\mu\nu}$ with $(\mathfrak{m}' = \mathfrak{m} + 1, \mathfrak{p}' = \mathfrak{p} - 1)$. There we choose the representation such that

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathrm{I}, \qquad {\gamma_i}^\dagger = -\gamma_i, \quad i \in \{0, \ldots, \mathfrak{m} - 1\}, \qquad {\gamma_j}^\dagger = \gamma_j, \quad j \in \{\mathfrak{m}, \ldots, d - 1\}, \tag{F.17}$$

$$\{\gamma'_\mu, \gamma'_\nu\} = 2\eta'_{\mu\nu}\mathrm{I}, \qquad \gamma'_\mu = \begin{cases} \gamma_\mu, & \mu \in \{0, \ldots, \mathfrak{m} - 1, \mathfrak{m} + 1, \ldots, d - 1\}, \\ \mathrm{i}\gamma_\mathfrak{m}, & \mu = \mathfrak{m}. \end{cases} \tag{F.18}$$

We assume to have a spin metric h for a given $\varepsilon_{\mathfrak{p},d}$ on $\eta_{\mu\nu}$. By going from $(\mathfrak{m}, \mathfrak{p})$ to $(\mathfrak{m}', \mathfrak{p}')$ there are again two possibilities for $\varepsilon_{\mathfrak{p}',d}$. First there is $\varepsilon'_{\mathfrak{p},d} = \varepsilon_{\mathfrak{p},d}$, implying that d is even. It is straightforward to check that we then have $h' = \pm \mathrm{i}^{\varepsilon_{\mathfrak{p},d}+\mathfrak{p}} \cdot h\gamma_*\gamma_\mathfrak{m}$ and correspondingly

$$\det h' = (-1)^{(\varepsilon_{\mathfrak{p},d}+\mathfrak{p})\frac{d_\gamma}{2}} \cdot \det h, \qquad \mathfrak{m}' = \mathfrak{m} + 1, \quad \mathfrak{p}' = \mathfrak{p} - 1, \quad \varepsilon_{\mathfrak{p}',d} = \varepsilon_{\mathfrak{p},d}. \tag{F.19}$$

In the second case we have $\varepsilon_{\mathfrak{p}',d} = 1 - \varepsilon_{\mathfrak{p},d}$. Then we have $h' = \pm \mathrm{i}^{\varepsilon_{\mathfrak{p},d}+1} h\gamma_\mathfrak{m}$ with

$$\det h' = (-1)^{\varepsilon_{\mathfrak{p},d}\frac{d_\gamma}{2}} \cdot \det h, \qquad \mathfrak{m}' = \mathfrak{m} + 1, \quad \mathfrak{p}' = \mathfrak{p} - 1, \quad \varepsilon_{\mathfrak{p}',d} = 1 - \varepsilon_{\mathfrak{p},d}. \tag{F.20}$$

Taking the equations (F.15), (F.16), (F.19) and (F.20) together we can calculate the determinant for every signature and dimension. The result is

$$\det h = \begin{cases} -1, & (\mathfrak{p} - \mathfrak{m})(2\varepsilon_{\mathfrak{p},d} - 1) > 0, \ d \in \{2, 3\}, \\ 1, & \text{else}. \end{cases} \tag{F.21}$$

We continue with implementing the spin metric compatibility as expressed in equation (4.16). This tells us that

$$\Gamma_\mu + \bar{\Gamma}_\mu = h^{-1}\partial_\mu h \tag{F.22}$$

has to hold. Taking into account that, cf. equation (4.19),

$$(-1)^{1+\varepsilon_{\mathrm{p},d}} D_{(\mathrm{LC})_\mu} h\gamma^\nu h^{-1} = D_{(\mathrm{LC})_\mu} {\gamma_\nu}^\dagger = (D_{(\mathrm{LC})_\mu}\gamma^\nu)^\dagger = -[\hat{\Gamma}_\mu, \gamma^\nu]^\dagger, \tag{F.23}$$

we arrive at

$$\left[h^{-1}(\partial_\mu h) - \hat{\Gamma}_\mu - \bar{\hat{\Gamma}}_\mu, \gamma^\nu\right] = 0. \tag{F.24}$$

Because $\operatorname{tr}\hat{\Gamma}_\mu = 0$, this implies

$$\hat{\Gamma}_\mu + \bar{\hat{\Gamma}}_\mu = h^{-1}\partial_\mu h - \frac{1}{d_\gamma}\operatorname{tr}(h^{-1}\partial_\mu h)\,\mathrm{I}. \tag{F.25}$$

Now we use $\det h = \pm 1$ to deduce

$$0 = \frac{\partial_\mu \det h}{\det h} = \partial_\mu \ln(\det h) = \partial_\mu \operatorname{tr}(\ln h) = \operatorname{tr}(h^{-1}\partial_\mu h). \tag{F.26}$$

This leaves us with

$$\Gamma_\mu + \bar{\Gamma}_\mu = h^{-1}\partial_\mu h = \hat{\Gamma}_\mu + \bar{\hat{\Gamma}}_\mu, \tag{F.27}$$

which implies that

$$\operatorname{Re}\operatorname{tr}\Gamma_\mu = 0. \tag{F.28}$$

These two identities are used in section 4.2 to constrain the spin torsion.

G Reducible Representations

So far, our considerations have been based on the irreducible representation of the Clifford algebra characterized by $d_\gamma = 2^{\lfloor d/2 \rfloor}$. Nevertheless, sometimes reducible representations of the Clifford algebra are desired or even necessary for physical reasons, see, e.g., the 2+1 dimensional Thirring model [288, 289]. A generalization of our formalism to reducible representations is not completely trivial, since the construction of the spin connection makes explicit use of a particular complete basis of the Clifford algebra. The basis used above may not generalize straightforwardly to any reducible representation. Therefore, we confine ourselves to those reducible representations where the basis used so far is still sufficient.

Our construction leads to reducible representations with $d_\gamma = n2^{\lfloor d/2 \rfloor}$, for $n \in \mathbb{N}$. For this, we assume that the new Dirac matrices can be written as tensor product of a possibly spacetime dependent matrix $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ of dimension $n \times n$ and the Dirac matrices

$\gamma^\mu \in \mathrm{Mat}(2^{\lfloor d/2 \rfloor} \times 2^{\lfloor d/2 \rfloor}, \mathbb{C})$ of the irreducible representation used above,

$$\gamma_{(d_\gamma)}{}^\mu = A \otimes \gamma^\mu, \tag{G.1}$$

obviously implying that $d_\gamma = n 2^{\lfloor d/2 \rfloor}$. Of course, the set of $\gamma_{(d_\gamma)}{}^\mu$ shall also satisfy the Clifford algebra,

$$\{\gamma_{(d_\gamma)}{}_\mu, \gamma_{(d_\gamma)}{}_\nu\} = 2g_{\mu\nu}\mathrm{I}_{(d_\gamma)} = 2g_{\mu\nu}\mathrm{I}_{(n)} \otimes \mathrm{I}, \tag{G.2}$$

which tells us that A is idempotent,

$$A^2 = \mathrm{I}_{(n)}. \tag{G.3}$$

Analogously to our previous construction, we need a covariant derivative $\nabla_{(d_\gamma)}{}_\mu$ and a spin metric $h_{(d_\gamma)}$. We require the covariant derivative to factorize accordingly,

$$\nabla_{(d_\gamma)}{}_\mu A \otimes \gamma^\nu = (\nabla_{(n)}{}_\mu A) \otimes \gamma^\nu + A \otimes (\nabla_\mu \gamma^\nu), \tag{G.4}$$

where $\nabla_{(n)}{}_\mu$ acts on the "A-part" and ∇_μ is identical to the covariant derivative in irreducible representation. Additionally we demand a similar structure as in the irreducible case and hence also impose

$$\nabla_{(d_\gamma)}{}_\mu A \otimes \gamma^\nu = D_\mu(A \otimes \gamma^\nu) + [\Gamma_{(d_\gamma)}{}_\mu, A \otimes \gamma^\nu]. \tag{G.5}$$

This tells us that the spin connection has to read

$$\Gamma_{(d_\gamma)}{}_\mu = \Gamma_{(n)}{}_\mu \otimes \mathrm{I} + \mathrm{I}_{(n)} \otimes \Gamma_\mu. \tag{G.6}$$

Because the irreducible component already carries all relevant structures for general covariance, the A-part in its simplest form should be covariantly constant,

$$0 = \nabla_{(n)}{}_\mu A = \partial_\mu A + [\Gamma_{(n)}{}_\mu, A]. \tag{G.7}$$

We can rewrite this into a condition for the connection $\Gamma_{(n)}{}_\mu$ which has to satisfy

$$\Gamma_{(n)}{}_\mu = A\Gamma_{(n)}{}_\mu A^{-1} - (\partial_\mu A)A^{-1}. \tag{G.8}$$

For a given choice of A on a given spacetime, equation (G.8) may or may not have a solution in terms of a set of $\Gamma_{(n)}{}_\mu$. If a solution exists, it completes the definition of the spin connection for this reducible representation. For the simpler case of constant matrices A, a solution is always given by $\Gamma_{(d_\gamma)}{}_\mu = 0$.

The natural way to embed the spin-base transformations is given by the form

$$\mathcal{S}_{(d_\gamma)} = \mathrm{I}_{(n)} \otimes \mathcal{S}, \qquad\qquad \mathcal{S} \in \mathrm{SL}(2^{\lfloor d/2 \rfloor}, \mathbb{C}). \tag{G.9}$$

The corresponding transformation law for the spin connection then reads

$$\Gamma_{(d_\gamma)}{}_\mu \to \mathcal{S}_{(d_\gamma)} \Gamma_{(d_\gamma)}{}_\mu \mathcal{S}_{(d_\gamma)}{}^{-1} - (\partial_\mu \mathcal{S}_{(d_\gamma)}) \mathcal{S}_{(d_\gamma)}{}^{-1} = \Gamma_{(d_\gamma)}{}_\mu \otimes \mathrm{I} + \mathrm{I}_{(n)} \otimes (\mathcal{S} \Gamma_\mu \mathcal{S}^{-1} - (\partial_\mu \mathcal{S}) \mathcal{S}^{-1}).$$

It is worthwhile to emphasize that the choice of the embedding (G.9) is not unique. Reducible representations of the Clifford algebra have a much larger symmetry of $\mathrm{SL}(d_\gamma = n 2^{\lfloor d/2 \rfloor}, \mathbb{C})$, such that there are typically many more options of embedding $\mathrm{SL}(2^{\lfloor d/2 \rfloor}, \mathbb{C})$ into $\mathrm{SL}(d_\gamma = n 2^{\lfloor d/2 \rfloor}, \mathbb{C})$. The present choice is motivated by the similarity to the embedding of local spin transformations that we would encounter in the corresponding vielbein formalism. On the level of Dirac matrices we have, cf. equation (3.41),

$$\gamma_{(e)}{}_\mu \to \mathcal{S}_{\mathrm{Lor}} \gamma_{(e)}{}_\mu \mathcal{S}_{\mathrm{Lor}}^{-1}. \tag{G.10}$$

Vielbeins transform under the corresponding Lorentz transformations, cf. equation (2.33) and (3.41),

$$e_\mu{}^a \to e_\mu{}^b \Lambda_{\mathrm{Lor}}{}^a{}_b. \tag{G.11}$$

The spin transformation $\mathcal{S}_{\mathrm{Lor}}$ is parametrized by the real coefficients $\omega_{ab} = -\omega_{ba}$,

$$\mathcal{S}_{\mathrm{Lor}} = \exp\left(\frac{\omega_{ab}}{8} [\gamma_{(\mathrm{f})}{}^a, \gamma_{(\mathrm{f})}{}^b] \right), \tag{G.12}$$

and the corresponding Lorentz transformation $\Lambda_{\mathrm{Lor}}{}^a{}_b$ is then given by

$$\Lambda_{\mathrm{Lor}}{}^a{}_b = (\mathrm{e}^{-\eta\omega})^a{}_b = \delta^a_b - \omega^a{}_b + \sum_{m=2}^{\infty} \frac{(-1)^m}{m!} \omega^a{}_{c_1} \dots \omega^{c_{m-1}}{}_b, \qquad \omega^a{}_b = \eta^{ac} \omega_{cb}. \tag{G.13}$$

Promoting the (fixed) Dirac matrices to the reducible representation given above, the spin transformation reads

$$\mathcal{S}_{\mathrm{Lor}(d_\gamma)} = \exp\left(\frac{\omega_{ab}}{8} [A \otimes \gamma_{(\mathrm{f})}{}^a, A \otimes \gamma_{(\mathrm{f})}{}^b] \right) \equiv \mathrm{I}_n \otimes \mathcal{S}_{\mathrm{Lor}}, \tag{G.14}$$

which is structurally identical to our choice for the embedding of equation (G.9).

Finally, we also need the spin metric for the reducible representation, which has to satisfy

$$\gamma^\dagger_{(d_\gamma)}{}_\mu = (-1)^{1+\varepsilon_{\mathfrak{p},d}} h_{(d_\gamma)} \gamma_{(d_\gamma)}{}_\mu h_{(d_\gamma)}{}^{-1}. \tag{G.15}$$

It is obvious that this condition is satisfied by

$$h_{(d_\gamma)} = A \otimes h, \qquad\qquad A^\dagger = A, \tag{G.16}$$

demanding that A is Hermitean in order to have $h_{(d_\gamma)}$ anti-Hermitean. Of course also the absolute value of the determinant is equal to one as required, since

$$\left|\det h_{(d_\gamma)}\right| = |\det A \otimes h| = \left|(\det A)^{d_\gamma} \cdot (\det h)^n\right| = 1. \tag{G.17}$$

This completes the construction of a generalization to particularly simple reducible representations of the Clifford algebra. Again, the embedding (G.16) may not be unique. The present choice is intuitive, because in conventional choices for the flat spacetime Dirac matrices, the spin metric is simply given by $\gamma_{(\mathrm{f})0}$. In the corresponding reducible representation, the "new" $\gamma_{(d_\gamma)0}$ would read

$$\gamma_{(d_\gamma)0} = A \otimes \gamma_{(\mathrm{f})0}, \tag{G.18}$$

matching precisely with our extended spin metric.

Let us emphasize again that the straightforwardly induced symmetries of the present construction may not exhaust the full invariance of the reducible Clifford algebra. For instance, one can immediately verify that our construction is invariant under local $\mathrm{SU}(n) \otimes \mathrm{SL}(2^{\lfloor d/2 \rfloor}, \mathbb{C})$ transformations, which is in general only a subgroup of the $\mathrm{SL}(d_\gamma, \mathbb{C})$ invariance of the Clifford algebra in reducible representation.

H Gauge Fields

In the preceding sections, we have set a possible trace part of the spin connection Γ_μ to zero, as such a trace part proportional to the identity in Dirac space $\sim \mathrm{I}$ does not transform the Dirac matrices nontrivially, cf. equation (4.4). If we allow for this generalization, the symmetry group can be extended to $\mathcal{G} \otimes \mathrm{SL}(d_\gamma, \mathbb{C})$, where $\mathcal{G}$ denotes the symmetry group of the trace part. The Clifford algebra is, of course, also invariant under this larger group, since the Dirac matrices and thus the geometry do not transform under $\mathfrak{t} \in \mathcal{G}$.

To construct a connection $\Gamma_{(\mathcal{G}\otimes\mathrm{SL})\mu}$ for this larger group, we consider symmetry transformations $\mathfrak{t}\otimes\mathcal{S} \in \mathcal{G} \otimes \mathrm{SL}(d_\gamma, \mathbb{C})$ and find, analogously to equation (4.14),

$$\Gamma_{(\mathcal{G}\otimes\mathrm{SL})\mu} \to \mathfrak{t}\otimes\mathcal{S}\, \Gamma_{(\mathcal{G}\otimes\mathrm{SL})\mu} (\mathfrak{t}\otimes\mathcal{S})^{-1} - \big(\partial_\mu (\mathfrak{t}\otimes\mathcal{S})\big)(\mathfrak{t}\otimes\mathcal{S})^{-1} \tag{H.1}$$

as the transformation property of the spin connection. Here we can use the product rule for the derivative and expand the inhomogeneous part,

$$\big(\partial_\mu(\mathfrak{t}\otimes\mathcal{S})\big)(\mathfrak{t}\otimes\mathcal{S})^{-1}=\big((\partial_\mu\mathfrak{t})\mathfrak{t}^{-1}\big)\otimes\mathrm{I}+\mathrm{I}_{(\mathcal{G})}\otimes\big((\partial_\mu\mathcal{S})\mathcal{S}^{-1}\big), \tag{H.2}$$

where $\mathrm{I}_{(\mathcal{G})}$ is the unit element of $\mathcal{G}$. Because of this behavior, it is sufficient to consider connections with the property

$$\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu}=\Gamma_{(\mathcal{G})_\mu}\otimes\mathrm{I}+\mathrm{I}_{(\mathcal{G})}\otimes\Gamma_\mu, \tag{H.3}$$

where $\Gamma_{(\mathcal{G})_\mu}$ is the connection for the group $\mathcal{G}$ and Γ_μ is the traceless connection for the $\mathrm{SL}(d_\gamma,\mathbb{C})$ part, i.e. the Γ_μ from the previous sections. Obviously, the Dirac trace part of $\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu}$ accommodates the connection for the group $\mathcal{G}$.

Similarly, a straightforward generalization of the spin metric suggests the form

$$h_{(\mathcal{G}\otimes\mathrm{SL})}=\mathrm{I}_{(\mathcal{G})}\otimes h, \tag{H.4}$$

with the corresponding transformation law

$$h_{(\mathcal{G}\otimes\mathrm{SL})}\to(\mathfrak{t}^\dagger\otimes\mathcal{S}^\dagger)^{-1}h_{(\mathcal{G}\otimes\mathrm{SL})}(\mathfrak{t}\otimes\mathcal{S})^{-1} \tag{H.5}$$

under a $\mathfrak{t}\otimes\mathcal{S}$ transformation. Requiring the transformation (H.5) to preserve equation (H.4), the elements of $\mathcal{G}$ need to be unitary,

$$\mathfrak{t}^{-1}=\mathfrak{t}^\dagger. \tag{H.6}$$

If we now additionally demand spin metric compatibility, cf. equation (4.16), we get

$$\mathrm{I}_{(\mathcal{G})}\otimes\big(h^{-1}(\partial_\mu h)\big)=\Gamma_{(\mathcal{G}\otimes\mathrm{SL})_\mu}+\mathrm{I}_{(\mathcal{G})}\otimes h^{-1}\,\Gamma^\dagger_{(\mathcal{G}\otimes\mathrm{SL})_\mu}\,\mathrm{I}_{(\mathcal{G})}\otimes h, \tag{H.7}$$

from which we deduce with regard to equation (H.3) that the connection of $\mathcal{G}$ needs to be anti-Hermitean,

$$\Gamma^\dagger_{(\mathcal{G})_\mu}=-\Gamma_{(\mathcal{G})_\mu}. \tag{H.8}$$

Here we also used equation (4.32). This justifies to introduce the gauge field $\mathcal{A}_\mu$ by

$$\Gamma_{(\mathcal{G})_\mu}=\mathrm{i}\mathcal{A}_\mu, \tag{H.9}$$

which is associated with the $\mathcal{G}$ symmetry. This field can in general be non-Abelian but is always Hermitean as is conventional in ordinary gauge field theory.

To summarize, the inclusion of a trace part in the spin connection Γ_μ can be viewed as an extension of the symmetry group from $\mathrm{SL}(d_\gamma,\mathbb{C})$ to $\mathcal{G}\otimes\mathrm{SL}(d_\gamma,\mathbb{C})$, with $\mathcal{G}$ being a unitary group.

The spin connection can then be decomposed as

$$\Gamma_{(\mathcal{G}\otimes\mathrm{SL})\mu} = \mathrm{i}\mathcal{A}_\mu \otimes \mathrm{I} + \mathrm{I}_{(\mathcal{G})} \otimes (\hat{\Gamma}_\mu + \Delta\Gamma_\mu), \tag{H.10}$$

or in short

$$\Gamma_\mu = \mathrm{i}\mathcal{A}_\mu + \hat{\Gamma}_\mu + \Delta\Gamma_\mu, \tag{H.11}$$

as it is understood and used in the following. Within the physical context of fermions in curved space, the $\mathrm{SL}(d_\gamma, \mathbb{C})$ part of the connection is always present in covariant derivatives of spinor fields, since it carries the information about how fermions evolve dynamically in a given curved space. By contrast, the gauge part of the connection may or may not be present depending on whether a fermion is charged under the group $\mathcal{G}$. Technically, the distinction among differently charged fermions may be parametrized by a charge matrix as a factor inside $\mathcal{A}_\mu$.

I Spin Base Path Integral

As an application of the spin-base invariant formalism, let us discuss possible implications for quantizing gravity within a path integral framework. Of course, the question as to whether such a path integral exists is far from being settled. For the purpose of the following discussion, we simply assume that there is such a path integral possibly regularized in a symmetry-preserving way and possibly amended with a suitable gauge fixing procedure. For simplicity, we consider the case of vanishing spin torsion, spacetime torsion and gauge fields

$$\Delta\Gamma_\mu = 0, \quad C^\kappa{}_{\mu\nu} = 0, \quad \mathcal{A}_\mu = 0, \tag{I.1}$$

even though the following considerations will not interfere with any of these quantities.

In the main text, we took the viewpoint that the spacetime-dependent Dirac matrices γ_μ are the basic objects encoding the essential properties of the spacetime. In fact, given a set of Dirac matrices, we can compute the metric,

$$g_{\mu\nu} = \frac{1}{4}\,\mathrm{tr}(\gamma_\mu \gamma_\nu). \tag{I.2}$$

Also the spin metric necessary for including fermionic Dirac degrees of freedom is fixed (up to a sign) by the conditions

$$h^\dagger = -h, \qquad \bar{\gamma}^\mu = -\gamma^\mu, \qquad |\det h| = 1, \tag{I.3}$$

see appendix F. The Dirac matrices also determine the spin connection (up to spin torsion), cf. appendix E, and all these ingredients suffice to define a classical theory of gravity including dynamical fermions. One is hence tempted to base a quantized theory also on the Dirac matrices as the fundamental degree of freedom. This would be analogous to quantizing gravity in terms of a vielbein. Whereas this is certainly a valid and promising option, we show in the following that this Dirac matrix/vielbein quantization is actually not necessary.

Demanding that quantization preserves the local Clifford algebra constraint also off shell

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathrm{I}, \qquad\qquad \gamma^\mu \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C}), \tag{I.4}$$

(for a correspondingly off-shell metric), the Weldon theorem (4.21) (see also appendix A) already tells us that a fluctuation of the Dirac matrices can always be decomposed into a metric fluctuation and an $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ fluctuation,

$$\delta\gamma^\mu = \frac{1}{2}(\delta g^{\mu\nu})\gamma_\nu + [\delta\mathcal{S}_\gamma, \gamma^\mu]. \tag{I.5}$$

Hence, we do not attempt to construct an integration measure for Dirac matrices "$\mathcal{D}\gamma$", satisfying the Dirac algebra constraint. Instead, it appears more natural to integrate over metrics and $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ fluctuations. In the following, we show that the $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ fluctuations factor out of the path integral because of spin-base invariance, such that a purely metric-based quantization scheme appears sufficient also in the presence of dynamical fermions.

The crucial starting point of our line of argument is the fact that all possible sets of Dirac matrices compatible with a given metric are connected with each other via $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ transformations and in odd dimensions via a sign flip [128, 129]. In odd dimensions the sign flip changes the connected component of the Clifford algebra. As we have already discussed, the choice of the connected component can be important (see section 3.2). Hence, we assume to be fixed on one connected component. This means that we can cover the space of Dirac matrices by (i) choosing an arbitrary mapping $\tilde{\gamma}^\mu$ of the metric into the space of Dirac matrices satisfying the Clifford algebra

$$g_{\mu\nu} \to \tilde{\gamma}_\mu = \tilde{\gamma}_\mu(g), \tag{I.6}$$

and (ii) performing $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ transformations $\mathrm{e}^{\mathcal{M}}$ of this mapping

$$\gamma_\mu(g) = \gamma_\mu\big(\tilde{\gamma}(g), \mathcal{M}(g)\big) = \mathrm{e}^{\mathcal{M}(g)}\tilde{\gamma}_\mu(g)\mathrm{e}^{-\mathcal{M}(g)} \tag{I.7}$$

where $\mathcal{M}$ is an arbitrary tracefree matrix which can be spanned by the generators of $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ transformations. This matrix $\mathcal{M}$ may even depend on the metric if we demand $\gamma_\mu(g)$ to be a particular Dirac matrix compatible with the Clifford algebra independently of the choice of the representative Dirac matrices $\tilde{\gamma}_\mu$.

Equation (I.7) emphasizes the fact that every possible set of Dirac matrices yielding a given metric $g_{\mu\nu}$ can be constructed by this mapping.

The variation of the resulting Dirac matrices under an infinitesimal variation in terms of the metric $\delta g_{\mu\nu}$ can be represented analogously to the Weldon theorem:

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + \left[G^{\rho\lambda}\,\delta g_{\rho\lambda}, \gamma_\mu\right], \tag{I.8}$$

where the tensor $G_{\rho\lambda} \in \mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$ is tracefree and depends on the actual choice of $\tilde{\gamma}_\mu(g)$ and $\mathcal{M}(g)$. $G^{\rho\lambda}$ can be calculated from

$$\left[[G^{\rho\lambda}, \gamma_\mu], \gamma^\mu\right] = \left[\frac{\partial\gamma_\mu(g)}{\partial g_{\rho\lambda}}, \gamma^\mu\right]. \tag{I.9}$$

The infinitesimal $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ fluctuation $\delta\mathcal{S}_\gamma$ acting on the Dirac matrices, as it occurs in the Weldon theorem, is obviously given by

$$\delta\mathcal{S}_\gamma = G^{\rho\lambda}\,\delta g_{\rho\lambda}. \tag{I.10}$$

Now, the microscopic actions subject to quantization are considered to be functionals of the fermions and the Dirac matrices, $S[\psi, \bar{\psi}, \gamma]$. From our construction given above, the Dirac matrices arise from a representative Dirac matrix $\tilde{\gamma}^\mu(g)$ which is related to the metric by an arbitrary but fixed *bijection*, $g_{\mu\nu} \leftrightarrow \tilde{\gamma}^\mu$. The Dirac matrix γ^μ occurring in the action is then obtained via the $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ transformation governed by $\mathcal{M}$, cf. equation (I.7). Therefore, it is useful to think of the action as a functional of the metric and of $\mathcal{M}$, $S[\psi, \bar{\psi}, g; \mathcal{M}]$. In particular, the freedom to choose $\mathcal{M}$ (or the corresponding $\mathrm{SL}(d_\gamma, \mathbb{C})_\gamma$ group element) guarantees that the space of all possible Dirac matrices compatible with a given metric can be covered – for any choice of the representative $\tilde{\gamma}^\mu(g)$.

In addition to diffeomorphism invariance, we demand that the actions under consideration are invariant under spin-base transformations

$$S[\psi, \bar{\psi}, g; \mathcal{M}] \to S[\mathcal{S}\psi, \bar{\psi}\mathcal{S}^{-1}, g; \ln(\mathcal{S}\mathrm{e}^{\mathcal{M}})] \equiv S[\psi, \bar{\psi}, g; \mathcal{M}]. \tag{I.11}$$

Especially we may always choose

$$\mathcal{S} = \mathrm{e}^{-\mathcal{M}}, \tag{I.12}$$

such that

$$S[\psi, \bar{\psi}, g; \mathcal{M}] = S[\psi', \bar{\psi}', g; 0], \;\; \psi' = \mathrm{e}^{-\mathcal{M}}\psi, \;\; \bar{\psi}' = \bar{\psi}\mathrm{e}^{\mathcal{M}}. \tag{I.13}$$

The essential ingredient for a path integral quantization is the choice of the measure. As argued above, the present construction suggests, to integrate over metrics g and successively over $\mathcal{M}$ to cover the space of all Dirac matrices.

More specifically, let us study the expectation value of an operator $\hat{\mathcal{O}}_{\mathrm{Ob}}(\psi,\bar{\psi},g;\mathcal{M})$ which is a scalar under spin-base transformations. For illustrative purposes, let us first consider only the functional integrations over the fermionic and metric degrees of freedom:

$$\mathcal{O}_{\mathrm{Ob}}[\mathcal{M}] = \langle \hat{\mathcal{O}}_{\mathrm{Ob}}(\psi,\bar{\psi},g;\mathcal{M})\rangle = \int \mathcal{D}g\mathcal{D}\psi\mathcal{D}\bar{\psi}\,\hat{\mathcal{O}}_{\mathrm{Ob}}(\psi,\bar{\psi},g;\mathcal{M})\,\mathrm{e}^{\mathrm{i}S[\psi,\bar{\psi},g;\mathcal{M}]}, \tag{I.14}$$

with suitable measures $\mathcal{D}g\mathcal{D}\psi\mathcal{D}\bar{\psi}$. The following argument only requires that the measure transforms in a standard manner under a *change of variables*

$$\mathcal{D}\psi = \mathcal{D}\psi' \left(\det \frac{\delta\psi'}{\delta\psi}\right)^{-1}. \tag{I.15}$$

As a consequence, $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is invariant under spin-base transformations, since the Jacobians from $\mathcal{D}\psi$ and from $\mathcal{D}\bar{\psi}$ are inverse to each other

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \mathcal{D}(\mathcal{S}\psi)\mathcal{D}(\bar{\psi}\mathcal{S}^{-1}). \tag{I.16}$$

Because $\hat{\mathcal{O}}_{\mathrm{Ob}}$ is a scalar in Dirac space, it also needs to be invariant under spin-base transformations

$$\hat{\mathcal{O}}_{\mathrm{Ob}}(\psi,\bar{\psi},g;\mathcal{M}) \to \hat{\mathcal{O}}_{\mathrm{Ob}}\big(\mathcal{S}\psi,\bar{\psi}\mathcal{S}^{-1},g;\ln(\mathcal{S}\mathrm{e}^{\mathcal{M}})\big) \equiv \hat{\mathcal{O}}_{\mathrm{Ob}}(\psi,\bar{\psi},g;\mathcal{M}). \tag{I.17}$$

Now it is easy to see, that $\mathcal{O}_{\mathrm{Ob}}[\mathcal{M}]$ is actually independent of the choice of $\mathcal{M}(g)$

$$\begin{aligned}\mathcal{O}_{\mathrm{Ob}}[\mathcal{M}] = &\int \mathcal{D}g\mathcal{D}\psi\mathcal{D}\bar{\psi}\,\hat{\mathcal{O}}_{\mathrm{Ob}}(\psi,\bar{\psi},g;\mathcal{M})\,\mathrm{e}^{\mathrm{i}S[\psi,\bar{\psi},g;\mathcal{M}]} = \int \mathcal{D}g\mathcal{D}\psi'\mathcal{D}\bar{\psi}'\,\hat{\mathcal{O}}_{\mathrm{Ob}}(\psi',\bar{\psi}',g;0)\,\mathrm{e}^{\mathrm{i}S[\psi',\bar{\psi}',g;0]}\\ = &\,\mathcal{O}_{\mathrm{Ob}}[0].\end{aligned} \tag{I.18}$$

Therefore, every set of Dirac matrices compatible with a given metric contributes identically to such an expectation value. Hence, we may choose any convenient spin basis to simplify explicit computations. From another viewpoint, an additional functional integration over $\mathrm{SL}(d_\gamma,\mathbb{C})_\gamma$ with a suitable measure $\mathcal{D}\mathcal{M}$ would have factored out of the path integral and thus can be included trivially in its normalization.

This concludes our argument that a quantization of interacting theories of fermions and gravity may be solely based on a quantization of the metric together with the fermions. The spin-base invariant formulation given here suggests that this quantization scheme is natural. A quantization in terms of vielbeins/Dirac matrices – though perhaps legitimate – is not mandatory.

With hindsight, our results rely crucially on the constraint that the fluctuations of the Dirac matrices satisfy the Clifford algebra equation (I.4) also off-shell. If this assumption is relaxed, e.g., if the anticommutator of two Dirac matrices in the path integral is no longer bound to be proportional to the identity, a purely metric-based quantization scheme may no longer be possible.

With this settled it is tempting to think, that instead of the metric, we could have chosen the vielbein and then performed the same steps to arrive at a vielbein path integral. In the following we investigate what happens, if one tries to give the Dirac matrix path integral a meaning using the vielbein formulation. In other words, we aim at decomposing an arbitrary Dirac matrix fluctuation $\delta\gamma_\mu$ (compatible with the Clifford algebra) uniquely into a vielbein fluctuation $\delta {e_\mu}^a$ and a fluctuation of some other quantity $\delta\mathcal{S}_e$. With such a pair we then could integrate over Dirac matrices in terms of the vielbein $\delta e_\mu a$ and the residual fluctuations $\delta\mathcal{S}_e$. However, for the standard procedure we need that $\delta\mathcal{S}_e$ corresponds to a representation of an algebra, e.g., in the metric case we had $\delta\mathcal{S}_\gamma \in \mathfrak{sl}(d_\gamma, \mathbb{C})$.

We begin by assuming the existence of a vielbein degree of freedom ${e_\mu}^a$ in the spin base formalism. Then we can define a set of spacetime dependent flat Dirac matrices ${\gamma_{(\mathrm{f})}}_a$ by

$$ {\gamma_{(\mathrm{f})}}_a = {e^\mu}_a \gamma_\mu, \tag{I.19} $$

satisfying a Clifford algebra for the flat metric η_{ab}. Hence, we can express the Dirac matrices as $\gamma_\mu = {e_\mu}^a {\gamma_{(\mathrm{f})}}_a$. A general vielbein fluctuation can be decomposed uniquely into a metric fluctuation $\delta g_{\mu\nu}$ and a Lorentz fluctuation $\delta {\Lambda^a}_b$

$$ \delta {e_\mu}^a = \frac{1}{2}(\delta g_{\mu\nu}) e^{\nu a} + {e_\mu}^b \delta {\Lambda^a}_b, \tag{I.20} $$

where

$$ \delta g_{\mu\nu} = (\delta {e_\mu}^a) e_{\nu a} + (\delta {e_\nu}^a) e_{\mu a}, \qquad \delta {\Lambda^a}_b = \frac{1}{2}(\delta {e_\rho}^a) {e^\rho}_b - \frac{1}{2}(\delta {e_\rho}^c)\, \eta_{cb}\, {e^\rho}_d\, \eta^{da}. \tag{I.21} $$

From a given Dirac matrix fluctuation $\delta\gamma_\mu$ we can read off the corresponding metric fluctuation $\delta g_{\mu\nu}$ and the corresponding spin-base fluctuation $\delta\mathcal{S}_\gamma$ from the Weldon theorem,

$$ \delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + [\delta\mathcal{S}_\gamma, \gamma_\mu], \tag{I.22} $$

cf. appendix A. On the other hand we can calculate the fluctuations of the decomposition (I.19)

$$ \delta\gamma_\mu = (\delta {e_\mu}^a) \cdot {\gamma_{(\mathrm{f})}}_a + {e_\mu}^a \cdot (\delta {\gamma_{(\mathrm{f})}}_a), \tag{I.23} $$

where $\delta {e_\mu}^a$ is the vielbein fluctuation and $\delta {\gamma_{(\mathrm{f})}}_a$ is a fluctuation of a flat Dirac matrix. Here we can decompose the vielbein fluctuation like in equation (I.20). Additionally we know that the ${\gamma_{(\mathrm{f})}}_a$ have to satisfy the flat Clifford algebra. Therefore the fluctuation $\delta {\gamma_{(\mathrm{f})}}_a$ has to be a pure spin-base fluctuation $\delta\mathcal{S}_{(\mathrm{f})}$

$$\delta\gamma_{(\mathrm{f})a} = [\delta\mathcal{S}_{(\mathrm{f})}, \gamma_{(\mathrm{f})a}]. \tag{I.24}$$

Then we arrive at

$$\delta\gamma_\mu = \frac{1}{2}(\delta g_{\mu\nu})\gamma^\nu + \left[\delta\mathcal{S}_{(\mathrm{f})} + \frac{1}{8}(\delta\Lambda^b{}_c)[\gamma_{(\mathrm{f})b}, \gamma_{(\mathrm{f})}{}^c], \gamma_\mu\right], \tag{I.25}$$

where we have used the identity $\big[[\gamma_{(\mathrm{f})b}, \gamma_{(\mathrm{f})}{}^c], \gamma_{(\mathrm{f})a}\big] = 4\delta^c_a\gamma_{(\mathrm{f})b} - 4\eta_{ab}\gamma_{(\mathrm{f})}{}^c$. Here we see that $\delta\gamma_\mu$ fixes the metric fluctuation part $\delta g_{\mu\nu}$ of the vielbein fluctuation $\delta e_\mu{}^a$. Incidentally, it follows that

$$\delta\mathcal{S}_\gamma = \delta\mathcal{S}_{(\mathrm{f})} + \frac{1}{8}(\delta\Lambda^b{}_c)[\gamma_{(\mathrm{f})b}, \gamma_{(\mathrm{f})}{}^c]. \tag{I.26}$$

This implies that there are in principle infinitely many possible Lorentz fluctuations $\delta\Lambda^a{}_b$ for a given spin-base fluctuation $\delta\mathcal{S}_\gamma$. In order to cure this ambiguity, we have to find a way to extract a unique Lorentz fluctuation from $\delta\mathcal{S}_\gamma$. The simplest way is to restrict $\delta\mathcal{S}_e$ such that it does not mix with $\delta\Lambda^a{}_b[\gamma_{(\mathrm{f})a}, \gamma_{(\mathrm{f})b}]$, i.e. $\delta\mathcal{S}_e \in \mathfrak{sl}(d_\gamma, \mathbb{C})\backslash\{\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu] : \omega_{\mu\nu} \in \mathbb{R}\}$. Then the decomposition

$$\delta\mathcal{S}_\gamma = \delta\mathcal{S}_e + \frac{1}{8}(\delta\Lambda^b{}_c)[\gamma_{(\mathrm{f})b}, \gamma_{(\mathrm{f})}{}^c]. \tag{I.27}$$

is bijective. Note that the set $\{[\gamma^\mu, \gamma^\nu]\}$ is a subset of the Dirac matrix base for $\mathrm{Mat}(d_\gamma \times d_\gamma, \mathbb{C})$. Unfortunately the residual degree of freedom $\delta\mathcal{S}_e$ then is not an element of an algebra, since $\delta\mathcal{S}_e \in \mathfrak{sl}(d_\gamma, \mathbb{C})\backslash\{\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu] : \omega_{\mu\nu} \in \mathbb{R}\}$ does not form an algebra. Hence, the construction of a meaningful integral for $\delta\mathcal{S}_e$ is an open problem.

In conclusion, we found that if one insists on integrating over Dirac matrices in terms of a vielbein, it will be difficult to define a meaningful residual quantity, necessary to cover all possible Dirac matrices. Either the remaining quantity will not be an element of an algebra, leading to a complicated construction of the corresponding path integral; or one already needs a revised and presumably inconvenient way of assigning a vielbein fluctuation to the Dirac matrix fluctuation.

J Flow Equations for Quantum Einstein Gravity

In this section, we display the right hand side of the Wetterich equation for general Regulators $R_k[\Delta]$ and in dimension $d = 4$. For simplicity, we introduce the anomalous dimension $\eta = (\dot{g} - 2g)/g$, and refer to terms linear in η as "η-terms". Let us start with the contribution from the TT-mode:

$$\mathcal{S}^{\mathrm{TT}} = \frac{5}{2}Q_2\left[\frac{\dot{R}_k - \eta R_k}{\Delta_k - 2\lambda(1-\tau)}\right] - \frac{5}{12}R\left(Q_1\left[\frac{\dot{R}_k - \eta R_k}{\Delta_k - 2\lambda(1-\tau)}\right] + (4-3\tau)Q_2\left[\frac{\dot{R}_k - \eta R_k}{\big(\Delta_k - 2\lambda(1-\tau)\big)^2}\right]\right), \tag{J.1}$$

where $\Delta_k = \Delta + R_k$ and the Q functionals are defined in terms of Mellin transforms [7]. For the transverse vector, and without field redefinition, let us define

$$\mathcal{G}_n^{1\mathrm{T}} = \left[-(\dot{R}_k - \eta R_k)\left(2\lambda(1-\tau) - \frac{1}{\alpha}(\Delta_k + \Delta)\right) - 2(\dot{\lambda} + 2\lambda)(1-\tau)R_k\right]\left(\Delta_k\left(\frac{\Delta_k}{\alpha} + 2\lambda(1-\tau)\right)\right)^{-n}. \tag{J.2}$$

With that, we have

$$\mathcal{S}^{1\mathrm{T}} = \frac{3}{2}Q_2\big[\mathcal{G}_1^{1\mathrm{T}}\big] + R\left(\frac{1}{4}Q_1\big[\mathcal{G}_1^{1\mathrm{T}}\big] + \frac{3}{2}(1 - \alpha(1-\tau))Q_3\big[\mathcal{G}_2^{1\mathrm{T}}\big] - \frac{3}{4}\lambda(1-\tau)Q_2\big[\mathcal{G}_2^{1\mathrm{T}}\big]\right). \tag{J.3}$$

On the other hand, the contribution with field redefinition reads,

$$\mathcal{S}_{\mathrm{fr}}^{1\mathrm{T}} = \frac{3}{2}Q_2\left[\frac{\dot{R}_k - \eta R_k}{\Delta_k - 2\alpha\lambda(1-\tau)}\right] + \frac{R}{8}Q_1\left[\frac{\dot{R}_k - \eta R_k}{\Delta_k - 2\alpha\lambda(1-\tau)}\right] + \frac{3R\big(1 - 2\alpha(1-\tau)\big)}{8}Q_2\left[\frac{\dot{R}_k - \eta R_k}{\big(\Delta_k - 2\alpha\lambda(1-\tau)\big)^2}\right]. \tag{J.4}$$

For the scalar contribution, we first define

$$\pi^\sigma = -3\lambda(1-\tau)\Delta_k^2 - \frac{3(\alpha-3)}{4\alpha}\Delta_k^3,\ \ \pi^{\mathrm{h}} = -\frac{1}{16\alpha}\big(-4\alpha\lambda(1+\tau) + (3\alpha - \beta^2)\Delta_k\big),\ \ \pi^{\mathrm{x}} = -\frac{3}{8\alpha}(\alpha-\beta)\Delta_k^2, \tag{J.5}$$

as well as

$$\begin{aligned}
\rho^\sigma &= -3(1-\tau)\big[(\dot{\lambda} + (2-\eta)\lambda)(\Delta_k + \Delta)R_k + 2\lambda\Delta_k\dot{R}_k\big] - \frac{3(\alpha-3)}{4\alpha}\big(3\Delta_k^2\dot{R}_k + (3\Delta^2 + 3\Delta R_k + R_k^2)\eta R_k\big),\\
\rho^{\mathrm{h}} &= -\frac{1}{16\alpha}(3\alpha - \beta^2)(\dot{R}_k - \eta R_k), \quad \rho^{\mathrm{x}} = -\frac{3}{8\alpha}(\alpha-\beta)\big(2\dot{R}_k\Delta_k - \eta R_k(\Delta_k + \Delta)\big).
\end{aligned} \tag{J.6}$$

The contribution is

$$S^{\sigma \mathrm{h}} = \frac{1}{2}Q_2\left[\frac{\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{nfr}}}{\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2}\right]+R\Bigg\{\frac{1}{12}Q_1\left[\frac{\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{nfr}}}{\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2}\right]-\frac{3}{4\alpha}(6-\alpha(4-3\tau))Q_4\left[\frac{\rho^\mathrm{h}}{\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2}\right]$$
$$+\lambda(1-\tau)Q_3\left[\frac{\rho^\mathrm{h}}{\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2}\right]-\frac{\tau}{32}Q_2\left[\frac{\rho^\sigma}{\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2}\right]-\frac{\alpha-\beta}{4\alpha}Q_3\left[\frac{\rho^\mathrm{x}}{\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2}\right]$$
$$+\frac{3}{4\alpha}(6-\alpha(4-3\tau))Q_4\left[\frac{\pi^\mathrm{h}\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{nfr}}}{\left(\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2\right)^2}\right]+\lambda(1-\tau)Q_3\left[\frac{\pi^\mathrm{h}\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{nfr}}}{\left(\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2\right)^2}\right]$$
$$+\frac{\tau}{32}Q_2\left[\frac{\pi^\sigma\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{nfr}}}{\left(\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2\right)^2}\right]+\frac{\alpha-\beta}{16\alpha}Q_2\left[\frac{\pi^\mathrm{x}\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{nfr}}}{\left(\pi^\sigma\pi^\mathrm{h}-(\pi^\mathrm{x})^2\right)^2}\right]\Bigg\}, \tag{J.7}$$

where $\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{nfr}} = \pi^\sigma\rho^\mathrm{h}+\pi^\mathrm{h}\rho^\sigma-2\pi^\mathrm{x}\rho^\mathrm{x}$. With field redefinition, we define

$$\pi^\sigma_{\mathrm{fr}} = -3\lambda(1-\tau)-\frac{3(\alpha-3)}{4\alpha}\Delta_k, \quad \pi^\mathrm{h}_{\mathrm{fr}} = \frac{1}{4}\lambda(1+\tau)-\frac{3\alpha-\beta^2}{16\alpha}\Delta_k, \quad \pi^\mathrm{x}_{\mathrm{fr}} = -\frac{3}{8\alpha}(\alpha-\beta)\Delta_k, \tag{J.8}$$

$$\rho^\sigma_{\mathrm{fr}} = \frac{3}{4\alpha}(3-\alpha)(\dot{R}_k-\eta R_k), \quad \rho^\mathrm{h}_{\mathrm{fr}} = -\frac{3\alpha-\beta^2}{16\alpha}(\dot{R}_k-\eta R_k), \quad \rho^\mathrm{x}_{\mathrm{fr}} = -\frac{3}{8\alpha}(\alpha-\beta)(\dot{R}_k-\eta R_k). \tag{J.9}$$

With $\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{fr}} = \pi^\sigma_{\mathrm{fr}}\rho^\mathrm{h}_{\mathrm{fr}}+\pi^\mathrm{h}_{\mathrm{fr}}\rho^\sigma_{\mathrm{fr}}-2\pi^\mathrm{x}_{\mathrm{fr}}\rho^\mathrm{x}_{\mathrm{fr}}$ the scalar contribution is

$$S^{\sigma\mathrm{h}}_{\mathrm{fr}} = \frac{1}{2}Q_2\left[\frac{\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{fr}}}{\pi^\sigma_{\mathrm{fr}}\pi^\mathrm{h}_{\mathrm{fr}}-(\pi^\mathrm{x}_{\mathrm{fr}})^2}\right]+R\Bigg\{\frac{1}{12}Q_1\left[\frac{\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{fr}}}{\pi^\sigma_{\mathrm{fr}}\pi^\mathrm{h}_{\mathrm{fr}}-(\pi^\mathrm{x}_{\mathrm{fr}})^2}\right]-\frac{3}{8\alpha}(1-\alpha(1-\tau))Q_2\left[\frac{\rho^\mathrm{h}_{\mathrm{fr}}}{\pi^\sigma_{\mathrm{fr}}\pi^\mathrm{h}_{\mathrm{fr}}-(\pi^\mathrm{x}_{\mathrm{fr}})^2}\right]$$
$$-\frac{\tau}{32}Q_2\left[\frac{\rho^\sigma_{\mathrm{fr}}}{\pi^\sigma_{\mathrm{fr}}\pi^\mathrm{h}_{\mathrm{fr}}-(\pi^\mathrm{x}_{\mathrm{fr}})^2}\right]-\frac{\alpha-\beta}{16\alpha}Q_2\left[\frac{\rho^\mathrm{x}_{\mathrm{fr}}}{\pi^\sigma_{\mathrm{fr}}\pi^\mathrm{h}_{\mathrm{fr}}-(\pi^\mathrm{x}_{\mathrm{fr}})^2}\right]+\frac{3}{8\alpha}(1-\alpha(1-\tau))Q_2\left[\frac{\pi^\mathrm{h}_{\mathrm{fr}}\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{fr}}}{\left(\pi^\sigma_{\mathrm{fr}}\pi^\mathrm{h}_{\mathrm{fr}}-(\pi^\mathrm{x}_{\mathrm{fr}})^2\right)^2}\right]$$
$$+\frac{\tau}{32}Q_2\left[\frac{\pi^\sigma_{\mathrm{fr}}\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{fr}}}{\left(\pi^\sigma_{\mathrm{fr}}\pi^\mathrm{h}_{\mathrm{fr}}-(\pi^\mathrm{x}_{\mathrm{fr}})^2\right)^2}\right]+\frac{\alpha-\beta}{16\alpha}Q_2\left[\frac{\pi^\mathrm{x}_{\mathrm{fr}}\mathcal{F}^{\sigma\mathrm{h}}_{\mathrm{fr}}}{\left(\pi^\sigma_{\mathrm{fr}}\pi^\mathrm{h}_{\mathrm{fr}}-(\pi^\mathrm{x}_{\mathrm{fr}})^2\right)^2}\right]\Bigg\}. \tag{J.10}$$

Further, the ghost contribution reads without field redefinition,

$$S^{\mathrm{gh}} = -5Q_2\left[\frac{\dot{R}_k}{\Delta_k}\right]-R\left(\frac{7}{12}Q_1\left[\frac{\dot{R}_k}{\Delta_k}\right]+\frac{3}{4}Q_2\left[\frac{\dot{R}_k}{\Delta_k^2}\right]+\frac{4}{3-\beta}Q_3\left[\frac{\dot{R}_k}{\Delta_k^3}\right]\right). \tag{J.11}$$

With field redefinition, it is

$$S^{\mathrm{gh}}_{\mathrm{fr}} = -4Q_2\left[\frac{\dot{R}_k}{\Delta_k}\right]-R\left(\frac{5}{12}Q_1\left[\frac{\dot{R}_k}{\Delta_k}\right]+\left(\frac{3}{4}+\frac{1}{3-\beta}\right)Q_2\left[\frac{\dot{R}_k}{\Delta_k^2}\right]\right). \tag{J.12}$$

Finally, the contribution of the Jacobian for the case without field redefinition is

$$\mathcal{S}^{\text{Jac}} = \frac{1}{2}\, S^{\text{gh}}\big|_{\beta=0} + Q_2\left[\frac{\dot{R}_k}{\Delta_k}\right] + \frac{1}{6} R Q_1 \left[\frac{\dot{R}_k}{(\Delta + R_k)^2}\right]. \tag{J.13}$$

With field redefinition, all Jacobians cancel, at least on maximally symmetric backgrounds, which is sufficient for the truncation considered here [262].

K Beta Function in the Gross-Neveu Model

Here we give some details on the derivation of equation (7.6) from section 7.1. We use the flow equation (7.2) (cf. also section 5.1) and work in field space parametrized by the collective fields

$$\varphi = \begin{pmatrix} \psi \\ \bar{\psi}^{\mathrm{T}} \end{pmatrix}, \qquad\qquad \bar{\varphi} = (\bar{\psi}, \psi^{\mathrm{T}}), \tag{K.1}$$

representing Graßmann-valued functions on the manifold, reminiscent of Nambu-Gorkov spinors. Note that the fermions still have an internal flavor index, $\psi = (\psi^i)$. For instance, the representation of the unit element in field space becomes rather intuitive,

$$\mathbb{1}(x,y) = \varphi(x)\frac{\overleftarrow{\delta}}{\delta\varphi(y)} = \frac{\overrightarrow{\delta}}{\delta\bar{\varphi}(x)}\varphi(y) = \begin{pmatrix} \delta(x,y) & 0 \\ 0 & \delta(y,x)^{\mathrm{T}} \end{pmatrix}. \tag{K.2}$$

By $\delta(x,y)$, we denote the spin-valued delta distribution, which satisfies, cf. equation (5.15),

$$\psi(\boldsymbol{x}) = \int_y \delta(\boldsymbol{x},\boldsymbol{y})\psi(\boldsymbol{y}), \qquad\qquad \bar{\psi}(\boldsymbol{x}) = \int_y \bar{\psi}(\boldsymbol{y})\delta(\boldsymbol{y},\boldsymbol{x}). \tag{K.3}$$

With this notation, we indicate that $\delta(\boldsymbol{x},\boldsymbol{y})$ transforms as a spinor in $\boldsymbol{x}$ and a Dirac-conjugated spinor in $\boldsymbol{y}$. Therefore, the spin covariant derivative $\nabla^{(x)}_\mu$ with respect to x^μ acts like

$$\nabla^{(x)}_\mu \delta(\boldsymbol{x},\boldsymbol{y}) = \partial^{(x)}_\mu \delta(\boldsymbol{x},\boldsymbol{y}) + \hat{\Gamma}_\mu(\boldsymbol{x})\delta(\boldsymbol{x},\boldsymbol{y}). \tag{K.4}$$

We remark that there is a difference between $\delta(\boldsymbol{x},\boldsymbol{y})$ and $\delta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}}$ in the spin structure, since

$$\nabla^{\mathrm{T}(x)}_\mu \delta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}} = \partial^{(x)}_\mu \delta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}} - \hat{\Gamma}^{\mathrm{T}}_\mu \delta(\boldsymbol{y},\boldsymbol{x})^{\mathrm{T}} = \left(\nabla^{(x)}_\mu \delta(\boldsymbol{y},\boldsymbol{x})\right)^{\mathrm{T}}. \tag{K.5}$$

For the evaluation of the flow equation (7.2), we proceed in a standard fashion. Since we are only interested in the four fermion coupling, where no derivatives are involved, we can drop every covariant derivative acting on the fields $\nabla_\mu\psi(\boldsymbol{x}) \to 0$, as these terms correspond to other action monomials. Then we can choose constant fields $\psi(\boldsymbol{x}) = \Psi$, $\partial_\mu\psi(\boldsymbol{x}) = 0$ (i.e. they are not covariantly constant, $\nabla_\mu\psi \neq 0$). Furthermore we decompose the full regularized inverse propagator, $\Gamma_k^{(2)}[\bar{\Psi},\Psi] + \mathcal{R}_k(\nabla\!\!\!/) = \mathcal{F}_k[\bar{\Psi},\Psi] + \mathcal{P}_k(\nabla\!\!\!/)$, into a field-dependent part $\mathcal{F}_k[\bar{\Psi},\Psi]$,

$$\mathcal{F}_k[\bar{\Psi},\Psi] = -\frac{\bar{\lambda}_k}{N_\mathrm{f}}\begin{pmatrix} -[(\bar{\Psi}\Psi)\mathrm{I} + \Psi\bar{\Psi}] & \Psi\Psi^\mathrm{T} \\ \bar{\Psi}^\mathrm{T}\bar{\Psi} & [(\bar{\Psi}\Psi)\mathrm{I} + \Psi\bar{\Psi}]^\mathrm{T}\end{pmatrix}\mathbb{1}, \tag{K.6}$$

and a field-independent part $\mathcal{P}_k(\nabla\!\!\!/)$,

$$\mathcal{P}_k(\nabla\!\!\!/) = \begin{pmatrix} \nabla\!\!\!/\big(\mathrm{I} + r_k^\psi(\tau)\big) & 0 \\ 0 & \nabla\!\!\!/^\mathrm{T}\big(\mathrm{I} + r_k^\psi(\tau^\mathrm{T})\big)\end{pmatrix}\mathbb{1}. \tag{K.7}$$

This enables us to expand the Wetterich equation in powers of the fields,

$$\partial_k\Gamma_k[\bar{\Psi},\Psi] = \frac{\mathrm{i}}{2}\sum_{n=0}^{\infty}(-1)^n\,\mathrm{STr}\Big(\big[\mathcal{P}_k^{-1}(\nabla\!\!\!/)\,\big(\partial_k\mathcal{R}_k(\nabla\!\!\!/)\big)\big]\,\big(\mathcal{P}_k^{-1}(\nabla\!\!\!/)\mathcal{F}_k[\bar{\Psi},\Psi]\big)^n\Big). \tag{K.8}$$

The left hand-side boils down to

$$\partial_k\Gamma_k[\bar{\Psi},\Psi] = \frac{\partial_k\bar{\lambda}_k}{2N_\mathrm{f}}(\bar{\Psi}\Psi)^2\Omega, \tag{K.9}$$

where $\Omega = \int_x 1$ is the spacetime volume. Here we observe that only the term corresponding to $n=2$ in equation (K.8) can contribute to the flow of $\bar{\lambda}_k$. The first part on the right-hand side of equation (K.8) reads

$$\mathcal{P}_k^{-1}(\nabla\!\!\!/)\partial_k\mathcal{R}_k(\nabla\!\!\!/) = \frac{2}{k}\begin{pmatrix} \big(\mathrm{I}{+}r_k^\psi(\tau)\big)^{-1}\tau\, r_k^{\psi\prime}(\tau) & 0 \\ 0 & \big(\mathrm{I}{+}r_k^\psi(\tau^\mathrm{T})\big)^{-1}\tau^\mathrm{T}\, r_k^{\psi\prime}(\tau^\mathrm{T})\end{pmatrix}\mathbb{1},\quad r_k^{\psi\prime}(x) = \frac{\mathrm{d}}{\mathrm{d}x}r_k^\psi(x), \tag{K.10}$$

and therefore only the diagonal of $\big(\mathcal{P}_k^{-1}(\nabla\!\!\!/)\mathcal{F}_k[\bar{\Psi},\Psi]\big)^2$ is required. In the limit $N_\mathrm{f}\to\infty$, only the following terms remain:

$$\Big[\big(\mathcal{P}_k^{-1}(\nabla\!\!\!/)\mathcal{F}_k[\bar{\Psi},\Psi]\big)^2\Big]_{11}(\boldsymbol{x},\boldsymbol{y}) = -\frac{\bar{\lambda}_k^2}{N_\mathrm{f}^2k^2}(\bar{\Psi}\Psi)^2\Big[\tau\big(\mathrm{I} + r_k^\psi(\tau)\big)^2\Big]^{-1}\delta(\boldsymbol{x},\boldsymbol{y}), \tag{K.11}$$

$$\Big[\big(\mathcal{P}_k^{-1}(\nabla\!\!\!/)\mathcal{F}_k[\bar{\Psi},\Psi]\big)^2\Big]_{22}(\boldsymbol{x},\boldsymbol{y}) = \Big[\big(\mathcal{P}_k^{-1}\mathcal{F}_k[\bar{\Psi},\Psi]\big)^2\Big]_{11}^\mathrm{T}(\boldsymbol{y},\boldsymbol{x}). \tag{K.12}$$

Finally, the relevant part of the right-hand side of equation (K.8) yields

$$\frac{\mathrm{i}}{2}\,\mathrm{STr}\big[(\mathcal{P}_k^{-1}\partial_k R_k)(\mathcal{P}_k^{-1}\mathcal{F}_k)^2\big] = \mathrm{i}\frac{2\bar{\lambda}_k^2}{N_\mathrm{f}^2k^3}(\bar{\Psi}\Psi)^2\,\mathrm{STr}\left[\frac{r'(\tau)}{\big(\mathrm{I}+r(\tau)\big)^3}\frac{\delta(x,y)}{\sqrt{-g}}\right]. \tag{K.13}$$

Inserting the Callan-Symanzik regulator, cf. equation (7.5), we end up with equation (7.6) of the main text.

L Derivation of the Fermionic Heat Kernel on AdS$_3$

For the evaluation of the "STr" on AdS$_3$ in section 7.1.1 we need the heat kernel $\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})$ corresponding to the Laplacean $\boldsymbol{\Delta}_E = -\nabla^\mu\nabla_\mu + \frac{R}{4}\mathrm{I}$. Following [297], we choose the ansatz

$$\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s}) = f\big(\sigma(\boldsymbol{x},\boldsymbol{y}),\mathfrak{s}\big)\cdot\mathfrak{U}(\boldsymbol{x},\boldsymbol{y}), \tag{L.1}$$

where $\mathfrak{U}(\boldsymbol{x},\boldsymbol{y})$ is the parallel propagator (cf. equation (7.11)) and $f(\sigma,\mathfrak{s})$ is a scalar function of $\sigma(\boldsymbol{x},\boldsymbol{y})$ (i.e. one half the square of the geodesic distance from $\boldsymbol{x}$ to $\boldsymbol{y}$) and the proper time $\mathfrak{s}$. Plugging this ansatz into equation (7.9) while using the abbreviations $\big(w(\boldsymbol{x},\boldsymbol{y})\big)^2 = \frac{|R|\sigma(\boldsymbol{x},\boldsymbol{y})}{12}$, $n_\mu(\boldsymbol{x},\boldsymbol{y}) = \partial^{(x)}_\mu\sigma(\boldsymbol{x},\boldsymbol{y})$, $A(w) = w\coth(2w)$ and $B(w) = \frac{w}{\sqrt{8}}\tanh w$ together with the identities

$$\begin{aligned}
&n_\mu(\boldsymbol{x},\boldsymbol{y})n^\mu(\boldsymbol{x},\boldsymbol{y}) = 2\sigma(\boldsymbol{x},\boldsymbol{y}), \quad \nabla_\mu\mathfrak{U}(\boldsymbol{x},\boldsymbol{y}) = \frac{B\big(w(\boldsymbol{x},\boldsymbol{y})\big)}{2\sigma(\boldsymbol{x},\boldsymbol{y})}[\gamma_\mu(\boldsymbol{x}),\gamma_\nu(\boldsymbol{x})]n^\nu(\boldsymbol{x},\boldsymbol{y})\mathfrak{U}(\boldsymbol{x},\boldsymbol{y}),\\
&D_{(\mathrm{LC})_\mu}n_\nu(\boldsymbol{x},\boldsymbol{y}) = A\big(w(\boldsymbol{x},\boldsymbol{y})\big)g_{\mu\nu}(\boldsymbol{x}) + \frac{1-A\big(w(\boldsymbol{x},\boldsymbol{y})\big)}{2\sigma(\boldsymbol{x},\boldsymbol{y})}n_\mu(\boldsymbol{x},\boldsymbol{y})n_\nu(\boldsymbol{x},\boldsymbol{y}),
\end{aligned} \tag{L.2}$$

cf. [297], we get

$$0 = \big(3w^2 - 4B(w)^2\big)f(\sigma,\mathfrak{s}) + \big(2\sigma A(w)+\sigma\big)f'(\sigma,\mathfrak{s}) + 2\sigma^2 f''(\sigma,\mathfrak{s}) + \mathrm{i}\sigma\dot{f}(\sigma,\mathfrak{s}), \tag{L.3}$$

with $w^2 = \frac{|R|\sigma}{12}$, $f'(\sigma,\mathfrak{s}) = \partial_\sigma f(\sigma,\mathfrak{s})$ and $\dot{f}(\sigma,\mathfrak{s}) = \partial_\mathfrak{s} f(\sigma,\mathfrak{s})$. Because of the boundary condition for $\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x},\boldsymbol{y};\mathfrak{s})$ and the regularity of $\mathfrak{U}(\boldsymbol{x},\boldsymbol{y}\to\boldsymbol{x}) = \mathrm{I}$ we observe that

$$\lim_{\mathfrak{s}\searrow 0} f\big(\sigma(\boldsymbol{x},\boldsymbol{y}),\mathfrak{s}\big) = \frac{\delta^{(d)}(x^\mu - y^\mu)}{\sqrt{\mathfrak{g}}}, \tag{L.4}$$

has to hold, with $\delta^{(d)}(x^\mu - y^\mu) = \prod_{\mu=0}^{d-1}\delta(x^\mu - y^\mu)$ representing the standard delta distribution in terms of the coordinates x^μ. One suitable representation of the delta distribution in AdS$_3$ in the limit $\mathfrak{s}\searrow 0$ is, cf. equation (5.24),

$$\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{(4\pi\mathfrak{s})^{\frac{3}{2}}}\exp\left(\mathrm{i}\frac{\sigma(\boldsymbol{x},\boldsymbol{y})}{2\mathfrak{s}}\right) \overset{\mathfrak{s}\searrow 0}{\to} \frac{\delta^{(d)}(x^\mu - y^\mu)}{\sqrt{\mathfrak{g}}}. \tag{L.5}$$

Next, we factorize $f(\sigma,\mathfrak{s})$ into the "delta part" and an auxiliary function $p(\sigma,\mathfrak{s})$,

$$f(\sigma,\mathfrak{s}) = p(\sigma,\mathfrak{s})\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{(4\pi)^{\frac{3}{2}}\sqrt{\mathfrak{s}}}\exp\left(\mathrm{i}\frac{\sigma}{2s}\right). \tag{L.6}$$

Assuming that we can expand $p(\sigma, \mathfrak{s})$ in powers of $\frac{1}{\mathfrak{s}}$ and using the small $\mathfrak{s}$ behavior of $f(\sigma, \mathfrak{s})$ leads to

$$p(\sigma, \mathfrak{s}) = \frac{1}{\mathfrak{s}} p_1(\sigma) + \mathrm{i} p_0(\sigma). \tag{L.7}$$

Then we plug this into equation (L.3) and find

$$\begin{aligned}
&\text{(i)}\ 0 = -\big(1 - A(w)\big) p_1(\sigma) + 2\sigma p_1'(\sigma), \\
&\text{(ii)}\ 0 = \sigma A(w) p_0(\sigma) + 2\sigma^2 p_0'(\sigma) + \big(4B(w)^2 - 3w^2\big) p_1(\sigma) - \big(2\sigma A(w) + \sigma\big) p_1'(\sigma) - 2\sigma^2 p_1''(\sigma), \\
&\text{(iii)}\ 0 = \big(4B(w)^2 - 3w^2\big) p_0(\sigma) - \big(2\sigma A(w) + \sigma\big) p_0'(\sigma) - 2\sigma^2 p_0''(\sigma),
\end{aligned} \tag{L.8}$$

with the boundary condition $p_1(0) = 1$. From equation (i) and the boundary condition, we see

$$p_1(\sigma) = \frac{2w}{\sinh(2w)}. \tag{L.9}$$

Plugging this into (ii) gives

$$p_0(\sigma) = \frac{|R|}{12 \cosh^2 w}, \tag{L.10}$$

where we have eliminated the constant of integration using equation (iii). From this, we finally get the heat kernel,

$$\mathfrak{K}_{\boldsymbol{\Delta}_E}(\boldsymbol{x}, \boldsymbol{y}; \mathfrak{s}) = \frac{\mathrm{e}^{\mathrm{i}\frac{\sigma(\boldsymbol{x},\boldsymbol{y})}{2\mathfrak{s}}}}{(4\pi\mathfrak{s})^{\frac{3}{2}}} \left(\frac{w(\boldsymbol{x}, \boldsymbol{y})}{\sinh\big(w(\boldsymbol{x}, \boldsymbol{y})\big)} + \mathrm{i} \frac{\mathfrak{s}\,|R|}{12 \cosh\big(w(\boldsymbol{x}, \boldsymbol{y})\big)} \right) \frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{\cosh\big(w(\boldsymbol{x}, \boldsymbol{y})\big)} \mathfrak{U}(\boldsymbol{x}, \boldsymbol{y}). \tag{L.11}$$

M Curvature Expansion of β_λ on the Lobachevsky Plane

For the detailed analysis of the spatially curved case in section 7.2.2, the integral representation of the running coupling (7.25) can be studied in various limits analytically. More specifically, $\mathfrak{I}(\alpha)$ as defined in equation (7.23) can be expanded for small and for large values of α. Let us first consider an expansion of this function about $\alpha = 0$, starting with an expansion of the integrand,

$$\alpha v \coth\frac{\alpha v}{2} \simeq 2 + \frac{1}{6}(\alpha v)^2 - \frac{1}{360}(\alpha v)^4 + \mathcal{O}\big((\alpha v)^6\big). \tag{M.1}$$

Using the standard integral

$$\int_0^\infty \mathrm{d}v\, v^{2k} K_0(v) = 2^{2k-1}\, \Gamma\left(\frac{2k+1}{2}\right)^2 = \frac{\pi}{2^{2k+1}} \left(\frac{(2k)!}{k!}\right)^2, \tag{M.2}$$

the small α expansion of $\mathfrak{I}(\alpha)$ can be computed to any order. To order α^4, we find

$$\mathfrak{I}(\alpha) \simeq 1 + \frac{\alpha^2}{12} - \frac{\alpha^4}{80}. \tag{M.3}$$

Due to the factorial growth of the coefficients, cf. equation (M.2), the expansion is an asymptotic series. By comparison with the numerical result, the accuracy of equation (M.3) turns out to be above 99% up to $\alpha \simeq 1$.

A similar approximation can be done for large α by expanding

$$\coth\frac{\alpha v}{2} = \frac{1+\mathrm{e}^{-\alpha v}}{1-\mathrm{e}^{-\alpha v}} = 1 + 2\sum_{n=1}^{\infty} \mathrm{e}^{-n\alpha v}, \tag{M.4}$$

which holds for any $\alpha v > 0$. Next, we employ

$$\int_0^\infty \mathrm{d}v\, v K_0(v) \mathrm{e}^{-n\alpha v} = \frac{-1}{(n\alpha)^2 - 1} + \frac{n\alpha \operatorname{arcosh}(n\alpha)}{\big((n\alpha)^2 - 1\big)^{\frac{3}{2}}}, \tag{M.5}$$

which can be used for $n\alpha > -1$. For $-1 < n\alpha < 1$, an analytic continuation into the complex is implicitly understood, leading to a replacement of the term $\operatorname{arcosh}(n\alpha)\big((n\alpha)^2-1\big)^{-3/2}$ by $[-\arccos(n\alpha)]\big(1-(n\alpha)^2\big)^{-3/2}$ here and in the following. This leads to the convergent series

$$\mathfrak{I}(\alpha) = \frac{\alpha}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{-\alpha}{(n\alpha)^2 - 1} + \frac{n\alpha^2 \operatorname{arcosh}(n\alpha)}{\big((n\alpha)^2 - 1\big)^{\frac{3}{2}}}\right] \tag{M.6}$$

for any $\alpha \geq 0$. Neglecting orders higher than $\frac{1}{\alpha}$ and using $\operatorname{arcosh}(n\alpha) \to \ln(2n\alpha)$ for $n\alpha \to \infty$, we arrive at

$$\mathfrak{I}(\alpha) \simeq \frac{\alpha}{\pi} + \frac{\pi}{3}\frac{\ln\alpha}{\alpha} + \frac{\pi}{3}\left(1 + \gamma - \ln\frac{A^{12}}{\pi}\right)\frac{1}{\alpha}, \tag{M.7}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant and $A \approx 1.282$ is the Glaisher-Kinkelin constant. The accuracy of this result is above 99% for $\alpha > 5$.

With the series (M.6) it is possible to find a series representation for the required integral (7.33)

$$\mathfrak{F}(\alpha) = \frac{\ln\alpha}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\ln(2n\alpha) - \frac{n\alpha \operatorname{arcosh}(n\alpha)}{\sqrt{(n\alpha)^2 - 1}}\right]. \tag{M.8}$$

This series is convergent for any $\alpha > 0$. The expansions for small and large α are

$$\alpha \to 0: \quad \mathfrak{F}(\alpha) \simeq -\frac{1}{\alpha} + c + \frac{\alpha}{12} - \frac{\alpha^3}{240}, \tag{M.9}$$

$$\alpha \to \infty: \quad \mathfrak{F}(\alpha) \simeq \frac{\ln\alpha}{\pi} - \frac{\pi}{6}\frac{\ln\alpha}{\alpha^2}, \tag{M.10}$$

where c is defined as the limit

$$c = \lim_{\alpha\to 0}\left(\mathfrak{F}(\alpha) + \frac{1}{\alpha}\right) \approx 0.364. \tag{M.11}$$

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Veröffentlichungen

H. Gies, B. Knorr and S. Lippoldt,
"Generalized Parametrization Dependence in Quantum Gravity,"
Phys. Rev. D **92**, no. 8, 084020 (2015) [arXiv:1507.08859 [hep-th]]

S. Lippoldt,
"Spin-base invariance of Fermions in arbitrary dimensions,"
Phys. Rev. D **91**, no. 10, 104006 (2015) [arXiv:1502.05607 [hep-th]]

H. Gies and S. Lippoldt,
"Global surpluses of spin-base invariant fermions,"
Phys. Lett. B **743**, 415 (2015) [arXiv:1502.00918 [hep-th]]

H. Gies and S. Lippoldt,
"Fermions in gravity with local spin-base invariance,"
Phys. Rev. D **89**, no. 6, 064040 (2014) [arXiv:1310.2509 [hep-th]]

H. Gies and S. Lippoldt,
"Renormalization flow towards gravitational catalysis in the 3d Gross-Neveu model,"
Phys. Rev. D **87**, no. 10, 104026 (2013) [arXiv:1303.4253 [hep-th]]

Ausgewählte Vorträge

April 2015	"Relativity Lunch" (Gruppenseminar am Imperial College), London (Großbritannien), (eingeladen), "What fermions can tell us about quantum gravity"
März 2015	RTG Networking Workshop (zwischen dem "Graduiertenkolleg Bremen" und dem "Graduiertenkolleg Jena"), Bremen (Deutschland), "Fermions in gravity with local spin-base invariance"
Februar 2015	International Skype-Seminar on Asymptotic Safety, (eingeladen), "Global surpluses of spin-base invariant Fermions"
November 2014	"Kalter Quanten Kaffee" (Gruppenseminar der Heidelberg Graduate School of Fundamental Physics), Heidelberg (Deutschland), (eingeladen), "Fermions in gravity with local spin-base invariance"
September 2014	7th International Conference on the Exact Renormalization Group, Lefkada (Griechenland), "Fermions in gravity with local spin-base invariance"
September 2014	Conceptual and Technical Challenges for Quantum Gravity 2014, Rom (Italien), "Fermions in gravity with local spin-base invariance"
Juli 2014	Workshop of the DFG Research Group FOR 723, Functional Renormalization Group for Correlated Fermion Systems, Wien (Österreich), "Fermions in gravity with local spin-base invariance"
April 2014	Quantum Gravity Group Meeting, Perimeter Institute (Kanada), (eingeladen), "Fermions in gravity with local spin-base invariance"
April 2014	Renormalization Group Approaches to Quantum Gravity, Perimeter Institute (Kanada), "Why we don't need a vierbein"
Januar 2014	Gruppenseminar von Prof. Dr. M. Reuter, Mainz (Deutschland), (eingeladen), "Fermions in gravity with local spin-base invariance"
November 2013	International Skype-Seminar on Asymptotic Safety, (eingeladen), "Fermions in gravity with local spin-base invariance"

Februar 2013	DPG Frühjahrstagung, Jena (Deutschland), "Fermions in curved space"
Oktober 2012	Monitoring Workshop 2012 (zwischen dem "Doktoratskolleg Graz" und dem "Graduiertenkolleg Jena"), Graz (Österreich), "Fermions without Vierbeins"

Schulen und Konferenzen

April 2015	Quantum Gravity in Paris, Paris (Frankreich)
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December 2012 to date	Graduate student under supervision of Prof. Dr. H. Gies “Quantum theory of fermions in curved spacetimes”
December 2012 to date	Member of DFG research training group GRK 1523 “Quantum and Gravitational Fields”
October 2009 to date	Tutor for different under graduate courses in theoretical physics: • advanced seminar on quantum field theory in curved spacetimes (Prof. Dr. A. Wipf) • classical electrodynamics (Prof. Dr. K.-H. Lotze) • mathematical methods of physics (Prof. Dr. K.-H. Lotze) • quantum mechanics (Prof. Dr. A. Wipf) • relativistic physics (Prof. Dr. R. Meinel) • quantum field theory (Prof. Dr. M. Ammon)

Publications

H. Gies, B. Knorr and S. Lippoldt,
“Generalized Parametrization Dependence in Quantum Gravity,”
Phys. Rev. D **92**, no. 8, 084020 (2015) [arXiv:1507.08859 [hep-th]]

S. Lippoldt,
“Spin-base invariance of Fermions in arbitrary dimensions,”
Phys. Rev. D **91**, no. 10, 104006 (2015) [arXiv:1502.05607 [hep-th]]

H. Gies and S. Lippoldt,
“Global surpluses of spin-base invariant fermions,”
Phys. Lett. B **743**, 415 (2015) [arXiv:1502.00918 [hep-th]]

H. Gies and S. Lippoldt,
“Fermions in gravity with local spin-base invariance,”
Phys. Rev. D **89**, 064040 (2014) [arXiv:1310.2509 [hep-th]]

H. Gies and S. Lippoldt,
“Renormalization flow towards gravitational catalysis in the 3d Gross-Neveu model,”
Phys. Rev. D **87**, 104026 (2013) [arXiv:1303.4253 [hep-th]]

Selected Talks

April 2015	"Relativity Lunch" (group seminar at the Imperial College), London (Great Britain), (invited), "What fermions can tell us about quantum gravity"
March 2015	RTG Networking Workshop (between the "Graduiertenkolleg Bremen" and the "Graduiertenkolleg Jena"), Bremen (Germany), "Fermions in gravity with local spin-base invariance"
February 2015	International Skype-Seminar on Asymptotic Safety, (invited), "Global surpluses of spin-base invariant Fermions"
November 2014	"Cold Quantum Coffee" (group seminar of the Heidelberg Graduate School of Fundamental Physics), Heidelberg (Germany), (invited), "Fermions in gravity with local spin-base invariance"
September 2014	7th International Conference on the Exact Renormalization Group, Lefkada Island (Greece), "Fermions in gravity with local spin-base invariance"
September 2014	Conceptual and Technical Challenges for Quantum Gravity 2014, Rome (Italy), "Fermions in gravity with local spin-base invariance"
July 2014	Workshop of the DFG Research Group FOR 723, Functional Renormalization Group for Correlated Fermion Systems, Vienna (Austria), "Fermions in gravity with local spin-base invariance"
April 2014	Quantum Gravity Group Meeting, Perimeter Institute (Canada), (invited), "Fermions in gravity with local spin-base invariance"
April 2014	Renormalization Group Approaches to Quantum Gravity, Perimeter Institute (Canada), "Why we don't need a vierbein"
January 2014	Group Seminar of Prof. Dr. M. Reuter, Mainz (Germany), (invited), "Fermions in gravity with local spin-base invariance"
November 2013	International Skype-Seminar on Asymptotic Safety, (invited), "Fermions in gravity with local spin-base invariance"

February 2013	DPG Frühjahrstagung (Spring Meeting of the German Physical Society), Jena (Germany), “Fermions in curved space”
October 2012	Monitoring Workshop 2012 (between the “Doktoratskolleg Graz” and the “Graduiertenkolleg Jena”), Graz (Austria), “Fermions without Vierbeins”

Schools and Conferences

April 2015	Quantum Gravity in Paris, Paris (France)
March 2015	Probing the Fundamental Nature of Spacetime with the Renormalization Group, Stockholm (Sweden)
March 2015	RTG Networking Workshop (between the “Graduiertenkolleg Bremen” and the “Graduiertenkolleg Jena”), Bremen (Germany)
October 2014	Monitoring Workshop 2014 (between the “Doktoratskolleg Graz” and the “Graduiertenkolleg Jena”), Jena (Germany)
September 2014	7th International Conference on the Exact Renormalization Group, Lefkada Island (Greece)
September 2014	Conceptual and Technical Challenges for Quantum Gravity 2014, Rome (Italy)
July 2014	Workshop of the DFG Research Group FOR 723, Functional Renormalization Group for Correlated Fermion Systems, Vienna (Austria)
April 2014	Renormalization Group Approaches to Quantum Gravity, Perimeter Institute (Canada)
February 2014	Annual Meeting GRK 1523 “Quantum and Gravitational Fields”, Oppurg (Germany)
November 2013	Workshop on Strongly-Interacting Field Theories 2013, Jena (Germany)
October 2013	Monitoring Workshop 2013 (between the “Doktoratskolleg Graz” and the “Graduiertenkolleg Jena”), Jena (Germany)
September 2013	‘Saalburg’ Summer School on “Foundations and New Methods in Theoretical Physics”, Saalburg (Germany)

April 2013	Spring School "Relativistic Fermion Systems", Regensburg (Germany)
February 2013	DPG Frühjahrstagung (Spring Meeting of the German Physical Society), Jena (Germany)
November 2012	Workshop on Strongly-Interacting Field Theories 2012, Jena (Germany)
October 2012	Monitoring Workshop 2012 (between the "Doktoratskolleg Graz" and the "Graduiertenkolleg Jena"), Graz (Austria)

Place, Date   Signature of the Author

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbstständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

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Weitere Personen waren an der inhaltlich-materiellen Erstellung der vorliegenden Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten (Promotionsberater oder andere Personen) in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen.

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Die geltende Promotionsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt.

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Ort, Datum Unterschrift des Verfassers

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